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The opportunistic replacement problem: analysis and case studies

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Abstract

We consider an optimization model for determining optimal opportunistic maintenance (that is, component replacement) schedules when data is deterministic. This problem generalizes that of Dickman, Epstein, and Wilamowsky [21] and is a natural starting point for the modelling of replacement schedules when component lives are non-deterministic. We show that this basic opportunistic replacement problem is NP-hard. We show that the convex hull of the set of feasible replacement schedules is full-dimensional, and that all the necessary inequalities also are facet-inducing. We show that when maintenance occasions are fixed, the remaining problem can be stated as a linear program; when maintenance costs are monotone with time, the latter is solvable through a greedy procedure. Results from a series of case studies performed in the areas of aircraft engine and wind turbine maintenance are also reported. These illustrate the advantages of utilizing opportunistic maintenance activities based on a complete optimization model, as compared to simpler policies.

1 Introduction

The importance of performing maintenance operations well—and of improving the state of the art—seems to be impossible to overestimate: according to [39, Ch. 1], maintenance costs in plants in the US alone accounted for more than \$600 billion ($\$600 \cdot 10^9$) in 1981, more than \$800 billion in 1991, and were then projected to increase to become more than \$1200 billion by the year 2000. It is stated that these evaluations indicate that on average one third, or \$250 billion, of all maintenance dollars are wasted through ineffective maintenance management methods. According to a recent study (made by Forum Vision Instandhaltung, Germany), maintenance costs in the manufacturing industry within the EU amount to roughly \$2000 billion per year. Studies over the last 20 years have indicated that around Europe, the direct cost of maintenance is equivalent to between 4% and 8% of total sales turnover. Also in these cases, it is quite natural to assume that not all the money spent is spent well: according to information gathered by the Swedish Center for Maintenance Management, maintenance is often performed in an uncoordinated and/or corrective only (that is, after failure has occured) fashion, resulting in too frequently needing to shut down production; surprisingly often equipment failure is triggered

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by inspections and the condition monitoring itself. According to a study on fossil power plants ([16]) 56% of the forced outages occured within one week from an intrusive maintenance task. One objective with constructing and studying mathematical models for the optimization of the scheduling of maintenance and inspection activities is to mitigate some of these problems, and to thereby contribute to a shift of focus from considering maintenance as mainly a cost-inducing activity to that of an investment in availability improvement.

One strategy for planning maintenance activities is so called opportunistic maintenance, in which a mathematical model is utilized to decide whether, at a (possibly already planned) maintenance occasion, more than the necessary maintenance activities should be performed; we may refer to this as preventive maintenance activities at an opportunity. According to Dickman et al. [21], Jorgenson and Radner [36] introduced the original opportunistic replacement/maintenance problem. They considered a system of stochastically failing components, which incur extensive maintenance costs upon failure, that is, for shutting down and disassembling the system. When the system is down for whatever reason, components may be replaced at no additional maintenance cost. Thereby, opportunities arise to trade off remaining life of components in order to avoid maintenance costs associated with component failure, perhaps already in the near future. This is their main motive for studying the problem.

Our original motivation for studying the replacement problem was a project concerning the optimization of jet engine maintenance schedules at Volvo Aero Corporation (VAC). An aircraft engine consists of thousands of parts. Some of the parts are safety-critical, which means that if they fail there will be an engine breakdown, possibly with catastrophic consequences. Therefore, the safety-critical parts have fixed life limits (before which the probability of failure is effectively zero), and must be replaced before they are reached. Hence, we consider, as does VAC, the safety-critical parts as having deterministic lives. (The corresponding situation is present, for example, in nuclear power plants; see [30, 17].) All other parts of the engine are considered to have stochastic lives; therefore, their life limits need to be estimated, which in turn makes it much more difficult to compute a reliable replacement schedule. For some of these parts failure distributions may be computed from historical data and monitoring observations. This information could then be discretized and used as an input into optimization models. This was the subject of two PhD projects (see [4, 55]).

Taking into account parts that are either deterministic or stochastic in a unified model is quite a lot more complex than what has been studied in the past; even stochastic models found in the literature typically do not incorporate failure distributions but failure intensities only, and solution approaches provide simple maintenance policies for infinite horizon problems; see further the survey in Section 3.

The purpose of the present paper is to initiate a detailed mathematical study of a model of the opportunistic replacement problem, to be defined below. In the near future we will consider several extensions thereof. In a recent case study at VAC, the structure of the jet engine, and in particular the disassembly of its parts, has been taken better into account through detailed cost dependencies between components. Further, recent applications of opportunistic maintenance optimization to the generation of wind and nuclear power (e.g., [10]) have resulted in the study of stochastic programming models, properly incorporating stochastic information about the remaining lives of components.

2 The opportunistic replacement model

Consider a set $\mathcal{N}$ of components; let $N = |\mathcal{N}|$. Consider also a set $\mathcal{T} = \{1, \ldots, T\}$ of times, with $T \geq 2$. Suppose a new component $i \in \mathcal{N}$ has a (deterministic) life of T_i time steps. (Without loss of generality, $2 \leq T_i \leq T$.) The purchase cost at time $t \in \mathcal{T}$ for component i is $c_{it} > 0$. There is a fixed cost of $d_t > 0$ associated with a maintenance occasion at time t, independent of

the number of parts replaced.

The objective is to minimize the cost of having a working system between times 1 and T. Formally, we define the opportunistic replacement problem as follows.

Definition 1 (opportunistic replacement problem)**.** Given a fixed cost d_t for a maintenance occasion and a cost c_{it} for replacing a component $i \in \mathcal{N}$ at time $t \in \mathcal{T}$, find a maintenance schedule over the period $\mathcal{T}$ that minimizes the total maintenance cost and in which, for each component $i \in \mathcal{N}$, no period without replacement longer than the component's life T_i exist. □

Letting

$$z_t = \begin{cases} 1, & \text{if maintenance shall occur at time } t, \\ 0, & \text{otherwise,} \end{cases} \qquad t \in \mathcal{T},$$

$$x_{it} = \begin{cases} 1, & \text{if component } i \text{ shall be replaced at time } t, \\ 0, & \text{otherwise,} \end{cases} \qquad i \in \mathcal{N}, \quad t \in \mathcal{T},$$

the *opportunistic replacement model* is defined as that to

$$\begin{aligned}
\underset{(x,z)}{\text{minimize}} \quad & \sum_{t \in \mathcal{T}} \left(\sum_{i \in \mathcal{N}} c_{it} x_{it} + d_t z_t \right), && && && \text{(1a)} \\
\text{subject to} \quad & \sum_{\substack{t=\ell+1 \\ \ell+T_i}} x_{it} \geq 1, && \ell = 0, \ldots, T - T_i, && i \in \mathcal{N}, && \text{(1b)} \\
& x_{it} \leq z_t, && t \in \mathcal{T}, && i \in \mathcal{N}, && \text{(1c)} \\
& x_{it} \geq 0, && t \in \mathcal{T}, && i \in \mathcal{N}, && \text{(1d)} \\
& z_t \leq 1, && t \in \mathcal{T}, && && \text{(1e)} \\
& x_{it} \in \{0, 1\}, && t \in \mathcal{T}, && i \in \mathcal{N}. && \text{(1f)} \\
& z_t \in \{0, 1\}, && t \in \mathcal{T}. && && \text{(1g)}
\end{aligned}$$

The constraints (1b) ensure that each part is replaced before the end of its life; the constraints (1c) enforce the payment of the fixed maintenance cost d_t whenever any part is replaced at time t, and, once this cost is paid, induces maintenance opportunities at no extra maintenance cost. The remaining constraints are definitional; the removal of (1f)–(1g) amounts to a continuous relaxation of the problem.

This model stems from [21]; the model in [4] replaces the original constraints $\sum_{i \in \mathcal{N}} x_{it} \leq N z_t$, $t \in \mathcal{T}$, in [21] with the equivalent but stronger constraints (1c); the model (1), in turn, generalizes the cost function in [4] to allow for time dependency.

As a numerical illustration, we consider an instance of (1) with $T = 60$, $N = 4$, $T_1 = 13$, $T_2 = 19$, $T_3 = 34$, $T_4 = 18$, $c_{1t} = 80$, $c_{2t} = 185$, $c_{3t} = 160$, and $c_{4t} = 125$ for all $t \in \mathcal{T}$. The data is chosen so that the relations between the lives and the costs are similar to those for the fan module of the RM12 engine, maintained at VAC. The model is solved for $d_t = 0$, 10, and 1000 for all t (where $d_t = 10$ represents the most reasonable value in the maintenance situation at VAC). For $d_t = 0$, the optimal total number of replacement occasions is 11 and there is no advantage with replacing a component before its life limit is reached. Increasing the value of d_t from 0 to 10 decreases the optimal total number of replacement occasions from 11 to five. It is now beneficial to replace the components in larger groups and they are often replaced before their respective life limits are reached. (Notice that the optimal solution obtained for $d_t = 10$ is, in fact, optimal also for the case of $d_t = 0$.) For $d_t = 1000$ it is very important to utilize the opportunity to replace several components at the same time. The optimal total number of replacement occasions is four (which is the minimum number of replacement occasions for this instance).

Figure 1 shows optimal maintenance schedules for each of the three cases. The horizontal axis represents the 60 time steps and each maintenance occasion is represented by a vertical bar, where a dot at a certain height represents a component of the corresponding type being replaced. The figure clearly illustrates how opportunistic replacement becomes more beneficial with an increasing fixed maintenance cost.

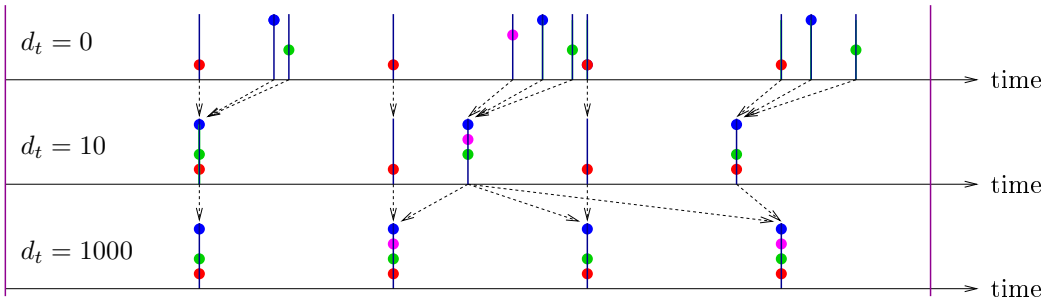


Figure 1: *Optimal maintenance schedules for* $d_t = 0$, 10, *and* 1000 *for all* t. *When* d_t *increases from* 0 *to* 10 *the replacement occasions 1–3, 5–7, and 9–11, are grouped into one occasion each. When* d_t *is increased from* 10 *to* 1000, *the last four maintenance occasions are rearranged into three occasions, also resulting in several more component replacements.*

The remainder of the paper is organized as follows. Section 3 contains a survey of the most relevant literature on maintenance optimization. In Section 4, we establish that the opportunistic replacement problem is NP-hard, based on a reduction from the set covering problem. Section 5 presents some properties characterizing an optimal maintenance schedule. We show that if the variables z_t are fixed to binary values, then the polyhedron arising from the continuous relaxation of the variables x_{it} is integral (i.e., possesses integral extreme points); in other words, the integrality restrictions (1f) may be dropped. Moreover, we provide results, in part reached in [21], on the possibility to a priori remove some maintenance occasions from consideration. In Section 6 we perform a polyhedral study of the convex hull of the set of feasible solutions to the model (1), referred to as the *replacement polytope*. We show that the replacement polytope is full-dimensional under natural assumptions and that the necessary inequality constraints (1b)–(1e) in the original formulation (1) are facet-defining. Further, we show that they are not sufficient to completely describe the replacement polytope. In Section 7 we present results from numerical case studies of problems with stochastic and deterministic lives, originating from the aircraft and wind power industries. We conclude with remarks on current and planned research endeavours.

