Thesis for the Degree of Licentiate of Philosophy

Tensor products of highest weight representations and skew-symmetric matrix equations $A + B + C = 0$

Ida Säfström

CHALMERS | GÖTEBORGS UNIVERSITET

Division of Mathematics
Department of Mathematical Sciences
Chalmers University of Technology
and
University of Gothenburg
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Ida Säfström

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Department of Mathematical Sciences
Chalmers University of Technology
and
University of Gothenburg
SE-412 96 GÖTEBORG, Sweden
Phone: +46 (0)31-772 10 00

Author e-mail: `safstrom@chalmers.se`

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Abstract

The question of characterizing the eigenvalues for the sum of two Hermitian matrices, was solved in 1999, after almost a century of efforts. The saturation conjecture for $GL_{\mathbb{C}}(n)$ was proven by Knutson and Tao, filling in the last gap in Horn's conjecture. Under certain conditions, this problem is equivalent to decomposing the tensor product of two finite dimensional irreducible highest weight representations of $GL_{\mathbb{C}}(n)$.

In the first part of this thesis we use the methods of moment maps and coadjoint orbits to find equivalence between the eigenvalue problem for skew-symmetric matrices and the decomposition of tensor products of irreducible highest weight representations of $SO_{\mathbb{C}}(2k)$. We characterize the eigenvalues in the cases $k = 2, 3$, where we can take advantage of Lie algebra isomorphisms.

In the second part, we consider irreducible, infinite dimensional, unitary highest weight representations of $GL_{\mathbb{C}}(n+1)$ as representations on spaces of vector valued polynomials, and we find irreducible factors in the tensor product of two such representations.

Keywords: Highest weight respresentation, tensor product decomposition, orthogonal algebra, skew-symmetric matrix, Horn's conjecture, moment map, coadjoint orbit, flag manifold, general linear algebra, infinite dimensional unitary representation.

Tack

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Contents

Introduction

When writing any text in mathematics, one should always ask oneself who one is writing for, or in other words: what am I to assume known? Since this thesis is not a collection of articles published in specialized journals, and since it borrows results from diverse fields, I found it reasonable to assume little more than a general knowledge of Lie and representation theory, algebraic and Riemannian geometry.[1] Picturing my reader as someone with a grounding schooling in algebra and geometry - i.e. as myself when I started working on this project - or someone belonging to one field only, I will try to state and give references for proofs for all results used in what follows.

This thesis is divided into two parts where at first sight the questions may seem vastly different. Part I, investigating a skew version of Horn's conjecture from 1962, asks whether we are able to classify triples of eigenvalues belonging to skew-symmetric matrices $A + B = C$, whereas Part II obtains a set of irreducible factors in a tensor product of infinite dimensional polynomial representations of $\mathfrak{gl}_{\mathbb{C}}(n+1)$. Though, the amount of concepts needed to formulate each problem give no hint of the methods needed to solve them; you will see that Part I travels through a number of deep results from various fields of mathematics, whereas Part II mostly consists of, perhaps snarling but rather elementary, calculations.

The connection between these two parts is the point where we end up, and to some extent what is left out in the two treatments, therefore I will say a few words on this connection here. For more details, see [Olv99].

Preceding abstract algebra was the study of transformations on vector spaces. Considering some class of functions on a vector space and what happens to them under the action of a group of transformations, is known as **classical invariant theory**. More precisely, a G-invariant function on the space X, on which the group G acts, is a function

$$I : X \longrightarrow \mathbb{R} \,:\, I(g.x) = I(x) \quad \forall g \in G.$$

[1] In need of reviving these subjects, confer [FH91] or [Kna86] on representation theory, [GH78] on algebraic geometry and [Boo03] on Riemannian geometry.

Oftentimes one considers spaces of polynomials of certain degree, e.g. binary forms Q. This was what Alfred Clebsch and Paul Gordan were occupied by in the late 19th century Germany, giving names to problems as well as methods for solving them.

If G is a linear group we nowadays call the action of G on X a representation of G. Any invariant I will give rise to an invariant subspace $\{x : I(x) = 0\} \subseteq X$. In terms of representations this is a subrepresentation of X. Finding some sort of basis for the invariants, e.g. a Hilbert basis $I_1, \ldots, I_m$ for which every other invariant can be written as a polynomial $p(I_1, \ldots, I_m)$, is then closely related to finding irreducible subrepresentations of X for G.

If we already have all irreducible representations, we shift our attention to figuring out which ones occur in more complicated representations, and a natural starting point is to attack the tensor product of two irreducibles. This problem is known as the **Clebsch-Gordan problem**, and the coefficients searched for, i.e. $c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}}$ in the equation:

$$V_{\overline{\alpha}} \otimes V_{\overline{\beta}} = \bigoplus_{\overline{\gamma}} c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} V_{\overline{\gamma}}.$$

given by $c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} = \dim \operatorname{Hom}_G(V_{\overline{\gamma}}, V_{\overline{\alpha}} \otimes V_{\overline{\beta}})$, the dimension of the space of G-equivariant linear transformations, are related to the **Clebsch-Gordan coefficients**, appearing in formulas for writing products of spherical harmonics as a linear sum. A similiar question is asking for which $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$

$$(V_{\overline{\alpha}} \otimes V_{\overline{\beta}} \otimes V_{\overline{\gamma}})^G > 0.$$

In words: when the triple tensor product contains a G-invariant factor, i.e. when $c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} \neq 0$. This is the formulation preferred by Allen Knutson and Terrence Tao in [KT99]. We will revisit this triple tensor product in part I, where it shows up in the Kirwan-Ness theorem, belonging in **geometrical invariant theory**, a field where we will refer to its founder David Mumford [MFK94]. This will be the end point of part I: the reformulation of the eigenvalue problem as a Clebsch-Gordan type problem.

These problems may obviously be of varying difficulty, depending on for which Lie algebra to solve it. A popular exercise in introductory Lie theory textbooks, see e.g. [Hum72], is to consider the $\mathfrak{sl}_{\mathbb{C}}(2)$ case, by noting that every representation of this algebra can be realized as the space of homogeneous polynomials in two variables x, y, of degree equalling the highest weight of the representation. If, for example, we have a representations of highest weight α, we have the single diagonal element H acting on the variables as

$x \mapsto x$ and $y \mapsto -y$, yielding x^α as highest weight vector, and the action on general monomials to be:

$$H(x^{\alpha-k}y^k) = (\alpha - 2k)x^{\alpha-k}y^k.$$

Tensoring two representations V_α and V_β will correspond to multiplying polynomials, and it is then easily seen that:

$$V_\alpha \otimes V_\beta = V_{\alpha+\beta} \oplus V_{\alpha+\beta-2} \oplus \cdots \oplus V_{|\alpha-\beta|}.$$

The reason for lingering on this small example is for you to keep it in mind until we refer to it in the end of part I; getting there through the general setting requires quite some work.

Another well-known example is the $GL_{\mathbb{C}}(n)$ case, where the coefficients are usually called **Littlewood-Richardson coefficients**. The usual way to define them are in terms of partitions and Young diagrams. Regarding $\overline{\alpha} = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n)$, $\overline{\beta} = (\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n)$ and $\overline{\gamma} = (\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n)$ with $\sum \gamma_i = \sum \alpha_i + \sum \beta_i$ as partitions, $c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}}$ is the number of ways to fill the Young diagram $\overline{\gamma} \setminus \overline{\alpha}$ with β_i i:s, so that every row is weakly decreasing from left to right and every column is strictly decreasing from top to bottom, and so that when you have filled the boxes you can read a lattice word [2]. When filled, the diagram is called a Young tableau.

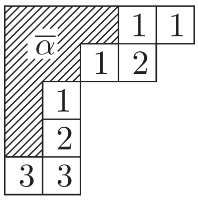


Figure 1: A possible Young tableau for $\overline{\alpha} = (3, 2, 1, 1, 0)$, $\overline{\beta} = (4, 2, 2, 0, 0)$ and $\overline{\gamma} = (5, 4, 2, 2, 2)$. The lattice word in this case is 11121233.

The Littlewood-Richardson coefficients show up in the proof of Horn's conjecture, the Hermitian eigenvalue problem, and they will pay a visit to us in an example in the end of part I. A special case will also appear in the construction of factors of the infinite representations of $\mathfrak{gl}_{\mathbb{C}}(n+1)$ in part II.

The classical problem being decomposing the tensor product of two finite dimensional representations of $GL_{\mathbb{C}}(n)$, we will in this thesis find results for problems related in two directions: finite dimensional, irreducible representations for $SO_{\mathbb{C}}(2k)$ in Part I, and infinite dimensional, irreducible representations of $GL_{\mathbb{C}}(n)$ in Part II. As the classical case may be formulated in terms

[2] This means that for any k, $1 \leq k \leq \sum \beta_i$ and i, $1 \leq i \leq n$, the i:s in the k first boxes are at least as many as the $i+1$:s.

of eigenvalues for sums of hermitian matrices, and the $SO_{\mathbb{C}}(2k)$-case in terms of eigenvalues for sums of skew-hermitian matrices, it would be interesting to see whether the infinite dimensional case of $GL_{\mathbb{C}}(n)$ is related to a eigenvalue problem, but this is yet to be investigated.

Notation

$G, H, \ldots$	groups, mostly Lie groups
$\mathfrak{g}, \mathfrak{h}$	their Lie algebras
$\mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{S}$	sets, sometimes even manifolds
$\mathcal{V}, \mathcal{W}$	vectors spaces
$\mathcal{M}(n)$	the set of matrices of dimension $n \times n$ (base field will be evident)
g, h	elements in Lie groups
A, B, C, X, Y	elements/matrices in Lie algebras
$\mathbb{u}, \mathbb{v}, \mathbb{x}, \mathbb{z}, \ldots$	column vectors
$\mathbb{v}^{\mathsf{T}}$	the transpose of $\mathbb{v}$
$\mathbb{v}^*$	the conjugate transpose of $\mathbb{v}$
$\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \underline{k}, \underline{l}$	eigenvalues, eigen-k-tuples, weights
0	the zero vector, the zero matrix
$\mathbb{e}_i$	the i:th basis vector of a vector space
I_n	the $n \times n$ identity matrix
E_{ij}	the matrix in $\mathcal{M}(n)$ with 1 as (i, j)-entry and zero otherwise
$\mathcal{E}_{ij}$	the dual basis
$\operatorname{diag}(a_i)$	the (block) diagonal matrix with entries (or blocks) a_i down the diagonal

Part I

Chapter 1

Prerequisites

A considerable amount is written on problems relating to Horn's conjecture, one way to start reading is [Ful00], and one of the main contributions is made by [Knu00], and [KT99] by proving the saturation conjecture, formulated in section 2.3. They hence managed to prove Horn's classification of triples of eigenvalues $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ belonging to hermitian matrices $A + B = C$. Various generalizations have been made after their articles, but one quite natural question has not been specifically handled: what happens in the skew-symmetric case?

Denoting the dimension of the matrices by n, it is obvious that the set of skew triples $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$, $\mathcal{E}^n_{ss}$, will be a subset of the hermitian set $\mathcal{E}^n_{he}$ (simply multiply the matrices by i). When it comes to real, skew-symmetric matrices of even dimension $n = 2k$, we know that the eigenvalues are of the form $(\pm i\alpha_1, \ldots, \pm i\alpha_k)$, where α_i are real, positive numbers, and therefore we will often consider the k-tuple $(\alpha_1, \ldots, \alpha_k)$ rather that the set of eigenvalues. We will refer to this k-tuple as the **eigen-k-tuple** of the skew-symmetric matrix A.

Knowing this, we may add the obvious condition that each set of eigenvalues must have elements which are pairwise zero, writing:

$$\mathcal{E}^n_{ss} \subseteq (\mathcal{E}^n_{he})_{ss}.$$

We readily realize that these two sets are not generally the same by taking a look at $n = 2$:

$$\{(\alpha, \beta, |\alpha \pm \beta|)\} \subsetneq \{(\alpha, \beta, \gamma) : |\alpha - \beta| \leq \gamma \leq \alpha + \beta\}.$$

When n is odd, 0 is an eigenvalue for every skew-symmetric matrix, and the others are pairwise zero. A basic calculation shows that in the first interesting case, $n = 3$, we have equality of the sets:

$$\mathcal{E}^3_{ss} = (\mathcal{E}^3_{he})_{ss}.$$

In the following text, however, we will concentrate on the even dimensional case, and already when $n = 4$ any attempt to get result from elementary calculations seems fruitless. We need to turn to the language and methods of [Knu00], and to be able to use these, we need to formalize our setting.

Let $n = 2k$ and $\mathcal{M}_{ss}(n)$ be the set of $n \times n$-dimensional, skew-symmetric matrices. For any matrix $A \in \mathcal{M}_{ss}(n)$ we have a unique eigen-k-tuple $\overline{\alpha} = (\alpha_1 \geq \cdots \geq \alpha_k)$, such that $\pm i(\alpha_1 \geq \cdots \geq \alpha_k)$ are the eigenvalues of A. On the other hand, for every such $\overline{\alpha}$ we have a set of skew-symmetric matrices:

$$\mathcal{O}_{\overline{\alpha}} := \{A \in \mathcal{M}_{ss}(n) : \mathrm{eig}(A) = \pm i\overline{\alpha}\}.$$

A representative element in this set is the block-diagonal matrix

$$B_{\overline{\alpha}} := \begin{pmatrix} \begin{smallmatrix} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{smallmatrix} 0 & \alpha_k \\ -\alpha_k & 0 \end{smallmatrix} \end{pmatrix},$$

since every matrix in $A \in \mathcal{O}_{\overline{\alpha}}$ can be expressed as $gB_{\overline{\alpha}}g^{\intercal}$ for some $g \in O(2k)$, the set of real orthogonal matrices of dimension $2k \times 2k$. It is then clear that:

$$\mathcal{O}_{\overline{\alpha}} \cong O(2k)/S(B_{\overline{\alpha}}),$$

where $S(B_{\overline{\alpha}}) = \{g \in O(2k) : gB_{\overline{\alpha}}g^{\intercal} = B_{\overline{\alpha}}\}$, the stabilizer subgroup of $O(2k)$ for $B_{\overline{\alpha}}$, with respect to the conjugate action of $O(2k)$ on our matrices.

The plan, following [Knu00] closely, is now to identify the set $\mathcal{O}_{\overline{\alpha}}$ as, on one hand, a symplectic manifold, and on the other hand a complex flag manifold which we can embed in projective space, so that we by use of Kirwan-Ness's theorem on equivalence between symplectic and geometric invariant quotients, can relate the problem concerning eigenvalues of sums of matrices to tensor product decomposition of representations of $SO_{\mathbb{C}}(2k)$.

In chapter 2 we will study the general case, and in section 2.4 what further conclusions can be drawn in some special cases, but before we begin we will introduce some necessary concepts and results from a variety of areas: symplectic geometry, complex algebraic geometry and representation theory.

1.1 Symplectic manifolds and moment maps

We are all familiar with Riemannian geometry and the notion of symmetric inner products on the tangent spaces of manifolds, for which we often require positive definiteness. In symplectic geometry, however, we instead have an

antisymmetric inner product ω, for which such a property is impossible. We then rather require ω to be **nondegenerate**, i.e. for every tangent vector $\mathbb{v}_1$ there is an other tangent vector $\mathbb{v}_2$ such that $\omega(\mathbb{v}_1, \mathbb{v}_2) \neq 0$.

Example 1.1. If $n = 2k$ we can equip $\mathbb{R}^n$ (being its own tangent space) with the standard nondegenerate antisymmetric form ω_k defined by $\omega_k(\mathbb{e}_i, \mathbb{e}_j) = \omega_k(\mathbb{e}_{k+i}, \mathbb{e}_{k+j}) = 0$ and $\omega_k(\mathbb{e}_i, \mathbb{e}_{k+j}) = -\omega_k(\mathbb{e}_{k+j}, \mathbb{e}_i) = \delta_{ij}$. In other words:

$$\omega_k = \sum_{i=1}^{k} dx_i \wedge dx_{k+i}.$$

It is then natural to define symplectic manifolds as follows:

Definition 1.1. Let $(\mathcal{M}, \omega)$ be a pair consisting of a manifold $\mathcal{M}$ and a nondegenerate, antisymmetric 2-form ω. If it is locally isomorphic to the pair $(\mathbb{R}^{2k}, \omega_k)$, for some k, then $\mathcal{M}$ is called a **symplectic manifold**, and ω is called its **symplectic form**.

