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A New Robustness Index for Multi-Objective Optimization based on a User Perspective

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Abstract

Solving practical optimization problems that are sensitive to small changes in the variables or model parameters requires special attention regarding the robustness of solutions. We present a definition of a new robustness index for multi-objective optimization problems. The definition is based on an approximation of the underlying utility function for a single decision maker. We further demonstrate an efficient computational procedure to evaluate robustness. The procedure is applied to two numerical examples: one analytic test problem and one real-world problem in antenna design. The results show that the robustness varies over the Pareto front and that it can be improved if the decision maker is willing to sacrifice in optimality of the solution.

Keywords: multi-objective optimization, vector optimization, robustness, multi-criteria decision making, utility theory

1 Introduction

Many applications of optimization comprise several more or less conflicting objectives, such as cost versus quality and expected return versus risk. These are to be optimized simultaneously and the aim is to find the most appropriate balance between all of the objectives. Mathematically, such a problem is denoted a *multi-objective optimization problem* (MOOP) and is formulated as that to

$$\underset{\mathbf{x}\in X}{\text{minimize}} \ \ (f_1(\mathbf{x}), \dots, f_k(\mathbf{x}))\,. \tag{1}$$

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Here, $\mathbf{x} \in \mathbb{R}^n$ denotes a vector of decision variables, $X \subseteq \mathbb{R}^n$ is the feasible decision space, and each function $f_i : X \to \mathbb{R},\ i = 1, \dots, k$, is an objective function to be minimized. Since minimization of a vector in general is not well defined, the notion of optimality for multi-objective problems is different compared to that of single-objective problems. Optimality is here based on dominance, and the following standard definition is used.

DEFINITION 1.1 (Pareto Optimality) *A feasible solution* $\hat{\mathbf{x}} \in X$ *is called* Pareto optimal *if there exists* no *vector* $\mathbf{x} \in X$ *such that* $f_i(\mathbf{x}) \leq f_i(\hat{\mathbf{x}}),\ i = 1, \dots, k$, *with at least one inequality holding strictly. The set of all Pareto optimal solutions is denoted* $\mathcal{P} \subseteq X$.

It is reasonable to assume that a decision maker (DM) prefers solution in this set, since for every feasible decision vector outside of $\mathcal{P}$, there is a solution in $\mathcal{P}$ which is at least as good in all objectives (and strictly better in at least one).

The perhaps most intuitive method for solving a MOOP, i.e., to find $\mathcal{P}$ or at least a good approximation of $\mathcal{P}$, is to solve a sequence of standard optimization problems of the form

$$\underset{\mathbf{x} \in X}{\text{minimize}} \sum_{i=1}^{k} w_i f_i(\mathbf{x}), \tag{2}$$

where the multiple objectives are transformed to single objectives by varying the weight vector $\mathbf{w}$ within the set $\{\mathbf{v} \in \mathbb{R}^k \mid \sum_{i=1}^{k} v_i = 1,\ v_i \geq 0,\ i = 1, \dots, k\}$. This solution strategy suffers from serious limitations, such as that it is possible to find only the subset of $\mathcal{P}$ which is mapped onto the convex part of the Pareto front [31], and also that the mapping between $\mathbf{w}$ and the optimal values to (2), i.e., $\mathbb{R}^k \ni \mathbf{w} \mapsto \min_{\mathbf{x} \in X} \sum_{i=1}^{k} w_i f_i(\mathbf{x}) \in \mathbb{R}^k$, is non-linear and strongly depends on the properties of the functions involved [10]. To avoid finding *weakly Pareto optimal solutions* (where the strict inequality requirement in Def. 1.1 is dropped), the weights must be strictly positive. Despite of its limitations, the weighting strategy is fundamental, and it is used as a starting point for the definition of robustness presented in this paper.

Observe that a DM seeks *one* final solution which is preferred or optimal to him / her in the sense of balancing the different criteria. The reason for using a multi-objective formulation is to push forward the decisions until more knowledge is revealed about the characteristics and the limitations of the problem at hand. With this in mind, a multi-objective problem can often be viewed as a single-objective optimization problem with the caveat that the single objective function is not explicitly known. Although the multi-objective formulation is “DM independent”, the corresponding single objective problem is not.

This single-objective optimization problem can be formulated as that to

$$\underset{\mathbf{x} \in X}{\text{minimize}}\ u\left(f_1(\mathbf{x}), \dots, f_k(\mathbf{x})\right), \tag{3}$$

where $u : \mathbb{R}^k \to \mathbb{R}$, the *utility* or *value function*, is the single objective. The ideas developed in this paper is based on this formulation. Note that the utility function is used to quantitatively capture the preferences of the DM with a scalar function, thus enabling a total ordering of the feasible decision vectors for each DM. In this paper the convention that a smaller utility value is better than a larger is used. See [28] for extensive material on decision problems and utility functions.

Incidentally, compared to the value function method optimization [31], our approach can be seen as an inverse problem: we start with a solution on the Pareto front and seek an objective u such that its minimum would result in our initial solution.

1.1 Robustness in single- and multi-objective optimization

An optimal solution which is sensitive to perturbations in the data is often not useful in a practical application. A natural approach to deal with this situation is to incorporate the uncertainty into the model. This approach is used in Stochastic Programming (SP) (cf. [8, 27]) and Robust Optimization (RO) (cf. [5]). In SP, the objective function is typically the expected value over all uncertain parameters, which implies that an optimal solution is good on average. In RO, feasibility is required for all outcomes of the uncertain parameters, which produces a "conservative" optimal solution. Although most RO theory is restricted to convex problems with an explicit objective function, there are some recent development of RO methods also for nonconvex as well as simulation-based optimization problems (cf. [6, 7]).

There are, however, situations where it is not suitable or even possible to remodel the problem including robustness directly, but where there is an interest in assessing the robustness of an optimal solution in a post-process. This could be the case, e.g., if the risk aversion of the DM is unknown, or if the mathematical modeling of the uncertainty is not very clear. This opens up the question of how robustness is evaluated. Considering a single-objective problem, we can use the sensitivity of the objective value at an optimal solution as a measure of robustness, but for multi-objective problems this is less straightforward. For such problems we have to quantify the uncertain responses in the objective space; see Figure 1 for an illustration.

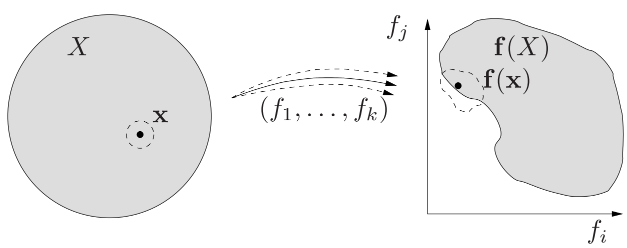


Figure 1: Uncertainties in $\mathbf{x}$ (e.g. implementation precision) and in $\mathbf{f} = (f_1, \ldots, f_k)$ (e.g. in model parameters) lead to uncertain responses $\mathbf{f}(\mathbf{x})$ in the objective space.

Among the many papers published on robust optimization, only few concerns multi-objective optimization. One has to distinguish between robust multi-objective optimization for which robustness is one objective and performance is the other, i.e., single-objective problems where the robustness of the solutions is taken as an objective in itself (cf. [9, 25]), and our interpretation of robust multi-objective optimization where the wish is to find robust solutions to a multi-objective optimization problem. For the latter, Deb and Gupta [11, 12, 13] have made a direct extension of SP by using averaged values of the objective functions over a small ball around the intended value in the decision space to define a robust Pareto front (cf. Section 4.1). The work of Deb and Gupta may be seen as the current reference work in this area of research. In this framework, solutions are classified as robust or not, without any further grading. This is noted by Barrico and Antunes [2, 3, 4]. The authors introduce the concept of *degree of robustness*, with which decision vectors are graded with respect to how the corresponding objective vectors are affected by small variations of the values of the decision vectors and / or small variations of model parameters. Gunawan and Azarm [19] use a similar idea for robustness that relates the change in the objective value due to changes in the model parameters. This measure is also used, in a slightly different way, by Li et al. [30]. In another recent work, Gaspar-Cunha and Covas [17] also use a graded measure of the robustness, here with the aim of finding the most robust parts of the Pareto front. The robustness of a decision vector is measured by the expectation and the variance of the fitness (in an evolutionary algorithm) of the decision vector. In [20], Hassan and Clark note that the relative positions of objective vectors might alter when uncertain parameters take on different values, but that a DM should be interested in a solution which maintains the characteristics of its objectives such that it still fulfills his / her preferences. Therefore, the authors define robustness by measuring how well points stay in certain clusters based on the current objectives and how well they keep their rank w.r.t. the different objectives. The method is described in a setting for a specific application, but the main idea is similar to ours. The way of measuring the characteristics of the objective functions, and thus the defintion of robustness, however differs. The method by Hassan and Clark may lead to bad decisions since objective vectors will not necessarily keep their utility properties.

Compared to the current methods in the literature, a weakness of our definition of robustness is the fact that the robustness cannot be computed until the (unperturbed) Pareto front is known, thus implying that it cannot be used as an objective in itself during the optimization. The major strength, however, is that the measure is more elaborate than the ones previously suggested, and it should in a better way capture the true preferences of the DM and thus in a better way measure what he / she cares about.