3 Literature overview

Major research efforts on the mathematical modelling of, and methods for, maintenance and replacement scheduling were initiated during WWII at the military institute RAND at Santa Monica, CA, USA. (Prior to this effort isolated research can be traced back at least to the 1930s; see the historical review in [8].) The group at RAND included Richard Bellman, whose invention dynamic programming was also the first efficient solution method applied in the area ([3, 53, 9, 23]). A later development took place at Stanford University, where H. M. Wagner and co-workers developed preventive maintenance and replacement models, starting from their integer programming work on scheduling (e.g., [59]). Maintenance planning models and methods also found a central place in OR text books around this time; cf. [1, 58]. (The perhaps first mention of maintenance planning in text book form is found in the OR text by Morse and Kimball [41].)

Several of these models can be found as applications of dynamic programming in Wagner's OR text book ([58]). In this early development, manpower planning was as an important part of the problem as was the replacement part, as is evidenced in [41, p. 78], [58] as well as in the book by Morse [40, Ch. 11].

Common themes in this development are a focus on an infinite planning horizon, few parts (often only one or two), and a quest for obtaining a simple maintenance/replacement policy. Perhaps more than anything, it reflects the fact that (mixed) integer and combinatorial optimization was not yet well developed. It also reflects the fact that all problems were stochastic, and the modelling and methodology development took place in true OR fashion in a world where mathematical statistics and mathematical programming are joint research fields. A good example of this interplay is the work of Morse, Barlow, Hunter, and others: Morse [40] analyzed a preventive maintenance model based on queueing theory; Barlow and Hunter [7] later provided a policy based on this work, focusing on "reliability". Further developments later lead to the classic mathematical statistics book on reliability by Barlow and Proschan [8].

Opportunistic maintenance models are less frequently found in the literature, compared to the preventive case. Sasieni [53] presents a policy that includes opportunities, for a Markov based problem concerning two parts. Campbell [13] is an exceptional, early paper from 1941. It concerns the replacement of lamps, e.g. along a city street. Two policies can be utilized, where the first is to replace each lamp when it breaks, and the other is to replace all lamps as soon as one breaks. (The latter therefore constitutes an early opportunistic maintenance policy.) The research question is when to go from the first policy to the second. The paper is also exceptional in that it discusses not only the infinite horizon case, but also a finite planning horizon. The first major developments on opportunistic maintenance following the work by Sasieni were made at the beginning of the 1960s by RAND researchers (e.g., [36, 49, 50, 38]), in particular characterizations of optimal policies for certain problems. (See [60] for a thorough account on policies, and [35] for the early work done at RAND.)

In many ways, later development has followed a similar path, incorporating more parts, more advanced failure models and system states (e.g., [57]), and also in combination with, for example, production planning (see, e.g., [11]). As stated in the surveys [47, 44], the infinite horizon case is still the one mainly treated, operations research methods are still not well developed, and test cases are also usually few and seldom realistic. The development of finite horizon models is in [44] deemed essential for the maintenance of multi-component systems to become operational. Our ambition is to contribute to an improvement in the analysis as well as utilization of maintenance models, starting from the deterministic model studied here.

The basic model (1) is developed from the one in [21], and it is therefore instrumental to investigate the relation of the latter reference to the existing literature. We first trace its history. Epstein and Wilamowsky provide in [24] a simple policy for the maintenance of one life-limited part (a jet engine compressor unit) and an exponentially failing system (disks in the engine compressor); they extend this policy to multiple life limited components in [25]. In [27] they establish the optimality of their policy wrt. the utilization of individual disks. In [26] they isolate the life limited part of the problem, and study the deterministic problem for the case of two parts. Without providing an optimization model for their problem, they establish the special case of Proposition 2 below to the case of time-independent costs, namely that maintenance need only to be considered at points of failure of at least one of the parts. They are also able to further limit the number of interesting maintenance occasions, and show that the time interval between optimal replacements for a given part is non-increasing. (Discussions on the difficulties in extending policies to more than two parts can be found, for example, in [19, 61].) The above references concern the infinite time horizon case. Dickman et al. [22] consider also the finite horizon case of the deterministic problem with two parts, and extend the results in [26] concerning patterns in optimal maintenance schedules. Their problem formulation has integer

variables that correspond to the actual maintenance times, and it is limited to the two-part case. The conference proceedings paper [20] presents a 0/1 integer formulation, which, however, is nonlinear. The paper by Dickman et al. [21], finally, reaches a linear 0/1 formulation for the general N-component case; it is a special case of the problem (1), as discussed in Section 2. They also establish a version of Proposition 2, and that the integrality of the z_t variables can be relaxed; the latter is done using a greedy argument, similar to that utilized in Proposition 3.

A citation search on these papers in February 2010 resulted in the following: The paper [25] is cited in [26], in three surveys on maintenance scheduling ([15, 18, 19]), and in a 1990 paper on group policies. The paper [26] is cited by four papers on policies, none published later than 1995, and the surveys [19, 56, 45]. Our main source, the paper [21], is cited only once, in [54] as a general reference to deterministic maintenance models. The very limited number of citations shows that although the authors tackle a quite interesting problem, they have found no followers prior to this work.

Also references to the theoretical complexity of maintenance optimization problems are scarce. In [54], maintenance optimization is described as being possible to state as a partitioning problem; it is then erroneously concluded that optimal replacement therefore is an NP-hard problem. In [31] a maintenance problem is studied where it is to be decided upon an optimal cyclic (periodic) maintenance scheduling pattern; the cycle length itself is assumed fixed. Referring to the periodic maintenance problem in [5], where the cycle length is free, and the NP-hardness proof of that problem in [6], they conclude that their problem is also NP-hard. (The proof is not complete, however, and, indeed, in the survey [44] the authors state that the problem in [54] "appears to be NP-hard.") Their analysis cannot be utilized in our setting, as we do not seek periodic scheduling patterns. In [12] the preventive maintenance of a railway system is studied. Both (short) routine activities and (long) unique projects must be scheduled in a certain period. Two versions of the problem are studied, one with fixed intervals between two consecutive executions of the same routine work, and one with only a maximum interval; the problem resembles in this sense that in [31], and the reduction from graph colouring utilized in [12] can not be used for our problem.

For recent accounts of maintenance modelling in engineering and industry, see [32, 2, 43, 52] (plants), [33, 28] (infrastructure), [34] (electrical networks), and [51, 14, 37] (production systems).

4 Complexity analysis

Theorem 1 (problem reduction)**.** *The set covering problem is polynomially reducible to the opportunistic replacement problem.*

Proof. Let $\{\mathcal{A}_t\}_{t=1}^m$ be a given collection of nonempty subsets of the finite set $\{1,\dots,n\}$ such that $\cup_{t=1}^m \mathcal{A}_t = \{1,\dots,n\}$. Letting $a_{it} = 1$ if $i \in \mathcal{A}_t$ and 0 otherwise, the *set covering* problem is represented as the following optimization model:

$$\text{minimize} \quad \sum_{t=1}^{m} y_t, \tag{2a}$$

$$\text{subject to} \quad \sum_{t=1}^{m} a_{it} y_t \geq 1, \qquad i = 1,\dots,n, \tag{2b}$$

$$y_t \in \{0,1\}, \quad t = 1,\dots,m. \tag{2c}$$

Consider then an instance of the program (1) such that $N = n$, $T = m$, $d_t = 1$, $T_i = m$, and $c_{it} = 2(1 - a_{it})$ for all $i = 1,\dots,n$ and $t = 1,\dots,m$. Since $T_i = T = m$, each component must be replaced once between the times 1 and T, and one replacement is always enough (for feasibility). Furthermore, in every optimal solution and for each i and t such that $a_{it} = 0$, $x_{it} = 0$ holds since

$c_{it} = 2 > d$ and there exists a $\tilde{t} \in \mathcal{T}$ with $a_{i\tilde{t}} = 1$, which implies that $c_{i\tilde{t}} = 0$. Hence, this specific instance of (1) can be reformulated as the problem to

$$\text{minimize} \left\{ \sum_{t=1}^{m} z_t \;\middle|\; \sum_{t=1}^{m} a_{it} x_{it} \geq 1,\ i = 1, \ldots, n, \text{ and (1c)–(1g) hold} \right\}. \tag{3}$$

An optimal solution (x^*, z^*) to (3) is given by

$$z^* \in \operatorname*{argmin}_{z \in \{0,1\}^m} \left\{ \sum_{t=1}^{m} z_t \;\middle|\; \sum_{t=1}^{m} a_{it} z_t \geq 1,\ i = 1, \ldots, n \right\} \tag{4}$$

and $x^*_{it} = a_{it} z^*_t$, $i = 1, \ldots, n$, $t = 1, \ldots, m$. The result then follows, since the program (4) is equivalent to (2). □

Since the set covering decision problem is an NP-complete problem (see [29]), it follows that the set covering optimization problem is an NP-hard problem and thus the opportunistic replacement problem is NP-hard.

It should be mentioned that the complexity of the instance of the opportunistic replacement problem for which the costs c_{it} are non-increasing with time (i.e., $c_{i,t+1} \leq c_{it}$ for all i and t) is still an open question; this includes the interesting special case for which the costs are constant over time (i.e., $c_{it} = c_i$ and $d_t = d$ for all i and t), as originally studied in [21] and [4].

5 Special properties of optimal solutions

We here present some special properties of the opportunistic replacement model (1). First we show that the integrality constraints on the variables x_{it} can be relaxed. Then we review a result from [21] and show that for instances of the model where costs are monotone with time the replacement activities will only occur at times that are sums of positive integer multiples of life limits. Finally, we show that, again for monotone costs and given fixed binary values of the z_t variables, the optimal x_{it} values can be obtained by a greedy algorithm.

5.1 Integrality property

The following proposition concerns integrality properties of the polyhedron in $\mathbb{R}^{N \times T}$ defined by (1b)–(1d), when the variables z_t, $t \in \mathcal{T}$, are fixed to binary values. Accordingly, we let $\tilde{z}_t \in \{0, 1\}$, $t \in \mathcal{T}$, and define $\widetilde{\mathcal{T}} = \{t \in \mathcal{T} \mid \tilde{z}_t = 1\}$.

Proposition 1 (integral polyhedron). *The polyhedron defined by* (1b) *and*

$$x_{it} \leq 1, \qquad t \in \widetilde{\mathcal{T}}, \tag{5a}$$

$$x_{it} \leq 0, \qquad t \in \mathcal{T} \setminus \widetilde{\mathcal{T}}, \tag{5b}$$

for $i \in \mathcal{N}$, is integral.

Proof. Observe that the constraint matrix A corresponding to the system of inequalities defined by (1b) and (5) has the consecutive ones property (that is, for all rows i, if $a_{ik} = a_{ij} = 1$ then $a_{il} = 1$ for all $k < l < j$). Hence, [42, p. 544, Cor. 2.10] implies that the transpose of the constraint matrix A^T is TU, and [42, p. 540, Prop. 2.1] in turn implies that the constraint matrix A is TU. Since the right-hand sides of (1b) and (5) are all integral it follows from [42, p. 541, Prop. 2.2] that the corresponding polyhedron is integral. □

The result of Proposition 1 implies that the binary requirements (1f) on the variables x_{it} can be relaxed, provided that the model (1) is to be solved using an algorithm that detects extreme optimal solutions to linear programming subproblems.

5.2 Monotone costs

The results presented in this section are derived for instances of the model (1) for which the costs are non-increasing with time (that is, $c_{i,t+1} \leq c_{it}$ and $d_{t+1} \leq d_t$ for all i and t). For any problem with costs that are non-decreasing with time (that is, $c_{i,t+1} \geq c_{it}$ and $d_{t+1} \geq d_t$ for all i and t) the variable transformation $\hat{x}_{it} = x_{i,T+1-t}$ and $\hat{z}_t = z_{T+1-t}$ for all $t \in \mathcal{T}$ and $i \in \mathcal{N}$ results in an equivalent problem with costs being non-increasing with time. Therefore, analogous properties hold for the latter case.

The next proposition extends the statement of [21, Thm. 2] from three to N components. It states that we may a priori set $z_t = 0$ in (1) for each t not being a non-negative sum of lives.