This definition is equivalent to the classical formulation ([Ber01] Definition 2.1) by Darboux's theorem ([Ber01] Corollary 2.7). For any function f on a symplectic manifold, we may define a gradient:

Definition 1.2. The vector field S_f, associated to the function f, which is uniquely defined by

$$\partial_{\mathbb{v}(\mathbb{m})} f = \omega(\mathbb{v}(\mathbb{m}), S_f(\mathbb{m})) \quad \forall \mathbb{m} \in \mathcal{M}, \mathbb{v}(\mathbb{m}) \in T(\mathbb{m}),$$

is called the **symplectic gradient** of f.

If we have a smooth action of a Lie group G on a symplectic manifold $(\mathcal{M}, \omega)$, we have each element X in the Lie algebra $\mathfrak{g}$ of G, giving a vector field $\tilde{X}$ on $\mathcal{M}$, by $\tilde{X}(\mathbb{m}) = \frac{d}{dt}(\exp(tX))\mathbb{m}$. Letting $\{\, ,\, \}$ denote the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$, we are able to introduce the next, useful concept.

Definition 1.3. A map $\Phi : \mathcal{M} \to \mathfrak{g}^*$ is a **moment map** for the action of G on $\mathcal{M}$ if

1. $\forall g \in G, \mathbb{m} \in \mathcal{M}, \Phi(g\mathbb{m}) = Ad(g)(\Phi(\mathbb{m}))$ (*equivariance of* Φ)
2. $\forall X \in \mathfrak{g}\ S_{\{X,\Phi\}} = \tilde{X}$

In this case, the G-action is called a **Hamiltonian action**, and $\mathcal{M}$ is called a **Hamiltonian G-manifold**.

For moment maps, there is a number of nice results, which will assist us later. We start by three small facts:

Fact 1.1. ([GS84b] Proposition 25.2) If $\mathcal{M} = Ad^*(G)B$ is a coadjoint orbit for some $B \in \mathfrak{g}^*$, i.e. an orbit of the group G:s action on $\mathfrak{g}^*$, then $\mathcal{M}$ is a symplectic G-manifold, and its moment map is the inclusion $\mathcal{M} \hookrightarrow \mathfrak{g}^*$.

Example 1.2. ([Ber01] 2.6 iv) $\mathbb{P}^n_{\mathbb{C}}$ is a coadjoint orbit for $SU(n+1)$. $\mathfrak{su}(n+1)$ is the Lie algebra of traceless, skew-hermitian matrices ($A^* = -A$, $\mathrm{tr}A = 0$), and its dual $\mathfrak{su}^*(n+1)$ is the Lie algebra of traceless Hermitian matrices. For $B = \begin{pmatrix} \frac{1}{n}I_n & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{su}^*(n+1)$, the coadjoint orbit $Ad^*(SU(n+1))B = SU(n+1)/U(n) = \mathbb{P}^n_{\mathbb{C}}$. We may also regard $\mathbb{P}^n_{\mathbb{C}}$ as the coadjoint orbit

$$Ad^*(U(n+1))\mathrm{diag}(1, 0, \ldots, 0) = U(n+1)/(U(1) \times U(n)).$$

As a Hamiltonian $U(n+1)$-manifold, it has moment map:

$$[\mathbb{v}] \longmapsto \text{ the rank one projection onto } \mathbb{C}\mathbb{v}, \tag{1.1}$$

since $\mathfrak{u}^*(n+1)$ is the space of Hermitian matrices, with possibly non-zero trace.

Fact 1.2. If $\mathcal{M}$ is a Hamiltonian G-manifold with moment map Φ_G, and we have a Lie group homomorphism $\rho : H \to G$, then the induced action of H on $\mathcal{M}$ is also Hamiltonian, and its moment map is $\Phi_H = (d\rho)^* \circ \Phi_G$.

Example 1.3. When ρ is the inclusion of a subgroup H, the moment map will be surjective.

Fact 1.3. If $\mathcal{M}$ is a Hamiltonian G-manifold with moment map Φ and $\mathcal{N}$ a Hamiltonian H-manifold with moment map Ψ, then the direct sum $\mathcal{M} \times \mathcal{N}$ is a Hamiltonian $G \times H$-manifold, with moment map $\Phi \times \Psi$.

We also need a few somewhat deeper results, beginning with the Atiyah-Guillemin-Sternberg theorem and Kirwan's theorem:

Theorem 1.1. *If T is a torus, and $\mathcal{M}$ a compact, connected Hamiltonian T-manifold with moment map $\Phi : \mathcal{M} \to \mathfrak{t}^*$, then the image of Φ is a certain convex polytope, namely the convex hull of the image of the T-fixed points on $\mathcal{M}$.*

Example 1.4. The $U(n)$-case is known as the Schur-Horn theorem, stating that the image of the map taking a Hermitian matrix with spectrum $\overline{\alpha}$ to its diagonal entries, is the convex hull of the $n!$ permutations of $\overline{\alpha}$. See further [Knu00].

Theorem 1.2. *Let $\mathcal{M}$ be compact, connected Hamiltonian G-manifold with moment map Φ. The image of the composite map $\Psi \circ \Phi$, where $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*_+$ sends an element to the unique point in its G-orbit lying inside the positive Weyl chamber $\mathfrak{g}^*_+$, is a convex polytope.*

In [MFK94] these are stated together as Theorem 8.9, where there is also references for proofs, e.g. [GS82] and [GS84a]. Since Kirwan's theorem tells nothing of the extreme points of the convex polytope in question, we fill in with the following proposition:

Proposition 1.1. *Let $\mathcal{M}$ be a Hamiltonian G-manifold with moment map $\Phi : \mathcal{M} \to \mathfrak{g}^*$. For every boundary point to Im Φ, $P \in \mathfrak{g}^*$, the fiber $\Phi^{-1}(P)$ is stabilized by at least a circle subgroup of G.*

Proof. See corollary in section 2.4 of [Knu00] . □

For any Lie group G and an element $X \in \mathfrak{g}^*$, the isotropy group $G_X = \{g \in G | \mathrm{Ad}^*(g^{-1})X = X\}$ is a closed subgroup of G, and hence a Lie group itself; especially $G_0 = G$. For a moment map Φ for some action of G on a manifold $\mathcal{M}$, it is then meaningful to define the **symplectic quotient** (also called symplectic reduction and Marsden-Weinstein quotient) as

$$\Phi^{-1}(X)/G_X.$$

These quotients are themselves symplectic manifolds ([Ber01] Theorem 4.24), and we have the following theorem due to Kirwan and Ness:

Theorem 1.3. *Let G be a compact subgroup of $U(n)$ acting on a graded ring $R = \mathbb{C}[x_1, \ldots, x_n]/I$ for some ideal I, where all generators x_i are of degree 1. Then Proj R is a subvariety of $\mathbb{P}^{n-1}_{\mathbb{C}}$, and if it is smooth even a Hamiltonian G-manifold, with moment map:*

$$\Phi : \mathit{Proj}\, R \hookrightarrow \mathbb{P}^{n-1}_{\mathbb{C}} \to \mathfrak{u}^*(n) \to \mathfrak{g}^*,$$

and we have a natural identification between the symplectic and the geometric invariant quotients, i.e.

$$\Phi^{-1}(0)/G \cong \mathit{Proj}\,(R^{G^{\mathbb{C}}}),$$

$R^{G^{\mathbb{C}}}$ being the $G^{\mathbb{C}}$-invariant polynomials in R.

Proof. Note that $\mathbb{P}^{n-1}_{\mathbb{C}} \longrightarrow \mathfrak{g}^*$ is a moment map by example 1.1 and Fact 1.1. The corresponding sympletic structure will be inherited by *Proj* R. The full proof is found in [MFK94] (Theorem 8.3). □

We let this theorem end this section, as the presence of *Proj* leads us naturally into the next.

1.2 Projective varieties

Later on, in Chapter 2, we will need to deal with flag manifolds as complex, projective varieties, therefore we will prepare ourselves by discussing a few things about varieties.

If we let $\mathbb{P}(\mathcal{V})$ be the projective space of a finite dimensional vector space $\mathcal{V}$, the symmetric algebra

$$\mathrm{Sym}^{\bullet}\mathcal{V}^* = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i \mathcal{V}^*,$$

is the algebra of polynomial forms on $\mathbb{P}(\mathcal{V})$, the so called **homogeneous coordinate ring** of $\mathbb{P}(\mathcal{V})$. If $\mathcal{X}$ is a subvariety of $\mathbb{P}(\mathcal{V})$ with ideal $I(\mathcal{X})$, $\mathrm{Sym}^{\bullet}\mathcal{V}^*/I(\mathcal{X})$ is the homogeneous coordinate ring of $\mathcal{X}$.

For a graded ring $R = \bigoplus_{i\geq 0} R_i$, the set $Proj\, R$ of homogeneous prime ideals in R not containing $R_+ = \bigoplus_{i>0} R_i$, is a variety. Though there exists much more structure on $Proj\, R$, we leave it be, as it is not needed here. We simply need to note that if we have a variety $\mathcal{X}$ with homogeneous coordinate ring R, we may write $\mathcal{X} = Proj\, R$.[1]

Over $\mathbb{P}(\mathcal{V})$ we may define the tautological line bundle $\mathcal{O}_{\mathcal{V}}(-1)$, by letting the fibre of a point in $\mathbb{P}(\mathcal{V})$ be the line in $\mathcal{V}$ defining it. If we rather let the fibre over $[L]$ be the dual line L^*, we get the line bundle $\mathcal{O}_{\mathcal{V}}(1)$. The holomorphic sections of this bundle are defined by:

$$[L] \longmapsto \mathbb{v}_L^* = \mathbb{v}^*\big|_L \text{ for some } \mathbb{v}^* \in \mathcal{V}^*,$$

which are merely the coordinates on $\mathbb{P}(\mathcal{V})$. If we restrict $\mathcal{O}_{\mathcal{V}}(1)$ to a subvariety $\mathcal{X}$ of $\mathbb{P}(\mathcal{V})$, the section of the restricted bundle will be coordinates on $\mathcal{X}$, i.e. the space of holomorphic sections $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) = \mathrm{Sym}^1\mathcal{V}/I(X)$.

If we denote by $\mathcal{O}_{\mathcal{V}}(n)$ the n:th tensor power of $\mathcal{O}_{\mathcal{V}}(1)$, we have that for a subvariety $\mathcal{X} \in \mathbb{P}(\mathcal{V})$, where $\mathcal{V} = \otimes_{i=1}^k \mathrm{Sym}^{a_i}\mathcal{V}_i$,

$$\mathcal{O}_{\mathcal{X}}(1) = (pr_1)^*\mathcal{O}_{\mathcal{V}_1}(a_1) \otimes \cdots \otimes (pr_k)^*\mathcal{O}_{\mathcal{V}_k}(a_k),$$

where pr_i is the projection from $\mathcal{X}$ to the i:th factor. To stress the specific embedding, we call this bundle $\mathcal{O}_{\mathcal{X}}(a_1, \ldots, a_k)$.

[1] Going in to details, it is in fact a bit more complicated: If $\mathcal{X}$ is a projective variety with homogeneous coordinate ring R, *the scheme* associated to $\mathcal{X}$ via the categorial functor from varieties to schemes, is isomorphic to $Proj\, R$. See [Har77] Exercise 2.14 d. For our purposes though, we may regard them as the same.

1.3 Homogeneous flag manifolds

Remember that if we decompose a Lie algebra $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$ by its root spaces, i.e. $\mathfrak{h}$ is the Cartan subalgebra, and $\mathfrak{n}^+$ is the spaces corresponding to positive roots, then every highest weight representation of $\mathfrak{g}$ gives a character on the Borel subgroup $B = HN^+ = \exp(\mathfrak{h} + \mathfrak{n}^+)$. Assume

$$\tau : \mathfrak{g} \longrightarrow \mathrm{End}(\mathcal{V}_{\overline{\alpha}})$$

to be an irreducible representation of highest weight $\overline{\alpha}$. We have $\overline{\alpha} : \mathfrak{h} \to \mathbb{C}$, and hence the character:

$$\chi_{\overline{\alpha}} = e^{\overline{\alpha}} : H \longrightarrow \mathbb{C}^\times.$$

We may extend it to the Borel subgroup by letting $\chi_{\overline{\alpha}}$ act trivially on N^+.

Furthermore, the character gives rise to a line bundle $\mathcal{O}(\overline{\alpha})$ on the homogeneous space G/B, for which the fibre over the point gB is the line $\mathbb{C}\chi_{-\overline{\alpha}}$, i.e. the line defining the point. By definition, G acts on the space of holomorphic sections of this bundle, denoted $\Gamma(G/B, \mathcal{O}(\overline{\alpha}))$. We have the Borel-Weil theorem:

Theorem 1.4. *([Kna86] Theorem 0.9) For a complex, compact, connected Lie group G with Borel subgroup B, the irreducible representation of highest weight $\overline{\alpha}$ is realized as the space of holomorphic sections of the line bundle given by $\overline{\alpha}$, i.e.*

$$\Gamma(G/B, \mathcal{O}(\overline{\alpha})) \cong \mathcal{V}_{\overline{\alpha}}.$$

Remark. The mapping $\mathcal{V}_{\overline{\alpha}} \to \Gamma(G/B, \mathcal{O}(\overline{\alpha}))$ is not hard to find. From [Kna86] we know that the sections are the holomorphic functions $f : G \to \mathbb{C}$ such that $f(gb) = \chi_{\overline{\alpha}}(b^{-1})f(g)\ \forall g \in G, b \in B$. Given any representation $(\mathcal{V}, \tau)$ of G, there is a contragredient representation $(\mathcal{V}^*, \tau^*)$ on the dual space, by:

$$(\tau^*(g)f)(\mathbb{v}) = f(\tau(g)^{-1}\mathbb{v}) \quad \forall f \in \mathcal{V}^*, \mathbb{v} \in \mathcal{V}.$$

If $(\mathcal{V}, \tau)$ is a highest weight representation, then also $(\mathcal{V}^*, \tau^*)$ is a highest weight representation, with highest weight vector, say $\mathbb{v}_\tau^*$. Thus, to any $\mathbb{v} \in \mathcal{V}_{\overline{\alpha}}$ we can associate the holomorphic function

$$f_{\mathbb{v}} : g \longmapsto (\tau^*(g)\mathbb{v}_\tau^*)\mathbb{v}.$$

Borel-Weil says that the map $\mathbb{v} \mapsto f_{\mathbb{v}}$ is, in fact, surjective.

1.4 Lie theory for the orthogonal algebra

This section follows Lecture 18 in [FH91].

Let us start with the Lie group $SO_{\mathbb{C}}(2k)$. The usual way to describe this group is to say that it consists of the special orthogonal matrices, i.e. matrices M such that $\det(M) = 1$ and $M^{\intercal}M = MM^{\intercal} = I_{2k}$, in other words: matrices preserving the bilinear form $\mathbb{x}^{\intercal}\mathbb{y}$ on $\mathbb{C}^{2k}$. However, we could also give a more general description, saying that $SO_{\mathbb{C}}(2k)$ consists of the matrices preserving a given nondegenerate, symmetric bilinear form Q, letting us specify Q as is convenient:

$$SO_{\mathbb{C}}(2k, Q) = \{M \in \mathcal{M}(2k) | Q(M\mathbb{x}, M\mathbb{y}) = Q(\mathbb{x}, \mathbb{y}), \det(M) = 1\}.$$

All the groups $SO_{\mathbb{C}}(2k, Q)$ are conjugate, and hence isomorphic to each other, therefore we simply denote them $SO_{\mathbb{C}}(2k)$, independently of choice of Q.

