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*PREPRINT 2011:20*

# On the Caffarelli-Kohn-Nirenberg type inequalities involving Critical and Supercritical Weights. Revised version

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Gothenburg Sweden 2011

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Gothenburg, August 2011

Preprint 2011:20
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2011

# On the Caffarelli-Kohn-Nirenberg type inequalities involving Critical and Supercritical Weights Revised version

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**Abstract**

[*] The main purpose of this article is to establish the CKN-type inequalities for all $\alpha \in \mathbf{R}$ and to study the relating matters systematically. Roughly speaking, we shall discuss about the characterizations of the CKN-type inequalities for all $\alpha \in \mathbf{R}$ as the variational problems, the existence and nonexistence of the extremal solutions to these variational problems in proper spaces, the exact values and the assymptotic behaviors of the best constants in both the noncritical case and the critical case.

In the study of the CKN-type inequalities, the presence of weight functions in the both sides prevents us from employing effectively the so-called spherically symmetric rearrangement. Further the invariance of $\mathbf{R}^n$ by the group of dilatations creates some possible loss of compactness. As a result we will see that the existence of extremals and the values of best constants and their asymptotic behaviors essentially depend upon the relations among parameters in the inequality.

## Contents



*2000 *Mathematics Subject Classification.* Primary 35J70; Secondary 35J60.
*Key words and phrases.* CKN-type inequality, Hardy-Sobolev inequality, Weighted Hardy inequality, Degenerate elliptic equation, Best constant.

# 1 Introduction and Histrical remarks

## 1.1 Introduction

We shall begin with recalling the classical weighted Sobolev inequalities (1.1), which are often called the Caffarelli-Kohn-Nirenberg type inequalities (the CKN-type inequalities).

There is a positive number $S$ depending only on $p, q, \alpha, \beta$ and $n$ such that we have

$$\int_{\mathbf{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \geq S\left(\int_{\mathbf{R}^n} |u|^q |x|^{\beta q}\, dx\right)^{p/q}, \qquad \text{for any } u \in C_0^\infty(\mathbf{R}^n), \tag{1.1}$$

where $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots \frac{\partial u}{\partial x_n})$ and $|\nabla u| = \left(\sum_{k=1}^n |\frac{\partial u}{\partial x_k}|^2\right)^{1/2}$.
Here $n \geq 1$, $1 \leq p < +\infty$ and $q, \alpha, \beta$ are real numbers satisfying

$$\begin{cases} \alpha > 1 - \frac{n}{p}, \\ (1-\alpha+\beta)p < n, \\ 0 \leq 1/p - 1/q = (1-\alpha+\beta)/n, \\ \beta \leq \alpha. \end{cases} \tag{1.2}$$

The main purpose of this article is not only to establish the CKN-type inequalities for all $\alpha \in \mathbf{R}$ but also to study the relating matters systematically. Roughly speaking, we shall discuss about the characterizations of the imbeddings as the variational problems, the existence and nonexistence of the extremal solutions to these variational problems in proper spaces, the exact values and the asymptotic behaviors of the best constants.

Now we introduce a crucial parameter $\gamma$ as follows.

**Definition 1.1** *For $1 \leq p < +\infty$, in (1.2) let us set*

$$\gamma = \alpha - 1 + \frac{n}{p} = \beta + \frac{n}{q}. \tag{1.3}$$

Under the condition (1.2), we have $0 < \gamma$ as well. By noting that $\alpha p = p(1+\gamma) - n, \beta = \gamma q - n$, after all we can rewrite (1.1) and (1.2 ) to obtain the followings:

$$\int_{\mathbf{R}^n} |\nabla u|^p |x|^{p(1+\gamma)-n}\, dx \geq S\left(\int_{\mathbf{R}^n} |u|^q |x|^{\gamma q - n}\, dx\right)^{p/q}, \qquad \text{for any } u \in C_0^\infty(\mathbf{R}^n), \tag{1.4}$$

where $n \geq 1$, $1 \leq p < +\infty$ and $q, \gamma$ are real numbers satisfying

$$\begin{cases} \gamma > 0, \\ q < +\infty, \\ 0 \leq 1/p - 1/q \leq 1/n. \end{cases} \tag{1.5}$$

Throughout the present article we shall work with a parameter $\gamma \in \mathbf{R}$ instead of $\alpha$ and $\beta$, so that most of our results become symmetric in $\gamma$ with repect to $\gamma = 0$.

Furthermore we classify the CKN-type inequalities according to the range of the parameter $\gamma$ into the three cases. Namely

**Definition 1.2** *The parameter $\gamma$ is said to be subcritical, critical and supercritical if $\gamma$ satisfies $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$ respectively.*

**Remark 1.1**
1. *Here we note that the conditions $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$ are equaivalent to $\alpha > 1 - \frac{n}{p}$, $\alpha = 1 - \frac{n}{p}$ and $\alpha < 1 - \frac{n}{p}$ respectively.*
2. *In the classical CKN-type inequalities (1.1), it follows from the subcritical condition $\gamma > 0$ that we have $\beta q > -n$, hence the weight functions in the both sides are locally integrable on $\mathbf{R}^n$. By this reason these inequalities (1.1) are classified into the subcritical case of the CKN-type inequalities in this article.*

## 1.2 Histrical remarks

Before we go further into our main results on the CKN-type inequalities involving critical and supercritical cases, we give a brief historical review here. As we have already mentioned, the inequality (1.1) for $\gamma = \alpha - 1 + \frac{n}{p} > 0$ is often called the Caffarelli-Kohn-Nirenberg type (the CKN-type inequalities). In fact in [CKN] they established general multiplicative inequalities including these types. In [Ho1] we have also studied these inequalities among more general imbedding theorems on the weighted Sobolev spaces, where the weights are powers of distance from a given closed set $F$.

It was also very interesting for us to study further the properties of the imbedding operators obtained there. But for a general $F$ it seemed not easy to study these problems in a detailed way. By this reason, in [Ho2] we restricted ourselves on the simplest case that $F$ consists of a single point, namely, the origin. In this particular case we have studied the relating problems in a various aspect and obtained interesting results such as the exact values of the best constant $S = S(p, q, \alpha)$ in certain cases, the existence and nonexistence of the extremals and so on.

Recently we have revisited the weighted Hardy-Sobolev inequality in [AH2]. It is easy to see that the classical CKN-type inequality coincides with the weighted Hardy-Sobolev inequality if $\beta = \alpha - 1$, or equivalently $p = q$. To our surprise it was shown that the weighted Hardy-Sobolev inequalities themselves hold for all $\gamma \in \mathbf{R}$ ( or equivalently all $\alpha \in \mathbf{R}$) with some modifications. In fact, even if $\gamma = \alpha - 1 + \frac{n}{p} = 0$ holds, the sharp inequality of the Hardy type remains valid as long as the whole space $\mathbf{R}^n$ is replaced by a bounded domain containing the origin and the weight functions in the right hand side are replaced by the logarismic ones. Moreover we have successfully improved those weighted Hardy-Sobolev inequalities by finding out sharp missing terms, as a result they turned out to be very useful in many aspects. For the improved inequalities, see Proposition1.2 below. ( For the complete argument and the related applications see [AH2].)

On the other hand, the counterpart in the CKN-type inequalities to the weighted Hardy-Sobolev inequalities in [AH2] seems to be unknown so far. But it seems reasonable for us to expect that the CKN-type inequalities should remain valid for all $\gamma \in \mathbf{R}$ ( $\alpha \in \mathbf{R}$ ) with a similar modification as was performed in the weighted Hardy-Sobolev inequalities. In this spirit we shall establish the CKN type inequalities for all $\gamma \in \mathbf{R}$ ( $\alpha \in \mathbf{R}$ ) and we shall further study them systematically in the present paper.

In order to emphasize the meaning of this classification of the CKN-type inequalities and our motivation in this paper, let us recall the results on the weighed Hardy-Sobolev inequalities as the necessary background.

We first review as Proposition 1.1 the classical weighted Hardy-Sobolev inequalities in the noncritical case, and then we also recall as Proposition 1.2 the improved weighted Hardy-Sobolev inequalities with sharp missing terms in [AH2]. It follows from these results that the weighted Hardy-Sobolev inequalities are valid for all $\gamma \in \mathbf{R}$ and Definition1.2 should be natural for us to study the CKN-type inequalities based on the (improved) weighted Hardy-Sobolev inequalities.

**Proposition 1.1** *Let $n \geq 1$, $0 \in \Omega$ and $\Omega$ is a domain of $\mathbf{R}^n$. Assume that $1 < p < +\infty$ and $\gamma \neq 0$. Then we have*

$$\int_\Omega |\nabla u|^p |x|^{(1+\gamma)p-n}\, dx \geq |\gamma|^p \int_\Omega |u(x)|^p |x|^{\gamma p - n}\, dx \tag{1.6}$$

*for any $u \in C_0^\infty(\Omega \setminus \{0\})$.*

In this inequalities (1.6), the domain $\Omega$ may be unbounded and the best constant $|\gamma|^p$ is apparently independent of the shape of domains. In particular we can put $\Omega = \mathbf{R}^n$.

**Proposition 1.2** *Let $n \geq 1$, $0 \in \Omega$ and $\Omega$ is a bounded domain of $\mathbf{R}^n$.*

*1.* Subcritical case ($\gamma > 0$, $1 < p < +\infty$),

*There exist* $K = K(n) > 1$ *and* $C = C(n) > 0$ *such that if* $R > K \sup_\Omega |x|$ *then*

$$\begin{aligned}\int_\Omega |\nabla u|^p |x|^{(1+\gamma)p-n}\, dx \geq & |\gamma|^p \int_\Omega |u(x)|^p |x|^{\gamma p - n}\, dx \\ & + C \int_\Omega |u(x)|^p \left(\log \frac{R}{|x|}\right)^{-2} |x|^{\gamma p-n}\, dx\end{aligned} \tag{1.7}$$

*for any* $u \in C_0^\infty(\Omega)$.

*2.* Critical case ($\gamma = 0$, $1 < p < +\infty$),

*Then there exist* $K = K(n) > 1$ *and* $C = C(n) > 0$ *such that if* $R > K \sup_\Omega |x|$ *then*

$$\begin{aligned}\int_\Omega |\nabla u|^p |x|^{p-n}\, dx \geq & \frac{1}{(p')^p} \int_\Omega \frac{|u(x)|^p}{|x|^n} \left(\log \frac{R}{|x|}\right)^{-p} dx \\ & + C \int_\Omega \frac{|u(x)|^p}{|x|^n} \left(\log \frac{R}{|x|}\right)^{-p} \left(\log\left(\log \frac{R}{|x|}\right)\right)^{-2} dx\end{aligned} \tag{1.8}$$

*for any* $u \in C_0^\infty(\Omega)$. *Here* $p' = \frac{p}{p-1}$.

*3.* Supercritical case ($\gamma < 0$, $1 < p < +\infty$),

*Then there exist* $K = K(n) > 0$ *and* $C = C(n) > 0$ *such that if* $R > K \sup_\Omega |x|$ *then*

$$\begin{aligned}\int_\Omega |\nabla u|^p |x|^{(1+\gamma)p-n}\, dx \geq & |\gamma|^p \int_\Omega |u(x)|^p |x|^{\gamma p - n}\, dx \\ & + C \int_\Omega |u(x)|^p \left(\log \frac{R}{|x|}\right)^{-2} |x|^{\gamma p-n}\, dx\end{aligned} \tag{1.9}$$

*for any* $u \in C_0^\infty(\Omega \setminus \{0\})$.

**Remark 1.2** *1. If we replace a bounded domain* $\Omega$ *by the whole space* $\mathbf{R}^n$*, then in general we can not expect any improved weighted Hardy-Sobolev inequalities with a missing term.*

*2. If* $\gamma = 0$ *(the critical case) and* $\Omega = \mathbf{R}^n$*, then one can show from a capacitary argument that for any compact set* $K \subset \mathbf{R}^n$

$$\inf \left[\int_{\mathbf{R}^n} |\nabla u|^p |x|^{p-n}\, dx : u \in C_0^\infty(\mathbf{R}^n), u \geq 1 \text{ on } K\right] = 0.$$

*Therefore we can not expect the weighted Hardy inequality in the whole space* $\mathbf{R}^n$.

# 2 Main results

## 2.1 The CKN-type inequalities

In the subsequent we employ the following notations:

$$p' = \frac{p}{p-1},\ p^* = \frac{np}{(n-p)_+} \quad \text{for } 1 \leq p \leq \infty. \tag{2.1}$$

Here we set $t_+ = \max\{0, t\}$ and $1/0 = \infty$.

As we have already mentioned in §1, for fixed $p, q$, instead of parameters $\alpha, \beta$ in the CKN-type inequalities we work with a new parameter

$$\gamma = \alpha - 1 + \frac{n}{p} = \beta + \frac{n}{q}. \tag{2.2}$$

Then the range for $p, q, \gamma$ becomes

$$1 \le p \le q < \infty,\ (0 \le)\ \tau_{p,q} = \frac{1}{p} - \frac{1}{q} \le \frac{1}{n},\ \gamma \in \mathbf{R}. \tag{2.3}$$

From these conditions we obtain for a fixed $p$

$$p \le q \le p^* = \frac{np}{n-p} \quad \text{if } 1 \le p < n\ ;\ \ p \le q < p^* = \infty \quad \text{if } n \le p < \infty. \tag{2.4}$$

We recall that the subcritical condition, the critical condition and the subcritical condition simply correspond to $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$ respectively.

We prepare more notations below.

**Definition 2.1** *For $\alpha \in \mathbf{R}$ and $R \ge 1$ we set*

$$I_\alpha(x) = I_\alpha(|x|) = \frac{1}{|x|^{n-\alpha}} \qquad \text{for } x \in \mathbf{R}^n \setminus \{0\}, \tag{2.5}$$

$$A_{1,R}(x) = A_{1,R}(|x|) = \begin{cases} \log \dfrac{R}{|x|} & \text{for } x \in \overline{B_1} \setminus \{0\}, \\ \log(R|x|) & \text{for } x \in \mathbf{R}^n \setminus B_1 \end{cases} \tag{2.6}$$

*When $0 < \alpha < n$ holds, $I_\alpha$ is called a Riesz kernel of order $\alpha$.*

Under these notations the CKN-type inequalities have the following forms:

If $\gamma \neq 0$, then

$$\int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \ge S\Big(\int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x)\,dx\Big)^{p/q}, \tag{2.7}$$

If $\gamma = 0$, then for $R > 1$

$$\int_{B_1} |\nabla u(x)|^p I_p(x)\,dx \ge C\Big(\int_{B_1} |u(x)|^q \frac{I_0(x)}{A_{1,R}(x)^{1+q/p'}}\,dx\Big)^{p/q}. \tag{2.8}$$

Now we introduce function spaces and relating norms below.

**Definition 2.2** *Let $1 \le p \le q < \infty$, $\gamma \in \mathbf{R}$ and $R \ge 1$. Let $\Omega$ be a domain of $\mathbf{R}^n$ and let $u : \Omega \to \mathbf{R}$.*

1. *For $w : \Omega \to \mathbf{R}$ satisfying $w \ge 0$ a.e. on $\Omega$, we set*

$$\|u\|_{L^q(\Omega\,;w)} = \Big(\int_\Omega |u(x)|^q w(x)\,dx\Big)^{1/q}. \tag{2.9}$$

2. *Under the above notation we set*

$$\|u\|_{L^q_\gamma(\Omega)} = \|u\|_{L^q(\Omega\,;I_{q\gamma})},\ \|\nabla u\|_{L^p_{1+\gamma}(\Omega)} = \||\nabla u|\|_{L^p_{1+\gamma}(\Omega)}, \tag{2.10}$$
$$\|u\|_{L^q_{p;R}(\Omega)} = \|u\|_{L^q(\Omega\,;I_0/A_{1,R}^{1+q/p'})}.$$

3. $L^q_\gamma(\Omega) = \{u : \Omega \to \mathbf{R} \mid \|u\|_{L^q_\gamma(\Omega)} < \infty\}$, $L^q_{p;R}(\Omega) = \{u : \Omega \to \mathbf{R} \mid \|u\|_{L^q_{p;R}(\Omega)} < \infty\}$.

4. *By $W^{1,p}_{\gamma,0}(\Omega)$ we denote the completion of $C_c^\infty(\Omega \setminus \{0\})$ with respect to the norm*

$$u \mapsto \|\nabla u\|_{L^p_{1+\gamma}(\Omega)}.$$

5. *Let $\Omega$ be a radially symmetric domain. For any function space $V(\Omega)$ on $\Omega$, we set*

$$V(\Omega)_{\text{rad}} = \{ u \in V(\Omega) \mid u \text{ is radial} \}. \tag{2.11}$$

Here we remark the following fundamental properties concerning with the density of smooth functions. ( The proof is given in §8. )

**Proposition 2.1** *Assume that $1 < p < \infty$ and $\gamma \in \mathbf{R}$.*

1. *If $\gamma > 0$, then $C_c^\infty(\mathbf{R}^n) \subset W_{\gamma,0}^{1,p}(\mathbf{R}^n)$ and $C_c^\infty(\mathbf{R}^n)$ is densely contained in $W_{\gamma,0}^{1,p}(\mathbf{R}^n)$.*
2. *If $\gamma < 0$, then $C_c^\infty(\mathbf{R}^n) \not\subset W_{\gamma,0}^{1,p}(\mathbf{R}^n)$.*
3. *If $\gamma = 0$, then $C_c^\infty(B_1) \subset W_{0,0}^{1,p}(B_1)$ and $C_c^\infty(B_1)$ is densely contained in $W_{0,0}^{1,p}(B_1)$.*

Then the CKN-type inequalities are simply represented as follows:

If $\gamma \neq 0$, then

$$\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \geq S \|u\|^p_{L^q_\gamma(\mathbf{R}^n)} \quad \text{for } u \in W_{\gamma,0}^{1,p}(\mathbf{R}^n). \tag{2.12}$$

If $\gamma = 0$ , then for $R > 1$

$$\|\nabla u\|^p_{L^p_1(B_1)} \geq C \|u\|^p_{L^q_{p;R}(B_1)} \quad \text{for } u \in W_{0,0}^{1,p}(B_1). \tag{2.13}$$

**Remark 2.1** 1. *When $p = q$ holds, these two inequalities are called the Hardy-Sobolev inequalities. It is known that the best constants $S$ of (2.12) and $C$ of (2.13) coincide with the ones restricted in the radial functional spaces $W_{\gamma,0}^{1,p}(\mathbf{R}^n)_{\text{rad}}$ and $W_{0,0}^{1,p}(B_1)_{\text{rad}}$ respectively, and hence we have*

$$S = S^{p,p;\gamma} = \gamma^p, \; C = C^{p,p;R} = \frac{1}{(p')^p}. \tag{2.14}$$

2. *It follows from the Hardy-Sobolev inequalities that if $\gamma > 0$, then the space $W_{\gamma,0}^{1,p}(\mathbf{R}^n)$ coincides with the completion of $C_c^\infty(\mathbf{R}^n \setminus \{0\})$ with respect to the norm*

$$\|u\|_{W_\gamma^{1,p}(\mathbf{R}^n)} = \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|u\|_{L^p_\gamma(\mathbf{R}^n)} \tag{2.15}$$

*and if $\gamma = 0$, then the space $W_{0,0}^{1,p}(B_1)$ coincides with the completion of $C_c^\infty(B_1 \setminus \{0\})$ with respect to the norm*

$$\|u\|_{W_{0;R}^{1,p}(B_1)} = \|\nabla u\|_{L^p_1(B_1)} + \|u\|_{L^p_{p;R}(B_1)} \quad \textit{with } R > 1. \tag{2.16}$$

Here we note that if $\gamma = 0$, then the weight function in the right-hand side of the CKN-type inequality (2.13) is sharp in the following sense. ( The proof is given in §8. )

**Proposition 2.2** *Let $1 < p \leq q < \infty$, $\tau_{p,q} \leq 1/n$ and $R > 1$. Assume that $w \in C(\overline{B_1} \setminus \{0\})$ satisfies*

$$w(x) \geq 0 \quad \text{for } x \in \overline{B_1} \setminus \{0\}, \quad \frac{A_{1,R}(x)^{1+q/p'}}{I_0(x)} w(x) \to \infty \quad \text{as } x \to 0,$$

*then we have*

$$\inf \left\{ \left( \frac{\|\nabla u\|_{L^p_1(B_1)}}{\|u\|_{L^q(B_1;w)}} \right)^p \;\middle|\; u \in W_{0,0}^{1,p}(B_1) \setminus \{0\} \right\} = 0.$$

In the subsequent we study the validity of the CKN-type inequalities and the behavior of the best constants precisely when the parameters enjoy $1 < p \leq q < \infty$, $\tau_{p,q} \leq 1/n$, and in addition the cases that $\gamma < 0$ and $R = 1$ are considered. Moreover when $\gamma = 0$, we also establish the CKN-type inequality in the exterior domain $\mathbf{R}^n \setminus \overline{B_1}$ such that

$$\|\nabla u\|^p_{L^p_1(\mathbf{R}^n \setminus \overline{B_1})} \geq \overline{C} \|u\|^p_{L^q_{p;R}(\mathbf{R}^n \setminus \overline{B_1})} \quad \text{for } u \in W_{0,0}^{1,p}(\mathbf{R}^n \setminus \overline{B_1}). \tag{2.17}$$

## 2.2 Main results in the noncritical case

In this subsection we describe the results when $\gamma \neq 0$.

**Definition 2.3** *Let $1 \le p \le q < \infty$ and $\gamma \neq 0$.*

*1.*

$$E^{p,q;\gamma}[u] = \left( \frac{\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}}{\|u\|_{L^q_\gamma(\mathbf{R}^n)}} \right)^p \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\backslash\{0\}. \tag{2.18}$$

*2.*

$$\begin{aligned} S^{p,q;\gamma} &= \inf\{ E^{p,q;\gamma}[u] \mid u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\backslash\{0\}\} \\ &= \inf\{ E^{p,q;\gamma}[u] \mid u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n \backslash \{0\})\backslash\{0\}\}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} S^{p,q;\gamma}_{\mathrm{rad}} &= \inf\{ E^{p,q;\gamma}[u] \mid u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\backslash\{0\}\} \\ &= \inf\{ E^{p,q;\gamma}[u] \mid u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n \backslash \{0\})_{\mathrm{rad}}\backslash\{0\}\}. \end{aligned} \tag{2.20}$$

First of all we state the CKN-type inequalities in the noncritical case.

**Theorem 2.1** *Assume that $1 < p \le q < \infty$, $\tau_{p,q} \le 1/n$ and $\gamma \neq 0$. Then, we have $S^{p,q;\gamma}_{\mathrm{rad}} \ge S^{p,q;\gamma} > 0$ and the following inequalities.*

$$\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \ge S^{p,q;\gamma} \|u\|^p_{L^q_\gamma(\mathbf{R}^n)} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n), \tag{2.21}$$

$$\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \ge S^{p,q;\gamma}_{\mathrm{rad}} \|u\|^p_{L^q_\gamma(\mathbf{R}^n)} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}. \tag{2.22}$$

This follows from the assertions 1-4 of Theorem 2.2. Let us introduce more notations.

**Definition 2.4** *For $1 < p \le q < \infty$, we set*

$$\gamma_{p,q} = \frac{n-1}{1+q/p'}, \; S_{p,q} = \begin{cases} (p')^{p-2+p/q} q^{p/q} \left( \dfrac{\omega_n}{\tau_{p,q}} \mathrm{B}\left( \dfrac{1}{p\tau_{p,q}}, \dfrac{1}{p'\tau_{p,q}} \right) \right)^{1-p/q} & \text{if } p < q, \\ 1 & \text{if } p = q \end{cases} \tag{2.23}$$

*Here $\mathrm{B}(\cdot,\cdot)$ is a beta function.*

**Remark 2.2** *1. It holds that*

$$\mathrm{B}\left( \frac{1}{p\tau}, \frac{1}{p'\tau} \right)^\tau \to \frac{1}{p^{1/p}(p')^{1/p'}} \quad \text{as } \tau \to 0. \tag{2.24}$$

*In fact for $0 < \tau < \min\{1/p, 1/p'\}$, we see that*

$$t^{1/p-\tau}(1-t)^{1/p'-\tau} \le \frac{1}{(1-2\tau)^{1-2\tau}} \left( \frac{1}{p} - \tau \right)^{1/p-\tau} \left( \frac{1}{p'} - \tau \right)^{1/p'-\tau} \quad \text{for } 0 \le t \le 1, \tag{2.25}$$

*hence we have*

$$\begin{aligned} \mathrm{B}\left( \frac{1}{p\tau}, \frac{1}{p'\tau} \right)^\tau &= \left( \int_0^1 (t^{1/p-\tau}(1-t)^{1/p'-\tau})^{1/\tau} dt \right)^\tau \\ &\le \frac{1}{(1-2\tau)^{1-2\tau}} \left( \frac{1}{p} - \tau \right)^{1/p-\tau} \left( \frac{1}{p'} - \tau \right)^{1/p'-\tau} \to \frac{1}{p^{1/p}(p')^{1/p'}} \quad \text{as } \tau \to 0, \end{aligned}$$

$$\mathrm{B}\left(\frac{1}{p\tau},\frac{1}{p'\tau}\right)^{\tau} \ge \left(\int_0^1 (t^{1/p}(1-t)^{1/p'})^{1/\tau}dt\right)^{\tau}$$
$$\to \max_{0\le t\le 1} t^{1/p}(1-t)^{1/p'} = \frac{1}{p^{1/p}(p')^{1/p'}} \quad \text{as } \tau \to 0.$$

2. *Since* $\tau_{p,q} \to 0$ *as* $q \to p$, *it follows from the argument of 1 that we have*

$$S_{p,q} = \frac{(p')^{p-1-p\tau_{p,q}}}{(1/p-\tau_{p,q})^{1-p\tau_{p,q}}}\left(\frac{\omega_n}{\tau_{p,q}}\mathrm{B}\left(\frac{1}{p\tau_{p,q}},\frac{1}{p'\tau_{p,q}}\right)\right)^{p\tau_{p,q}} \to 1 = S_{p,p} \quad \text{as } q \to p. \tag{2.26}$$

Under these preparation we can compute the best constant $S^{p,q;\gamma}_{\mathrm{rad}}$ of the CKN-type inequality in the radial function space to obtain the exact representation. In the next we describe important relations among the best constants $S^{p,q;\gamma}_{\mathrm{rad}}$ and $S^{p,q;\gamma}$.

**Theorem 2.2** *Assume that* $1 < p \le q < \infty$ *and* $\tau_{p,q} \le 1/n$. *Then it holds that:*

1. $S^{p,q;\gamma} = S^{p,q;-\gamma}$, $S^{p,q;\gamma}_{\mathrm{rad}} = S^{p,q;-\gamma}_{\mathrm{rad}}$ *for* $\gamma \ne 0$.
2. $S^{p,q;\gamma}_{\mathrm{rad}} = S_{p,q}|\gamma|^{p(1-\tau_{p,q})}$ *for* $\gamma \ne 0$.
3. $S^{p,q;\gamma} = S^{p,q;\gamma}_{\mathrm{rad}} = S_{p,q}|\gamma|^{p(1-\tau_{p,q})}$ *for* $0 < |\gamma| \le \gamma_{p,q}$.
4. $\left|\frac{\gamma}{\overline{\gamma}}\right|^{p(1-\tau_{p,q})} S^{p,q;\overline{\gamma}} \le S^{p,q;\gamma} \le \left|\frac{\overline{\gamma}}{\gamma}\right|^{p\tau_{p,q}} S^{p,q;\overline{\gamma}}$ *for* $0 < |\gamma| \le |\overline{\gamma}|$.
5. $\frac{1}{(2-\gamma_{p,p^*}/\gamma)^p} S^{p,p^*;\gamma_{p,p^*}} \le S^{p,p^*;\gamma} \le S^{p,p^*;\gamma_{p,p^*}} = S^{p,p^*;\gamma_{p,p^*}}_{\mathrm{rad}}$ *for* $|\gamma| \ge \gamma_{p,p^*} = \frac{n-p}{p}$ *if* $p < n$.
6. $S^{2,2^*;\gamma} = S^{2,2^*;\gamma_{2,2^*}} = S^{2,2^*;\gamma_{2,2^*}}_{\mathrm{rad}}$ *for* $|\gamma| \ge \gamma_{2,2^*} = \frac{n-2}{2}$ *if* $p = 2 < n$.
7. $S^{p,q;\gamma} \ge (|\gamma|^{p\tau_{q,\overline{q}}}(S^{p,\overline{q};\gamma})^{\tau_{p,q}})^{1/\tau_{p,\overline{q}}}$ *for* $\gamma \ne 0$.
   *In particular,*
   $S^{p,q;\gamma} \ge |\gamma|^{p(1-n\tau_{p,q})}(S^{p,p^*;\gamma})^{n\tau_{p,q}}$ *for* $\gamma \ne 0$ *if* $p < n$.

**Remark 2.3** 1. *The assertions1-4 are proved in §§3 and 4, and the assertions 5-7 are establised in §6.*

2. *It follows from Remark (2.1) and Proposition 3.1 that we have*

$$S^{p,p;\gamma} = S^{p,p;\gamma}_{\mathrm{rad}} = |\gamma|^p \quad \text{for } \gamma \ne 0. \tag{2.27}$$

3. *For* $1 < p < n$, *the number;*

$$S^{p,p^*;\gamma_{p,p^*}} = S^{p,p^*;\gamma_{p,p^*}}_{\mathrm{rad}} = n\left(\frac{n-p}{p-1}\right)^{p-1}\left(\frac{\omega_n}{p'}\mathrm{B}\left(\frac{n}{p},\frac{n}{p'}\right)\right)^{p/n} \tag{2.28}$$

*coincides with the classical best constant of the Sobolev inequality;*

$$\|\nabla u\|^p_{L^p(\mathbf{R}^n)} = \|\nabla u\|^p_{L^p_{1+\gamma_{p,p^*}}(\mathbf{R}^n)} \ge S\|u\|^p_{L^{p^*}_{\gamma_{p,p^*}}(\mathbf{R}^n)} = S\|u\|^p_{L^{p^*}(\mathbf{R}^n)} \quad \text{for } u \in W^{1,p}_{\gamma_{p,p^*},0}(\mathbf{R}^n).$$

*In particular for* $n \ge 3$ *and* $p = 2$, *we see that*

$$S^{2,2^*;\gamma_{2,2^*}} = S^{2,2^*;\gamma_{2,2^*}}_{\mathrm{rad}} = n(n-2)\left(\frac{\omega_n}{2}\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)\right)^{2/n} = n(n-2)\left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{2/n}\pi. \tag{2.29}$$

*Here,* $\Gamma(\cdot)$ *is a gamma function.*

Moreover the best constant $S^{p,q;\gamma}$ is a continuous function of the parameters $q$ and $\gamma$. Namely we have the following that is established in §6.

**Theorem 2.3** *For $1 < p < \infty$, the maps*

$$([p, p^*]\setminus\{\infty\})\times(\mathbf{R}\setminus\{0\}) \ni (q\,;\gamma) \mapsto S^{p,q;\gamma},\ S^{p,q;\gamma}_{\mathrm{rad}} \in \mathbf{R} \tag{2.30}$$

*are continuous. In particular, it holds that*

$$S^{p,q;\gamma} \to S^{p,p;\gamma} = |\gamma|^p \quad \text{as } q \to p. \tag{2.31}$$

In the next we describe results on the existence and non-existence of extremal functions which attain the best constants of the CKN-type inequalities. Shortly speaking, the best constant $S^{p,q;\gamma}$ is attained by some element in $W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ provided that $p < q < p^*$ is satisfied. On the other hand if $q = p$, then the corresponding CKN-type inequalities are reduced to the Hardy-Sobolev inequalities and therefore no extremal function exists. When $q = p^*$ holds, then $S^{p,p^*;\gamma}$ is attained provided that $0 < |\gamma| \le (n-p)/p = \gamma_{p,p^*}$, but in the case that $|\gamma| > (n-p)/p$, it is unkown in general except for the case $p = 2$, whether $S^{p,p^*;\gamma}$ is achieved by some element or not. If $p = 2$ is assumed, then it is shown that no extremal exists provided that $|\gamma| > (n-2)/2$ holds.

**Theorem 2.4** *Assum that $1 < p \le q < \infty$, $\tau_{p,q} \le 1/n$ and $\gamma \ne 0$. Then we have the followings:*

1. *If $p < q$, then $S^{p,q;\gamma}_{\mathrm{rad}}$ is achieved in $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\setminus\{0\}$.*
2. *If $p < q < p^*$, then $S^{p,q;\gamma}$ is achieved in $W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$.*
3. *If $p < n$, $q = p^*$ and $|\gamma| \le (n-p)/p = \gamma_{p,p^*}$, then $S^{p,p^*;\gamma} = S^{p,p^*;\gamma}_{\mathrm{rad}}$ is achieved in $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\setminus\{0\}$.*
4. *If $p = 2 < n$, $q = 2^* = 2n/(n-2)$ and $|\gamma| > (n-2)/2 = \gamma_{2,2^*}$, then $S^{2,2^*;\gamma} = S^{2,2^*;\gamma_{2,2^*}}_{\mathrm{rad}}$ holds and $S^{2,2^*;\gamma}$ is not achieved in $W^{1,2}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$.*

**Remark 2.4** *The assertions 1 and 3 is proved in §4. On the other hand the assertions 2 and 4 are established in §7 and §8 respectively.*

**Proposition 2.3** *If $1 < p = q < \infty$, $\gamma \ne 0$, then $S^{p,p;\gamma}$ and $S^{p,p;\gamma}_{\mathrm{rad}}$ are not achieved in $W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ and $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\setminus\{0\}$ respectively.*

This is proved in §8.

## 2.3 Main results in the critical case

In this subsection we state the results in the case of $\gamma = 0$. Let us begin with defining various functionals and best constants.