Proposition 2 (a priori variable elimination for non-increasing costs)**.** *For all instances of* (1) *with costs fulfilling $c_{i,t+1} \leq c_{it}$ and $d_{t+1} \leq d_t$ for all i and t, an optimal solution exists with $z_t = 0$ for every $t \in \mathcal{T}$ which is not a sum of non-negative integer multiples of the life limits (that is, for every $t \in \mathcal{T}$ such that $\left\{\ell \in \mathbb{Z}_+^N \mid \sum_{i\in\mathcal{N}} \ell_i T_i = t\right\} = \emptyset$).*

Proof. Consider a feasible solution to (1) with $z_t = 1$ for some t that is *not* a positive sum of lives T_i, $i \in \mathcal{N}$, and with objective value f. Assume, without loss of generality, that t is the earliest time with such a property (i.e., all previous replacement times are positive sums of lives). This implies that all parts have remaining lives $\tau_i > 0$ at time t. We can therefore postpone all replacements made at t to $\tilde{t} = t + \min_{i\in\mathcal{N}} \tau_i$. The time $\tilde{t}$ equals a positive sum of lives T_i. The adjusted solution, with $z_t = 0$ and $z_{\tilde{t}} = 1$, is feasible in (1) and its corresponding objective value $\tilde{f}$ fulfills $\tilde{f} \leq f$. Apply this procedure to all t that are not positive sums of lives and for which $z_t = 1$. The result follows. □

If the variables z_t, $t \in \mathcal{T}$, are assigned binary values, $\tilde{z}_t \in \{0, 1\}$, the remaining optimization model separates over the components $i \in \mathcal{N}$ and the corresponding constraint matrix is TU. For each component $i \in \mathcal{N}$ this model is thus given by

$$\underset{x_i}{\text{minimize}} \left\{ \sum_{t\in\mathcal{T}} c_{it} x_{it} \;\middle|\; \sum_{t=\ell+1}^{\ell+T_i} x_{it} \geq 1, \ell = 0, \ldots, T - T_i;\; 0 \leq x_{it} \leq \tilde{z}_t, t \in \mathcal{T} \right\}. \tag{6}$$

Using Algorithm 1 component i is replaced as late as possible within its life and among the time points $t \in \mathcal{T}$ with $\tilde{z}_t = 1$.

Algorithm 1 (non-increasing cost greedy rule for component $i \in \mathcal{N}$)

$\widetilde{\mathcal{T}} \leftarrow \left\{t \in \mathcal{T} \,\middle|\, \tilde{z}_t = 1\right\} \cup \{T+1\}$; $\tilde{x}_{it} \leftarrow 0\ \forall t \in \mathcal{T}$; $\tilde{t} \leftarrow \min\left\{t \,\middle|\, t \in \widetilde{\mathcal{T}}\right\}$; $s \leftarrow 0$; $\widetilde{\mathcal{T}} \leftarrow \widetilde{\mathcal{T}} \setminus \{\tilde{t}\}$;
while $\widetilde{\mathcal{T}} \neq \emptyset$ **do**
  $\hat{t} \leftarrow \min\left\{t \,\middle|\, t \in \widetilde{\mathcal{T}}\right\}$;
  if $T_i < \hat{t} - s$ **then** $\tilde{x}_{i\tilde{t}} \leftarrow 1$; $s = \tilde{t}$ **end if**
  $\tilde{t} \leftarrow \hat{t}$; $\widetilde{\mathcal{T}} \leftarrow \widetilde{\mathcal{T}} \setminus \{\tilde{t}\}$
end while
return $\tilde{x}_{it}\ \forall t \in \mathcal{T}$

The next proposition shows that for non-increasing costs and binary values for z_t, $t \in \mathcal{T}$, Algorithm 1 yields an optimal solution to (6).

Proposition 3 (non-increasing greedy rule yields optimum)**.** *Assume that $c_{i,t+1} \leq c_{it}$ holds, $i \in \mathcal{N}$, $t \in \mathcal{T} \setminus \{T\}$. Let $\tilde{z}_t \in \{0, 1\}$, $t \in \mathcal{T}$, and assume that the set $\widetilde{\mathcal{T}} = \{t \in \mathcal{T} \mid \tilde{z}_t = 1\}$ is such that for each $t \in \widetilde{\mathcal{T}} \cup \{0\}$ there is an $s \in \widetilde{\mathcal{T}} \cup \{T+1\}$ with $1 \leq s - t \leq \min_{i\in\mathcal{N}} T_i$. Then, Algorithm 1 produces an optimal solution to the model* (6).

Proof. By assumption, $\tilde{x}_i$ is feasible in (6). Let $\bar{x}_i \neq \tilde{x}_i$ be feasible in (6). Postpone, where possible, replacements corresponding to $\bar{x}_i$ to the next time point in $\widetilde{\mathcal{T}} \cup \{T+1\}$. This will transform $\bar{x}_i$ to $\tilde{x}_i$ without introducing any additional replacements and at a non-increasing cost. Hence, $\sum_{t\in\mathcal{T}} c_{it}(\tilde{x}_{it} - \bar{x}_{it}) \leq 0$ holds; the result follows. □

6 The replacement polytope

We let the set $S \subset \mathbb{R}^{N\times T} \times \{0,1\}^T$ be defined by the values of the variables (x, z) that fulfil the constraints (1b)–(1e), (1g). The convex hull of S, denoted $\operatorname{conv} S$, is called the *replacement polytope*. By studying the facial structure of $\operatorname{conv} S$ and thereby describing it by a finite set of linear inequalities, it is possible to solve the problem using linear programming techniques. Our ambition here is to take the first steps towards such a complete linear description of the replacement polytope.

We compute the dimension of the replacement polytope and show, under weak and natural assumptions, that all the necessary inequalities in (1b)–(1e) define facets of the same. However, by an example we show that these basic inequalities do not completely define $\operatorname{conv} S$.

Proposition 4 (dimension of the replacement polytope)**.** *If $T_i \geq 2$ for all $i \in \mathcal{N}$, then the dimension of* $\operatorname{conv} S$ *is $(N+1)T$, that is,* $\operatorname{conv} S$ *is full-dimensional.*

Proof. First note that since $S \subseteq \mathbb{R}^{(N+1)T}$ it holds that $\dim(\operatorname{conv} S) \leq (N+1)T$. Let the vectors $(x^k, z^k) \in \mathbb{B}^{(N+1)T}$, $k \in \{0,\dots,(N+1)T\}$, be given by the following. For $i \in \mathcal{N}$ and $t \in \mathcal{T}$, let $x^k_{it} = 0$ if $k \in \{(N+1)(t-1)+i, (N+1)t\}$ and $x^k_{it} = 1$ otherwise. For $t \in \mathcal{T}$, let $z^k_t = 0$ if $k = (N+1)t$ and $z^k_t = 1$ otherwise. Since $T_i \geq 2$ for $i \in \mathcal{N}$ it holds that $\sum_{t=\ell+1}^{\ell+T_i} x^k_{it} \geq 1$ for all $i \in \mathcal{N}$, all $\ell \in \{0,\dots,T-T_i\}$, and all $k \in \{0,\dots,(N+1)T\}$.

Moreover, for all $t \in \mathcal{T}$ and $k \in \{0,\dots,(N+1)T\}$ such that $z^k_t = 0$ it holds that $x^k_{it} = 0$, $i \in \mathcal{N}$; it follows that $(x^k, z^k) \in S$. It can be verified that the only solution to the system

$$\sum_{k=0}^{(N+1)T} x^k_{it}\alpha_k = 0, \quad i \in \mathcal{N}, \qquad \sum_{k=0}^{(N+1)T} z^k_t\alpha_k = 0, \quad t \in \mathcal{T}, \qquad \sum_{k=0}^{(N+1)T} \alpha_k = 0,$$

is $\alpha_k = 0$, $k \in \{0,\dots,(N+1)T\}$, which implies that the vectors (x^k, z^k), $k \in \{0,\dots,(N+1)T\}$, are affinely independent. Hence, it holds that $\dim(\operatorname{conv} S) \geq (N+1)T$, thus implying that $\dim(\operatorname{conv} S) = (N+1)T$. The proposition follows. □

The replacement polytope is *not* full-dimensional if $T_i = 1$ for some $i \in \mathcal{N}$, since it then holds that $x_{it} = z_t = 1$, $t \in \mathcal{T}$, for all $(x, z) \in \operatorname{conv} S$. Letting $A^=$ denote the matrix corresponding to the equality subsystem of $\operatorname{conv} S$, this would yield that $\operatorname{rank} A^= \geq 2T$ and thus that $\dim(\operatorname{conv} S) \leq (N-1)T$. However, the case that $T_i = 1$ is not interesting in practice since it would mean that component i must be replaced—and thus maintenance must be performed—at *every* time step.

The following result from polyhedral combinatorics ([42, Thm. 3.6 of Ch. I.4]) is utilized to determine facets of $\operatorname{conv} S$.

Theorem 2 (characterization of facets)**.** *Let P be a full-dimensional polyhedron and let $F = \{x \in P \mid \pi^{\mathrm{T}}x = \pi_0\}$ be a proper face of P (i.e., $\emptyset \neq F \subset P$). The following two statements are equivalent:*

1. *F is a facet of P.*
2. *If $\lambda^{\mathrm{T}}x = \lambda_0$ for all $x \in F$ then $(\lambda, \lambda_0) = \alpha(\pi, \pi_0)$ for some $\alpha \in \mathbb{R}$.* □

Proposition 5 (the inequalities (1b) define facets). *If $T_i \geq 2$ for all $i \in \mathcal{N}$, then each of the inequalities $\sum_{t=\ell+1}^{\ell+T_i} x_{it} \geq 1$, $\ell = 0, \ldots, T - T_i$, $i \in \mathcal{N}$, defines a facet of* $\operatorname{conv} S$.

Proof. Since $T_i \geq 2$ for $i \in \mathcal{N}$, $\operatorname{conv} S$ is full-dimensional (cf. Proposition 4). Hence, we can use the uniqueness characterization of the facet description from Theorem 2 to show the proposition.

For each $r \in \mathcal{N}$ and each $\ell \in \{0, \ldots, T - T_r\}$, let $\widehat{F}_{r\ell} = \left\{ (x, z) \in \operatorname{conv} S \mid \sum_{t=\ell+1}^{\ell+T_r} x_{rt} = 1 \right\}$. Further, let $x^0_{it} = z^0_t = 1$, $i \in \mathcal{N}$, $t \in \mathcal{T}$. Since $T_i \geq 2$ it follows that $(x^0, z^0) \in S \setminus \widehat{F}_{r\ell}$. Then, defining the vector $(x^{\mathrm{A}}, z^{\mathrm{A}})$ as $x^{\mathrm{A}}_{it} = 0$ if $i = r$ and $t \in \{\ell + 2, \ldots, \ell + T_r\}$, $x^{\mathrm{A}}_{it} = 1$ otherwise, and $z^{\mathrm{A}}_t = 1$, $t \in \mathcal{T}$, it follows that $\left(x^{\mathrm{A}}, z^{\mathrm{A}}\right) \in \widehat{F}_{r\ell}$ and hence that $\widehat{F}_{r\ell}$ is a proper face of $\operatorname{conv} S$. Moreover, there exist values of $\lambda \in \mathbb{R}^{N \times T}$, $\mu \in \mathbb{R}^T$, and $\rho \in \mathbb{R}$ such that the equation

$$\sum_{t \in \mathcal{T}} \left(\sum_{i \in \mathcal{N}} \lambda_{it} x_{it} + \mu_t z_t \right) = \rho \tag{7}$$

is satisfied for all $(x, z) \in \widehat{F}_{r\ell}$. We will show that for any value of $\alpha \in \mathbb{R}$, in a solution to (7) it holds that $\lambda_{it} = \alpha$ if $i = r$ and $t \in \{\ell + 1, \ldots, \ell + T_r\}$, $\lambda_{it} = 0$ otherwise, $\mu_t = 0$, $t \in \mathcal{T}$, and $\rho = \alpha$.

Choose any $i \in \mathcal{N} \setminus \{r\}$ and any $t \in \mathcal{T}$. Let, for $j \in \mathcal{N}$ and $k \in \mathcal{T}$, $x^1_{jk} = 0$ if $j = i$ and $k = t$, $x^1_{jk} = x^{\mathrm{A}}_{jk}$ otherwise, and let $z^1 = z^{\mathrm{A}}$. It follows that $(x^1, z^1) \in \widehat{F}_{r\ell}$. The vectors $\left(x^{\mathrm{A}}, z^{\mathrm{A}}\right)$ and $\left(x^1, z^1\right)$, respectively, inserted in (7) then yield that $\lambda_{it} = 0$. It follows that $\lambda_{it} = 0$ for all $i \in \mathcal{N} \setminus \{r\}$ and all $t \in \mathcal{T}$.