The Lie algebra of $SO_{\mathbb{C}}(2k)$, $\mathfrak{so}_{\mathbb{C}}(2k) = \mathfrak{o}_{\mathbb{C}}(2k)$, is then the set of matrices such that:

$$Q(A\mathbb{x}, \mathbb{y}) + Q(\mathbb{x}, A\mathbb{y}) = 0.$$

Already when we find the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}_{\mathbb{C}}(2k)$, we realize that we can choose a more feasible Q than the standard one, since with the standard Q, $\mathfrak{h}$ is spanned by the matrices $E_{i,i+1} - E_{i+1,i}$, $i = 1, 3, \ldots, 2k-1$. Usually we have the diagonal matrices forming the Cartan subalgebra, but in this case, of course, there are no diagonal matrices. Therefore, we come up with the idea of choosing Q in the following way:

$$Q(\mathbb{x}, \mathbb{y}) = \mathbb{x}^{\intercal} \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix} \mathbb{y},$$

yielding $\mathfrak{so}_{\mathbb{C}}(2k)$ to be matrices of the form:

$$\begin{pmatrix} A & B \\ C & -A^{\intercal} \end{pmatrix} \text{ with } B, C \in \mathcal{M}_{ss}(k),$$

which indeed includes diagonal matrices. The Cartan subalgebra $\mathfrak{h}$ is then generated by the matrices $H_i = E_{ii} - E_{i+k,i+k}$, for $i = 1, \ldots, k$. The dual basis for $\mathfrak{h}^*$ we denote $\{\mathcal{H}_i\}$. This implies that we have three types of root spaces of $\mathfrak{so}_{\mathbb{C}}(2k)$, characterized by which type of matrix they are generated by:

$$\begin{aligned} &E_{ij} - E_{k+i,k+j} \text{ with root } \mathcal{H}_i - \mathcal{H}_j \\ &E_{i,k+j} - E_{j,k+i} \text{ with root } \mathcal{H}_i + \mathcal{H}_j \\ &E_{k+i,j} - E_{k+j,i} \text{ with root } -\mathcal{H}_i - \mathcal{H}_j. \end{aligned}$$

Choosing as positive roots:

$$R^+ = \{\mathcal{H}_i + \mathcal{H}_j\} \cup \{\mathcal{H}_i - \mathcal{H}_j\} \quad i < j,$$

and corresponding simple roots:

$$\mathcal{H}_1 - \mathcal{H}_2, \mathcal{H}_2 - \mathcal{H}_3 \ldots, \mathcal{H}_{k-1} - \mathcal{H}_k, \mathcal{H}_{k-1} + \mathcal{H}_k,$$

the positive Weyl chamber is the set:

$$\{\sum_{i=1}^{k} a_i \mathcal{H}_i | a_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq |a_k|\}.$$

Chapter 2

Sums of skew-symmetric matrices

As we saw in the introduction, the set of skew-symmetric matrices sharing a certain eigen-k-tuple could be seen as a quotient between the Lie group $O(2k)$ and the stabilizer subgroup $S(B_{\overline{\alpha}})$. From now on we will always assume the eigen-k-tuple (or equally the eigenvalues) to be without repetitions, i.e. $\overline{\alpha} = (\alpha_1 > \cdots > \alpha_k > 0)$. In this setting we have the following lemma:

Lemma 2.1. *For $\overline{\alpha} = (\alpha_1 > \cdots > \alpha_k > 0)$, the stabilizer subgroup $S(B_{\overline{\alpha}})$ is $SO(2)^k$.*

Proof. First observe that for any $g \in SO(2)^k$, $gB_{\overline{\alpha}}g^{\intercal} = B_{\overline{\alpha}}$, i.e. $SO(2)^k \subseteq S(B_{\overline{\alpha}})$.

On the other hand, the condition for $g \in O(2k)$ to lie in $S(B_{\overline{\alpha}})$, can be written as $L := gB_{\overline{\alpha}} = B_{\overline{\alpha}}g =: R$. For $1 \leq i, j \leq k$, denote the 2×2-submatrices of L and R as follows:

$$L_{ij} = \begin{pmatrix} l_{2i-1,2j-1} & l_{2i-1,2j} \\ l_{2i,2j-1} & l_{2i,2j} \end{pmatrix} \text{ and } R_{ij} = \begin{pmatrix} r_{2i-1,2j-1} & r_{2i-1,2j} \\ r_{2i,2j-1} & r_{2i,2j} \end{pmatrix}.$$

Because of $B_{\overline{\alpha}}$:s block-diagonal form, we realize that for $g = (g_{ij})$ we have:

$$L_{ij} = \alpha_j \begin{pmatrix} -g_{2i-1,2j} & g_{2i-1,2j-1} \\ -g_{2i,2j} & g_{2i,2j-1} \end{pmatrix} \text{ and } R_{ij} = \alpha_i \begin{pmatrix} g_{2i,2j-1} & g_{2i,2j} \\ -g_{2i-1,2j-1} & -g_{2i-1,2j} \end{pmatrix},$$

and hence $L = R$ boils down to the equations:

$$\begin{aligned} -\alpha_j^2 g_{2i-1,2j} = \alpha_i\alpha_j g_{2i,2j-1} = -\alpha_i^2 g_{2i-1,2j} \\ \alpha_j^2 g_{2i-1,2j-1} = \alpha_i\alpha_j g_{2i,2j} = \alpha_i^2 g_{2i-1,2j-1}. \end{aligned}$$

Since $\overline{\alpha}$ has no repeated elements, this means that when $i \neq j$, $g_{ij} = 0$, and the diagonal blocks g_{ii} are of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Since $g \in O(2k)$ every such block must have determinant 1, and hence $g \in SO(2)^k$, and $S(B_{\overline{\alpha}}) \subseteq SO(2)^k$ □

We are now ready to change to the symplectic geometry language and, with the help of some facts from section 1.1, prove the following theorem:

Theorem 2.1. *$\mathcal{O}_{\overline{\alpha}}$ is a symplectic manifold.*

Proof. We have already seen that $\mathcal{O}_{\overline{\alpha}}$ is an orbit of the adjoint action of $O(2k)$ on the element $B_{\overline{\alpha}}$, which lies in the Lie algebra of $O(2k)$, $\mathfrak{o}(2k)$. We write this as:

$$Ad(O(2k))B_{\overline{\alpha}}.$$

Since we have the identification $\mathfrak{o}(2k) = \mathfrak{o}^*(2k)$, given by the pairing

$$\text{a linear functional } \tau \in \mathfrak{o}^*(2k) \longleftrightarrow \text{a matrix } Y \in \mathfrak{o}(2k), \tag{2.1}$$

where Y is the matrix defined by $\tau(X) = \mathrm{tr}(YX)$ for any $X \in \mathfrak{o}(2k)$, we may see $B_{\overline{\alpha}}$ as an element of $\mathfrak{o}^*(2k)$. Furthermore, we realize that the operator Ad^*, being defined by the condition:

$$\mathrm{tr}(Ad^*(g)(Y)X) = \mathrm{tr}(Y Ad(g^{-1})X),$$

is merely the usual Ad, since:

$$\mathrm{tr}(Y Ad(g^{-1})X) = \mathrm{tr}(Yg^{-1}Xg) = tr(Ad(g)(Y)X).$$

This means that $\mathcal{O}_{\overline{\alpha}}$ is the coadjoint orbit $Ad^*(O(2k))B_{\overline{\alpha}}$, and as such a symplectic manifold by Fact 1.1. □

Describing our set as a symplectic manifold in this way seems good, but as we saw in section 1.1, many results in symplectic geometry merely holds for connected manifolds, and since $O(2k)$ is not connected, neither is our manifold. Therefore, it is better to describe it as a union of orbits for a connected group; a given candidate being $SO(2k)$, one of the connected components of $O(2k)$ with the same Lie algebra: $\mathfrak{so}^*(2k) = \mathfrak{o}(2k)$. We note that:

$$\mathcal{O}_{\overline{\alpha}} = Ad(SO(2k))B_{\overline{\alpha}} \cup Ad(SO(2k))\hat{I}B_{\overline{\alpha}}\hat{I} =: \mathcal{O}^+_{\overline{\alpha}} \cup \mathcal{O}^-_{\overline{\alpha}}, \tag{2.2}$$

where $\hat{I} = \text{diag}(1, \dots, 1, -1)$. This union will prove itself useful, so we will stick to this description of $\mathcal{O}_{\overline{\alpha}}$ from hereon.

Glancing at AGS (Theorem 1.1), we see that these two coadjoint orbits, being Hamiltonian $SO(2k)$-manifolds, will also be H-manifolds for any subgroup H of $SO(2k)$ by Fact 1.2. Especially this holds for the torus $T = SO(2)^k$. To be able to formulate our precise version of AGS, we need to find the fix points of this torus. We have the lemma:

Lemma 2.2. *Let $SO(2)^k$ act on $\mathcal{O}_{\overline{\alpha}} = \mathcal{O}^+_{\overline{\alpha}} \cup \mathcal{O}^-_{\overline{\alpha}}$ by the conjugate action. The fix points of this action are 2×2-block diagonal matrices with blocks of type:*

$$B_i^+ = \begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix} \quad \text{or} \quad B_i^- = \begin{pmatrix} 0 & -\alpha_i \\ \alpha_i & 0 \end{pmatrix}.$$

Furthermore, if A is a fix point belonging to $\mathcal{O}^+_{\overline{\alpha}}$, the number of B_i^--blocks in A is even, and if $A \in \mathcal{O}^-_{\overline{\alpha}}$, the number of B_i^--blocks in A is odd.

Proof. First we prove that any matrix $A = (a_{ij}) \in \mathcal{M}_{ss}(2k)$ that is fixed by the conjugate action of $SO(2)^k$, has to be 2×2-block diagonal. Every element in $SO(2)^k$ is on the form

$$T = \text{diag}\left(\begin{pmatrix} \cos\vartheta_i & -\sin\vartheta_i \\ \sin\vartheta_i & \cos\vartheta_i \end{pmatrix}\right).$$

As in the proof of Lemma 2.1, we prefer to work with the equality $TA = AT$, which yields four categories of equations, corresponding to the four entries in every 2×2-block. To have any chance to overview these equations we substitute:

$$\begin{pmatrix} a_{2i-1,2j-1} & a_{2i-1,2j} \\ a_{2i,2j-1} & a_{2i,2j} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and get the four equations:

$$\begin{aligned} a\cos\vartheta_i - c\sin\vartheta_i &= a\cos\vartheta_j + b\sin\vartheta_j \\ b\cos\vartheta_i - d\sin\vartheta_i &= -a\sin\vartheta_j + b\cos\vartheta_j \\ a\sin\vartheta_i + c\cos\vartheta_i &= c\cos\vartheta_j + d\sin\vartheta_j \\ b\sin\vartheta_i + d\cos\vartheta_i &= -c\sin\vartheta_j + d\cos\vartheta_j, \end{aligned}$$

which are to hold for every pair ϑ_i, ϑ_j. If we first consider the off-diagonal blocks, we can thus choose e.g. $\vartheta_i = 0$ and $\vartheta_j = \pi$, giving:

$$a = -a \quad b = -b \quad c = -c \quad d = -d,$$

meaning that every off-diagonal block in A must be zero. On the other hand, for the diagonal blocks $i = j$ and $a = d = 0$, $c = -b$, leaving the four equations true for all ϑ_i, and we see that the fix points of the conjugate action of $SO(2k)$ are precisely the 2×2-block diagonal matrices.

Returning to our specific setting, the only block diagonal matrices possibly lying in $\mathcal{O}^{+}_{\overline{\alpha}}$ or $\mathcal{O}^{-}_{\overline{\alpha}}$, are those with some permutation of the blocks $\left(\begin{smallmatrix} 0 & \pm\alpha_i \\ \mp\alpha_i & 0 \end{smallmatrix}\right)$ down the diagonal. To prove the second statement, we would like to find out which of these lies in which set, i.e. given a fixpoint $A = PB_{\overline{\alpha}}P^{\intercal}$ for $P \in O(2k)$, is the determinant of P positive or negative?

First note that any P permuting the order of the blocks in A will be of determinant 1, thus we can assume the blocks to be in order. If we consider a matrix A with a number of blocks flipped, i.e. with $\left(\begin{smallmatrix} 0 & -\alpha_i \\ \alpha_i & 0 \end{smallmatrix}\right)$ instead of $\left(\begin{smallmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{smallmatrix}\right)$ for $i \in S \subseteq \{1, \ldots, k\}$, we realize that the matrix P differing from the identity matrix by replacing the i:th $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $i \in S$ will fulfill $A = PB_{\overline{\alpha}}P^{\intercal}$. Every $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ contributes to the determinant by changing its sign, hence an even number of flips places $A \in \mathcal{O}^{+}_{\overline{\alpha}}$, and an odd number places $A \in \mathcal{O}^{-}_{\overline{\alpha}}$. Since the subgroup stabilizing A is $SO(2)^k$, there is no other possibility. $\square$

The above proofs as well as later computations can presumably be simplified by using diagonalization in the complexified space $\mathbb{C}^{2k}$, but we will stay with 'real' computations here.

Since we have introduced the notion of eigen-k-tuples, this lemma gives us the possibility to regard $\mathcal{O}^{+}_{\overline{\alpha}}$ as skew-symmetric matrices with eigen-k-tuple $(\alpha_1, \ldots, \alpha_k)$, whereas $\mathcal{O}^{-}_{\overline{\alpha}}$ may be seen as skew-symmetric matrices with eigen-k-tuple $(\alpha_1, \ldots, \alpha_{k-1}, -\alpha_k)$, even though their eigenvalues are the same. This idea will come in handy in Proposition 2.3 and section 2.4.

However, we are now ready to formulate our own AGS-type theorem:

Theorem 2.2. *Let $\mathcal{O}_{\overline{\alpha}}$ be the set of skew-symmetric matrices with eigenvalues $\pm i\overline{\alpha}$, and let $\Phi : \mathcal{O}_{\overline{\alpha}} \to \mathbb{R}^k$ be the map taking a matrix $A = (a_{ij})$ to the upper right elements of its diagonal 2×2-blocks*

$$\begin{pmatrix} 0 & \boxed{a_{12}} & a_{13} & \cdots & & & \\ -a_{12} & 0 & a_{23} & & & & \\ \vdots & -a_{23} & 0 & \boxed{a_{34}} & & & \\ & & -a_{34} & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \boxed{a_{2k-1,2k}} \\ & & & & & -a_{2k-1,2k} & 0 \end{pmatrix} \overset{\Phi}{\longmapsto} (a_{12}, a_{34}, \ldots, a_{2k-1,2k}).$$

Then Φ is a moment map for the action of $SO(2)^k$ on $\mathcal{O}_{\overline{\alpha}}$, and its image the union of two convex polytopes, one which is the convex hull of all permutations of $\{(\pm\alpha_1, \ldots, \pm\alpha_k)\}$ with an even number of minus signs, the other *being the convex hull of permutations of* $\{(\pm\alpha_1, \dots, \pm\alpha_k)\}$ *having an odd number of minus signs.*

Proof. We have already seen that the conditions for applying Theorem 1.1 are fulfilled, but we need to determine the moment map explicitly, identifying it with Φ.