**Definition 2.5** *Let $1 \le p \le q < \infty$ and $R \ge 1$.*

1.
$$F^{p,q;R}[u] = \left( \frac{\|\nabla u\|_{L^p_1(B_1)}}{\|u\|_{L^q_{p;R}(B_1)}} \right)^p \qquad \text{for } u \in W^{1,p}_{0,0}(B_1)\setminus\{0\}. \tag{2.32}$$

*2.*

$$C^{p,q;R} = \inf\{ F^{p,q;R}[u] \mid u \in W^{1,p}_{0,0}(B_1)\backslash\{0\}\} \tag{2.33}$$
$$= \inf\{ F^{p,q;R}[u] \mid u \in C^\infty_{\mathrm{c}}(B_1\backslash\{0\})\backslash\{0\}\},$$

$$C^{p,q;R}_{\mathrm{rad}} = \inf\{ F^{p,q;R}[u] \mid u \in W^{1,p}_{0,0}(B_1)_{\mathrm{rad}}\backslash\{0\}\} \tag{2.34}$$
$$= \inf\{ F^{p,q;R}[u] \mid u \in C^\infty_{\mathrm{c}}(B_1\backslash\{0\})_{\mathrm{rad}}\backslash\{0\}\}.$$

*3.*

$$\overline{F}^{p,q;R}[u] = \left( \frac{\|\nabla u\|_{L^p_1(\mathbf{R}^n\setminus\overline{B_1})}}{\|u\|_{L^q_{p;R}(\mathbf{R}^n\setminus\overline{B_1})}} \right)^p \quad \textit{for } u \in W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})\backslash\{0\}. \tag{2.35}$$

*4.*

$$\overline{C}^{p,q;R} = \inf\{ \overline{F}^{p,q;R}[u] \mid u \in W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})\backslash\{0\}\} \tag{2.36}$$
$$= \inf\{ \overline{F}^{p,q;R}[u] \mid u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\overline{B_1})\backslash\{0\}\},$$

$$\overline{C}^{p,q;R}_{\mathrm{rad}} = \inf\{ \overline{F}^{p,q;R}[u] \mid u \in W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})_{\mathrm{rad}}\backslash\{0\}\} \tag{2.37}$$
$$= \inf\{ \overline{F}^{p,q;R}[u] \mid u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\overline{B_1})_{\mathrm{rad}}\backslash\{0\}\}.$$

When $R > 1$, we have the next.

**Theorem 2.5** *Assume that* $1 < p \le q < \infty$, $\tau_{p,q} \le 1/n$ *and* $R > 1$. *Then, we have* $C^{p,q;R}_{\mathrm{rad}} \ge C^{p,q;R} > 0$, $\overline{C}^{p,q;R}_{\mathrm{rad}} \ge \overline{C}^{p,q;R} > 0$ *and the following inequalities:*

$$\|\nabla u\|^p_{L^p_1(B_1)} \ge C^{p,q;R}\|u\|^p_{L^q_{p;R}(B_1)} \quad \textit{for } u \in W^{1,p}_{0,0}(B_1), \tag{2.38}$$
$$\|\nabla u\|^p_{L^p_1(B_1)} \ge C^{p,q;R}_{\mathrm{rad}}\|u\|^p_{L^q_{p;R}(B_1)} \quad \textit{for } u \in W^{1,p}_{0,0}(B_1)_{\mathrm{rad}}, \tag{2.39}$$
$$\|\nabla u\|^p_{L^p_1(\mathbf{R}^n\setminus\overline{B_1})} \ge \overline{C}^{p,q;R}\|u\|^p_{L^q_{p;R}(\mathbf{R}^n\setminus\overline{B_1})} \quad \textit{for } u \in W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1}), \tag{2.40}$$
$$\|\nabla u\|^p_{L^p_1(\mathbf{R}^n\setminus\overline{B_1})} \ge \overline{C}^{p,q;R}_{\mathrm{rad}}\|u\|^p_{L^q_{p;R}(\mathbf{R}^n\setminus\overline{B_1})} \quad \textit{for } u \in W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})_{\mathrm{rad}}. \tag{2.41}$$

**Remark 2.5** *If* $p \ge n$, *these imbedding inequalities follow from the assertions 3 and 4 of Theorem 2.7. On the other hand if* $1 < p < n$, *then these are established in §5 by using the so-called nonlinear potential theory.*

When $R = 1$ holds, we have the next result which is established in §4 and partly in §8.

**Theorem 2.6** *Assume that* $1 < p \le q < \infty$, $\tau_{p,q} \le 1/n$ *and* $R = 1$. *Then we have the followings:*

1. *If* $n = 1$, *then* $C^{p,q;1}_{\mathrm{rad}} \ge C^{p,q;1} > 0$ *and* $\overline{C}^{p,q;1}_{\mathrm{rad}} \ge \overline{C}^{p,q;1} > 0$ *hold. Further the inequalities in Theorem 2.5 are valid with* $R = 1$.
2. *If* $n \ge 2$, *then* $C^{p,q;1}_{\mathrm{rad}} > 0$ *and* $\overline{C}^{p,q;1}_{\mathrm{rad}} > 0$ *hold. Further the inequalities in Theorem 2.5 are valid with* $R = 1$, *and* $C^{p,q;1} = \overline{C}^{p,q;1} = 0$ *holds.*

Now we introduce more notations.

**Definition 2.6** *For $1 < p \le q < \infty$ we set*

$$R_{p,q} = \exp \frac{1 + q/p'}{(n-1)p'} \quad \text{if } n \ge 2, \qquad C_{p,q} = \frac{S_{p,q}}{(p')^{p(1-\tau_{p,q})}}. \tag{2.42}$$

By virtue of these we can represent in a concrete way $C_{\text{rad}}^{p,q;R}$ and $\overline{C}_{\text{rad}}^{p,q;R}$ which are the best constants in the radial function spaces.

**Theorem 2.7** *Assume that $1 < p \le q < \infty$ and $\tau_{p,q} \le 1/n$. Then we have the followings:*

1. $C^{p,q;R} = \overline{C}^{p,q;R}$, $C_{\text{rad}}^{p,q;R} = \overline{C}_{\text{rad}}^{p,q;R}$ *for* $R \ge 1$.
2. $C_{\text{rad}}^{p,q;R} = \overline{C}_{\text{rad}}^{p,q;R} = C_{p,q}$ *for* $R \ge 1$.
3. $C^{p,q;R} = C_{\text{rad}}^{p,q;R} = \overline{C}^{p,q;R} = \overline{C}_{\text{rad}}^{p,q;R} = C_{p,q}$ *for* $R \ge \begin{cases} 1 & \text{if } n = 1, \\ R_{p,q} & \text{if } p \ge n \ge 2. \end{cases}$
4. $C^{p,q;R} = \overline{C}^{p,q;R} \le C^{p,q;\overline{R}} = \overline{C}^{p,q;\overline{R}} \le \left(\dfrac{\log \overline{R}}{\log R}\right)^p C^{p,q;R} = \left(\dfrac{\log \overline{R}}{\log R}\right)^p \overline{C}^{p,q;R}$
   *for* $1 < R \le \overline{R}$.

**Remark 2.6** 1. *The assertions 1 and 4 are established in §3 and the rests are done in §4.*

2. $C_{p,q} \to \dfrac{1}{(p')^p} = C_{p,p}$ *as* $q \to p$. *Namely we have the next which is established in §6.*
3. *From Remark 2.1 and Proposition 3.1 we obtain*

$$C^{p,p;R} = C_{\text{rad}}^{p,p;R} = \overline{C}^{p,p;R} = \overline{C}_{\text{rad}}^{p,p;R} = \frac{1}{(p')^p} = C_{p,p} \quad \text{for } R > 1. \tag{2.43}$$

Further the best constant $C^{p,q;R}$ is a continuous function of the parameters $q, R$.

**Theorem 2.8** 1. *For $1 < p < \infty$, the maps*

$$([p, p^*] \setminus \{\infty\}) \times (1, \infty) \ni (q; R) \mapsto C^{p,q;R} = \overline{C}^{p,q;R}, \; C_{\text{rad}}^{p,q;R} = \overline{C}_{\text{rad}}^{p,q;R} \in \mathbf{R} \tag{2.44}$$

*are continuous.*

2. *For $n = 1$ and $1 < p < \infty$, the maps*

$$[p, \infty) \times [1, \infty) \ni (q; R) \mapsto C^{p,q;R} = C_{\text{rad}}^{p,q;R} = \overline{C}^{p,q;R} = \overline{C}_{\text{rad}}^{p,q;R} \in \mathbf{R} \tag{2.45}$$

*are continuous.*

On the existence of extremal functions we have the next theorem which is proved in §4. When $n \ge 2$, $p < q$ and $R > 1$ hold, we do not know so far if $C^{p,q;R}$ and $\overline{C}^{p,q;R}$ are achieved by any extremals or not.

**Theorem 2.9** *Assume that $1 < p < q < \infty$, $\tau_{p,q} \le 1/n$ and $R \ge 1$. Then we have the followings.*

1. *For $R = 1$, $C_{\text{rad}}^{p,q;1}$ and $\overline{C}_{\text{rad}}^{p,q;1}$ are achieved in $W_{0,0}^{1,p}(B_1)_{\text{rad}} \setminus \{0\}$ and $W_{0,0}^{1,p}(\mathbf{R}^n \setminus \overline{B_1})_{\text{rad}} \setminus \{0\}$ respectively.*

2. *For* $n=1$, $R=1$, $C^{p,q;1}=C^{p,q;1}_{\rm rad}$ *and* $\overline{C}^{p,q;1}=\overline{C}^{p,q;1}_{\rm rad}$ *are achieved in* $W^{1,p}_{0,0}((-1,1))_{\rm rad}\backslash\{0\}$ *and* $W^{1,p}_{0,0}(\mathbf{R}\backslash[-1,1])_{\rm rad}\backslash\{0\}$ *respectively.*

3. *For* $R>1$, $C^{p,q;R}_{\rm rad}$ *and* $\overline{C}^{p,q;R}_{\rm rad}$ *are not achieved in* $W^{1,p}_{0,0}(B_1)_{\rm rad}\backslash\{0\}$ *and* $W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})_{\rm rad}\backslash\{0\}$ *respectively.*

We also have the next which will be proved in §8.2 together with the assertion 4 of Theorem 2.4.

**Proposition 2.4** *Let* $1<p=q<\infty$ *and* $\tau_{p,q}\le 1/n$. *If* $R>1$ *is sufficiently large, then* $C^{p,p;R}$, $C^{p,p;R}_{\rm rad}$, $\overline{C}^{p,p;R}$ *and* $\overline{C}^{p,p;R}_{\rm rad}$ *are not achieved in* $W^{1,p}_{0,0}(B_1)\backslash\{0\}$, $W^{1,p}_{0,0}(B_1)_{\rm rad}\backslash\{0\}$, $W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})\backslash\{0\}$ *and* $W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})_{\rm rad}\backslash\{0\}$ *respectively.*

# 3 Change of variables and the best constants

Here we see the relations among the best constants by the method of change of variables.

**Definition 3.1** *For* $\beta>0$ *and* $R\ge 1$, *we set the followings:*

1. $\overline{Y}(y)=\dfrac{y}{|y|^2}$ *for* $y\in\mathbf{R}^n\setminus\{0\}$.
2. $Y_\beta(y)=|y|^{\beta-1}y$ *for* $y\in\mathbf{R}^n$.
3. $\tilde{Y}_R(y)=R\exp\left(-\dfrac{1}{|y|}\right)\dfrac{y}{|y|}$ *for* $y\in\mathbf{R}^n$.

**Remark 3.1** *For* $\beta>0$ *and* $R\ge 1$, *we have the inverse maps as follows:*

1. $\overline{Y}^{-1}(x)=\overline{Y}(x)=\dfrac{x}{|x|^2}$ *for* $x\in\mathbf{R}^n\setminus\{0\}$.
2. $Y_\beta^{-1}(x)=Y_{1/\beta}(x)=|x|^{1/\beta-1}x$ *for* $x\in\mathbf{R}^n$.
3. $\tilde{Y}_R^{-1}(x)=\dfrac{1}{\log(R/|x|)}\dfrac{x}{|x|}$ *for* $x\in B_R$.

In the next we define various operators which are fundamental in the present paper.

**Definition 3.2** *Let* $\beta>0$ *and* $R\ge 1$. *Let* $\Omega$ *be a domain of* $\mathbf{R}^n$ *and* $u:\Omega\to\mathbf{R}$.

1. $\overline{T}u(y)=u(\overline{Y}(y))=u\left(\dfrac{y}{|y|^2}\right)$ *for* $y\in\overline{Y}^{-1}(\Omega\backslash\{0\})$.
2. $T_\beta u(y)=u(Y_\beta(y))=u(|y|^{\beta-1}y)$ *for* $y\in Y_{1/\beta}^{-1}(\Omega)$.
3. *For* $\Omega\subset B_R$,

   $\tilde{T}_Ru(y)=u(\tilde{Y}_R(y))=u\left(R\exp\left(-\dfrac{1}{|y|}\right)\dfrac{y}{|y|}\right)$ *for* $y\in\tilde{Y}_R^{-1}(\Omega)$.

We begin with studying the operator $\overline{T}$. By a direct calculation we have

$$\det(\delta_{ij}+ax_ix_j)_{1\le i,j\le n}=1+a|x|^2\quad\text{for } x\in\mathbf{R}^n,\ a\in\mathbf{R}.\tag{3.1}$$

Since the Jacobi determinant of the change of variables defined by $x=\overline{Y}(y)=y/|y|^2$ is

$$\det D\overline{Y}(y)=\det\left(\frac{1}{|y|^2}\left(\delta_{ij}-2\frac{y_iy_j}{|y|^2}\right)\right)_{1\le i,j\le n}=-\frac{1}{|y|^{2n}},\tag{3.2}$$

we have the next.

**Lemma 3.1** *Assume that* $1 \le p \le q < \infty$, $\gamma \neq 0$ *and* $R \ge 1$. *Then we have the followings:*

1. $\|u\|_{L^q_\gamma(\mathbf{R}^n)} = \|\overline{T}u\|_{L^q_{-\gamma}(\mathbf{R}^n)}$ *for* $u \in L^q_\gamma(\mathbf{R}^n)$,
   $\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = \|\nabla[\overline{T}u]\|_{L^p_{1-\gamma}(\mathbf{R}^n)}$ *for* $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)$.
2. $\|u\|_{L^q_{p;R}(B_1)} = \|\overline{T}u\|_{L^q_{p;R}(\mathbf{R}^n\setminus\overline{B_1})}$ *for* $u \in L^q_{p;R}(B_1)$,
   $\|\nabla u\|_{L^p_1(B_1)} = \|\nabla[\overline{T}u]\|_{L^p_1(\mathbf{R}^n\setminus\overline{B_1})}$ *for* $u \in W^{1,p}_{0,0}(B_1)$.

For the proof of this, it suffice to note that for $x = \frac{y}{|y|^2}$ we have

$$\left|(\nabla_x u)\left(\frac{y}{|y|^2}\right)\right|^2 = |y|^4\left|\nabla_y(\overline{T}u(y))\right|^2, \qquad \text{for } y \in \mathbf{R}\setminus\{0\}.$$

As a direct consequence of this we have the next proposition which proves the assertion 1 of Theorem 2.2 and the assertion 1 of Theorem 2.7. Further we see that in the proofs of Theorems 2.1–2.4, it suffices to assume that $\gamma > 0$, and it suffices to establish the proofs of Theorems 2.5–2.9 in a unit ball $B_1$ instead of a general domain.

**Proposition 3.1** *Assume that* $1 \le p \le q < \infty$, $\gamma \neq 0$ *and* $R \ge 1$. *Then we have the followings:*

1. $S^{p,q;\gamma} = S^{p,q;-\gamma}$, $S^{p,q;\gamma}_{\mathrm{rad}} = S^{p,q;-\gamma}_{\mathrm{rad}}$.
2. $C^{p,q;R} = \overline{C}^{p,q;R}$, $C^{p,q;R}_{\mathrm{rad}} = \overline{C}^{p,q;R}_{\mathrm{rad}}$.

**Proof:** From Lemma 3.1 we see that

$$E^{p,q;\gamma}[u] = E^{p,q;-\gamma}[\overline{T}u] \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}, \tag{3.3}$$
$$F^{p,q;R}[u] = \overline{F}^{p,q;R}[\overline{T}u] \quad \text{for } u \in W^{1,p}_{0,0}(B_1)\setminus\{0\}, \tag{3.4}$$

hence the assertions follow. □

In the next we consider the operators $T_\beta$, $\tilde{T}_R$. By $\Delta_{S^{n-1}}$ we denote the Laplace-Beltrami operator on a unit sphere $S^{n-1}$. Then a gradient operator $\Lambda$ on $S^{n-1}$ is defined by

$$\int_{S^{n-1}}(-\Delta_{S^{n-1}}u)v\,dS = \int_{S^{n-1}}\Lambda u\cdot\Lambda v\,dS \quad \text{for } u,v \in C^2(S^{n-1}). \tag{3.5}$$

Here we note that

$$\Delta u = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\Delta_{S^{n-1}}u,\ |\nabla u|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}|\Lambda u|^2, \tag{3.6}$$

where

$$r(x) = |x|,\ \frac{\partial u}{\partial r}(x) = \frac{x}{|x|}\cdot\nabla u(x). \tag{3.7}$$

The Jacobi determinant of the change of variables $x = Y_\beta(y) = |y|^{\beta-1}y$ is given by

$$\det DY_\beta(y) = \det\left(|y|^{\beta-1}\left(\delta_{ij} + (\beta-1)\frac{y_iy_j}{|y|^2}\right)\right)_{1\le i,j\le n} = \beta|y|^{n(\beta-1)}. \tag{3.8}$$

Hence by calculations we have the next lemma.

**Lemma 3.2** *Assume that* $1 \le p \le q < \infty$, $\gamma > 0$, $R \ge 1$ *and* $\beta > 0$. *Then we have the followings:*

1. $\|u\|_{L^q_\gamma(\mathbf{R}^n)} = \beta^{1/q}\|T_\beta u\|_{L^q_{\beta\gamma}(\mathbf{R}^n)}$ *for* $u \in L^q_\gamma(\mathbf{R}^n)$,

$$\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = \frac{1}{\beta^{1/p'}}\Big\|\Big(\Big|\frac{\partial}{\partial r}[T_\beta u]\Big|^2 + \frac{\beta^2}{r^2}|\Lambda[T_\beta u]|^2\Big)^{1/2}\Big\|_{L^p_{1+\beta\gamma}(\mathbf{R}^n)} \quad \textit{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n).$$

2. $\|u\|_{L^q_{p;R}(B_1)} = \dfrac{1}{\beta^{1/p'}}\|T_\beta u\|_{L^q_{p;R^{1/\beta}}(B_1)}$ *for* $u \in L^q_{p;R}(B_1)$,

$$\|\nabla u\|_{L^p_1(B_1)} = \frac{1}{\beta^{1/p'}}\Big\|\Big(\Big|\frac{\partial}{\partial r}[T_\beta u]\Big|^2 + \frac{\beta^2}{r^2}|\Lambda[T_\beta u]|^2\Big)^{1/2}\Big\|_{L^p_1(B_1)} \quad \textit{for } u \in W^{1,p}_{0,0}(B_1).$$

As a consequence we have the next proposition which assures the assertion 4 of Theorem 2.2 and the assertion 4 of Theorem 2.7 as well.

**Proposition 3.2** *Assume that* $1 \le p \le q < \infty$*. Then we have the followings:*

1. $\Big(\dfrac{\gamma}{\overline{\gamma}}\Big)^{p(1-\tau_{p,q})} S^{p,q;\overline{\gamma}} \le S^{p,q;\gamma} \le \Big(\dfrac{\overline{\gamma}}{\gamma}\Big)^{p\tau_{p,q}} S^{p,q;\overline{\gamma}},\ S^{p,q;\gamma}_{\mathrm{rad}} = \Big(\dfrac{\gamma}{\overline{\gamma}}\Big)^{p(1-\tau_{p,q})} S^{p,q;\overline{\gamma}}_{\mathrm{rad}}$ *for* $0 < \gamma \le \overline{\gamma}$.

   *In particular, there is a constant* $\hat{S}_{p,q} \ge 0$ *such that we have*

$$S^{p,q;\gamma}_{\mathrm{rad}} = \hat{S}_{p,q}\gamma^{p(1-\tau_{p,q})} \quad \textit{for } \gamma > 0.$$

2. $C^{p,q;R} \le C^{p,q;\overline{R}} \le \Big(\dfrac{\log \overline{R}}{\log R}\Big)^p C^{p,q;R},\ C^{p,q;R}_{\mathrm{rad}} = C^{p,q;\overline{R}}_{\mathrm{rad}}$ *for* $1 < R \le \overline{R}$.

   *In particular, there is a constant* $\hat{C}_{p,q} \ge 0$ *such that we have*

$$C^{p,q;R}_{\mathrm{rad}} = \hat{C}_{p,q} \quad \textit{for } R > 1.$$

**Proof:** Let us note that by Remark 3.1, $u = T_{\frac{1}{\beta}} v$ holds for $v = T_\beta u$. Then it follows from the assertion 1 of Lemma 3.2 with $\beta = \dfrac{\gamma}{\overline{\gamma}}$ that we have

$$\Big(\frac{\gamma}{\overline{\gamma}}\Big)^{p(1-\tau_{p,q})} E^{p,q;\overline{\gamma}}[T_{\overline{\gamma}/\gamma}u] \le E^{p,q;\gamma}[u] \le \Big(\frac{\overline{\gamma}}{\gamma}\Big)^{p\tau_{p,q}} E^{p,q;\overline{\gamma}}[T_{\overline{\gamma}/\gamma}u] \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\backslash\{0\},$$

$$E^{p,q;\gamma}[u] = \Big(\frac{\gamma}{\overline{\gamma}}\Big)^{p(1-\tau_{p,q})} E^{p,q;\overline{\gamma}}[T_{\overline{\gamma}/\gamma}u] \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\backslash\{0\}.$$

From the assertion 2 with $\beta = \dfrac{\log \overline{R}}{\log R}$, we have

$$F^{p,q;R}[T_{\log \overline{R}/\log R}u] \le F^{p,q;\overline{R}}[u] \le \Big(\frac{\log \overline{R}}{\log R}\Big)^p F^{p,q;R}[T_{\log \overline{R}/\log R}u] \quad \text{for } u \in W^{1,p}_{0,0}(B_1)\backslash\{0\},$$

$$F^{p,q;\overline{R}}[u] = F^{p,q;R}[T_{\log \overline{R}/\log R}u] \quad \text{for } u \in W^{1,p}_{0,0}(B_1)_{\mathrm{rad}}\backslash\{0\}.$$

Thus the desired assertions follow. □

Further from Proposition 3.2 we have the next one.

**Proposition 3.3** *Assume that* $1 \le p \le q < \infty$*,* $\overline{\gamma} > 0$ *and* $\underline{R} > 1$*. Then we have the followings:*

1. *If* $S^{p,q;\overline{\gamma}} = S^{p,q;\overline{\gamma}}_{\mathrm{rad}}$ *holds, then*

$$S^{p,q;\gamma} = S^{p,q;\gamma}_{\mathrm{rad}} = \hat{S}_{p,q}\gamma^{p(1-\tau_{p,q})} \quad \textit{for } 0 < \gamma \le \overline{\gamma}.$$

*2. If* $C^{p,q;\underline{R}} = C^{p,q;\underline{R}}_{\rm rad}$ *holds, then*

$$C^{p,q;R} = C^{p,q;R}_{\rm rad} = \hat{C}_{p,q} \qquad \textit{for } R \geq \underline{R}.$$

Lastly we have the next lemma, noting that the Jacobi determinant of the change of variables $x = \tilde{Y}_R(y) = R\exp(-1/|y|)y/|y|$ is given by

$$\begin{aligned} \det D\tilde{Y}_R(y) &= \det \Big( \frac{R}{|y|} \exp\Big(-\frac{1}{|y|}\Big)\Big(\delta_{ij} + \Big(\frac{1}{|y|} - 1\Big)\frac{y_i y_j}{|y|^2}\Big)\Big)_{1\leq i,j\leq n} \\ &= R^n \exp\Big(-\frac{n}{|y|}\Big)\frac{1}{|y|^{n+1}}. \end{aligned} \tag{3.9}$$

**Lemma 3.3** *Assume that* $1 \leq p \leq q < \infty$, $R \geq 1$. *Then we have the followings:*

$$\|u\|_{L^q_{p;R}(B_1)} = \|\tilde{T}_R u\|_{L^q_{1/p'}(B_{1/\log R})} \qquad \textit{for } u \in L^q_{p;R}(B_1),$$

$$\|\nabla u\|_{L^p_1(B_1)} = \Big\| \Big( \Big| \frac{\partial}{\partial r}[\tilde{T}_R u]\Big|^2 + \frac{1}{r^4}|\Lambda[\tilde{T}_R u]|^2\Big)^{1/2}\Big\|_{L^p_{1+1/p'}(B_{1/\log R})} \qquad \textit{for } u \in W^{1,p}_{0,0}(B_1).$$

Conbining this with the assertion 2 of Proposition 3.2 we have the next.

**Proposition 3.4** *For* $1 \leq p \leq q < \infty$ *we have*

$$C^{p,q;R}_{\rm rad} = S^{p,q;1/p'}_{\rm rad} = \frac{\hat{S}_{p,q}}{(p')^{p(1-\tau_{p,q})}} \quad \textit{for } R \geq 1. \tag{3.10}$$

**Proof:** It follows from Lemma 3.3 that we have

$$F^{p,q;R}[u] = E^{p,q;1/p'}[\tilde{T}_R u] \quad \text{for } u \in W^{1,p}_{0,0}(B_1)_{\rm rad}\backslash\{0\}. \tag{3.11}$$

Here we note that the operator $\tilde{T}_R u$ is an extension of $T_R u$ to the whole $\mathbf{R}^n$ by setting $\tilde{T}_R u = 0$ on $\mathbf{R}^n \backslash B_{1/\log R}$. Then we immediately have $C^{p,q;1}_{\rm rad} = S^{p,q;1/p'}_{\rm rad}$. Form the assertion 2 of Proposition 3.2 we also have

$$\begin{aligned} \hat{C}_{p,q} &= \inf_{R>1} C^{p,q;R}_{\rm rad} = \inf_{R>1} \inf\{ F^{p,q;R}[u] \mid u \in C^\infty_{\rm c}(B_1\backslash\{0\})_{\rm rad}\backslash\{0\}\} \\ &= \inf_{R>1} \inf\{ E^{p,q;1/p'}[\tilde{T}_R u] \mid u \in C^\infty_{\rm c}(B_1\backslash\{0\})_{\rm rad}\backslash\{0\}\} \\ &= \inf\{ E^{p,q;1/p'}[v] \mid v \in C^\infty_{\rm c}(\mathbf{R}^n\backslash\{0\})_{\rm rad}\backslash\{0\}\} = S^{p,q;1/p'}_{\rm rad}. \end{aligned}$$

The assertion follows from this together with the assertions 1 and 2 of Proposition 3.2. □

# 4 Relations among $S^{p,q;\gamma}_{\rm rad}$, $C^{p,q;R}_{\rm rad}$ , $S^{p,q;\gamma}$ and $C^{p,q;R}$

In this section we exactly determine the best constant $S^{p,q;\gamma}_{\rm rad}$, $C^{p,q;R}_{\rm rad}$ in the radial function spaces, and we study when $S^{p,q;\gamma}_{\rm rad}$ and $C^{p,q;R}_{\rm rad}$ should coincide with $S^{p,q;\gamma}$ and $C^{p,q;R}$ respectively.

## 4.1 Variational problems in radially symmetric spaces

In this subsection we determine the best constants $S_{\rm rad}^{p,q;\gamma}$ and $C_{\rm rad}^{p,q;R}$ for $p < q$ by solving corresponding variational problems in radially symmetric spaces employing Talenti's result in an essential way. We begin with introducing variational problems and solutions.

**Definition 4.1** *Let* $1 < p < q < \infty$ *and* $a, b > 0$.

1. $C^1_{p,q}((0,\infty)) = \Big\{ u \in C^1((0,\infty)) \;\Big|\; \int_0^\infty |u'(r)|^p r^{1/\tau_{p,q}-1} dr < \infty,\ u(r) \to 0 \quad \text{as } r \to \infty \Big\}.$

2. $J^{p,q}[u] = \dfrac{\left(\int_0^\infty |u'(r)|^p r^{1/\tau_{p,q}-1} dr\right)^{1/p}}{\left(\int_0^\infty |u(r)|^q r^{1/\tau_{p,q}-1} dr\right)^{1/q}}$ *for* $u \in C^1_{p,q}((0,\infty)) \backslash \{0\}$.

3. $\varphi_0(x) = \varphi_0(|x|) = \dfrac{1}{(a + b|x|^{p'})^{p/(q-p)}}$ *for* $x \in \mathbf{R}^n \backslash \{0\}$.

   *(In the subsequent* $\varphi_0$ *is also regarded as a function of* $r = |x|$ *on* $(0, \infty)$. *)*

The next lemma is essentially due to G. Talenti.

**Lemma 4.1** *For* $1 < p < q < \infty$, *we have*

$$J^{p,q}[u] \geq J^{p,q}[\varphi_0] \quad \text{for } u \in C^1_{p,q}((0,\infty)) \backslash \{0\}. \tag{4.1}$$

Noting that

$$\int_0^\infty \frac{t^{\alpha-1}}{(1+t)^\beta} dt = \mathrm{B}(\alpha, \beta - \alpha) \quad \text{for } 0 < \alpha < \beta, \tag{4.2}$$

we have

$$\int_0^\infty |\varphi_0(r)|^q r^{1/\tau_{p,q}-1} dr = \frac{1}{(a^{1/p} b^{1/p'})^{1/\tau_{p,q}}} \frac{1}{p'} \mathrm{B}\left(\frac{1}{p\tau_{p,q}}, \frac{1}{p'\tau_{p,q}}\right), \tag{4.3}$$

$$\int_0^\infty |\varphi_0'(r)|^p r^{1/\tau_{p,q}-1} dr = \frac{1}{(a^{1/p} b^{1/p'})^{p/(q\tau_{p,q})}} \frac{(p')^{p-1}}{(q\tau_{p,q})^p} \mathrm{B}\left(\frac{1}{p\tau_{p,q}} - 1, \frac{1}{p'\tau_{p,q}} + 1\right) \tag{4.4}$$

$$= \frac{1}{(a^{1/p} b^{1/p'})^{p/(q\tau_{p,q})}} \frac{(p')^{p-2}}{q^{p-1}\tau_{p,q}^p} \mathrm{B}\left(\frac{1}{p\tau_{p,q}}, \frac{1}{p'\tau_{p,q}}\right).$$

Hence we have

$$J^{p,q}[\varphi_0] = \frac{(p')^{1/p' - \tau_{p,q}}}{q^{1/p'}\tau_{p,q}} \mathrm{B}\left(\frac{1}{p\tau_{p,q}}, \frac{1}{p'\tau_{p,q}}\right)^{\tau_{p,q}}. \tag{4.5}$$

First of all, for $\gamma > 0$, we have the next proposition and then the assertion 1 of Theorem 2.4 follows. Moreover combining it with Proposition 3.2, the assertion 2 of Theorem 2.2 follows.

**Proposition 4.1** *Assume that* $1 < p < q < \infty$ *and* $\gamma > 0$. *Then we have the followings:*

1. *The infimum of* $S_{\rm rad}^{p,q;\gamma}$ *in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\rm rad} \backslash \{0\}$ *is attained by* $u^{p,q;\gamma} = T_{q\tau_{p,q}\gamma}\varphi_0$.

2. *In the assertion 1 of Proposoition 3.2,*

$$\hat{S}_{p,q} = (\omega_n^{\tau_{p,q}} (q\tau_{p,q})^{1-\tau_{p,q}} J^{p,q}[\varphi_0])^p = S_{p,q}.$$

**Proof:** 1. It follows from Lemma 3.2 that we have for $u \in C_c^\infty(\mathbf{R}^n \setminus \{0\})_{\rm rad}$,

$$\|u\|_{L^q_\gamma(\mathbf{R}^n)} = \frac{1}{(q\tau_{p,q}\gamma)^{1/q}} \|T_{1/(q\tau_{p,q}\gamma)} u\|_{L^q_{1/(q\tau_{p,q})}(\mathbf{R}^n)}, \tag{4.6}$$

$$\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = (q\tau_{p,q}\gamma)^{1/p'} \|\nabla[T_{1/(q\tau_{p,q}\gamma)} u]\|_{L^p_{1+1/(q\tau_{p,q})}(\mathbf{R}^n)}. \tag{4.7}$$

Then we have

$$E^{p,q;\gamma}[u] = (\omega_n^{\tau_{p,q}}(q\tau_{p,q})^{1-\tau_{p,q}} J^{p,q}[T_{1/(q\tau_{p,q}\gamma)} u])^p \gamma^{p(1-\tau_{p,q})} \quad \text{for } u \in C_c^\infty(\mathbf{R}^n \setminus \{0\})_{\rm rad} \setminus \{0\},$$

hence the assertion follows from Lemma 4.1.

2. This is clear from the previous result 1, (2.23) and the assertion 1 of Proposition 3.2. □

Let us proceed to the case $\gamma = 0$. In this case we have the next proposition, from which Theorem 2.6 and the assertions 1 and 3 of Theorem 2.9 follow. Moreover combining it with the assertion 2 of Proposition 3.2, the assertion 2 of Theorem 2.7 follows.

**Proposition 4.2** *Assume that $1 < p < q < \infty$, $\gamma = 0$ and $R \geq 1$. Then we have the followings:*

1. *If $R = 1$, the infimum of $C_{\rm rad}^{p,q;1}$ in $W_{0,0}^{1,p}(B_1)_{\rm rad} \setminus \{0\}$ is attained by $\tilde{u}^{p,q;1} = \tilde{T}_1^{-1}[T_{q\tau_{p,q}/p'}\varphi_0]$.*

2. *In the assertion of Proposition 3.2, it holds that*

$$\hat{C}_{p,q} = \Big(\omega_n^{\tau_{p,q}} \Big(\frac{q\tau_{p,q}}{p'}\Big)^{1-\tau_{p,q}} J^{p,q}[\varphi_0]\Big)^p = C_{p,q}$$

3. *If $R > 1$, then the infimum of $C_{\rm rad}^{p,q;R}$ is not attained in $W_{0,0}^{1,p}(B_1)_{\rm rad} \setminus \{0\}$.*

**Proof:** 1. From Lemmas 3.2 and 3.3, we have for $u \in C_c^\infty(B_1 \setminus \{0\})_{\rm rad}$

$$\|u\|_{L^q_{p;R}(B_1)} = \|\tilde{T}_1 u\|_{L^q_{1/p'}(\mathbf{R}^n)} = \Big(\frac{p'}{q\tau_{p,q}}\Big)^{1/q} \|T_{p'/(q\tau_{p,q})}[\tilde{T}_1 u]\|_{L^q_{1/(q\tau_{p,q})}(\mathbf{R}^n)}, \tag{4.8}$$

$$\|\nabla u\|_{L^p_1(B_1)} = \|\nabla[\tilde{T}_1 u]\|_{L^p_{1+1/p'}(\mathbf{R}^n)} = \Big(\frac{q\tau_{p,q}}{p'}\Big)^{1/p'} \|\nabla[T_{p'/(q\tau_{p,q})}[\tilde{T}_1 u]]\|_{L^p_{1+1/(q\tau_{p,q})}(\mathbf{R}^n)}, \tag{4.9}$$

and we have

$$F^{p,q;1}[u] = \Big(\omega_n^{\tau_{p,q}} \Big(\frac{q\tau_{p,q}}{p'}\Big)^{1-\tau_{p,q}} J^{p,q}[T_{p'/(q\tau_{p,q})}[\tilde{T}_1 u]]\Big)^p \quad \text{for } u \in C_c^\infty(B_1 \setminus \{0\})_{\rm rad} \setminus \{0\}.$$

Hence from Lemma 4.1 the desired assertion follows.

