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PREPRINT 2011:3

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Gothenburg Sweden 2011

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Gothenburg, January 2011

Preprint 2011:3
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2011

A POSTERIORI ERROR ESTIMATES FOR A COUPLED WAVE SYSTEM WITH A LOCAL DAMPING

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ABSTRACT. In this paper we study a finite element method applied to a system of coupled wave equations, in a bounded smooth domain in $\mathbb{R}^d$, $d = 1, 2, 3$, associated with a locally distributed damping function. We start with a, *spatially continuous* finite element formulation allowing jump discontinuities in time. This approach yields, $L_2(L_2)$ and $L_\infty(L_2)$, *a posteriori* error estimates in terms of weighted residuals of the equation system. The proof of the a posteriori error estimates are based on the strong stability estimates for the corresponding adjoint equations. Optimal convergence rates are derived upon the maximal available regularity of the exact solution and justified through numerical examples.

1. INTRODUCTION

The finite element study for the system of one dimensional damped wave equation has been considered by several authors in various settings (see, e.g. [4], [5], and the references therein). The corresponding study for the multi-dimensional wave equation system is, however, more involved and a reasonable numerical analysis is possible only in very restrictive cases. Hence, the approximate solution of the wave propagation in an arbitrary domains in higher dimensions, especially in the system form, and with a rigorous error analysis is of vital interest. The advantages of a posteriori error bounds based on the fact that they are in terms of the residual of the computed (and therefore known) approximate solutions, rather than some norms of the unknown exact solution, which is the matter of the a priori error estimates. There are various approaches to a posteriori error estimates applied to a large number of problems (see, e.g. [1], [2]-[3], [7], and [9]-[11]).

In this paper, we consider a system of coupled, multidimensional, wave equations associated with locally damping terms. Then, introducing some vector quantities related to the solution, we reformulate this hyperbolic system as a new elliptic system of equations. We also formulate a *streamline diffusion method* (SDM) adequate for the finite element solution for the hyperbolic type pdes. This, however, will not be our main concern. Instead, we shall focus on a, spatially continuous, finite element scheme (with a *streamline-diffusion type* structure, but without the streamline-diffusion term) for the new elliptic system of equations where we allow

1991 *Mathematics Subject Classification.* 65M12, 65M15, 65M60, 35L53, 35K40.

Key words and phrases. coupled wave equation, finite element methods, a posteriori error estimate, streamline diffusion method, hyperbolic problem.

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jump discontinuities over certain time levels. For this system, we derive a posteriori error estimates in $L_2(L_2)$ and $L_\infty(L_2)$ norms. In our numerical examples we do insert the streamline diffusion term in the scheme.

Studies of this type are considered by Gergoulus and co-workers [7] where they used Galerkin finite element method for linear wave equation, without damping term, and obtained a posteriori error estimates in $L_\infty(L_2)$ norm. Johnson [12] proved existence of solution for the second order hyperbolic problems, and used discontinues Galerkin method to obtain a priori and a posteriori L_2 error estimates.

Our concern is a model problem, of interest in computational fluid mechanics and plasma physics, cf [8], formulated as follows: construct an algorithm for numerical solution of a coupled wave system with energy decay such that the error between the exact and computed solution, in a given norm, may be guaranteed to be below a given tolerance such that the computational work is nearly minimal. More specifically, we consider the following system of linear coupled wave equations:

$$\begin{cases} u_{tt} - \Delta u + \alpha(x)(u_t - v_t) = 0, & \text{in } \Omega \times [0,\infty), \\ v_{tt} - \Delta v + \alpha(x)(v_t - u_t) = 0, & \text{in } \Omega \times [0,\infty), \\ u = v = 0, & \text{in } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & \text{in } \Omega, \\ u_t(x,0) = u_1(x), \qquad v_t(x,0) = v_1(x), & \text{in } \Omega, \quad (u_t := \partial u/\partial t), \end{cases} \tag{1.1}$$

where, $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$; is a bounded domain with smooth boundary $\partial\Omega$ (for $d = 2,3$) and $\alpha \in W^{1,\infty}(\Omega)$, is a damping term, such that $\alpha(x) \geq 0$ in Ω, with

$$\alpha_0 := \int_\Omega \alpha(x)\,dx > 0.$$

Hence, $\alpha(x)$ may vanish at some points of the domain Ω, but its support has a positive measure. Here Δ denotes the Laplace operator in the space variable x.

For the existence and uniqueness of solution for the continuous problem (1.1), we refer to Raposo and Bastos, [14]. Komornik and Bopeng [13] proved that the solution for (1.1) has an exponentially decaying energy, associated with a locally distributed damping in a bounded, smooth, multidimensional domain. We use a vector form and reformulate the equation system (1.1) as *an abstract elliptic pde system*, viz

$$\begin{cases} \mathcal{L}u := \mathbf{u}_t + A\mathbf{u} = 0, & \text{in } \Omega \times [0,\infty), \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\mathbf{u}(x,t) = (u(x,t), \phi(x,t), v(x,t), \psi(x,t))^T$, with $\phi = u_t$, $\psi = v_t$ and the operator A is defined as

$$A : [H_0^1 \times L_2]^2 \longrightarrow [H_0^1 \times L_2]^2,$$

with the domain of definition $D(A) = [(H_0^1 \bigcap H^2) \times H_0^1]^2$, and the matrix-operator form

$$A = \begin{bmatrix} 0 & -\mathcal{I} & 0 & 0 \\ -\Delta & \alpha(x)\mathcal{I} & 0 & -\alpha(x)\mathcal{I} \\ 0 & 0 & 0 & -\mathcal{I} \\ 0 & -\alpha(x)\mathcal{I} & -\Delta & \alpha(x)\mathcal{I} \end{bmatrix},$$

where $\mathcal{I}$ is the identity operator. We also denote the initial data by $\mathbf{u}_0(x) = (u_0(x), u_1(x), v_0(x), v_1(x))^T$. Let now $L_2(\Omega \times [0,\infty)) := H^0(\Omega \times [0,\infty))$ be the usual Sobolev spaces of Lebesgue square integrable functions defined in $\Omega \times [0,\infty)$. By $H_0^1(\Omega \times [0,\infty))$ we shall mean a subspace of $H^1(\Omega \times [0,\infty))$, consisting of the functions vanishing on $\partial\Omega\times[0,\infty)$, where $H_0^1(\Omega\times[0,\infty))$ consists of all functions in $H^0(\Omega\times[0,\infty))$ having also all their first order partial derivatives in $H^0(\Omega\times[0,\infty))$.

The remaining part of this paper is organized as follows: In Section 2 we give the preliminaries and formulate the finite element method for (1.2), considering space-time slabs: $S_n := \Omega\times I_n$, with $I_n = (t_{n-1}, t_n)$, $n = 1, 2, \dots, n$, being subintervals of the time domain. In Section 3.1, we study a posteriori error estimates for (1.2), and derive optimal $L_2(L_2)$ and $L_\infty(L_2)$ norm error bounds. The main ingredients of the proof are through a duality argument. In Section 3.2, we introduce projection operators and, once again using duality, derive the interpolation estimates and complete the proofs of the a posteriori error bounds. Section 4 is devoted to the proofs of the strong stability estimates for the dual problems. Some computational results are given in Section 5.

2. Notations and preliminaries

In this section, we consider a time discontinuous Galerkin method for solving (1.2) that is based on using finite elements over the space-time domain $\Omega\times[0,T]$. To define this method, let $0 = t_0 < t_1 < \dots < t_N = T$ be a subdivision of the time interval $[0,T]$ into the subintervals $I_n = (t_n, t_{n+1})$, with the time steps $k_n = t_{n+1} - t_n$, $n = 0, 1, ..., N-1$, and introduce the corresponding space-time slabs:

$$S_n = \{\ (x,t):\ x\in\Omega,\ t_n < t < t_{n+1}\ \},\quad n = 0, 1, ..., N-1. \tag{2.1}$$

For notational convenience, we shall denote the mesh function for the time discretization by $k = k(t)$, where $k(t) = k_n$ for $t\in(t_n, t_{n+1})$. Further, we shall assume that, in the one-dimensional case, Ω is a bounded open interval, and for $d\geq 2$, Ω is an open bounded subset of $\mathbb{R}^d$ with a piecewise smooth boundary $\partial\Omega$. We shall use standard procedure partitioning Ω into subintervals ($d = 1$), quasiuniform triangular elements ($d = 2$), or tetrahedrons (with corresponding minimal vertex angle conditions) for $d = 3$.