We know from Fact 1.1 that it is the composition of the $SO(2k)$ moment map, which is merely inclusion, and $(d\rho)^*$ for our specific Lie group homomorphism, the inclusion $\rho : T = SO(2)^k \hookrightarrow SO(2k)$. The corresponding Lie algebra map $(d\rho) : \mathfrak{t} \twoheadrightarrow \mathfrak{o}(2k)$ is given by:

$$\rho(\exp(\underline{\vartheta})) = \exp(d\rho(\underline{\vartheta})) \quad \forall \underline{\vartheta} \in \mathfrak{t}. \tag{2.3}$$

Considering an element $\underline{\vartheta} \in \mathfrak{t}$ as a k-tuple $(\vartheta_1, \dots, \vartheta_k)$ we have that

$$\rho(\exp(\underline{\vartheta})) = \operatorname{diag}\left(\begin{pmatrix} \cos\vartheta_i & -\sin\vartheta_i \\ \sin\vartheta_i & \cos\vartheta_i \end{pmatrix}\right),$$

which is the same as

$$\exp\left(\operatorname{diag}\left(\begin{pmatrix} 0 & \vartheta_i \\ -\vartheta_i & 0 \end{pmatrix}\right)\right).$$

Hence by (2.3), $d\rho(\underline{\vartheta}) = \operatorname{diag}(\left(\begin{smallmatrix} 0 & \vartheta_i \\ -\vartheta_i & 0 \end{smallmatrix}\right))$. The dual map $(d\rho)^*$ is given by

$$(d\rho)^*(A)(\underline{\vartheta}) = \operatorname{tr}\left(A \cdot \operatorname{diag}\left(\begin{pmatrix} 0 & \vartheta_i \\ -\vartheta_i & 0 \end{pmatrix}\right)\right) = -\frac{1}{2}\sum_{1=0}^{k-1} a_{2i+1,2i+2}\vartheta_{i+1},$$

i.e. $(d\rho)^*$ takes a matrix $A = (a_{ij}) \in \mathfrak{o}^*(2k)$ to the k-tuple

$$(a_{12}, a_{34}, \dots, a_{2k-1,2k})^1.$$

So the map Φ is the moment map of the action of the torus $T = SO(2)^k$ on the two symplectic manifolds $\mathcal{O}^+_{\underline{\alpha}}$ and $\mathcal{O}^-_{\underline{\alpha}}$, which means that $\mathrm{Im}\Phi$, regarded as a map from $\mathcal{O}^+_{\underline{\alpha}} \cup \mathcal{O}^-_{\underline{\alpha}}$, is the union of the convex hulls of the image of the fix points under the action of the torus. By Lemma 2.2 these are as stated. ☐

[1] Note that these are the entries where we would have found the eigen-k-tuple if A would have been block diagonalized.

2.1 Returning to matrix sums

Knowing that the set of skew-symmetric matrices sharing an eigen-k-tuple is a union of Hamiltonian $SO(2)^k$-manifolds, whose moment map takes a matrix to the entries corresponding to the eigen-k-tuple, makes it not so preposterous to believe that we might be able to do the same for a pair of matrices, finding a moment map going to the entries in the sum. Making right use of the results at hand, we might even be able to make that moment map target the specific k-tuple which carries an 'eigen' prefix. That would mean that we have a map taking two matrices to the eigen-k-tuple of their sum.

This is precisely what Kirwan's result (Theorem 1.2) helps us accomplish:

Theorem 2.3. *Given two eigen-k-tuples, $\overline{\alpha} = (\alpha_1 > \cdots > \alpha_k)$ and $\overline{\beta} = (\beta_1 > \cdots > \beta_k)$, the set of eigen-$k$-tuples $\overline{\gamma} = (\gamma_1 > \cdots > \gamma_k)$ belonging to matrices $C = A + B$ where $A \in \mathcal{O}_{\overline{\alpha}}$ and $B \in \mathcal{O}_{\overline{\beta}}$, is the union of four convex polytopes, namely the images of four moment maps intersected with the half-space $\{\gamma_k \geq 0\}$.*

Proof. As before we prefer action of $SO(2k)$ rather than $O(2k)$, thus we are to consider four combinations of matrices, taking the pairs from $\mathcal{O}^+_{\overline{\alpha}} \times \mathcal{O}^+_{\overline{\beta}}$, $\mathcal{O}^+_{\overline{\alpha}} \times \mathcal{O}^-_{\overline{\beta}}$, $\mathcal{O}^-_{\overline{\alpha}} \times \mathcal{O}^+_{\overline{\beta}}$ and $\mathcal{O}^-_{\overline{\alpha}} \times \mathcal{O}^-_{\overline{\beta}}$ respectively. The reasoning is analogous in all cases, so we are content with merely looking at the first one.

The two sets $\mathcal{O}^+_{\overline{\alpha}}$ and $\mathcal{O}^+_{\overline{\beta}}$ are Hamiltonian $SO(2k)$-manifolds, thus by Fact 1.3, $\mathcal{O}^+_{\overline{\alpha}} \times \mathcal{O}^+_{\overline{\beta}}$ is a Hamiltonian $SO(2k) \times SO(2k)$-manifold. Placing $SO(2k)$ in this larger group by the diagonal inclusion $\rho : P \mapsto (P, P)$, Fact 1.1 tells us that $\mathcal{O}^+_{\overline{\alpha}} \times \mathcal{O}^+_{\overline{\beta}}$ is a $SO(2k)$-manifold as well. Furthermore, the map $(d\rho)^*$ in this case is summation, since $d\rho$ is the diagonal inclusion of Lie algebras $\mathfrak{o}(2k) \ni X \mapsto (X, X) \in \mathfrak{o}(2k) \times \mathfrak{o}(2k)$, the pairing between the direct sums of algebras is given by $\tau(X, Y) = \mathrm{tr}(AX) + \mathrm{tr}(BY) = (A, B)(X, Y)$ according to (2.1), and $(d\rho)^*$ is given by:

$$(d\rho)^*(A, B)(X) = \mathrm{tr}(AX) + \mathrm{tr}(BX) = \mathrm{tr}((A + B)X).$$

Therefore, the summation of a pair of matrices is a moment map for the action of $SO(2k)$, and we may use Theorem 1.2, composing the moment map with the map $\mathfrak{o}^*(2k) \to \mathcal{W}^+$, taking $A + B$ to the unique point common for its $SO(2k)$-orbit and the positive Weyl chamber, $\mathcal{W}^+$. Let us denote this composite map by e_{++}. The positive Weyl chamber for $SO(2k)$ is the set

$$\{\sum_{i=1}^{k} a_i \mathcal{H}_i | a_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq |a_k|\},$$

which means that e_{++} takes a pair of matrices (A, B) to the k-tuple

$$(\gamma_1, \ldots, \gamma_{k-1}, \pm\gamma_k),$$

where the sign of the last entry depends on whether the matrix $A + B$ lies in $\mathcal{O}^+_{\overline{\gamma}}$ or $\mathcal{O}^-_{\overline{\gamma}}$. The range of this map is therefore not precisely the possible eigen-k-tuples, but this is easily atoned for.

If we go back to considering all four maps, denoting them e_{++}, e_{+-}, e_{-+} and e_{--} respectively, we realize that if $(\gamma_1, \ldots, \gamma_{k-1}, -\gamma_k)$ lies in the image of one of the maps, then $(\gamma_1, \ldots, \gamma_{k-1}, \gamma_k)$ lies in the image of another - namely the one with opposite signs. This is so, since

$$\begin{aligned}(A, B) \mapsto (\gamma_1, \ldots, \gamma_{k-1}, -\gamma_k) &\Longleftrightarrow A + B \in \mathcal{O}^-_{\overline{\gamma}} \\ &\Longleftrightarrow A + B = P\hat{I}B_{\overline{\gamma}}\hat{I}P^\intercal,\end{aligned}$$

for some $P \in SO(2k)$, following the notation in (2.2). However, for each such P, there is a $\tilde{P} \in SO(2k)$ such that $P\hat{I} = \hat{I}\tilde{P}$ and $(\tilde{P})^\intercal = \widetilde{P^\intercal}$, implying:

$$A + B = \hat{I}\tilde{P}B_{\overline{\gamma}}\tilde{P}^\intercal\hat{I} \Longleftrightarrow \hat{I}A\hat{I} + \hat{I}B\hat{I} = \tilde{P}B_{\overline{\gamma}}\tilde{P}^\intercal,$$

where by repeating the argument $A \in \mathcal{O}^\pm_{\overline{\alpha}} \Longleftrightarrow \hat{I}A\hat{I} \in \mathcal{O}^\mp_{\overline{\alpha}}$, and analogous for B. The curious and careful reader might appreciate knowing that for $P = (p_{i,j})$, we have explicitly:

$$\tilde{P} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,n-1} & -p_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ p_{n-1,1} & \cdots & p_{n-1,n-1} & -p_{n-1,n} \\ -p_{n,1} & \cdots & -p_{n,n-1} & p_{n,n} \end{pmatrix}.$$

Hereby, we know that all possible eigenvalues are represented by its eigen-k-tuple in the image of one of the maps, and therefore we may compose them with the map $(x_1, \ldots, x_{k-1}, x_k) \mapsto (x_1, \ldots, x_{k-1}, |x_k|)$, yielding the total image to be the same as intersecting the union $\text{Im } e_{++} \cup \text{Im } e_{+-} \cup \text{Im } e_{-+} \cup \text{Im } e_{--}$ with the half space $x_k \geq 0$, obviously resulting in a union of four convex polytopes. □

So we know that the set of eigen-k-tuples is a union of convex polytopes, and hence is restricted by some set of inequalities, depending on $\overline{\alpha}$ and $\overline{\beta}$, and naturally, we would like to find these inequalities, or at least come closer to them. We have a result saying something about the boundary points of these polytopes, Proposition 1.1, but as we shall see, this does not get us very far:

Proposition 2.1. *The boundary points of the moment map polytopes of Theorem 2.3, are the images of pairs of matrices for which every 2×2-block is of the form:*

$$\begin{pmatrix} k & -m \\ m & k \end{pmatrix}.$$

Proof. By Proposition 1.1, any boundary point is stabilized by at least a circle subgroup of $SO(2k)$. Such a subgroup is given by a k-tuple $(s_1t, s_2t, \dots s_kt)$, corresponding to matrices of the form:

$$T = \begin{pmatrix} \begin{smallmatrix} \cos s_1t & -\sin s_1t \\ \sin s_1t & \cos s_1t \end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{smallmatrix} \cos s_kt & -\sin s_kt \\ \sin s_kt & \cos s_kt \end{smallmatrix} \end{pmatrix}.$$

For a matrix $A = (a_{ij})$ to be stabilized by this group, the equality $TA = AT$ needs to hold, or equivalently, for each 2×2- block, the (rather horrific) system of equations:

$$\begin{aligned}
a_{2i-1,2j-1}\cos s_it - a_{2i,2j-1}\sin s_it &= a_{2i-1,2j-1}\cos s_jt + a_{2i-1,2j}\sin s_jt \\
a_{2i-1,2j}\cos s_it - a_{2i,2j}\sin s_it &= -a_{2i-1,2j-1}\sin s_jt + a_{2i-1,2j}\cos s_jt \\
a_{2i-1,2j-1}\sin s_it + a_{2i,2j-1}\cos s_it &= a_{2i,2j-1}\cos s_jt + a_{2i,2j}\sin s_jt \\
a_{2i-1,2j}\sin s_it + a_{2i,2j}\cos s_it &= -a_{2i,2j-1}\sin s_jt + a_{2i,2j}\cos s_jt
\end{aligned}$$

$$\begin{aligned}
a_{2j-1,2i-1}\cos s_jt - a_{2j-1,2i}\sin s_jt &= a_{2j-1,2i-1}\cos s_it + a_{2j,2i-1}\sin s_it \\
a_{2j,2i-1}\cos s_jt - a_{2j-1,2i}\sin s_jt &= -a_{2j-1,2i-1}\sin s_it + a_{2j,2i-1}\cos s_it \\
a_{2j-1,2i-1}\sin s_jt + a_{2j,2i}\cos s_jt &= a_{2j-1,2i}\cos s_it + a_{2j,2i}\sin s_it \\
a_{2j,2i-1}\sin s_jt + a_{2j,2i}\cos s_jt &= -a_{2j-1,2i}\sin s_it + a_{2j,2i}\cos s_it,
\end{aligned}$$

where the first four comes from above the diagonal, and the last four from below. If we introduce the following substitutions:

$$\begin{aligned}
k &= a_{2i-1,2j-1} = -a_{2j-1,2i-1} \\
l &= a_{2i-1,2j} = -a_{2j,2i-1} \\
m &= a_{2i,2j-1} = -a_{2j-1,2i} \\
n &= a_{2i,2j} = -a_{2j,2i},
\end{aligned}$$

we get the more pleasant, equivalent system:

$$\begin{aligned} k(\cos s_i t - \cos s_j t) &= 0 \\ l(\cos s_i t - \cos s_j t) &= 0 \\ m(\cos s_i t - \cos s_j t) &= 0 \\ n(\cos s_i t - \cos s_j t) &= 0 \\ \\ l \sin s_i t + m \sin s_j t &= 0 \\ m \sin s_i t + l \sin s_j t &= 0 \\ n \sin s_i t - k \sin s_j t &= 0 \\ k \sin s_i t - n \sin s_j t &= 0. \end{aligned}$$

If all s_i are different, we immediately see from the first four equations that $k = l = m = n = 0$ and A must be block diagonal. However, if some of them are the same, the last four equations yield:

$$\begin{aligned} m + l &= 0 \\ n - k &= 0 \end{aligned}$$

for these specific i, j, creating a possibility for A to have non-zero off-diagonal blocks of the form given in the statement. □

Due to this abrupt end of progress - we have no way of generally finding the eigenvalues of the matrices in Proposition 2.1 - we are obliged to a new change of language, considering the matrix sets as complex flag manifolds.

2.2 Embedding the flag manifold

Let us begin by noting that formulating a problem symmetrically is often a beneficial choice. Therefore we would rather deal with matrix equations $A + B + C = 0$, than with $A + B = C$. In this setting we may consider the $SO(2k)$-manifold $\mathcal{O}^{\pm}_{\overline{\alpha}} \times \mathcal{O}^{\pm}_{\overline{\beta}} \times \mathcal{O}^{\pm}_{\overline{\gamma}}$[2] with moment map $\Phi : (A, B, C) \mapsto A+B+C$, for which the fiber $\Phi^{-1}(0)$ seems highly interesting. Our eigenvalue problem can then be formulated as the question: For which $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ is this fiber non-empty? We could of course equivalently consider the symplectic quotient $\Phi^{-1}(0)/SO(2k)$; preferable since it is so well-behaved, and since it is able to carry us into the world of representation theory, by Theorem 1.3. To be able

[2]For the moment, we choose to disregard $\mathcal{O}^{-}_{\overline{\alpha}} \times \mathcal{O}^{\pm}_{\overline{\beta}} \times \mathcal{O}^{\pm}_{\overline{\gamma}}$, $\mathcal{O}^{\pm}_{\overline{\alpha}} \times \mathcal{O}^{-}_{\overline{\beta}} \times \mathcal{O}^{\pm}_{\overline{\gamma}}$ and $\mathcal{O}^{\pm}_{\overline{\alpha}} \times \mathcal{O}^{\pm}_{\overline{\beta}} \times \mathcal{O}^{-}_{\overline{\gamma}}$ for clarity, since the argument is analogous.

to appreciate this result fully, we need to make sense of the right hand side of the identification:

$$\Phi^{-1}(0)/SO(2k) \cong \text{Proj } (R^{SO_{\mathbb{C}}(2k)}),$$

i.e. find an embedding of $\mathcal{O}^{+}_{\alpha}$ in projective space. The first step is to realize $\mathcal{O}^{+}_{\alpha}$ as a flag manifold, identifying a flag with the eigenspaces of a matrix in $\mathcal{O}^{+}_{\alpha}$.

One way to define $SO(2k)$ is to introduce a quadratic form Q on $\mathbb{C}^{2k}$, and letting $SO(2k)$ be all matrices of determinant 1 preserving this form:

$$Q(X\mathbb{u}, X\mathbb{v}) = Q(\mathbb{u}, \mathbb{v}).$$

The Lie algebra $\mathfrak{so}(2k)$, the skew-symmetric matrices, are then the ones fulfilling the differentiated condition:

$$Q(A\mathbb{u}, \mathbb{v}) + Q(\mathbb{u}, A\mathbb{v}) = 0.$$

We then choose Q convenient for our purposes, and not merely as the usual $\langle\, ,\, \rangle_{2k}$. However, regardless of which Q we choose, we see that if $\mathbb{v}$ is an eigenvector for A with eigenvalue αi, then:

$$Q(\mathbb{v}, \mathbb{v}) = \frac{1}{2\alpha i}(Q(A\mathbb{v}, \mathbb{v}) + Q(\mathbb{v}, A\mathbb{v})) = 0,$$

which in words reads: $\mathbb{v}$ is an isotropic vector, and the eigenspace for A are isotropic subspaces.