2. This is clear from the assertion 2 of Propositions 3.2 and Proposition 3.4.

3. If $u \in W_{0,0}^{1,p}(B_1)_{\rm rad} \setminus \{0\}$ for $R > 1$ achieves the infimum of $C_{\rm rad}^{p,q;R}$, then from the previous result we have

$$F^{p,q;R}[u] = C_{\rm rad}^{p,q;R} = C_{p,q}.$$

But we have $F^{p,q;R}[u] > F^{p,q;R'}[u] \geq C_{p,q}$ for any $1 < R' < R$, and this is a contradiction. □

## 4.2 A generalized rearrangement of functions

We introduce a rearrangement of functions with respect to general weight functions instead of Lebesugue measure to eatablish the validities of $S^{p,q;\gamma} = S_{\rm rad}^{p,q;\gamma}$ and $C^{p,q;R} = C_{\rm rad}^{p,q;R}$ under additional conditions. First we begin with studying a theory of generalized rearrangement of functions.

**Definition 4.2** *1. For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $f \geq 0$ a.e. on $\mathbf{R}^n$, let us set for a (Lebesgue) measurable set $A$*

$$\mu_f(A) = \int_A d\mu_f = \int_A f(x)dx. \tag{4.10}$$

*Then $\mu_f$ is said to be the measure determined by $f$.*

*2. $f$ is said to be admissible, if and only if $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n) \cap C(\mathbf{R}^n \setminus \{0\})_{\mathrm{rad}}$, $f \geq 0$ on $\mathbf{R}^n \setminus \{0\}$ and $f$ is non-increasing with respect to $r = |x|$. For $u : \mathbf{R}^n \to \mathbf{R}$ and $u \geq 0$ a.e. on $\mathbf{R}^n$, we set*

$$\mu_f[u](t) = \mu_f(\{u > t\}) = \int_{\{u>t\}} f(x)dx \qquad \text{for } t \geq 0, \tag{4.11}$$

$$\mathcal{R}_f[u](x) = \sup\{t \geq 0 \mid \mu_f[u](t) > \mu_f(B_{|x|})\} \quad \text{for } x \in \mathbf{R}^n \setminus \{0\}. \tag{4.12}$$

*Then $\mu_f[u]$ and $\mathcal{R}_f[u]$ are said to be the distribution function of $u$ and the rearrangement function of $u$ with respect to $f$ respectively.*

Direct from this definition we see the next proposition.

**Proposition 4.3** *Let $1 \leq p < \infty$ and assume that $f$ is admissible. Then, for $u : \mathbf{R}^n \to \mathbf{R}$ and $u \geq 0$ a.e. on $\mathbf{R}^n$, we have the followings:*

1. *$\mu_f[u](t) = \mu_f[\mathcal{R}_f[u]](t)$ for $t \geq 0$.*
2. *$\mathcal{R}_f[u^p](x) = \mathcal{R}_f[u](x)^p$ for $x \in \mathbf{R}^n \setminus \{0\}$.*
3. *If $u$ is radially symmetric and nonincreasing with respect to $r = |x|$, then*

$$\mathcal{R}_f[u](x) = u(x) \qquad \text{for } x \in \mathbf{R}^n \setminus \{0\}.$$

Further we have

**Proposition 4.4** *Let $1 \leq p < \infty$ and assume that $f$ is admissible. Then, for $u, v : \mathbf{R}^n \to \mathbf{R}$ and $u, v \geq 0$ a.e. on $\mathbf{R}^n$, we have the followings:*

1. $\displaystyle\int_{\mathbf{R}^n} u(x)^p f(x)dx = \int_{\mathbf{R}^n} \mathcal{R}_f[u](x)^p f(x)dx.$
2. $\displaystyle\int_{\mathbf{R}^n} u(x)v(x)f(x)dx \leq \int_{\mathbf{R}^n} \mathcal{R}_f[u](x)\mathcal{R}_f[v](x)f(x)dx.$

**Proof:** 1. Since $u(x)^p = p\int_0^\infty \chi_{\{u>t\}}(x)t^{p-1}dt$ for a.e. $x \in \mathbf{R}^n$, we see that

$$\begin{aligned}\int_{\mathbf{R}^n} u(x)^p f(x)dx &= p\int_{\mathbf{R}^n}\Big(\int_0^\infty \chi_{\{u>t\}}(x)t^{p-1}dt\Big)f(x)dx \\ &= p\int_0^\infty\Big(\int_{\{u>t\}} f(x)dx\Big)t^{p-1}dt = p\int_0^\infty \mu_f[u](t)t^{p-1}dt,\end{aligned} \tag{4.13}$$

and in a similar way

$$\int_{\mathbf{R}^n} \mathcal{R}_f[u](x)^p f(x)dx = p\int_0^\infty \mu_f[\mathcal{R}_f[u]](t)t^{p-1}dt. \tag{4.14}$$

Then the assertion follows from the assertion 1 of Proposition 4.3.

2. (a) First we show that

$$\mu_f(\{u>t\}\cap\{v>s\}) \le \mu_f(\{\mathcal{R}_f[u]>t\}\cap\{\mathcal{R}_f[v]>s\}) \quad \text{for } s,t\ge 0. \tag{4.15}$$

If $\mu_f(\{u>t\}) \le \mu_f(\{v>s\})$, then we have $\{\mathcal{R}_f[u]>t\}\subset\{\mathcal{R}_f[v]>s\}$. So it follows from the assertion 1 of Proposition 4.3 that we have

$$\mu_f(\{u>t\}\cap\{v>s\}) \le \mu_f(\{u>t\}) = \mu_f(\{\mathcal{R}_f[u]>t\}) = \mu_f(\{\mathcal{R}_f[u]>t\}\cap\{\mathcal{R}_f[v]>s\}).$$

If $\mu_f(\{v>s\}) \le \mu_f(\{u>t\})$, then we see $\{\mathcal{R}_f[v]>s\}\subset\{\mathcal{R}_f[u]>t\}$, hence in a similar way the desired assertion holds.

(b) In a similar way we see that

$$\begin{aligned}\int_{\mathbf{R}^n} u(x)v(x)f(x)dx &= \int_{\mathbf{R}^n}\Big(\int_0^\infty \chi_{\{u>t\}}(x)dt\Big)\Big(\int_0^\infty \chi_{\{v>s\}}(x)ds\Big)f(x)dx \\ &= \int_0^\infty\int_0^\infty\Big(\int_{\{u>t\}\cap\{v>s\}} f(x)dx\Big)dsdt = \int_0^\infty\int_0^\infty \mu_f(\{u>t\}\cap\{v>s\})dsdt\end{aligned} \tag{4.16}$$

and

$$\int_{\mathbf{R}^n}\mathcal{R}_f[u](x)\mathcal{R}_f[v](x)f(x)dx = \int_0^\infty\int_0^\infty \mu_f(\{\mathcal{R}_f[u]>t\}\cap\{\mathcal{R}_f[v]>s\})dsdt. \tag{4.17}$$

The assertion therefore follows from (a). □

Lastly we show the next. Here by $H^{n-1}$ we denote the $(n-1)-$dimensional Hausdorff measure.

**Proposition 4.5** *Let $1\le p<\infty$ and assume that $f$ is admissible. Then for $u\in C_c^\infty(\mathbf{R}^n)$ we have the followings:*

1. $\displaystyle\int_{\{\mathcal{R}_f[|u|]=t\}} dH^{n-1} \le \int_{\{|u|=t\}} dH^{n-1}.$
2. $\displaystyle\int_{\mathbf{R}^n} |\nabla[\mathcal{R}_f[|u|]](x)|^p \frac{1}{f(x)^{p-1}}dx \le \int_{\mathbf{R}^n}|\nabla u(x)|^p\frac{1}{f(x)^{p-1}}dx.$

For the proof we prepare two lemmas below.

**Lemma 4.2** *(The coarea formula) Let $1\le p<\infty$. For $u\in C_c^1(\mathbf{R}^n)$ and $g\in L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$, it holds that*

$$\int_{\mathbf{R}^n}|\nabla u(x)|^p g(x)dx = \int_{-\infty}^\infty\int_{\{u=s\}}|\nabla u(x)|^{p-1}g(x)dH^{n-1}(x)ds \tag{4.18}$$

**Remark 4.1** *We note that the boundedness of $g$ is not essential, in fact by a usual approximation argument we see that the formula remains valid if the left-hand side is finite. This is also valid under the assumption that $u$ is Lipschitz continuous and $\nabla u$ is integrable. For the proof of this formula, see [Ma; Theorem in §1.2.4] for example.*

In this formula, assuming that for an admissible $f$, $g=f$, $u\in C_0^\infty(\mathbf{R}^n)$ is nonnegative and

$$\Phi = \begin{cases} \frac{\chi_{\{u>t\}}(x)}{|\nabla u|^p}f(x), & \text{if } \nabla u\ne 0,\\ 0 & \text{if } \nabla u = 0,\end{cases}$$

we have

$$\mu_f[u](t) = \mu_f(\{u>t\}\cap\{\nabla u=0\}) + \int_t^\infty ds\int_{\{u=s\}}\frac{f(x)}{|\nabla u|}dH^{n-1}. \tag{4.19}$$

By Sard's lemma the set of critical values of $u \in C_0^\infty(\mathbf{R}^n)$ has a vanishing measure, hence we have

$$-\mu_f[u]'(t) = \int_{\{u=t\}} \frac{f(x)}{|\nabla u|}\, dH^{n-1}, \qquad \text{for almost all } t \in (0,\infty). \tag{4.20}$$

Now we replace $u$ by its rearrangement $\mathcal{R}_f[u]$ in (4.19). We recall that both $u$ and $\mathcal{R}_f[u]$ share the same distribution function, and $\mathcal{R}_f[u]$ is at least Lipschitz continuous as a rearrangement of a smooth $u$. If we admit the property (4.21);

$$\frac{d}{dt}\mu_f(\{\mathcal{R}_f[u] > t\} \cap \{\nabla \mathcal{R}_f[u] = 0\}) = 0, \qquad \text{for almost all } t \in (0,\infty), \tag{4.21}$$

then we see that

$$\mu_f[u]'(t) = -\frac{\int_{\{\mathcal{R}_f[u]=t\}} f\, dH^{n-1}}{|\nabla \mathcal{R}_f[u]|_{|\{\mathcal{R}_f[u]=t\}}}, \qquad \text{for almost all } t \in (0,\infty). \tag{4.22}$$

When $f \equiv 1$, the proof of (4.21) is seen in [CF;Lemma 2.4, Lemma 2.6]. By the same argument it is easy to see the validity of the property (4.21) for a general $f$. Here we shall give a brief explanation of (4.21) for the reader's convenience. Since $\mathcal{R}_f[u]$ and $f$ are radial, we may assume that $n = 1$ without loss of the generality. Then the property (4.21) follows from the next elementaly lemma due to [CF; p.12, lemma 2.4].

**Lemma 4.3** *Let $f$ be an admissible weight function on $\mathbf{R}$. Let $I = (a,b)$ be a bounded open interval of $\mathbf{R}$ and let $v$ be a cmpactly supported and Lipschitz continuous function in $I$. Then we have*

1. *There is a Borel set $N$ in $I$ with $|N| = 0$ such that $v(s)$ is differentiable for all $s \in I \setminus N$. Moreover we have*
$$|v(\{s \in I \setminus N : v'(s) = 0\})| = 0.$$
*Here by $v(A)$ with $A \subset I$ we denote the set defined by $v(A) = \{t \in \mathbf{R} : t = v(s), \text{for some } s \in A\}$.*
2. *The function $h : \mathbf{R} \to [0,\infty)$, defined as*
$$h(t) = \mu_f\left(\{s \in I \setminus N : v'(s) = 0\} \cap \{s \in I : v(s) > t\}\right) \tag{4.23}$$
*is non-increasing, right-continuous and $h'(t) = 0$ for a.e. $t \in \mathbf{R}$.*

**A sketch of the proof:** Let us set

$$|v'|((a,b)) = \sup\left\{\sum_{j=1}^{M-1} |v(s_{j+1}) - v(s_j)| : M \geq 2, a < s_1 < s_2 \cdots < s_M < b\right\}.$$

This is called the total variation of $v$ in $I = (a,b)$. Since $v$ is compactly supported and Lipschitz continuous, this quantity is finite. Then we see that $|v(I)| \leq |v'|((a,b)) = \int_a^b |v'(s)|\, ds$. By a similar argument we have

$$|v(\{s \in I \setminus N : v'(s) = 0\})| \leq |v'|(A)$$

for any Borel set $A$ containing $\{s \in I \setminus N : v'(s) = 0\}$. By taking a infimum with respect to $A$, we have

$$|v(\{s \in I \setminus N : v'(s) = 0\})| \leq \int_{\{s \in I \setminus N : v'(s)=0\}} |v'(s)|\, ds = 0$$

This proves the assertion 1.

From the definition of $h(t)$ we easily see that $h$ is non-increasing and right-continuous. Hence we have $h(t_2) - h(t_1) = h'((t_1, t_2])$ for any $t_1, t_1 \in \mathbf{R}$ with $t_1 < t_2$. Then for any $F \subset \mathbf{R}$

$$|h'|(F) = \int_{\{s \in I \setminus N : v'(s)=0\} \cap v^{-1}(F)} f(s)\, ds.$$

It follows from the assertion 1 that there is a Borel set $F_0$ such that

$$v(\{s \in I \setminus N : v'(s) = 0\}) \subset F_0, \quad \text{and} \quad |F_0| = 0.$$

Then we see that

$$|h'|(\mathbf{R} \setminus F_0) = \int_{\{s \in I \setminus N : v'(s)=0\} \cap v^{-1}(\mathbf{R} \setminus F_0)} f(s)\, ds = 0.$$

This means $|h'|$ is concentrated in $F_0$, and hence $h'(t) = 0$ for a.e. $t \in \mathbf{R}$. □

Now we give a proof to Proposition 4.5.

**Proof of Proposition 4.5 :**
**1.** Let $A$ be any Borel set such as $0 < |A| < \infty$. By $\mathcal{R}_f[A]$ we denote the rearrangement of $A$ with respect to an admissible weight $f$, namely, $\mathcal{R}_f[A]$ ia a ball centered at the origin satisfying

$$\mu_f(A) = \mu_f(\mathcal{R}_f[A]). \tag{4.24}$$

Let $r$ be a positive number such that

$$|B_r(0)| = |A|. \tag{4.25}$$

Since

$$\int_{B_r(0)} f(x)\, dx \geq \int_A f(x)\, dx, \tag{4.26}$$

we see that

$$\mathcal{R}_f[A] \subset B_r(0). \tag{4.27}$$

Therefore we conclude that

$$\int_{\partial A} dH^{n-1} \geq \int_{\partial B_r(0)} dH^{n-1} \geq \int_{\partial \mathcal{R}_f[A]} dH^{n-1}. \tag{4.28}$$

Since $A$ is an arbitrary Borel set, the assertion follows immediately.

**2.** Since $\mathcal{R}_f[u]$ is Lipschitz continuous, we can employ the coarea formula. Then we have

$$\int_{\mathbf{R}^n} |\nabla \mathcal{R}_f[u]|^p f(x)^{1-p}\, dx = \int_0^\infty dt \int_{\{\mathcal{R}_f[u]=t\}} |\nabla \mathcal{R}_f[u]|^{p-1} f(x)^{1-p}\, dH^{n-1}$$

$$= \int_0^\infty dt \frac{\left(\int_{\{\mathcal{R}_f[u]=t\}} dH^{n-1}\right)^p}{\left(\int_{\{\mathcal{R}_f[u]=t\}} \frac{f(x)}{|\nabla \mathcal{R}_f[u]|} dH^{n-1}\right)^{p-1}}$$

$$= \int_0^\infty dt \frac{\left(\int_{\{\mathcal{R}_f[u]=t\}} dH^{n-1}\right)^p}{(-\mu_f[u]'(t))^{p-1}} \qquad \text{(the property (4.22))}$$

$$\le \int_0^\infty dt \frac{\left(\int_{\{u=t\}} dH^{n-1}\right)^p}{(-\mu_f[u]'(t))^{p-1}} \qquad \text{(Assertion 1)}$$

$$= \int_0^\infty dt \frac{\left(\int_{\{u=t\}} dH^{n-1}\right)^p}{\left(\int_{\{u=t\}} \frac{f(x)}{|\nabla u|} dH^{n-1}\right)^{p-1}} \qquad \text{( the property (4.20))}$$

$$\le \int_0^\infty dt \int_{\{u=t\}} |\nabla u|^{p-1} f(x)^{1-p}\, dH^{n-1} \qquad \text{( Hölder inequality )}$$

$$= \int_{\mathbf{R}^n} |\nabla u|^p f(x)^{1-p}\, dx.$$

Clearly this proves the assertion. This proves the assertion. □

## 4.3 Application of the theory on rearrangement of functions

In this subsection we establish $S^{p,q;\gamma} = S^{p,q;\gamma}_{\rm rad}$ and $C^{p,q;R} = C^{p,q;R}_{\rm rad}$ under certain assumption using the theory of the generalized rearrangement of functions which was developed in the previous subsection.

First let us consider the case that $\gamma > 0$. Then we have the next proposition which proves the assertion 3 of Theorem 2.2. Further, making use of the assertion 4 of Theorem 2.2 at the same time, we see that the assertion 1 of Theorem 2.1 follows as well. Here we note that $I_\alpha$ is admissible if $0 < \alpha \le n$.

**Proposition 4.6** *For $1 \le p \le q < \infty$, $\tau_{p,q} \le 1/n$ and $0 < \gamma \le \gamma_{p,q}$, it holds that $S^{p,q;\gamma} = S^{p,q;\gamma}_{\rm rad}$.*

**Proof:** By virtue of the assertion 1 of Proposition 3.3, it suffices to consider the case that $\gamma = \gamma_{p,q} = (n-1)/(1+q/p')$. Since $0 < q\gamma_{p,q} < n$, by using the assertions 2 of Proposition 4.3, 1 of Proposition 4.4 and 2 of Proposition 4.5, we have for $u \in C^\infty_{\rm c}(\mathbf{R}^n \setminus \{0\})$

$$\begin{aligned}
\|u\|^q_{L^q_{\gamma_{p,q}}(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma_{p,q}}(x)dx = \int_{\mathbf{R}^n} \mathcal{R}_{I_{q\gamma_{p,q}}}[|u|^q](x) I_{q\gamma_{p,q}}(x)dx \\
&= \int_{\mathbf{R}^n} \mathcal{R}_{I_{q\gamma_{p,q}}}[|u|](x)^q I_{q\gamma_{p,q}}(x)dx = \|\mathcal{R}_{I_{q\gamma_{p,q}}}[|u|]\|^q_{L^q_{\gamma_{p,q}}(\mathbf{R}^n)}, \\
\|\nabla u\|^p_{L^p_{1+\gamma_{p,q}}(\mathbf{R}^n)} &= \|\nabla[|u|]\|^p_{L^p_{1+\gamma_{p,q}}(\mathbf{R}^n)} = \int_{\mathbf{R}^n} |\nabla[|u|](x)|^p \frac{1}{I_{q\gamma_{p,q}}(x)^{p-1}} dx \\
&\ge \int_{\mathbf{R}^n} |\nabla[\mathcal{R}_{I_{q\gamma_{p,q}}}[|u|]](x)|^p \frac{1}{I_{q\gamma_{p,q}}(x)^{p-1}} dx = \|\nabla[\mathcal{R}_{I_{q\gamma_{p,q}}}[|u|]]\|^p_{L^p_{1+\gamma_{p,q}}(\mathbf{R}^n)}.
\end{aligned}$$

Therefore

$$E^{p,q;\gamma_{p,q}}(u) = \left(\frac{\|\nabla u\|_{L^p_{1+\gamma_{p,q}}(\mathbf{R}^n)}}{\|u\|_{L^q_{\gamma_{p,q}}(\mathbf{R}^n)}}\right)^p \geq \left(\frac{\|\nabla[\mathcal{R}_{I_{q\gamma_{p,q}}}[|u|]]\|_{L^p_{1+\gamma_{p,q}}(\mathbf{R}^n)}}{\|\mathcal{R}_{I_{q\gamma_{p,q}}}[|u|]\|_{L^q_{\gamma_{p,q}}(\mathbf{R}^n)}}\right)^p \geq S^{p,q;\gamma_{p,q}}_{\mathrm{rad}} \quad (4.29)$$

$$\text{for } u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})\setminus\{0\}.$$

This proves the assertion. □

Now we consider the case $\gamma = 0$. Noting that the above argument works only when $p \geq n$, we have the following which assures the assertion 3 of Theorem 2.7.

**Proposition 4.7** *Let $n \geq 2$. Assume that $n \leq p \leq q < \infty$ and $R \geq R_{p,q}$, then it holds that $C^{p,q;R} = C^{p,q;R}_{\mathrm{rad}}$.*

**Proof:** When $R \geq R_{p,q} = \exp((1+q/p')/((n-1)p'))$ holds, $|x|^{-(n-1)p'}/A^{1+q/p'}_{1,R} : B_1\setminus\{0\} \to \mathbf{R}$ is positive and decreasing with respect to $r = |x|$. Then, noting that $0 < (n-1)p' < n$ and $I_0 = I_{(n-1)p'}|x|^{-(n-1)p'}$, it follows from the assertion 2 of Proposition 4.4, the assertions 2 and 3 of Proposition 4.3 that we have for $u \in C^\infty_{\mathrm{c}}(B_1\setminus\{0\})$

$$\begin{aligned}
\|u\|^q_{L^q_{p;R}(B_1)} &= \int_{B_1} |u(x)|^q \left[\frac{|x|^{-(n-1)p'}}{A^{1+q/p'}_{1,R}}\right](x) I_{(n-1)p'}(x)\,dx \\
&\leq \int_{B_1} \mathcal{R}_{I_{(n-1)p'}}[|u|^q](x)\mathcal{R}_{I_{(n-1)p'}}\left[\frac{|x|^{-(n-1)p'}}{A^{1+q/p'}_{1,R}}\right](x) I_{(n-1)p'}(x)\,dx \\
&= \int_{B_1} \mathcal{R}_{I_{(n-1)p'}}[|u|](x)^q \left[\frac{|x|^{-(n-1)p'}}{A^{1+q/p'}_{1,R}}\right](x) I_{(n-1)p'}(x)\,dx = \|\mathcal{R}_{I_{(n-1)p'}}[|u|]\|^q_{L^q_{p;R}(B_1)},
\end{aligned}$$

Since $(n-(n-1)p')(p-1) = p-n$, we have, using the assertion 2 of Proposition 4.5,

$$\begin{aligned}
\|\nabla u\|^p_{L^p_1(B_1)} = \|\nabla[|u|]\|^p_{L^p_1(B_1)} &= \int_{B_1} |\nabla[|u|](x)|^p \frac{1}{I_{(n-1)p'}(x)^{p-1}}\,dx \\
&\geq \int_{B_1} |\nabla[\mathcal{R}_{I_{(n-1)p'}}[|u|]](x)|^p \frac{1}{I_{(n-1)p'}(x)^{p-1}}\,dx = \|\nabla[\mathcal{R}_{I_{(n-1)p'}}[|u|]]\|^p_{L^p_1(B_1)}.
\end{aligned}$$

Therefore we see that

$$F^{p,q;R}(u) = \left(\frac{\|\nabla u\|_{L^p_1(B_1)}}{\|u\|_{L^q_{p;R}(B_1)}}\right)^p \geq \left(\frac{\|\nabla[\mathcal{R}_{I_{(n-1)p'}}[|u|]]\|_{L^p_1(B_1)}}{\|\mathcal{R}_{I_{(n-1)p'}}[|u|]\|_{L^q_{p;R}(B_1)}}\right)^p \geq C^{p,q;R}_{\mathrm{rad}}$$

$$\text{for } u \in C^\infty_{\mathrm{c}}(B_1\setminus\{0\})\setminus\{0\},$$

and this proves the assertion.

□

We remark that $I_{(n-1)p'} = I_0$ is not admissible if $n = 1$. Hence we can not apply the same method direct, but the assertion 2 of Theorem 2.9 follows from the next proposition.

**Proposition 4.8** *Let $n = 1$. If $1 < p \leq q < \infty$ and $R \geq 1$, then it holds that $C^{p,q;R} = C^{p,q;R}_{\mathrm{rad}}$.*

**Proof:** (a) Admitting that $(1+t^p)^{1/p} \geq (1+t^q)^{1/q}$ for $t \geq 0$, it holds that for any $u \in C^\infty_{\mathrm{c}}((-1,1)\setminus\{0\})$

$$(\|u\|^p_{L^q_{p;R}((-1,0))} + \|u\|^p_{L^q_{p;R}((0,1))})^{1/p} \geq (\|u\|^q_{L^q_{p;R}((-1,0))} + \|u\|^q_{L^q_{p;R}((0,1))})^{1/q} = \|u\|_{L^q_{p;R}((-1,1))}.$$

Then we also have

$$\frac{\|u'\|_{L_1^p((-1,1))}}{\|u\|_{L^q_{p;R}((-1,1))}} \geq \min\left\{\frac{\|u'\|_{L_1^p((-1,0))}}{\|u\|_{L^q_{p;R}((-1,0))}}, \frac{\|u'\|_{L_1^p((0,1))}}{\|u\|_{L^q_{p;R}((0,1))}}\right\} \quad \text{for } u \in C_c^\infty((-1,1)\setminus\{0\})\setminus\{0\}.$$

In fact, if $\|u'\|_{L_1^p((-1,0))}/\|u\|_{L^q_{p;R}((-1,0))} \geq \|u'\|_{L_1^p((0,1))}/\|u\|_{L^q_{p;R}((0,1))}$ holds, then we have

$$\begin{aligned}
\frac{\|u'\|_{L_1^p((-1,1))}}{\|u\|_{L^q_{p;R}((-1,1))}} &= \frac{(\|u'\|^p_{L_1^p((-1,0))} + \|u'\|^p_{L_1^p((0,1))})^{1/p}}{\|u\|_{L^q_{p;R}((-1,1))}} \\
&\geq \frac{1}{\|u\|_{L^q_{p;R}((-1,1))}}\left(\frac{\|u'\|^p_{L_1^p((0,1))}}{\|u\|^p_{L^q_{p;R}((0,1))}}\|u\|^p_{L^q_{p;R}((-1,0))} + \|u'\|^p_{L_1^p((0,1))}\right)^{1/p} \\
&= \frac{\|u'\|_{L_1^p((0,1))}}{\|u\|_{L^q_{p;R}((0,1))}} \frac{(\|u\|^p_{L^q_{p;R}((-1,0))} + \|u\|^p_{L^q_{p;R}((0,1))})^{1/p}}{\|u\|_{L^q_{p;R}((-1,1))}} \geq \frac{\|u'\|_{L_1^p((0,1))}}{\|u\|_{L^q_{p;R}((0,1))}}.
\end{aligned}$$

If $\|u'\|_{L_1^p((0,1))}/\|u\|_{L^q_{p;R}((0,1))} \geq \|u'\|_{L_1^p((-1,0))}/\|u\|_{L^q_{p;R}((-1,0))}$, then in a similar way we see

$$\frac{\|u'\|_{L_1^p((-1,1))}}{\|u\|_{L^q_{p;R}((-1,1))}} \geq \frac{\|u'\|_{L_1^p((-1,0))}}{\|u\|_{L^q_{p;R}((-1,0))}}.$$

(b) Since we have

$$\begin{aligned}
C^{p,q;R}_{\mathrm{rad}} &= \inf\left\{\left(\frac{\|u'\|_{L_1^p((-1,1))}}{\|u\|_{L^q_{p;R}((-1,1))}}\right)^p \;\middle|\; u \in C_c^\infty((-1,1)\setminus\{0\})_{\mathrm{rad}}\setminus\{0\}\right\} \\
&= \inf\left\{\left(\frac{\|u'\|_{L_1^p((-1,0))}}{\|u\|_{L^q_{p;R}((-1,0))}}\right)^p \;\middle|\; u \in C_c^\infty((-1,1)\setminus\{0\})\setminus\{0\}\right\} \\
&= \inf\left\{\left(\frac{\|u'\|_{L_1^p((0,1))}}{\|u\|_{L^q_{p;R}((0,1))}}\right)^p \;\middle|\; u \in C_c^\infty((-1,1)\setminus\{0\})\setminus\{0\}\right\},
\end{aligned}$$

it follows fromt (a) that we have

$$F^{p,q;R}(u) = \left(\frac{\|u'\|_{L_1^p((-1,1))}}{\|u\|_{L^q_{p;R}((-1,1))}}\right)^p \geq \min\left\{\left(\frac{\|u'\|_{L_1^p((-1,0))}}{\|u\|_{L^q_{p;R}((-1,0))}}\right)^p, \left(\frac{\|u'\|_{L_1^p((0,1))}}{\|u\|_{L^q_{p;R}((0,1))}}\right)^p\right\} \geq C^{p,q;R}_{\mathrm{rad}}$$
$$\text{for } u \in C_c^\infty((-1,1)\setminus\{0\})\setminus\{0\}.$$

Thus the assertion follows. □

# 5 Application of Nonlinear potential theory

It follow from Proposition 4.7 and Proposition 4.8 that we have the assertion 3 of Theorem 2.7. Then, combining it with the assertion 4 of Theorem 2.7, Theorem 2.5 clearly follows provided that $p \geq n$. Therefore, it suffices to assume that $1 < p < n$ in the rest of the proof of Theorem 2.5. We finish this aim by employing the so-called nonlinear potential theory.

**Definition 5.1** *(Muckenhoupt $A_p$ class) Let $1 < p < \infty$. $w \in C(\mathbf{R}^n \setminus \{0\})$ is said to belong to $A_p$ class, if and only if $w > 0$ on $\mathbf{R}^n \setminus \{0\}$ and*

$$\sup_{x\in\mathbf{R}^n, r>0} \frac{n}{\omega_n r^n}\int_{B_r(x)} w(y)\,dy \left(\frac{n}{\omega_n r^n}\int_{B_r(x)} \frac{1}{w(y)^{1/(p-1)}}dy\right)^{p-1} < \infty \tag{5.1}$$

*is satisfied.*

*When $w$ belongs to $A_p$ class, simply we describe $w \in A_p(\mathbf{R}^n)$. Let us define*

$$J_p[w](x,r) = \int_r^\infty \left( \frac{n}{\omega_n t^n} \int_{B_t(x)} \frac{1}{w(y)^{1/(p-1)}} dy \right) \frac{1}{t^{1+\nu_p}} dt \quad \text{for } x \in \mathbf{R}^n,\ r > 0. \tag{5.2}$$

*Here, $\nu_p = (n-p)/(p-1)$.*

Under these notations we have the next lemma which is due to R. Adams [Ad].

**Lemma 5.1** *Let $1 < p < q < \infty$. Assume that $w \in A_p(\mathbf{R}^n)$, $g \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $g \geq 0$ a.e. on $\mathbf{R}^n$. Then, the following two assertions are equivalent to each other.*

*(a)* $$\sup_{x \in \mathbf{R}^n,\, r > 0} \mu_g(B_r(x)) J_p[w](x,r)^{q/p'} < \infty.$$

*(b) There is a positive number $C > 0$ such that we have*

$$\|I_1 * f\|_{L^q(\mathbf{R}^n; g)} \leq C \|f\|_{L^p(\mathbf{R}^n; w)} \quad \text{for any } f \in L^p(\mathbf{R}^n; w).$$

Using this we establish the next Proposition. Then, combining it with the assertion 4 of Theorem 2.7, we see that Theorem 2.5 is valid even when $1 < p < n$ holds.

**Proposition 5.1** *Assume that $1 < p < q < \infty$, $p < n$, $\tau_{p,q} \leq 1/n$ and $R > 3$, then we have $C^{p,q;R} > 0$.*

Introducing more notations, we verify this using Lemma 5.1.

**Definition 5.2** *For $1 < p < q < \infty$, $p < n$, $\tau_{p,q} \leq 1/n$ and $R > 1$, we set*

$$w_p(x) = w_p(|x|) = \max\{I_p(x), 1\} \qquad \text{for } x \in \mathbf{R}^n \setminus \{0\}, \tag{5.3}$$

$$g_{p,q;R}(x) = g_{p,q;R}(|x|) = \begin{cases} \dfrac{I_0(x)}{A_{1,R}(x)^{1+q/p'}} & \text{for } x \in \overline{B_1} \setminus \{0\}, \\ 0 & \text{for } x \in \mathbf{R}^n \setminus B_1. \end{cases} \tag{5.4}$$

In order to apply Lemma 5.1 to these weight functions, let us prepare more lemmas.

**Lemma 5.2** *For $1 < p < n$, it holds that $w_p \in A_p(\mathbf{R}^n)$.*

**Proof:** Let us set

$$\sigma_p[w_p](x,r) = \frac{n}{\omega_n r^n} \int_{B_r(x)} w_p(y) dy \left( \frac{n}{\omega_n r^n} \int_{B_r(x)} \frac{1}{w_p(y)^{1/(p-1)}} dy \right)^{p-1} \quad \text{for } x \in \mathbf{R}^n,\ r > 0 \tag{5.5}$$

and show it to be bounded.

(i) First we asssume that $0 \leq |x| \leq 1$.

(a) If $0 < r \leq \min\{|x|/2, 1 - |x|\}$, then we see $B_r(x) \subset B_{|x|+r} \setminus \overline{B_{|x|-r}} \subset B_1$, hence

$$\begin{aligned} \sigma_p[w_p](x,r) &\leq \frac{n}{\omega_n r^n} \int_{B_r(x)} w_p(|x| - r) dy \left( \frac{n}{\omega_n r^n} \int_{B_r(x)} \frac{1}{w_p(|x|+r)^{1/(p-1)}} dy \right)^{p-1} \\ &= \left( \frac{|x| + r}{|x| - r} \right)^{n-p} \leq \left( \frac{|x| + |x|/2}{|x| - |x|/2} \right)^{n-p} = 3^{n-p}. \end{aligned}$$

(b) If $1-|x| \le r \le |x|/2$, we see $|x| \ge 2/3$, hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_r(x)} w_p(|x|-r)\,dy \left(\frac{n}{\omega_n r^n}\int_{B_r(x)} dy\right)^{p-1} = \frac{1}{(|x|-r)^{n-p}} \\ &\le \frac{1}{(|x|-|x|/2)^{n-p}} = \left(\frac{2}{|x|}\right)^{n-p} \le 3^{n-p}.\end{aligned}$$

(c) If $|x|/2 \le r \le 1-|x|$ , then we see $B_r(x) \subset B_{|x|+r} \subset B_1$, and hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_{|x|+r}} w_p(y)\,dy \left(\frac{n}{\omega_n r^n}\int_{B_{|x|+r}} \frac{1}{w_p(y)^{1/(p-1)}}dy\right)^{p-1} \\ &= \frac{n}{p}\left(\frac{n'}{p'}\right)^{p-1}\left(\frac{|x|+r}{r}\right)^{np} \le \frac{n}{p}\left(\frac{n'}{p'}\right)^{p-1}\left(\frac{2r+r}{r}\right)^{np} = \frac{n}{p}\left(\frac{n'}{p'}\right)^{p-1} 3^{np}.\end{aligned}$$

(d) If $r \ge \max\{|x|/2, 1-|x|\}$, then $r \ge 1/3$ and $B_r(x) \subset B_{|x|+r}$. Hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_{|x|+r}} w_p(y)\,dy \left(\frac{n}{\omega_n r^n}\int_{B_{|x|+r}} \frac{1}{w_p(y)^{1/(p-1)}}dy\right)^{p-1} \\ &= \left(\left(\frac{|x|+r}{r}\right)^n + \left(\frac{n}{p}-1\right)\frac{1}{r^n}\right)\left(\left(\frac{|x|+r}{r}\right)^n - \left(1-\frac{n'}{p'}\right)\frac{1}{r^n}\right)^{p-1} \\ &\le \left(\left(\frac{2r+r}{r}\right)^n + \left(\frac{n}{p}-1\right)3^n\right)\left(\frac{2r+r}{r}\right)^{n(p-1)} = \frac{n}{p}3^{np}.\end{aligned}$$

(ii) Secondly we assume that $|x| \ge 1$.