2.1. The time discontinuous Galerkin scheme.

For each n let $\mathbf{U}^n$ be a finite element subspace of $[H_0^1(S_n)\times L_2(S_n)]^2$. On each slab S_n we formulate a spatially continuous problem as: for each $n = 0, ..., N-1$, find $\mathbf{u}^n\in\mathbf{U}^n$ such that

$$(\mathbf{u}_t^n + A\mathbf{u}^n, \mathbf{g})_n + \langle\mathbf{u}_+^n, \mathbf{g}_+\rangle_n = \langle\mathbf{u}_-^n, \mathbf{g}_+\rangle_n, \qquad\forall\quad \mathbf{g}\in\mathbf{U}^n, \tag{2.2}$$

where, we use the following notations:

$$(\mathbf{w},\mathbf{g})_n = \int_{S_n}\mathbf{w}^T\cdot\mathbf{g}\,dxdt,$$

$$\langle\mathbf{w},\mathbf{g}\rangle_n = \int_\Omega\mathbf{w}^T(x,t_n)\cdot\mathbf{g}(x,t_n)dx,$$

$$\mathbf{w}_\pm(x,t) = \lim_{s\to 0^\pm}\mathbf{w}(x,t+s),$$

The $\langle\cdot,\cdot\rangle$-term yields a jump which imposes a weakly enforced continuity condition across the slab interfaces at each time level $t = t_n$: a mechanism which governs the flow of information from one slab to adjacent one in the positive time direction. Note that, we have defined the inner product in the space $[H_0^1(S_n)\times L_2(S_n)]^2$, $n = 0, 1, ..., N-1$, and with

$$\mathbf{u}_j = (u_j, \phi_j, v_j, \psi_j)^T,\qquad j = 1, 2,$$

by

$$(\mathbf{u}_1, \mathbf{u}_2)_n = \int_{S_n} (\nabla u_1 \cdot \nabla u_2 + \nabla v_1 \cdot \nabla v_2 + \phi_1 \phi_2 + \psi_1 \psi_2)\, dx\, dt. \tag{2.3}$$

Summing over n, we get the function space $\mathbf{U} := \prod_{n=0}^{N-1} \mathbf{U}^n$. Thus we may rewrite (2.2) in a more concise form as follows: find $\tilde{\mathbf{u}} \in \mathbf{U}$ such that

$$B(\tilde{\mathbf{u}}, \mathbf{g}) = L(\mathbf{g}), \qquad \forall g \in \mathbf{U}, \tag{2.4}$$

where the bilinear form $B(\cdot, \cdot)$ and the linear form $L(\cdot)$ are define by

$$B(\tilde{\mathbf{u}}, \mathbf{g}) = \sum_{n=0}^{N-1} (\mathbf{u}_t^n + A\mathbf{u}^n, g)_n + \sum_{n=1}^{N-1} \langle [\mathbf{u}^n], \mathbf{g}_+ \rangle_n + \langle \mathbf{u}_+^n, \mathbf{g}_+ \rangle_0, \tag{2.5}$$

and

$$L(g) = \langle \mathbf{u}_0, \mathbf{g}_+ \rangle_0, \tag{2.6}$$

respectively. A corresponding weak variational formulation for the continuous problem (1.2) would be as

$$B(\mathbf{u}, \mathbf{g}) = L(\mathbf{g}), \qquad \forall g \in [H_0^1(\Omega) \times L_2(\Omega)]^2, \tag{2.7}$$

were in (2.5) we replace $\tilde{\mathbf{u}}$ and $\mathbf{u}$ by $\mathbf{u}$ and put the jumps $[\mathbf{u}] \equiv 0$. We let now $\mathbf{u}^n = (u^n, \phi^n, v^n, \psi^n)^T$ and introduce the jump

$$[\mathbf{u}^n] = ([u^n], [\phi^n], [v^n], [\psi^n])^T,$$

where for $q = u^n, \phi^n, v^n, \psi^n$ we have $[q] = q_+ - q_-$. Finally, we let $\mathcal{T}_h$ be a partition of Ω into a quasiuniform triangular ($d = 2$) or tetrahedral ($d = 3$) domains of the maximal diameter h (the mesh size) and introduce

$$\mathbf{U}_h^n = \{\mathbf{u}^n \in [H_0^1(S_n) \times L_2(S_n)]^2 : \mathbf{u}^n|_K \in [P_\ell(K) \times P_\ell(K)]^2 \;\text{ for } K \in \mathcal{T}_h\}.$$

Where $P_\ell(K)$ denotes the set of polynomials in K of degree less than or equal ℓ and we define the discrete function space $\mathbf{U}_h$, by

$$\mathbf{U}_h = \prod_{n=0}^{N-1} \mathbf{U}_h^n.$$

Thus (2.4) can be reformulated as follows: find $\mathbf{u}_h \in \mathbf{U}_h$ such that

$$B(\mathbf{u}_h, \mathbf{g}) = L(\mathbf{g}), \qquad \forall \mathbf{g} \in \mathbf{U}_h. \tag{2.8}$$

Finally, subtracting (2.8) from (2.7), for $\mathbf{g} \in \mathbf{U}_h$, we end up with the Galerkin orthogonality relation

$$B(\mathbf{e}, \mathbf{g}) = 0, \qquad \forall \mathbf{g} \in \mathbf{U}_h, \tag{2.9}$$

where $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$, represents the error.

3. A posteriori error analysis

In this section, we estimate the error of a particular approximation of solution, in some weighted norms, by using the information from computation. The procedure is split in the following two steps.

3.1. **Dual problem, stability and error representation formula in $L_2(L_2)$.** In this part, we shall state the dual problem for the weak (variational) formulation given for the continuous problem (1.2), with jump discontinuities across time levels $t = t_n$. Find $\mathbf{u}_h \in \mathbf{U}_h$, such that for $n = 0, 1, \ldots, N-1$:

$$\sum_{n=0}^{N-1} (\mathbf{u}_{h,t}^n + A\mathbf{u}_h^n, \mathbf{g})_n + \sum_{n=1}^{N-1} \langle [\mathbf{u}_h^n], \mathbf{g}_+ \rangle_n + \langle \mathbf{u}_{h,+}^n, \mathbf{g}_+ \rangle_0 = \langle \mathbf{u}_0, \mathbf{g}_+ \rangle_0, \tag{3.1}$$

where $g \in \mathbf{U}_h$ and $\mathbf{u}_{h,-}^0 = \mathbf{u}_0$. In order to obtain a representation of the error, we consider the following dual problem: find $\Phi \in [H_0^1(\Omega \times [0, \infty)) \times L_2(\Omega \times [0, \infty))]^2$, such that

$$\begin{cases} \mathcal{L}^*\Phi \equiv -\Phi_t + A^T\Phi = \Psi^{-1}\mathbf{e}, & \text{in } \Omega, \\ \Phi(x,t)\Big|_{t=T} = 0, & x \in \Omega, \end{cases} \tag{3.2}$$

where $\mathcal{L}^*$ denotes the adjoint of the differential operator $\mathcal{L}$ describing the left hand side of the first equation in (1.2), and Ψ is a positive weight function. Note that this problem is computed "backward", but there is a corresponding change in sign. In what follows, we use the following notation of weighted L_2 norm:

$$\| \mathbf{u} \|_{L_2^\psi(\Omega)} = (\mathbf{u}, \Psi\mathbf{u})_\Omega^{1/2}. \tag{3.3}$$

Multiplying (3.2) by $\mathbf{e}$ and integrating by parts over Ω yields the following error representation formula:

$$\| \mathbf{e} \|_{L_2^{\Psi^{-1}}(\Omega)}^2 = (\mathbf{e}, \Psi^{-1}\mathbf{e})_\Omega = (\mathbf{e}, \mathcal{L}^*\Phi)_\Omega = \sum_{n=0}^{N-1} (\mathbf{e}, -\Phi_t)_n + \sum_{n=0}^{N-1} (\mathbf{e}, A^T\Phi)_n. \tag{3.4}$$