1.2 Outline

In Section 2, we discuss and construct utility functions for specific DMs evaluating the objectives from his / her point of view. We also present some mathematical properties of these functions and define two measures of robustness based on them. In Section 3, we then discuss the computation of these robustness measures, or indices. Depending on the properties of the problem in terms of constraints and differentiability, we suggest two approaches to compute approximations of these measures. We also deal with the search for robust solutions. Instead of just assessing the robustness of the Pareto solutions, we state an optimization problem with the goal to find robust, near-optimal solutions. In Section 4 we present two numerical examples. The first uses a known multi-objective test problem and the second considers a real-world problem instance in antenna design. Finally, in Section 5, we summarize the article and suggest some future work.

1.3 Summarizing the main idea

To quantify the change in the objective space due to uncertainties in the decision space and in the objective functions, we find an approximate utility function for each potential DM capturing his / her preferences. The robustness of a solution is then measured using this function. With this approach, the computation of robustness must be considered as a post-process, since the preferences of the DM depend on the Pareto front.

The idea is to present a set of candidate solutions that are robust and constitute a reasonable approximation of the Pareto front. This implies that robustness can be treated as an objective in itself, which is natural in a multi-objective setting.

2 Robustness based on a utility function

This section begins with a technical note on the requirements that we pose on a utility function. As we shall see, these requirements ensure that the function accurately captures a reasonable DM. The section then continues with the presentation of two robustness indices based on the utility functions.

2.1 Properties of the utility function

We begin with a definition whose requirements should be fulfilled for a reasonable utility function. In the literature, this definition coincides with the notion of a utility function being consistent with the preference structure (which in our case is given by the $\leq$ relation) [28].

DEFINITION 2.1 (rationality) *A utility function* $u : \mathbb{R}^k \to \mathbb{R}$ *is* rational *if for* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\mathbf{y})$ *implies that* $(u \circ \mathbf{f})(\mathbf{x}) \leq (u \circ \mathbf{f})(\mathbf{y})$. *A DM is* rational *if his/her associated utility function is rational.*

Rationality means that if a point $\mathbf{y}$ is dominated by a point $\mathbf{x}$, then $\mathbf{x}$ must be appreciated as at least as good as $\mathbf{y}$ by the DM.

With the above definition of rationality, the following proposition shows how rational utility functions can be characterized.

PROPOSITION 2.2 *The utility function* u *is rational if and only if* $u(f_1, \ldots, f_k)$ *is monotonically increasing with each* $f_i,\ i = 1, \ldots, k$.

Proof If u is monotonically increasing in each argument it holds that $u(\mathbf{f}(\mathbf{x})) \leq u(\mathbf{f}(\mathbf{y}))$ whenever $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\mathbf{y})$, i.e., u is rational. Suppose now that u is rational, but not monotonically increasing, i.e., $\exists \mathbf{f} \in \mathbb{R}^k$, $j \in \{1, \ldots, k\}$ and $\varepsilon > 0$ such that $u(f_1, \ldots, f_j, \ldots, f_k) > u(f_1, \ldots, f_j + \varepsilon, \ldots, f_k)$. But u is rational and hence, since $(f_1, \ldots, f_j, \ldots, f_k) \leq (f_1, \ldots, f_j + \varepsilon, \ldots, f_k)$, it holds that $u(f_1, \ldots, f_j, \ldots, f_k) \leq u(f_1, \ldots, f_j + \varepsilon, \ldots, f_k)$. This is a contradiction, whence u must be monotonically increasing.

We make the following assumption on the function values of the Pareto solutions.

Assumption A The Pareto set satisfies $\mathbf{f}(\mathcal{P}) \subseteq (0, 1]^k$.

If the range of f over X is bounded, it is always possible to scale the objectives such that this assumption holds true.

The utility functions that we will assign to potential DMs are assumed to have the following additive form:

$$u(\mathbf{f}) := \sum_{i=1}^{k} w_i f_i^{\alpha}, \tag{4}$$

where $\mathbf{w} > \mathbf{0}_+^k$ are weights and $\alpha \geq 1$ is a parameter that controls the curvature of the function. Although simple, this expression does have some attractive properties as we shall see.

Before continuing, we remark that the general design of utility functions is a subject of multi-attribute utility theory (MAUT). For a background on this subject we refer to [28, 34]. The utility function designed in this paper follows a different approach than the traditional in MAUT since it is based on the Pareto front entirely, and does not require any direct interactions with a DM.

We define a family of utility functions:

DEFINITION 2.3 (attainable utility functions) *A family $\mathcal{U}$ of* attainable *utility functions is defined as*

$$\mathcal{U} = \left\{ \sum_{i=1}^{k} w_i f_i^{\alpha} \;\middle|\; w_i > 0,\ i = 1, \ldots, k;\ \alpha \in [1, \infty) \right\}. \tag{5}$$

We associate a utility function to each candidate vector $\bar{\mathbf{x}} \in X$, i.e., to any solution that a DM might be interested in. If $\bar{\mathbf{x}} \in \mathcal{P}$, then $\mathbf{w}$ and α are chosen such that α is as small as possible and that $\bar{\mathbf{x}} \in \arg\min \{u \circ \mathbf{f}(\mathbf{x}) \mid \|\nabla_f (u \circ \mathbf{f})(\mathbf{x})\|_1 = 1\}$. This means that $\mathbf{w}$ is selected such that u is minimized at $\bar{\mathbf{x}}$, and that thereafter u is scaled to be of similar value for different values of α. Note that for $\alpha = 1$, the reasonable requirement $\sum_i w_i = 1$ is implied.

In the following, we present a few properties of the family of utility functions in (5). The main goal is to show that the family is rational, and complete with respect to certain Pareto optimal points in a sense to be defined below. These are points that can be reached using a utility function in the family $\mathcal{U}$, and we will use the notion of *proper Pareto optimality* to identify them.

We first define completeness for a general family of utility functions.

DEFINITION 2.4 (completeness) *A family of utility functions* $\mathcal{U}$ *is* complete with respect to a set $\hat{\mathcal{P}} \subseteq \mathcal{P}$ *if for every* $\mathbf{x}^* \in \hat{\mathcal{P}}$ *there exists a* $u \in \mathcal{U}$ *such that*

$$\mathbf{x}^* \in \arg\min_{\mathbf{x}\in X} u\big(f_1(\mathbf{x}), \dots, f_k(\mathbf{x})\big).$$

This means that, in a complete family, for each $\mathbf{x}^* \in \hat{\mathcal{P}} \subseteq \mathcal{P}$ there is at least one utility function that evaluates $\mathbf{x}^*$ as a best one. A *good* family of utility functions is both rational and complete with respect to a set which is a close approximation to $\mathcal{P}$. We will show that the family (5) is good.

PROPOSITION 2.5 *The family of utility functions defined by* (4) *is rational.*

Proof Since $\mathbf{w} \geq \mathbf{0}^k$ and $\alpha \geq 1$, all $u \in \mathcal{U}$ are monotonically increasing in all their arguments; the result then follows immediately from Prop. 2.2.

Geoffrion [18] introduced the notion of *proper* Pareto optimality to exclude some Pareto optimal solutions that are insensible to reasonable DMs.

DEFINITION 2.6 (proper Pareto optimality) *A feasible solution* $\hat{\mathbf{x}} \in X$ *to* (1) *is called* proper Pareto optimal in the sense of Geoffrion *if it is Pareto optimal in* (1) *and if there exists a number* $M > 0$ *such that for each* $i \in \{1, \dots, k\}$ *and each* $\mathbf{x} \in X$ *satisfying* $f_i(\mathbf{x}) < f_i(\hat{\mathbf{x}})$, *there exists a* $j \in \{1, \dots, k\} \backslash \{i\}$ *such that* $f_j(\hat{\mathbf{x}}) < f_j(\mathbf{x})$ *and*

$$\frac{f_i(\hat{\mathbf{x}}) - f_i(\mathbf{x})}{f_j(\mathbf{x}) - f_j(\hat{\mathbf{x}})} \leq M. \tag{6}$$

We denote the set of all proper Pareto vectors in the sense of Geoffrion by $\mathcal{P}_{\mathrm{p}}$.

A vector $\mathbf{x}$ is properly Pareto optimal in the sense of Geoffrion if it has finite trade-offs between the objectives. We make a somewhat different definition of proper Pareto optimality based on the family of utility functions (5).

DEFINITION 2.7 (firmly proper Pareto optimality) *A feasible solution to* (1) *is called* firmly proper Pareto optimal *if it is the minimizer of* (3) *for some utility function* u *in the family* $\mathcal{U}$ *defined in* (5)*. We denote the set of all firmly proper Pareto vectors by* $\mathcal{P}_{\mathrm{fp}}$.

Figure 2 illustrates some firmly proper, proper and non-proper Pareto optimal solutions. The definition of firmly proper Pareto optimal points implies that the family of utility functions is complete with respect to these points. The question now is which points that are firmly proper.

The two following propositions show that firmly proper Pareto optimal solutions are indeed Pareto optimal, and that these solutions are also proper in the sense of Geoffrion.

PROPOSITION 2.8 *Under Assumption* A*, each firmly proper Pareto optimal solution to* (1) *is a Pareto optimal solution, i.e.,* $\mathcal{P}_{\text{fp}} \subseteq \mathcal{P}$.