(a) If $0 < r \le |x|/2$, then $B_r(x) \subset B_{|x|+r} \backslash \overline{B_{|x|-r}}$, hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_r(x)} w_p(|x|-r)\,dy \left(\frac{n}{\omega_n r^n}\int_{B_r(x)} dy\right)^{p-1} = \frac{1}{(|x|-r)^{n-p}} \\ &\le \frac{1}{(|x|-|x|/2)^{n-p}} = \left(\frac{2}{|x|}\right)^{n-p} \le 2^{n-p}.\end{aligned}$$

(b) If $r \ge |x|/2$, then $r \ge 1/2$ and $B_r(x) \subset B_{|x|+r}$, hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_{|x|+r}} w_p(y)\,dy \left(\frac{n}{\omega_n r^n}\int_{B_{|x|+r}} \frac{1}{w_p(y)^{1/(p-1)}}dy\right)^{p-1} \\ &= \left(\left(\frac{|x|+r}{r}\right)^n + \left(\frac{n}{p}-1\right)\frac{1}{r^n}\right)\left(\left(\frac{|x|+r}{r}\right)^n - \left(1-\frac{n'}{p'}\right)\frac{1}{r^n}\right)^{p-1} \\ &\le \left(\left(\frac{2r+r}{r}\right)^n + \left(\frac{n}{p}-1\right)2^n\right)\left(\frac{2r+r}{r}\right)^{n(p-1)} = \left(3^n + \left(\frac{n}{p}-1\right)2^n\right)3^{np}.\end{aligned}$$

□

**Lemma 5.3** *For $1 < p < n$ and $R > 3$, there exist positive numbers $c_p$ and $c_{p;R} > 0$ such that we have the followings:*

1. $J_p[w_p](x,r) \le \dfrac{1}{\nu_p}\dfrac{1}{r^{\nu_p}}$ *for $x \in \mathbf{R}^n$, $r > 0$.*

*2.* $J_p[w_p](x,r) \le c_p \left(1 + \log\frac{1}{r} + \left(\frac{|x|}{r}\right)^{\nu_p}\right)$ *if* $|x| + r \le 1$.

*3.* $J_p[w_p](x,r) \le c_{p;R} \left(A_{1,R}(\min\{1,|x|\}) + \left(\frac{\min\{1,|x|\}}{r}\right)^{\nu_p}\right)$ *if* $0 < r \le \frac{|x|}{2}$.

**Proof:** Let us note that

$$\frac{n}{\omega_n t^n}\int_{B_t(x)} \frac{1}{w_p(y)^{1/(p-1)}} dy \le \min\{1, (t+|x|)^{\nu_p}\} \le 1 \quad \text{for } x \in \mathbf{R}^n,\ t > 0. \tag{5.6}$$

Then

1. 
$$\begin{aligned} J_p[w_p](x,r) &= \int_r^\infty \left(\frac{n}{\omega_n t^n}\int_{B_t(x)} \frac{1}{w_p(y)^{1/(p-1)}} dy\right) \frac{1}{t^{1+\nu_p}} dt \le \int_r^\infty \frac{1}{t^{1+\nu_p}} dt \\ &= \frac{1}{\nu_p}\frac{1}{r^{\nu_p}} \quad \text{for } x \in \mathbf{R}^n,\ r > 0. \end{aligned}$$

2. If $|x| + r \le 1$, then we see $r \le 1$, and hence

$$\begin{aligned} J_p[w_p](x,r) &\le \int_r^\infty \min\{1, (t+|x|)^{\nu_p}\} \frac{1}{t^{1+\nu_p}} dt \le \int_1^\infty \frac{1}{t^{1+\nu_p}} dt + \int_r^1 (t+|x|)^{\nu_p} \frac{1}{t^{1+\nu_p}} dt \\ &= \frac{1}{\nu_p} + \int_r^1 \left(1 + \frac{|x|}{t}\right)^{\nu_p} \frac{1}{t} dt \le \frac{1}{\nu_p} + 2^{(\nu_p-1)_+}\int_r^1 \left(1 + \left(\frac{|x|}{t}\right)^{\nu_p}\right) \frac{1}{t} dt \\ &= \frac{1}{\nu_p} + 2^{(\nu_p-1)_+}\left(\log\frac{1}{r} + \frac{|x|^{\nu_p}}{\nu_p}\left(\frac{1}{r^{\nu_p}} - 1\right)\right) \le \frac{1}{\nu_p} + 2^{(\nu_p-1)_+}\left(\log\frac{1}{r} + \frac{1}{\nu_p}\left(\frac{|x|}{r}\right)^{\nu_p}\right). \end{aligned}$$

3. (a) If $|x| + r \le 1$ and $0 < r \le \frac{|x|}{2}$, then $|x| \le 1$ and $|x|/r \ge 2$. From the argument of 2 and

$$1 + \log t \le \tilde{c}_p t^{\nu_p} \quad \text{for } t \ge 1 \tag{5.7}$$

it holds that

$$\begin{aligned} J_p[w_p](x,r) &\le c_p\left(1 + \log\frac{1}{r} + \left(\frac{|x|}{r}\right)^{\nu_p}\right) \le c_p\left(1 + \log\frac{|x|}{r} + \log\frac{R}{|x|} + \left(\frac{|x|}{r}\right)^{\nu_p}\right) \\ &\le c_p\left(A_{1,R}(x) + (1+\tilde{c}_p)\left(\frac{|x|}{r}\right)^{\nu_p}\right). \end{aligned}$$

(b) If $|x| + r \ge 1$, then from 1 we see that

$$\begin{aligned} J_p[w_p](x,r) &\le \frac{1}{\nu_p}\frac{1}{r^{\nu_p}} \le \frac{1}{\nu_p}\left(\frac{|x|+r}{r}\right)^{\nu_p} \le \frac{2^{(\nu_p-1)_+}}{\nu_p}\left(1 + \left(\frac{|x|}{r}\right)^{\nu_p}\right) \\ &\le \frac{2^{(\nu_p-1)_+}}{\nu_p}\left(\frac{A_{1,R}(x)}{\log R} + \left(\frac{|x|}{r}\right)^{\nu_p}\right) \qquad \text{if } |x| \le 1, \\ J_p[w_p](x,r) &\le \frac{1}{\nu_p}\frac{1}{r^{\nu_p}} \le \frac{1}{\nu_p}\left(A_{1,R}(1) + \left(\frac{1}{r}\right)^{\nu_p}\right) \qquad \text{if } |x| \ge 1. \end{aligned}$$

□

**Lemma 5.4** *For* $1 < p < q < \infty$, $p < n$, $\tau_{p,q} \le 1/n$ *and* $R > 3$, *there exists a positive number* $c_{p,q;R} > 0$ *such that we have*

$$\mu_{g_{p,q;R}}(B_r(x)) \le \begin{cases} c_{p,q;R}\, g_{p,q;R}(\min\{1,|x|\}) r^n & \textit{if } 0 < r \le \dfrac{1}{2}\min\{1,|x|\}, \\ c_{p,q;R} \dfrac{1}{A_{1,R}(r)^{q/p'}} & \textit{if } \dfrac{|x|}{2} \le r \le \dfrac{1}{2}, \\ c_{p,q;R} & \textit{if } r \ge \dfrac{1}{2}. \end{cases} \tag{5.8}$$

**Proof:** First we note that for $1 < \underline{R} < R$

$$\log \frac{\underline{R}}{r} \ge \frac{\log \underline{R}}{\log R} \log \frac{R}{r} \quad \text{for } 0 < r \le 1\, (< \underline{R}). \tag{5.9}$$

By the definition we have

$$\begin{aligned} \mu_{g_{p,q;R}}(B_r(x)) &= \int_{B_r(x)} g_{p,q;R}(y)\,dy \qquad (5.10)\\ &= \int_{B_r(x)\cap B_1} \frac{1}{(\log(R/|y|))^{1+q/p'}} \frac{1}{|y|^n} dy \quad \text{for } x \in \mathbf{R}^n,\ r > 0. \end{aligned}$$

(a) If $0 < r \le |x|/2 \le 1/2$, then $|x|/2 \le |x| - r \le |y| \le |x| + r \le 3|x|/2$ for $y \in B_r(x)$, hence we have, using (5.9) with $\underline{R} = 2R/3$,

$$\begin{aligned} \mu_{g_{p,q;R}}(B_r(x)) &\le \int_{B_r(x)} \frac{1}{(\log(2R/(3|x|)))^{1+q/p'}} \left(\frac{2}{|x|}\right)^n dy \\ &= \frac{\omega_n r^n}{n} \frac{1}{(\log(2R/(3|x|)))^{1+q/p'}} \left(\frac{2}{|x|}\right)^n \le 2^n \frac{\omega_n}{n} \left(\frac{\log R}{\log(2R/3)}\right)^{1+q/p'} g_{p,q;R}(x) r^n. \end{aligned}$$

(b) If $0 < r \le 1/2 \le |x|/2$, then $1/2 \le |x|/2 \le |x| - r \le |y|$ for $y \in B_r(x)$, hence we have

$$\mu_{g_{p,q;R}}(B_r(x)) \le \int_{B_r(x)} \frac{1}{(\log R)^{1+q/p'}} 2^n dy = 2^n \frac{\omega_n}{n} g_{p,q;R}(1) r^n.$$

(c) If $1/2 \ge r \ge |x|/2$, then $B_r(x) \subset B_{3r} \subset B_R$, and hence we have, using (5.9) with $\underline{R} = R/3$,

$$\begin{aligned} \mu_{g_{p,q;R}}(B_r(x)) &\le \int_{B_{3r}} \frac{1}{(\log(R/|y|))^{1+q/p'}} \frac{1}{|y|^n} dy = \omega_n \frac{p'}{q} \frac{1}{(\log(R/(3r)))^{q/p'}} \\ &\le \omega_n \frac{p'}{q} \left(\frac{\log R}{\log(R/3)}\right)^{q/p'} \frac{1}{A_{1,R}(r)^{q/p'}}. \end{aligned}$$

(d) If $r \ge 1/2$, then we have

$$\mu_{g_{p,q;R}}(B_r(x)) \le \int_{B_1} \frac{1}{(\log(R/|y|))^{1+q/p'}} \frac{1}{|y|^n} dy = \omega_n \frac{p'}{q} \frac{1}{(\log R)^{q/p'}}.$$

□

After all we have the following.

**Lemma 5.5** *For $1 < p < q < \infty$, $p < n$, $\tau_{p,q} \le 1/n$ and $R > 3$, it holds that*

$$\sup_{x \in \mathbf{R}^n, r > 0} \mu_{g_{p,q;R}}(B_r(x)) J_p[w_p](x,r)^{q/p'} < \infty. \tag{5.11}$$

**Proof:** (a) If $r \ge 1/2$, it follows from the assertion 1 of Lemma 5.3 and Lemma 5.4 that we have

$$\mu_{g_{p,q;R}}(B_r(x)) J_p[w_p](x,r)^{q/p'} \le c_{p,q;R} \frac{1}{\nu_p} \frac{1}{r^{\nu_p}} \le c_{p,q;R} \frac{2^{\nu_p}}{\nu_p}.$$

(b) For $0 < r \le \min\{1, |x|\}/2$, it follows from the assertion 3 of Lemma 5.3 and Lemma 5.4 that we have

$$\begin{aligned}
&\mu_{g_{p,q;R}}(B_r(x)) J_p[w_p](x,r)^{q/p'} \\
&\le c_{p,q;R}\, g_{p,q;R}(\min\{1,|x|\}) r^n \left( c_{p;R} \left( A_{1,R}(\min\{1,|x|\}) + \left( \frac{\min\{1,|x|\}}{r} \right)^{\nu_p} \right) \right)^{q/p'} \\
&= \frac{c_{p,q;R}\, c_{p;R}^{q/p'}}{A_{1,R}(\min\{1,|x|\})} \left( \frac{r}{\min\{1,|x|\}} \right)^{nq(1/n - \tau_{p,q})} \left( \left( \frac{r}{\min\{1,|x|\}} \right)^{\nu_p} + \frac{1}{A_{1,R}(\min\{1,|x|\})} \right)^{q/p'} \\
&\le \frac{c_{p,q;R}\, c_{p;R}^{q/p'}}{A_{1,R}(1)} \frac{1}{2^{nq(1/n - \tau_{p,q})}} \left( \frac{1}{2^{\nu_p}} + \frac{1}{A_{1,R}(1)} \right)^{q/p'}.
\end{aligned}$$

(c) Assume that $|x|/2 \le r \le 1/2$. First we deal with the case $|x| + r \le 1$. Then, from the assertion 2 of Lemma 5.3 and Lemma 5.4 we have

$$\begin{aligned}
&\mu_{g_{p,q;R}}(B_r(x)) J_p[w_p](x,r)^{q/p'} \le c_{p,q;R} \frac{1}{A_{1,R}(r)^{q/p'}} \left( c_p \left( 1 + \log \frac{1}{r} + \left( \frac{|x|}{r} \right)^{\nu_p} \right) \right)^{q/p'} \\
&\le c_{p,q;R}\, c_p^{q/p'} \left( \frac{1 + 2^{\nu_p} + \log(1/r)}{\log R + \log(1/r)} \right)^{q/p'} \le c_{p,q;R}\, c_p^{q/p'} \left( \max \left\{ 1, \frac{1 + 2^{\nu_p}}{\log R} \right\} \right)^{q/p'}.
\end{aligned}$$

If $|x| + r > 1$, then we have $r > 1/3$. Hence, from the assertion 1 of Lemma 5.3 and Lemma 5.4 we have

$$\begin{aligned}
&\mu_{g_{p,q;R}}(B_r(x)) J_p[w_p](x,r)^{q/p'} \le c_{p,q;R} \frac{1}{A_{1,R}(r)^{q/p'}} \left( \frac{1}{\nu_p} \frac{1}{r^{\nu_p}} \right)^{q/p'} \\
&\le c_{p,q;R} \frac{1}{A_{1,R}(1/2)^{q/p'}} \left( \frac{3^{\nu_p}}{\nu_p} \right)^{q/p'}.
\end{aligned}$$

□

In addition we use the next.

**Lemma 5.6** *For $u \in C_c^\infty(\mathbf{R}^n)$, it holds that*

$$u(x) = \frac{1}{\omega_n} \int_{\mathbf{R}^n} \frac{\nabla u(y) \cdot (x - y)}{|x - y|^n} dy \quad \textit{for } x \in \mathbf{R}^n. \tag{5.12}$$

*In particular*

$$|u(x)| \le \frac{1}{\omega_n} I_1 * [|\nabla u|](x) \quad \textit{for } x \in \mathbf{R}^n. \tag{5.13}$$

**Proof:** Noting that

$$u(x) = -\int_0^\infty \nabla u(x+t\omega)\cdot\omega\, dt \quad \text{for } \omega \in S^{n-1}, \tag{5.14}$$

we immediately have

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{\nabla u(y)\cdot(x-y)}{|x-y|^n} dy &= -\int_{\mathbf{R}^n} \frac{\nabla u(x+y)\cdot y}{|y|^n} dy = -\int_{S^{n-1}}\int_0^\infty \frac{\nabla u(x+t\omega)\cdot t\omega}{t^n} t^{n-1} dt dS(\omega) \\ &= -\int_{S^{n-1}}\int_0^\infty \nabla u(x+t\omega)\cdot\omega\, dt dS(\omega) = \int_{S^{n-1}} u(x) dS(\omega) = \omega_n u(x).\end{aligned}$$

□

Now we are in a position to establish Proposition 5.1.

**Proof of Proposition 5.1:** It follows from Lemma 5.2, Lemma 5.5 and Lemma 5.1 that there exists a positive number $C_{p,q;R} > 0$ such that we have

$$\|I_1 * f\|_{L^q(\mathbf{R}^n;\, g_{p,q;R})} \le C_{p,q;R} \|f\|_{L^p(\mathbf{R}^n;\, w_p)} \quad \text{for } f \in L^p(\mathbf{R}^n; w_p).$$

Then, from Lemma 5.6 we have

$$\begin{aligned}\|u\|_{L^q_{p;R}(B_1)} &= \|u\|_{L^q(\mathbf{R}^n;\, g_{p,q;R})} \le \frac{1}{\omega_n} \|I_1 * [|\nabla u|]\|_{L^q(\mathbf{R}^n;\, g_{p,q;R})} \le \frac{C_{p,q;R}}{\omega_n} \|\nabla u\|_{L^p(\mathbf{R}^n;\, w_p)} \\ &= \frac{C_{p,q;R}}{\omega_n} \|\nabla u\|_{L^p_1(B_1)} \quad \text{for } u \in C_{\mathrm{c}}^\infty(B_1 \setminus \{0\}).\end{aligned}$$

□

# 6 Continuity of the best constants on parameters

In this section we prove that the best constants $S^{p,q;\gamma}$ and $C^{p,q;R}$ are continuous on parameters with $p$ being arbitrarily fixed and we also establish some relating estimates. It is clear from the assertion 2 of Theorem 2.2 and the assertion 2 of Theorem 2.7 that the best constants in radial spaces $S^{p,q;\gamma}_{\mathrm{rad}}$ and $C^{p,q;R}_{\mathrm{rad}}$ are continuous functions of $q, \gamma, R$ as well.

## 6.1 The noncritical case ($\gamma \ne 0$)

First in the case $\gamma > 0$, we study the continuity of $S^{p,q;\gamma}$ on $q, \gamma$. Let us introduce the next transformation.

**Definition 6.1** *Let $1 < p < \infty$ and $\gamma > 0$. For $u : \mathbf{R}^n \to \mathbf{R}$, we set*

$$\hat{T}_\gamma v(x) = \frac{1}{|x|^\gamma} v(x) \quad \textit{for } x \in \mathbf{R}^n \setminus \{0\}$$

*and set*

$$\Phi^{p;\gamma}[v] = \int_{\mathbf{R}^n} \left| \nabla v(x) - \gamma v(x) \frac{x}{|x|^2} \right|^p I_p(x) dx.$$

Then, it follows from diect calculations and triangle inequalities that we have the next.

**Lemma 6.1** *For $1 < p \le q < \infty$, $\tau_{p,q} \le 1/n$, $\gamma, \overline{\gamma} > 0$, we have the followings:*

1. $\|\hat{T}_\gamma v\|^q_{L^q_\gamma(\mathbf{R}^n)} = \|v\|^q_{L^q_0(\mathbf{R}^n)}$, $\|\nabla[\hat{T}_\gamma v]\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = \Phi^{p;\gamma}[v]$ *for $v \in \hat{T}_\gamma^{-1}(W^{1,p}_{\gamma,0}(\mathbf{R}^n))$.*

*2.*

$$S^{p,q;\gamma} = \inf\Big\{ \frac{\Phi^{p;\gamma}[v]}{\|v\|^p_{L^q_0(\mathbf{R}^n)}} \;\Big|\; v \in \hat{T}_\gamma^{-1}(W^{1,p}_{\gamma,0}(\mathbf{R}^n))\setminus\{0\}\Big\}$$
$$= \inf\Big\{ \frac{\Phi^{p;\gamma}[v]}{\|v\|^p_{L^q_0(\mathbf{R}^n)}} \;\Big|\; v \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})\setminus\{0\}\Big\}.$$

*3.*

$$S^{p,q;\gamma}\|v\|^p_{L^q_0(\mathbf{R}^n)} \le \Phi^{p;\gamma}[v] \quad \text{for } v \in \hat{T}_\gamma^{-1}(W^{1,p}_{\gamma,0}(\mathbf{R}^n)).$$

*In particular*

$$\gamma^p\|v\|^p_{L^p_0(\mathbf{R}^n)} \le \Phi^{p;\gamma}[v] \quad \text{for } v \in \hat{T}_\gamma^{-1}(W^{1,p}_{\gamma,0}(\mathbf{R}^n)).$$

*4.* $|\Phi^{p;\gamma}[v]^{1/p} - \Phi^{p;\overline{\gamma}}[v]^{1/p}| \le |\gamma - \overline{\gamma}|\|v\|_{L^p_0(\mathbf{R}^n)}$ *for* $v \in \hat{T}_\gamma^{-1}(W^{1,p}_{\gamma,0}(\mathbf{R}^n)) \cap \hat{T}_{\overline{\gamma}}^{-1}(W^{1,p}_{\overline{\gamma},0}(\mathbf{R}^n))$.

Now let us state a crucial lemma.

**Lemma 6.2** *Let* $1 < p \le q < \infty$, $\tau_{p,q} \le 1/n$ *an* $\gamma > 0$. *Assume that* $\{q_j\}_{j=1}^\infty \subset (p,p^*)$ *satisfies*

$$q_j \to q \quad \text{as } j \to \infty.$$

*If* $\{v_j\}_{j=1}^\infty \subset C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})$ *and* $\{\Phi^{p;\gamma}[v_j]\}_{j=1}^\infty$ *is bounded, then it holds that*

$$\limsup_{j\to\infty}(\|v_j\|^{q_j}_{L^{q_j}_0(\mathbf{R}^n)} - \|v_j\|^q_{L^q_0(\mathbf{R}^n)}) \le 0.$$

**Proof:** For $p < \underline{q} < \overline{q} < \tilde{q} < p^*$, let us note that

$$0 \le t^{\underline{q}}\log\frac{1}{t} \le \frac{1}{e(\underline{q}-p)}t^p \quad \text{for } 0 < t \le 1, \quad 0 \le t^{\overline{q}}\log t \le \frac{1}{e(\tilde{q}-\overline{q})}t^{\tilde{q}} \quad \text{for } t \ge 1. \tag{6.1}$$

(a) When $p < q < p^*$ holds, we choose $\underline{q}$, $\overline{q}$ and $\tilde{q}$ such as $p < \underline{q} \le q_j \le \overline{q} < \tilde{q} < p^*$ for $j \ge 1$. Then it follows from Lemma 6.1 that we have

$$\begin{aligned}
&\|v_j\|^{q_j}_{L^{q_j}_0(\mathbf{R}^n)} - \|v_j\|^q_{L^q_0(\mathbf{R}^n)} = \int_{\mathbf{R}^n}(|v_j(x)|^{q_j} - |v_j(x)|^q)I_0(x)dx \\
&= \int_{\mathbf{R}^n}\Big((q_j-q)\int_0^1 |v_j(x)|^{\theta q_j + (1-\theta)q}\log|v_j(x)|d\theta\Big)I_0(x)dx \\
&\le |q_j - q|\Big(\int_{\{|v_j|\le 1\}}|v_j(x)|^{\underline{q}}\Big(\log\frac{1}{|v_j(x)|}\Big)I_0(x)dx + \int_{\{|v_j|\ge 1\}}|v_j(x)|^{\overline{q}}(\log|v_j(x)|)I_0(x)dx\Big) \\
&\le |q_j - q|\Big(\frac{1}{e(\underline{q}-p)}\int_{\{|v_j|\le 1\}}|v_j(x)|^p I_0(x)dx + \frac{1}{e(\tilde{q}-\overline{q})}\int_{\{|v_j|\ge 1\}}|v_j(x)|^{\tilde{q}}I_0(x)dx\Big) \\
&\le |q_j - q|\Big(\frac{1}{e(\underline{q}-p)}\|v_j\|^p_{L^p_0(\mathbf{R}^n)} + \frac{1}{e(\tilde{q}-\overline{q})}\|v_j\|^{\tilde{q}}_{L^{\tilde{q}}_0(\mathbf{R}^n)}\Big) \\
&\le |q_j - q|\Big(\frac{1}{e(\underline{q}-p)}\frac{1}{\gamma^p}\Phi^{p;\gamma}[v_j] + \frac{1}{e(\tilde{q}-\overline{q})}\Big(\frac{1}{S^{p,\tilde{q};\gamma}}\Phi^{p;\gamma}[v_j]\Big)^{\tilde{q}/p}\Big) \to 0 \quad \text{as } j \to \infty.
\end{aligned}$$

(b) When $q = p$ holds, we choose $\bar{q}$ and $\tilde{q}$ such as $p < q_j \le \bar{q} < \tilde{q} < p^*$ for $j \ge 1$. Then in a similar way as the argument in (a), we have

$$\begin{aligned}
&\|v_j\|^{q_j}_{L_0^{q_j}(\mathbf{R}^n)} - \|v_j\|^{p}_{L_0^{p}(\mathbf{R}^n)} \le \int_{\{|v_j|\ge 1\}} (|v_j(x)|^{q_j} - |v_j(x)|^p) I_0(x)\,dx \\
&= \int_{\{|v_j|\ge 1\}} \Big((q_j - p)\int_0^1 |v_j(x)|^{\theta q_j + (1-\theta)p} \log|v_j(x)|\,d\theta\Big) I_0(x)\,dx \\
&\le (q_j - p)\frac{1}{e(\tilde{q}-\bar{q})}\Big(\frac{1}{S^{p,\tilde{q};\gamma}}\Phi^{p;\gamma}[v_j]\Big)^{\tilde{q}/p} \to 0 \quad \text{as } j \to \infty
\end{aligned}$$

(c) When $q = p^* < \infty$ holds, we choose $\underline{q}$ such as $p < \underline{q} \le q_j < p^*$ for $j \ge 1$. Then in a similar way as the argument in (a), we have

$$\begin{aligned}
&\|v_j\|^{q_j}_{L_0^{q_j}(\mathbf{R}^n)} - \|v_j\|^{p^*}_{L_0^{p^*}(\mathbf{R}^n)} \le \int_{\{|v_j|\le 1\}} (|v_j(x)|^{q_j} - |v_j(x)|^{p^*}) I_0(x)\,dx \\
&= \int_{\{|v_j|\ge 1\}} \Big((p^* - q_j)\int_0^1 |v_j(x)|^{\theta q_j + (1-\theta)p^*} \log\frac{1}{|v_j(x)|}\,d\theta\Big) I_0(x)\,dx \\
&\le (p^* - q_j)\frac{1}{e(\underline{q}-p)}\frac{1}{\gamma^p}\Phi^{p;\gamma}[v_j] \to 0 \quad \text{as } j \to \infty.
\end{aligned}$$

□

Then we have the following that assures Theorem 2.3.

**Proposition 6.1** *Let* $1 < p \le q < \infty$, $\tau_{p,q} \le 1/n$ *and* $\gamma > 0$. *Assume that* $\{(q_j;\gamma_j)\}_{j=1}^\infty \subset (p,p^*)\times(0,\infty)$ *satisfies*

$$q_j \to q,\ \gamma_j \to \gamma \quad \textit{as } j \to \infty.$$

*Then, it holds that*

$$S^{p,q_j;\gamma_j} \to S^{p,q;\gamma} \quad \textit{as } j \to \infty.$$

**Proof:** (a) We begin with showing

$$\limsup_{j\to\infty} S^{p,q_j;\gamma_j} \le S^{p,q;\gamma}.$$

For $\varepsilon > 0$, it follows from the assertion 2 of Lemma 6.1 that there exists $v_\varepsilon \in C_c^\infty(\mathbf{R}^n\setminus\{0\})\setminus\{0\}$ such that

$$\frac{\Phi^{p;\gamma}[v_\varepsilon]}{\|v_\varepsilon\|^p_{L_0^q(\mathbf{R}^n)}} \le S^{p,q;\gamma} + \frac{\varepsilon}{2}.$$

By the Lebesgue convergence theorem we have

$$\|v_\varepsilon\|^{q_j}_{L_0^{q_j}(\mathbf{R}^n)} \to \|v_\varepsilon\|^{q}_{L_0^{q}(\mathbf{R}^n)},\ \Phi^{p;\gamma_j}[v_\varepsilon] \to \Phi^{p;\gamma}[v_\varepsilon] \quad \text{as } j \to \infty.$$

Hence for some $j_\varepsilon \in \mathbf{N}$, we have

$$\frac{\Phi^{p;\gamma_j}[v_\varepsilon]}{\|v_\varepsilon\|^p_{L_0^{q_j}(\mathbf{R}^n)}} - \frac{\Phi^{p;\gamma}[v_\varepsilon]}{\|v_\varepsilon\|^p_{L_0^q(\mathbf{R}^n)}} < \frac{\varepsilon}{2} \quad \text{for } j \ge j_\varepsilon.$$

We therefore have

$$S^{p,q_j;\gamma_j} \le \frac{\Phi^{p;\gamma_j}[v_\varepsilon]}{\|v_\varepsilon\|^p_{L_0^{q_j}(\mathbf{R}^n)}} \le \frac{\Phi^{p;\gamma}[v_\varepsilon]}{\|v_\varepsilon\|^p_{L_0^q(\mathbf{R}^n)}} + \frac{\varepsilon}{2} \le S^{p,q;\gamma} + \varepsilon \quad \text{for } j \ge j_\varepsilon.$$

(b) Secondly we show that

$$S^{p,q;\gamma} \le \liminf_{j\to\infty} S^{p,q_j;\gamma_j}.$$

By the assertion 2 of Lemma 6.1 there exists $\{v_j\}_{j=1}^\infty \subset C_c^\infty(\mathbf{R}^n\setminus\{0\})\setminus\{0\}$ such that we have

$$\Phi^{p;\gamma}[v_j] = 1,\ \frac{\Phi^{p;\gamma_j}[v_j]}{\|v_j\|^p_{L_0^{q_j}(\mathbf{R}^n)}} \le S^{p,q_j;\gamma_j} + \frac{1}{j} \quad \text{for } j \ge 1.$$

Then from the assertions 3 and 4 of Lemma 6.1 we have

$$\begin{aligned}\Phi^{p;\gamma_j}[v_j]^{1/p} &\ge \Phi^{p;\gamma}[v_j]^{1/p} - |\gamma_j-\gamma|\|v_j\|_{L_0^p(\mathbf{R}^n)} \ge \Phi^{p;\gamma}[v_j]^{1/p} - \frac{|\gamma_j-\gamma|}{\gamma}\Phi^{p;\gamma}[v_j]^{1/p}\\ &= 1-\frac{|\gamma_j-\gamma|}{\gamma} \quad \text{for } j\ge 1.\end{aligned}$$

Combining with (a), there exist $j_1 \in \mathbf{N}$ and $c > 0$ such that we have,

$$\|v_j\|^p_{L_0^{q_j}(\mathbf{R}^n)} \ge \frac{\Phi^{p;\gamma_j}[v_j]}{S^{p,q_j;\gamma_j}+1/j} \ge c \quad \text{for } j \ge j_1.$$

Letting $\varepsilon$ satisfy $0<\varepsilon<c$, it follows from Lemma 6.2 that there exists $j_\varepsilon \ge j_1$ such that

$$\|v_j\|^p_{L_0^{q_j}(\mathbf{R}^n)} \le \|v_j\|^p_{L_0^{q}(\mathbf{R}^n)} + \varepsilon \quad \text{for } j \ge j_\varepsilon.$$

Then from the assertions 3 and 4 of Lemma 6.1 we have

$$\begin{aligned}S^{p,q;\gamma} &\le \frac{\Phi^{p;\gamma}[v_j]}{\|v_j\|^p_{L_0^{q}(\mathbf{R}^n)}} \le \frac{1}{\|v_j\|^p_{L_0^{q_j}(\mathbf{R}^n)}-\varepsilon}(\Phi^{p;\gamma_j}[v_j]^{1/p} + |\gamma_j-\gamma|\|v_j\|_{L_0^p(\mathbf{R}^n)})^p\\ &= \frac{1}{1-\varepsilon/\|v_j\|^p_{L_0^{q_j}(\mathbf{R}^n)}}\left(\left(\frac{\Phi^{p;\gamma_j}[v_j]}{\|v_j\|^p_{L_0^{q_j}(\mathbf{R}^n)}}\right)^{1/p} + |\gamma_j-\gamma|\frac{\|v_j\|_{L_0^p(\mathbf{R}^n)}}{\|v_j\|_{L_0^{q_j}(\mathbf{R}^n)}}\right)^p\\ &\le \frac{1}{1-\varepsilon/c}\left(\left(S^{p,q_j;\gamma_j}+\frac{1}{j}\right)^{1/p} + \frac{|\gamma_j-\gamma|}{c^{1/p}\gamma}\right)^p \quad \text{for } j \ge j_\varepsilon,\end{aligned}$$

and this proves the assertion. □

## 6.2 The critical case ($\gamma = 0$)

In this subsection we study the continuity of $C^{p,q;R}$ on the parameters $q, R$. Let us introduce the next transformation.

**Definition 6.2** *Let $1<p<\infty$, $R>0$. For $u : B_1 \to \mathbf{R}$, we set*

$$\hat{T}_{p;R}v(x) = A_{1,R}(x)^{1/p'}v(x) \quad \textit{for } x \in B_1\setminus\{0\}$$

*and set*

$$\Psi^{p;R}[v] = \int_{B_1}\left|A_{1,R}(x)\nabla v(x) - \frac{1}{p'}v(x)\frac{x}{|x|^2}\right|^p \frac{I_p(x)}{A_{1,R}(x)}dx.$$

It follows from direct calculations together with triangle inequalities that we have

**Lemma 6.3** *For $1<p\le q<\infty$, $\tau_{p,q}\le 1/n$ and $R>1$ it holds that*

1. $\|\hat{T}_{p;R}v\|^q_{L^q_{p;R}(B_1)} = \|v\|^q_{L^q_{1;R}(B_1)}$, $\|\nabla[\hat{T}_{p;R}v]\|^p_{L^p_1(B_1)} = \Psi^{p;R}[v]$ *for* $v \in \hat{T}^{-1}_{p;R}(W^{1,p}_{0,0}(B_1))$.