Further partial integration in t yields

$$(\mathbf{e}, -\Phi_t)_n = -\int_\Omega \left(\mathbf{e}^T(x,t) \cdot \Phi(x,t)\Big|_{t=t_n}^{t=t_{n+1}} \right) dx + (\mathbf{e}_t, \Phi)_n. \tag{3.5}$$

We recall that $\mathbf{e} = \mathbf{e}(x,t) = (e_1, e_2, e_3, e_4)^T$ and $\Phi = \Phi(x,t) = (\phi_1, \phi_2, \phi_3, \phi_4)^T$, moreover for $n = 0, 1, ..., N-1$:

$$\begin{aligned} (\mathbf{e}, A^T\Phi)_n &= \int_{S_n} \mathbf{e}^T \begin{bmatrix} 0 & -\Delta & 0 & 0 \\ -\mathcal{I} & \alpha(x)\mathcal{I} & 0 & -\alpha(x)\mathcal{I} \\ 0 & 0 & 0 & -\Delta \\ 0 & -\alpha(x)\mathcal{I} & -\mathcal{I} & \alpha(x)\mathcal{I} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} dxdt \\ &= \int_{S_n} [e_1, e_2, e_3, e_4] \begin{bmatrix} -\Delta\phi_2 \\ -\phi_1 + \alpha(x)\phi_2 - \alpha(x)\phi_4 \\ -\Delta\phi_4 \\ -\alpha(x)\phi_2 - \phi_3 + \alpha(x)\phi_4 \end{bmatrix} dxdt. \end{aligned} \tag{3.6}$$

Hence

$$
\begin{aligned}
&(\mathbf{e}, A^T\Phi)_n = \\
&= \int_{S_n} (-e_1\Delta\phi_2) + e_2(-\phi_1 + \alpha(x)(\phi_2 - \phi_4)) - e_3\Delta\phi_4 + e_4(-\phi_3 + \alpha(x)(\phi_4 - \phi_2)) \\
&= \int_{S_n} (\nabla e_1 \cdot \nabla\phi_2 - e_2\phi_1 + \alpha(x)(e_2\phi_2 - e_2\phi_4 - e_4\phi_2 + e_4\phi_4) + \nabla e_3 \cdot \nabla\phi_4 - e_4\phi_3) \\
&= \int_{S_n} (-\Delta e_1\phi_2 - e_2\phi_1 + \alpha(x)(e_2\phi_2 - e_2\phi_4 - e_4\phi_2 + e_4\phi_4) - \Delta e_3\phi_4 - e_4\phi_3)dxdt \\
&= \int_{S_n} (A\mathbf{e})^T \cdot \Phi dxdt = (A\mathbf{e}, \Phi)_n.
\end{aligned}
$$

Now we compute the sum of the jumps appearing on the right hand side of (3.5),

$$
\begin{aligned}
J &= \sum_{n=0}^{N-1} \int_\Omega \Big(\mathbf{e}^T(x, t_{n+1}) \cdot \Phi(x, t_{n+1}) - \mathbf{e}^T(x, t_n) \cdot \Phi(x, t_n) \Big)\, dx \\
&= (\langle \mathbf{e}_-, \Phi_- \rangle_1 - \langle \mathbf{e}_+, \Phi_+ \rangle_0) + (\langle \mathbf{e}_-, \Phi_- \rangle_2 - \langle \mathbf{e}_+, \Phi_+ \rangle_1) + \ldots + \\
&\quad + (\langle \mathbf{e}_-, \Phi_- \rangle_{N-1} - \langle \mathbf{e}_+, \Phi_+ \rangle_{N-2}) + (\langle \mathbf{e}_-, \Phi_- \rangle_N - \langle \mathbf{e}_+, \Phi_+ \rangle_{N-1}).
\end{aligned}
$$

We rearrange the above sum writing $\Phi_-^n = \Phi_-^n - \Phi_+^n + \Phi_+^n$, for $n = 1, ..., N-1$. Then, we may write

$$
-J = \langle \mathbf{e}_-, \Phi_- \rangle_N + \langle \mathbf{e}_+, \Phi_+ \rangle_0 + \sum_{n=0}^{N-1} \langle [\mathbf{e}], \Phi_+ \rangle_n + \sum_{n=0}^{N-1} \langle \mathbf{e}_-, [\Phi] \rangle_n.
$$

According to (3.2), $\Phi(\cdot, t_N = T) = 0$ and since $\mathbf{e}_-^0 = [\mathbf{u}_0] = 0$, we get

$$
J = \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], \Phi_+ \rangle_n. \tag{3.7}
$$

Hence, using (3.5)- (3.7) in (3.4), yields

$$
\begin{aligned}
\| \mathbf{e} \|^2_{L_2^{\Psi^{-1}}(\Omega)} &= \sum_{n=0}^{N-1} (\mathbf{e}_t, \Phi)_n + \sum_{n=0}^{N-1} (A\mathbf{e}, \Phi)_n - \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], \Phi_+ \rangle_n \\
&= \sum_{n=0}^{N-1} ((\mathbf{u} - \mathbf{u}_h)_t + A(\mathbf{u} - \mathbf{u}_h), \Phi)_n - \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], \Phi_+ \rangle_n \\
&= \sum_{n=0}^{N-1} (-\mathbf{u}_{h,t} - A\mathbf{u}_h, \Phi)_n - \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], \Phi_+ \rangle_n.
\end{aligned}
$$

So that recalling (3.1) and using the Galerkin orthogonality (2.9), we obtain the final form of *the error representation formula*

$$
\| \mathbf{e} \|^2_{L_2^{\Psi^{-1}}(\Omega)} = \sum_{n=0}^{N-1} (\mathbf{u}_{h,t} + A\mathbf{u}_h, \hat{\Phi} - \Phi)_n + \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], (\hat{\Phi} - \Phi)_+ \rangle_n \equiv I + II, \tag{3.8}
$$

where $\hat{\Phi} \in \mathbf{U}_h$ is an interpolant of Φ. The idea is now to estimate $\hat{\Phi} - \Phi$ in terms of $\Psi^{-1}\mathbf{e}$ using strong stability estimates for the solution Φ of the dual problem (3.2).

3.2. **A posteriori error estimates for the dual solution in $L_2(L_2)$.** Here, for the interpolant $\hat{\Phi} \in \mathbf{U}_h$ in (3.8), we may consider a certain space-time L_2-projection of Φ. To this end first we define the projections:

$$P_n : [H_0^1 \times L_2]^2 \Longrightarrow \mathbf{U}_h^n,$$

and the, local, time averages

$$\pi_n : [L_2(S_n)]^4 \longrightarrow \Pi_{0,n} = \{\mathbf{u} \in [L_2(S_n)]^4 : \mathbf{u}(x,.) \text{ is constant on } I_n, \;\; x \in \Omega\},$$

satisfying

$$\int_\Omega (P_n\Phi)^T \cdot \mathbf{u}\, dx = \int_\Omega \Phi^T \cdot \mathbf{u}\, dx, \qquad \forall \mathbf{u} \in \mathbf{U}_h^n,$$
$$\pi_n \mathbf{u} \mid_{S_n} = \frac{1}{k_n} \int_{I_n} \mathbf{u}(\cdot, t)\, dt, \qquad \forall \mathbf{u} \in \Pi_{0,n}.$$

Then, we define $\hat{\Phi} \mid_{S_n} \in \mathbf{U}_h^n$ as

$$\hat{\Phi} \mid_{S_n} = P_n \pi_n \Phi = \pi_n P_n \Phi \in \mathbf{U}_h^n,$$

where $\Phi = \Phi \mid_{S_n}$. Further, if we introduce P and π defined by

$$(P\Phi) \mid_{S_n} = P_n(\Phi \mid_{S_n}),$$

and

$$(\pi\Phi) \mid_{S_n} = \pi_n(\Phi \mid_{S_n}),$$

respectively, then we can choose $\hat{\Phi} \in \mathbf{U}_h$ to be

$$\hat{\Phi} = P\pi\Phi = \pi P\Phi.$$

Now, we define the residuals for the computed solution $\mathbf{u}_h$ by

$$R_0 = \mathbf{u}_{h,t} + A\mathbf{u}_h, \quad R_1 = (\mathbf{u}_{h,+}^n - \mathbf{u}_{h,-}^n)/k_n \;\text{ on } S_n, \quad R_2 = (P_n - \mathcal{I})\mathbf{u}_{h,-}^n/k_n \;\text{ on } S_n,$$

where $\mathcal{I}$ is the identity operator. Below, in our a posteriori approach we shall see how these residuals will appear in a natural way.