For each $t \in \mathcal{T} \setminus \{\ell + 1, \ldots, \ell + T_r + 1\}$, let, for $i \in \mathcal{N}$ and $k \in \mathcal{T}$, $x^2_{ik} = 0$ if $i = r$ and $k = t$, $x^2_{ik} = x^{\mathrm{A}}_{ik}$ otherwise, and let $z^2 = z^{\mathrm{A}}$. It follows that $\left(x^2, z^2\right) \in \widehat{F}_{r\ell}$. The vectors $\left(x^{\mathrm{A}}, z^{\mathrm{A}}\right)$ and $\left(x^2, z^2\right)$, respectively, inserted in (7) then yield that $\lambda_{rt} = 0$ for all $t \in \mathcal{T} \setminus \{\ell + 1, \ldots, \ell + T_r + 1\}$.

Further, let, for $i \in \mathcal{N}$, $x^{\mathrm{B}}_{it} = 0$ if $i = r$ and $t = \ell + 1$, $x^{\mathrm{B}}_{it} = 1$ if $i = r$ and $t = \ell + T_r$, $x^{\mathrm{B}}_{it} = x^{\mathrm{A}}_{it}$ otherwise, and let $z^{\mathrm{B}} = z^{\mathrm{A}}$. Moreover, let, for $i \in \mathcal{N}$, $x^3_{it} = 0$ if $i = r$ and $t = \ell + T_r + 1$, $x^3_{it} = x^{\mathrm{B}}_{it}$ otherwise, and let $z^3 = z^{\mathrm{B}}$. It follows that $\left(x^3, z^3\right) \in \widehat{F}_{r\ell}$. The vectors $\left(x^{\mathrm{B}}, z^{\mathrm{B}}\right)$ and $\left(x^3, z^3\right)$, respectively, inserted in (7) then yield that $\lambda_{r,\ell+T_r+1} = 0$. The equation (7) can then be rewritten as

$$\sum_{t \in \mathcal{T}} \mu_t z_t + \sum_{t=\ell+1}^{\ell+T_r} \lambda_{rt} x_{rt} = \rho. \tag{8}$$

For each $t \in \mathcal{T} \setminus \{\ell + 1, \ell + T_r + 1\}$, let, for $i \in \mathcal{N}$, $x^4_{ik} = 0$ if $k = t$, $x^4_{ik} = x^{\mathrm{A}}_{ik}$ otherwise, and let $z^4_k = 0$ if $k = t$, and $z^4_k = z^{\mathrm{A}}_k$ otherwise. It follows that $(x^4, z^4) \in \widehat{F}_{r\ell}$. The vectors $\left(x^{\mathrm{A}}, z^{\mathrm{A}}\right)$ and $\left(x^4, z^4\right)$, respectively, inserted in (8) then yield that $\mu_t = 0$ for all $t \in \mathcal{T} \setminus \{\ell + 1, \ell + T_r + 1\}$.

Further, for each $t \in \{\ell + 1, \ell + T_r + 1\}$, let, for $i \in \mathcal{N}$, $x^5_{ik} = 0$ if $k = t$, $x^5_{ik} = x^{\mathrm{B}}_{ik}$ otherwise, and let $z^5_k = 0$ if $k = t$, and $z^5_k = z^{\mathrm{B}}_k$ otherwise. It follows that $\left(x^5, z^5\right) \in \widehat{F}_{r\ell}$. The vectors $\left(x^{\mathrm{B}}, z^{\mathrm{B}}\right)$ and $\left(x^5, z^5\right)$, respectively, inserted in (8) then yield that $\mu_{\ell+1} = \mu_{\ell+T_r+1} = 0$. Equation (8) can then be rewritten as

$$\sum_{t=\ell+1}^{\ell+T_r} \lambda_{rt} x_{rt} = \rho. \tag{9}$$

For each $t \in \{\ell+2, \ldots, \ell+T_r\}$, let for $i \in \mathcal{N}$ and $k \in \mathcal{T}$, $x^6_{ik} = 0$ if $i = r$ and $k = \ell+1$, $x^6_{ik} = 1$ if $i = r$ and $k = t$, and $x^6_{ik} = x^{\mathrm{A}}_{ik}$ otherwise, and let $z^6 = z^{\mathrm{A}}$. It follows that $\left(x^6, z^6\right) \in \widehat{F}_{r\ell}$. The vectors $\left(x^{\mathrm{A}}, z^{\mathrm{A}}\right)$ and $\left(x^6, z^6\right)$, respectively, inserted in (9) then yield that $\lambda_{r,\ell+1} = \lambda_{rt}$. Hence, λ_{rt} is constant over $t \in \{\ell + 1, \ldots, \ell + T_r\}$ and we define $\lambda_{rt} = \lambda$, $t \in \{\ell + 1, \ldots, \ell + T_r\}$. Since $\left(x^{\mathrm{A}}, z^{\mathrm{A}}\right) \in \widehat{F}_{r\ell}$ it follows that $\lambda = \rho$. Letting $\alpha = \rho$, the equation (9) can be written as

$\sum_{t=\ell+1}^{\ell+T_r} \alpha x_{rt} = \alpha$. From [42, pp. 91–92] then follows that the inequality $\sum_{t=\ell+1}^{\ell+T_r} x_{rt} \geq 1$ defines a facet of $\operatorname{conv} S$.

□

The technique used to prove Proposition 5 can be applied to Propositions 6–8 below, whose proofs are given in Appendix A.

Proposition 6 (the inequalities (1c) define facets)**.** *If $T_i \geq 2$ for all $i \in \mathcal{N}$, then each of the inequalities $x_{it} \leq z_t$, $i \in \mathcal{N}$, $t \in \mathcal{T}$, defines a facet of* $\operatorname{conv} S$. □

Proposition 7 (the inequalities (1d) define facets)**.** *If $T_i \geq 2$ for all $i \in \mathcal{N}$, then each of the inequalities $x_{kt} \geq 0$, $k \in \mathcal{N} : T_k \geq 3$, $t \in \mathcal{T}$, defines a facet of* $\operatorname{conv} S$. □

The inequalities $x_{kt} \geq 0$, $t \in \mathcal{T}$ (cf. Proposition 7) do *not* define facets for any $k \in \mathcal{N}$ such that $T_k = 2$ since the constraints (1d) are then implied by (1b)–(1c), (1e) according to $x_{k,t+1} \geq 1 - x_{kt} \geq 1 - z_t \geq 0$, $t \in \mathcal{T} \setminus \{T\}$, and $x_{k1} \geq 1 - x_{k2} \geq 1 - z_2 \geq 0$. Hence, the constraints (1d) need to be defined only for $i \in \mathcal{N}$ such that $T_i \geq 3$.

Proposition 8 (the inequalities (1e) define facets)**.** *If $T_i \geq 2$ for all $i \in \mathcal{N}$, then each of the inequalities $z_t \leq 1$, $t \in \mathcal{T}$, defines a facet of* $\operatorname{conv} S$. □

It follows from Propositions 5–8 that all of the inequalities necessary in the description of the set S define facets of its convex hull. A natural question then arises: Is $\operatorname{conv} S$ completely described by the system (1b)–(1e)? The answer to this question is "no", which becomes apparent by the following example.

Example 1 (continuous relaxation)**.** Consider a system with $N = 2$, $T_1 = 2$, $T_2 = 3$ and $T = 8$. Let the costs be $c_{it} = 1$ and $d_t = 1$ for all $i \in \mathcal{N}$ and $t \in \mathcal{T}$. An optimal solution to model (1) is

t	1	2	3	4	5	6	7	8
x_{1t}	0	1	0	1	0	1	1	0
x_{2t}	0	1	0	1	0	1	0	0
z_t	0	1	0	1	0	1	1	0

with the objective value of 11. After relaxing the integrality constratints (1f) and (1g) an optimal solution is

t	1	2	3	4	5	6	7	8
x_{1t}	1/2	1/2	1/2	1/2	1/2	1/2	1	0
x_{2t}	0	1/2	1/2	0	1/2	1/2	1/2	0
z_t	1/2	1/2	1/2	1/2	1/2	1/2	1	0

with the objective value of 10.5. Hence the convex hull of the set of feasible solutions to the system (1b)–(1g) is not completely defined by the inequalities therein. □

According to the Propositions 5–8, all of the necessary inequalities define facets of $\operatorname{conv} S$. Since, by Proposition 4, $\operatorname{conv} S$ is full-dimensional (under reasonable assumptions) the minimal description of $\operatorname{conv} S$ is unique. Therefore, all of these facets are necessary in the description of $\operatorname{conv} S$.

Example 1 shows that the inequalities (1b)–(1e) are not sufficient to describe $\operatorname{conv} S$. To completely describe $\operatorname{conv} S$ we hence need also other facets; facet-generating procedures will be presented in forthcoming work.

7 Case studies

In this section we present results from numerical case studies of replacement problems with both stochastic and deterministic component lives. These problems originate from the aircraft engine and wind power industries. Here, all costs are time-independent, i.e., $c_{it} = c_i$, $i \in \mathcal{N}$, and $d_t = d$, $t \in \mathcal{T}$. We compare the results from using solutions to the opportunistic replacement problem to that of two maintenance policies and of performing no opportunistic maintenance. We also investigate how the maintenance occasion cost d affects the relative performance of the methods.

7.1 Deterministic and stochastic opportunistic replacement problems

For the cases in which the lives of all the components are deterministic, an optimal maintenance schedule is found by solving the opportunistic replacement problem (see Def. 1). Many maintenance problems, however, include components with stochastic lives, and we wish to apply our model to these problems as well. When dealing with stochastic lives, an optimal maintenance schedule for the entire planning period can not be determined; the actual failure of components will provide new information, which in turn will affect the decisions to be taken in the future. Therefore, we aim at finding a maintenance policy, being a function that is called upon failure of some component of the system in order to determine which components to replace.

Definition 2 (maintenance policy)**.** Given the cost d of a maintenance occasion, the replacement cost c_i, the age a_i and the life T_i (or life distribution) of each component $i \in \mathcal{N}$, and the remaining planning horizon T; decide which component(s) to replace at the current maintenance occasion.

We use the following definition of a *stochastic opportunistic replacement* problem.

Definition 3 (stochastic opportunistic replacement problem)**.** Given the cost d of a maintenance occasion, the replacement cost c_i, the age and the life distribution of each component $i \in \mathcal{N}$; find a policy that minimizes the expected cost for maintenance over the planning period from 0 to T.

The mean maintenance cost resulting from using a certain policy over a large number of life scenarios reflects how well the policy solves the stochastic opportunistic replacement problem. A *scenario* for a stochastic opportunistic replacement problem is defined as a sequence $\{T_i^k\}_{k=1}^K$ (where $K \in \mathbb{Z}_+$ is large enough) of lives for each component $i \in \mathcal{N}$. These sequences are drawn from the components' life distributions; for components with deterministic lives, $T_i^k = T_i$ for all k. We calculate the performance of a maintenance policy for a specific scenario according to Alg. 2; the parameters τ_i and a_i denote the remaining life and age, respectively, of component $i \in \mathcal{N}$.

Algorithm 2 (total maintenance cost from using a policy for a given scenario)

$t \leftarrow 0$; $\texttt{cost} \leftarrow 0$
for $i \in \mathcal{N}$ **do** $\tau_i \leftarrow T_i^0$; $a_i \leftarrow 0$; $k_i \leftarrow 1$ **end for**
repeat
  $\bar{\tau} \leftarrow \min\{\, \tau_i \mid i \in \mathcal{N} \,\}$; $t \leftarrow t + \bar{\tau}$; $\texttt{cost} \leftarrow \texttt{cost} + d$
  for $i \in \mathcal{N}$ **do** $\tau_i \leftarrow \tau_i - \bar{\tau}$; $a_i \leftarrow a_i + \bar{\tau}$ **end for**
  Apply a maintenance policy to decide which components to replace, say $\mathcal{R} \subseteq \mathcal{N}$
  for $i \in \mathcal{R}$ **do** $\tau_i \leftarrow T_i^{k_i}$; $a_i \leftarrow 0$; $k_i \leftarrow k_i + 1$; $\texttt{cost} \leftarrow \texttt{cost} + c_i$ **end for**
until $t \geq T$
return cost

In all the tests described in Sections 7.3 and 7.4, every stochastic component has a Weibull

distributed life with the probability density function f defined by

$$f(x;\alpha,\beta) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left(\left(-\frac{x}{\alpha}\right)^{\beta}\right), & \text{if } x \geq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

with the scale parameter $\alpha > 0$ and the shape parameter $\beta > 0$ varying over the components. Weibull distributions are common for modeling lives of components (see e.g. [8, Ch. 2] and [55]). The methodology developed for this case study can, however, be applied to systems with arbitrary distributions for the components' lives.