2.

$$C^{p,q;R} = \inf\Big\{ \frac{\Psi^{p;R}[v]}{\|v\|^p_{L^q_{1;R}(B_1)}} \Bigm| v \in \hat{T}^{-1}_{p;R}(W^{1,p}_{0,0}(B_1))\backslash\{0\}\Big\}$$
$$= \inf\Big\{ \frac{\Psi^{p;R}[v]}{\|v\|^p_{L^q_{1;R}(B_1)}} \Bigm| v \in C^\infty_{\rm c}(B_1\backslash\{0\})\backslash\{0\}\Big\}.$$

3.

$$C^{p,q;R}\|v\|^p_{L^q_{1;R}(B_1)} \le \Psi^{p;R}[v] \quad \text{for } v \in \hat{T}^{-1}_{p;R}(W^{1,p}_{0,0}(B_1)).$$

*In particular*

$$\frac{1}{(p')^p}\|v\|^p_{L^p_{1;R}(B_1)} \le \Psi^{p;R}[v] \quad \text{for } v \in \hat{T}^{-1}_{p;R}(W^{1,p}_{0,0}(B_1)).$$

4.

$$\int_{B_1} |\nabla v(x)|^p A_{1,R}(x)^{p-1} I_p(x)dx \le 2^p \Psi^{p;R}[v] \quad \text{for } v \in \hat{T}^{-1}_{p;R}(W^{1,p}_{0,0}(B_1)).$$

Further we show

**Lemma 6.4** *For* $1 < p \le \bar{q} < \infty$, $\tau_{p,\bar{q}} \le 1/n$ *and* $\underline{R} > 1$, *there exists positive numbers* $c_{p;\underline{R}}, c_{p,\bar{q};\underline{R}} > 0$ *such that for* $p \le q \le \bar{q}$, $\underline{R} \le R \le \overline{R}$ *we have the followings:*

1. $\Psi^{p;R}[v]^{1/p} \le \Big(1 + \Big(\frac{\log \overline{R}}{\log R}\Big)^{1/p}\Big)\Psi^{p;\overline{R}}[v]^{1/p}$ *for* $v \in C^\infty_{\rm c}(B_1\backslash\{0\})$.
2. $|\Psi^{p;\overline{R}}[v]^{1/p} - \Psi^{p;R}[v]^{1/p}| \le c_{p;\underline{R}}(\overline{R} - R)\Psi^{p;\underline{R}}[v]$ *for* $v \in C^\infty_{\rm c}(B_1\backslash\{0\})$.
3. $|\|v\|^q_{L^q_{1;\overline{R}}(B_1)} - \|v\|^q_{L^q_{1;R}(B_1)}| \le c_{p,\bar{q};\underline{R}}(\overline{R} - R)\Psi^{p;\underline{R}}[v]^{q/p}$ *for* $v \in C^\infty_{\rm c}(B_1\backslash\{0\})$.

**Proof:** First we have

$$A_{1,R}(x) \le A_{1,\overline{R}}(x) \le \frac{\log \overline{R}}{\log R}A_{1,R}(x),$$
$$A_{1,\overline{R}}(x)^{1/p'} - A_{1,R}(x)^{1/p'} = \int_0^1 \frac{1}{p'}\frac{\overline{R}-R}{\theta\overline{R} + (1-\theta)R}\frac{1}{A_{1,\theta\overline{R}+(1-\theta)R}(x)^{1/p}}d\theta$$
$$\le \frac{\overline{R}-R}{p'\underline{R}}\frac{1}{A_{1,\underline{R}}(x)^{1/p}}.$$

In a similar way,

$$\frac{1}{A_{1,R}(x)^{1/p}} - \frac{1}{A_{1,\overline{R}}(x)^{1/p}} \le \frac{\overline{R}-R}{p\underline{R}}\frac{1}{A_{1,\underline{R}}(x)^{1+1/p}},$$
$$\frac{1}{A_{1,R}(x)} - \frac{1}{A_{1,\overline{R}}(x)} \le \frac{\overline{R}-R}{\underline{R}}\frac{1}{A_{1,\underline{R}}(x)^2} \quad \text{for } x \in B_1\backslash\{0\}.$$

1. From the assertion 3 of Lemma 6.3 we have

$$\Psi^{p;R}[v]^{1/p} = \Big\{\int_{B_1}\Big|\frac{A_{1,R}(x)}{A_{1,\overline{R}}(x)}\Big(A_{1,\overline{R}}(x)\nabla v(x) - \frac{1}{p'}v(x)\frac{x}{|x|^2}\Big)$$
$$- \frac{1}{p'}v(x)\Big(1 - \frac{A_{1,R}(x)}{A_{1,\overline{R}}(x)}\Big)\frac{x}{|x|^2}\Big|^p \frac{I_p(x)}{A_{1,R}(x)}dx\Big\}^{1/p}$$

$$\le \Big(\int_{B_1}\Big|A_{1,\overline{R}}(x)\nabla v(x)-\frac{1}{p'}v(x)\frac{x}{|x|^2}\Big|^p\frac{I_p(x)}{A_{1,\overline{R}}(x)}dx\Big)^{1/p}+\frac{1}{p'}\Big(\int_{B_1}|v(x)|^p\frac{I_0(x)}{A_{1,R}(x)}dx\Big)^{1/p}$$
$$\le \Psi^{p;\overline{R}}[v]^{1/p}+\frac{1}{p'}\Big(\frac{\log\overline{R}}{\log R}\Big)^{1/p}\|v\|_{L^p_{1;\overline{R}}(B_1)}\le\Big(1+\Big(\frac{\log\overline{R}}{\log R}\Big)^{1/p}\Big)\Psi^{p;\overline{R}}[v]^{1/p}.$$

2. From the assertions 3 and 4 of Lemma 6.3 we have

$$|\Psi^{p;\overline{R}}[v]^{1/p}-\Psi^{p;R}[v]^{1/p}|$$
$$\le\Big\{\int_{B_1}\Big|(A_{1,\overline{R}}(x)^{1/p'}-A_{1,R}(x)^{1/p'})\nabla v(x)$$
$$-\frac{1}{p'}\Big(\frac{1}{A_{1,\overline{R}}(x)^{1/p}}-\frac{1}{A_{1,R}(x)^{1/p}}\Big)v(x)\frac{x}{|x|^2}\Big|^pI_p(x)dx\Big\}^{1/p}$$
$$\le\Big(\int_{B_1}\Big(\frac{\overline{R}-R}{p'\underline{R}}\frac{1}{A_{1,\underline{R}}(x)^{1/p}}|\nabla v(x)|\Big)^pI_p(x)dx\Big)^{1/p}$$
$$+\frac{1}{p'}\Big(\int_{B_1}\Big(\frac{\overline{R}-R}{p\underline{R}}\frac{1}{A_{1,\underline{R}}(x)^{1+1/p}}|v(x)|\Big)^pI_0(x)dx\Big)^{1/p}$$
$$\le\frac{\overline{R}-R}{p'\underline{R}\log\underline{R}}\Big(\Big(\int_{B_1}|\nabla v(x)|^pA_{1,\underline{R}}(x)^{p-1}I_p(x)dx\Big)^{1/p}+\frac{1}{p}\Big(\int_{B_1}|v(x)|^p\frac{I_0(x)}{A_{1,\underline{R}}(x)}dx\Big)^{1/p}\Big)$$
$$\le\frac{\overline{R}-R}{p'\underline{R}\log\underline{R}}\Big(2+\frac{p'}{p}\Big)\Psi^{p;\underline{R}}[v]^{1/p}.$$

3. Using that $t^t\le\max\{1,\bar{t}^{\bar{t}}\}$ for $0<t\le\bar{t}$ and $\omega_n/(\bar{q}/q)'\le\omega_n/(p/q)'$, we have

$$\Big(\frac{\omega_n}{(\bar{q}/q)'}\Big)^{1/(\bar{q}/q)'}\le\max\Big\{1,\Big(\frac{\omega_n}{(p/q)'}\Big)^{1/(p/q)'}\Big\}.$$

Then by the Hölder inequality and the assertion 3 of Lemma 6.3, it holds that

$$|\|v\|^q_{L^q_{1;\overline{R}}(B_1)}-\|v\|^q_{L^q_{1;R}(B_1)}|=\int_{B_1}|v(x)|^q\Big(\frac{1}{A_{1,R}(x)}-\frac{1}{A_{1,\overline{R}}(x)}\Big)I_0(x)dx$$
$$\le\frac{\overline{R}-R}{\underline{R}}\int_{B_1}|v(x)|^q\frac{1}{A_{1,\underline{R}}(x)}\frac{I_0(x)}{A_{1,\underline{R}}(x)}dx$$
$$\le\frac{\overline{R}-R}{\underline{R}}\Big(\int_{B_1}(|v(x)|^q)^{\bar{q}/q}\frac{I_0(x)}{A_{1,\underline{R}}(x)}dx\Big)^{q/\bar{q}}\Big(\int_{B_1}\Big(\frac{1}{A_{1,\underline{R}}(x)}\Big)^{(\bar{q}/q)'}\frac{I_0(x)}{A_{1,\underline{R}}(x)}dx\Big)^{1/(\bar{q}/q)'}$$
$$=\frac{\overline{R}-R}{\underline{R}\log\underline{R}}\Big(\frac{\omega_n}{(\bar{q}/q)'}\Big)^{1/(\bar{q}/q)'}\|v\|^q_{L^{\bar{q}}_{1;\underline{R}}(B_1)}$$
$$\le\frac{\overline{R}-R}{\underline{R}\log\underline{R}}\max\Big\{1,\Big(\frac{\omega_n}{(p/q)'}\Big)^{1/(p/q)'}\Big\}\Big(\frac{1}{C^{p,\bar{q};\underline{R}}}\Psi^{p;\underline{R}}[v]\Big)^{q/p}.$$

□

In a quite similar way as the argument in Lemma 6.2 we can show the next.

**Lemma 6.5** *Let* $1<p\le q<\infty$, $\tau_{p,q}\le 1/n$ *and* $R>1$. *Assume that* $\{q_j\}_{j=1}^\infty\subset(p,p^*)$ *satisfies*

$$q_j\to q\quad as\ j\to\infty.$$

*If* $\{v_j\}_{j=1}^\infty\subset C_c^\infty(B_1\backslash\{0\})$ *and* $\{\Psi^{p;R}[v_j]\}_{j=1}^\infty$ *is bounded, then it holds that*

$$\limsup_{j\to\infty}(\|v_j\|^{q_j}_{L^{q_j}_{1;R}(B_1)}-\|v_j\|^q_{L^q_{1;R}(B_1)})\le 0.$$

By using these we have the following proposition that assures Theorem 2.8.

**Proposition 6.2** *Let* $1 < p \le q < \infty$, $\tau_{p,q} \le 1/n$ *and* $R > 1$. *Assume that* $\{(q_j\,;R_j)\}_{j=1}^{\infty} \subset (p,p^*)\times(1,\infty)$ *satisfies*

$$q_j \to q,\ R_j \to R \quad \text{as } j \to \infty.$$

*Then it holds that*

$$C^{p,q_j;R_j} \to C^{p,q;R} \quad \text{as } j \to \infty.$$

**Proof:** (a) In a similar way as the argument in Proposition 6.1;(a), we have

$$\limsup_{j\to\infty} C^{p,q_j;R_j} \le C^{p,q;R}.$$

(b) In the next we show that

$$C^{p,q;R} \le \liminf_{j\to\infty} C^{p,q_j;R_j}.$$

To this end, let us take $\bar{q}$ and $\underline{R}$ such that

$$p \le q_j \le \bar{q} \begin{cases} \le p^* & \text{if } p < n, \\ < \infty & \text{if } p \ge n, \end{cases} \quad 1 < \underline{R} \le R_j \quad \text{for } j \ge 1.$$

It follows from the assertion 2 of Lemma 6.3 that there exists $\{v_j\}_{j=1}^{\infty} \subset C_c^{\infty}(B_1\setminus\{0\})\setminus\{0\}$ such that

$$\Psi^{p;\underline{R}}[v_j] = 1,\ \frac{\Psi^{p;R_j}[v_j]}{\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)}} \le C^{p,q_j;R_j} + \frac{1}{j} \quad \text{for } j \ge 1.$$

Since $\underline{R} \le R$ holds, it follows from the assertion 1 of Lemma 6.4 that we have

$$1 = \Psi^{p;\underline{R}}[v_j] \le \left(1 + \left(\frac{\log R}{\log \underline{R}}\right)^{1/p}\right)^p \Psi^{p;R}[v_j] \quad \text{for } j \ge 1.$$

Using the assertions 2 and 3 of Lemma 6.4 we also have

$$\begin{aligned} &\Psi^{p;R}[v_j]^{1/p} \le \Psi^{p;R_j}[v_j]^{1/p} + c_{p;\underline{R}}|R_j - R|\Psi^{p;\underline{R}}[v_j]^{1/p} = \Psi^{p;R_j}[v_j]^{1/p} + c_{p;\underline{R}}|R_j - R|, \\ &\|v_j\|^{q_j}_{L^{q_j}_{1;R_j}(B_1)} \le \|v_j\|^{q_j}_{L^{q_j}_{1;R}(B_1)} + c_{p,\bar{q};\underline{R}}|R_j - R| \quad \text{for } j \ge 1. \end{aligned}$$

Combining with (a), there exist $j_1 \in \mathbf{N}$ and $c > 0$ such that we have

$$\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)} \ge \frac{\Psi^{p;R_j}[v_j]}{C^{p,q_j;R_j} + 1/j} \ge c,\ |R_j - R| \le \frac{c^{q_j/p}}{c_{p,\bar{q};\underline{R}}} \quad \text{for } j \ge j_1.$$

Now let $\varepsilon$ satisfy $0 < \varepsilon < c$. Then it follows from Lemma 6.5 that there exists $j_\varepsilon \ge j_1$ such that we have

$$\|v_j\|^p_{L^{q_j}_{1;R}(B_1)} \le \|v_j\|^p_{L^{q}_{1;R}(B_1)} + \varepsilon \quad \text{for } j \ge j_\varepsilon.$$

Then we see

$$
\begin{aligned}
&C^{p,q;R}\Big(\Big(1-\frac{c_{p,\overline{q};\underline{R}}}{c^{q_j/p}}|R_j-R|\Big)^{p/q_j}-\frac{\varepsilon}{c}\Big)\\
&\le C^{p,q;R}\Big(\Big(1-\frac{c_{p,\overline{q};\underline{R}}}{\|v_j\|^{q_j}_{L^{q_j}_{1;R_j}(B_1)}}|R_j-R|\Big)^{p/q_j}-\frac{\varepsilon}{\|v_j\|^{p}_{L^{q_j}_{1;R_j}(B_1)}}\Big)\\
&=\frac{C^{p,q;R}}{\|v_j\|^{p}_{L^{q_j}_{1;R_j}(B_1)}}((\|v_j\|^{q_j}_{L^{q_j}_{1;R_j}(B_1)}-c_{p,\overline{q};\underline{R}}|R_j-R|)^{p/q_j}-\varepsilon)\le \frac{C^{p,q;R}}{\|v_j\|^{p}_{L^{q_j}_{1;R_j}(B_1)}}(\|v_j\|^{p}_{L^{q_j}_{1;R}(B_1)}-\varepsilon)\\
&\le \frac{C^{p,q;R}\|v_j\|^{p}_{L^{q}_{1;R}(B_1)}}{\|v_j\|^{p}_{L^{q_j}_{1;R_j}(B_1)}}\le \frac{\Psi^{p;R}[v_j]}{\|v_j\|^{p}_{L^{q_j}_{1;R_j}(B_1)}}\le \frac{1}{\|v_j\|^{p}_{L^{q_j}_{1;R_j}(B_1)}}(\Psi^{p;R_j}[v_j]^{1/p}+c_{p;\underline{R}}|R_j-R|)^p\\
&=\frac{\Psi^{p;R_j}[v_j]}{\|v_j\|^{p}_{L^{q_j}_{1;R_j}(B_1)}}\Big(1+\frac{c_{p;\underline{R}}}{\Psi^{p;R_j}[v_j]^{1/p}}|R_j-R|\Big)^p\\
&\le \Big(C^{p,q_j;R_j}+\frac{1}{j}\Big)\Big(1+c_{p;\underline{R}}\Big(1+\Big(\frac{\log R_j}{\log \underline{R}}\Big)^{1/p}\Big)|R_j-R|\Big)^p \quad \text{for } j\ge j_\varepsilon,
\end{aligned}
$$

and the assertion is thus established. □

## 6.3 Some estimates for the best constants

In this subsection we establish the assertions 5, 6 and 7 of Theorem 2.2. First the assertion 7 of Theorem 2.2 follows from the next proposition.

**Proposition 6.3** *Assume that* $1<p\le q\le \overline{q}<\infty$ *and* $\tau_{p,\overline{q}}\le 1/n$, *then we have*

$$S^{p,q;\gamma}\ge (\gamma^{p\tau_{q,\overline{q}}}(S^{p,\overline{q};\gamma})^{\tau_{p,q}})^{1/\tau_{p,\overline{q}}} \quad \textit{for } \gamma>0.$$

For the proof we employ the lemma below.

**Lemma 6.6** *Let* $1\le p\le q\le \overline{q}<\infty$, $\gamma>0$ *and let* $\Omega$ *be a domain of* $\mathbf{R}^n$. *Then we have*

$$\|u\|^{\tau_{p,\overline{q}}}_{L^q_\gamma(\Omega)}\le \|u\|^{\tau_{q,\overline{q}}}_{L^p_\gamma(\Omega)}\|u\|^{\tau_{p,q}}_{L^{\overline{q}}_\gamma(\Omega)} \quad \textit{for } u\in L^p_\gamma(\Omega)\cap L^{\overline{q}}_\gamma(\Omega).$$

**Proof:** Noting that $q\tau_{q,\overline{q}}/(p\tau_{p,\overline{q}})+q\tau_{p,q}/(\overline{q}\tau_{p,\overline{q}})=1$, we have

$$\|u\|^q_{L^q_\gamma(\Omega)}=\int_\Omega (|u(x)||x|^\gamma)^{q\tau_{q,\overline{q}}/\tau_{p,\overline{q}}}(|u(x)||x|^\gamma)^{q\tau_{p,q}/\tau_{p,\overline{q}}}I_0(x)dx.$$

Then the assertion easily follows from this by the aid of the Hölder inequality. □

**Proof of Proposition 6.3 :** For $\varepsilon>0$, there exists a $u_\varepsilon\in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})\setminus\{0\}$ such that we have

$$\|u_\varepsilon\|^q_{L^q_\gamma(\mathbf{R}^n)}=1,\ S^{p,q;\gamma}\le \|\nabla u_\varepsilon\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\le S^{p,q;\gamma}+\varepsilon.$$

Then, by Lemma 6.6 and Theorem 2.1 we have

$$
\begin{aligned}
1&=\|u_\varepsilon\|^{p\tau_{p,\overline{q}}}_{L^q_\gamma(\mathbf{R}^n)}\le \|u_\varepsilon\|^{p\tau_{q,\overline{q}}}_{L^p_\gamma(\mathbf{R}^n)}\|u_\varepsilon\|^{p\tau_{p,q}}_{L^{\overline{q}}_\gamma(\mathbf{R}^n)}\le \Big(\frac{1}{\gamma^p}\|\nabla u_\varepsilon\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\Big)^{\tau_{q,\overline{q}}}\Big(\frac{1}{S^{p,\overline{q};\gamma}}\|\nabla u_\varepsilon\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\Big)^{\tau_{p,q}}\\
&\le \frac{1}{\gamma^{p\tau_{p,\overline{q}}}(S^{p,\overline{q};\gamma})^{\tau_{p,q}}}(S^{p,q;\gamma}+\varepsilon)^{\tau_{p,\overline{q}}},
\end{aligned}
$$

and this proves the assertion. □

In order to prove the assertions 5 and 6 of Theorem 2.2, we establish the next proposition. Noting the assertion 3 of Theorem 2.2, the assertion 5 and the assertion 6 of Theorem 2.2 follow from the assertions 1, 2 and from the assertions 2, 3 of the next proposition respectively.

**Proposition 6.4** *Let $n \geq 2$, $1 < p < n$ and $q = p^*$. Then we have the followings:*

1.
$$S^{p,p^*;\gamma_{p,p^*}} \leq \Big(2 - \frac{\gamma_{p,p^*}}{\gamma}\Big)^p S^{p,p^*;\gamma} \quad \text{for } \gamma \geq \gamma_{p,p^*}.$$

2.
$$S^{p,p^*;\gamma} \leq S^{p,p^*;\gamma_{p,p^*}} \qquad \text{for } \gamma \geq \gamma_{p,p^*}.$$

3. *When $p = 2$,*
$$S^{2,2^*;\gamma} \leq S^{2,2^*;\overline{\gamma}} \qquad \text{for } 0 < \gamma \leq \overline{\gamma}.$$

**Proof:** 1. For $\varepsilon > 0$, there exists a $u_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}$ such that we have

$$\|u_\varepsilon\|^{p^*}_{L^{p^*}_\gamma(\mathbf{R}^n)} = 1, \; S^{p,p^*;\gamma} \leq \|\nabla u_\varepsilon\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \leq S^{p,p^*;\gamma} + \varepsilon.$$

Since $n - \gamma_{p,p^*} p^* = 0$, it holds that

$$\|\hat{T}_{\gamma_{p,p^*}-\gamma} u_\varepsilon\|^{p^*}_{L^{p^*}(\mathbf{R}^n)} = \|\hat{T}_{\gamma_{p,p^*}-\gamma} u_\varepsilon\|^{p^*}_{L^{p^*}_{\gamma_{p,p^*}}(\mathbf{R}^n)} = \|u_\varepsilon\|^{p^*}_{L^{p^*}_\gamma(\mathbf{R}^n)} = 1.$$

Noting that $n - (1 + \gamma_{p,p^*})p = 0$ and $n - p(1 + \gamma) = (\gamma_{p,p^*} - \gamma)p$, by the Sobolev inequality and the Hardy-Sobolev inequality we have

$$\begin{aligned}
(S^{p,p^*;\gamma_{p,p^*}})^{1/p} &\leq \|\nabla[\hat{T}_{\gamma_{p,p^*}-\gamma} u_\varepsilon]\|_{L^p_{1+\gamma_{p,p^*}}(\mathbf{R}^n)} = \|\nabla[\hat{T}_{\gamma_{p,p^*}-\gamma} u_\varepsilon]\|_{L^p(\mathbf{R}^n)} \\
&= \Big(\int_{\mathbf{R}^n} \Big|\nabla u_\varepsilon(x) + (\gamma - \gamma_{p,p^*}) u_\varepsilon(x) \frac{x}{|x|^2}\Big|^p I_{p(1+\gamma)}(x)\, dx\Big)^{1/p} \\
&\leq \|\nabla u_\varepsilon\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + (\gamma - \gamma_{p,p^*}) \|u_\varepsilon\|_{L^p_\gamma(\mathbf{R}^n)} \\
&\leq \|\nabla u_\varepsilon\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + (\gamma - \gamma_{p,p^*}) \frac{1}{\gamma} \|\nabla u_\varepsilon\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \leq \Big(2 - \frac{\gamma_{p,p^*}}{\gamma}\Big)(S^{p,p^*;\gamma} + \varepsilon)^{1/p}.
\end{aligned}$$

2. Let $u \in C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}$ and $e_1 = (1, 0, \ldots, 0) \in \mathbf{R}^n$. Since $p^*\gamma \geq n$, $p(1+\gamma) \geq n$ hold, we have

$$\varepsilon^{p^*\gamma - n} \Big\|u\Big(\cdot - \frac{e_1}{\varepsilon}\Big)\Big\|^{p^*}_{L^{p^*}_\gamma(\mathbf{R}^n)} = \int_{\mathbf{R}^n} |u(x)|^{p^*} I_{p^*\gamma}(\varepsilon x + e_1)\, dx \to \int_{\mathbf{R}^n} |u(x)|^{p^*} dx = \|u\|^{p^*}_{L^{p^*}_{\gamma_{p,p^*}}(\mathbf{R}^n)},$$

$$\begin{aligned}
\varepsilon^{p(1+\gamma)-n} \Big\|\nabla\Big[u\Big(\cdot - \frac{e_1}{\varepsilon}\Big)\Big]\Big\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(\varepsilon x + e_1)\, dx \\
&\to \int_{\mathbf{R}^n} |\nabla u(x)|^p dx = \|\nabla u\|^p_{L^p_{1+\gamma_{p,p^*}}(\mathbf{R}^n)} \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
S^{p,p^*;\gamma} \leq E^{p,p^*;\gamma}\Big[u\Big(\cdot - \frac{e_1}{\varepsilon}\Big)\Big] &= \frac{\varepsilon^{p(1+\gamma)-n} \Big\|\nabla\Big[u\Big(\cdot - \frac{e_1}{\varepsilon}\Big)\Big]\Big\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}}{\Big(\varepsilon^{p^*\gamma-n} \Big\|u\Big(\cdot - \frac{e_1}{\varepsilon}\Big)\Big\|^{p^*}_{L^{p^*}_\gamma(\mathbf{R}^n)}\Big)^{p/p^*}} \\
&\to E^{p,p^*;\gamma_{p,p^*}}[u] \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

and this proves the assertion.

3. (a) For $u \in C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}$, we set

$$\zeta[u](\gamma) = E^{2,2^*;\gamma}[u I_{n-\gamma}] \quad \text{for } \gamma > 0.$$

If we note that

$$2\int_{\mathbf{R}^n} u(x)(x\cdot\nabla u(x))I_0(x)\,dx = \int_{S^{n-1}}\int_0^\infty \frac{\partial}{\partial r}[u^2](r\omega)\,dr\,dS(\omega) = 0,$$

then we obtain

$$\begin{aligned}\zeta[u](\gamma) &= \frac{\|\nabla[uI_{n-\gamma}]\|^2_{L^2_{1+\gamma}(\mathbf{R}^n)}}{\|uI_{n-\gamma}\|^2_{L^{2^*}_\gamma(\mathbf{R}^n)}}\\ &= \frac{1}{\|u\|^2_{L^{2^*}_0(\mathbf{R}^n)}}\int_{\mathbf{R}^n}(\gamma^2u(x)^2 - 2\gamma u(x)(x\cdot\nabla u(x)) + |x|^2|\nabla u(x)|^2)I_0(x)\,dx\\ &= \frac{1}{\|u\|^2_{L^{2^*}_0(\mathbf{R}^n)}}(\gamma^2\|u\|^2_{L^2_0(\mathbf{R}^n)} + \|\nabla u\|^2_{L^2_1(\mathbf{R}^n)}) \quad \text{for } \gamma > 0,\end{aligned}$$

and so, we see that $\zeta[u]$ is non-decreasing with respect to $\gamma$.

(b) For $0 < \gamma \le \overline{\gamma}$, it follows from (a) that we have

$$S^{2,2^*;\gamma} \le E^{2,2^*;\gamma}\Big[\frac{u}{I_{n-\overline{\gamma}}}I_{n-\gamma}\Big] = \zeta\Big[\frac{u}{I_{n-\overline{\gamma}}}\Big](\gamma) \le \zeta\Big[\frac{u}{I_{n-\overline{\gamma}}}\Big](\overline{\gamma}) = E^{2,2^*;\overline{\gamma}}[u] \quad \text{for } u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})\setminus\{0\}.$$

This clearly proves the assertion. □

# 7 Existence of minimizers for the best constants

In this subsection we prove existence of minimizers for $S^{p,q;\gamma}$ by the effective use of the so-called concentration compactness principle when $p < q < p^*$ and $\gamma > 0$. We begin with preparing some notations.

**Definition 7.1** *(i) Let $\psi_1, \rho_1 \in C^\infty_{\mathrm{c}}(\mathbf{R}^n)_{\mathrm{rad}}$ and $\overline{\rho}_1 \in C^\infty(\mathbf{R}^n)_{\mathrm{rad}}$ satisfy*

$$0 \le \psi_1 \le 1,\ \rho_1 \ge 0,\ \overline{\rho}_1 > 0 \quad \text{on } \mathbf{R}^n, \quad \psi_1 = 1 \quad \text{on } \overline{B_{1/2}}, \quad \psi_1 = \rho_1 = 0 \quad \text{on } \mathbf{R}^n\setminus B_1,$$

$$\psi_1' = \frac{\partial\psi_1}{\partial r} \le 0 \quad \text{on } \mathbf{R}^n\setminus\{0\}, \quad \|\nabla\psi_1\|_{L^\infty(\mathbf{R}^n)} \le 3,\ \|\rho_1\|_{L^1(\mathbf{R}^n)} = \|\overline{\rho}_1\|_{L^1(\mathbf{R}^n)} = 1.$$

*(ii) For $\varepsilon > 0$*

$$\psi_\varepsilon(x) = \psi_\varepsilon(|x|) = \psi_1\Big(\frac{x}{\varepsilon}\Big),\ \tilde{\psi}_\varepsilon(x) = \tilde{\psi}_\varepsilon(|x|) = -[\psi_1']\Big(\frac{|x|}{\varepsilon}\Big) = \Big|[\psi_1']\Big(\frac{|x|}{\varepsilon}\Big)\Big|,$$

$$\rho_\varepsilon(x) = \rho_\varepsilon(|x|) = \frac{1}{\varepsilon^n}\rho_1\Big(\frac{x}{\varepsilon}\Big),\ \overline{\rho}_\varepsilon(x) = \overline{\rho}_\varepsilon(|x|) = \frac{1}{\varepsilon^n}\overline{\rho}_1\Big(\frac{x}{\varepsilon}\Big) \quad \text{for } x \in \mathbf{R}^n.$$

## 7.1 Preliminaries

In this subsection we prepare some well-known properties in the theory of concentration compactness due to P. L. Lions, which are useful in the proof of existence of minimizer of the best constant $S^{p,q;\gamma}$. We admit the next lemma without a proof, see §1.3 of [Li1] for the detail.