To estimates I and II we shall use stability estimates based on the following interpolation estimate for the projection operator P.

Lemma 3.1. *There is a constant C such that for a given residual $R \in L_2(\Omega)$,*

$$| (R, \Phi - P\Phi)_\Omega | \leq C \parallel h^2(\mathcal{I} - P)R \parallel_{L_2^{\Psi^{-1}}(\Omega)} \parallel \Phi \parallel_{\dot{H}^{2,\Psi}(\Omega)}, \tag{3.9}$$

where $\dot{H}^{2,\Psi}$ is, the Ψ-weighted, seminorm.

Proof. The proof is an extension of the one space dimensional approach given in [11] (see also [8]), the details are lengthy and therefore omitted. □

Now, we prove the a posteriori error estimates by bounding the terms I and II in the error representation formula (3.8). To this end we introduce the stability factors (see [4] and [6]) associated with discretization in time and space, defined by

$$\gamma_{\mathbf{e}}^t = \frac{\parallel \Phi_t \parallel_{L_2^{\Psi}(\Omega)}}{\parallel \mathbf{e} \parallel_{L_2^{\Psi^{-1}}(\Omega)}}, \tag{3.10}$$

and

$$\gamma_{\mathbf{e}}^x = \frac{\parallel \Phi \parallel_{\dot{H}^{2,\Psi}(\Omega)}}{\parallel \mathbf{e} \parallel_{L_2^{\Psi^{-1}}(\Omega)}} \tag{3.11}$$

respectively. We now combine (3.8), the interpolation estimate (3.9) and the strong stability factors (3.10) and (3.11), and derive the $L_2(L_2)$ a posteriori error estimates for the finite element scheme (3.1).

Theorem 1. *Let* $\mathbf{u}$ *be the solution of the continuous problem* (1.2) *and* $\mathbf{u}_h$ *its finite element approximation given by* (3.1). *Then, the error* $\mathbf{e} := \mathbf{u} - \mathbf{u}_h$*, satisfies*

$$\begin{aligned} \| \mathbf{e} \|_{L_2^{\Psi^{-1}}(\Omega)} \le C\gamma_{\mathbf{e}}^x \| h^2(\mathcal{I} - P)R_0 \|_{L_2^{\Psi^{-1}}(\Omega)} + C\gamma_{\mathbf{e}}^t \| k_n R_1 \|_{L_2^{\Psi^{-1}}(\Omega)} \\ + \gamma_{\mathbf{e}}^x \| h^2 R_2 \|_{L_2^{\Psi^{-1}}(\Omega)} + \gamma_{\mathbf{e}}^t \| k_n R_2 \|_{L_2^{\Psi^{-1}}(\Omega)} . \end{aligned} \tag{3.12}$$

Proof. Using the above notation, from (3.8) we have

$$\| \mathbf{e} \|^2_{L_2^{\Psi^{-1}}(\Omega)} = \sum_{n=0}^{N-1} (R_0, \hat{\Phi} - \Phi)_n + \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], (\hat{\Phi} - \Phi)_+ \rangle_n := I + II.$$

We shall estimate I and II separately. Writing $\hat{\Phi} - \Phi = \hat{\Phi} - P\Phi + P\Phi - \Phi$ and using $\hat{\Phi}_n = \pi_n P\Phi$, we have

$$\begin{aligned} I &= \sum_{n=0}^{N-1} (R_0, \hat{\Phi}_n - P\Phi + P\Phi - \Phi)_n = \sum_{n=0}^{N-1} (R_0, (\pi_n - \mathcal{I})P\Phi)_n + \sum_{n=0}^{N-1} (R_0, P\Phi - \Phi)_n \\ &\le C \| h^2(\mathcal{I} - P)R_0 \|_{L_2^{\Psi^{-1}}(\Omega)} \| \Phi \|_{\dot{H}^{2,\Psi}(\Omega)}, \end{aligned}$$

where we have used the fact that R_0 is constant in time and by the definition of the projections we can easily get that the contribution form the first term in the first sum is zero. To estimate II we use (3.9) and the identity

$$\Phi_+^n(x) = \Phi(x,t) - \int_{t_n}^t \frac{\partial}{\partial \tau} \Phi(x,\tau)\, d\tau,$$

so that integrating over I_n yields

$$k_n \Phi_+^n(x) = \int_{I_n} \Phi(x,t)dt - \int_{I_n} \int_{t_n}^t \Phi_\tau(x,\tau) d\tau dt, \tag{3.13}$$

where $\Phi_\tau = \frac{\partial \Phi}{\partial \tau}$ and $\hat{\Phi}_n = \hat{\Phi}(\cdot, t_n)$. Now we rewrite II as

$$\begin{aligned} II &= \sum_{n=0}^{N-1} \left\langle k_n \frac{[\mathbf{u}_h]}{k_n}, (\hat{\Phi} - \Phi)_+ \right\rangle_n = \sum_{n=0}^{N-1} \left\langle k_n \frac{[\mathbf{u}_h]}{k_n}, (\hat{\Phi}_n - P\Phi + P\Phi - \Phi)_+ \right\rangle_n \\ &= \sum_{n=0}^{N-1} \left\langle k_n \frac{[\mathbf{u}_h]}{k_n}, (\hat{\Phi}_n - P\Phi)_+ \right\rangle_n + \sum_{n=0}^{N-1} \left\langle k_n \frac{[\mathbf{u}_h]}{k_n}, (P\Phi - \Phi)_+ \right\rangle_n := II_1 + II_2. \end{aligned}$$

To estimate II_1, we use (3.13) to get

$$\begin{aligned} II_1 &= \sum_{n=0}^{N-1} \langle k_n R_1 , (\hat{\Phi}_n)_+ - P\Phi_+ \rangle_n = \sum_{n=0}^{N-1} \langle R_1 , k_n \hat{\Phi}_n - P k_n \Phi_+ \rangle_n \\ &= \sum_{n=0}^{N-1} \left\langle R_1 , k_n \hat{\Phi}_n - \int_{I_n} P\Phi(\cdot, t)dt + \int_{I_n} \int_{t_n}^t P\Phi_\tau(\cdot, \tau) d\tau dt \right\rangle_n \\ &= \sum_{n=0}^{N-1} \int_{I_n} \int_{t_n}^t \langle R_1 , P\Phi_\tau(\cdot, \tau) \rangle_n d\tau dt \\ &\le \| k_n R_1 \|_{L_2^{\Psi^{-1}}(\Omega_T)} \| P\Phi_t \|_{L_2^{\Psi}(\Omega_T)} \le \| k_n R_1 \|_{L_2^{\Psi^{-1}}(\Omega_T)} \| \Phi_t \|_{L_2^{\Psi}(\Omega_T)}, \end{aligned}$$

were $\Omega_T := \Omega\times[0,T]$. As for the II_2-terms we can write

$$\begin{aligned}
II_2 &= \sum_{n=0}^{N-1}\Big\langle k_n\frac{[\mathbf{u}_h]}{k_n}\,,\,(P\Phi-\Phi)_+\Big\rangle_n = \sum_{n=0}^{N-1}\Big\langle \frac{\mathbf{u}^n_{h,+}-\mathbf{u}^n_{h,-}}{k_n}\,,\,(P_n-I)k_n\Phi_+\Big\rangle_n\\
&= \sum_{n=0}^{N-1}\Big\langle \frac{P_n\mathbf{u}^n_{h,-}-\mathbf{u}^n_{h,-}}{k_n}\,,\,(P_n-I)\Big(\int_{I_n}\Phi(.,t)dt-\int_{I_n}\int_{t_n}^t\Phi_\tau(.,\tau)d\tau dt\Big)\Big\rangle_n\\
&= \sum_{n=0}^{N-1}\int_{I_n}\Big\langle \frac{(P_n-I)\mathbf{u}^n_{h,-}}{k_n}\,,\,(P_n-I)\Phi(.,t)\Big\rangle_n dt\\
&\quad -\sum_{n=0}^{N-1}\int_{I_n}\int_{t_n}^t\Big\langle \frac{(P_n-I)\mathbf{u}^n_{h,-}}{k_n}\,,\,(P_n-I)\Phi_\tau(.,t)d\tau dt\Big\rangle_n\\
&\le \| k_nR_2\|_{L_2^{\Psi^{-1}}(\Omega_T)}\|\Phi\|_{\dot H^{2,\Psi}(\Omega_T)}+\| k_nR_2\|_{L_2^{\Psi^{-1}}(\Omega_T)}\|\Phi_t\|_{L_2^{\Psi}(\Omega_T)}\,.
\end{aligned}$$