Furthermore, if $\mathbb{v}$ is an eigenvector with eigenvalue αi, then:

$$A(\text{Re } \mathbb{v} + i\text{Im } \mathbb{v}) = \alpha i\text{Re } \mathbb{v} - \alpha\text{Im } \mathbb{v},$$

and since A is a matrix with real entries we must have:

$$\begin{aligned} A(\text{Re } \mathbb{v}) &= -\alpha\text{Im } \mathbb{v} \\ A(\text{Im } \mathbb{v}) &= \alpha\text{Re } \mathbb{v}, \end{aligned}$$

and obviously $\bar{\mathbb{v}}$ is an eigenvector with eigenvalue $-\alpha i$. If we have an eigen-k-tuple and a corresponding flag $\{\mathcal{V}_i\}_{i=1}^{k}$, with $\mathcal{V}_i$ the sum of the eigenspaces for $\alpha_1, \ldots, \alpha_i$, we are hence able to reconstruct A:

$$A = \sum_{i=1}^{k} \alpha_i i \left(P(V_{i-1}^{\perp} \cap \mathcal{V}_i) - P(\overline{\mathcal{V}_{i-1}^{\perp} \cap \mathcal{V}_i}) \right),$$

if we let $P(V)$ be the projection on V and $\overline{\mathcal{V}} = \{\bar{\mathbb{v}} | \mathbb{v} \in \mathcal{V}\}$. Thus we have $1-1$-correspondence between skew-symmetric matrices sharing a certain eigen-k-tuple, and the set of full, isotropic flags in $\mathbb{C}^{2k}$:

$$\mathcal{O}^{\pm}_{\bar{\alpha}} \longleftrightarrow \mathcal{F}_{\mathcal{ISO}}(\mathbb{C}^{2k}),$$

where

$$\mathcal{F}_{\mathcal{ISO}}(\mathbb{C}^{2k}) = \{0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_k \subset \mathbb{C}^{2k} | Q(\mathcal{V}_k, \mathcal{V}_k) = 0, \forall i \ \dim(\mathcal{V}_i/\mathcal{V}_{i+1}) = 1\}.$$

This set of flags, called a flag manifold, we can also describe as a quotient:

Proposition 2.2. *$\mathcal{F}_{\mathcal{ISO}}(\mathbb{C}^{2k}) \cong SO_{\mathbb{C}}(2k)/B$, where B is the Borel subgroup of $SO_{\mathbb{C}}(2k)$.*

Proof. First let us decide on a quadratic form Q. If we let $Q(\mathbb{x}, \mathbb{y}) = \mathbb{x}^* \left(\begin{smallmatrix} 0 & I_k \\ I_k & 0 \end{smallmatrix}\right) \mathbb{y}$, the standard flag

$$\{\mathbb{C}^i\} = \mathbb{C}\mathbb{e}_1 \subset \mathbb{C}\{\mathbb{e}_1, \mathbb{e}_2\} \subset \cdots \subset \mathbb{C}\{\mathbb{e}_1, \ldots, \mathbb{e}_k\}$$

will be isotropic. We proceed by showing that given an isotropic flag $\{V_i\}$ there is an element $M \in SO_{\mathbb{C}}(2k)$ such that $M(\mathbb{C}^i) = V_i \ \forall i$. We may assume that we have an orthonormal basis $\mathbb{v}_1, \ldots, \mathbb{v}_k$ such that $V_i = \text{Span}(\mathbb{v}_1, \ldots, \mathbb{v}_i)$. If we name the upper half and the lower half of these vectors:

$$\mathbb{v}_i = \begin{pmatrix} \mathbb{s}_i \\ \mathbb{t}_i \end{pmatrix},$$

we may write the isotropy condition for $\{V_i\}$ as:

$$\mathbb{s}_i^* \mathbb{t}_j + \mathbb{t}_i^* \mathbb{s}_j = 0 \quad \forall i, j,$$

and the orthonormal condition as:

$$\mathbb{s}_i^* \mathbb{s}_j + \mathbb{t}_i^* \mathbb{t}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j, \end{cases}$$

or equivalently, for the matrices $S = (\mathbb{s}_i)$ and $T = (\mathbb{t}_i)$:

$$\begin{aligned} S^*T + T^*S &= 0 \\ S^*S + T^*T &= I_k. \end{aligned} \tag{2.4}$$

Obviously, what we are looking for must be a matrix:

$$M = \begin{pmatrix} S & * \\ T & * \end{pmatrix} : M^* \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}.$$

But then $\left(\begin{smallmatrix} S & T \\ T & S \end{smallmatrix}\right)$ will do, since:

$$\begin{pmatrix} S^* & T^* \\ T^* & S^* \end{pmatrix} \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix} \begin{pmatrix} S & T \\ T & S \end{pmatrix} = \begin{pmatrix} S^*T + T^*S & S^*S + T^*T \\ T^*T + S^*S & T^*S + S^*T \end{pmatrix} = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix},$$

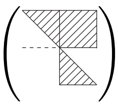
by (2.4). The last step is to show that the subgroup stabilizing an element in $\mathcal{F}_{\mathcal{JSO}}(\mathbb{C}^{2k})$, e.g. the standard flag $\{\mathbb{C}^i\}$, is precisely the Borel group $B = \exp(\mathfrak{b})$. The Borel subalgebra is the direct sum of the positive weight spaces and $\mathfrak{so}_{\mathbb{C}}(2k)$:s Cartan subalgebra $\mathfrak{h}$. With respect to our choice of Q, we may choose, as seen in section 1.4:

$$R^+ = \{\mathcal{H}_i + \mathcal{H}_j\} \cup \{\mathcal{H}_i - \mathcal{H}_j\} \quad i < j,$$

as positive roots, corresponding to the part of $\mathfrak{o}_{\mathbb{C}}(2k)$ spanned by:

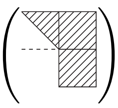
$$\begin{aligned} &E_{ij} - E_{k+j,k+i} \quad i < j \\ &E_{i,k+j} - E_{j,k+i} \quad i < j. \end{aligned}$$

Together with $\mathfrak{h} = \text{Span}\{E_{i,i} - E_{k+i,k+i}\}$ this gives all matrices of the form:



,

which shape is preserved when taking powers, and hence by the exponential map, i.e. the Borel subgroup also consists of the matrices of this form. On the other hand, any matrix stabilizing $\{\mathbb{C}^i\}_{i=1}^k$ must be of the form:

$M =$  $=: \begin{pmatrix} S & T \\ 0 & V \end{pmatrix},$

and as a subgroup of $SO_{\mathbb{C}}(2k)$, the following must hold:

$$\begin{pmatrix} S & T \\ 0 & V \end{pmatrix}^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} S & T \\ 0 & V \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \Longleftrightarrow S^*V = I_k \text{ and } T^*V + V^*T = 0,$$

which forces V to be lower triangular, and imposes no further condition on T, i.e. $M \in B$. This concludes the proof and we have, indeed, that

$$\mathcal{F}_{\mathcal{JSO}}(\mathbb{C}^{2k}) \cong SO_{\mathbb{C}}(2k)/B$$

□

We may now, by a series of well-known inclusions[3], embed our flag manifold in a complex projective space. First we may map $\mathcal{F}_{\mathcal{ISO}}(\mathbb{C}^{2k})$ into a product of isotropic Grassmannians by:

$$\mathcal{F}_{\mathcal{ISO}}(\mathbb{C}^{2k}) \ni \{\mathcal{V}_i\} \mapsto \prod \mathcal{V}_i \in \prod_{i=1}^{k} \mathrm{Gr}_{\mathcal{ISO}}(i, \mathbb{C}^{2k}).$$

This product we may in turn send along with the Plücker embedding:

$$\mathrm{Gr}_{\mathcal{ISO}}(i, \mathbb{C}^{2k}) \hookrightarrow \mathbb{P}(\wedge^i \mathbb{C}^{2k}),$$

which maps an i-dimensional vector space in $\mathbb{C}^{2k}$ to a one-dimensional subspace in $\wedge^i \mathbb{C}^{2k}$, sending a basis for this space to its wedge product:

$$\{\mathbb{v}_i\} \mapsto \mathbb{v}_1 \wedge \mathbb{v}_2 \wedge \cdots \wedge \mathbb{v}_i.$$

A change of basis merely changes the image element by multiplication of a scalar factor, justifying the definition.

It is now time to remind ourselves that the set $\mathcal{O}^{\pm}_{\overline{\alpha}}$ corresponds to *the pair* $\mathcal{F}_{\mathcal{ISO}}(\mathbb{C}^{2k})$ together with the eigen-k-tuple $\overline{\alpha}$. If $\overline{\alpha}$ consists of integers, and we let:

$$\begin{aligned} a_1 &= \alpha_1 - \alpha_2 \\ &\vdots \\ a_{k-1} &= \alpha_{k-1} - \alpha_k \\ a_k &= \alpha_k, \end{aligned} \tag{2.5}$$

we can send any vector, or wedge product of vectors, to its symmetric tensor power: $\omega \mapsto \omega^{\odot a_i}$. Therefore we have another inclusion, the so called a_i:th Veronese embedding:

$$\mathbb{P}(\wedge^i \mathbb{C}^{2k}) \mapsto \mathbb{P}(\mathrm{Sym}^{a_i}(\wedge^i \mathbb{C}^{2k})).$$

Summarizing we have the inclusion:

$$(\mathcal{F}_{\mathcal{ISO}}(\mathbb{C}^{2k}), \overline{\alpha}) \hookrightarrow \prod_{i=1}^{k} \mathbb{P}(\mathrm{Sym}^{a_i}(\wedge^i \mathbb{C}^{2k})), \tag{2.6}$$

which finally yields:

$$(\mathcal{F}_{\mathcal{ISO}}(\mathbb{C}^{2k}), \overline{\alpha}) \hookrightarrow \mathbb{P}(\otimes_{i=1}^{k} \mathrm{Sym}^{a_i}(\wedge^i \mathbb{C}^{2k})), \tag{2.7}$$

[3]You find more about these in e.g. [GH78]

by composing (2.6) with Segre embeddings $\mathbb{P}(\mathcal{V}) \times \mathbb{P}(\mathcal{W}) \hookrightarrow \mathbb{P}(\mathcal{V} \otimes \mathcal{W})$.

We note that $SO_{\mathbb{C}}(2k)$ acts on $\otimes_{i=1}^{k}(\wedge^i \mathbb{C}^{2k})^{a_i}$ by tensoring the defining action on $\mathbb{C}^{2k}$, and the vectors $\otimes_{i=1}^{k}(\mathbb{v}_1 \wedge \cdots \wedge \mathbb{v}_i)^{a_i}$ in the lines in $\mathbb{P}(\otimes_{i=1}^{k}\mathrm{Sym}^{a_i}(\wedge^i \mathbb{C}^{2k}))$ generate an irreducible representation of $SO_{\mathbb{C}}(2k)$ with highest weight $\overline{\alpha}$. We will however realize the representations as a subspace of the coordinate ring, which is another formulation of Borel-Weil (Theorem 1.4).

Having embedded $\mathcal{O}^{\pm}_{\overline{\alpha}} = SO_{\mathbb{C}}(2k)/B$ in projective space as above, we know that it has an ideal $I(SO_{\mathbb{C}}(2k)/B)$, and the homogeneous coordinate ring:

$$\bigoplus_{d=0}^{\infty} R_d^{\overline{\alpha}} = \bigoplus_{d=0}^{\infty} \mathrm{Sym}^d \left(\bigotimes_{i=1}^{k} \mathrm{Sym}^{a_i}(\wedge^i \mathbb{C}^k) \right) / I(SO_{\mathbb{C}}(2k)/B).$$

Since $SO_{\mathbb{C}}(2k)$ acts on the projective variety, it also acts on its homogeneous coordinate ring, making each graded piece of it a representation. From section 1.2 we know that the 1-graded piece of this ring is the space of holomorphic sections of the line bundle $\mathcal{O}_{SO_{\mathbb{C}}(2k)/B}(a_1, \ldots, a_k)$, and for the embedding (2.7) of $SO_{\mathbb{C}}(2k)/B$, this is precisely the bundle $\mathcal{O}(\overline{\alpha})$ defined in section 1.3 (see [Ful97] p. 143).

By Theorem 1.4 we then have that $R_1^{\overline{\alpha}}$ is the irreducible representation $\mathcal{V}_{\overline{\alpha}}$ of $\mathfrak{so}_{\mathbb{C}}(2k)$, of highest weight $\overline{\alpha}$. Since we may write

$$\bigoplus_{d=0}^{\infty} R_d^{\overline{\alpha}} = \bigoplus_{d=0}^{\infty} \left(\bigotimes_{i=1}^{k} \mathrm{Sym}^{da_i}(\wedge^i \mathbb{C}^k) \right) / I(SO_{\mathbb{C}}(2k)/B),$$

and by applying theorem 1.4 to the 1-graded pieces for each $d\overline{\alpha}$, we get:

$$\bigoplus_{d=0}^{\infty} R_d^{\overline{\alpha}} = \bigoplus_{d=0}^{\infty} \mathcal{V}_{d\overline{\alpha}}.$$

As noted in section 1.2, we this have that $\mathcal{O}^{\pm}_{\overline{\alpha}} = \mathrm{Proj}(\bigoplus_{d=0}^{\infty} \mathcal{V}_{d\overline{\alpha}})$, and considering the product space $\mathcal{O}^{\pm}_{\overline{\alpha}} \times \mathcal{O}^{\pm}_{\overline{\beta}} \times \mathcal{O}^{\pm}_{\overline{\gamma}}$, its homogeneous coordinate ring will be:

$$\bigoplus_{d=0}^{\infty} (\mathcal{V}_{d\overline{\alpha}} \otimes \mathcal{V}_{d\overline{\beta}} \otimes \mathcal{V}_{d\overline{\gamma}}),$$

by applying the Segre embedding again. With help from Theorem 1.3 we arrive at:

Proposition 2.3. *If there is an $SO_{\mathbb{C}}(2k)$-invariant vector in the tensor product $\mathcal{V}_{\overline{\alpha}} \otimes \mathcal{V}_{\overline{\beta}} \otimes \mathcal{V}_{\overline{\gamma}}$ of irreducible representations, then there are skew-symmetric matrices $A + B + C = 0$ with corresponding eigen-k-tuples $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$.*

On the other hand, if there are matrices $A+B+C=0$ in $\mathcal{O}^+_{\overline{\alpha}} \times \mathcal{O}^+_{\overline{\beta}} \times \mathcal{O}^+_{\overline{\gamma}}$ with eigen-k-tuples $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$, then for some $N > 0$ there is a tensor product of irreducible representations $\mathcal{V}_{d\overline{\alpha}} \otimes \mathcal{V}_{d\overline{\beta}} \otimes \mathcal{V}_{d\overline{\gamma}}$ which has an $SO_{\mathbb{C}}(2k)$-invariant vector.