**Lemma 7.1** *Assume that $\{Q_j\}_{j=1}^\infty$ is a sequence of uniformly bounded and non-decreasing functions on $[1,\infty)$. Then, there exist a subsequence $\{Q_{j_k}\}_{k=1}^\infty$ and a non-decreasing function $Q$ on $[1,\infty)$ such that we have*

$$Q_{j_k}(t) \to Q(t) \quad \text{as } k \to \infty \quad \text{for } t > 1.$$

It follows from the Hölder inequality that we have

**Lemma 7.2** *For $1 < p \le q < \infty$, $\gamma > 0$ and $R > 0$, we have*

$$\|u\|_{L^p_\gamma(B_{2R}\setminus\overline{B_R})} \le (\omega_n \log 2)^{\tau_{p,q}} \|u\|_{L^q_\gamma(B_{2R}\setminus\overline{B_R})} \quad \text{for } u \in L^q_\gamma(B_{2R}\setminus\overline{B_R}).$$

The proof is omitted. It follows from the Rellich lemma that we have

**Lemma 7.3** *For $1 < p \le q < \infty$, $\tau_{p,q} < 1/n$ and $\gamma > 0$, assume that $\Omega$ is a bounded domain of $\mathbf{R}^n$ and $\partial\Omega$ is smooth. Then, the imbedding $W^{1,p}_{\gamma,0}(\Omega) \subset L^q_{\gamma+1-n\tau_{p,q}}(\Omega)$ is compact.*

**Proof:** For $u \in C^\infty_{\mathrm{c}}(\Omega\setminus\{0\})$, we have

$$\nabla[uI_{1+\gamma+n/p'}](x) = I_{p(1+\gamma)}(x)^{1/p}\nabla u(x) + \Big(1+\gamma-\frac{n}{p}\Big)I_{p\gamma}(x)^{1/p}u(x)\frac{x}{|x|} \quad \text{for } x \in \Omega.$$

Hence we have

$$\begin{aligned}\|\nabla[uI_{1+\gamma+n/p'}]\|_{L^p(\Omega)} &\le \|I^{1/p}_{p(1+\gamma)}\nabla u\|_{L^p(\Omega)} + \Big|1+\gamma-\frac{n}{p}\Big|\|I^{1/p}_{p\gamma}u\|_{L^p(\Omega)} \\ &= \|\nabla u\|_{L^p_{1+\gamma}(\Omega)} + \Big|1+\gamma-\frac{n}{p}\Big|\|u\|_{L^q_\gamma(\Omega)} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\Omega).\end{aligned}$$

Therefore, if $\{u_j\}_{j=1}^\infty$ is bounded in $W^{1,p}_{\gamma,0}(\Omega)$, then $\{u_jI_{1+\gamma+n/p'}\}_{j=1}^\infty$ should be bounded in $W^{1,p}_0(\Omega)$ ( a classical Sobolev space without a weight), and by the Rellich lemma $\{u_jI_{1+\gamma+n/p'}\}_{j=1}^\infty$ has a subseuence $\{u_{j_k}I_{1+\gamma+n/p'}\}_{k=1}^\infty$ which converges in $L^q(\Omega)$. Noting that $n-(1+\gamma+n/p') = (n-(1+\gamma)p)/p$ and $(n-(1+\gamma)p)q/p = n-q(1+\gamma-n\tau_{p,q})$, we get $\{u_{j_k}\}_{k=1}^\infty$ converges in $L^q_{\gamma+1-n\tau_{p,q}}(\Omega)$ as well. □

Let us recall a sharp Fatou's lemma, which is essentially due to H. Brézis and E. Lieb [BL]. ( See also [LL] )

**Lemma 7.4** *For $1 < q < \infty$ and $\gamma > 0$, assume that $\{u_j\}_{j=1}^\infty$ is bounded in $L^q_\gamma(\mathbf{R}^n)$ and assume that*

$$u_j \to u \quad \text{a.e. on } \mathbf{R}^n \quad \text{as } j \to \infty.$$

*Then, we have $u \in L^q_\gamma(\mathbf{R}^n)$ and*

$$\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u_j-u\|^q_{L^q_\gamma(\mathbf{R}^n)} \to \|u\|^q_{L^q_\gamma(\mathbf{R}^n)} \quad \text{as } j \to \infty.$$

**Proof:** For $0 < \varepsilon < 1$, there exists a positive number $\hat{c}_{q;\varepsilon} > 0$ such that we have

$$||s+t|^q - |s|^q - |t|^q| \le \varepsilon|s|^q + \hat{c}_{q;\varepsilon}|t|^q \quad \text{for } s,t \in \mathbf{R}. \tag{7.1}$$

Since $|u_j|I_{q\gamma} \to |u|I_{q\gamma}$ a.e. on $\mathbf{R}^n$ as $j \to \infty$ by the hypothesis, it follows from Fatou's lemma that

$$\|u\|^q_{L^q_\gamma(\mathbf{R}^n)} \le \liminf_{j\to\infty}\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} \le \sup_{j\ge1}\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} < \infty,$$

hence we see $u \in L^q_\gamma(\mathbf{R}^n)$. Then we have $|u|^qI_{q\gamma} \in L^1(\mathbf{R}^n)$ and

$$(||u_j|^q - |u_j-u|^q - |u|^q| - \varepsilon|u_j-u|^q)_+ I_{q\gamma} \le \hat{c}_{q;\varepsilon}|u|^qI_{q\gamma} \quad \text{a.e. on } \mathbf{R}^n \quad \text{for } j \ge 1.$$

Using Lebesgue's convergence theorem, we have

$$\int_{\mathbf{R}^n}[(||u_j|^q - |u_j-u|^q - |u|^q| - \varepsilon|u_j-u|^q)_+I_{q\gamma}](x)dx \to 0 \quad \text{as } j \to \infty.$$

After all we have

$$
\begin{aligned}
&|\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u_j-u\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u\|^q_{L^q_\gamma(\mathbf{R}^n)}| \le \int_{\mathbf{R}^n} [||u_j|^q - |u_j-u|^q - |u|^q| I_{q\gamma}](x)dx \\
&= \int_{\mathbf{R}^n} [(||u_j|^q - |u_j-u|^q - |u|^q| - \varepsilon|u_j-u|^q) I_{q\gamma}](x)dx + \varepsilon\|u_j-u\|^q_{L^q_\gamma(\mathbf{R}^n)} \\
&\le \int_{\mathbf{R}^n} [(||u_j|^q - |u_j-u|^q - |u|^q| - \varepsilon|u_j-u|^q)_+ I_{q\gamma}](x)dx + \varepsilon\Big(2\sup_{j\ge1}\|u_j\|_{L^q_\gamma(\mathbf{R}^n)}\Big)^q \\
&\to \varepsilon\Big(2\sup_{j\ge1}\|u_j\|_{L^q_\gamma(\mathbf{R}^n)}\Big)^q \quad \text{as } j\to\infty.
\end{aligned}
$$

Thus the assertion is established. □

**Lemma 7.5** *For $1 < p \le q < \infty$, $\tau_{p,q} < 1/n$ and $\gamma > 0$, there exists a positive number $\bar{c}_{p,q;\gamma} > 0$ such that we have*

$$
\|u\|^p_{L^q_\gamma(B_{|y|/4}(y))} \le \bar{c}_{p,q;\gamma}(\|\nabla u\|^p_{L^p_{1+\gamma}(B_{|y|/2}(y))} + \|u\|^p_{L^p_\gamma(B_{|y|/2}(y))}) \quad \textit{for } y\in\mathbf{R}^n\setminus\{0\},\ u\in W^{1,p}_{\gamma,0}(\mathbf{R}^n).
$$

**Proof:** For $y\in\mathbf{R}^n\setminus\{0\}$ and $u\in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})$ we set

$$
K_y u(x) = \psi_{|y|/2}(x-y)u(x) \quad \text{for } x\in\mathbf{R}^n. \tag{7.2}
$$

By differentiation we have

$$
\nabla[K_y u](x) = \psi_{|y|/2}(x-y)\nabla u(x) - \frac{2}{|y|}\tilde{\psi}_{|y|/2}(x-y)u(x)\frac{x-y}{|x-y|} \quad \text{for } x\in\mathbf{R}^n.
$$

Since $\mathrm{supp}(K_y u)\subset\overline{B_{|y|/2}(y)}$ and $|x|\le 3|y|/2$ for $x\in B_{|y|/2}(y)$, it holds that

$$
|\nabla[K_y u](x)| \le \Big(|\nabla u(x)| + \frac{9}{|x|}|u(x)|\Big)\chi_{B_{|y|/2}(y)}(x) \quad \text{for } x\in\mathbf{R}^n\setminus\{0\}.
$$

Noting that

$$
|u(x)|\chi_{B_{|y|/4}(y)}(x) \le |K_y u(x)| \quad \text{for } x\in\mathbf{R}^n,
$$

we have

$$
\begin{aligned}
\|u\|^p_{L^q_\gamma(B_{|y|/4}(y))} &= \|u\chi_{B_{|y|/4}(y)}\|^p_{L^q_\gamma(\mathbf{R}^n)} \le \|K_y u\|^p_{L^q_\gamma(\mathbf{R}^n)} \\
&\le \frac{1}{S^{p,q;\gamma}}\|\nabla[K_y u]\|^p_{L^q_{1+\gamma}(\mathbf{R}^n)} \le \frac{1}{S^{p,q;\gamma}}\Big\||\nabla u| + 9\frac{|u|}{|\cdot|}\Big\|^p_{L^q_{1+\gamma}(B_{|y|/2}(y))} \\
&\le \frac{1}{S^{p,q;\gamma}}(\|\nabla u\|_{L^p_{1+\gamma}(B_{|y|/2}(y))} + 9\|u\|_{L^q_\gamma(B_{|y|/2}(y))})^p \quad \text{for } u\in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\}),
\end{aligned}
$$

and hence the assertion follows. □

**Lemma 7.6** *Let us take $\{z^k\}_{k=1}^\infty\subset\mathbf{R}^n\setminus\{0\}$ and $L\in\mathbf{N}$ such that*

$$
\bigcup_{k=1}^\infty B_{|z^k|/4}(z^k) = \mathbf{R}^n\setminus\{0\},\ L = \sup_{x\in\mathbf{R}^n\setminus\{0\}} \sharp\{k\in\mathbf{N}\mid x\in B_{|z^k|/2}(z^k)\} < \infty.
$$

*Then, for $1<q<\infty$ and $\gamma>0$ we have*

$$
\|u\|^q_{L^q_\gamma(\mathbf{R}^n)} \le \sum_{k=1}^\infty \|u\|^q_{L^q_\gamma(B_{|z^k|/4}(z^k))} \le \sum_{k=1}^\infty \|u\|^q_{L^q_\gamma(B_{|z^k|/2}(z^k))} \le L\|u\|^q_{L^q_\gamma(\mathbf{R}^n)} \quad \textit{for } u\in L^q_\gamma(\mathbf{R}^n).
$$

**Proof :** By the assumption on $\{z^k\}_{k=1}^\infty$ and $L$, it holds that

$$1 \le \sum_{k=1}^\infty \chi_{B_{|z^k|/4}(z^k)}(x) \le \sum_{k=1}^\infty \chi_{B_{|z^k|/2}(z^k)}(x) \le L \quad \text{for } x \in \mathbf{R}^n\backslash\{0\},$$

and this proves the assertion. $\square$

Now we verify the following.

**Lemma 7.7** *Assume that* $1 < p \le q < \infty$, $p \le \tilde{q} < \infty$, $\tau_{p,q} < 1/n$, $\tau_{p,\tilde{q}} < 1/n$ *and* $\gamma > 0$. *Then, there exists positive numbers* $\theta_{p,q,\tilde{q}} \in (0,1)$ *and* $\bar{c}_{p,q,\tilde{q};\gamma} > 0$ *such that we have*

$$\|u\|_{L^q_\gamma(\mathbf{R}^n)} \le \bar{c}_{p,q,\tilde{q};\gamma} \|\nabla u\|^{\theta_{p,q,\tilde{q}}}_{L^p_{1+\gamma}(\mathbf{R}^n)} \Big( \sup_{y\in\mathbf{R}^n\backslash\{0\}} \|u\|_{L^{\tilde{q}}_\gamma(B_{|y|/4}(y))}\Big)^{1-\theta_{p,q,\tilde{q}}} \quad \textit{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n).$$

**Proof:** (i) Assume that $\tilde{q} < q$. Noting that $1/p - (q/p-1)/\tilde{q} - 1/q = (1/p - 1/q)(1-q/\tilde{q}) < 0$ we choose $\overline{q} = \overline{q}_{p,q,\tilde{q}}$ such that

$$\max\Big\{\frac{1}{p^*}, \frac{1}{p} - \Big(\frac{q}{p} - 1\Big)\frac{1}{\tilde{q}}\Big\} < \frac{1}{\overline{q}} = \frac{1}{\overline{q}_{p,q,\tilde{q}}} < \frac{1}{q},$$

and then we put

$$\theta = \theta_{p,q,\tilde{q}} = \frac{1/\tilde{q} - 1/q}{1/\tilde{q} - 1/\overline{q}} = \frac{1/\tilde{q} - 1/q}{1/\tilde{q} - 1/\overline{q}_{p,q,\tilde{q}}}.$$

Then, noting that $\tilde{q} < q < \overline{q}$, $q\theta > p$ and $\tau_{p,\overline{q}} < 1/n$, it follows from Lemmas 6.6, 7.5 and 7.6 that we have

$$\begin{aligned}
&\|u\|^q_{L^q_\gamma(\mathbf{R}^n)} \le \sum_{k=1}^\infty (\|u\|^{1/\tilde{q}-1/\overline{q}}_{L^q_\gamma(B_{|z^k|/4}(z^k))})^{q/(1/\tilde{q}-1/\overline{q})} \\
&\le \sum_{k=1}^\infty (\|u\|^{1/q-1/\overline{q}}_{L^{\tilde{q}}_\gamma(B_{|z^k|/4}(z^k))} \|u\|^{1/\tilde{q}-1/q}_{L^{\overline{q}}_\gamma(B_{|z^k|/4}(z^k))})^{q/(1/\tilde{q}-1/\overline{q})} \\
&\le \sum_{k=1}^\infty \|u\|^{q(1-\theta)}_{L^{\tilde{q}}_\gamma(B_{|z^k|/4}(z^k))} \|u\|^{q\theta}_{L^{\overline{q}}_\gamma(B_{|z^k|/4}(z^k))} \\
&\le \sum_{k=1}^\infty \Big(\sup_{y\in\mathbf{R}^n\backslash\{0\}} \|u\|_{L^{\tilde{q}}_\gamma(B_{|y|/4}(y))}\Big)^{q(1-\theta)} \|u\|^{q\theta-p}_{L^{\overline{q}}_\gamma(\mathbf{R}^n)} \|u\|^p_{L^{\overline{q}}_\gamma(B_{|z^k|/4}(z^k))} \\
&\le \Big(\sup_{y\in\mathbf{R}^n\backslash\{0\}} \|u\|_{L^{\tilde{q}}_\gamma(B_{|y|/4}(y))}\Big)^{q(1-\theta)} \|u\|^{q\theta-p}_{L^{\overline{q}}_\gamma(\mathbf{R}^n)} \sum_{k=1}^\infty \bar{c}_{p,\overline{q};\gamma} (\|\nabla u\|^p_{L^p_{1+\gamma}(B_{|y|/2}(y))} + \|u\|^p_{L^p_\gamma(B_{|y|/2}(y))}) \\
&\le L\bar{c}_{p,\overline{q};\gamma} \Big(\sup_{y\in\mathbf{R}^n\backslash\{0\}} \|u\|_{L^{\tilde{q}}_\gamma(B_{|y|/4}(y))}\Big)^{q(1-\theta)} \|u\|^{q\theta-p}_{L^{\overline{q}}_\gamma(\mathbf{R}^n)} (\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|u\|^p_{L^p_\gamma(\mathbf{R}^n)}) \\
&\le L\bar{c}_{p,\overline{q};\gamma} \Big(\sup_{y\in\mathbf{R}^n\backslash\{0\}} \|u\|_{L^{\tilde{q}}_\gamma(B_{|y|/4}(y))}\Big)^{q(1-\theta)} \\
&\quad \cdot \Big(\frac{1}{(S^{p,\overline{q};\gamma})^{1/p}} \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\Big)^{q\theta-p} \Big(\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} + \frac{1}{\gamma^p}\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\Big) \\
&= \frac{L\bar{c}_{p,\overline{q};\gamma}}{(S^{p,\overline{q};\gamma})^{q\theta/p-1}} \Big(1 + \frac{1}{\gamma^p}\Big) \|\nabla u\|^{q\theta}_{L^p_{1+\gamma}(\mathbf{R}^n)} \Big(\sup_{y\in\mathbf{R}^n\backslash\{0\}} \|u\|_{L^{\tilde{q}}_\gamma(B_{|y|/4}(y))}\Big)^{q(1-\theta)} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n).
\end{aligned}$$

(ii) Assume that $q \le \tilde{q}$. Let us take $\overline{q} = \overline{q}_{p,q,\tilde{q}}$ such that it satisfies $\tilde{q} \le \overline{q} = \overline{q}_{p,q,\tilde{q}} < \infty$ and $\tau_{p,\overline{q}} < 1/n$. Then it follows from (i) that there exist positive numbers $\theta_{p,\overline{q},\tilde{q}} \in (0,1)$ and $\bar{c}_{p,\overline{q},\tilde{q};\gamma} > 0$ such that we have

$$\|u\|_{L^{\overline{q}}_\gamma(\mathbf{R}^n)} \le \bar{c}_{p,\overline{q},\tilde{q};\gamma} \|\nabla u\|^{\theta_{p,\overline{q},\tilde{q}}}_{L^p_{1+\gamma}(\mathbf{R}^n)} \Big(\sup_{y\in\mathbf{R}^n\backslash\{0\}} \|u\|_{L^{\tilde{q}}_\gamma(B_{|y|/4}(y))}\Big)^{1-\theta_{p,\overline{q},\tilde{q}}} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n).$$

Then from Lemma 6.6 we have

$$\|u\|_{L^q_\gamma(\mathbf{R}^n)}^{1/p-1/\overline{q}} \leq \|u\|_{L^p_\gamma(\mathbf{R}^n)}^{1/q-1/\overline{q}}\|u\|_{L^{\overline{q}}_\gamma(\mathbf{R}^n)}^{1/p-1/q} \leq \frac{1}{\gamma^{1/q-1/\overline{q}}}\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}^{1/q-1/\overline{q}}\|u\|_{L^{\overline{q}}_\gamma(\mathbf{R}^n)}^{1/p-1/q} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n).$$

Therefore we have the desired estimate with

$$\theta_{p,q,\tilde{q}} = \frac{1/q - 1/\overline{q} + \theta_{p,\overline{q},\tilde{q}}(1/p - 1/q)}{1/p - 1/\overline{q}}.$$

□

## 7.2 Some properties of minimizing sequences

In this subsection we study minimizing sequences for the best constants $S^{p,q;\gamma}$ by using the concentration compactness principle on annular doamains.

**Definition 7.2** *Let $1 < p \leq q < \infty$ and $\gamma > 0$. For $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)$ we set*

$$\rho^{p,q;\gamma}[u] = |u|^q I_{q\gamma} + |\nabla u|^p I_{p(1+\gamma)}, \tag{7.3}$$

$$Q^{p,q;\gamma}[u](t) = \sup_{r>0} \|\rho^{p,q;\gamma}[u]\|_{L^1(B_{tr}\setminus\overline{B_r})} \quad \textit{for } t > 0. \tag{7.4}$$

First of all we show that there exists a minimazing sequence for $S^{p,q;\gamma}$ which does not vanish.

**Proposition 7.1** *Assume that $1 < p < q < \infty$, $\tau_{p,q} < 1/n$ and $\gamma > 0$. Then, there exist $\{u_j\}_{j=1}^\infty \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$, a non-decreasing function $Q : (1,\infty) \to \mathbf{R}$ and positive numbers $\underline{\lambda}, \lambda$ satisfying*

$$0 < \underline{\lambda} \leq \lambda \leq 1 + S^{p,q;\gamma}$$

*such that:*

1. $\|u_j\|_{L^q_\gamma(\mathbf{R}^n)}^q = 1$ *for* $j \geq 1$, $\|\nabla u_j\|_{L^p_{1+\gamma}(\mathbf{R}^n)}^p \to S^{p,q;\gamma}$ *as* $j \to \infty$.
2. $\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{5/4}\setminus\overline{B_{3/4}})} \geq \|u_j\|_{L^q_\gamma(B_{5/4}\setminus\overline{B_{3/4}})}^q \geq \underline{\lambda}$ *for* $j \geq 1$.
3. $Q^{p,q;\gamma}[u_j](t) \to Q(t)$ *as* $j \to \infty$ *for* $t > 1$ ; $Q(t) \to \lambda$ *as* $t \to \infty$.

**Proof:** 1–2: From definition 2.3, there exists a sequence $\{v_j\}_{j=1}^\infty \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ such that

$$\|v_j\|_{L^q_\gamma(\mathbf{R}^n)}^q = 1 \quad \text{for } j \geq 1, \quad \|\nabla v_j\|_{L^p_{1+\gamma}(\mathbf{R}^n)}^p \to S^{p,q;\gamma} \quad \text{as } j \to \infty. \tag{7.5}$$

Then, from Lemma 7.7 with $\tilde{q} = q$, we have

$$\liminf_{j\to\infty} \sup_{y\in\mathbf{R}^n\setminus\{0\}} \|v_j\|_{L^q_\gamma(B_{|y|/4}(y))} 0, \tag{7.6}$$

therefore there exist $\underline{\lambda} > 0$ and $\{y^j\}_{j=1}^\infty \subset \mathbf{R}^n\setminus\{0\}$ such that

$$\|v_j\|_{L^q_\gamma(B_{|y^j|/4}(y^j))}^q \underline{\lambda} \quad \text{for } j \geq 1. \tag{7.7}$$

Now putting

$$u_j(x) = |y^j|^\gamma v_j(|y^j|x) \quad \text{for } x \in \mathbf{R}^n,\ j \geq 1, \tag{7.8}$$

we see that

$$\|u_j\|^q_{L^q_\gamma(B_{5/4}\setminus\overline{B_{3/4}})} \geq \|u_j\|^q_{L^q_\gamma(B_{1/4}(y^j/|y^j|))} = \|v_j\|^q_{L^q_\gamma(B_{|y^j|/4}(y^j))}\underline{\lambda}, \tag{7.9}$$

$$\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} = \|v_j\|^q_{L^q_\gamma(\mathbf{R}^n)} = 1 \quad \text{for } j \geq 1, \tag{7.10}$$

$$\|\nabla u_j\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} = \|\nabla v_j\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \to S^{p,q;\gamma} \quad \text{as } j \to \infty. \tag{7.11}$$

(iii) We see that each $Q^{p,q;\gamma}[u_j]$ is non-decreasing on $(1,\infty)$ and that $\{Q^{p,q;\gamma}[u_j]\}_{j=1}^\infty$ is uniformly bounded on $(1,\infty)$. Therefore, it follows from Lemma 7.1 that there exist, by taking a subsequence if necessary, a non-decreasing function $Q : (1,\infty) \to \mathbf{R}$ and a positive number $\lambda \in \mathbf{R}$ such that

$$Q^{p,q;\gamma}[u_j](t) \to Q(t) \quad \text{as } j \to \infty \quad \text{for } t > 1 \;\; ; \;\; Q(t) \to \lambda \quad \text{as } t \to \infty. \tag{7.12}$$

Noting that

$$Q^{p,q;\gamma}[u_j]\Big(\frac{5}{3}\Big) \geq \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{5/4}\setminus\overline{B_{3/4}})} \geq \|u_j\|^q_{L^q_\gamma(B_{5/4}\setminus\overline{B_{3/4}})}\underline{\lambda} \quad \text{for } j \geq 1, \tag{7.13}$$

we have

$$\underline{\lambda} < Q^{p,q;\gamma}[u_j]\Big(\frac{5}{3}\Big) \leq Q^{p,q;\gamma}[u_j](t) \leq \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)} \quad \text{for } t \geq \frac{5}{3},\ j \geq 1. \tag{7.14}$$

Letting $j \to \infty$, we have

$$\underline{\lambda} \leq Q\Big(\frac{5}{3}\Big) \leq Q(t) \leq 1 + S^{p,q;\gamma} \quad \text{for } t \geq \frac{5}{3}. \tag{7.15}$$

Then by letting $t \to \infty$, we reach to the desired estimate $\underline{\lambda} \leq \lambda \leq 1 + S^{p,q;\gamma}$. □

In order to show that no dichotomy occurs in the minimizing sequence which has been chosen in Proposition 7.1, we prepare the following.

**Proposition 7.2** *Assume that* $1 < p < q < \infty$, $\tau_{p,q} < 1/n$ *and* $\gamma > 0$. *Let* $\{u_j\}_{j=1}^\infty \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ *satisfy the properties 1, 2 and 3 in Proposition 7.1. Then for an arbtrary* $\varepsilon > 0$, *there exist* $\{v_{\varepsilon,j}\}_{j=1}^\infty \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)$, $j_\varepsilon \in \mathbf{N}$ *and* $\tilde{\varepsilon}_{p,q;\varepsilon} > 0$ *such that we have*

$$|\|\rho^{p,q;\gamma}[v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)} - \lambda| \leq \tilde{\varepsilon}_{p,q;\varepsilon},\ |\|\rho^{p,q;\gamma}[u_j - v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)} - (1 + S^{p,q;\gamma} - \lambda)| \leq \tilde{\varepsilon}_{p,q;\varepsilon}, \tag{7.16}$$
$$0 \leq 1 - \|v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u_j - v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} < 2\varepsilon \quad \textit{for } j \geq j_\varepsilon.$$

*Further it holds that* $\tilde{\varepsilon}_{p,q;\varepsilon} \to 0$ *as* $\varepsilon \to 0$.

**Proof:** Let $\varepsilon > 0$ .

(a) From the assertion 3 of Proposition 7.1, there exists $t_\varepsilon > 1$ such that we have

$$\lambda - \frac{\varepsilon}{2} < Q(t) \leq \lambda \quad \text{for } t \geq t_\varepsilon. \tag{7.17}$$

Also from Definition 7.2 there exist $\{r_{\varepsilon,j}\}_{j=1}^\infty \cup \{R_{\varepsilon,j}\}_{j=1}^\infty \subset (0,\infty)$ such that we have

$$Q^{p,q;\gamma}[u_j](t_\varepsilon) \leq \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} + \frac{\varepsilon}{4},\ R_{\varepsilon,j} = t_\varepsilon r_{\varepsilon,j} \quad \text{for } j \geq 1. \tag{7.18}$$

Further from the assertions 1 and 2 of Proposition 7.1, there exists$j_\varepsilon \in \mathbf{N}$ such that we have

$$0 \leq \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)} - (1 + S^{p,q;\gamma}) < \varepsilon, \tag{7.19}$$
$$|Q^{p,q;\gamma}[u_j](t_\varepsilon) - Q(t_\varepsilon)| < \frac{\varepsilon}{4},\ |Q^{p,q;\gamma}[u_j](4t_\varepsilon) - Q(4t_\varepsilon)| < \varepsilon \quad \text{for } j \geq j_\varepsilon.$$

(b) Since

$$\lambda - \frac{\varepsilon}{2} < Q(t_\varepsilon) < Q^{p,q;\gamma}[u_j](t_\varepsilon) + \frac{\varepsilon}{4} < \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} + \frac{\varepsilon}{2}, \tag{7.20}$$
$$\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} \le Q^{p,q;\gamma}[u_j](t_\varepsilon) < Q(t_\varepsilon) + \frac{\varepsilon}{4} < \lambda + \frac{\varepsilon}{2} \quad \text{for } j \ge j_\varepsilon,$$

we see that

$$\big|\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} - \lambda\big| < \varepsilon \quad \text{for } j \ge j_\varepsilon. \tag{7.21}$$

Hence we see

$$\begin{aligned} &\big|\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n\setminus\overline{B_{R_{\varepsilon,j}}})} + \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{r_{\varepsilon,j}})} - (1+S^{p,q;\gamma}-\lambda)\big| \\ &= \big|\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)} - \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} - (1+S^{p,q;\gamma}) + \lambda\big| \\ &\le \big|\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)} - (1+S^{p,q;\gamma})\big| + \big|\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} - \lambda\big| < 2\varepsilon \quad \text{for } j \ge j_\varepsilon. \end{aligned} \tag{7.22}$$

Since

$$\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{2R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})} \le Q^{p,q;\gamma}[u_j](4t_\varepsilon) \le Q(4t_\varepsilon) + \varepsilon \le \lambda + \varepsilon \quad \text{for } j \ge j_\varepsilon, \tag{7.23}$$

we have

$$\begin{aligned} &\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}})} + \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})} \\ &= \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{2R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})} - \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} < (\lambda+\varepsilon) - (\lambda-\varepsilon) = 2\varepsilon \quad \text{for } j \ge j_\varepsilon. \end{aligned} \tag{7.24}$$

(c) Let us set $v_{\varepsilon,j}(x) = \psi_{2R_{\varepsilon,j}}(x)(1-\psi_{r_{\varepsilon,j}}(x))u_j(x)$ for $x \in \mathbf{R}^n$, $j \ge 1$. Then from Lemma 7.2 and elementary inequalities;

$$(1+t)^p \le 2^{p-1}(1+t^p),\ 1+t^{p/q} \le 2^{1-p/q}(1+t)^{p/q} \quad \text{for } t \ge 0, \tag{7.25}$$

we have

$$\begin{aligned} &\left|\|\rho^{p,q;\gamma}[v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)} - \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})}\right| \\ &= \int_{B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}}} \Bigg\{ |\psi_{2R_{\varepsilon,j}}(x)u_j(x)|^q I_{q\gamma}(x) \\ &\qquad + \left| -\frac{1}{2R_{\varepsilon,j}}\tilde{\psi}_{2R_{\varepsilon,j}}(x)u_j(x)\frac{x}{|x|} + \psi_{2R_{\varepsilon,j}}(x)\nabla u_j(x)\right|^p I_{p(1+\gamma)}(x)\Bigg\} dx \\ &\quad + \int_{B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}}} \Bigg\{ |(1-\psi_{r_{\varepsilon,j}}(x))u_j(x)|^q I_{q\gamma}(x) \\ &\qquad + \left|\frac{1}{r_{\varepsilon,j}}\tilde{\psi}_{r_{\varepsilon,j}}(x)u_j(x)\frac{x}{|x|} + (1-\psi_{r_{\varepsilon,j}}(x))\nabla u_j(x)\right|^p I_{p(1+\gamma)}(x)\Bigg\} dx \\ &\le \int_{B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}}} \left( |u_j(x)|^q I_{q\gamma}(x) + 2^{p-1}\left( \left(\frac{3|x|}{2R_{\varepsilon,j}}|u_j(x)|\right)^p I_{p\gamma}(x) + |\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\right)\right) dx \\ &\quad + \int_{B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}}} \left( |u_j(x)|^q I_{q\gamma}(x) + 2^{p-1}\left( \left(\frac{3|x|}{r_{\varepsilon,j}}|u_j(x)|\right)^p I_{p\gamma}(x) + |\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\right)\right) dx \end{aligned} \tag{7.26}$$

$$
\begin{aligned}
&\le 2^{p-1}\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}})}+\frac{6^p}{2}\|u_j\|^p_{L^p_\gamma(B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}})}\\
&\quad+2^{p-1}\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})}+\frac{6^p}{2}\|u_j\|^p_{L^p_\gamma(B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})}\\
&\le 2^{p-1}\cdot 2\varepsilon+\frac{6^p}{2}(((\omega_n\log 2)^{\tau_{p,q}}\|u_j\|_{L^q_\gamma(B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}})})^p+((\omega_n\log 2)^{\tau_{p,q}}\|u_j\|_{L^q_\gamma(B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})})^p)\\
&\le 2^p\varepsilon+\frac{1}{2}(6(\omega_n\log 2)^{\tau_{p,q}})^p 2^{1-p/q}(\|u_j\|^q_{L^q_\gamma(B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}})}+\|u_j\|^q_{L^q_\gamma(B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})})^{p/q}\\
&\le 2^p\varepsilon+\frac{1}{2^{p/q}}(6(\omega_n\log 2)^{\tau_{p,q}})^p(2\varepsilon)^{p/q}=2^p\varepsilon+(6(\omega_n\log 2)^{\tau_{p,q}})^p\varepsilon^{p/q}\quad\text{for } j\ge j_\varepsilon.
\end{aligned}
$$

In a similar way we have

$$
\begin{aligned}
&|\|\rho^{p,q;\gamma}[u_j-v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)}-(\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n\setminus\overline{B_{R_{\varepsilon,j}}})}+\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{r_{\varepsilon,j}})})| \qquad (7.27)\\
&=\Big|\int_{B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}}}\Big\{(|u_j(x)|^q-|(1-\psi_{2R_{\varepsilon,j}}(x))u_j(x)|^q)I_{q\gamma}(x)\\
&\qquad+\Big(|\nabla u_j(x)|^p-\Big|\frac{1}{2R_{\varepsilon,j}}\tilde\psi_{2R_{\varepsilon,j}}(x)u_j(x)\frac{x}{|x|}+(1-\psi_{2R_{\varepsilon,j}}(x))\nabla u_j(x)\Big|^p\Big)I_{p(1+\gamma)}(x)\Big\}dx\\
&\quad+\int_{B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}}}\Big\{(|u_j(x)|^q-|\psi_{r_{\varepsilon,j}}(x)u_j(x)|^q)I_{q\gamma}(x)\\
&\qquad+\Big(|\nabla u_j(x)|^p-\Big|-\frac{1}{r_{\varepsilon,j}}\tilde\psi_{r_{\varepsilon,j}}(x)u_j(x)\frac{x}{|x|}+\psi_{r_{\varepsilon,j}}(x)\nabla u_j(x)\Big|^p\Big)I_{p(1+\gamma)}(x)\Big\}dx\Big|\\
&\le\int_{B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}}}\Big\{|u_j(x)|^qI_{q\gamma}(x)+|\nabla u_j(x)|^pI_{p(1+\gamma)}(x)\\
&\qquad+2^{p-1}\Big(\Big(\frac{3|x|}{2R_{\varepsilon,j}}|u_j(x)|\Big)^pI_{q\gamma}(x)+|\nabla u_j(x)|^pI_{p(1+\gamma)}(x)\Big)\Big\}dx\\
&\quad+\int_{B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}}}\Big\{|u_j(x)|^qI_{q\gamma}(x)+|\nabla u_j(x)|^pI_{p(1+\gamma)}(x)\\
&\qquad+2^{p-1}\Big(\Big(\frac{3|x|}{r_{\varepsilon,j}}|u_j(x)|\Big)^pI_{q\gamma}(x)+|\nabla u_j(x)|^pI_{p(1+\gamma)}(x)\Big)\Big\}dx\\
&\le 2(2^{p-1}+1)\varepsilon+(6(\omega_n\log 2)^{\tau_{p,q}})^p\varepsilon^{p/q}\quad\text{for } j\ge j_\varepsilon.
\end{aligned}
$$

(d) From (7.21), (7.22), (7.26) and (7.27) in (b) and (c), we have

$$
\begin{aligned}
&|\|\rho^{p,q;\gamma}[v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)}-\lambda|\le 2^p\varepsilon+(6(\omega_n\log 2)^{\tau_{p,q}})^p\varepsilon^{p/q}+\varepsilon, \qquad (7.28)\\
&|\|\rho^{p,q;\gamma}[u_j-v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)}-(1+S^{p,q;\gamma}-\lambda)|\\
&\le 2(2^{p-1}+1)\varepsilon+(6(\omega_n\log 2)^{\tau_{p,q}})^p\varepsilon^{p/q}+2\varepsilon\quad\text{for } j\ge j_\varepsilon.
\end{aligned}
$$

Noing that

$$
\theta^q+(1-\theta)^q\le 1\quad\text{for } 0\le\theta\le 1, \qquad (7.29)
$$

we have

$$
\begin{aligned}
&0\le 1-(\psi_{2R_{\varepsilon,j}}(x)(1-\psi_{r_{\varepsilon,j}}(x)))^q-(1-\psi_{2R_{\varepsilon,j}}(x)(1-\psi_{r_{\varepsilon,j}}(x)))^q\\
&\le\chi_{B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}}}(x)+\chi_{B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}}}(x)\quad\text{for } x\in\mathbf{R}^n,\ j\ge j_\varepsilon.
\end{aligned}
$$

Then from this inequality and (b) we have

$$
\begin{aligned}
&0\le 1-\|v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}-\|u_j-v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}=\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)}-\|v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}-\|u_j-v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}\\
&\le\|u_j\|^q_{L^q_\gamma(B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}})}+\|u_j\|^q_{L^q_\gamma(B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})}\\
&\le\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}})}+\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})}<2\varepsilon\quad\text{for } j\ge j_\varepsilon.
\end{aligned}
$$

□

**Proposition 7.3** *Assume that* $1 < p < q < \infty$, $\tau_{p,q} < 1/n$ *and* $\gamma > 0$. *Assume that* $\{u_j\}_{j=1}^{\infty} \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)\backslash\{0\}$ *satisfies the property 1 of Proposition 7.1. Then we have* $\lambda = 1 + S^{p,q;\gamma}$.

**Proof:** On the contrary we assume that $\lambda \neq 1 + S^{p,q;\gamma}$. Then from Proposition 7.1 we should have $0 < \lambda < 1 + S^{p,q;\gamma}$. Let us retain the notations in Proposition 7.2.