The final estimate is obtained by collecting the terms and using the definition of the stability factors (3.10) and (3.11). □

3.3. **An a posteriori error estimates in $L_\infty(L_2)$.** We shall now derive *a posteriori error bounds* in the $L_\infty(L_2)$-norm for the scheme (3.1). To this end, we introduce the dual problem

$$\tag{3.14}\begin{cases} L^*\Phi\equiv-\Phi_t+A^T\Phi=0, & \text{in } \Omega,\quad 0<t<T,\\ \Phi(x,T)=E, & x\in\Omega,\end{cases}$$

where E satisfies the Poisson equation

$$\tag{3.15} -\Delta E=\mathbf{e},\qquad \text{with}\quad e(x)=\mathbf{u}(x)-\mathbf{u}_h(x),\quad x\in\Omega.,$$

We define the *energy* norm

$$||\mathbf{e}||_{L_2(\Omega)}=(\nabla E(T),\nabla E(T))_\Omega^{1/2}.$$

Then, using (3.15) and partial integration, we get

$$\begin{aligned}
||\mathbf{e}||^2_{L_2(\Omega)} &= ||\nabla_x E||^2_{L_2(\Omega)}\\
&= \langle\mathbf{e}_-\,,\,\Phi\rangle_N+\sum_{n=0}^{N-1}(\mathbf{e},L^*\Phi)_n=\langle\mathbf{e}_-\,,\,\Phi\rangle_N+\sum_{n=0}^{N-1}(\mathbf{e}\,,\,-\Phi_t+A^T\Phi)_n\\
&= \langle\mathbf{e}_-\,,\,\Phi\rangle_N-\sum_{n=0}^{N-1}\mathbf{e}^T.\Phi|_{t_n}^{t_{n+1}}+\sum_{n=0}^{N-1}(\mathbf{e}_t\,,\,\Phi)_n+\sum_{n=0}^{N-1}(A\mathbf{e},\Phi)_n\\
&= \sum_{n=0}^{N-1}(\mathbf{e}_t\,,\,\Phi)_n+\sum_{n=0}^{N-1}(A\mathbf{e}\,,\,\Phi)_n-\sum_{n=0}^{N-1}\langle[\mathbf{u}_h],\Phi_+\rangle_n\\
&= \sum_{n=0}^{N-1}(-\mathbf{u}_{h,t}-A\mathbf{u}_h\,,\,\Phi)_n-\sum_{n=0}^{N-1}\langle[\mathbf{u}_h]\,,\,\Phi_+\rangle_n.
\end{aligned}$$

By using the Galerkin orthogonality (2.3), we may subtract the interpolant $\hat\Phi\in\mathbf{U}_h$, from Φ on the right hand side above, without changing the norm:

$$\tag{3.16} ||\mathbf{e}||^2_{L_2(\Omega)}=\sum_{n=0}^{N-1}(\mathbf{u}_{h,t}+A\mathbf{u}_h\,,\,\hat\Phi-\Phi)_n+\sum_{n=0}^{N-1}\langle[\mathbf{u}_h]\,,\,(\hat\Phi-\Phi)_+\rangle_n.$$

Here, once again we need to introduce stability factors (see (3.10)-(3.11)), this time in modified norms, adequate in the study of the fully discrete (space-time discretization) problem in the L_∞-norm:

$$\gamma_E^t = \frac{\| \Phi_t \|_{L_1(L_2(\Omega))}}{\| \mathbf{e} \|_{L_2(\Omega)}}, \tag{3.17}$$

and

$$\gamma_E^x = \frac{\| \Phi \|_{L_1(\dot{H}^2(\Omega))}}{\| \mathbf{e} \|_{L_2(\Omega)}}. \tag{3.18}$$

Now using interpolation estimates (3.9) and a similar argument as in the proof of the Theorem 3.1, we get the following $L_\infty(L_2)$ estimate.

Theorem 2. *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be as in the Theorem 3.1. Then for the error* $\mathbf{e} : \mathbf{u} - \mathbf{u}_h$*, we have,*

$$\begin{aligned} \| \mathbf{e} \|_{L_2(\Omega)} \leq C\gamma_E^x \| h^2(\mathcal{I} - P)R_0 \|_{L_\infty(L_2(\Omega))} + C\gamma_E^t \| k_n R_1 \|_{L_\infty(L_2(\Omega))} \\ + \gamma_E^x \| h^2 R_2 \|_{L_\infty(L_2(\Omega))} + \gamma_E^t \| k_n R_2 \|_{L_\infty(L_2(\Omega))}, \end{aligned} \tag{3.19}$$

where

$$||\Phi||_{L_\infty(L_2(\Omega))} = \sup_{0<t<T} ||\Phi(\cdot, t)||_{L_2(\Omega)},$$

$$||\Phi||_{L_1(L_2(\Omega))} = \int_0^T ||\Phi(\cdot, t)||_{L_2(\Omega)} dt.$$

Proof. The proof of this theorem is a modified version of that of the previous one and therefore is omitted. The only difference is in the use of Hölder's inequality $\|fg\|_1 \leq \|f\|_p \|g\|_q$, $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$ (in the Theorem 1, $p = q = 2$ whereas in the Theorem 2, $p = 1$, $q = \infty$). □

4. Analytical strong stability estimates in $L_2(L_2)$

We need to estimate the strong stability factors that used in the previous sections. Let us consider the a posteriori error estimate of the type (3.12) in Theorem 1, which is based on using the following dual problem

$$\begin{cases} L^*\Phi \equiv -\Phi_t + A^T\Phi = \Psi^{-1}\mathbf{e}, & \text{in } \Omega, \\ \Phi(x, T) = 0, & x \in \Omega. \end{cases} \tag{4.1}$$

We prove the following strong stability estimate for dual problem (4.1).

Theorem 3. *For a given positive weight function* $\Psi(x, t)$*, the solution* Φ *of the dual problem (4.1) satisfies the following estimate*

$$||\Psi^{1/2}(\Phi_t - A^T\Phi)||_\Omega = ||\mathbf{e}||_{L_2^{\Psi^{-1}}(\Omega)}.$$

Proof. We multiply the equation (4.1) by $-\Psi(\Phi_t - A^T\Phi)$ and integrate over Ω to get

$$\int_\Omega \Psi(\Phi_t - A^T\Phi)^2 dx = -\int_\Omega \mathbf{e}(\Phi_t - A^T\Phi) dx \leq \frac{1}{2}||\Psi^{-1/2}\mathbf{e}||_\Omega^2 + \frac{1}{2}||\Psi^{1/2}(\Phi_t - A^T\Phi)||_\Omega^2.$$

This yields

$$||\Psi^{1/2}(\Phi_t - A^T\Phi)||_\Omega^2 \leq ||\Psi^{-1/2}\mathbf{e}||_\Omega^2. \tag{4.2}$$

Similarly, multiplying the equation (4.1) by $\mathbf{e}$ and integrating over Ω, yields

$$\int_\Omega \mathbf{e}^2 \Psi^{-1} dx = ||\Psi^{-1/2}\mathbf{e}||_\Omega^2 = -\int_\Omega \mathbf{e}(\Phi_t - A^T\Phi)^2 dx$$
$$\leq \frac{1}{2}||\Psi^{-1/2}\mathbf{e}||_\Omega^2 + \frac{1}{2}||\Psi^{1/2}(\Phi_t - A^T\Phi)||_\Omega^2.$$

So that

$$|||\Psi^{-1/2}\mathbf{e}||_\Omega^2 \leq ||\Psi^{1/2}(\Phi_t - A^T\Phi)||_\Omega^2. \tag{4.3}$$