Proof Suppose that $\mathbf{x} \in X \backslash \mathcal{P}$. Then, $\exists \mathbf{y} \in X$ such that $f_i(\mathbf{y}) \leq f_i(\mathbf{x}),\ i = 1, \ldots, k$, with $f_j(\mathbf{y}) < f_j(\mathbf{x})$ for some index j. It follows that $u(\mathbf{f}(\mathbf{y})) = \sum_{i=1}^k w_i f_i(\mathbf{y})^\alpha < \sum_{i=1}^k w_i f_i(\mathbf{x})^\alpha = u(\mathbf{f}(\mathbf{x}))$ since $w_j > 0$, $f_j(\mathbf{y}) < f_j(\mathbf{x})$, $\alpha \geq 1$, and $\mathbf{f}(\mathbf{y}) > \mathbf{0}^k$. Thus $\mathbf{x} \notin \mathcal{P}_{\text{fp}}$ and the proposition follows.

PROPOSITION 2.9 *Under Assumption* A*, each firmly proper Pareto optimal solution to* (1) *is a proper Pareto optimal solution, i.e.,* $\mathcal{P}_{\text{fp}} \subseteq \mathcal{P}_{\text{p}}$.

Proof Choose a vector $\mathbf{w} > \mathbf{0}^k$ and an $\alpha \geq 1$. We define $\mathcal{K} = \{1, \ldots, k\}$ and let $\mathbf{x}^* \in \arg\min_{\mathbf{x} \in X} \sum_{k \in \mathcal{K}} w_i f_i(\mathbf{x})^\alpha$. From Definition 2.7 follows that $\mathbf{x}^* \in \mathcal{P}_{\text{fp}}$ and Proposition 2.8 then implies that $\mathbf{x}^* \in \mathcal{P}$.

Now, assume that $\mathbf{x}^* \notin \mathcal{P}_{\text{p}}$. Then for every $M > 0$, there exists an $\bar{\imath} = i(M) \in \mathcal{K}$ and an $\bar{\mathbf{x}} = \mathbf{x}(M) \in X$, with $f_{\bar{\imath}}(\bar{\mathbf{x}}) < f_{\bar{\imath}}(\mathbf{x}^*)$ such that $\frac{f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})}{f_j(\bar{\mathbf{x}}) - f_j(\mathbf{x}^*)} > M$ for every $j \in \mathcal{K} \setminus \{\bar{\imath}\}$ for which the inequality $f_j(\mathbf{x}^*) < f_j(\bar{\mathbf{x}})$ holds. We will show that this assumption leads to the conclusion that $\mathbf{x}^* \notin \mathcal{P}_{\text{fp}}$, which contradicts the above definition of $\mathbf{x}^*$.

From the characterization of $\bar{\mathbf{x}}$ and $\bar{\imath}$ it follows that $f_j(\bar{\mathbf{x}}) < f_j(\mathbf{x}^*) + \frac{1}{M}[f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})]$, $j \in \mathcal{K} \setminus \{\bar{\imath}\}$. By the definition (4) it then follows that

$$\begin{aligned} u(\mathbf{f}(\mathbf{x}^*)) - u(\mathbf{f}(\bar{\mathbf{x}})) &= \sum_{j \in \mathcal{K}} w_j \left[f_j(\mathbf{x}^*)^\alpha - f_j(\bar{\mathbf{x}})^\alpha \right] \\ &> w_{\bar{\imath}} \left[f_{\bar{\imath}}(\mathbf{x}^*)^\alpha - f_{\bar{\imath}}(\bar{\mathbf{x}})^\alpha \right] + \sum_{j \in \mathcal{K} \setminus \{\bar{\imath}\}} \left(w_j \left[f_j(\mathbf{x}^*)^\alpha - \left(f_j(\mathbf{x}^*) + \frac{1}{M} \left[f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}}) \right] \right)^\alpha \right] \right) \\ &= w_{\bar{\imath}} \left[f_{\bar{\imath}}(\mathbf{x}^*)^\alpha - f_{\bar{\imath}}(\bar{\mathbf{x}})^\alpha \right] + \sum_{j \in \mathcal{K} \setminus \{\bar{\imath}\}} \left(w_j f_j(\mathbf{x}^*)^\alpha \left[1 - \left(1 + \frac{f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})}{M f_j(\mathbf{x}^*)} \right)^\alpha \right] \right). \end{aligned}$$

Due to Assumption A and the definitions of $\mathbf{x}^*$, $\bar{\mathbf{x}}$, and $\bar{\imath}$, the strict inequalities $0 < f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}}) < 1$ hold. Therefore, for every $M \geq M^* := \left(\min_{j \in \mathcal{K}} \{ f_j(\mathbf{x}^*) \} \right)^{-1}$ it holds that $0 < \frac{f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})}{M f_j(\mathbf{x}^*)} < 1$. We can thus utilize the exact Taylor expansion

$$(1 + y)^\alpha = 1 + \alpha(1 + \theta y)^{\alpha - 1} y, \quad \text{for all } y \in (-1, 1) \text{ for some } \theta = \theta(y) \in (0, 1), \tag{7}$$

to conclude that the relations

$$\begin{aligned} & u(\mathbf{f}(\mathbf{x}^*)) - u(\mathbf{f}(\bar{\mathbf{x}})) \\ &> w_{\bar{\imath}} \left[f_{\bar{\imath}}(\mathbf{x}^*)^\alpha - f_{\bar{\imath}}(\bar{\mathbf{x}})^\alpha \right] + \sum_{j \in \mathcal{K} \setminus \{\bar{\imath}\}} \left(-w_j f_j(\mathbf{x}^*)^\alpha \alpha \left[1 + \theta \frac{f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})}{M f_j(\mathbf{x}^*)} \right]^{\alpha - 1} \left[\frac{f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})}{M f_j(\mathbf{x}^*)} \right] \right) \\ &= \left[f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}}) \right] \left(w_{\bar{\imath}} \frac{f_{\bar{\imath}}(\mathbf{x}^*)^\alpha - f_{\bar{\imath}}(\bar{\mathbf{x}})^\alpha}{f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})} - \sum_{j \in \mathcal{K} \setminus \{\bar{\imath}\}} \left[\frac{w_j f_j(\mathbf{x}^*)^{\alpha - 1} \alpha}{M} \left(1 + \theta \frac{f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})}{M f_j(\mathbf{x}^*)} \right)^{\alpha - 1} \right] \right) \end{aligned}$$

hold for all $M \geq M^*$. The presumptions that $\mathbf{w} > \mathbf{0}^k$ and $\alpha \geq 1$, Assumption A (implying that $0 < f_j(\mathbf{x}^*)^{\alpha-1} \leq 1$), and the conclusion that $\mathbf{x}^* \in \mathcal{P}$ (implying that $f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}}) < 1$) then yields that the strict inequality

$$u(\mathbf{f}(\mathbf{x}^*)) - u(\mathbf{f}(\bar{\mathbf{x}})) > \\ \left[f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})\right] \left(w_{\bar{\imath}} \frac{f_{\bar{\imath}}(\mathbf{x}^*)^\alpha - f_{\bar{\imath}}(\bar{\mathbf{x}})^\alpha}{f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})} - \frac{\alpha}{M} \sum_{j \in \mathcal{K} \setminus \{\bar{\imath}\}} w_j \left[1 + \frac{1}{M f_j(\mathbf{x}^*)}\right]^{\alpha-1} \right),$$

holds for all $M \geq M^*$. Utilizing the fact that the inequality $\frac{f_{\bar{\imath}}(\mathbf{x}^*)^\alpha - f_{\bar{\imath}}(\bar{\mathbf{x}})^\alpha}{f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})} \geq f_{\bar{\imath}}(\mathbf{x}^*)^{\alpha-1}$ holds whenever $0 \leq f_{\bar{\imath}}(\bar{\mathbf{x}}) < f_{\bar{\imath}}(\mathbf{x}^*)$ and $\alpha \geq 1$, it then follows that

$$\begin{aligned} & u(\mathbf{f}(\mathbf{x}^*)) - u(\mathbf{f}(\bar{\mathbf{x}})) \\ & > \left[f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})\right] \left(w_{\bar{\imath}} f_{\bar{\imath}}(\mathbf{x}^*)^{\alpha-1} - \frac{\alpha}{M} \sum_{j \in \mathcal{K} \setminus \{\bar{\imath}\}} \left[w_j \left(1 + \frac{1}{M \min_{i \in \mathcal{K}} \{f_i(\mathbf{x}^*)\}}\right)^{\alpha-1} \right] \right) \\ & \geq \left[f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}})\right] \left(\min_{j \in \mathcal{K}} \left\{ w_j f_j(\mathbf{x}^*)^{\alpha-1} \right\} - \frac{\alpha}{M} \left(1 + \frac{M^*}{M}\right)^{\alpha-1} \sum_{j \in \mathcal{K} \setminus \{\bar{\imath}\}} w_j \right), \end{aligned}$$

for all $M \geq M^*$. By the assumptions made, the strict inequality $f_{\bar{\imath}}(\mathbf{x}^*) - f_{\bar{\imath}}(\bar{\mathbf{x}}) > 0$ holds for all values of $M > 0$. Further, the strict inequalities $w_j f_j(\mathbf{x}^*)^{\alpha-1} > 0$, $j \in \mathcal{K}$, and $\sum_{j \in \mathcal{K} \setminus \{\bar{\imath}\}} w_j > 0$ hold irrespective of the value of $M > 0$. Moreover, since $\left\{ \frac{\alpha}{M} \left(1 + \frac{M^*}{M}\right)^{\alpha-1} \right\} \to 0$ as $M \to \infty$, it follows that $u(\mathbf{f}(\mathbf{x}^*)) - u(\mathbf{f}(\bar{\mathbf{x}})) > 0$, for some $\bar{\mathbf{x}} = \mathbf{x}(M)$, when $M \geq M^*$ is sufficiently large, i.e., the utility of some $\bar{\mathbf{x}}$ is strictly better than that of $\mathbf{x}^*$, which implies that $\mathbf{x}^* \notin \mathcal{P}_{\text{fp}}$. The result follows by contradiction.