7.2 The maintenance policies compared in this study

We next describe the specific maintenance policies considered in the case study. These policies are applied to both deterministic and stochastic problems. The simplest policy is to make no coordination of component replacements.

Definition 4 (non-opportunistic maintenance policy)**.** Replace failed components only.

In the maintenance literature age replacement policies are common; see e.g. [8, Ch. 3].

Definition 5 (age policy)**.** Given age limits $\hat{a}_i$ for all components $i \in \mathcal{N}$, a component $i \in \mathcal{N}$ is replaced if its age $a_i \geq \hat{a}_i$.

Finding optimal values for the age limits $\hat{a}_i$ in an age policy is computationally demanding; we have implemented the heuristic procedure of Alg. 3; the value of the parameter $\Delta > 0$ is chosen such that the calculations become manageable. For all components $i \in \mathcal{N}$ let $\hat{T}_i = T_i$ if the problem is deterministic and $\hat{T}_i = \text{mean}(T_i)$ if the problem is stochastic.

Algorithm 3 (heuristic for computing age limits $\hat{a}_i$, $i \in \mathcal{N}$)

```
mincost ← Σ_{i∈N} ⌊T/T̂_i⌋(d + c_i)
for r ∈ {0, 1, ..., ⌊Δ^{-1}T⌋} do
    for i ∈ N do a_i ← max(0, T̂_i − rΔ) end for
    Apply Alg. 2 with the age policy of Definition 5 and T_i^k = T̂_i, i ∈ N, k = 1, ..., K.
    if cost < mincost then δ ← rΔ; mincost ← cost end if
end for
return max(0, T̂_i − δ), i ∈ N.
```

We have constructed a value policy to resemble the behavior of the decision methodology used at Volvo Aero Corporation (VAC), and which is there combined with some manual adjustments.

Definition 6 (value policy)**.** Each component $i \in \mathcal{N}$ with $c_i > d$ is assigned the value $v_i = c_i \cdot \tau_i / \hat{T}_i$, where τ_i is the (expected) remaining life of the component. An age limit $T_{\min} \leq T$ is given. A component $i \in \mathcal{N}$ is replaced if either $c_i > d \geq v_i$ holds or $c_i \leq d$ holds and $a_i \geq T_{\min}$.

For the aircraft engines at VAC the age limit $T_{\min}$ is set to 150 flight hours, which is around 20% of the shortest component life. Also for the wind turbine study we set $T_{\min}$ to 20% of the shortest component life. Notice that the value policy can be interpreted as an age policy, for which $\hat{a}_i = T_i(1 - d/c_i)$ if $c_i \geq d$ and $\hat{a}_i = T_{\min}$ otherwise.

The deterministic optimization model (1) cannot be directly applied to a stochastic problem. Instead, we introduce the *optimization policy* that utilizes the following extension of the model (1): Introduce the time step 0 and the binary variables x_{i0}, $i \in \mathcal{N}$, representing opportunistic replacements of the respective components at the current maintenance occasion, which is triggered by the failure of some component. Hence, an opportunistic replacement of component i at

time 0 generates the replacement cost c_i but not the maintenance occasion cost d. The objective (1a) is thus modified to

$$\underset{(x,z)}{\text{minimize}} \quad \sum_{t=0}^{T} \sum_{i \in \mathcal{N}} c_i x_{it} + \sum_{t=1}^{T} d z_t. \tag{11a}$$

Since, typically, the components are not new, their (expected) remaining lives τ_i fulfil $\tau_i \leq \hat{T}_i$, $i \in \mathcal{N}$, which is accommodated by the constraints

$$\sum_{t=0}^{\tau_i} x_{it} \geq 1, \qquad i \in \mathcal{N}, \tag{11b}$$

$$x_{i0} \in \{0,1\}, \quad i \in \mathcal{N}. \tag{11c}$$

We refer to the model composed by the variables x_{it}, $i \in \mathcal{N}$, z_t, $t \in \mathcal{T}$, the additional variables x_{i0}, $i \in \mathcal{N}$, the objective function (11a), and the constraints (1b)–(1g), (11b)–(11c), as the *extended opportunistic replacement model*.

Definition 7 (optimization policy)**.** Solve the extended opportunistic replacement model with T_i being the (expected) value of the life of component $i \in \mathcal{N}$. Replace components according to the optimal solution at time 0, i.e. the optimal values of x_{i0}, $i \in \mathcal{N}$.

The optimization models are implemented in the modelling language `AMPL` (version 11.1) and solved by the mixed integer programming solver `CPLEX` (version 11.1). The policies and the scenario generation are implemented in `MATLAB` (version 7.5). All the tests are performed on a Linux double processor unit; each integer programming problem in this case study was solved in between 0.2 and 1 CPU-seconds.

7.3 Aircraft engines

When an aircraft engine is removed for overhaul it needs to be replaced by a spare engine so that the aircraft can stay in service during the maintenance period. This generates a large maintenance occasion cost which is independent of the actual maintenance that is to be performed. The sources of the maintenance occasion cost d are the cost for hiring a spare engine and the work, transportation, inspection, and administration costs associated with the engine exchange. The cost for purchasing a component $i \in \mathcal{N}$ and the work cost associated with its replacement constitute the cost c_i. An aircraft engine consists of components with stochastic and/or deterministic lives. Some components are safety critical, which means that their failure may lead to a catastrophic outcome. Such components are therefore assigned age limits—in terms of numbers of flight hours—before which they must be replaced. The probability that a failure occurs before this limit is very low. We may therefore consider the lives of these components as *deterministic*. The non-safety critical components are replaced "on condition", i.e., if they fail during operation or if—at an inspection—they are found to be (almost) failed. We call these components *stochastic*, and assume that they possess Weibull distributed lives, as suggested in [55]. (Non-safety critical components are replaced when crack lengths above certain limits are observed; the case study in [55] on survival estimation models for an application to the crack growth in the nozzle component of a low pressure turbine indicated that a non-stationary renewal process with Weibull distributed lives is a good model for the conditional life distribution.)

The RM12 engine of the military aircraft JAS39 Gripen consists of modules which are composed by components; a module must be removed before any of its components can be replaced. Since this structure is more complex than the system considered in the model (1) it cannot be applied to the whole engine. Thus, we here consider one engine module at a time, namely the high and low pressure turbines; a mathematical model comprising the entire RM12 engine is the subject of a forthcoming article.

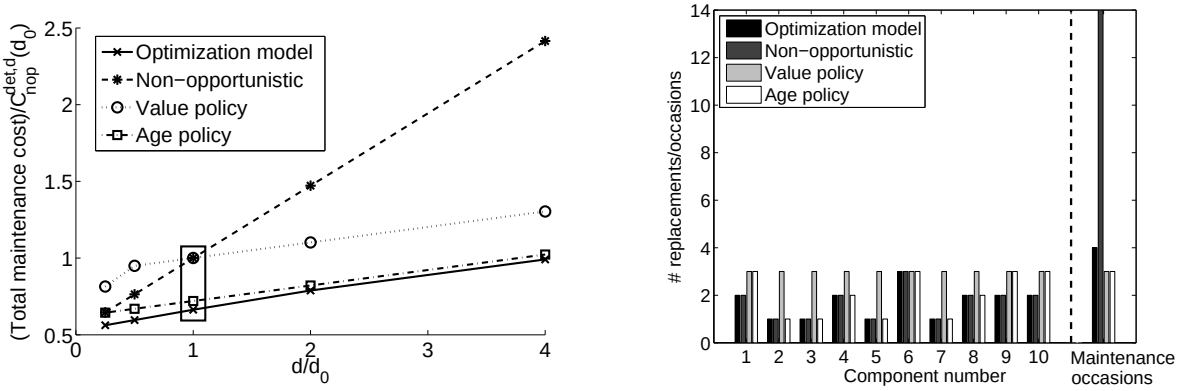
The data used for our tests originate from VAC; since the RM12 data are confidential the true values of the costs and lives of components are not revealed. We let the maintenance occasion cost d include the cost of removing the module; the true value of d is denoted d_0. When the parameter d is varied, all resulting total maintenance costs are divided by (the mean value of) the cost of non-opportunistic maintenance obtained at $d = d_0$; this value is denoted $C_{\text{nop}}^{\text{det},d}(d_0)$ and $C_{\text{nop}}^{\text{sto},d}(d_0)$, respectively. When the parameter β is varied, all resulting total maintenance costs are divided by the mean value of the cost of non-opportunistic maintenance obtained at $\beta = 4$ (denoted $C_{\text{nop}}^{\text{sto},\beta}(4)$). The planning horizon T corresponds to 5000 flight hours. The optimization model (1) and the optimization policy employ time steps of 50 flight hours. For the value policy, the parameter $T_{\min}$ corresponds to 150 flight hours.

We evaluate the policies for both deterministic and stochastic opportunistic replacement problems. The expected lives of the stochastic (on condition) components are known but not the corresponding distributions. The deterministic problems use the expected values for the stochastic components' lives. For the stochastic problems, simulations are performed with different values of the shape parameter β in (10); for each component and each value of β, the parameter α is chosen such that the expected life equals the known value.

7.3.1 The low pressure turbine

The low pressure turbine (LPT) consists of 10 components, of which six are on condition and four are safety critical. For the age policy, the parameter δ corresponds to 1050 flight hours; this value was chosen by Alg. 3 with $\Delta = 50$ flight hours and $d = d_0$.

Figure 2 shows the results from the tests on the deterministic problem. Figure 2(a) shows that, for $d = d_0$ the total maintenance cost of using the optimization model is 34% lower than that of using the non-opportunistic policy. Furthermore, as the maintenance occasion cost d increases, all the policies improve compared to the non-opportunistic policy. Figure 2(b) shows that, although the number of maintenance occasions resulting from the optimization model is about a third compared to the non-opportunistic policy, the number of replacements of each of the components is equal. The value and age policies result in even fewer maintenance occasions, but at the price of replacing more components.



(a) (b)

Figure 2: LPT—the deterministic problem solved by the optimization model and the three policies: (a) Resulting total maintenance costs for different values of d. The box corresponds to the actual maintenance occasion cost d_0 at VAC. (b) The number of replacements of the respective components for $d = d_0$. The rightmost set of bars shows the number of maintenance occasions.

The tests on the stochastic opportunistic replacement problems are reported in Figure 3. Figure 3(a) shows the mean of the resulting total cost for maintenance when d is varied. During these tests the stochastic component lives were assigned Weibull distributions according to: $\beta = 2$ for components 1 and 5, $\beta = 4$ for components 4 and 9, and $\beta = 6$ for components 6 and 10. Observe that the optimization policy performs well for all values of d. For $d = d_0$ the mean total maintenance cost of using the optimization policy is 17% lower than that of using the non-opportunistic policy. For the lowest values of d the optimization policy is, however, slightly worse than the non-opportunistic policy. The results illustrated in Figure 3(b) resemble those of Figure 2(b). However, for the stochastic problem the optimization policy yields slightly more component replacements than the non-opportunistic policy. In Figure 3(c) the stochastic components' life distribution parameter β is varied (equally over the six components having stochastic lives). Clearly, the optimization policy performs better than all the other policies. Moreover, the difference between the optimization and non-opportunistic policies grows as the uncertainty decreases (i.e., the value of β increases). Note that the value $\beta = 1$ corresponds to the exponential distribution; since this means that the stochastic components do not age, the optimal policy for these components would be non-opportunistic. Nonetheless, since some components in the LPT have deterministic lives, the optimization policy may yield a lower cost also for this case.

7.3.2 The high pressure turbine

The high pressure turbine (HPT) consists of 9 components, of which five are on condition and four are safety critical. For the age policy, the parameter value δ is set to 250 flight hours; this value was chosen by Alg. 3 with $\Delta = 50$ flight hours and $d = d_0$.