Combining (4.2) and (4.3) completes the proof. □

Theorem 4. *If* $\Psi(x,t)$ *is a positive weight function satisfying*

$$\Psi_t + A^T\Psi \geq -\Psi, \quad \textit{in } \Omega \tag{4.4}$$

then the solution Φ *for* (4.1) *satisfies*

$$||\Psi^{1/2}\Phi||_\Omega \leq C_T||\mathbf{e}||_{L_2^{\Psi^{-1}}(\Omega_T)}, \qquad C_T = e^T.$$

Proof. First, we multiply the equation (4.1) by $\Psi\Phi$ and integrate over Ω:

$$-(\Phi_t, \Psi\Phi(t)) + (A^T\Phi, \Psi\Phi(t)) = (\mathbf{e}, \Phi(t)).$$

This can be rewritten as

$$-\frac{1}{2}\frac{d}{dt}||\Psi^{1/2}\Phi(t)||^2 + \frac{1}{2}(\Psi_t, \Phi^2(t)) + (A^T\Phi, \Psi\Phi(t)) = (\mathbf{e}, \Phi(t)).$$

We integrate by parts in space and then use (4.1) and Cauchy-Schwarz inequality to get

$$-\frac{1}{2}\frac{d}{dt}||\Psi^{1/2}\Phi(t)||^2 + \frac{1}{2}(\Psi_t + A^T\Psi, \Phi^2(t)) \leq ||\Psi^{-1/2}\mathbf{e}||||\Psi^{1/2}\Phi||$$
$$\leq \frac{1}{2}||\Psi^{-1/2}\mathbf{e}||^2 + \frac{1}{2}||\Psi^{1/2}\Phi||^2.$$

Hence, using (4.4), yields

$$-\frac{1}{2}\frac{d}{dt}||\Psi^{1/2}\Phi(t)||^2 - \frac{1}{2}(\Psi\,,\,\Phi^2(t)) \leq \frac{1}{2}||\Psi^{-1/2}\mathbf{e}||^2 + \frac{1}{2}||\Psi^{1/2}\Phi||^2,$$

Now, we integrate in time over the interval (t,T), and use the fact that $\Psi(\cdot,T) = 0$, to get that

$$||\Psi^{1/2}\Phi(t)||^2 \leq ||\Psi^{-1/2}\mathbf{e}||_{\Omega_T}^2 + 2\int_t^T ||\Psi^{1/2}\Phi(t)||^2 ds.$$

So that, by the Gronwall 's inequality, we get the desired result

$$||\Psi^{1/2}\Phi(t)||^2 \leq \mathbf{e}^{2T}||\Psi^{-1/2}\mathbf{e}||_{\Omega_T}^2.$$

□

We omit the proof for the analytical strong stability estimates in $L_\infty(L_2)$ because it is similar to the one for the $L_2(L_2)$ case above.

5. NUMERICAL COMPUTATIONS

In this section we consider a general numerical scheme: *the streamline diffusion (SD) method*, introduced and developed by Hughes and Brooks, see e.g. [9]. The SD method is a modified version of Galerkin's method designed, basically, for numerical investigations of hyperbolic type problems (the system (1.1) rather than (1.2)). Roughly speaking, compared to the standard Galerkin method (SG), the SD method is a Petrov-Galerkin type method, with modified test functions, that combines high accuracy and good stability properties. Generally, the convergence rate for the SG method for the hyperbolic problems is of one order lower than the corresponding elliptic and parabolic cases. In a SD scheme the appropriate choice of the test functions gives rise to a, weakly imposed, diffusion term which improves the convergence rate of the SG method for hyperbolic problems by $\sim \mathcal{O}(h^{1/2})$.

We shall implement a general scheme (both with and without SD modification) applied to solve a one-dimensional time dependent coupling of two hyperbolic equations. For simplifying we shall use the notation: $\Upsilon^T := (1,1,1,1)$, $U := (u_1,u_2,u_3,u_4)^T$, $V := (v_1,v_2,v_3,v_4)^T$. Further, for an arbitrary operator Γ we define a product vector and a componentwise product as

$$\Gamma(U \otimes \Upsilon) = \Gamma U = (\Gamma u_1, \Gamma u_2, \Gamma u_3, \Gamma u_4)^T,$$

and

$$U \odot V = (u_1v_1, u_2v_2, u_3v_3, u_4v_4)^T,$$

respectively. Below we shall discuss computational aspects of the approximate solution for (1.1), through using the SD method for (1.2): for $n = 0, ..., N-1$, find $\mathbf{u}^n \in \mathbf{U}^n$ such that

$$(\mathbf{u}^n_{h,t} + A\mathbf{u}^n_h \, , \, \mathbf{g}_h + \delta(\mathbf{g}_{h,t} + A\mathbf{g}_h))_n + \langle \mathbf{u}^n_{h,+} \, , \, \mathbf{g}_{h,+}\rangle_n = \langle \mathbf{u}^n_{h,-} \, , \, \mathbf{g}_{h,+}\rangle_n, \tag{5.1}$$

Here δ is the SD parameter (usually $\delta \sim h$). Since (1.2) is a parabolic problem, the improving potential of the δ-term ($\delta \neq 0$ in (5.1)) is rather minimal. Nevertheless, the scheme removes oscillatory behavior near the boundary layers. We use finite element approximation on a space-time slab with the trial functions being piecewise polynomials in space and piecewise linear in time; that is, for $(x,t) \in S_n$. We seek the approximate solution

$$\begin{aligned} \mathbf{u}^n_h(x,t) &= \sum_{i=1}^{M} \left\{ \varphi_i(x)(\theta_1(t)\tilde{\mathbf{u}}^n_i + \theta_2(t)\mathbf{u}^{n+1}_i) \right\} \otimes \Upsilon \\ &= \begin{pmatrix} u^n_h(x,t) = \sum_{i=1}^M \varphi_i(x)(\theta_1(t)\tilde{u}^n_i + \theta_2(t)u^{n+1}_i) \\ \phi^n_h(x,t) = \sum_{i=1}^M \varphi_i(x)(\theta_1(t)\tilde{\phi}^n_i + \theta_2(t)\phi^{n+1}_i) \\ v^n_h(x,t) = \sum_{i=1}^M \varphi_i(x)(\theta_1(t)\tilde{v}^n_i + \theta_2(t)v^{n+1}_i) \\ \psi^n_h(x,t) = \sum_{i=1}^M \varphi_i(x)(\theta_1(t)\tilde{\psi}^n_i + \theta_2(t)\psi^{n+1}_i) \end{pmatrix}, \end{aligned} \tag{5.2}$$

where $\varphi_i(x_j) = \delta_{ij}, (j = 1, \ldots, M)$ is the spatial shape function at node i, $i = 1, \ldots, M$, and $\theta_1(t)$ and $\theta_2(t)$ are the piecewise linear bases functions for the subinterval $(t_n, t_{n+1}]$ in the time discretization:

$$\theta_1(t) = \frac{t_{n+1} - t}{t_{n+1} - t_n} = \frac{t_{n+1} - t}{k},$$

and

$$\theta_2(t) = \frac{t - t_n}{t_{n+1} - t_n} = \frac{t - t_n}{k}.$$

Also, the nodal value of $\mathbf{u}$ for node i at $(t_n)_+$ and $(t_{n+1})_-$ are denoted by $\tilde{\mathbf{u}}_i^n$ and $\mathbf{u}_i^{n+1}$, respectively. Then, on each slab S_n, the test functions $\mathbf{g}_h^n$ are defined as a linear combination of $\varphi_j(x)\theta_1(t)$ and $\varphi_j(x)\theta_2(t)$, for $j = 1, .., M$. Then, (5.1) is equivalent to the following system of equations with the unknowns $\tilde{\mathbf{u}}_i^n$ and $\mathbf{u}_i^{n+1}$: for $n = 0, 1, \dots, N-1$, and for all $j = 0, 1, \dots M$, find $\tilde{\mathbf{u}}_i^n$ and $\mathbf{u}_i^{n+1}$ such that