(a) Since $\tilde{\varepsilon}_{p,q;\varepsilon} \to 0$ as $\varepsilon \to 0$, there exists some $\varepsilon_0 > 0$ such that

$$0 < \tilde{\varepsilon}_{p,q;\varepsilon} < \frac{1}{2}\min\{\lambda, 1 + S^{p,q;\gamma} - \lambda\} \quad \text{for } 0 < \varepsilon < \varepsilon_0. \tag{7.30}$$

Then, from Proposition 7.2 and Theorem 2.1 we have

$$\begin{aligned}
&\frac{1}{2}\lambda \le \lambda - \tilde{\varepsilon}_{p,q;\varepsilon} \le \|\rho^{p,q;\gamma}[v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)} \le \Big(\frac{1}{S^{p,q;\gamma}}\|\nabla v_{\varepsilon,j}\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\Big)^{q/p} + \|\nabla v_{\varepsilon,j}\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)},\\
&\frac{1}{2}(1 + S^{p,q;\gamma} - \lambda) \le 1 + S^{p,q;\gamma} - \lambda - \tilde{\varepsilon}_{p,q;\varepsilon} \le \|\rho^{p,q;\gamma}[u_j - v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)}\\
&\le \Big(\frac{1}{S^{p,q;\gamma}}\|\nabla[u_j - v_{\varepsilon,j}]\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\Big)^{q/p} + \|\nabla[u_j - v_{\varepsilon,j}]\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \text{ for } j \ge j_\varepsilon,\ 0 < \varepsilon < \varepsilon_0.
\end{aligned}$$

Hence, for some $\beta > 0$, we have

$$\|\nabla v_{\varepsilon,j}\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \ge \beta,\ \|\nabla[u_j - v_{\varepsilon,j}]\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \ge \beta \quad \text{for } j \ge j_\varepsilon,\ 0 < \varepsilon < \varepsilon_0. \tag{7.31}$$

(b) Choose a sequence $\{\varepsilon_k\}_{k=1}^{\infty} \subset (0, \varepsilon_0)$ satifying $\varepsilon_k \to 0$ as $k \to \infty$. Then from Proposition 7.2 we have

$$0 \le 1 - \|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} \le 2\varepsilon_k \quad \text{for } j \ge j_{\varepsilon_k},\ k \ge 1, \tag{7.32}$$

and we see that $\{\|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}\}_{k=1}^{\infty}$ and $\{\|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}\}_{k=1}^{\infty}$ are bounded. Hence, by choosing a subsequence with respect to $j$, there exist $\{\overline{\sigma}_k\}_{k=1}^{\infty} \cup \{\underline{\sigma}_k\}_{k=1}^{\infty} \subset [0,1]$ such that we have

$$\|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} \to \overline{\sigma}_k,\ \|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} \to \underline{\sigma}_k \quad \text{as } j \to \infty \quad \text{for } k \ge 1. \tag{7.33}$$

Since $0 \le 1 - \overline{\sigma}_k - \underline{\sigma}_k \le 2\varepsilon_k$ for $k \ge 1$, by choosing a subsequence with respect to $k$, there exists $\sigma \in [0,1]$ such that we have

$$\overline{\sigma}_k \to \sigma,\ \underline{\sigma}_k \to 1 - \sigma \quad \text{as } k \to \infty. \tag{7.34}$$

(c) From (a), Proposition 7.2 and Theorem 2.1, we have

$$\begin{aligned}
&\max\{S^{p,q;\gamma}(\|v_{\varepsilon_k,j}\|^p_{L^q_\gamma(\mathbf{R}^n)} + \|u_j - v_{\varepsilon_k,j}\|^p_{L^q_\gamma(\mathbf{R}^n)}), \beta + S^{p,q;\gamma}\|u_j - v_{\varepsilon_k,j}\|^p_{L^q_\gamma(\mathbf{R}^n)},\\
&\qquad \beta + S^{p,q;\gamma}\|v_{\varepsilon_k,j}\|^p_{L^q_\gamma(\mathbf{R}^n)}\}\\
&\le \|\nabla v_{\varepsilon_k,j}\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|\nabla[u_j - v_{\varepsilon_k,j}]\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\\
&= \|\rho^{p,q;\gamma}[v_{\varepsilon_k,j}]\|_{L^1(\mathbf{R}^n)} + \|\rho^{p,q;\gamma}[u_j - v_{\varepsilon_k,j}]\|_{L^1(\mathbf{R}^n)} - (\|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} + \|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)})\\
&\le (\lambda + \tilde{\varepsilon}_{p,q;\varepsilon_k}) + (1 + S^{p,q;\gamma} - \lambda + \tilde{\varepsilon}_{p,q;\varepsilon_k}) - \|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}\\
&= S^{p,q;\gamma} + 1 - \|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} + 2\tilde{\varepsilon}_{p,q;\varepsilon_k} \quad \text{for } j \ge j_{\varepsilon_k},\ k \ge 1.
\end{aligned}$$

Therefore, letting $j \to \infty$, $k \to \infty$ and using (b), we have

$$\max\{S^{p,q;\gamma}(\sigma^{p/q} + (1-\sigma)^{p/q}), \beta + S^{p,q;\gamma}(1-\sigma)^{p/q}, S^{p,q;\gamma}\sigma^{p/q} + \beta\} \le S^{p,q;\gamma}, \tag{7.35}$$

and we have $\sigma^{p/q} + (1-\sigma)^{p/q} \leq 1$. If we note that

$$\theta^{p/q} + (1-\theta)^{p/q} > 1 \quad \text{for } 0 < \theta < 1, \tag{7.36}$$

we have $\sigma \in \{0,1\}$. Then it holds that $\beta \leq 0$, and this is a contradiction. □

Then we have the following.

**Proposition 7.4** *Assume that* $1 < p < q < \infty$, $\tau_{p,q} < 1/n$ *and* $\gamma > 0$. *Assume that* $\{u_j\}_{j=1}^{\infty} \subset W_{\gamma,0}^{1,p}(\mathbf{R}^n)\setminus\{0\}$ *satisfies the properties of Proposition 7.1. Then,* $\{\rho^{p,q;\gamma}[u_j]\}_{j=1}^{\infty}$ *is tight. Namely, for an arbitrary* $\varepsilon > 0$, *there exists a constant* $R_\varepsilon > 0$ *such that we have*

$$\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n\setminus\overline{B_{R_\varepsilon}})} < \varepsilon \quad \textit{for } j \geq 1. \tag{7.37}$$

*In particular, both* $\{|u_j|^q I_{q\gamma}\}_{j=1}^{\infty}$ *and* $\{|\nabla u_j|^p I_{p(1+\gamma)}\}_{j=1}^{\infty}$ *are tight as well.*

**Proof:** Let $0 < \varepsilon < \underline{\lambda}$. (a) From Proposition 7.3 we see that $\lambda = 1 + S^{p,q;\gamma}$, hence there exists $t_\varepsilon > 1$ such that we have

$$1 + S^{p,q;\gamma} - \frac{\varepsilon}{4} < Q(t) \leq 1 + S^{p,q;\gamma} \quad \text{for } t \geq t_\varepsilon. \tag{7.38}$$

From the assertions 1 and 3 of Proposition 7.1 there exists $j_\varepsilon \in \mathbf{N}$ such that we have

$$\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)} < 1 + S^{p,q;\gamma} + \frac{\varepsilon}{4},\ |Q^{p,q;\gamma}[u_j](t_\varepsilon) - Q(t_\varepsilon)| < \frac{\varepsilon}{4} \quad \text{for } j \geq j_\varepsilon. \tag{7.39}$$

Further, by Definition 7.2 there exist $\{r_{\varepsilon,j}\}_{j=1}^{\infty} \cup \{R_{\varepsilon,j}\}_{j=1}^{\infty} \subset (0,\infty)$ such that we have

$$\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} Q^{p,q;\gamma}[u_j](t_\varepsilon) - \frac{\varepsilon}{4},\ R_{\varepsilon,j} = t_\varepsilon r_{\varepsilon,j} \quad \text{for } j \geq 1. \tag{7.40}$$

Therefore it holds that

$$\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} Q^{p,q;\gamma}[u_j](t_\varepsilon) - \frac{\varepsilon}{4} Q(t_\varepsilon) - \frac{\varepsilon}{2} 1 + S^{p,q;\gamma} - \frac{3}{4}\varepsilon \quad \text{for } j \geq j_\varepsilon. \tag{7.41}$$

Then we have

$$r_{\varepsilon,j} \leq \frac{5}{4} \quad \text{for } j \geq j_\varepsilon. \tag{7.42}$$

In fact, if not, we have

$$(B_{5/4}\setminus\overline{B_{3/4}}) \cap (B_{R_{\varepsilon,j_0}}\setminus\overline{B_{r_{\varepsilon,j_0}}}) = \phi \quad \text{for some } j_0 \geq j_\varepsilon, \tag{7.43}$$

and hence

$$\begin{aligned} 1 + S^{p,q;\gamma} + \frac{\varepsilon}{4} & \|\rho^{p,q;\gamma}[u_{j_0}]\|_{L^1(\mathbf{R}^n)} \\ & \geq \|\rho^{p,q;\gamma}[u_{j_0}]\|_{L^1(B_{5/4}\setminus\overline{B_{3/4}})} + \|\rho^{p,q;\gamma}[u_{j_0}]\|_{L^1(B_{R_{\varepsilon,j_0}}\setminus\overline{B_{r_{\varepsilon,j_0}}})} \\ & \underline{\lambda} + 1 + S^{p,q;\gamma} - \frac{3}{4}\varepsilon. \end{aligned} \tag{7.44}$$

Then we have $\underline{\lambda} < \varepsilon$, but this is a contradiction.

(b) Let us take a number $R_\varepsilon > 0$ such that

$$R_\varepsilon > \frac{5}{4} t_\varepsilon, \quad \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n\setminus\overline{B_{R_\varepsilon}})} < \varepsilon \quad \text{for } 1 \leq j \leq j_\varepsilon - 1. \tag{7.45}$$

Since $B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}} \subset B_{R_\varepsilon}$ for $j \geq j_\varepsilon$, we have

$$\begin{aligned} \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n\setminus\overline{B_{R_\varepsilon}})} & \leq \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)} - \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} \\ & < 1 + S^{p,q;\gamma} + \frac{\varepsilon}{4} - \Big(1 + S^{p,q;\gamma} - \frac{3}{4}\varepsilon\Big) = \varepsilon \quad \text{for } j \geq j_\varepsilon, \end{aligned} \tag{7.46}$$

and this proves the assertion. □

## 7.3 Convergence of minimizing sequence

In this subsection we investigate the minimizing sequence $\{u_j\}_{j=1}^{\infty}$ for $S^{p,q;\gamma}$, which are introduced in Proposition 7.1, and we finally prove the existence of minimizer. To this end we employ the following lemma that is an easy corollary to [Lions, Lemma 2.1]. Here, by $\mathcal{B}(\mathbf{R}^n)$ we denote a set of all finite Borel measures on $\mathbf{R}^n$, and by $\delta_0$ we denote a Dirac measure with a unit mass at the origin. In a canonical way we see that $L^1(\mathbf{R}^n) \subset \mathcal{B}(\mathbf{R}^n)$. For $\nu \in \mathcal{B}(\mathbf{R}^n)$, by $\nu_{\rm ac}$ and $\nu_{\rm s}$ we denote an absolutely continuous part and a singular part of $\nu$ with respect to Lebesgue measure respectively. In this notation we see that $\nu_{\rm ac} \in L^1(\mathbf{R}^n)$ and $\nu = \nu_{\rm ac} + \nu_{\rm s}$.

**Lemma 7.8** *(Lions) Assume that* $1 < p \le q < \infty$, $\mu, \nu \in \mathcal{B}(\mathbf{R}^n)$, $\mu, \nu \ge 0$, $\operatorname{supp} \nu_{\rm s} \subset \{0\}$ *and* $S > 0$. *Assume that*

$$S\Big(\int_{\mathbf{R}^n} |\phi(x)|^q d\nu(x)\Big)^{p/q} \le \int_{\mathbf{R}^n} |\phi(x)|^p d\mu(x) \quad \text{for } \phi \in C_{\rm c}^{\infty}(\mathbf{R}^n \setminus \{0\}). \tag{7.47}$$

*Then there exists a constant* $a_0 \in [0, \infty)$ *such that we have*

$$\nu = a_0 \delta_0, \;\; \mu \ge (S a_0^{p/q}) \delta_0. \tag{7.48}$$

For reader's convenience, let us briefly recall a notion of weak convergence of a sequence of measures. Let us denote by $BC(\mathbf{R}^n)$ a set of all bounded, continuous functions on $\mathbf{R}^n$, then $\mathcal{B}(\mathbf{R}^n)$ is regarded as a subspace of $BC(\mathbf{R}^n)'$, which is a dual of $BC(\mathbf{R}^n)$. A sequence $\{\nu_j\}_{j=1}^{\infty} \subset \mathcal{B}(\mathbf{R}^n)$ is said to converge weakly to $\nu$ in $BC(\mathbf{R}^n)'$, if $\{\nu_j\}_{j=1}^{\infty}$ converges in a weak $*$ topology to $\nu$ in $BC(\mathbf{R}^n)'$, that is to say,

$$\int_{\mathbf{R}^n} \phi(x) d\nu_j(x) \to \int_{\mathbf{R}^n} \phi(x) d\nu(x) \quad \text{as } j \to \infty \quad \text{for any } \phi \in BC(\mathbf{R}^n). \tag{7.49}$$

When $\{\nu_j\}_{j=1}^{\infty} \subset \mathcal{B}(\mathbf{R}^n)$ converges weakly to $\nu$ in $BC(\mathbf{R}^n)'$, we simply write

$$\nu_j \rightharpoonup \nu \quad \text{weakly as } j \to \infty. \tag{7.50}$$

We employ the following lemma. (The proof is omitted.)

**Lemma 7.9** *Assume that* $\{\nu_j\}_{j=1}^{\infty}$ *is bounded in* $\mathcal{B}(\mathbf{R}^n)$. *If* $\{\nu_j\}_{j=1}^{\infty}$ *is tight, then* $\{\nu_j\}_{j=1}^{\infty}$ *contains a weakly convergent subsequence.*

If $\{u_j\}_{j=1}^{\infty}$ satisfies the assertions of Proposition 7.1, then from Proposition 7.4 we see that both $\{|u_j|^q I_{q\gamma}\}_{j=1}^{\infty}$ and $\{|\nabla u_j|^p I_{p(1+\gamma)}\}_{j=1}^{\infty}$ are tight. Hence from Lemma 7.9 they contain weakly convergent subsequences respectively. Further, from Rellich's therem and Lemma 7.3 we have the following:

**Proposition 7.5** *Assume that* $1 < p < q < \infty$, $\tau_{p,q} < 1/n$ *and* $\gamma > 0$. *Then there exist* $\{u_j\}_{j=1}^{\infty} \subset W_{\gamma,0}^{1,p}(\mathbf{R}^n) \setminus \{0\}$ *and* $u \in W_{\gamma,0}^{1,p}(\mathbf{R}^n) \setminus \{0\}$, $\mu, \nu \in \mathcal{B}(\mathbf{R}^n)$ *such that we have the followings:*

1. $\|u_j\|_{L_\gamma^q(\mathbf{R}^n)}^q = 1$ *for* $j \ge 1$, $\|\nabla u_j\|_{L_{1+\gamma}^p(\mathbf{R}^n)}^p \to S^{p,q;\gamma}$ *as* $j \to \infty$.
2. $u_j \rightharpoonup u$ *weakly in* $W_{\gamma,0}^{1,p}(\mathbf{R}^n)$, $u_j \to u$ *in* $L_{\rm loc}^q(\mathbf{R}^n \setminus \{0\}) \cap (L_{\gamma+1-n\tau_{p,q}}^q)_{\rm loc}(\mathbf{R}^n)$,
   $u_j \to u$ *a.e. on* $\mathbf{R}^n$ *as* $j \to \infty$.
3. $|u_j|^q I_{q\gamma} \rightharpoonup \nu$, $|\nabla u_j|^p I_{p(1+\gamma)} \rightharpoonup \mu$ *weakly as* $j \to \infty$.
4. $\nu_{\rm ac} = |u|^q I_{q\gamma}$ *a.e. on* $\mathbf{R}^n$, $\operatorname{supp} \nu_{\rm s} \subset \{0\}$.

**Proof:** We prove the assertion 4 only. For $\varepsilon > 0$ it follows from the assertios 2 and 3 that

$$\int_{\mathbf{R}^n} \phi(x)|\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\,dx \to \int_{\mathbf{R}^n} \phi(x)\,d\mu(x), \tag{7.51}$$

$$\int_{\mathbf{R}^n} \phi(x)|u_j(x)|^q I_{q\gamma}(x)\,dx \to \int_{\mathbf{R}^n} \phi(x)\,d\nu(x) \quad \text{as } j \to \infty \quad \text{for } \phi \in C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \overline{B_\varepsilon}), \tag{7.52}$$

hence it holds that

$$\int_{\mathbf{R}^n} \phi(x)(|u(x)|^q I_{q\gamma}(x) - \nu_{\mathrm{ac}}(x))\,dx = \int_{\mathbf{R}^n} \phi(x)\,d\nu_{\mathrm{s}}(x) \quad \text{for } \phi \in C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \overline{B_\varepsilon}). \tag{7.53}$$

Therefore, $|u|^q I_{q\gamma} - \nu_{\mathrm{ac}}$ coincides with $\nu_{\mathrm{s}}$ as measures on $\mathbf{R}^n \setminus \overline{B_\varepsilon}$. Since they are absolutely continuous and singular with respect to Lebesgue measure respectively, they should be vanishig as measures on $\mathbf{R}^n \setminus \overline{B_\varepsilon}$. Hence we have

$$|u|^q I_{q\gamma} - \nu_{\mathrm{ac}} = 0 \quad \text{a.e. on } \mathbf{R}^n \setminus \overline{B_\varepsilon}, \quad \operatorname{supp} \nu_{\mathrm{s}} \subset \overline{B_\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$|u|^q I_{q\gamma} - \nu_{\mathrm{ac}} = 0 \quad \text{a.e. on } \mathbf{R}^n, \quad \operatorname{supp} \nu_{\mathrm{s}} \subset \{0\}.$$

□

Let us define the next.

**Definition 7.3** *For $\phi \in BC(\mathbf{R}^n)$ satisfying $\phi > 0$ on $\mathbf{R}^n$, we set*

$$\|u\|_{W_\gamma^{1,p}[\phi](\mathbf{R}^n)} = \Big(\int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x)\phi(x)\,dx\Big)^{1/p}. \tag{7.54}$$

*By $W_{\gamma,0}^{1,p}[\phi](\mathbf{R}^n)$ we denote the completion of $C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \{0\})$ with respect to the norm $\|\cdot\|_{W_\gamma^{1,p}[\phi](\mathbf{R}^n)}$.*

In this definition we have

$$\|u\|_{W_\gamma^{1,p}[\phi](\mathbf{R}^n)} \le \|\phi\|_{L^\infty(\mathbf{R}^n)} \|\nabla u\|_{L_{1+\gamma}^p(\mathbf{R}^n)} \quad u \in W_{\gamma,0}^{1,p}(\mathbf{R}^n), \tag{7.55}$$

hence we have a continuous imbedding $W_{\gamma,0}^{1,p}(\mathbf{R}^n) \subset W_{\gamma,0}^{1,p}[\phi](\mathbf{R}^n)$. From this fact we have the next.

**Lemma 7.10** *For $1 < p < \infty$ and $\gamma > 0$, assume that $\{u_j\}_{j=1}^\infty \subset W_{\gamma,0}^{1,p}(\mathbf{R}^n)$, $u \in W_{\gamma,0}^{1,p}(\mathbf{R}^n)$ and $\mu \in \mathcal{B}(\mathbf{R}^n)$ satisfy*

$$u_j \rightharpoonup u \quad \textit{weakly in } W_{\gamma,0}^{1,p}(\mathbf{R}^n), \quad |\nabla u_j|^p I_{p(1+\gamma)} \rightharpoonup \mu \quad \textit{weakly as } j \to \infty. \tag{7.56}$$

*Then, we have*

$$|\nabla u|^p I_{p(1+\gamma)} \le \mu. \tag{7.57}$$

**Proof:** For $\phi \in C_{\mathrm{c}}(\mathbf{R}^n)$ with $\phi \ge 0$ on $\mathbf{R}^n$, it suffices to show that

$$\int_{\mathbf{R}^n} \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \le \int_{\mathbf{R}^n} \phi(x)\,d\mu(x). \tag{7.58}$$

(a) First we show this inequality to be valid assuming that $\phi \in BC(\mathbf{R}^n)$ satifies $\phi > 0$ on $\mathbf{R}^n$. Since the imbedding $W_{\gamma,0}^{1,p}(\mathbf{R}^n) \subset W_{\gamma,0}^{1,p}[\phi](\mathbf{R}^n)$ is continuous, we see that

$$u_j \rightharpoonup u \quad \text{weakly in } W_{\gamma,0}^{1,p}[\phi](\mathbf{R}^n) \quad \text{as } j \to \infty.$$

Therefore we have

$$\int_{\mathbf{R}^n} \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx = \|u\|^p_{W^{1,p}_\gamma[\phi](\mathbf{R}^n)} \le \liminf_{j\to\infty} \|u_j\|^p_{W^{1,p}_\gamma[\phi](\mathbf{R}^n)} \tag{7.59}$$
$$= \lim_{j\to\infty} \int_{\mathbf{R}^n} \phi(x)|\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\,dx = \int_{\mathbf{R}^n} \phi(x)\,d\mu(x).$$

(b) Secondly we consider the case that $\phi \in C_c(\mathbf{R}^n)$ and $\phi \ge 0$ on $\mathbf{R}^n$. For $\varepsilon > 0$ it holds that $\overline{\rho}_\varepsilon * \phi \in BC(\mathbf{R}^n)$ and $\overline{\rho}_\varepsilon * \phi > 0$ on $\mathbf{R}^n$. Then, from (a) we have

$$\int_{\mathbf{R}^n} \overline{\rho}_\varepsilon * \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \le \int_{\mathbf{R}^n} \overline{\rho}_\varepsilon * \phi(x)\,d\mu(x)$$ for $\varepsilon > 0$. Here noting that $\phi$ is uniformly continuous on $\mathbf{R}^n$, for any $\eta > 0$ there exists a number $r_\eta > 0$ such that we have

$$|\phi(x-y) - \phi(x)| < \eta \quad \text{for } x \in \mathbf{R}^n,\ y \in B_{r_\eta}. \tag{7.60}$$

Then

$$\Big|\int_{\mathbf{R}^n} \overline{\rho}_\varepsilon * \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx - \int_{\mathbf{R}^n} \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx\Big|$$
$$= \Big|\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \overline{\rho}_\varepsilon(y)(\phi(x-y) - \phi(x))|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dy\,dx\Big|$$
$$\le \int_{\mathbf{R}^n}\Big(\int_{B_{r_\eta}} \eta\overline{\rho}_\varepsilon(y)\,dy + \int_{\mathbf{R}^n\setminus\overline{B_{r_\eta}}} 2\|\phi\|_{L^\infty(\mathbf{R}^n)}\overline{\rho}_\varepsilon(y)\,dy\Big)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx$$
$$\le (\eta + 2\|\phi\|_{L^\infty(\mathbf{R}^n)}\|\overline{\rho}_1\|_{L^1(\mathbf{R}^n\setminus\overline{B_{r_\eta/\varepsilon}})})\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \to \eta\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \quad \text{as } \varepsilon \to 0,$$

hence

$$\int_{\mathbf{R}^n} \overline{\rho}_\varepsilon * \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \to \int_{\mathbf{R}^n} \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \quad \text{as } \varepsilon \to 0. \tag{7.61}$$

In a similar way we have

$$\int_{\mathbf{R}^n} \overline{\rho}_\varepsilon * \phi(x)\,d\mu(x) \to \int_{\mathbf{R}^n} \phi(x)\,d\mu(x) \quad \text{as } \varepsilon \to 0, \tag{7.62}$$

and the assertion follows. □

Then we have

**Proposition 7.6** *Assume that $1 < p < q < \infty$, $\tau_{p,q} < 1/n$ and $\gamma > 0$. Then, in Proposition 7.5, there exists a constant $a_0 \in [0,\infty)$ such that we have*

$$\nu = |u|^q I_{q\gamma} + a_0\delta_0,\ \ \mu \ge |\nabla u|^p I_{p(1+\gamma)} + (S^{p,q;\gamma} a_0^{p/q})\delta_0. \tag{7.63}$$

**Proof:** (a) Taking an arbitrary $\phi \in C_c^\infty(\mathbf{R}^n)$ with satisfying $\operatorname{supp}\phi \subset B_R$. Then it follows from Lemma 7.4 that we have

$$\|\phi u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|\phi(u_j - u)\|^q_{L^q_\gamma(\mathbf{R}^n)} \to \|\phi u\|^q_{L^q_\gamma(\mathbf{R}^n)} \quad \text{as } j \to \infty, \tag{7.64}$$

and from the assertion 3 of Proposition 7.5 we have

$$\|\phi u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} = \int_{\mathbf{R}^n} |\phi(x)|^q |u_j(x)|^q I_{q\gamma}(x)\,dx \to \int_{\mathbf{R}^n} |\phi(x)|^q\,d\nu(x) \quad \text{as } j \to \infty. \tag{7.65}$$

Hence we have

$$\|\phi(u_j-u)\|^q_{L^q_\gamma(\mathbf{R}^n)} \to \int_{\mathbf{R}^n}|\phi(x)|^q d\nu(x) - \|\phi u\|^q_{L^q_\gamma(\mathbf{R}^n)} \quad \text{as } j\to\infty. \tag{7.66}$$

Since $1/p=\tau_{p,q}+1/q$, from Hölder's inequality and the assertion 2 of Proposition 7.5 we have

$$\begin{aligned}
&\|(u_j-u)\nabla\phi\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = \||\nabla\phi|(u_j-u)I_{1+\gamma+n/p'}\|_{L^p(B_R)} \\
&\le \|\nabla\phi\|_{L^{1/\tau_{p,q}}(\mathbf{R}^n)}\|(u_j-u)I_{1+\gamma+n/p'}\|_{L^q(B_R)} = \|\nabla\phi\|_{L^{1/\tau_{p,q}}(\mathbf{R}^n)}\|u_j-u\|_{L^q_{\gamma+1-n\tau_{p,q}}(\mathbf{R}^n)} \\
&\to 0 \quad \text{as } j\to\infty.
\end{aligned}\tag{7.67}$$

Here we used the relations; $p(n-(1+\gamma+n/p'))=n-p(1+\gamma)$ and $q(n-(1+\gamma+n/p'))=n-q(1+\gamma-n\tau_{p,q})$. By the assertion 3 of Proposition 7.5 we have

$$\|\phi\nabla u_j\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} = \int_{\mathbf{R}^n}|\phi(x)|^p|\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\,dx \to \int_{\mathbf{R}^n}|\phi(x)|^p d\mu(x) \quad \text{as } j\to\infty. \tag{7.68}$$

Then, letting $j\to\infty$ in the inequality below

$$\begin{aligned}
&(S^{p,q;\gamma})^{1/p}\|\phi(u_j-u)\|_{L^q_\gamma(\mathbf{R}^n)} \le \|\nabla[\phi(u_j-u)]\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \\
&\le \|\phi\nabla[u_j-u]\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|(u_j-u)\nabla\phi\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \\
&\le 2^{1/p'}(\|\phi\nabla u_j\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|\phi\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)})^{1/p} + \|(u_j-u)\nabla\phi\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \quad \text{for } j\ge 1,
\end{aligned}\tag{7.69}$$

we get

$$\begin{aligned}
&(S^{p,q;\gamma})^{1/p}\Big(\int_{\mathbf{R}^n}|\phi(x)|^q d\nu(x) - \int_{\mathbf{R}^n}|\phi(x)|^q|u(x)|^q I_{q\gamma}(x)\,dx\Big)^{1/q} \\
&\qquad \le 2^{1/p'}\Big(\int_{\mathbf{R}^n}|\phi(x)|^p d\mu(x) + \int_{\mathbf{R}^n}|\phi(x)|^p|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx\Big)^{1/p} \quad \text{for } \phi\in C^\infty_{\mathrm{c}}(\mathbf{R}^n).
\end{aligned}\tag{7.70}$$

Since $\operatorname{supp}(\nu-|u|^q I_{q\gamma})_{\mathrm{s}}\subset\{0\}$ by the assertion 4 of Proposition 7.5, it follows from Lemma 7.8 that we have for some $a_0\in[0,\infty)$

$$\nu-|u|^q I_{q\gamma} = a_0\delta_0. \tag{7.71}$$

Further by letting $j\to\infty$ in the inequality below

$$(S^{p,q;\gamma})^{1/p}\|\phi u_j\|_{L^q_\gamma(\mathbf{R}^n)} \le \|\nabla[\phi u_j]\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \le \|\phi\nabla u_j\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|u_j\nabla\phi\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \quad \text{for } j\ge 1,$$

we have

$$(S^{p,q;\gamma})^{1/p}(\|\phi u\|^q_{L^q_\gamma(\mathbf{R}^n)} + a_0|\phi(0)|^q)^{1/q} \le \Big(\int_{\mathbf{R}^n}|\phi(x)|^p d\mu(x)\Big)^{1/p} + \|u\nabla\phi\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \quad \text{for } \phi\in C^\infty_{\mathrm{c}}(\mathbf{R}^n). \tag{7.72}$$

(b) Let $\varepsilon>0$ and let $\psi_\varepsilon$ be given in Definition 7.1. Noting that $1/p=\tau_{p,q}+1/q$, by Hölder's inequality we have

$$\begin{aligned}
\|u\nabla\psi_\varepsilon\|_{L^p_{1+\gamma}(\mathbf{R}^n)} &= \frac{1}{\varepsilon}\Big(\int_{B_\varepsilon}(|\tilde\psi_\varepsilon(x)||x|\cdot|u(x)||x|^\gamma)^p I_0(x)\,dx\Big)^{1/p} \\
&\le \frac{1}{\varepsilon}\Big(\int_{\mathbf{R}^n}(|\tilde\psi_\varepsilon(x)||x|)^{1/\tau_{p,q}} I_0(x)\,dx\Big)^{\tau_{p,q}}\Big(\int_{B_\varepsilon}(|u(x)||x|^\gamma)^q I_0(x)\,dx\Big)^{1/q} \\
&= \|\tilde\psi_1\|_{L^{1/\tau_{p,q}}(\mathbf{R}^n)}\|u\|_{L^q_\gamma(B_\varepsilon)}.
\end{aligned}\tag{7.73}$$

Hence, by virtue of (a) we have

$$(S^{p,q;\gamma})^{1/p} a_0^{1/q} \le (S^{p,q;\gamma})^{1/p} (\|\psi_\varepsilon u\|^q_{L^q_\gamma(\mathbf{R}^n)} + a_0)^{1/q} \tag{7.74}$$
$$\le \left( \int_{\mathbf{R}^n} |\psi_\varepsilon(x)|^p d\mu(x) \right)^{1/p} + \|u \nabla \psi_\varepsilon\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \le \left( \int_{B_\varepsilon} d\mu(x) \right)^{1/p} + \|\tilde{\psi}_1\|_{L^{1/\tau_{p,q}}(\mathbf{R}^n)} \|u\|_{L^q_\gamma(B_\varepsilon)}$$
$$= \mu(B_\varepsilon)^{1/p} + \|\tilde{\psi}_1\|_{L^{1/\tau_{p,q}}(\mathbf{R}^n)} \|u\|_{L^q_\gamma(B_\varepsilon)} \to \mu(\{0\})^{1/p} \quad \text{as } \varepsilon \to 0,$$

hence

$$\mu(\{0\}) \ge S^{p,q;\gamma} a_0^{p/q}, \ \mu \ge (S^{p,q;\gamma} a_0^{p/q}) \delta_0. \tag{7.75}$$

On the other hand, by Lemma 7.10 $|\nabla u|^p I_{p(1+\gamma)} \le \mu$ holds, and we have

$$\mu \ge |\nabla u|^p I_{p(1+\gamma)} + (S^{p,q;\gamma} a_0^{p/q}) \delta_0. \tag{7.76}$$

□

After all we have the following that proves the assertion 2 of Theorem 2.4.

**Proposition 7.7** *Assume that $1 < p < q < \infty$, $\tau_{p,q} < 1/n$ and $\gamma > 0$. Then, in Proposition 7.6, it holds that $a_0 = 0$ and*

$$\|u\|^q_{L^q_\gamma(\mathbf{R}^n)} = 1, \ \|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} = S^{p,q;\gamma}. \tag{7.77}$$

**Proof:** By the assertion 3 of Proposition 7.5 we have

$$\int_{\mathbf{R}^n} |u_j(x)|^q I_{q\gamma}(x) dx \to \int_{\mathbf{R}^n} d\nu(x), \ \int_{\mathbf{R}^n} |\nabla u_j(x)|^p I_{p(1+\gamma)}(x) dx \to \int_{\mathbf{R}^n} d\mu(x) \quad \text{as } j \to \infty.$$

Combining the assertion 1 of Proposition 7.5 with Proposition 7.6 we have

$$1 = \int_{\mathbf{R}^n} d\nu(x) = \int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x) dx + a_0 > a_0, \ S^{p,q;\gamma} = \int_{\mathbf{R}^n} d\mu(x). \tag{7.78}$$

Moreover by Proposition 7.6 and Theorem 2.1 we have

$$S^{p,q;\gamma} = \int_{\mathbf{R}^n} d\mu(x) \ge \int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x) dx + S^{p,q;\gamma} a_0^{p/q} \tag{7.79}$$
$$\ge S^{p,q;\gamma} \left( \left( \int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x) dx \right)^{p/q} + a_0^{p/q} \right) = S^{p,q;\gamma} ((1 - a_0)^{p/q} + a_0^{p/q}),$$

and then $(1 - a_0)^{p/q} + a_0^{p/q} \le 1$ and $a_0 = 0$ follow. In particular we have

$$1 = \int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x) dx, \ S^{p,q;\gamma} = \int_{\mathbf{R}^n} d\mu(x) \ge \int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x) dx, \tag{7.80}$$

and this proves the assertion. □

# 8 Proofs of Propositions 2.1, 2.2 and some assertions

In this section we establish Proposition 2.1, Proposition 2.2 and the propositions on non-existence of minimizers and failure of some imbedding inequalities, whose proofs have been postponed.

## 8.1 Proofs of Propositions 2.1 and 2.2

In order to prove Propositions 2.1 and 2.2, we introduce a cut-off function.

**Definition 8.1** *For $0<\varepsilon<1$ and $0<\eta<1/4$ we set*

$$\phi_{\varepsilon,\eta}(x)=\phi_{\varepsilon,\eta}(|x|)=\begin{cases} 1 & \text{for } x\in\overline{B_{3\varepsilon\eta}},\\ \dfrac{\log(\varepsilon(1-\eta)/|x|)}{\log((1-\eta)/(3\eta))} & \text{for } x\in\overline{B_{\varepsilon(1-\eta)}}\setminus B_{3\varepsilon\eta},\\ 0 & \text{for } x\in\mathbf{R}^n\setminus B_{\varepsilon(1-\eta)},\end{cases} \tag{8.1}$$

*and we set*

$$\psi_{\varepsilon,\eta}(x)=\psi_{\varepsilon,\eta}(|x|)=\phi_{\varepsilon,\eta}*\rho_{\varepsilon\eta}(x) \quad \text{for } x\in\mathbf{R}^n. \tag{8.2}$$

**Lemma 8.1** *Assume that $1<p\le q<\infty$, $\gamma\ge 0$, $R\ge 1$ and $0<\alpha<1/p'$. Then there exist positive numbers $c_{p;\gamma}, \bar{c}_{p;\alpha}, \underline{c}_{p,q;\alpha}>0$ such that we have for $0<\varepsilon<1$ and $0<\eta<1/8$ the followings:*

1. $\psi_{\varepsilon,\eta}\in C_c^\infty(\mathbf{R}^n)_{\mathrm{rad}}$, $0\le\psi_{\varepsilon,\eta}\le 1$ *on* $\mathbf{R}^n$, $\psi_{\varepsilon,\eta}=1$ *on* $\overline{B_{2\varepsilon\eta}}$, $\psi_{\varepsilon,\eta}=0$ *on* $\mathbf{R}^n\setminus B_\varepsilon$.
2. $\|\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\le c_{p;\gamma}\,\varepsilon^{1+\gamma}$, $\|\nabla\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\le\begin{cases}\dfrac{c_{p;\gamma}\varepsilon^\gamma}{\log(1/\eta)} & \text{if } \gamma>0,\\ \dfrac{c_{p;0}}{(\log(1/\eta))^{1/p'}} & \text{if } \gamma=0.\end{cases}$
3. $\|\nabla[A_{1,R}^\alpha\psi_{\varepsilon,\eta}]\|_{L^p_1(B_1)}\le\bar{c}_{p;\alpha}A_{1,R}(\varepsilon)^\alpha\left(\dfrac{1}{(\log(1/\eta))^{1/p'}}+\dfrac{1}{A_{1,R}(\varepsilon)^{1/p'}}\right)$,
   $\|A_{1,R}^\alpha\psi_{\varepsilon,\eta}\|_{L^q_{p;R}(B_1)}\ge\dfrac{\underline{c}_{p,q;\alpha}}{A_{1,R}(2\varepsilon\eta)^{1/p'-\alpha}}$.