Remark. If e.g. A lies in $\mathcal{O}^-_{\overline{\alpha}}$ instead, we will have a corresponding statement, substituting $\mathcal{V}_{d\overline{\alpha}}$ with $\mathcal{V}_{d(\alpha_1,\ldots,\alpha_{k-1},-\alpha_k)}$

Proof. Theorem 1.3 tell us that the symplectic quotient at level 0 of the Hamiltonian $SO(2k)$-manifold $\mathcal{O}^+_{\overline{\alpha}} \times \mathcal{O}^+_{\overline{\beta}} \times \mathcal{O}^+_{\overline{\gamma}}$, equals the geometric invariant quotient, i.e.

$$\Phi^{-1}(0)/SO(2k) = \mathrm{Proj}\left(\bigoplus_{d=0}^{\infty} (\mathcal{V}_{d\overline{\alpha}} \otimes \mathcal{V}_{d\overline{\beta}} \otimes \mathcal{V}_{d\overline{\gamma}})^{SO_{\mathbb{C}}(2k)}\right).$$

If there is an invariant vector in $\mathcal{V}_{\overline{\alpha}} \otimes \mathcal{V}_{\overline{\beta}} \otimes \mathcal{V}_{\overline{\gamma}}$, then the right hand side is non-empty, hence so is the left hand side, which is just the space of matrices $A+B+C=0$ quoted by $SO(2k)$. If we start at the left hand side, assuming that there are matrices with the required eigen-k-tuples and of zero sum, then the right hand side cannot be empty, and hence there must be some degree N for which the N-graded piece is non-zero. ☐

2.3 Discussing saturation

Having reformulated the question about triples of eigen-k-tuples as a Clebsch-Gordan type problem, and there being much written and known about coefficients for tensor product decomposition $V_{\overline{\alpha}} \otimes V_{\overline{\beta}} = \bigoplus_{\overline{\gamma}} c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} V_{\overline{\gamma}}$, the next task seems to be to jettison the uninvited N:s. For Hermitian matrices this last step was called the saturation conjecture, and since 1999 it is a theorem:

Theorem 2.4. *[KT99] If $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ is a triple of integral dominant weights of $GL_{\mathbb{C}}(n)$ and there is a number $N > 0$ such that:*

$$\left(V_{N\overline{\alpha}} \otimes V_{N\overline{\beta}} \otimes V_{N\overline{\gamma}}\right)^{GL_{\mathbb{C}}(n)} > 0 \quad \textit{then also} \quad \left(V_{\overline{\alpha}} \otimes V_{\overline{\beta}} \otimes V_{\overline{\gamma}}\right)^{GL_{\mathbb{C}}(n)} > 0.$$

The proof by Knutson and Tao is in the language of honeycombs. In [Ful00] the reader will find references to other proofs using other methods, such as hives or representations of quivers. In this article the saturation theorem takes the following, alternative form:

Theorem 2.5. *If $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ are a triple of partitions such that the Littlewood-Richardson coefficient $c^{N\overline{\gamma}}_{N\overline{\alpha}N\overline{\beta}} \neq 0$ then $c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} \neq 0$.*

Trying to prove a $SO_{\mathbb{C}}(2k)$-version of this theorem would probably demand getting into an additional language, e.g. that of honeycombs or hives, which lies beyond the scope of this thesis. The conjecture made in [KT99], p. 1085 about connected, complex, semisimple Lie groups also suggests that the situation might not be as plain as in the $GL_{\mathbb{C}}(n)$-case. We will however unravel what happens in two small dimensional cases, where we are able to exploit isomorphism with other Lie groups.

2.4 Low dimensions

Not being able to find a set of inequalities for the general problem, we want to make amends by including two illustrative (or even illustrational) examples. As we saw earlier, already for $k = 1$ the skew-symmetric answer is different from the Hermitian one. Whereas for Hermitian matrices, the possible γ:s for given α, β lay on the line segment $[|\alpha - \beta|, \alpha + \beta]$, the only skew γ are the end points $|\alpha \pm \beta|$. If we go on looking at $k = 2$, we may also in this case compare the Hermitian and skew answers.

As we saw in section 1.4, the simple roots of the Lie algebra $\mathfrak{so}_{\mathbb{C}}(4)$ are $\mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_1 - \mathcal{H}_2$, i.e. the root system is a product of two smaller root systems, each corresponding to the Lie algebra $\mathfrak{sl}_{\mathbb{C}}(2)$, by

$$(\mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_1 - \mathcal{H}_2) \mapsto (2\mathcal{H}_1, 2\mathcal{H}_2).$$

We therefore have the isomorphism $\mathfrak{so}_{\mathbb{C}}(4) \cong \mathfrak{sl}_{\mathbb{C}}(2) \times \mathfrak{sl}_{\mathbb{C}}(2)$, where the eigen-2-tuple (α_1, α_2) of a matrix $A \in \mathfrak{so}_{\mathbb{C}}(4)$ is mapped to the pair of eigenvalues $(\alpha_1 + \alpha_2, \alpha_1 - \alpha_2)$. On the other hand, a pair of eigenvalues (a_1, a_2) for $(A_1, A_2) \in \mathfrak{sl}_{\mathbb{C}}(2) \times \mathfrak{sl}_{\mathbb{C}}(2)$ is mapped to $(\frac{a_1+a_2}{2}, \frac{|a_1-a_2|}{2})$.

Since the decomposition of tensor products of highest weight representations of $\mathfrak{sl}_{\mathbb{C}}(2)$ takes on such a simple form, as seen in the introduction, it is obvious that this Lie algebra, and hence $\mathfrak{so}_{\mathbb{C}}(4)$ is saturated. The image of two eigen-k-tuples (α_1, α_2) and (β_1, β_2) is then the set of $(\frac{c_1+c_2}{2}, \frac{|c_1-c_2|}{2})$ where:

$$\begin{aligned} c_1 &= \alpha_1 + \alpha_2 + \beta_1 + \beta_2, \ldots, |\alpha_1 + \alpha_2 - \beta_1 - \beta_2| \\ c_2 &= \alpha_1 - \alpha_2 + \beta_1 - \beta_2, \ldots, |\alpha_1 - \alpha_2 - \beta_1 + \beta_2.| \end{aligned}$$

If we wish to account for all skew-symmetric matrices, we must also consider what happens to the pair $(\alpha_1, -\alpha_2)$, (β_1, β_2), corresponding to $\mathcal{O}^-_{\overline{\alpha}} \times \mathcal{O}^+_{\overline{\beta}}$. The image are the eigen-k-tuples $(\frac{c_1+c_2}{2}, \frac{|c_1-c_2|}{2})$, with:

$$\begin{aligned} c_1 &= \alpha_1 - \alpha_2 + \beta_1 + \beta_2, \ldots, |\alpha_1 - \alpha_2 - \beta_1 - \beta_2| \\ c_2 &= \alpha_1 + \alpha_2 + \beta_1 - \beta_2, \ldots, |\alpha_1 + \alpha_2 - \beta_1 + \beta_2|. \end{aligned}$$

In contrast, the set $(\mathcal{E}^4_{he})_{ss}$ is defined by the inequalities:

$$\begin{aligned}
\max\left\{\begin{array}{c}\alpha_1-\beta_1\\ |\alpha_2-\beta_2|\end{array}\right\} &\leq \gamma_1 \leq & \alpha_1+\beta_1\\
\max\left\{\begin{array}{c}\alpha_2-\beta_1\\ 0\end{array}\right\} &\leq \gamma_2 \leq & \min\left\{\begin{array}{c}\alpha_2+\beta_1\\ \alpha_1+\beta_2\end{array}\right\}\\
\max\left\{\begin{array}{c}|\alpha_1+\alpha_2-\beta_1-\beta_2|\\ |\alpha_1-\alpha_2-\beta_1+\beta_2|\end{array}\right\} &\leq \gamma_1+\gamma_2 \leq & \alpha_1+\alpha_2+\beta_1+\beta_2\\
\max\left\{\begin{array}{c}\alpha_1-\alpha_2-\beta_1-\beta_2\\ 0\end{array}\right\} &\leq \gamma_1-\gamma_2 \leq & \min\left\{\begin{array}{c}\alpha_1+\alpha_2+\beta_1-\beta_2\\ \alpha_1-\alpha_2+\beta_1+\beta_2\end{array}\right\},
\end{aligned}$$

found in e.g. [Ful00] p. 212. Picturing this, we see that the skew set is a union of four convex polytopes inside the convex polytope constituting the hermitian set, as assumed. For $\overline{\alpha} = (7,1), \overline{\beta} = (4,3)$ we have:

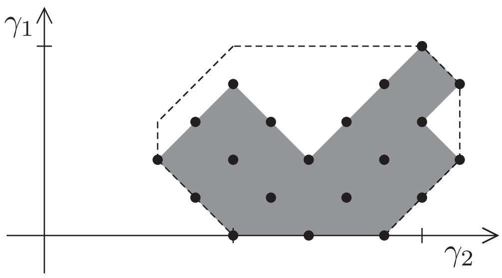


Figure 2.1: The dashed line is the boundary of $(\mathcal{E}^4_{he})_{ss}$, in this case restricted by $3 \leq \gamma_1 \leq 11$, $0 \leq \gamma_2 \leq 5$, $5 \leq \gamma_1+\gamma_2 \leq 15$ and $0 \leq \gamma_1-\gamma_2 \leq 9$. The black dots are the possible integer $\overline{\gamma}$:s for skew-symmetric matrices.

We summarize our findings in a proposition:

Proposition 2.4. *If $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are positive integers, such that $(\gamma_1, \gamma_2) = (\frac{c_1+c_2}{2}, \frac{|c_1-c_2|}{2})$ for some c_1, c_2 with either:*

$$\begin{aligned}
c_1 &\in \{\alpha_1+\alpha_2+\beta_1+\beta_2, \ldots, |\alpha_1+\alpha_2-\beta_1-\beta_2|\}\\
c_2 &\in \{\alpha_1-\alpha_2+\beta_1-\beta_2, \ldots, |\alpha_1-\alpha_2-\beta_1+\beta_2|\},
\end{aligned}$$

or:

$$\begin{aligned}
c_1 &\in \{\alpha_1-\alpha_2+\beta_1+\beta_2, \ldots, |\alpha_1-\alpha_2-\beta_1-\beta_2|\}\\
c_2 &\in \{\alpha_1+\alpha_2+\beta_1-\beta_2, \ldots, |\alpha_1+\alpha_2-\beta_1+\beta_2|\},
\end{aligned}$$

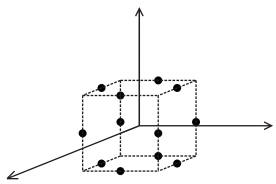
then there exist matrices $A, B, C \in \mathcal{M}_{ss}(4)$, such that $A + B = C$ with eigenvalues $\pm i(\alpha_1, \alpha_2)$, $\pm i(\beta_1, \beta_2)$ and $\pm i(\gamma_1, \gamma_2)$ respectively.

On the other hand, if $A + B = C$ for $A, B, C \in \mathcal{M}_{ss}(4)$, have eigenvalues $\pm i(\alpha_1, \alpha_2), \pm i(\beta_1, \beta_2), \pm i(\gamma_1, \gamma_2) \in i\mathbb{Z}^4$, then $(\gamma_1, \gamma_2) = (c_1, c_2)$ for some c_1, c_2 as above.

In the case $k = 3$ we also have a beneficial isomorphism. The simple roots for $\mathfrak{so}_{\mathbb{C}}(6)$ are

$$\mathcal{H}_1 - \mathcal{H}_2, \mathcal{H}_2 - \mathcal{H}_3, \mathcal{H}_2 + \mathcal{H}_3,$$

placing the root system in three dimensional space. Placing $\mathcal{H}_i$ at the i:th axis, the roots land on the centers of the edges of a cube of edge length 2, centered around the origin:



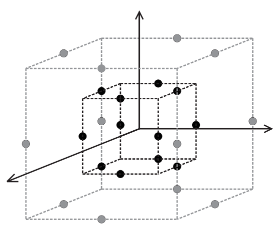
The picture is the same for the Lie algebra $\mathfrak{sl}_{\mathbb{C}}(4)$. The root lattice for this algebra lies in the set:

$$\mathbb{C}\{\tilde{\mathcal{H}}_1, \tilde{\mathcal{H}}_2, \tilde{\mathcal{H}}_3, \tilde{\mathcal{H}}_4\}/(\tilde{\mathcal{H}}_1 + \tilde{\mathcal{H}}_2 + \tilde{\mathcal{H}}_3 + \tilde{\mathcal{H}}_4 = 0),$$

also visualizable in three dimensional space. Since the Cartan subalgebra $\mathfrak{h}$ acts on $\mathfrak{sl}_{\mathbb{C}}(4)$ as diagonal matrices on traceless ones, the roots of $\mathfrak{sl}_{\mathbb{C}}(4)$ are the pairwise differences $\tilde{\mathcal{H}}_i - \tilde{\mathcal{H}}_j$, each corresponding to the basis vector E_{ij}. If we place:

$$\begin{aligned} \tilde{\mathcal{H}}_1 \text{ at } (-1,1,1) \quad & \tilde{\mathcal{H}}_2 \text{ at } (1,1,-1) \\ \tilde{\mathcal{H}}_3 \text{ at } (-1,-1,-1) \quad & \tilde{\mathcal{H}}_4 \text{ at } (1,-1,1), \end{aligned}$$

they are of equal length and zero sum, and the roots end up at the center of the edges of the cube of edge length 4, superimposed on the image above we get:



We may choose as simple roots for $\mathfrak{sl}_{\mathbb{C}}(4)$:

$$\begin{aligned} \tilde{\mathcal{H}}_1 - \tilde{\mathcal{H}}_2 &= -2(\mathcal{H}_1 - \mathcal{H}_3) \\ \tilde{\mathcal{H}}_2 - \tilde{\mathcal{H}}_3 &= 2(\mathcal{H}_1 + \mathcal{H}_2) \\ \tilde{\mathcal{H}}_3 - \tilde{\mathcal{H}}_4 &= -2(\mathcal{H}_1 + \mathcal{H}_3). \end{aligned}$$

This means that the eigenvalues $(\alpha_1, \alpha_2, \alpha_3)$ of a skew symmetric matrix are sent to the eigenvalues $(-2(\alpha_1+\alpha_2), 2(\alpha_1+\alpha_2+\alpha_3), -2(\alpha_1+\alpha_3), 2\alpha_1)$ of a traceless matrix; the last eigenvalue being determined by the tracelessness. All together, we have the following:

Proposition 2.5. *Assume $\overline{\alpha}, \overline{\beta}, \overline{\gamma} \in \mathbb{N}^3$. Then there exist matrices $A, B, C \in \mathcal{M}_{ss}(6)$ with eigenvalues $\pm\overline{\alpha}i$, $\pm\overline{\beta}i$ and $\pm\overline{\gamma}i$ respectively, such that $A+B=C$, if and only if there exist traceless matrices $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{M}(4)$, with $\tilde{A}+\tilde{B}=\tilde{C}$, for which*

$$\begin{aligned} eig(\tilde{A}) &= \{-2(\alpha_1+\alpha_2), 2(\alpha_1+\alpha_2\pm\alpha_3), -2(\alpha_1\pm\alpha_3), 2\alpha_1\} \\ eig(\tilde{B}) &= \{-2(\beta_1+\beta_2), 2(\beta_1+\beta_2\pm\beta_3), -2(\beta_1\pm\beta_3), 2\beta_1\} \\ eig(\tilde{C}) &= \{-2(\gamma_1+\gamma_2), 2(\gamma_1+\gamma_2\pm\gamma_3), -2(\gamma_1\pm\gamma_3), 2\gamma_1\}. \end{aligned} \tag{2.8}$$

If we, for the triple $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ defined by (2.8), have the additional condition that for the corresponding irreducible representations of $\mathfrak{sl}_{\mathbb{C}}(4)$:

$$\left(V_{\tilde{\alpha}} \otimes V_{\tilde{\beta}} \otimes V_{\tilde{\gamma}}\right)^{\mathfrak{sl}_{\mathbb{C}}(4)} > 0 \text{ when } \left(V_{N\tilde{\alpha}} \otimes V_{N\tilde{\beta}} \otimes V_{N\tilde{\gamma}}\right)^{\mathfrak{sl}_{\mathbb{C}}(4)} > 0,$$ [4]

then the possible $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ are precisely the ones for which the Littlewood-Richardson coefficient $c_{\tilde{\alpha}\tilde{\beta}}^{\tilde{\gamma}} \neq 0$, i.e. there will be skew-symmetric 6×6-matrices with zero sum and eigen-k-tuples $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$, whenever there are Hermitian 4×4-matrices with eigenvalues $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$.

[4] This imposes an additional condition on $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$, see [KT99] Section 7.