Figure 4 shows results from our tests on the deterministic problem. Figure 4(a) reveals trends for the age policy and the optimization model similar to those for the LPT. The difference between the optimization model and the non-opportunistic policy is, however, smaller. For $d = d_0$ the total maintenance cost of using the optimization model is 9% lower than that of using the non-opportunistic policy. Figure 4(b) shows that the number of maintenance occasions is equal for the optimization model and the age policy; this is 40% lower than that of the non-opportunistic policy. The number of component replacements are equal for using the optimization model and the non-opportunistic and age policies, except that the age policy employs one additional replacement of component 2.

Figure 5 shows results from the tests on the stochastic problem. Figure 5(a) shows the mean of the resulting total maintenance cost when d is varied. For these tests the stochastic component lives were assigned Weibull distributions according to: $\beta = 2$ for components 4 and 6, $\beta = 4$ for components 5 and 9, and $\beta = 6$ for component 7. For $d = d_0$ the mean total maintenance cost of using the optimization policy is 4% lower than that of using the non-opportunistic policy. Observe that the optimization policy performs the best for high values of d; for low values of d, however, it performs slightly worse than all the other policies. Figure 5(b) shows that the optimization policy produces slightly more component replacements than the non-opportunistic policy. In Figure 5(c) the stochastic components' life distributions are varied. Here, the optimization policy performs slightly worse than the age policy, but better than the value and non-opportunistic policies. As for the LPT, the difference between the optimization and non-opportunistic policies grows when the uncertainty decreases (i.e., when increasing the value of β).

7.4 Wind turbines

The data used for the wind turbine case study is based on the report [48, pp. D-18–D-20] and originates from a land based 2.5 MW wind turbine unit. We only consider types of maintenance

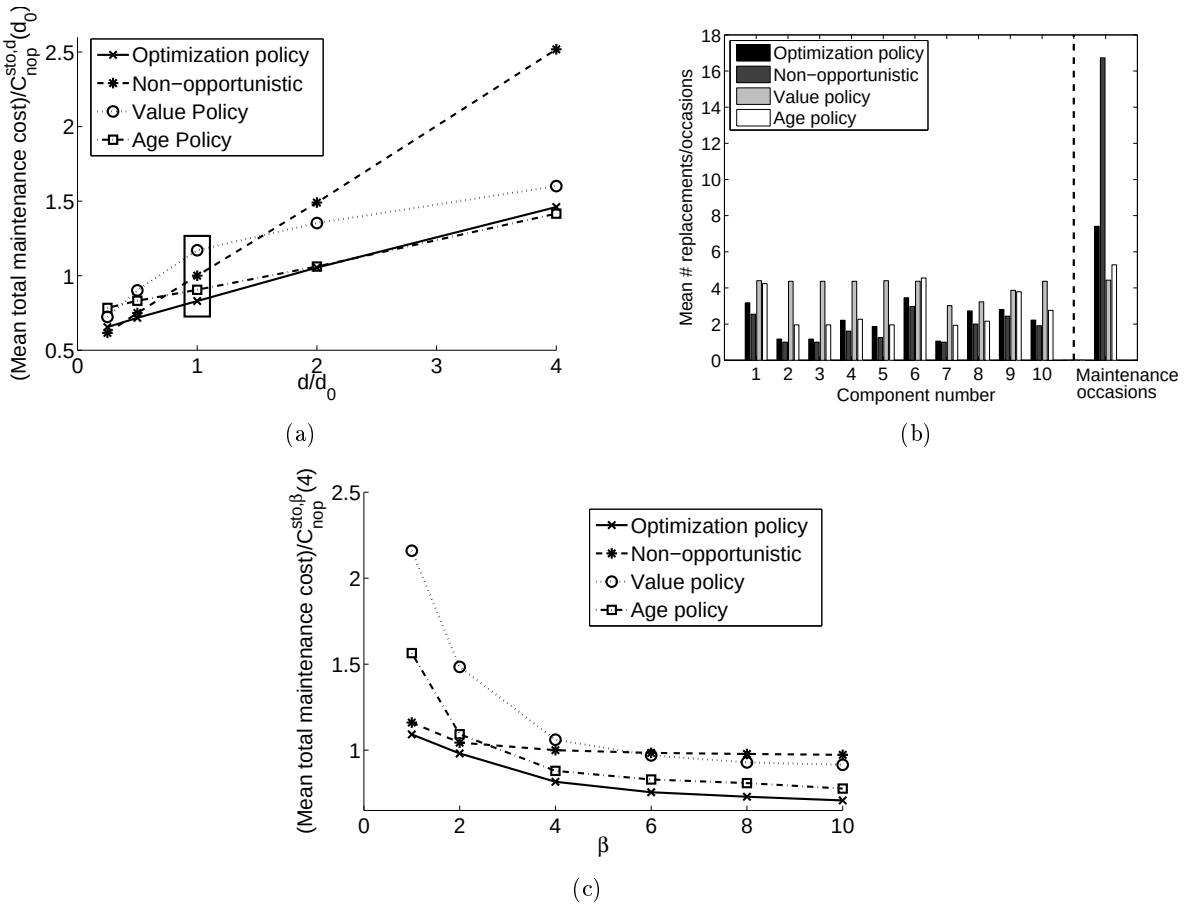
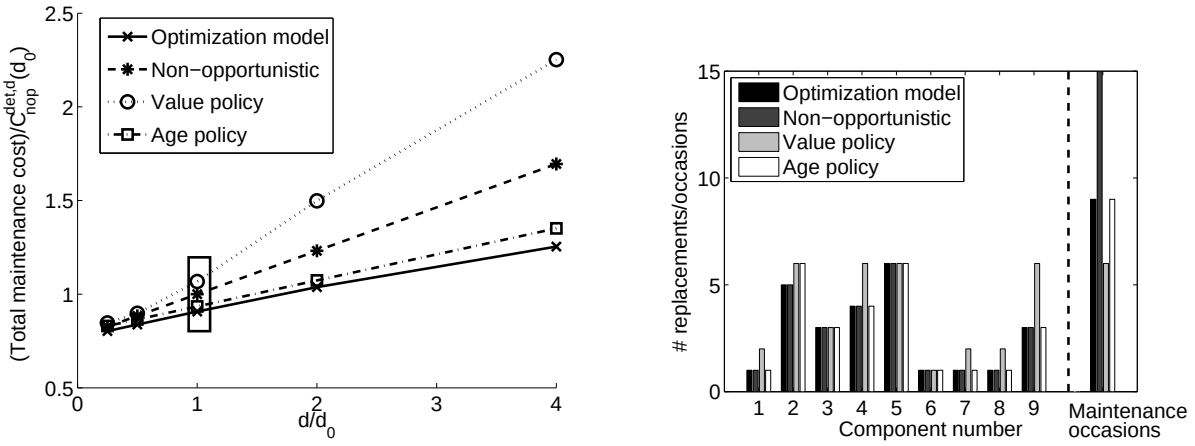


Figure 3: LPT—the stochastic problem (components 1, 4, 5, 6, 9, and 10 having stochastic lives) solved by the four policies: (a) Mean values of the resulting total maintenance costs for different values of d. The box corresponds to the actual maintenance occasion cost d_0 at VAC. (b) Mean number of replacements of the respective components when $\beta = 4$ for all stochastic components and $d = d_0$. The rightmost set of bars shows the mean of the number of maintenance occasions. (c) Mean values of the resulting total maintenance costs for different values of β.

that require the use of a large construction crane. The mobilization cost during three days of this construction crane is the main bulk of the maintenance occasion cost. The report only provides a total crane cost for a set of maintenance activities that varies between \$39 000 and \$84 000, which implies that the mobilization cost is at most \$39 000. After consulting wind power experts the value $d_0 = \$30\,000$ was chosen. The maintenance occasion cost for a wind turbine does, however, depend on the distance between the wind farm and the crane depot, whether the wind turbine is land based or offshore, and whether costs connected with production losses are included or not; this cost may therefore very well vary by a couple of orders of magnitude.

The wind turbine includes five components that require a construction crane for maintenance: blades, pitch bearing, main bearing, gearbox, and generator. The maintenance activities on these components are listed in Table 1 and each *activity* is regarded as a *component* $i \in \mathcal{N}$ (in the remainder of this section, these terms will also be used interchangeably). Note that some components are identical, for instance components $i \in \{5, 6, 7\}$ are all pitch bearings. The replacement cost c_i for each activity $i \in \mathcal{N}$ was calculated according to: $c_i =$ (material cost) + (total crane cost) $- d_0 +$ (labour hours)$\times$(labour hour cost), where the labour hour cost was set to \$50. Most wind turbines are currently at the beginning of their life span, which implies that reliable failure data is scarce. Therefore, many wind power farms employ non-opportunistic



(a) (b)

Figure 4: HPT—the deterministic problem solved by the optimization model and the three policies: (a) Resulting total maintenance costs for different values of d. The box corresponds to the actual maintenance occasion cost d_0 at VAC. (b) The number of replacements of the respective components at $d = d_0$. The rightmost set of bars shows the number of maintenance occasions.

maintenance planning. We consider one wind turbine[1] and assume that reliable distributions of component lives are available. The component lives are assigned Weibull distributions; the respective values of the parameters α and β are shown in Table 1. Note that most components have exponential life distributions (i.e., $\beta = 1$).

no.	component	material cost (k$)	total crane cost (k$)	labour need (hours)	replacement cost, c_i (k$)	α (years)	β
1–3	blades: structural maint.	89	39	49	101	400	1.0
4	blades: non-structural maint.	27	39	246	48	20	1.0
5–7	pitch bearing	31	39	69	43	400	1.0
8	main bearing	30	84	147	91	400	1.0
9	gearbox: gear	122	30	0	122	400	1.0
10	gearbox: regular bearings	81	30	71	85	20	3.5
11	gearbox: high speed bearings	81	84	46	137	20	3.5
12	generator: rotor	95	30	14	96	400	1.0
13–14	generator: bearings	6	60	10	36	17	3.5

Table 1: The components/maintenance activities of the wind turbine problem. The total crane cost is the cost of mobilization and use of crane during the maintenance activity. Labour hours is the number of working hours of external personal required for the maintenance activity.

According to [48, p. A-3], non-structural repair of blades is always performed simultaneously on all three blades; it is hence considered as one activity. On some components, more than one maintenance activity can be performed: structural and non-structural maintenance of the

[1]It would be more beneficial to consider maintenance planning for an entire wind farm and also to include production planning and costs in the mathematical model. As for a complete aircraft engine, this would, however, require a more complex model and is a topic for future research.