(5.3)
$$\sum_{i=1}^{M} \int_{S_n} \Big\{ \Big[(\varphi_i(x)(\frac{\mathbf{u}_i^{n+1} - \tilde{\mathbf{u}}_i^n}{k}) \odot \Upsilon) + A\left(\varphi_i(x)(\theta_1(t)\tilde{\mathbf{u}}_i^n + \theta_2(t)\mathbf{u}_i^{n+1}) \odot \Upsilon\right) \Big]$$
$$\odot \Big[(\varphi_j(x)\theta_1(t))\Upsilon + (\delta((\frac{-1}{k})\varphi_j(x))\Upsilon + A(\varphi_j(x)\theta_1(t)\Upsilon) \Big] \odot (dxdt\Upsilon) \Big\} \otimes \Upsilon = 0,$$

and,

$$\sum_{i=1}^{M} \int_{S_n} \Big\{ \Big[\varphi_i(x)(\frac{\mathbf{u}_i^{n+1} - \tilde{\mathbf{u}}_i^n}{k}) \odot \Upsilon + A(\varphi_i(x)(\theta_1(t)\tilde{\mathbf{u}}_i^n + \theta_2(t)\mathbf{u}_i^{n+1}) \odot \Upsilon) \Big] \odot$$
$$\Big[\varphi_j(x)\theta_2(t)\Upsilon + \delta((\frac{1}{k})\varphi_j(x))\Upsilon + A(\varphi_j(x)\theta_2(t)\Upsilon) \Big] \odot (dxdt\Upsilon) \Big\} \otimes \Upsilon \tag{5.4}$$
$$+ \sum_{i=1}^{M} \int_{\Omega} \{ \varphi_j(x)\varphi_i(x)(\tilde{\mathbf{u}}_i^n - \mathbf{u}_i^n) \odot (\Upsilon dx) \} \otimes \Upsilon = 0.$$

We choose φ_i as the *hat-functions* (a set of bases functions for piecewise linears)

$$\varphi_i(x) = \frac{1}{h} \begin{cases} x - x_{i-1}, & x \in [x_{i-1}, x_i], \\ x_{i+1} - x, & x \in [x_i, x_{i+1}], \\ 0, & \text{elsewhere}, \end{cases}$$

defined on a uniform partition $\mathcal{T}_h$ of $\Omega = [a, b]$, with the mesh size $h := x_{i+1} - x_i$. Thus we can compute the entries of the coefficient matrices as

$$M_{ij} = \Big(\int_{\Omega} \varphi_i(x)\varphi_j(x)dx \Big)\Upsilon = \frac{h}{6}\Upsilon \begin{cases} 4, & j = i, \\ 1, & j = i+1, j = i-1, \\ 0, & \text{elsewhere}, \end{cases}$$

$$B_{ij} = \int_{\Omega} (A(\varphi_i\varphi_j\Upsilon) \odot (dx\Upsilon)) \otimes \Upsilon = \int_{\Omega} \{(\varphi_i\Upsilon) \odot A(\varphi_j\Upsilon) \odot (\Upsilon dx)\} \otimes \Upsilon,$$

and

$$F_{ij} = \int_{\Omega} \{A(\varphi_i\Upsilon) \odot A(\varphi_j\Upsilon) \odot (dx\Upsilon)\} \otimes \Upsilon.$$

Further, using the trivial identities

$$\int_{t_n}^{t_{n+1}} \theta_1(t)\theta_2(t)dt = \frac{k}{6}, \quad \int_{t_n}^{t_{n+1}} \theta_i^2(t)dt = \frac{k}{3}, \quad \int_{t_n}^{t_{n+1}} \theta_i(t)dt = \frac{k}{2}, \qquad i = 1, 2,$$

we get the following equivalent forms for (5.3) and (5.4),

$$\sum_{i=1}^{M} \Big[(\frac{1}{2} - \frac{\delta}{k^2})M_{ij} + \frac{k}{6}B_{ij} + \frac{\delta k}{6}F_{ij} \Big]^T \mathbf{u}_i^{n+1} \tag{5.5}$$
$$+ \sum_{i=1}^{M} \Big[(\frac{\delta}{k^2} - \frac{1}{2})M_{ij} + (\frac{k}{3} + \delta)B_{ij} + \frac{k\delta}{3}F_{ij} \Big]^T \tilde{\mathbf{u}}_i^n = 0,$$

and

$$\begin{aligned} &\sum_{i=1}^{M}\Big[(\frac{1}{2}+\frac{\delta}{k^2})M_{ij}+(\frac{k}{3}+\delta)B_{ij}+\frac{\delta k}{3}F_{ij}\Big]^T\mathbf{u}_i^{n+1} \\ &\quad+\sum_{i=1}^{M}\Big[(\frac{1}{2}-\frac{\delta}{k^2})M_{ij}+\frac{k}{6}B_{ij}+\frac{k\delta}{6}F_{ij}\Big]^T\tilde{\mathbf{u}}_i^n+\sum_{i=1}^{M}M_{ij}^T\mathbf{u}_i^n=0. \end{aligned} \tag{5.6}$$

We may rewrite the above equations in the matrix form, for $n = 0, 1, ..., N-1$, as

$$\Big[(\frac{1}{2}-\frac{\delta}{k^2})M+\frac{k}{6}B+\frac{\delta k}{6}F\Big]\mathbf{U}^{n+1}+\Big[(\frac{\delta}{k^2}-\frac{1}{2})M+(\frac{k}{3}+\delta)B+\frac{k\delta}{3}F\Big]\tilde{\mathbf{U}}^n=0, \tag{5.7}$$

and

$$\Big[(\frac{1}{2}+\frac{\delta}{k^2})M+(\frac{k}{3}+\delta)B+\frac{\delta k}{3}F\Big]\mathbf{U}^{n+1}+\Big[(\frac{1}{2}-\frac{\delta}{k^2})M+\frac{k}{6}B+\frac{k\delta}{6}F\Big]\tilde{\mathbf{U}}^n+M\mathbf{U}^n=0, \tag{5.8}$$

where

$$\mathbf{U}^n=[\mathbf{u}_1^n,\dots,\mathbf{u}_M^n]^T,\quad \mathbf{U}^{n+1}=[\mathbf{u}_1^{n+1},\dots,\mathbf{u}_M^{n+1}]^T,\quad \text{and}\quad \tilde{\mathbf{U}}^n=[\tilde{\mathbf{u}}_1^n,\dots,\tilde{\mathbf{u}}_M^n].$$

5.1. **Test Problem.** We carry out experimental computations to solve (5.7) and (5.8), by an AMD Opteron computer with 15 Gigabytes RAM memory with 2.2 GHz CPU. For each slab S_n, we choose a partition of the spatial interval into the subinterval $J_i^n = (x_{i-1}^n, x_i^n)$, with $h_i^n = x_i^n - x_{i-1}^n$. For $h > 0$, small. We let T_h^n be a triangulation of the slab S_n into, a quasiuniform space-time triangular elements K (cf. Fig. 1. below), satisfying a minimum angle condition. The triangulation for S_n may be chosen independently of that of S_{n-1}, then a projection (from one face to the other) will be necessary. Here, for the sake of simplicity we assume same quasi-uniformity in all slabs and small shape variations. We shall use finite element approximation on a space time with the trial functions being piecewise polynomials in space and piecewise linear in time. First we compute the numerical solution for a given δ, and with $\Delta x := h = 0.01$, $\Delta t := k = 0.0005$, and discretize (1.1) assuming $\Omega = [-1, 1]$, and $\alpha(x) = x^2$,

$$u_0=v_0=\begin{cases} 0, & |x|\ge 1,\\ \frac{x(x+1)}{2+x}, & -1\le x\le 0,\\ \frac{x(x-1)}{2+x}, & 0\le x\le 1. \end{cases}$$

In the Figure 2 below, we verify numerically the rate of convergence of the error, in the L_2-norm, for u and v, i.e, for $\|u - u_h\|$ and $\|v - v_h\|$. The results are shown in even time steps at $t_j = j \times \Delta t$ and a uniform partition of the spatial domain $\Omega = [-1, 1]$, viz $x_0 = -1.00$, $x_i = x_0 + i \times \Delta x$, partition . For example, with the above choice, the time is between $t_0 = 0$ and $t_N = 200 \times \Delta t = 10^{-1}$. We plot the absolute error for $u_{ij} := u(i\Delta x, j\Delta t)$ and $v_{ij} := v(i\Delta x, j\Delta t)$, in a grid with the discrete times $t = 0 \times \Delta t$, $10^{-3} = 2 \times \Delta t, \dots, 10^{-1} = 200 \times \Delta t$ and the spatial nodes $x_0 = -1.00$, $x_1 = -0.99, ..., x_M = 0$, $x_{M+1} = 0.01, ..., x_{2M} = 1.00$, (see Fig. 2). Finally, in Tables 1 and 2, we show the error for the approximate solution for (5.1). The order of error is computed using the logarithmic division:

$$\text{Order of error for } w \approx \ln\frac{E_{h_i}(w)}{E_{h_{i+1}}(w)},\qquad w=u,v,$$

where $E_{h_i}(w) = \|w(x,0) - w_{h_i}(x,0)\|_\infty$, $w = u, v$, $i = 1, 2, 3, 4, 5$. The small values for the errors are indicating the efficiency of the method. We may also observe the behavior in initial error (the error made by the approximation in the initial data $u(x,0)$ and $v(x,0)$) in two independent variables x and δ, see Figures 3 and 4.