Part II

Chapter 3

Unitary highest weight representations of $\mathfrak{gl}_{\mathbb{C}}(n+1)$

In this part we will study tensor products of a certain kind of infinite dimensional, irreducible representations of $\mathfrak{gl}_{\mathbb{C}}(n+1)$ - those which arise from finite dimensional representations of the Lie group $U(n)$, as representations on polynomial spaces. Tensor products of infinite dimensional highest weight representations have been studied extensively, see e.g. [Rep79], [KV78] and [PZ04]. As mentioned in the introduction, it would be highly interesting to understand if the tensor products of infinite dimensional representations are related to moment mappings and eigenvalue problems.

We will soon see how the representations are constructed explicitly, but first we need to set some notation.

When we speak of $\mathbb{C}^n$, we assume it equipped with the Euclidian metric $\langle\, , \rangle_n$. We will consider polynomials in the vector variable $\mathbb{z}$, scalar-valued and vector-valued, which we will write:

$$\mathbb{p}(\mathbb{z}) = \sum_{k_1,\ldots,k_n} \mathbb{v}_{k_1,\ldots,k_n} \mathbb{z}^{k_1,\ldots,k_n},$$

where $\mathbb{v}_{k_1,\ldots,k_n}$ will lie in some representation space for $U(n)$, and for $\mathbb{z} = (z_1, \ldots, z_n)$ we use the short notation:

$$\mathbb{z}^{k_1,\ldots,k_n} := \prod_{i=1}^{n} z_i^{k_i}.$$

However, $\mathbb{C}^{n+1}$ will be equipped with the Hermitian metric:

$$(\mathbb{z}, \mathbb{z}) = |z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2 = \langle J\mathbb{z}, \mathbb{z}\rangle_{n+1},$$

where J is the matrix $\begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$. The Lie group $G_0 = U(n,1)$ is the subgroup of $GL_{\mathbb{C}}(n+1)$ preserving $(\, , \,)$, or equally

$$G_0 = \{g : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1} | g^* J g = J\},$$

implying its Lie algebra to be

$$\mathfrak{g}_0 = \mathfrak{u}(n,1) = \{X \in \mathrm{End}(\mathbb{C}^{n+1}) | X^* J + JX = 0\}.$$

The complexification $\mathfrak{g}_0^{\mathbb{C}}$ is evidently $\mathfrak{gl}_{\mathbb{C}}(n+1)$. According to the decomposition $\mathbb{C}^{n+1} = \mathbb{C}^n \oplus \mathbb{C}$, we write elements in these algebras as:

$$\begin{pmatrix} A & \mathbb{b} \\ \mathbb{c}^{\mathsf{T}} & d \end{pmatrix}$$

Letting $K_0 = G_0 \cap U(n+1)$ we get

$$K_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & e^{i\varphi} \end{pmatrix} | A \in U(n) \right\},$$

and the quotient space G_0/K_0 can be identified with the unit ball B^n, by letting G_0 act on B^n as rational linear transformations:

$$g = \begin{pmatrix} A & \mathbb{b} \\ \mathbb{c}^{\mathsf{T}} & d \end{pmatrix} \text{ acting by } \mathbb{z} \in B^n \mapsto \frac{A\mathbb{z} + \mathbb{b}}{\mathbb{c}^{\mathsf{T}}\mathbb{z} + d} \in B^n, \tag{3.1}$$

and observing that $K_0 = \{g \in G_0 | g0 = 0\}$.

We consider the defining action τ of $G \subseteq GL_{\mathbb{C}}(n)$ on $\mathbb{C}^n$, and the dual, or contragredient, action τ^* on the dual space $(\mathbb{C}^n)^*$, concretely:

$$\tau(g)\mathbb{v} = g\mathbb{v}$$

$$\tau^*(g)\mathbb{v} = (g^{-1})^{\mathsf{T}}\mathbb{v}.$$

The push forward $g_*(\mathbb{z}) : T_{\mathbb{z}}B^n \longrightarrow T_{g\mathbb{z}}B^n$ of the rational transformation (3.1) is simply:

$$\tau(g'(\mathbb{z})) : \mathbb{v} \mapsto ((\mathbb{c}^{\mathsf{T}}\mathbb{z} + d)A - (A\mathbb{z} + \mathbb{b})\mathbb{c}^{\mathsf{T}})\mathbb{v}(\mathbb{c}^{\mathsf{T}}\mathbb{z} + d)^{-2},$$

with Jacobian:

$$J_g(\mathbb{z}) = \det g_*(\mathbb{z}) = (\mathbb{c}^{\mathsf{T}}\mathbb{z} + d)^{-(n+1)}.$$

3.1 Construction of infinite dimensional representations

Now let $\{\mathcal{H}_i\}$ be dual basis for $(V_{\underline{k}}, \tau)$ be a representation of $U(n)$ with highest weight $\underline{k} = (k_1, \dots, k_n)$, and fix a number $\kappa > n+1$. We can define an infinite dimensional representation $(\mathcal{H}^\kappa_{\underline{k}}, \pi_{\kappa,\tau})$ on G_0, consisting of the weighted Bergman space $\mathcal{H}^\kappa_{\underline{k}}$ of all holomorphic functions $f : B^n \longrightarrow V_{\underline{k}}$ such that

$$\int_{B^n} \langle \tau((I - \mathbb{z}\mathbb{z}^*)^\intercal (1 - |\mathbb{z}|^2)) f(\mathbb{z}), f(\mathbb{z}) \rangle (1 - |\mathbb{z}|^2)^{\kappa - n - 1} dm(\mathbb{z}) < \infty,$$

which for $\kappa > n+1$ is non-trivial, and the endomorphism

$$\pi_{\kappa,\tau}(g) f(\mathbb{z}) = \tau((g_*^{-1}(\mathbb{z}))^\intercal f(g^{-1}\mathbb{z})(\mathbb{c}^\intercal \mathbb{z} + d)^{-\kappa} \tag{3.2}$$

Here, however, we are interested in regarding these representations as representations of the Lie algebra $\mathfrak{g}_0$ or its complexification $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{gl}_{\mathbb{C}}(n+1)$, and therefore we need to find a dense subspace of $\mathcal{H}^\kappa_{\underline{k}}$, stable under the action of $\mathfrak{g}_0$. The natural choice is the space of $V_{\underline{k}}$-valued polynomials:

$$\mathcal{P}^\kappa_{\underline{k}} = \mathcal{P} \otimes V_{\underline{k}},$$

where $\mathcal{P}$ is the scalar valued polynomials on $\mathbb{C}^n$, on which $\mathfrak{g}_0$ acts by differentiation of (3.2):

$$\pi_{\kappa,\underline{k}}(X) f = \frac{d}{d\epsilon}(\pi_{\kappa,\tau}(\exp(\epsilon X)) f)\Big|_{\epsilon=0},$$

where $X \in \mathfrak{g}_0$ is of the form $\left(\begin{smallmatrix} A & \mathbb{b} \\ \mathbb{c}^\intercal & d \end{smallmatrix}\right)$ with $A^* = -A$, $d^* = -d$, $\mathbb{b} = \mathbb{c}$. For our purposes we need to calculate this explicitly. For any $\epsilon > 0$ we see that:

$$\exp(-\epsilon X) = I - \epsilon X + o(\epsilon^2) = \left(\begin{smallmatrix} I - \epsilon A & -\epsilon \mathbb{b} \\ -\epsilon \mathbb{c}^\intercal & 1 - \epsilon d \end{smallmatrix}\right) + o(\epsilon^2) =: \left(\begin{smallmatrix} A_\epsilon & \mathbb{b}_\epsilon \\ \mathbb{c}_\epsilon^\intercal & d_\epsilon \end{smallmatrix}\right) =: g_\epsilon^{-1},$$

and hence that:

$$\begin{aligned} \frac{d}{d\epsilon}(\pi_{\kappa,\tau}(g_\epsilon) f)\Big|_{\epsilon=0} &= \frac{d}{d\epsilon}(\tau(g_{\epsilon *}^{-1}(\mathbb{z}))^\intercal)\Big|_{\epsilon=0} f(g_\epsilon^{-1}\mathbb{z})(\mathbb{c}_\epsilon^\intercal \mathbb{z} + d_\epsilon)^{-\kappa} \\ &+ \tau(g_{\epsilon *}^{-1}(\mathbb{z}))^\intercal \frac{d}{d\epsilon}(f(g_\epsilon^{-1}\mathbb{z}))\Big|_{\epsilon=0} (\mathbb{c}_\epsilon^\intercal \mathbb{z} + d_\epsilon)^{-\kappa} \\ &+ \tau(g_{\epsilon *}^{-1}(\mathbb{z}))^\intercal f(g_\epsilon^{-1}\mathbb{z}) \frac{d}{d\epsilon}((\mathbb{c}_\epsilon^\intercal \mathbb{z} + d_\epsilon)^{-\kappa})\Big|_{\epsilon=0}. \end{aligned} \tag{3.3}$$

This leaves us with six terms to look at more closely. The undifferentiated terms in this expression is easily calculated:

$$\begin{aligned} f(g_\epsilon^{-1}\mathbb{z}) &= f((A_\epsilon\mathbb{z}+\mathbb{b}_\epsilon)(\mathbb{c}_\epsilon^\intercal\mathbb{z}+d_\epsilon)^{-1}) = \\ &= f(((I-\epsilon A)\mathbb{z}-\epsilon\mathbb{b})(-\epsilon\mathbb{c}^\intercal\mathbb{z}+1-\epsilon d)^{-1}), \end{aligned}$$

which for $\epsilon = 0$ is merely $f(\mathbb{z})$. Furthermore

$$(\mathbb{c}_\epsilon^\intercal\mathbb{z}+d_\epsilon)^{-\kappa} = (-\epsilon\mathbb{c}^\intercal\mathbb{z}+1-\epsilon d)^{-\kappa},$$

which equals 1 when $\epsilon = 0$. We see that

$$\begin{aligned} \tau(g_{\epsilon*}^{-1}(\mathbb{z}))\mathbb{v} &= \tau(g_\epsilon^{-1}(\mathbb{z})'_{\mathbb{z}})\mathbb{v} \\ &= ((-\epsilon\mathbb{c}^\intercal\mathbb{z}+1-\epsilon d)(I-\epsilon A)-((I-\epsilon A)\mathbb{z}-\epsilon\mathbb{b})(-\epsilon\mathbb{c}^\intercal))\mathbb{v}\cdot \\ &\quad\cdot(-\epsilon\mathbb{c}^\intercal\mathbb{z}+1-\epsilon d)^{-2}+o(\epsilon^2) \\ &= (I-\epsilon(\mathbb{c}^\intercal\mathbb{z}+d+A-\mathbb{z}\mathbb{c}^\intercal))\mathbb{v}(1-\epsilon(\mathbb{c}^\intercal\mathbb{z}+d))^{-2}+o(\epsilon^2) \quad (3.4) \end{aligned}$$

which for $\epsilon = 0$ becomes simply $\mathbb{v}$, i.e. $\tau(g_{\epsilon*}^{-1}(\mathbb{z}))$ acts on any vector as the identity matrix. This simplifies (3.3) to:

$$\frac{d}{d\epsilon}(\tau(g_{\epsilon*}^{-1}(\mathbb{z}))^\intercal)f(\mathbb{z}) + \frac{d}{d\epsilon}(f(g_\epsilon^{-1}\mathbb{z})) + f(\mathbb{z})\,\frac{d}{d\epsilon}((\mathbb{c}_\epsilon^\intercal\mathbb{z}+d_\epsilon)^{-\kappa})\Big|_{\epsilon=0}.$$

The last term is easily calculated to be $\kappa(d+\mathbb{c}^\intercal\mathbb{z})f(\mathbb{z})$, and the middle term we leave as it is for now.

Looking again at $\tau(g_{\epsilon*}^{-1}(\mathbb{z}))$:s action on any vector $\mathbb{v}$, and making use of the Taylor expansion of $(1-x)^{-2}$, we see that:

$$\begin{aligned} (3.4) &= (I-\epsilon(\mathbb{c}^\intercal\mathbb{z}+d+A-\mathbb{z}\mathbb{c}^\intercal))\mathbb{v}(1+\epsilon(\mathbb{c}^\intercal\mathbb{z}+d))+o(\epsilon^2) = \\ &= \mathbb{v}+2\epsilon(\mathbb{c}^\intercal\mathbb{z}+d)\mathbb{v}-\epsilon(\mathbb{c}^\intercal\mathbb{z}+d+A-\mathbb{z}\mathbb{c}^\intercal)\mathbb{v}+o(\epsilon^2) = \\ &= \mathbb{v}+\epsilon(\mathbb{c}^\intercal\mathbb{z}+d-A+\mathbb{z}\mathbb{c}^\intercal)\mathbb{v}+o(\epsilon^2). \end{aligned}$$

Differentiating this and setting $\epsilon = 0$ this is just the action of $\tau(((d+\mathbb{c}^\intercal\mathbb{z})I + (\mathbb{z}\mathbb{c}^\intercal - A))^\intercal)$ on $\mathbb{v}$. Summarizing, we have that:

$$\begin{aligned} \pi_{\kappa,\underline{k}}(X)f := \frac{d}{d\epsilon}(\pi_{\kappa,\tau}(g_\epsilon^{-1})f)\Big|_{\epsilon=0} &= \\ = \tau(((d+\mathbb{c}^\intercal\mathbb{z})I+(\mathbb{z}\mathbb{c}^\intercal-A))^\intercal)f(\mathbb{z}) &+ \frac{d}{d\epsilon}(f(g_\epsilon^{-1}\mathbb{z}))\Big|_{\epsilon=0} \\ +\kappa(d+\mathbb{c}^\intercal\mathbb{z})f(\mathbb{z}). \end{aligned}$$

Proposition 3.1. *For any $\underline{k} = (k_1,\ldots,k_n)$, and a fix number $\kappa > n+1$, $\mathcal{P}_{\underline{k}}^\kappa$ is a highest weight representation of $\mathfrak{gl}_\mathbb{C}(n+1)$ with highest weight*

$$(-k_1,\ldots,-k_n,-|\underline{k}|-\kappa).$$

Proof. Remember that if we let $\mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$ be the root space decomposition of $\mathfrak{gl}_{\mathbb{C}}(n+1)$, every $N \in \mathfrak{n}^+$ and $H \in \mathfrak{h}$ are of the respective forms:

$$N = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ & & 0 & a_{n-1,n} & b_n \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} a_1 & 0 & \cdots & & 0 \\ 0 & a_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \\ & & & a_n & 0 \\ 0 & \cdots & & 0 & d \end{pmatrix}$$

Any highest weight vector $\mathbb{v}$ for the contragredient representation $\tilde{V}_{\underline{k}}$, seen as a constant polynomial $p(\mathbb{z}) = \mathbb{v}$, is just as well a highest weight vector for $\mathcal{P}^{\kappa}_{\underline{k}}$, since, for any $N \in \mathfrak{n}^+$:

$$\pi_{\kappa,\underline{k}}(N)\mathbb{v} = \tau(-A^{\intercal})\mathbb{v} = 0,$$

and for any $H \in \mathfrak{h}$:

$$\begin{aligned} \pi_{\kappa,\underline{k}}(H)\mathbb{v} =& \tau((dI - A)^{\intercal})\mathbb{v} + \kappa d\mathbb{v} \\ =& k_1(d - a_1) + \cdots + k_n(d - a_n) + \kappa d\mathbb{v}. \end{aligned}$$

Since the last basis vector in $\mathfrak{h}^*$ acts as $\mathcal{H}_{n+1}(H) = -d$, this implies that $\mathbb{v}$ is of the stated weight. $\square$