We will next identify which points on the Pareto front that are firmly proper. It turns out that for convex multi-objective problems, i.e., with all f_i convex and X convex, it is sufficient with $\alpha = 1$ and $\mathbf{w} \in \mathbb{R}^k_+$ in (4) to make the family (2.3) complete with respect to $\mathcal{P}_{\text{fp}}$. Since we require that the weights are strictly positive, there may be a few non-proper solutions; however, almost all Pareto optimal points to convex problems are firmly proper.

In the following proposition and corollary we show that also certain non-convex multi-objective problems have Pareto fronts consisting of only firmly proper Pareto points. The proposition is similar to what is shown in [29]; however, we assume that the objectives are scaled such that $\mathbf{f}(\mathcal{P}) \subseteq (0,1]^k$. This enables another line of arguments, leading to a significantly shorter proof.

PROPOSITION 2.10 (convexification) *Consider the problem* (1). *Let the Pareto front* $\mathbf{f}(\mathcal{P})$ *be parameterized by* $f_k = \phi(f_1, \ldots, f_{k-1})$. *Assume that the local trade-offs on the Pareto front between the pairs of objectives are continuous , and assume that* ϕ *is twice continuously differentiable. Then, for a sufficiently large* $p \in [0, \infty)$, *the Pareto front of the problem* $\min_{\mathbf{x} \in X} (f_1(\mathbf{x})^p, \ldots, f_k(\mathbf{x})^p)$ *is convex.*

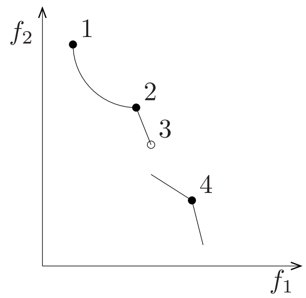


Figure 2: An illustration of a Pareto optimal set for a problem with two objectives. All points except the four marked are proper Pareto optimal points. Points 1 and 2 are not proper, and point 3 is not even Pareto optimal. Point 4 is proper but not firmly proper. Note that point 4 has points arbitrary close on both sides with different values of the trade-offs; this point can therefore be seen as insensible to a reasonable DM.

Proof Let $\bar{\mathbf{f}} = \{f_1, \ldots, f_{k-1}\}$ and $h(\bar{\mathbf{f}}) = \phi(\bar{\mathbf{f}})^p$, and let $\mathbf{x}^* \in \mathcal{P}$ arbitrarily. We will show that $\nabla^2_{(\bar{\mathbf{f}}^p)^2} h$ is positive semi-definite at $\mathbf{f}(\mathbf{x}^*)$ and hence that the Pareto front is convex. From the chain rule, we have that $\frac{\partial h}{\partial (f_j^p)} = \frac{\partial h}{\partial f_j} \frac{1}{p f_j^{p-1}}$ and that

$$\begin{aligned} \frac{\partial^2 h}{\partial (f_j^p)^2} &= \frac{\partial^2 h}{\partial f_j^2} \frac{1}{p^2 (f_j^{p-1})^2} - \frac{p-1}{p^2 f_j^{2p-1}} \frac{\partial h}{\partial f_j}, \\ \frac{\partial^2 h}{\partial f_j \partial f_i} &= \frac{\partial^2 h}{\partial f_j \partial f_i} \frac{1}{p^2 f_j^{p-1} f_i^{p-1}}. \end{aligned}$$

We define the exponent of a vector to be component-wise, and introduce the matrices $D = \mathrm{diag}(\bar{\mathbf{f}}^{p-1})^{-1}$ and $E = D^{p-1}$. We then have that

$$\nabla_{\bar{\mathbf{f}}^p} h = \frac{1}{p} D \nabla_{\bar{\mathbf{f}}} h, \tag{8a}$$

$$\nabla_{(\bar{\mathbf{f}}^p)^2} h = \frac{1}{p^2} D \nabla^2_{\bar{\mathbf{f}}^2} h D - \frac{p-1}{p} D \,\mathrm{diag}(\nabla_{\bar{\mathbf{f}}} h) D E. \tag{8b}$$

Now, since $\frac{\partial h}{\partial f_j} = p\phi^{p-1} \frac{\partial \phi}{\partial f_j}$ and $\frac{\partial^2 h}{\partial f_j^2} = p(p-1)\phi^{p-2} (\frac{\partial \phi}{\partial f_j})^2 + p\phi^{p-1} \frac{\partial^2 \phi}{\partial f_j^2}$, we get

$$\nabla^2_{\bar{\mathbf{f}}^2} h = p(p-1)\phi^{p-2} \nabla_{\bar{\mathbf{f}}} \phi (\nabla_{\bar{\mathbf{f}}} \phi)^{\mathsf{T}} + p\phi^{p-1} \nabla^2_{\bar{\mathbf{f}}^2} \phi.$$

Finally, by inserting the above expression into (8) we get

$$\nabla_{(\bar{\mathbf{f}}^p)^2} h = \frac{p-1}{p} \phi^{p-2} D \nabla_{\bar{\mathbf{f}}} \phi (\nabla_{\bar{\mathbf{f}}} \phi)^{\mathsf{T}} D + \frac{1}{p} \phi^{p-1} D \nabla^2_{\bar{\mathbf{f}}^2} \phi D - \frac{(p-1)^2}{p^2} \phi^{p-1} D \,\mathrm{diag}(\nabla_{\bar{\mathbf{f}}} \phi) D E.$$

The first term is positive semi-definite, and since $\nabla_{\bar{\mathbf{f}}}\phi \leq \mathbf{0}^{k-1}$ (ϕ is a parameterization of a Pareto front), so is the last term. As $p \to \infty$, the second term tends to zero faster than the first term, wherefore the result is proved.

COROLLARY 2.11 *Let the Pareto front be parameterized by $f_k = \phi(f_1, \ldots, f_{k-1})$, and let $\mathbf{x}^* \in \mathcal{P}$. Assume that the local trade-offs on the Pareto front between all pairs of objectives are continuous at $\mathbf{f}(\mathbf{x}^*)$, and assume that ϕ is twice continuously differentiable. Then, each proper Pareto optimal point is a firmly proper Pareto optimal point, and therefore $\mathcal{P}_{\mathrm{p}} = \mathcal{P}_{\mathrm{fp}}$.*

Proof It is well known (cf. [15], Thm. 3.11) that all proper Pareto optimal points to convex multi-objective optimization problems can be found using the standard weighting method with non-negative weights. Thus $\mathcal{P}_{\mathrm{p}} \subseteq \mathcal{P}_{\mathrm{fp}}$ and the result then follows from Prop. 2.9.

The corollary implies that all points on sufficiently smooth Pareto fronts are firmly proper, i.e., for problems with such Pareto fronts, the family $\mathcal{U}$ of utility functions is complete with respect to the entire set $\mathcal{P}$.

To conclude this subsection, we have shown that the family $\mathcal{U}$ of utility functions is rational and complete with respect to almost all Pareto solutions arising from convex problems and all Pareto fronts that are smooth enough.

2.2 The robustness index

We present two definitions of robustness for a given decision vector: absolute robustness and relative robustness. Both measures are based on the utility function (4), and for both of them a smaller value means a more robust point.

DEFINITION 2.12 (absolute robustness index) *Let $\bar{\mathbf{x}} \in \mathbb{R}^n$ be the point whose robustness is to be measured, and let $\boldsymbol{\eta} \in \Omega \subseteq \mathbb{R}^m$ be a stochastic variable with mean $\boldsymbol{\eta}_0$. Suppose that $u(\cdot)$ is the utility function associated with $\bar{\mathbf{x}}$. The absolute robustness index of $\bar{\mathbf{x}}$ is defined as*

$$R_A(\bar{\mathbf{x}}) = \mathbb{E}\left[(u \circ \mathbf{f})(\bar{\mathbf{x}}, \boldsymbol{\eta}) - (u \circ \mathbf{f})(\bar{\mathbf{x}}, \boldsymbol{\eta}_0)\right]. \tag{9}$$

REMARK 2.13 *If $(u \circ \mathbf{f})(\bar{\mathbf{x}}, \cdot)$ is convex, then the absolute robustness index is non-negative, since Jensen's inequality (cf. [16], Prop. 12) implies that*

$$R_A(\bar{\mathbf{x}}) \geq (u \circ \mathbf{f})(\bar{\mathbf{x}}, \mathbb{E}[\boldsymbol{\eta}]) - (u \circ \mathbf{f})(\bar{\mathbf{x}}, \boldsymbol{\eta}_0) = 0.$$

DEFINITION 2.14 (relative robustness index) *Let $\bar{\mathbf{x}} \in \mathbb{R}^n$ be the point whose robustness is to be measured, and let $\boldsymbol{\eta} \in \Omega \subseteq \mathbb{R}^m$ be a stochastic variable with mean $\boldsymbol{\eta}_0$. Suppose that $u(\cdot)$ is the utility function associated with $\bar{\mathbf{x}}$ and that $\mathbf{x}^*(\boldsymbol{\eta}) \in X \cap \arg\min (u \circ \mathbf{f})(\mathbf{x}, \boldsymbol{\eta})$. The relative robustness index of $\bar{\mathbf{x}}$ is defined as*

$$R(\bar{\mathbf{x}}) = \mathbb{E}\left[(u \circ \mathbf{f})(\bar{\mathbf{x}}, \boldsymbol{\eta}) - (u \circ \mathbf{f})(\mathbf{x}^*(\boldsymbol{\eta}), \boldsymbol{\eta})\right]. \tag{10}$$

In contrast to absolute robustness, relative robustness is not necessarily affected by large changes in the objective space due to different outcomes of $\boldsymbol{\eta}$, since it measures the relative loss to an optimal solution for each $\boldsymbol{\eta}$; see Figure 3.