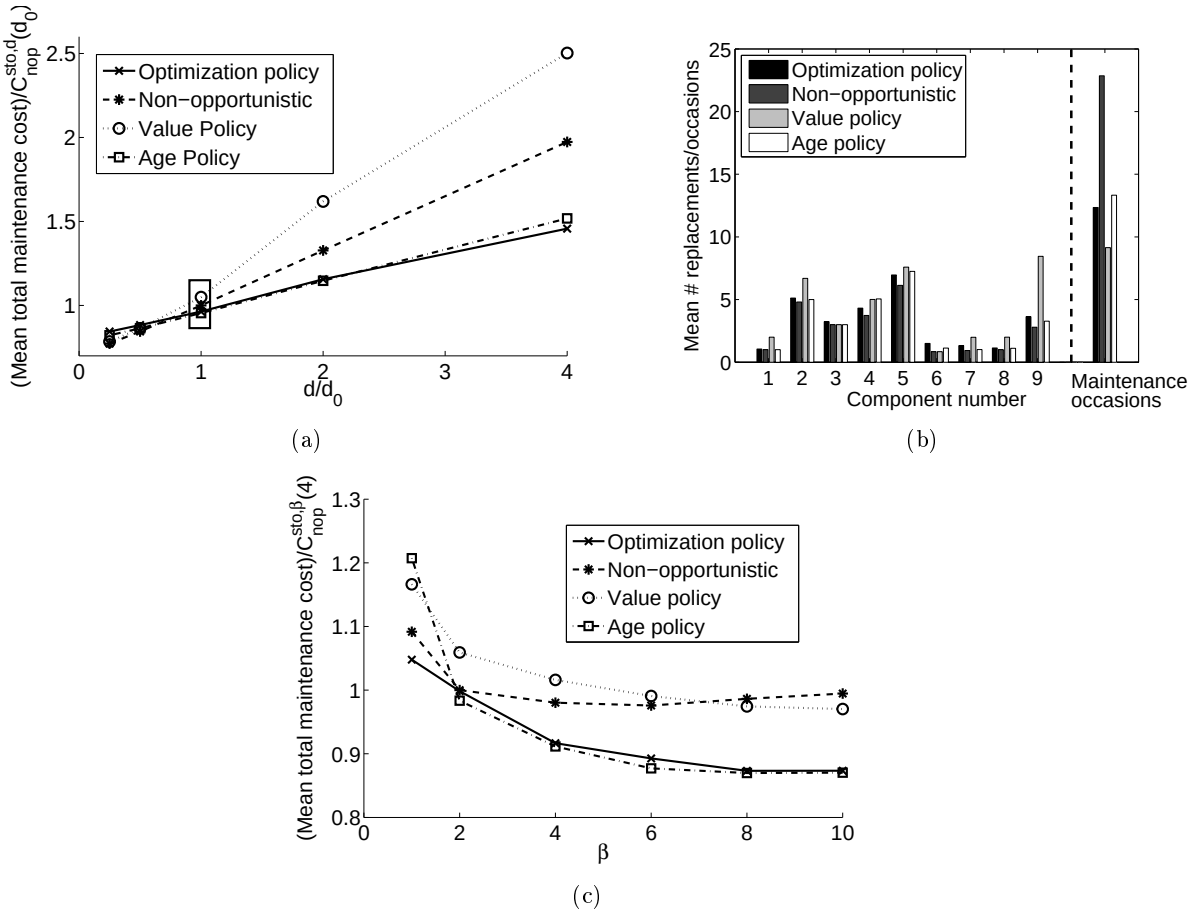


Figure 5: HPT—the stochastic problem (the components 4, 5, 6, 7, and 9 having stochastic lives) solved by the four policies: (a) Mean values of the resulting total maintenance costs for different values of d. The box corresponds to the actual maintenance occasion cost d_0 at VAC. (b) Mean number of replacements of the respective components for $\beta = 4$ and $d = d_0$. The rightmost set of bars shows the mean value of the number of maintenance occasions. (c) Mean values of the resulting total maintenance costs for different values of β.

blades, the replacement of the gear, the high speed bearings, and the regular bearings of the gearbox, and the replacement of the rotor and the bearings of the generator. Unfortunately, not all possible maintenance activities are listed in the report; some are only listed together with other maintenance activities.[2] In order to adapt[3] the problem to the form used in model (1) we need data for each individual maintenance activity. The data was transformed according to the following: Let A_1 and A_2 be two maintenance activities and assume that data for A_1 and $A_1 \cup A_2$ is available. Let k_A be the cost of the component (or the number of labour hours associated to the performance of the maintenance activity) A; then, $k_{A_2} = k_{A_1 \cup A_2} - k_{A_1}$. Let $\hat{k}_A$ be the total crane cost for activity A. Since this includes a mobilization cost d_0, we instead obtain $\hat{k}_{A_2} = \hat{k}_{A_1 \cup A_2} - \hat{k}_{A_1} + d_0$. The distribution of failures demanding the activity A_2 to be performed is assumed to equal that of failures demanding the performance of activity $A_1 \cup A_2$. We have used the original data for the structural and non-structural repair of blades, since the risk of a failure that demands these maintenance activities is not affected by the age of the blades (since

[2]For instance, for the gearbox the replacement of both the gear and the bearings and that of the bearings only are listed, but not the replacement of the gear only.

[3]We could easily adapt the model (1) to include such dependencies by introducing additional variables and constraints, but since the topic of the article is the model itself we choose to adapt the problem data.

the time points for these types of failures are exponentially distributed).

The deterministic problem is obtained by replacing the lives of all components by their respective expected values. The time horizon is set to 25 years, which corresponds to the expected technical life of a wind turbine. For the extended opportunistic replacement model we use time steps of 0.25 years. For the value policy the parameter $T_{\min}$ represents three years. For the age policy the parameter δ represents 5 years; this value was chosen by Alg. 3 with $\Delta = 0.25$ years and $d = d_0$.

Figure 6 shows the results of the test on the deterministic problem. The problem is rather trivial, since it comprises nine components whose lives are longer than the time horizon and five components which will all fail exactly once during the life of the turbine. The optimal solution is to replace all of these five components at the occasion of the first failure; the remaining nine components do not require any replacement. Figure 6(b) shows that for $d = d_0$ all the policies except the non-opportunistic policy find the optimal solution. Figure 6(a) shows that the value of the optimal solution at $d = d_0$ is 13% lower than that produced by the non-opportunistic policy. The age policy always finds the optimal solution; the value policy, however, fails to do so for values of $d > d_0$.

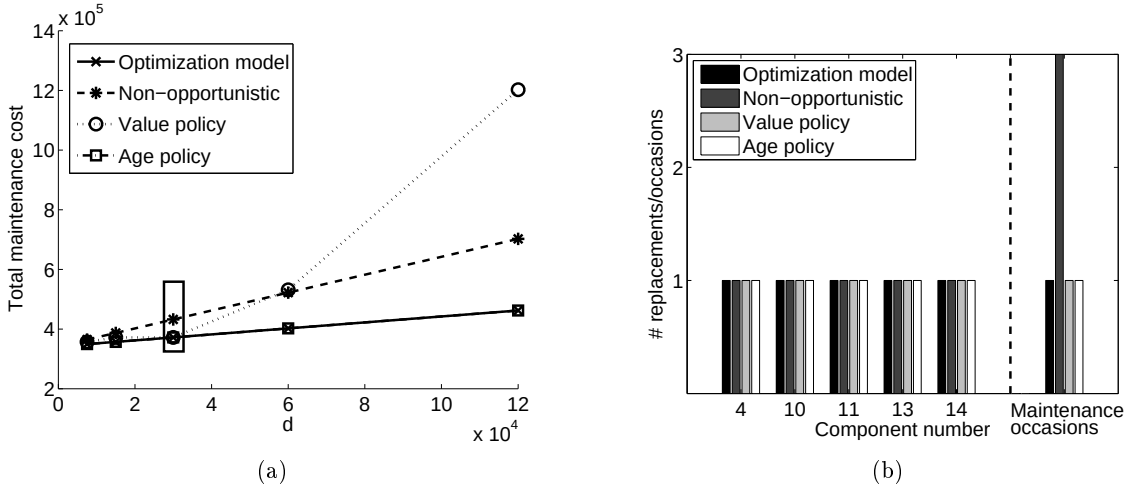


Figure 6: Wind turbine—the deterministic problem solved by the optimization model and the three policies: (a) Resulting total maintenance costs for different values of d (in \$). The box corresponds to the actual value $d_0 = \$30\,000$. (b) The number of replacements of the components with lives shorter than the time horizon at $d = d_0$. The rightmost set of bars shows the number of maintenance occasions.

Figure 7 shows results from the tests on the stochastic problem. Note that only components $i \in \{4, 10, 11, 13, 14\}$ have expected lives shorter than the planning horizon.[4] Figure 7(a) shows the results from varying the maintenance occasion cost d. The optimization and age policies perform better than the non-opportunistic policy for $d \in \{\$60000, \$120000\}$. For the lower values of d the non-opportunistic policy is better than or at least as good as the other policies. For $d = d_0$ the mean total maintenance cost of using the optimization policy is 4% higher than that of using the non-opportunistic policy. Figure 7(b) reveals that the number of maintenance occasions resulting from using the optimization policy is lower than those of the age and non-opportunistic policies. The number of individual component replacements is, however, higher for some components. The number of maintenance occasions is lowest for the value policy, but the corresponding numbers of replacements of components 13 and 14 are much higher than those

[4]The remaining nine components have exponentially distributed lives with expected values much larger than the horizon. These components have only a marginal effect on the mean value of the total replacement cost.

resulting from using the other policies. In Figure 7(c) the parameter β is varied equally for the non-exponentially distributed lives of components 10, 11, 13, and 14, while the lives of the other ten components stay exponentially distributed (i.e., $\beta = 1$). Observe that for higher values of β the optimization and age policies outperform the non-opportunistic and value policies. This is expected, since the life distributions then tend to deterministic ones. Note that, for $\beta = 1$ none of the components age, whence the non-opportunistic policy is optimal.

This case study shows that a simple deterministic problem may become much more difficult when the component lives are stochastic; it motivates the development of a replacement model based on stochastic programming (see [46]).

8 Conclusions and future research

The opportunistic replacement model is shown to have a nice inherent structure, in that while the problem is NP-hard, the model reduces to a linear program once the maintenance occasions are fixed; the latter can in some cases even be solved through a greedy procedure. Also, all the necessary linear constraints define facets of the convex hull of the set of feasible schedules. We have recently identified new classes of facets; their application will be reported in the near future.

The numerical case studies performed on applications from the wind power and aircraft engine industries show that the optimization model can be utilized to reduce costs in comparison to using simpler maintenance policies. The study also shows that the model can be used for maintenance scheduling of components with non-deterministic lives; the cost reduction tends to increase with the maintenance occasion cost and lower levels of uncertainty regarding component lives.

Work in progress include the optimization of maintenance decisions when component lives are non-deterministic through the use of a stochastic programming model. Even in the case when costs are independent of time, we have already shown that such a stochastic extension of the current problem is NP-hard. In order to provide a computationally feasible model we will therefore also investigate how to best define an accurate enough scenario representation of the component lives. Further, we intend to study models comprising successive improvements of life distribution estimates through the addition of measurement-based information about the condition of the system.

The opportunistic replacement model (1) is utilized in further studies of maintenance planning optimization at Volvo Aero as well as in the nuclear and wind power industries. In order to incorporate requirements specific to the application (such as spare component replacement and redundancies within the system) extensions of the model are made. In the near future, experiences from these activities will be reported.

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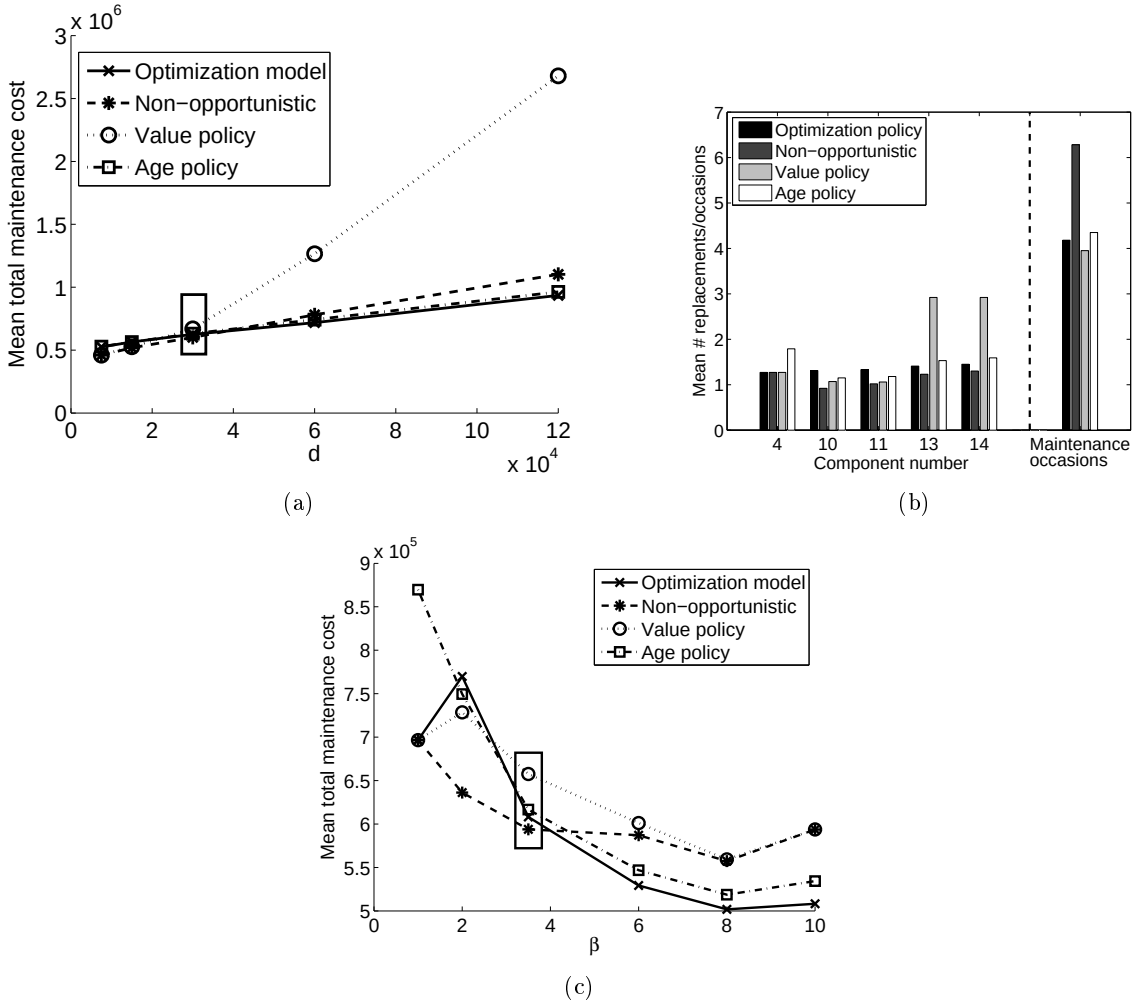