3.2 Tensor product decomposition

We are interested in finding irreducible factors of the tensor product of two such representations, when $\underline{k} = (0, \ldots, 0, k)$, $\underline{l} = (0, \ldots, 0, l)$. We write:

$$\mathcal{P}^{\kappa}_{k} \otimes \mathcal{P}^{\lambda}_{l}.$$

From [Pee94] we know that, restricting to scalar valued polynomials, we can factorize the tensor product:

$$\mathcal{P}^{\kappa} \otimes \mathcal{P}^{\lambda} = \mathcal{P}^{\kappa+\lambda} \otimes \bigoplus_{s=0}^{\infty} \mathcal{P}_s,$$

where $\mathcal{P}_s$ is the set of polynomials of degree s in the variable $\mathbb{z} - \mathbb{w}$; a finite dimensional representation of $\mathfrak{gl}_{\mathbb{C}}(n)$ of highest weight s. The $\mathfrak{gl}_{\mathbb{C}}(n+1)$-action π_κ is merely:

$$\pi_\kappa(X)f = \frac{d}{d\epsilon}(f(g_\epsilon^{-1}\mathbb{z}))\bigg|_{\epsilon=0} + \kappa(d + \mathbb{c}^{\intercal}\mathbb{z})f(\mathbb{z}). \tag{3.5}$$

In this case $(z_n - w_n)^s$ is a highest weight vector of highest weight

$$(0, \dots, 0, -s, -(\kappa + \lambda + s)).$$

Since $\mathcal{P}^\kappa_k = \mathcal{P}^\kappa \otimes \bigodot^k (\mathbb{C}^n)'$ - and we know all about tensor products of finite representations - it might be a fruitful idea to consider:

$$\mathcal{P}^\kappa_k \otimes \mathcal{P}^\lambda_l = (\mathcal{P}^\kappa \otimes \mathcal{P}^\lambda) \otimes (\bigodot^k (\mathbb{C}^n)' \otimes \bigodot^l (\mathbb{C}^n)'),$$

factorizing the "polynomial part" and the "vector space part" first. We know that

$$\bigodot^k (\mathbb{C}^n)' \otimes \bigodot^l (\mathbb{C}^n)' = \bigoplus_{t=0}^{\min\{k,l\}} V_{(0,\dots,0,t,m+l-t)},$$

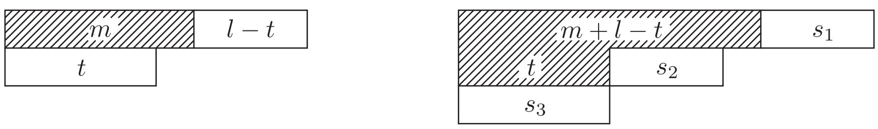
and if we tensor this sum by the highest weight representation of weight s we can factorize further:

$$\bigoplus_{t=0}^{\min\{k,l\}} V_{(0,\dots,0,t,k+l-t)} \otimes V_{(0,\dots,0,s)} = \bigoplus_{[V]} V_{(0,\dots,0,s_3,t+s_2,k+l+s_1-t)},$$

where the sum runs over the following parameters:

$$[V] = \begin{cases} 0 \le s \\ 0 \le t \le \min\{k, l\} \\ 0 \le s_3 \le \min\{t, s\} \\ 0 \le s_2 \le \min\{k + l - 2t, s\}, \end{cases}$$

which is easily seen by studying the Young diagrams below, which are to be filled with 1:s only.



We can even find highest weight vectors for these representations:

Lemma 3.1. *For*

$$\mathbb{v} := \odot^{k+l-s_2-2t} \mathbb{e}'_n \otimes (\mathbb{e}'_n \wedge \mathbb{e}'_{n-1})^{s_2-s_3+t} \otimes (\mathbb{e}'_n \wedge \mathbb{e}'_{n-1} \wedge \mathbb{e}'_{n-2})^{s_3},$$

we have that for any upper-triangular A with zero diagonal

$$\tau(-A^{\intercal})\mathbb{v} = 0,$$

and for every diagonal A

$$\tau((dI - A)^{\intercal})\mathbb{v} = (d(k + l + s_2 + s_3) - a_n(k + l - t) - a_{n-1}(s_2 + t) - a_{n-2}s_3)\mathbb{v}.$$

Proof. If A is upper-triangular, $-A^{\intercal}\mathbb{e}'_n = 0$, $-A^{\intercal}\mathbb{e}'_{n-1} = a_{1n}\mathbb{e}'_n$ and $-A^{\intercal}\mathbb{e}'_{n-2} = a_{2,n-1}\mathbb{e}'_{n-1} + a_{1,n-1}\mathbb{e}'_n$, we realize that no matter which term in the triple tensor product $\tau(-A^{\intercal})$ falls upon, the result will be zero.

Considering a diagonal A, we get:

$$\begin{aligned}\tau((dI - A)^{\intercal})\mathbb{v} =& (d - a_n)(k + l - s_2 - 2t)\mathbb{v} + (2d - a_n - a_{n-1})(s_2 - s_3 + t)\mathbb{v} \\ &+ (3d - a_n - a_{n-1} - a_{n-2})s_3\mathbb{v} \\ =& (d(k + l + s_2 + s_3) - a_n(k + l - t) - a_{n-1}(s_2 + t) - a_{n-2}s_3)\mathbb{v}\end{aligned}$$

□

We need two more lemmas to be able to prove our main result.

Lemma 3.2. *For any $N \in \mathfrak{n}^+$*

$$\pi_\kappa(N)\mathbb{z}^{0,\dots,0,i} = -ib_n\mathbb{z}^{0,\dots,0,i-1},$$

and for every $T \in \mathfrak{t}$

$$\pi_\kappa(T)\mathbb{z}^{0,\dots,0,i} = (\kappa d + (d - a_n)i)\mathbb{z}^{0,\dots,0,i}.$$

Proof. By (3.5) we have that:

$$\begin{aligned}\pi_\kappa(N)\mathbb{z}^{0,\dots,0,i} &= \frac{d}{d\epsilon}((g_\epsilon^{-1}\mathbb{z})^{0,\dots,0,i})\Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}((((I - \epsilon A)\mathbb{z} - \epsilon\mathbb{b})^{0,\dots,0,i})\Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}((z_n - \epsilon b_n)^i)\Big|_{\epsilon=0} \\ &= -ib_n z_n^{i-1},\end{aligned}$$

and that:

$$\begin{aligned}\pi_\kappa(T)\mathbb{z}^{0,\dots,0,i} &= \frac{d}{d\epsilon}\left(\left((I - \epsilon A)\mathbb{z}(1 - \epsilon d)^{-1}\right)^{0,\dots,0,i}\right)\Big|_{\epsilon=0} + \kappa d\mathbb{z}^{0,\dots,0,i} \\ &= \frac{d}{d\epsilon}\left(\left(\frac{(1 - \epsilon a_n)z_n}{1 - \epsilon d}\right)^i\right)\Big|_{\epsilon=0} + \kappa d z_n^i \\ &= i\left(\frac{(1 - \epsilon a_n)z_n}{1 - \epsilon d}\right)^{i-1}\left(\frac{-a_n z_n}{1 - \epsilon d} + \frac{(1 - \epsilon a_n)z_n d}{(1 - \epsilon)^2}\right)\Big|_{\epsilon=0} + \kappa d z_n^i \\ &= (i(d - a_n) + \kappa d)z_n^i.\end{aligned}$$

□

The second lemma we need is a minor binomial result:

Lemma 3.3. $\binom{s}{k} = \frac{s}{s-k}\binom{s-1}{k} = \frac{s}{k}\binom{s-1}{k-1}$.

Proof. By inspection. □

We are now able to prove our main theorem, which gives us a set of irreducible subrepresentations of the tensor product of two infinite dimensional, polynomial representations:

Theorem 3.1. *For every set of numbers $s_1 + s_2 + s_3 = s, t$ such that the conditions $[V]$ holds, the polynomial space*

$$\mathcal{P}^{\kappa+\lambda} \otimes \mathcal{P}_{s_1} \otimes \bigodot^{k+l-s_2-2t} (\mathbb{C}^n)' \otimes \bigodot^{s_2-s_3+t} ((\mathbb{C}^n)' \wedge (\mathbb{C}^n)') \otimes \bigodot^{s_3} ((\mathbb{C}^n)' \wedge (\mathbb{C}^n)' \wedge (\mathbb{C}^n)')$$

is an irreducible subrepresentation of $\mathcal{P}^\kappa_k \otimes \mathcal{P}^\lambda_l$, of highest weight

$$\underline{m}_{s_1,s_2,s_3,t} := (0, \dots, 0, -s_3, -(s_2 + t), -(k + l + s_1 - t), -(\kappa + \lambda + k + l + s)).$$

Proof. We will prove the theorem by showing that

$$\mathbb{p} = (z_n - w_n)^{s_1} \otimes \odot^{k+l-s_2-2t} \mathbb{e}'_n \otimes (\mathbb{e}'_n \wedge \mathbb{e}'_{n-1})^{s_2-s_3+t} \otimes (\mathbb{e}'_n \wedge \mathbb{e}'_{n-1} \wedge \mathbb{e}'_{n-2})^{s_3}$$

is a highest weight vector for $(\mathcal{P}^\kappa_k \otimes \mathcal{P}^\lambda_l, \pi)$, of the corresponding weight, i.e.:

$$\pi(X)\mathbb{p} = \begin{cases} 0 & \text{if } X \in \mathfrak{n}^+ \\ \underline{m}_{s_1,s_2,s_3,t}(X)\mathbb{p} & \text{if } X \in \mathfrak{h}. \end{cases}$$

First note that:

$$(z_n - w_n)^{s_1} = \sum_{i=0}^{s_1} (-1)^i \binom{s_1}{i} \mathbb{z}^{0,\dots,0,s_1-i} \otimes \mathbb{w}^{0,\dots,0,i},$$

so, with $\mathbb{v}$ as in Lemma 3.1 we may write

$$\mathbb{p} = \mathbb{v} \otimes \sum_{i=0}^{s_1} (-1)^i \binom{s_1}{i} \mathbb{z}^{0,\dots,0,s_1-i} \otimes \mathbb{w}^{0,\dots,0,i},$$

and hence that:

$$\begin{aligned} \pi(X)\mathbb{p} = &\sum_{i=0}^{s_1} (-1)^i \binom{s_1}{i} \pi_{\tau,\kappa}(X)(\mathbb{v} \otimes \mathbb{z}^{0,\dots,0,s_1-i}) \otimes \mathbb{w}^{0,\dots,0,i} \\ &+ \sum_{i=0}^{s_1} (-1)^i \binom{s_1}{i} \mathbb{v} \otimes \mathbb{z}^{0,\dots,0,s_1-i} \otimes \pi_{\tau,\lambda}(X)\mathbb{w}^{0,\dots,0,i}. \end{aligned}$$

For any $N \in \mathfrak{n}^+$ this implies:

$$\begin{aligned}\pi_{\tau,\kappa}(N)(\mathbb{v} \otimes \mathbb{z}^{0,\dots,0,s_1-i}) =&\tau(-A^\intercal)\mathbb{v} \otimes \mathbb{z}^{0,\dots,0,s_1-i} \\ &+\frac{d}{d\epsilon}(\mathbb{v} \otimes ((I-\epsilon A)\mathbb{z} - \epsilon\mathbb{b})^{0,\dots,0,s_1-i})|_{\epsilon=0}.\end{aligned}$$

Lemma 3.1 says that the first term of this expression zero, and since

$$((I-\epsilon A)\mathbb{z} - \epsilon\mathbb{b})^{0,\dots,0,s_1-i}) = (z_n - \epsilon b_n)^{s_1-i},$$

we see that the second term, and hence the whole expression, is:

$$-b_n(s_1 - i)\mathbb{v} \otimes \mathbb{z}^{0,\dots,0,s_1-i-1}.$$

Together with the first part of Lemma 3.2 this gives:

$$\begin{aligned}\pi(N)\mathbb{p} =&\sum_{i=0}^{s_1}(-1)^i\binom{s_1}{i}(-b_n(s_1-i)\mathbb{v} \otimes \mathbb{z}^{0,\dots,0,s_1-i-1}) \otimes \mathbb{w}^{0,\dots,0,i} \\ &+\sum_{i=0}^{s_1}(-1)^i\binom{s_1}{i}\mathbb{v} \otimes \mathbb{z}^{0,\dots,0,s_1-i} \otimes (-ib_n\mathbb{w}^{0,\dots,0,i-1}),\end{aligned}$$

which we can rewrite with the help of Lemma 3.3, and the fact that the first term equals zero when $i = s_1$, while the second term equals zero when $i = 0$, as

$$\begin{aligned}-b_n\mathbb{v}\otimes &\left(\sum_{i=0}^{s_1-1}(-1)^i\binom{s_1-1}{i}\mathbb{z}^{0,\dots,0,s_1-i-1} \otimes \mathbb{w}^{0,\dots,0,i}\right. \\ &\left. - \sum_{i=1}^{s_1}(-1)^{i-1}\binom{s_1-1}{i-1}\mathbb{z}^{0,\dots,0,s_1-i} \otimes \mathbb{w}^{0,\dots,0,i-1}\right).\end{aligned}$$

Now, simply changing the variable $j = i - 1$ in the second term, we realize that this is zero, in short:

$$\pi(N)\mathbb{p} = 0.$$

For any $H \in \mathfrak{h}$ we have that:

$$\begin{aligned}\pi_{\tau,\kappa}(H)(\mathbb{v} \otimes \mathbb{z}^{0,\dots,0,s_1-i}) =&\tau((dI-A)^\intercal)\mathbb{v} \otimes \mathbb{z}^{0,\dots,0,s_1-i} \\ &+\frac{d}{d\epsilon}(\mathbb{v} \otimes (((I-\epsilon A)\mathbb{z})(1-\epsilon d)^{-1})^{0,\dots,0,s_1-i})|_{\epsilon=0} \\ &+\kappa d(\mathbb{v} \otimes \mathbb{z}^{0,\dots,0,s_1-i}).\end{aligned} \tag{3.6}$$

Let us start by considering the second term. Noting that

$$((I-\epsilon A)\mathbb{z})(1-\epsilon d)^{-1})^{0,\dots,0,s_1-i} = \left(\frac{1-\epsilon a_n}{1-\epsilon d}z_n\right)^{s_1-i},$$

the term is easily calculated as

$$(s_1 - i)(d - a_n)(\mathbb{v} \otimes \mathbb{z}^{0,\ldots,0,s_1-i}).$$

The second statement in Lemma 3.1 tells us what the first term is, and summarizing the terms we get:

$$\begin{aligned}(3.6) = &(d(\kappa + k + l + s_2 + s_3) - a_n(m + l - t) - a_{n-1}(t + s_2) \\ &- a_{n-2}s_3 + (d - a_n)(s_1 - i))\,(\mathbb{v} \otimes \mathbb{z}^{0,\ldots,0,s_1-i}).\end{aligned}$$

Using the second part of Lemma 3.2, this leads us to

$$\begin{aligned}\pi(H)\mathbb{p} =&(d(\kappa + k + l) - a_n(m + l + s_2 + s_3 - t) - a_{n-1}(t + s_2) - a_{n-2}s_3)\mathbb{p} \\ &+ \sum_{i=0}^{s_1}(-1)^i\binom{s_1}{i}(s_1 - k)(d - a_n)\mathbb{v} \otimes \mathbb{z}^{0,\ldots,0,s_1-i} \otimes \mathbb{w}^{0,\ldots,0,i} \\ &+ \sum_{i=0}^{s_1}(-1)^i\binom{s_1}{i}k(d - a_n)\mathbb{v} \otimes \mathbb{z}^{0,\ldots,0,s_1-i} \otimes \mathbb{w}^{0,\ldots,0,i} + \lambda d\mathbb{p} \\ &= (d(\kappa + \lambda + k + l + s) - a_n(m + l + s_1 - t) - \\ &\qquad\qquad -a_{n-1}(t + s_2) - a_{n-2}s_3))\,\mathbb{p} \\ &= \underline{m}_{s_1,s_2,s_3,t}(H)\mathbb{p}.\end{aligned}$$

□

We have thus found a set of irreducible subrepresentations of the tensor product $\mathcal{P}_k^\kappa \otimes \mathcal{P}_l^\lambda$. A subsequent task is obviously to try to characterize all of them, decomposing the tensor product totally. From [Rep79] we get a recursive formula for the multiplicities, which might be helpful for groups of small rank. However, for general higher rank groups one has to develop new ideas.

An interesting question is whether it would be possible to elaborate the theory of honeycombs (see [KT99]) for infinite dimensional highest weight representations, and where this would take us.

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