Which robustness index should be used is a matter of choice for a DM, but practice may motivate the use of one before the other. For example, using relative robustness requires a minimization for each $\boldsymbol{\eta}$ which limits its practical use on some problems. In Section 3, we present procedures for computing approximations of the robustness indices.

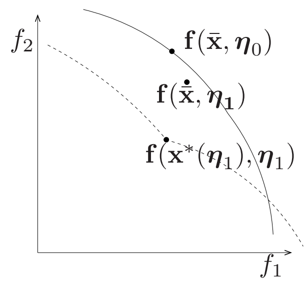


Figure 3: Two Pareto fronts for two realizations of the uncertainty parameter $\boldsymbol{\eta}$. There is a quality loss since the chosen candidate $\bar{\mathbf{x}}$ is not optimal for the outcome $\boldsymbol{\eta}_1$. This quality loss is measured in the relative robustness index.

3 Computation of the utility function and robustness index

In this section we present practical approaches for computing the utility function and the robustness indices. We start by noting that the computation of robustness is a post-process, since it requires a sufficient resolution of the Pareto front.

The computation of the robustness indices for a specific solution, which we call the candidate, is organized in a series of steps, summarized in Algorithm 1.

Algorithm 1 Calculate robustness index

Require: Candidate $\bar{\mathbf{x}}$, Pareto front $\mathbf{f}(\mathcal{P})$.

1. Approximate the Pareto front by a quadratic implicit curve around $\bar{\mathbf{x}}$.

2. Compute the utility function u for the candidate such that equations (11) and (12) are fulfilled.

3. Compute R or R_A according to the descriptions in subsections 3.2 and 3.3.

Ensure: R or R_A.

3.1 Computation of a utility function given a specific point $\bar{\mathbf{x}}$

We assume that the Pareto front $\mathbf{f}(\mathcal{P})$ is described by a level set of an implicit function $z(\mathbf{f}(\mathcal{P})) = 0$. By the definition of the utility function u, it is minimized by the candidate $\bar{\mathbf{x}}$. This implies the following two conditions which are also illustrated in Figure 4:

$$\nabla_f u = \gamma \nabla_f z, \tag{11}$$
$$\kappa(u) \geq \kappa(z), \tag{12}$$

where $\gamma \in \mathbb{R}_+$, and $\kappa(\cdot)$ is a measure of curvature (cf. (14)). We construct a quadratic function $Q(\mathbf{f})$ whose zero level set is fitted to the Pareto points within a ball of radius $\tau > 0$. In particular, given a candidate $\bar{\mathbf{x}}$ and the Pareto points $\mathbf{x}^j$ for $j = 1, \ldots, p$, within the ball $\mathcal{B}(\bar{\mathbf{x}}, \tau)$, we solve the following linear least-squares problem

$$\begin{aligned} \underset{c \in \mathbb{R}^k_+, b \in \mathbb{R}^k}{\text{m}inimize} \quad & \sum_{j=1}^{p} \sum_{i=1}^{k} \left(c_i f_i(\mathbf{x}^j)^2 + b_i f_i(\mathbf{x}^j) - 1 \right)^2, \\ \text{s}ubject\ to \quad & \sum_{i=1}^{k} c_i f_i(\bar{\mathbf{x}})^2 + b_i f_i(\bar{\mathbf{x}}) = 1, \end{aligned} \tag{13}$$

and set $Q(\mathbf{f}) := \frac{1}{2}\mathbf{f}^{\mathrm{T}} \operatorname{diag}(\mathbf{c})\, \mathbf{f} + \mathbf{b}^{\mathrm{T}}\mathbf{f} - 1$. The level set $Q(\mathbf{f}) = 0$ then yields an estimate of the normal and curvature of the front.

Since Q is quadratic with a diagonal Hessian, we can write an explicit expression for f_k in the remaining objectives $f_1, \ldots, f_{k-1}$ in the zero level set of Q. Here, (13) is used to decide on the correct solution of the two given by the resulting quadratic equation. Furthermore, the expression for f_k will be at least twice continously differentiable. Hence, there exists a twice continuously differentiable ϕ such that $f_k = \phi(f_1, \ldots, f_{k-1})$ and this implies, according to Prop. 2.10, that all points on the fitted surface are firmly proper. So even though the Pareto front may not be sufficiently smooth, we are always able to reach all points on the approximate front, and there always exists an α such that equations (11) and (12) hold. We use normal curvature in equation (12) and we define it, along with a vector d, to be

$$\kappa = \frac{d^{\mathrm{T}} H d}{\|d\|^2}, \tag{14}$$

where H is the Jacobian of the normal $N \in \mathbb{R}^k$ to the surface,

$$H = \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \cdots & \frac{\partial N_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial N_k}{\partial x_1} & \cdots & \frac{\partial N_k}{\partial x_k} \end{bmatrix}. \tag{15}$$

We note that in three dimensions, the principal curvatures are the two non-zero eigenvalues of the matrix H [1]. Equations (12) and (14) should hold for all directions $d \in \mathbb{R}^k$, although in practice we only consider a finite set of directions. For the quadratic implicit surface $Q(\mathbf{f}) = 0$, equation (15) reduces to (cf. [1, 21])

$$H = \frac{\nabla^2_{ff} Q}{||\nabla_f Q||} - \frac{(\nabla_f Q \nabla_f Q^{\mathrm{T}}) \nabla^2_{ff} Q}{||\nabla_f Q||^3}. \tag{16}$$

Equation (12) will be satisfied for a sufficiently high value of α. Therefore, a utility function u fulfilling the equations (11) and (12) can be found by iteratively increasing the curvature parameter $\alpha \geq 1$ such that it, together with an accompanying weight vector $\mathbf{w} > \mathbf{0}^k_+$, satisfies (11).

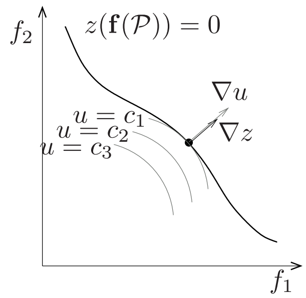


Figure 4: An illustration of the requirements (11) and (12) on the utility function u.

Given a candidate with a corresponding utility function, the next step is to compute R or R_A. This is described in the following subsections. For unconstrained problems with analytic objective functions, we present an approximate closed-form expression for relative robustness. For constrained problems, we show how a Monte-Carlo method can be used. Both methods are used in the numerical experiments in Section 4.

3.2 Robustness of $\bar{\mathrm{x}}$ – unconstrained problem

In addition to Assumption A of normalized objective function values, assumed to hold for $\mathbf{f}(\mathbf{x}, \boldsymbol{\eta}_0)$, we in this subsection also add the following:

Assumption B

(B1) The functions $f_i(\cdot, \cdot) > 0$, $i = 1, \ldots, k$, are twice continuously differentiable.

(B2) The feasible set is $X = \mathbb{R}^n$.

Under these assumptions, we can formulate a closed-form expression for an approximation of relative robustness. The approximation is based on the second-order Taylor expansion U of the utility function u. With $\hat{u}(\mathbf{x}, \boldsymbol{\eta}) := u(\mathbf{f}(\mathbf{x}, \boldsymbol{\eta}))$, we have

$$\begin{aligned} U(\mathbf{x}, \boldsymbol{\eta}) = {} & \hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0) + \nabla_x \hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0)^{\mathrm{T}} (\mathbf{x} - \bar{\mathbf{x}}) + \nabla_\eta \hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0)^{\mathrm{T}} (\boldsymbol{\eta} - \boldsymbol{\eta}_0) \\ & + (\mathbf{x} - \bar{\mathbf{x}})^{\mathrm{T}} \nabla^2_{x\eta} \hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0)(\boldsymbol{\eta} - \boldsymbol{\eta}_0) + \tfrac{1}{2} (\mathbf{x} - \bar{\mathbf{x}})^{\mathrm{T}} \nabla^2_{xx} \hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0)(\mathbf{x} - \bar{\mathbf{x}}) \\ & + \tfrac{1}{2} (\boldsymbol{\eta} - \boldsymbol{\eta}_0)^{\mathrm{T}} \nabla^2_{\eta\eta} \hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0)(\boldsymbol{\eta} - \boldsymbol{\eta}_0). \end{aligned}$$