Figure 7: Wind turbine—the stochastic problem solved by the four policies: (a) Mean values of the resulting total maintenance costs for different values of d (in \$). The box corresponds to the actual maintenance occasion cost $d_0 = \$30\ 000$. (b) Mean number of replacements of the components with expected lives shorter than the horizon for $d = d_0$; $\beta = 3.5$ for components 10, 11, 13, and 14; $\beta = 1$ for the remaining ten components. The rightmost set of bars shows the mean value of the number of maintenance occasions. (c) Mean values of the resulting total maintenance costs for different values of the parameter β (for components 10, 11, 13, and 14).

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A Proofs

Proof of Proposition 6
Since $T_i \geq 2$ for $i \in \mathcal{N}$, $\operatorname{conv} S$ is full-dimensional (cf. Proposition 4). Hence, we can use the uniqueness characterization of the facet description from Theorem 2 to show the proposition.

For each $r \in \mathcal{N}$ and each $s \in \mathcal{T}$, let $F_{rs} = \{\, (x,z) \in \operatorname{conv} S \mid x_{rs} = z_s \,\}$. Further, let, for $i \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^0_{it} = 0$ if $(i,t) = (r,s)$, $x^0_{it} = 1$ otherwise, and let $z^0_t = 1$, $t \in \mathcal{T}$. It follows that $(x^0, z^0) \in S \setminus F_{rs}$. Then, letting $x^{\mathrm{A}}_{it} = z^{\mathrm{A}}_t = 1$, $i \in \mathcal{N}$, $t \in \mathcal{T}$, it follows that $(x^{\mathrm{A}}, z^{\mathrm{A}}) \in F_{rs}$ and hence that F_{rs} is a proper face of $\operatorname{conv} S$.

Moreover, there exists values of $\lambda \in \mathbb{R}^{N \times T}$, $\mu \in \mathbb{R}^T$, and $\rho \in \mathbb{R}$ such that the equation (7) is satisfied for all $(x,z) \in F_{rs}$. We will show that for any value of $\mu_s \in \mathbb{R}$, in a solution to (7) the following hold: $\lambda_{it} = -\mu_s$ if $(i,t) = (r,s)$, $\lambda_{it} = 0$ otherwise; $\mu_t = 0$ for $t \in \mathcal{T} \setminus \{s\}$; $\rho = 0$.

For each $\ell \in \mathcal{T} \setminus \{s\}$, let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^1_{jt} = 0$ if $(j,t) = (r,\ell)$, $x^1_{jt} = x^{\mathrm{A}}_{jt}$ otherwise, and let $z^1 = z^{\mathrm{A}}$. It follows that $(x^1, z^1) \in F_{rs}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^1, z^1), respectively, inserted in (7) then yield that $\lambda_{r\ell} = 0$ for $\ell \in \mathcal{T} \setminus \{s\}$.

Similarly, for each $k \in \mathcal{N} \setminus \{r\}$ and each $\ell \in \mathcal{T}$, let for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^2_{jt} = 0$ if $(j,t) = (k,\ell)$, $x^2_{jt} = x^{\mathrm{A}}_{jt}$ otherwise, and let $z^2 = z^{\mathrm{A}}$. It follows that $(x^2, z^2) \in F_{rs}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^2, z^2), respectively, inserted in (7) then yield that $\lambda_{k\ell} = 0$ for $k \in \mathcal{N} \setminus \{r\}$ and $\ell \in \mathcal{T}$; hence, the equation (7) can be rewritten as

$$\lambda_{rs} x_{rs} + \sum_{t \in \mathcal{T}} \mu_t z_t = \rho. \tag{12}$$

For each $\ell \in \mathcal{T} \setminus \{s\}$, let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^3_{jt} = z^3_t = 0$ if $t = \ell$, $x^3_{jt} = z^3_t = 1$ otherwise. It follows that $(x^3, z^3) \in F_{rs}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^3, z^3), respectively, inserted in (12) then yields that $\mu_\ell = 0$ for $\ell \in \mathcal{T} \setminus \{s\}$. Equation (12) can now be rewritten as

$$\lambda_{rs} x_{rs} + \mu_s z_s = \rho. \tag{13}$$

Let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^4_{jt} = z^4_t = 0$ if $t = s$, $x^4_{jt} = z^4_t = 1$ otherwise. It follows that $(x^4, z^4) \in F_{rs}$. The vectors (x^4, z^4) and $(x^{\mathrm{A}}, z^{\mathrm{A}})$, respectively, inserted in (13) then yield that $0 = \rho = \lambda_{rs} + \mu_s$. The equation (13) can thus be rewritten as $\mu_s x_{rs} = \mu_s z_s$, and from [42, pp. 91–92] follows that the inequality $x_{rs} \leq z_s$ defines a facet of $\operatorname{conv} S$. □

Proof of Proposition 7
Since $T_i \geq 2$ for $i \in \mathcal{N}$, $\operatorname{conv} S$ is full-dimensional (cf. Proposition 4). Hence, we can use the uniqueness characterization of the facet description from Theorem 2 to show the proposition.

For each $r \in \mathcal{N}$ such that $T_r \geq 3$ and each $s \in \mathcal{T}$, let $\widetilde{F}_{rs} = \{\, (x,z) \in \operatorname{conv} S \mid x_{rs} = 0 \,\}$. Further, let $x^0_{it} = z^0_t = 1$, $i \in \mathcal{N}$, $t \in \mathcal{T}$. It follows that $(x^0, z^0) \in S \setminus \widetilde{F}_{rs}$. Then letting, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^{\mathrm{A}}_{jt} = 0$ if $(j,t) = (r,s)$, $x^{\mathrm{A}}_{jt} = 1$ otherwise, and letting $z^{\mathrm{A}}_t = 1$, $t \in \mathcal{T}$, it follows that $(x^{\mathrm{A}}, z^{\mathrm{A}}) \in \widetilde{F}_{rs}$ and hence that $\widetilde{F}_{rs}$ is a proper face of $\operatorname{conv} S$.

Moreover, there exists values of $\lambda \in \mathbb{R}^{N \times T}$, $\mu \in \mathbb{R}^T$, and $\rho \in \mathbb{R}$ such that the equation (7) is satisfied for all $(x,z) \in \widetilde{F}_{rs}$. We will show that for any value of $\lambda_{rs} \in \mathbb{R}$, in a solution to (7) the following hold: $\lambda_{it} = 0$ if $(i,t) \in \{\mathcal{N} \times \mathcal{T}\} \setminus \{(r,s)\}$; $\mu_t = 0$ for $t \in \mathcal{T}$; $\rho = 0$.

For each $i \in \mathcal{N}$ and each $t \in \mathcal{T}$, let for $j \in \mathcal{N}$ and $k \in \mathcal{T}$, $x^1_{jk} = 0$ if $(j,k) = (i,t)$, $x^1_{jk} = x^{\mathrm{A}}_{jk}$ otherwise, and let $z^1 = z^{\mathrm{A}}$. Since $T_r \geq 3$, it follows that $(x^1, z^1) \in \widetilde{F}_{rs}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^1, z^1), respectively, inserted in (7) then yield that $\lambda_{it} = 0$ for all $(i,t) \in \{\mathcal{N} \times \mathcal{T}\} \setminus \{(r,s)\}$. The equation (7) can then be rewritten as (12).

For each $t \in \mathcal{T}$, let, for $j \in \mathcal{N}$ and $k \in \mathcal{T}$, $x^2_{jk} = z^2_k = 0$ if $k = t$, $x^2_{jk} = x^{\mathrm{A}}_{jk}$ and $z^2_k = z^{\mathrm{A}}_k$ otherwise. Since $T_r \geq 3$, it follows that $(x^2, z^2) \in \widetilde{F}_{rs}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^2, z^2), respectively, inserted in (12) then yield that $\mu_t = 0$ for $t \in \mathcal{T}$.

Since $x_{rs} = 0$ for all $(x, z) \in \widetilde{F}_{rs}$ it follows that $\rho = 0$. Equation (12) can then be rewritten as $\lambda_{rs} x_{rs} = 0$, and from [42, pp. 91–92] follows that the inequality $x_{rs} \geq 0$ defines a facet of $\operatorname{conv} S$. □

Proof of Proposition 8

Since $T_i \geq 2$ for $i \in \mathcal{N}$, $\operatorname{conv} S$ is full-dimensional (cf. Proposition 4). Hence, we can use the uniqueness characterization of the facet description from Theorem 2 to show the proposition.

For each $s \in \mathcal{T}$, let $F_s = \{ (x, z) \in \operatorname{conv} S \mid z_s = 1 \}$. Further, let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^0_{jt} = z^0_t = 0$ if $t = s$, $x^0_{jt} = z^0_t = 1$, otherwise. It follows that $(x^0, z^0) \in S \setminus F_s$. Then, letting $x^{\mathrm{A}}_{it} = z^{\mathrm{A}}_t = 1$, $i \in \mathcal{N}$, $t \in \mathcal{T}$, it follows that $(x^{\mathrm{A}}, z^{\mathrm{A}}) \in F_s$ and that F_s is a proper face of $\operatorname{conv} S$.

Moreover, there exists values of $\lambda \in \mathbb{R}^{N \times T}$, $\mu \in \mathbb{R}^T$, and $\rho \in \mathbb{R}$ such that the equation (7) is satisfied for all $(x, z) \in F_s$. We will show that for any value of $\rho \in \mathbb{R}$, in a solution to (7) the following hold: $\lambda_{it} = 0$ for $i \in \mathcal{N}$ and $t \in \mathcal{T}$; $\mu_s = \rho$, $\mu_t = 0$ for $t \in \mathcal{T} \setminus \{s\}$.

For each $r \in \mathcal{N}$ and each $\ell \in \mathcal{T}$, let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^1_{jt} = 0$ if $(j, t) = (r, \ell)$, $x^1_{jt} = 1$ otherwise, and let $z^1 = z^{\mathrm{A}}$. It follows that $(x^1, z^1) \in F_s$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^1, z^1), respectively, inserted in (7) then yield that $\lambda_{r\ell} = 0$ for $r \in \mathcal{N}$ and $\ell \in \mathcal{T}$. Equation (7) can then be rewritten as

$$\sum_{t \in \mathcal{T}} \mu_t z_t = \rho. \tag{14}$$

For each $\ell \in \mathcal{T} \setminus \{s\}$, let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^2_{jt} = z^2_t = 0$ if $t = \ell$, $x^2_{jt} = z^2_t = 1$ otherwise. It follows that $(x^2, z^2) \in F_s$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^2, z^2), respectively, inserted in (14) then yield that $\mu_\ell = 0$ for $\ell \in \mathcal{T} \setminus \{s\}$. Equation (14) can then be rewritten as $\mu_s z_s = \rho$. Since $z_s = 1$ for all $(x, z) \in F_s$ it follows that $\mu_s = \rho$, which yields the equation $\rho z_s = \rho$. From [42, pp. 91–92] then follows that the inequality $z_s \leq 1$ defines a facet of $\operatorname{conv} S$. □