**Proof:** We see that $\phi_{\varepsilon,\eta}\in W^{1,\infty}(\mathbf{R}^n)$, and first derivatives of $\phi_{\varepsilon,\eta}$ in distribution sense are given by

$$\nabla\phi_{\varepsilon,\eta}(x)=-\frac{1}{\log((1-\eta)/(3\eta))}\chi_{B_{\varepsilon(1-\eta)}\setminus\overline{B_{3\varepsilon\eta}}}(x)\frac{x}{|x|^2} \quad \text{for a.e. } x\in\mathbf{R}^n. \tag{8.3}$$

Particularly we have

$$|\nabla\phi_{\varepsilon,\eta}(x)|\le\frac{1}{\log(1/(4\eta))}\frac{1}{|x|}\chi_{B_{\varepsilon(1-\eta)}\setminus\overline{B_{3\varepsilon\eta}}}(x) \quad \text{for a.e. } x\in\mathbf{R}^n, \tag{8.4}$$

$$|\nabla\phi_{\varepsilon,\eta}(x-y)|\le\frac{1}{\log(1/(4\eta))}\frac{1}{|x|-\varepsilon\eta}\chi_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}}(x) \quad \text{for a.e. } x\in\mathbf{R}^n,\ y\in B_{\varepsilon\eta}.$$

Here we note that

$$0\le\phi_{\varepsilon,\eta}(x)\le\chi_{B_{\varepsilon(1-\eta)}}(x),\ 0\le\phi_{\varepsilon,\eta}(x-y)\le\chi_{B_\varepsilon}(x) \quad \text{for a.e. } x\in\mathbf{R}^n,\ y\in B_{\varepsilon\eta}. \tag{8.5}$$

Since the assertion 1 is now clear, we prove the assertions 2 and 3 below.

2.

$$\|\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n}\phi_{\varepsilon,\eta}(x-y)\rho_{\varepsilon\eta}(y)\,dy\right)^p I_{p(1+\gamma)}(x)\,dx\right)^{1/p}$$

$$\leq \left(\int_{B_\varepsilon}\left(\int_{B_{\varepsilon\eta}}\rho_{\varepsilon\eta}(y)\,dy\right)^p I_{p(1+\gamma)}(x)\,dx\right)^{1/p} = \left(\int_{B_\varepsilon} I_{p(1+\gamma)}(x)\,dx\right)^{1/p} = \left(\frac{\omega_n}{p(1+\gamma)}\right)^{1/p}\varepsilon^{1+\gamma},$$

$$\|\nabla\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \leq \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n}|\nabla\phi_{\varepsilon,\eta}(x-y)|\rho_{\varepsilon\eta}(y)\,dy\right)^p I_{p(1+\gamma)}(x)\,dx\right)^{1/p}$$

$$\leq \left(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}}\left(\int_{B_{\varepsilon\eta}}\frac{1}{\log(1/(4\eta))}\frac{1}{|x|-\varepsilon\eta}\rho_{\varepsilon\eta}(y)\,dy\right)^p I_{p(1+\gamma)}(x)\,dx\right)^{1/p}$$

$$= \frac{1}{\log(1/(4\eta))}\left(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}}\frac{1}{(1-\varepsilon\eta/|x|)^p}I_{p\gamma}(x)\,dx\right)^{1/p} \leq \frac{2}{\log(1/(4\eta))}\left(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}} I_{p\gamma}(x)\,dx\right)^{1/p}$$

$$= \begin{cases} 2\left(\dfrac{\omega_n}{p\gamma}\right)^{1/p}\dfrac{\varepsilon^\gamma(1-(2\eta)^\gamma)}{\log(1/(4\eta))} \leq 2\left(\dfrac{\omega_n}{p\gamma}\right)^{1/p}\dfrac{\varepsilon^\gamma}{\log(1/(4\eta))} & \text{if } \gamma>0, \\ 2\omega_n^{1/p}\dfrac{(\log(1/(2\eta)))^{1/p}}{\log(1/(4\eta))} & \text{if } \gamma=0. \end{cases}$$

3.

$$\|A_{1,R}^\alpha\nabla\psi_{\varepsilon,\eta}\|_{L^p_1(B_1)} = \left(\int_{B_1}\left(\int_{\mathbf{R}^n}|\nabla\phi_{\varepsilon,\eta}(x-y)|\rho_{\varepsilon\eta}(y)\,dy\right)^p A_{1,R}(x)^{p\alpha}I_p(x)\,dx\right)^{1/p}$$

$$\leq \left(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}}\left(\int_{B_{\varepsilon\eta}}\frac{1}{\log(1/(4\eta))}\frac{1}{|x|-\varepsilon\eta}\rho_{\varepsilon\eta}(y)\,dy\right)^p A_{1,R}(x)^{p\alpha}I_p(x)\,dx\right)^{1/p}$$

$$= \frac{1}{\log(1/(4\eta))}\left(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}}\left(\frac{A_{1,R}(x)^\alpha}{1-\varepsilon\eta/|x|}\right)^p I_0(x)\,dx\right)^{1/p} \leq \frac{2A_{1,R}(\varepsilon)^\alpha}{\log(1/(4\eta))}\left(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}} I_0(x)\,dx\right)^{1/p}$$

$$= 2\omega_n^{1/p}A_{1,R}(\varepsilon)^\alpha\frac{(\log(1/(2\eta)))^{1/p}}{\log(1/(4\eta))},$$

$$\|\psi_{\varepsilon,\eta}\nabla[A_{1,R}^\alpha]\|_{L^p_1(B_1)} = \alpha\left(\int_{B_1}\psi_{\varepsilon,\eta}(x)^p\frac{I_0(x)}{A_{1,R}(x)^{p(1-\alpha)}}\,dx\right)^{1/p}$$

$$\leq \alpha\left(\int_{B_\varepsilon}\frac{I_0(x)}{A_{1,R}(x)^{p(1-\alpha)}}\,dx\right)^{1/p} = \alpha\left(\frac{\omega_n}{p(1/p'-\alpha)}\right)^{1/p}\frac{1}{A_{1,R}(\varepsilon)^{1/p'-\alpha}},$$

$$\|\psi_{\varepsilon,\eta}A_{1,R}^\alpha\|_{L^q_{p;R}(B_1)} = \left(\int_{B_1}\psi_{\varepsilon,\eta}(x)^q\frac{I_0(x)}{A_{1,R}(x)^{1+q(1/p'-\alpha)}}\,dx\right)^{1/q}$$

$$\geq \left(\int_{B_{2\varepsilon\eta}}\frac{I_0(x)}{A_{1,R}(x)^{1+q(1/p'-\alpha)}}\,dx\right)^{1/q} = \left(\frac{\omega_n}{q(1/p'-\alpha)}\right)^{1/q}\frac{1}{A_{1,R}(2\varepsilon\eta)^{1/p'-\alpha}}.$$

□

By using these we now verify Propositions 2.1 and 2.2.

**Proof of Proposition 2.1:** 1. For $\gamma>0$ it suffices to show $C_c^\infty(\mathbf{R}^n)\subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)$. Take and fix a $u\in C_c^\infty(\mathbf{R}^n)$. Then, for $0<\varepsilon<1$ and $0<\eta<1/8$ we see that $u(1-\psi_{\varepsilon,\eta})\in$

$C_c^\infty(\mathbf{R}^n\setminus\{0\})$ holds, hence by the assertion 2 of Lemma 8.1, we obtain

$$\begin{aligned}&\|\nabla[u(1-\psi_{\varepsilon,\eta})-u]\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = \|\nabla[u\psi_{\varepsilon,\eta}]\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \\ &\le \|\nabla u\|_{L^\infty(\mathbf{R}^n)}\|\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|u\|_{L^\infty(\mathbf{R}^n)}\|\nabla\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \\ &\le c_{p;\gamma}\left(\|\nabla u\|_{L^\infty(\mathbf{R}^n)}\varepsilon^{1+\gamma} + \|u\|_{L^\infty(\mathbf{R}^n)}\frac{\varepsilon^\gamma}{\log(1/\eta)}\right) \to 0 \quad \text{as } \varepsilon\to 0,\ \eta\to 0.\end{aligned} \tag{8.6}$$

The assertion 2 is now clear, hence we proceed to 3.

3. It suffices to prove $C_c^\infty(B_1)\subset W_{0,0}^{1,p}(B_1)$. Let $u\in C_c^\infty(B_1)$. Then, for $0<\varepsilon<1$ and $0<\eta<1/8$, we see that $u(1-\psi_{\varepsilon,\eta})\in C_c^\infty(B_1\setminus\{0\})$, and hence by the assertion 2 of Lemma 8.1 we have

$$\begin{aligned}&\|\nabla[u(1-\psi_{\varepsilon,\eta})-u]\|_{L^p_1(B_1)} = \|\nabla[u\psi_{\varepsilon,\eta}]\|_{L^p_1(B_1)} \\ &\le \|\nabla u\|_{L^\infty(B_1)}\|\psi_{\varepsilon,\eta}\|_{L^p_1(B_1)} + \|u\|_{L^\infty(B_1)}\|\nabla\psi_{\varepsilon,\eta}\|_{L^p_1(B_1)} \\ &\le c_{p;0}\left(\|\nabla u\|_{L^\infty(B_1)}\varepsilon + \|u\|_{L^\infty(B_1)}\frac{1}{(\log(1/\eta))^{1/p'}}\right) \to 0 \quad \text{as } \varepsilon\to 0,\ \eta\to 0.\end{aligned} \tag{8.7}$$

□

**Proof of Proposition 2.2:** (a) First we show that if $0<\alpha<1/p'$, it holds that

$$A_{1,R}^\alpha\psi_{\varepsilon,\eta}\in W_{0,0}^{1,p}(B_1) \quad \text{for } 0<\varepsilon<1,\ 0<\eta<\frac{1}{8}. \tag{8.8}$$

For $0<\delta<\min\{2\varepsilon\eta,1/8\}$, noting that $A_{1,R}^\alpha\psi_{\varepsilon,\eta}(1-\psi_{\delta,\delta})\in C_c^\infty(B_1\setminus\{0\})$ and $\psi_{\varepsilon,\eta}\psi_{\delta,\delta}=\psi_{\delta,\delta}$, it follows from the assertion 3 of Lemma 8.1 that we have

$$\begin{aligned}&\|\nabla[A_{1,R}^\alpha\psi_{\varepsilon,\eta}(1-\psi_{\delta,\delta})-A_{1,R}^\alpha\psi_{\varepsilon,\eta}]\|_{L^p_1(B_1)} = \|\nabla[A_{1,R}^\alpha\psi_{\varepsilon,\eta}\psi_{\delta,\delta}]\|_{L^p_1(B_1)} = \|\nabla[A_{1,R}^\alpha\psi_{\delta,\delta}]\|_{L^p_1(B_1)} \\ &\le \bar{c}_{p;\alpha}A_{1,R}(\delta)^\alpha\left(\frac{1}{(\log(1/\delta))^{1/p'}}+\frac{1}{A_{1,R}(\delta)^{1/p'}}\right)\to 0 \quad \text{as } \delta\to 0.\end{aligned}$$

(b) By the assumption, for an arbitrary $\varepsilon>0$ there exists $0<\eta_\varepsilon<1/8$ such that we have

$$\frac{I_0(x)}{A_{1,R}(x)^{1+q/p'}}\le\varepsilon w(x) \quad \text{for all } x\in\overline{B_{\eta_\varepsilon}}\setminus\{0\}. \tag{8.9}$$

Then, if $0<\eta<\eta_\varepsilon$, we see that

$$\|A_{1,R}^\alpha\psi_{\eta,\eta}\|_{L^q_{p;R}(B_1)} = \|A_{1,R}^\alpha\psi_{\eta,\eta}\|_{L^q_{p;R}(B_{\eta_\varepsilon})} \le \varepsilon\|A_{1,R}^\alpha\psi_{\eta,\eta}\|_{L^q(B_{\eta_\varepsilon};w)} = \varepsilon\|A_{1,R}^\alpha\psi_{\eta,\eta}\|_{L^q(B_1;w)}.$$

Hence using the assertion 3 of Lemma 8.1, we have

$$\begin{aligned}\frac{\|\nabla[A_{1,R}^\alpha\psi_{\eta,\eta}]\|_{L^p_1(B_1)}}{\|A_{1,R}^\alpha\psi_{\eta,\eta}\|_{L^q(B_1;w)}} &\le \frac{\|\nabla[A_{1,R}^\alpha\psi_{\eta,\eta}]\|_{L^p_1(B_1)}}{\|A_{1,R}^\alpha\psi_{\eta,\eta}\|_{L^q_{p;R}(B_1)}}\varepsilon \\ &\le \frac{\bar{c}_{p;\alpha}}{\underline{c}_{p,q;\alpha}}A_{1,R}(\eta)^\alpha\Big(\frac{1}{(\log(1/\eta))^{1/p'}}+\frac{1}{A_{1,R}(\eta)^{1/p'}}\Big)A_{1,R}(2\eta^2)^{1/p'-\alpha}\varepsilon \\ &\to 2^{1+1/p'-\alpha}\frac{\bar{c}_{p;\alpha}}{\underline{c}_{p,q;\alpha}}\varepsilon \quad \text{as } \eta\to 0.\end{aligned} \tag{8.10}$$

Thus the assertion follows. □

## 8.2 Non-existence of minimizers

In this subsection we verify the assertion 4 of Theorem 2.4, Proposition 2.3 and Proposition 2.4. We remark that both the assertion 4 of Theorem 2.4 and Proposition 2.4 follow from improved Hardy-Sobolev inequalities with sharp missing terms.

First the assertion 4 of Theorem 2.4 follows from the next whose proof is seen in [Ho2].

**Lemma 8.2** *(Horiuchi) Assume that $n \geq 3$, $p = 2 < q = 2^* = 2n/(n-2)$ and $\gamma > \gamma_{2,2^*} = (n-2)/2$, then it holds that*

$$\|\nabla u\|^2_{L^2_{1+\gamma}(\mathbf{R}^n)} \geq S^{2,2^*;\gamma_{2,2^*}} \|u\|^2_{L^{2^*}_\gamma(\mathbf{R}^n)} + (\gamma^2 - \gamma^2_{2,2^*})\|u\|^2_{L^2_\gamma(\mathbf{R}^n)} \quad \text{for } u \in W^{1,2}_{\gamma,0}(\mathbf{R}^n). \tag{8.11}$$

Proposition 2.4 follows from Lemma 8.3 below, which is seen in [AH2]. Here we put for $R > e$

$$A_{2,R}(x) = A_{2,R}(|x|) = \log A_{1,R}(x) = \log\Big(\log \frac{R}{|x|}\Big) \quad \text{for } x \in \overline{B_1}\setminus\{0\}. \tag{8.12}$$

**Lemma 8.3** *(Ando-Horiuchi) For $1 < p = q < \infty$ there exist positive numbers $R_p > 0$ and $C > 0$ such that we have for $R \geq R_p$*

$$\|\nabla u\|^p_{L^p_1(B_1)} \geq \frac{1}{(p')^p}\|u\|^p_{L^p_{p;R}(B_1)} + C\int_{B_1} |u(x)|^p \frac{I_0(x)}{A_{1,R}(x)^p A_{2,R}(x)^2}\,dx \quad \text{for } u \in W^{1,p}_{0,0}(B_1). \tag{8.13}$$

Now we proceed to the proof of Proposition 2.3. To this end we epmloy the next proposition.

**Proposition 8.1** *Let $1 < p = q < \infty$ and $\gamma > 0$. If $w \in C(\mathbf{R}^n\setminus\{0\})$ satisfies*

$$w(x) \geq 0 \quad \text{for } x \in \mathbf{R}^n\setminus\{0\}, \quad \frac{(\log(1/|x|))^p}{I_{p\gamma}(x)} w(x) \to \infty \quad \text{as } x \to 0, \tag{8.14}$$

*then it holds that*

$$\inf\left\{ \frac{\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} - (S^{p,p;\gamma})^{1/p}\|u\|_{L^p_\gamma(\mathbf{R}^n)}}{\|u\|_{L^p(\mathbf{R}^n;w)}} \;\middle|\; u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})\setminus\{0\} \right\} = 0. \tag{8.15}$$

**Proof:** (a) Assuming that $R > 1$, then it follows from the assumption that we have

$$\frac{A_{1,R}(x)^{1+p/p'}}{I_0(x)} \frac{w(x)}{I_{p\gamma+n}(x)} = \frac{A_{1,R}(x)^p}{I_{p\gamma}(x)} w(x) \to \infty \quad \text{as } x \to 0. \tag{8.16}$$

Hence by Proposition 2.2

$$\inf\left\{ \Big(\frac{\|\nabla v\|_{L^p_1(B_1)}}{\|v\|_{L^p(B_1;w/I_{p\gamma+n})}}\Big)^p \;\middle|\; v \in C^\infty_{\mathrm{c}}(B_1\setminus\{0\})\setminus\{0\} \right\} = 0. \tag{8.17}$$

(b) On the contrary we assume that the assertion is false. Since $S^{p,p;\gamma} = \gamma^p$ holds, there exists a number $C > 0$ such thta we have

$$\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \geq \gamma\|u\|_{L^p_\gamma(\mathbf{R}^n)} + C\|u\|_{L^p(\mathbf{R}^n;w)} \quad \text{for } u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\}). \tag{8.18}$$

Therefore

$$\begin{aligned}
\gamma\|v\|_{L^p_0(\mathbf{R}^n)} + C\|v\|_{L^q(\mathbf{R}^n;w/I_{p\gamma+n})} &= \gamma\|\hat{T}_\gamma v\|_{L^p_\gamma(\mathbf{R}^n)} + C\|\hat{T}_\gamma v\|_{L^p(\mathbf{R}^n;w)} \leq \|\nabla[\hat{T}_\gamma v]\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \\
&= \Big(\int_{\mathbf{R}^n}\Big|\nabla v(x) - \gamma v(x)\frac{x}{|x|^2}\Big|^p I_p(x)\,dx\Big)^{1/p} \\
&\leq \|\nabla v\|_{L^p_1(\mathbf{R}^n)} + \gamma\|v\|_{L^p_0(\mathbf{R}^n)} \quad \text{for } v \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\}),
\end{aligned}$$

$$\|\nabla v\|_{L^p_1(B_1)} \geq C\|v\|_{L^p(B_1;w/I_{p\gamma+n})} \quad \text{for } v \in C^\infty_{\mathrm{c}}(B_1\setminus\{0\}).$$

This contradicts to (a). □

Let us recall the result due to [AH2].

**Lemma 8.4** *(Ando–Horiuchi) Assume that* $1 < p = q < \infty$ *and* $\gamma > 0$. *If* $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ *is a minimizer for* $S^{p,p;\gamma}$, *then* $u$ *is radially symmetric with respect to the origin and has a constant sign. Moreover if* $u \ge 0$ *on* $\mathbf{R}^n$, *then* $u$ *is a monotonically decreasing function of* $r = |x|$ *and satisfies*

$$u(x)|x|^{\gamma} \to 0 \quad as\ |x| \to 0, \;\; u(x)|x|^{\gamma} \to 0 \quad as\ |x| \to \infty. \tag{8.19}$$

From this we have the next, from which Proposition 2.3 follows.

**Proposition 8.2** *Assume that* $1 < p = q < \infty$ *and* $\gamma > 0$. *Then, there exists no minimizer for* $S^{p,p;\gamma}$ *in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$.

**Proof:** (a) Assume that there exists a minimizer $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ for $S^{p,p;\gamma}$. Then it follows from variational principle that we have

$$\int_{\mathbf{R}^n} |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\phi(x) I_{p(1+\gamma)}(x)\,dx = \gamma^p \int_{\mathbf{R}^n} |u(x)|^{p-2}u(x)\phi(x) I_{p\gamma}(x)\,dx \tag{8.20}$$

$$\text{for } \phi \in W^{1,p}_{\gamma,0}(\mathbf{R}^n).$$

By Lemma 8.4 $u$ should be radially symmetric and satisfy

$$u > 0 \quad \text{on } \mathbf{R}^n, \quad \frac{\partial u}{\partial r} < 0 \quad \text{on } \mathbf{R}^n\setminus\{0\}. \tag{8.21}$$

Hence we have for $\phi \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}$

$$-\int_0^\infty \Big(-\frac{\partial u}{\partial r}(r)\Big)^{p-1}\frac{\partial \phi}{\partial r}(r) r^{p(1+\gamma)-1}dr = \gamma^p \int_0^\infty u(r)^{p-1}\phi(r) r^{p\gamma-1}dr. \tag{8.22}$$

Since $u : (0,\infty) \to (0, u(0))$ is surjective, we have the inverse $R : (0, u(0)) \to (0,\infty)$, and by Lemma 8.4 it holds that

$$R(\varepsilon)^{\gamma}\varepsilon = u(R(\varepsilon))R(\varepsilon)^{\gamma} \to 0 \quad \text{as } \varepsilon \to 0. \tag{8.23}$$

(b) For $0 < \varepsilon < u(0)$, we set

$$u_\varepsilon(x) = (u(x)-\varepsilon)_+ = \begin{cases} u(x)-\varepsilon & \text{for } x \in B_{R(\varepsilon)}, \\ 0 & \text{for } x \in \mathbf{R}^n \setminus B_{R(\varepsilon)}. \end{cases} \tag{8.24}$$

Then $u_\varepsilon \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}$ and its derivative in a sense of distribution is given by

$$\frac{\partial u_\varepsilon}{\partial r}(x) = \begin{cases} \dfrac{\partial u}{\partial r}(x) & \text{for } x \in B_{R(\varepsilon)}, \\ 0 & \text{for } x \in \mathbf{R}^n \setminus \overline{B_{R(\varepsilon)}}. \end{cases} \tag{8.25}$$

Therefore from (a) we have

$$-\int_0^\infty \Big(-\frac{\partial u}{\partial r}(r)\Big)^{p-1}\frac{\partial u_\varepsilon}{\partial r}(r) r^{p(1+\gamma)-1}dr = \gamma^p \int_0^\infty u(r)^{p-1}u_\varepsilon(r) r^{p\gamma-1}dr, \tag{8.26}$$

$$\int_0^{R(\varepsilon)} \Big(-\frac{\partial u}{\partial r}(r)\Big)^{p} r^{p(1+\gamma)-1}dr = \gamma^p \int_0^{R(\varepsilon)} u(r)^{p} r^{p\gamma-1}dr - \varepsilon\gamma^p \int_0^{R(\varepsilon)} u(r)^{p-1} r^{p\gamma-1}\,dr.$$

Setting

$$v(r) = u(r)r^{\gamma} \quad \text{for } r > 0, \tag{8.27}$$

we have

$$\int_0^{R(\varepsilon)} \Big(\gamma v(r) - \frac{\partial v}{\partial r}(r)\Big)^p \frac{1}{r} dr = \gamma^p \int_0^{R(\varepsilon)} (v(r) - \varepsilon r^{\gamma}) v(r)^{p-1} \frac{1}{r} dr \quad \text{for } 0 < \varepsilon < u(0), \tag{8.28}$$

$$v(R(\varepsilon)) = u(R(\varepsilon))R(\varepsilon)^{\gamma} \to 0 \quad \text{as } \varepsilon \to 0.$$

(c) Since there exists a number $c_p > 0$ such that

$$|1-t|^p - 1 + pt \geq c_p \frac{t^2}{1+t^2} \quad \text{for } t \in \mathbf{R}, \tag{8.29}$$

we have for $r > 0$

$$\Big(\gamma v(r) - \frac{\partial v}{\partial r}(r)\Big)^p \geq \gamma^p v(r)^p - p\gamma^{p-1} v(r)^{p-1} \frac{\partial v}{\partial r}(r) r + c_p \gamma^p \frac{v(r)^p \frac{\partial v}{\partial r}(r)^2 r^2}{\gamma^2 v(r)^2 + v(r)^p \frac{\partial v}{\partial r}(r)^2 r^2}.$$

By using Lemma 8.4, we have

$$\gamma^p \int_0^{R(\varepsilon)} (v(r) - \varepsilon r^{\gamma}) v(r)^{p-1} \frac{1}{r} dr = \int_0^{R(\varepsilon)} \Big(\gamma v(r) - \frac{\partial v}{\partial r}(r)\Big)^p \frac{1}{r} dr \tag{8.30}$$

$$\geq \int_0^{R(\varepsilon)} \Big(\gamma^p v(r)^p \frac{1}{r} - p\gamma^{p-1} v(r)^{p-1} \frac{\partial v}{\partial r}(r) + c_p \gamma^p \frac{v(r)^p \frac{\partial v}{\partial r}(r)^2 r}{\gamma^2 v(r)^2 + v(r)^p \frac{\partial v}{\partial r}(r)^2 r^2}\Big) dr$$

$$= \gamma^p \int_0^{R(\varepsilon)} \Big(v(r)^p \frac{1}{r} + c_p \frac{v(r)^p \frac{\partial v}{\partial r}(r)^2 r}{\gamma^2 v(r)^2 + v(r)^p \frac{\partial v}{\partial r}(r)^2 r^2}\Big) dr - \gamma^{p-1} v(R(\varepsilon))^p \quad \text{for } 0 < \varepsilon < u(0). \tag{8.31}$$

Therefore we have

$$0 \leq c_p \int_0^{R(\varepsilon)} \frac{v(r)^p \frac{\partial v}{\partial r}(r)^2 r}{\gamma^2 v(r)^2 + v(r)^p \frac{\partial v}{\partial r}(r)^2 r^2} dr \leq \frac{1}{\gamma} v(R(\varepsilon))^p - \varepsilon \int_0^{R(\varepsilon)} v(r)^{p-1} r^{\gamma-1} dr \tag{8.32}$$

$$\leq \frac{1}{\gamma} v(R(\varepsilon))^p \quad \text{for } 0 < \varepsilon < u(0),$$

and then, letting $\varepsilon \to 0$, it follows from (b) that

$$\int_0^{\infty} \frac{v(r)^p \frac{\partial v}{\partial r}(r)^2 r}{\gamma^2 v(r)^2 + v(r)^p \frac{\partial v}{\partial r}(r)^2 r^2} dr = 0. \tag{8.33}$$

Thus we have a constant $c$ such that

$$0 = v(r)^{p/2} \frac{\partial v}{\partial r}(r) = \frac{2}{p+2} \frac{\partial}{\partial r} [v^{(p+2)/2}](r), \;\; c = v(r) = u(r) r^{\gamma} \quad \text{for } r > 0,$$

and this contradicts to Lemma 8.4. □

### 8.3 Failure of imbedding inequalities

In this subsection we prove $C^{p,q;1}=0$ provided that $n\ge 2$ and $p<q$. Combining this fact with the assertions 1 and 2 of Proposition 4.2, we have the assertion 2 of Theorem 2.6.

**Proposition 8.3** *Assume that $n\ge 2$, $1<p<q<\infty$, $\tau_{p,q}\le 1/n$ and $R=1$. Then it holds that $C^{p,q;1}=0$.*

Let us set

$$B'_r=\{x'\in\mathbf{R}^{n-1}\mid |x'|<r\}\quad\text{for } r>0,\quad (B_1)_+=\{x=(x',x_n)\in B_1\mid x_n>0\},$$

and let us prepare the following.

**Lemma 8.5** *For $n\ge 2$, we set*

$$\varphi(x)=\varphi(x',x_n)=(x',\varphi_n(x)),\ \ \varphi_n(x)=\varphi_n(x',x_n)=(1-|x'|^2)^{1/2}-x_n \qquad \textit{for } x=(x',x_n)\in(B_1)_+. \tag{8.34}$$

*Then we have the followings:*

1. *$\varphi:(B_1)_+\to(B_1)_+$ is a diffeomorphism and $\varphi^{-1}=\varphi$ is valid. In particular we have*
$$\varphi_n(\varphi(x))=x_n\quad\textit{for } x\in(B_1)_+.$$
2. $\det D\varphi(x)=-1$ *for $x\in(B_1)_+$.*
3. $1-x_n\le|\varphi(x)|=(|x'|^2+\varphi_n(x)^2)^{1/2}\le 1+x_n$ *for $x\in(B_1)_+$.*

**Proof of Proposition 8.3:** (a) Let us fix an $\alpha>0$. For $0<\varepsilon<1/2$ we set

$$u_\varepsilon(x)=\begin{cases}\psi_\varepsilon(\varphi(x))\varphi_n(x)^{1+\alpha} & \text{for } x\in(B_1)_+,\\ 0 & \text{for } x\in B_1\backslash(B_1)_+.\end{cases} \tag{8.35}$$

Then, we see that $u_\varepsilon\in W^{1,p}_{0,0}(B_1)$ and

$$\partial_{x_n}|\varphi(x)|=-\frac{\varphi_n(x)}{|\varphi(x)|},\quad \partial_{x_j}|\varphi(x)|=\frac{x_jx_n}{|\varphi(x)|\sqrt{1-|x'|^2}}\qquad\text{for } 1\le j\le n-1.$$

Then we have

$$\begin{aligned}|\nabla u_\varepsilon(x)|^2&=\Big(\frac{1}{\varepsilon}\frac{\tilde\psi_\varepsilon(\varphi(x))}{|\varphi(x)|}\varphi_n(x)x_n+(1+\alpha)\psi_\varepsilon(\varphi(x))\Big)^2\varphi_n(x)^{2\alpha}\frac{|x'|^2}{1-|x'|^2}\\&\quad+\Big(\frac{1}{\varepsilon}\frac{\tilde\psi_\varepsilon(\varphi(x))}{|\varphi(x)|}\varphi_n(x)^2-(1+\alpha)\psi_\varepsilon(\varphi(x))\Big)^2\varphi_n(x)^{2\alpha}\\&\le 2\Big(\frac{1}{\varepsilon^2}\frac{\tilde\psi_\varepsilon(\varphi(x))^2}{|\varphi(x)|^2}\varphi_n(x)^2(|x'|^2x_n^2+(1-|x'|^2)\varphi_n(x)^2)+(1+\alpha)^2\psi_\varepsilon(\varphi(x))^2\Big)\frac{\varphi_n(x)^{2\alpha}}{1-|x'|^2}\\&\le 2\Big(\frac{1}{\varepsilon^2}\tilde\psi_\varepsilon(\varphi(x))^2\varphi_n(x)^2+(1+\alpha)^2\psi_\varepsilon(\varphi(x))^2\Big)\frac{\varphi_n(x)^{2\alpha}}{1-|x'|^2}\quad\text{for } x\in(B_1)_+.\end{aligned} \tag{8.36}$$

(b) By using (a) and Lemma 8.5 we have

$$\begin{aligned}&|u_\varepsilon(\varphi(y))|=\psi_\varepsilon(y)y_n^{1+\alpha},\\&|[\nabla u_\varepsilon](\varphi(y))|^2\le 2\Big(\frac{1}{\varepsilon^2}\tilde\psi_\varepsilon(y)^2y_n^2+(1+\alpha)^2\psi_\varepsilon(y)^2\Big)\frac{y_n^{2\alpha}}{1-|y'|^2}\\&\qquad\le\frac{8}{3}\Big(\frac{1}{\varepsilon^2}\tilde\psi_\varepsilon(y)^2y_n^2+(1+\alpha)^2\psi_\varepsilon(y)^2\Big)y_n^{2\alpha}\quad\text{for } y\in(B_1)_+\cap B_{1/2}.\end{aligned} \tag{8.37}$$

Noting the assertion 3 of Lemma 8.5 and

$$\frac{1}{t}\log\frac{1}{1-t} \le 2\log 2 \quad \text{for } 0 < t \le \frac{1}{2}, \tag{8.38}$$

we see that

$$\begin{aligned} I_0(\varphi(y)) &\ge \frac{1}{(1+1/2)^n} = \left(\frac{2}{3}\right)^n, \\ I_p(\varphi(y)) &\le \max\left\{\left(1+\frac{1}{2}\right)^{p-n}, \left(\frac{1}{1-1/2}\right)^{n-p}\right\} \le 2^{|n-p|}, \\ A_{1,1}(\varphi(y)) &= \log\frac{1}{|\varphi(y)|} \le \log\frac{1}{1-y_n} \le (2\log 2)y_n \quad \text{for } y \in (B_1)_+ \cap B_{1/2}. \end{aligned} \tag{8.39}$$

Then, we also have

$$\begin{aligned} \|u_\varepsilon\|^q_{L^q_{p;1}(B_1)} &= \int_{(B_1)_+} |u_\varepsilon(x)|^q \frac{I_0(x)}{A_{1,1}(x)^{1+q/p'}}dx = \int_{(B_1)_+} |u_\varepsilon(\varphi(y))|^q \frac{I_0(\varphi(y))}{A_{1,1}(\varphi(y))^{1+q/p'}}dy \\ &\ge \int_{(B_1)_+\cap B_{\varepsilon/2}} (\psi_\varepsilon(y)y_n^{1+\alpha})^q \frac{1}{((2\log 2)y_n)^{1+q/p'}}\left(\frac{2}{3}\right)^n dy \\ &\ge \frac{1}{(2\log 2)^{1+q/p'}}\left(\frac{2}{3}\right)^n \int_{B'_{\varepsilon/4}\times(0,\varepsilon/4)} y_n^{q(\alpha+1/p)-1}dy \\ &= \frac{1}{(2\log 2)^{1+q/p'}}\left(\frac{2}{3}\right)^n \frac{p}{q}\frac{\omega_{n-1}}{(n-1)(1+p\alpha)}\left(\frac{\varepsilon}{4}\right)^{n-1+q(\alpha+1/p)}, \end{aligned} \tag{8.40}$$

and

$$\begin{aligned} \|\nabla u_\varepsilon\|^p_{L^p_1(B_1)} &= \int_{(B_1)_+} |\nabla u_\varepsilon(x)|^p I_p(x)dx = \int_{(B_1)_+} |\nabla u_\varepsilon(\varphi(y))|^p I_p(\varphi(y))dy \\ &\le \int_{(B_1)_+\cap B_\varepsilon} \left(\frac{8}{3}\left(\frac{1}{\varepsilon^2}\tilde{\psi}_\varepsilon(y)^2 y_n^2 + (1+\alpha)^2\psi_\varepsilon(y)^2\right)y_n^{2\alpha}\right)^{p/2} 2^{|n-p|}dy \\ &\le 2^{|n-p|}\left(\frac{8}{3}(9+(1+\alpha)^2)\right)^{p/2}\int_{B'_\varepsilon\times(0,\varepsilon)} y_n^{p\alpha}dy \\ &= 2^{|n-p|}\left(\frac{8}{3}(9+(1+\alpha)^2)\right)^{p/2}\frac{\omega_{n-1}}{(n-1)(1+p\alpha)}\varepsilon^{n+p\alpha}. \end{aligned} \tag{8.41}$$

Since $n + p\alpha - (p/q)(n-1+q(\alpha+1/p)) = (n-1)(1-p/q) > 0$ holds, after all we have

$$F^{p,q;1}(u_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

□

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