Since the candidate $\bar{\mathbf{x}}$ is defined to minimize the utility function, the Hessian of u is positive semi-definite. If it is positive definite, and thus non-singular, we get an expression for the optimal solution $\mathbf{x}^*(\boldsymbol{\eta}) \in \arg\min_{\mathbf{x}\in X} U(\mathbf{x}, \boldsymbol{\eta})$ as a (linear) function of the uncertainty parameter $\boldsymbol{\eta}$:

$$\mathbf{x}^*(\boldsymbol{\eta}) = \bar{\mathbf{x}} - \nabla^2_{xx}\hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0)^{-1}\left[\nabla_x\hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0) + \nabla^2_{x\eta}\hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0)^{\mathrm{T}}(\boldsymbol{\eta}-\boldsymbol{\eta}_0)\right].$$

Inserting this into the definition (2.14) of robustness leads to a closed-form expression for the approximate relative robustness index:

$$\begin{aligned} R^U(\bar{\mathbf{x}}) &:= \mathbb{E}\left[\hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}) - \hat{u}(\mathbf{x}^*(\boldsymbol{\eta}), \boldsymbol{\eta})\right] \\ &= \mathbb{E}\left[\tfrac{1}{2}\nabla_x\hat{u}^{\mathrm{T}}\nabla^2_{xx}\hat{u}^{-1}\nabla_x\hat{u} + \nabla_x\hat{u}^{\mathrm{T}}\nabla^2_{xx}\hat{u}^{-1}\nabla^2_{\eta x}\hat{u}^{\mathrm{T}}(\boldsymbol{\eta}-\boldsymbol{\eta}_0)\right. \\ &\quad \left. + \tfrac{1}{2}(\boldsymbol{\eta}-\boldsymbol{\eta}_0)^{\mathrm{T}}\nabla^2_{\eta x}\hat{u}\nabla^2_{xx}\hat{u}^{-1}\nabla^2_{\eta x}\hat{u}^{\mathrm{T}}(\boldsymbol{\eta}-\boldsymbol{\eta}_0)\right]. \end{aligned} \tag{17}$$

Introducing Λ as the covariance matrix of $\boldsymbol{\eta}$, and noting that $\nabla_x\hat{u}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0) = 0$ since $\bar{\mathbf{x}}$ is the minimizer, the expression (17) reduces to

$$R^U(\bar{\mathbf{x}}) = \tfrac{1}{2}\mathrm{tr}\left(\Lambda\nabla^2_{\eta x}\hat{u}\nabla^2_{xx}\hat{u}^{-1}\nabla^2_{\eta x}\hat{u}^{\mathrm{T}}\right). \tag{18}$$

This expression only requires the solution of one linear equation with n unknowns and a few matrix-matrix multiplications, and is thus relatively fast to compute.

3.3 Robustness of $\bar{\mathbf{x}}$ – constrained problem

If any of the functions f_i are non-differentiable, if the problem includes constraints, or if analytic expressions of the functions f_i are not available, then the closed-form expression (18) does not apply. The robustness indices can however be computed using a Monte-Carlo method with randomized sampling. With N i.i.d. samples $\boldsymbol{\eta}_i,\ i = 1, \ldots, N$, of $\boldsymbol{\eta}$, we replace the expected value by the sample mean. We here only consider the absolute robustness index, since the relative index would require one minimization computation for each sample. The Monte-Carlo estimate is then given by

$$\hat{R}_A(\bar{\mathbf{x}}) = \frac{1}{N}\sum_{i=1}^{N}\left[u\big(\mathbf{f}(\bar{\mathbf{x}}, \boldsymbol{\eta}_i)\big) - u\big(\mathbf{f}(\bar{\mathbf{x}}, \boldsymbol{\eta}_0)\big)\right].$$

3.4 Search for robust solutions

We have assumed that the Pareto front is pre-computed, and that the computational procedures that were presented previously referred to candidates on the front. However, we may forsake optimality of a solution if robustness can be gained; that is, by moving away from a Pareto optimal solution $\bar{\mathbf{x}}$ on the front, we can search for solutions in its neighborhood and thereby presenting a possibly more robust alternative. Note that the parameters of the utility function is computed for the Pareto solution $\bar{\mathbf{x}}$, whence the utility function values for inner (non-Pareto) solutions depend on from which Pareto point one emanates. Let $\tau > 0$ be the radius of the ball around $\mathbf{f}(\bar{\mathbf{x}})$

used for the quadratic approximation Q of the front, and let $\varepsilon > 0$. We use the utility function u to define the neighborhood. For absolute robustness we formulate the optimization problem to

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & R_A(\mathbf{x}), \\ \text{subject to} \quad & u\left(\mathbf{f}(\mathbf{x})\right) - u\left(\mathbf{f}(\bar{\mathbf{x}})\right) \leq \varepsilon, \\ & \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\bar{\mathbf{x}})\| \leq \tau, \\ & \mathbf{x} \in X. \end{aligned} \tag{19}$$

The solution to (19) is the most robust point with at most a decrease of ε in utility compared to $\bar{\mathbf{x}}$ and which is sufficiently close to $\bar{\mathbf{x}}$ in the objective space such that the local approximation of the Pareto front remains valid. In Section 4.2, problem (19) is used to find alternative robust solutions.

For relative robustness, we have to take into account that inner solutions will have a lower utility value than the optimal solution $\mathbf{x}^*(\boldsymbol{\eta})$ for each realization of $\boldsymbol{\eta}$ (see the expression (10) and Figure 3). This difference has to be subtracted from the objective function in the optimization problem in order to not having inner solutions being punished due to a utility loss independent of the perturbation. Letting $\Delta u(\mathbf{x}) = u(\mathbf{x}, \boldsymbol{\eta}_0) - u(\bar{\mathbf{x}}, \boldsymbol{\eta}_0)$ denote the loss in utility at the unperturbed state, we formulate the optimization problem to

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & R(\mathbf{x}) - \Delta u(\mathbf{x}), \\ \text{subject to} \quad & u\left(\mathbf{f}(\mathbf{x})\right) - u\left(\mathbf{f}(\bar{\mathbf{x}})\right) \leq \varepsilon, \\ & \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\bar{\mathbf{x}})\| \leq \tau, \\ & \mathbf{x} \in X. \end{aligned} \tag{20}$$

Similar to above, the solution to (20) is the most (relatively) robust point which is sufficiently close to $\bar{\mathbf{x}}$ both in terms of function values and in terms of utility value.

4 Numerical Experiments

The ideas developed and the measures introduced in this article have been applied to both the analytical test functions constructed by Deb and Gupta [12] and on functions derived from real-world numerical data used for antenna optimization [23, 32].

The reader should note that the test functions in the first numerical example are designed to illustrate different principal cases when introducing uncertainty in multi-objective problems, and are not designed to imitate practical applications. Our intention with this example is to show how our definition of robustness compares to other already published ones.

The theory developed in this article poses no theoretical restriction on the number of objectives. In the numerical examples we, however, for illustrative reasons only consider bi-objective problems.

4.1 Analytical functions

Deb and Gupta [12] consider uncertainty in the decision space and formulates a program where each objective function is replaced by its respective average computed over a ball around the intended decision variable, i.e.,

$$f_i^{eff}(\mathbf{x}) = \frac{1}{|\mathcal{B}(\mathbf{x},\delta)|}\int_{\mathbf{y}\in\mathcal{B}(\mathbf{x},\delta)} f_i(\mathbf{y})d\mathbf{y}.$$

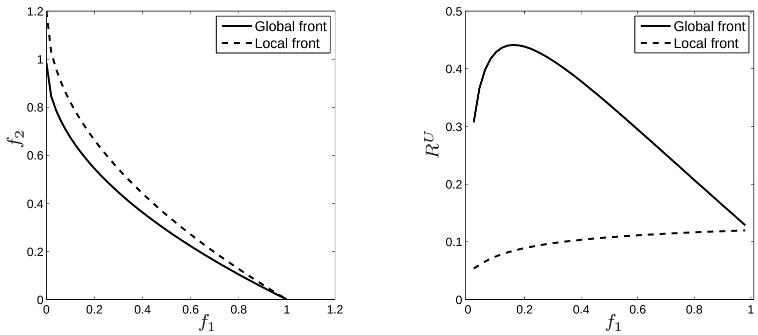
The radius of the ball is given by the parameter $\delta > 0$ which is varied in the numerical tests. A larger value of δ smoothens out the functions and makes sharp global optima less attractive. By using this framework, a "robust" Pareto optimal front is always found, but there is no distinction between the points on this front with respect to robustness. Furthermore, there is no continuous grading of robustness of the points that are not in the robust Pareto set. Deb and Gupta also present an alternative robustness model where they enforce robustness of the resulting solutions using a constraint. Here, the norm of the difference between the (unperturbed) function value and the averaged (or, the worst case) function value is required to be kept smaller than a certain threshold value. From our point of view, this formulation also suffers from the weakness that it just classifies solutions as robust Pareto optimal or not. It is also possible that large parts of the objective space will not contain any robust solutions if the effect of uncertainty is large. From now on, we will concentrate on Deb and Gupta's first formulation.

Since we derive robustness for the unperturbed front, and Deb and Gupta present a robust front possibly consisting of completely different solutions, it is difficult to directly compare the respective results.