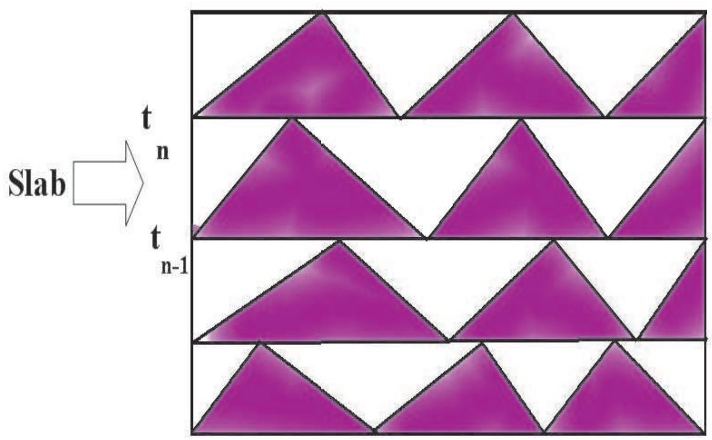

Figure 1. The slabs on Rectangle

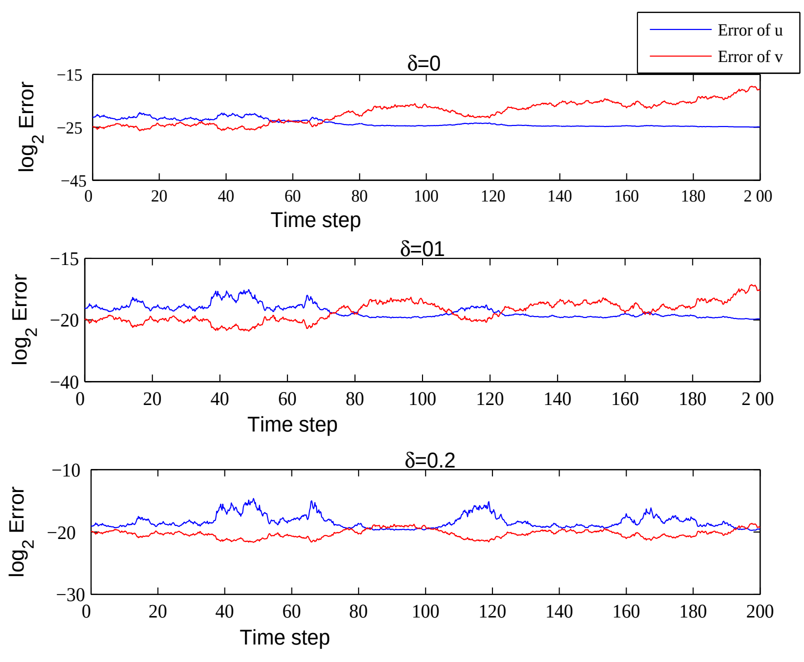

Figure 2: The behavior of error in time step for u and v. Here, we consider $x = -1.00, -0.99, \ldots, 0, 0.01, \ldots, 1.00$.

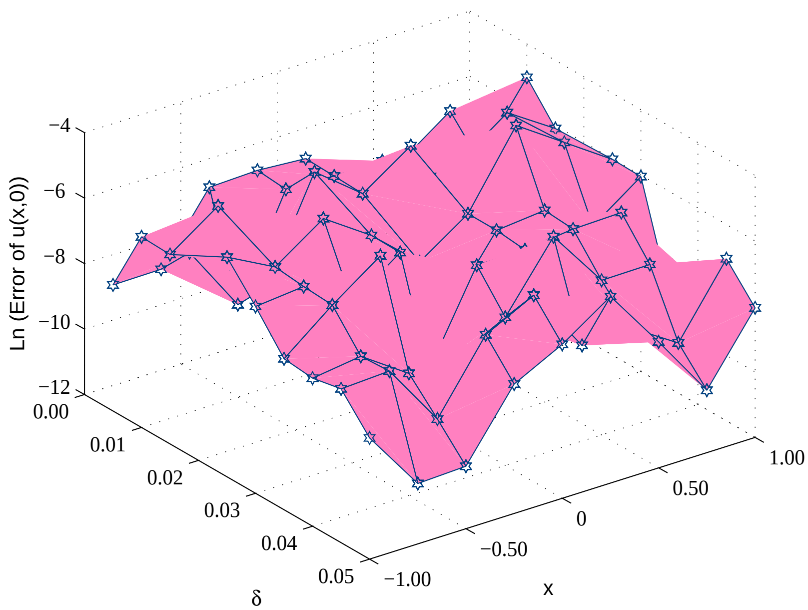

Figure 3: The error of SD method for $u(x, 0)$.

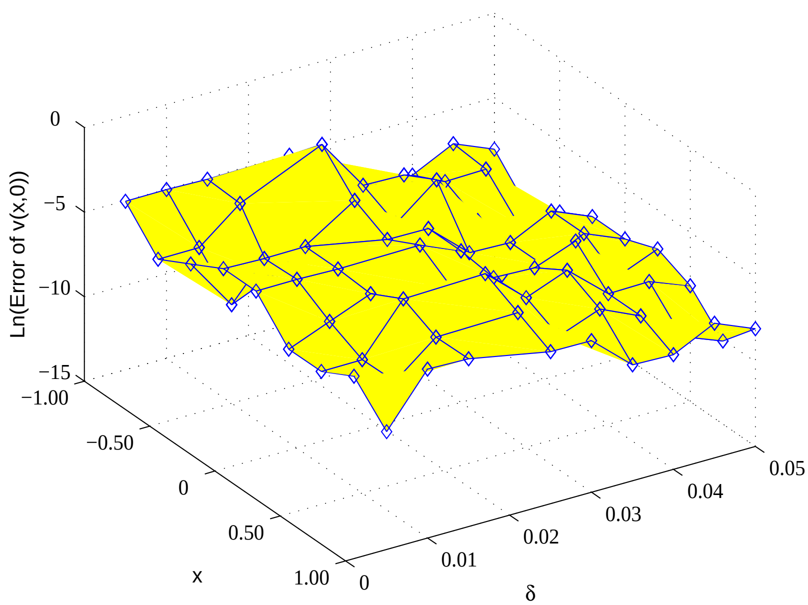

Figure 4: The error of SD method for $v(x, 0)$.

Table 1. $E_{h_i}(u)$ and order of error for u by SD method at $\delta = 0.1$ and $k = 0.01$

x	$h_1 = 0.15$	$h_2 = 0.10$	$h_3 = 0.05$	$h_4 = 0.01$	$h_5 = 0.005$
-1.0	0.481e-5	0.362e-6	0.431e-8	0.701e-8	0.401e-9
0.0	0.436e-6	0.911e-8	0.454e-7	0.983e-9	0.932e-12
1.0	0.734e-6	0.743e-7	0.713e-10	0.801e-9	0.210e-9
order	-	2.587	2.076	1.868	3.508

Table 2. $E_{h_i}(u)$ and order of error for v by SD method at $\delta = 0.1$ and $k = 0.01$

x	$h_1 = 0.15$	$h_2 = 0.10$	$h_3 = 0.05$	$h_4 = 0.01$	$h_5 = 0.005$
-1.0	0.4231e-5	0.362e-5	0.913e-7	0.634e-9	0.421e-9
0.0	0.206e-4	0.201e-6	0.934e-8	0.785e-9	0.762e-9
1.0	0.134e-6	0.176e-7	0.903e-8	0.701e-8	0.401e-10
order	-	1.739	3.680	2.569	2.812

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