We present numerical results for one test problem, DEBGUP3, which is one of four bi-objective problems from [12] which are also considered in [2, 3, 4]. The problem is to

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & (f_1(\mathbf{x}), f_2(\mathbf{x})) = \\ & \left(x_1, \left(2 - 0.8e^{-\left(\frac{x_2-0.35}{0.25}\right)^2} - e^{-\left(\frac{x_2-0.85}{0.03}\right)^2}\right)(1-\sqrt{x_1})\sum_{i=3}^{5} 50x_i^2\right), \\ \text{subject to} \quad & 0 \le x_i \le 1, \quad i = 1,2, \\ & -1 \le x_i \le 1, \quad i = 3,4,5. \end{aligned} \tag{DEBGUP3}$$

The uncertainty appears in the decision space, such that $\mathbf{x}$ is replaced by $\mathbf{x}+\boldsymbol{\eta}$ and $\boldsymbol{\eta}$ is drawn from a uniform distribution, $\boldsymbol{\eta} \in U([-0.03, 0.03]^5)$. A close study of the functions reveals that the unperturbed problem with $\boldsymbol{\eta} = \mathbf{0}^5$ has one local and one global Pareto optimal front (where a local Pareto front consists of points that are locally Pareto optimal, i.e., points $\hat{x}$ for which there is an $\varepsilon > 0$ such that $\hat{x}$ fulfilles Def. 1.1 with the decision space X replaced by a small ball $\mathcal{B}(\hat{x},\varepsilon)$). The fronts are shown in Figure 5(a). Figure 5(b) presents the approximation (18) of the relative robustness index (Def. 2.14) for the corresponding points. We have chosen to ignore the bounds when computing relative robustness which enabled the use of the closed-form expression (18). The implication of ignoring the bounds may be that the value of R^U is overestimated, i.e., the robustness value is underestimated. Note that the



(a) Function values (original functions).

(b) Robustness indices.

Figure 5: The global and the local Pareto fronts for the problem DEBGUP3. The robustness indices are shown for the corresponding points, parameterized by the values of the first objective.

robustness varies both along each single front, and also between the two fronts. For example, there are points at the global front at which the expected loss in utility due to perturbations are about four times as large compared to other points at the global front. The explanation for the relatively large values of the robustness indices, i.e., the expected utility loss (which for problems with convex Pareto fronts can be interpreted approximately as the expected increase in a weighted sum of the objectives), is that the constructed test example has a sharp local minimum in the second objective at the global Pareto front. This leads to that points at the front will be sensitive already for small perturbations. The local front is more robust than the global one as is expected from the results in [12]. Here, we can distinguish a difference between using the robustness index and using averaged objectives. Depending on the size of the radius δ, the robust front will equal either the local front, the global front, or a combination of these. It is possible to construct problems where the global front equals the robust Pareto front, but having a local Pareto front arbitrarily close and which has much better robustness indices according to our definition. The value of δ (which has to be selected prior to the analysis) highly determines which solutions are presented to a DM, whereas the idea in our paper is to partly push forward the decision of how much robustness is desired to the DM, and therefore present solutions of different robustness values. The robustness index may also show that robustness may vary along the Pareto front. With more complex objective functions found in real-world applications, we anticipate that there may be more dramatic changes in robustness between solutions close to each other on the Pareto front. In such cases, the DM may prefer a solution slightly off his/her ideal (optimal) solution if the robustness properties are better. This situation is presented in the following example.

4.2 A real-world example

Designing antennas typically involves a number of conflicting requirements. These may be based on spatial size, so called S-parameters related to electromagnetic properties, functions of the directivity of the antenna, band width, input impedance, or other characteristics of the antenna. In a joint project between the Fraunhofer–Chalmers Centre and the Antenna Research Centre at Ericsson AB a multi-objective optimization approach is taken on the antenna design problem, as described in [23, 32]. We have chosen to study this problem using a subset of the proposed objectives. The decision variables are the positions and geometrical dimensions of the antenna, and the objectives chosen are the maximum return loss ($|S_{11}|$) over the frequency band $[750, 850]$ MHz and the area of the hull of the antenna. An approximate Pareto front is shown in Figure 6, where it is clearly shown that the two objectives are conflicting. The objective functions are expensive to evaluate since they are outcomes of time-consuming computer simulations. For this reason, a surrogate modeling technique [24] is used, where approximate functions are constructed using the function values computed at a number of sample points. Jakobsson et al. [22, 23] have developed a new technique based on interpolation with rational radial basis functions to handle the sharp function behaviors around the resonance levels. The two objectives

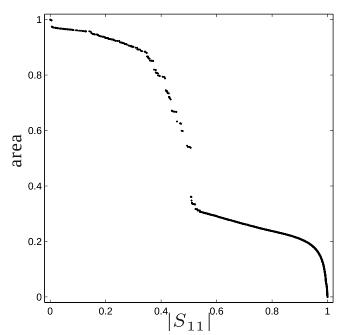


Figure 6: An approximate Pareto front (with the objectives scaled) to the problem found using the NSGA-II algorithm [14] with 400 generations and a population size of 196.

are interesting for a robustness study. Near resonance, small variations of the decision variables yield large differences in the function values. This is the case for many practical problems where resonance phenomena are part of the problem characteristics. We have noticed that the surrogate models are quite sensitive to the choice of sample points (and this choice is not obvious) and have constructed our numerical study based on this fact.

Originally, the decision space has been sampled at 2000 distinct points chosen using an ad-hoc design–of–experiments strategy, and the surrogate models (or, response surfaces) have been constructed using the rational RBF technique on the function evaluations at these points. In our numerical experiments, we have randomly selected 500 out of the 2000 points and constructed new response surfaces using only these. The uncertainty characteristics depend on which 500 of the 2000 points that are chosen, reflecting the fact that it is not clear from the start which sample points to choose. Obviously, a *robust* solution is a solution for which the randomness does not have a large effect, according to our definition of robustness.

To be able to handle robustness with respect to this type of uncertainty is a great advantage of our method as compared to the one by Deb and Gupta and others, where it is only possible to handle uncertainties in the precision of $\mathbf{x}$ and / or uncertainties in the parameter values of parameterized objective functions.

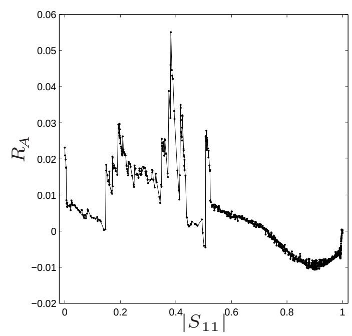
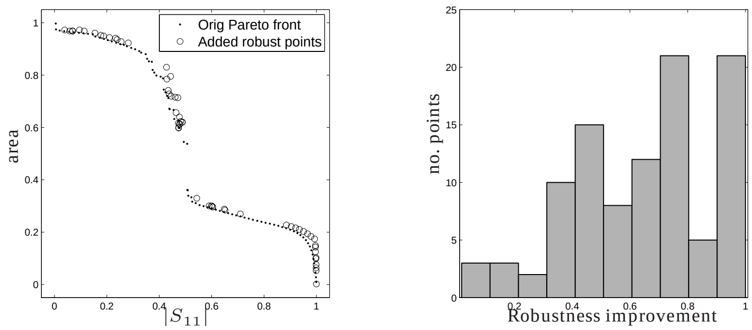


Figure 7: Absolute robustness for the points on the Pareto front, parameterized by the first objective.

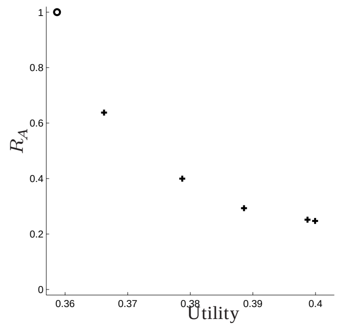
In Figure 7, the (absolute) robustness index is shown for the points on the Pareto front to the original problem, where the objective functions are the response surfaces constructed using all 2000 data points. The index varies substantially along the front, and for some Pareto points, there are other points on the front that are close in the objective space but with a very different robustness index. This opens up the possibility for a DM to choose a point which lies close to his / her ideal point with respect to the function values, but which are much more robust. Doing so will, on average, improve the utility. But since the front is only valid for the unperturbed problem, a DM could also be searching for a non-Pareto optimal solution since such a point can be even more robust; see Section 3.4. In Figure 8, we illustrate such a search. For each (unperturbed) Pareto optimal point, we search for optimal points according to the model (19) with the parameter values $\varepsilon = 0.01$ and $\tau = 0.1$. We use the global optimization algorithm `DIRECT` [26], implemented in `TOMLAB` [33]. We have also implemented a simple local search strategy to complement the algorithm. In the left figure, the points obtained are shown together with the original (approximate) Pareto front. The right figure shows a histogram of the size of the improvements in robustness when—for each unperturbed point—picking the corresponding robust alternative. One obvious conclusion is that for most Pareto optimal points, there are robust solutions that are close with respect to the value of the utility but with significantly better robustness indices (note the large portion of solutions with robustness improvements close to 1). This fact can be used by a DM, who then gets an option to balance between robustness and “optimality” for the unperturbed problem.



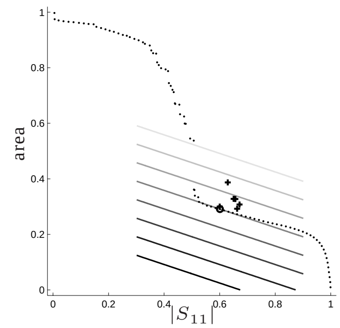
(a) (b)

Figure 8: In a), robust points are added to the (approximate) Pareto front. In b), the relative improvements in the robustness index for the points found are shown.

To further illustrate the framework, we consider the following scenario: Suppose we have presented a Pareto front corresponding to the unperturbed problem to a specific DM, and that he / she has located a candidate solution. Since the problem contains uncertain parameters, the DM is also interested in the robustness of this solution. We now solve problem (19) for varying values of the parameter ε. This will produce solutions that are more robust, but with lower utility values. These candidates are then presented to the DM, who gets the option to consider how much he / she values robustness considering how much utility is lost. In the spirit of multi-objective optimization, the decision of robustness versus optimality is thus left to the DM. Figure 9 shows the results for a specific candidate. In a), the robustness index is shown as a function of the utility for the alternative solutions. In b), the unperturbed Pareto front is shown along with level curves of the utility function for the specific candidate. The DM can clearly see that he / she can substantially improve the robustness if he / she is willing to sacrifice some utility.



(a)



(b)

Figure 9: Figure a) shows the utility function values and robustness index values for the alternative solutions. The robustness index is normalized by the original candidate. Figure b) shows the unperturbed function values for the candidates and level curves of the utility function. In both figures, the ring (o) corresponds to the unperturbed Pareto point originally chosen by the DM and the plus signs (+) correspond to the alternative points.

5 Summary and conclusions

We have presented a new definition of robustness for multi-objective problems based on an approximation of the utility function for each single decision maker and we present two robustness indices measuring the relative and absolute, respectively, expected loss in the utility function due to uncertainties. We have shown that the family of utility functions suggested has certain nice properties such as rationality and completeness. We have also presented procedures for computing the robustness indices and applied them to two numerical examples: one analytic test problem from [12], and one practical antenna optimization problem from [23, 32].

The formulation of robustness by Deb and Gupta [12] for multi-objective optimization, which consists of replacing the objectives by their respective expected values, is very natural and direct, and is suitable for many applications. In line with the main idea of multi-objective optimization, our approach, however, has the advantage that the decisions are moved to a later point in time at which more information about the problem is revealed. Also, our method produces a continuous measure of robustness and it does that to all points; it does not only tell whether a certain point is a robust Pareto optimal point or not. Furthermore, our method enables the handling of very general types of uncertainties.

In the future, our methodology will be applied to more numerical examples, including also problem instances with more than two objectives. The inclusion of constraints when considering relative robustness should be further developed. It would also be interesting to develop and apply other types of robustness measures based on utility functions.

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References

[1] B. R. ARAUJO AND J. A. P. JORGE, *Curvature dependent polygonization of implicit surfaces*, in Proceedings of the 17th Brazilian Symposium on Computer Graphics and Image Processing, Washington DC, USA, 2004, IEEE, pp. 266–273.

[2] C. BARRICO AND C. H. ANTUNES, *A new approach to robustness analysis in multi-objective optimization*, in Proceedings of the 7th International Conference on Multi-Objective Programming and Goal Programming (MOPGP'06), Loire Valley (City of Tours), France, 2006.

[3] ———, *Robustness analysis in multi-objective optimization using a degree of robustness concept*, in 2006 IEEE Congress on Evolutionary Computation (CEC'2006), Vancouver, BC, Canada, 2006, IEEE, pp. 6778–6783.

[4] ———, *An evolutionary approach for assessing the degree of robustness of solutions to multi-objective models*, in Evolutionary Computation in Dynamic and Uncertain Environments, S. Yang, Y. S. Ong, and Y. Jin, eds., Springer-Verlag, 2007, pp. 565–582.

[5] A. BEN-TAL AND A. NEMIROVSKI, *Robust optimization – methodology and applications*, Mathematical Programming, 92 (2002), pp. 453–480.

[6] D. BERTSIMAS, O. NOHADANI, AND K. M. TEO, *Robust nonconvex optimization for simulation-based problems*, Operations Research (to appear), (2008).

[7] ———, *Nonconvex robust optimization for problems with constraints*, Informs Journal on Computing (to appear), (2009).

[8] J. R. BIRGE AND F. LOUVEAUX, *Introduction to Stochastic Programming*, Springer Series in Operations Research, Springer-Verlag, New York, NY, USA, 1997.

[9] I. DAS, *Robustness optimization for constrained nonlinear programming problems*, Engineering Optimization, 32 (2000), pp. 585–618.

[10] I. DAS AND J. E. DENNIS JR, *A closer look at drawbacks of minimizing weighted sums of objectives for Pareto set generation in multicriteria optimization problems*, Structural Optimization, 14 (1997), pp. 63–69.

[11] K. DEB AND H. GUPTA, *Handling constraints in robust multi-objective optimization*, in Congress on Evolutionary Computation, vol. 1, 2005, pp. 25–32.

[12] ———, *Searching for robust Pareto-optimal solutions in multi-objective optimization*, in Evolutionary Multi-Criterion Optimization, 2005, pp. 150–164.

[13] ———, *Introducing robustness in multi-objective optimization*, Evolutionary Computation, 14 (2006), pp. 463–494.

[14] K. DEB, A. PRATAP, A. SAMEER, AND T. MEYARIVAN, *A fast elitist multi-objective genetic algorithm: NSGA-II*, IEEE Transactions on Evolutionary Computation, 6 (2000), pp. 182–197.

[15] M. EHRGOTT, *Multicriteria Optimization*, Springer-Verlag, Heidelberg, Germany, 2005.

[16] B. FRISTEDT AND L. GRAY, *A modern approach to probability theory*, Probability and its Applications, Birkhäuser Boston Inc., Boston, MA, USA, 1997.

[17] A. GASPAR-CUNHA AND J. A. COVAS, *Robustness in multi-objective optimization using evolutionary algorithms*, Computational Optimization and Applications, 39 (2008), pp. 75–96.

[18] A. GEOFFRION, *Proper efficiency and the theory of vector maximization*, Journal of Mathematical Analysis and Applications, 22 (1968), pp. 618–630.

[19] S. GUNAWAN AND S. AZARM, *Multi-objective robust optimization using a sensitivity region concept*, Structural and Multidisciplinary Optimization, 29 (2005), pp. 50–60.

[20] G. HASSAN AND C. D. CLACK, *Multiobjective robustness for portfolio optimization in volatile environments*, in GECCO '08: Proceedings of the 10th annual conference on Genetic and evolutionary computation, New York, NY, USA, 2008, ACM, pp. 1507–1514.

[21] J. HUGHES, *Differential geometry of implicit surfaces – a primer*, tech. report, Brown University, Computer Science, Providence, 2003.

[22] S. JAKOBSSON, B. ANDERSSON, AND F. EDELVIK, *Rational radial basis function interpolation with applications to antenna design*, tech. report, Fraunhofer–Chalmers Research Centre, Göteborg, Sweden, 2008.

[23] S. JAKOBSSON, B. ANDERSSON, AND F. EDELVIK, *Multiobjective optimization applied to design of PIFA antennas*, in Scientific Computing in Electrical Engineering SCEE 2008, vol. 14 of Mathematics in Industry, Springer-Verlag, Heidelberg, Germany, 2010, pp. 437–444.

[24] R. JIN, W. CHEN, AND T. W. SIMPSON, *Comparative studies of metamodeling techniques under multiple modeling criteria*, Journal of Structural Optimization, 23 (2001), pp. 1–13.

[25] Y. JIN AND B. SENDHOFF, *Trade-off between optimality and robustness: An evolutionary multi-objective approach, in evolutionary multi-criterion optimization*, in Proceedings of the Second International Conference on Evolutionary Multi-criterion Optimization, Springer-Verlag, 2003, pp. 237–251.

[26] D. R. JONES, *Direct global optimization algorithm*, in Encyclopedia of Optimization, 2009, pp. 725–735.

[27] P. KALL AND S. W. WALLACE, *Stochastic Programming*, John Wiley & Sons, Chichester, England, 1994.

[28] R. KEENEY AND H. RAIFFA, *Decisions with multiple objectives: Preferences and value trade-offs*, Cambridge University Press, Cambridge, UK, 1993.

[29] D. LI, *Convexification of a noninferior frontier*, Journal of Optimization Theory and Applications, 88 (1996), pp. 177–196.

[30] M. LI, S. AZARM, AND V. AUTE, *A multi-objective genetic algorithm for robust design optimization*, in GECCO '05: Proceedings of the 2005 Conference on Genetic and Evolutionary Computation, New York, NY, USA, 2005, ACM, pp. 771–778.

[31] K. MIETTINEN, *Nonlinear Multiobjective Optimization*, Kluwer Academic Publishers, Boston, MA, USA, 1998.

[32] A. STJERNMAN, A. DERNERYD, S. JAKOBSSON, B. ANDERSSON, AND F. EDELVIK, *Multiobjective optimization of MIMO antenna systems*, in Proceedings of the 3rd European Conference on Antennas and Propagation, Berlin, Germany, 2009.

[33] The TOMLAB optimization environment, http://tomopt.com/tomlab/.

[34] J. WALLENIUS, J. S. DYER, P. C. FISHBURN, R. E. STEUER, S. ZIONTS, AND K. DEB, *Multiple criteria decision making, multiattribute utility theory: Recent accomplishments and what lies ahead*, Management Science, 54 (2008), pp. 1336–1349.