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Abstract. We investigate the connection between representations of posets and those of the corresponding (bound) quivers. As far as quivers are concerned, we concentrate on unitarizable representations which are stable representations for some appropriately chosen slope function. They give rise to representations of the $*$-algebras corresponding to the related posets. In the case of primitive posets, this leads to an ADE-classification which describes the unitarization behaviour of its representations. If the poset is induced by a bound quiver, it is possible to construct unitarizable representations starting with polystable representations of some related poset which can be glued together with a suitable direct sum of simple representation.

Introduction

In this paper we continue the investigation of $*$-algebras, denoted by $\mathcal{A}_{\mathcal{N},\chi}$, associated to partially ordered sets $\mathcal{N}$ (posets in the sequel). Such algebras are generated by orthoprojections p_i (each projection corresponds to a point of the poset) satisfying the relations

$$p_j p_i = p_i p_j = p_i, \quad i \prec j;$$
$$\chi_1 p_1 + \ldots + \chi_n p_n = \chi_0 e,$$

for a given weight $\chi = (\chi_0; \chi_1, \ldots, \chi_n) \in \mathbb{R}_+^{|\mathcal{N}|+1}$. If the poset is primitive (that means that the corresponding quiver is star-shaped), the $*$-representations of the corresponding algebras are connected to Hermann Weyl's problem and its generalizations (see [5, 8, 1] and Section 1 for more details) and also to orthoscalar representations of quivers in the category of Hilbert spaces (see for example [15, 16]).

If one studies the representation theory of such algebras, two problems naturally arise: to find those weights χ for which the corresponding algebra has at least one representation; for appropriated χ, to describe all $*$-representations up to unitary equivalence. The second problem could turn out to be "hopeless", i.e. the isomorphism classes of $*$-representations can depend on arbitrarily many continuous parameters, or the algebra can be even $*$-wild (i.e. the classification problem contains the classification of two self-adjoint matrices up to unitary isomorphism as a subproblem).

Let $Q(\mathcal{N})$ be the (bound) quiver induced by $\mathcal{N}$. Every finite-dimensional $*$-representation of $\mathcal{A}_{\mathcal{N},\chi}$ gives rise to a strict representation of $Q(\mathcal{N})$. In order to build $*$-representations of $\mathcal{A}_{\mathcal{N},\chi}$, we can ask which strict representation $(V; V_i)$ of $Q(\mathcal{N})$ possesses a choice of a Hermitian structure in V in such a way that for the corresponding projections $P_i : V \to V_i$ the following equality holds

$$\chi_1 P_1 + \ldots + \chi_n P_n = \chi_0 I.$$

Those representations $(V; V_i)$ of $Q(\mathcal{N})$ are called *χ-unitarizable*, and $(V; V_i)$ is said to be *unitarizable* respectively if there exists at least one such weight.

In this paper we study unitarizable and non-unitarizable representations of posets. For this we use the fact that *unitarization* can be translated into the language of *stability* of quiver representations, see [14],[12] and others.

The article is organized as follows: in the beginning we recall some basic facts about $*$-algebras associated to posets and star-shaped graphs. Then we state some basics concerning to quivers

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and their stable representations. Afterwards we show that each general strict Schurian representation of a poset corresponding to an unbound quiver can be unitarized with some weight and hence gives rise to a $*$-representation of $\mathcal{A}_{\mathcal{N},\chi}$. Moreover, we specify those weights χ in terms of the dimension vector of the corresponding representation. Then we consider the rigid primitive case showing that the corresponding Hermitian operators are rigid in the sense of N.Katz (see [13]). We also give a classification of such posets with respect to the unitarization behavior of their representations. As far as posets are concerned which are induced by a bound quiver, we state a method how to construct unitarizable Schurian representations considering extensions of polystable representations of a related unbound quiver with some direct sum of simple representations. In this case, it is also possible to specify the weight in terms of the dimension vector.

Finally, we prove that, if $\mathcal{N}$ is a poset of wild type, there exist families of non-equivalent $*$-representations of $\mathcal{A}_{\mathcal{N},\chi}$ which depend on arbitrary many continuous parameters (which means that the corresponding algebras are not of $*$-tame representation type).

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1. Preliminaries

1.1. **$*$-Algebras associated to posets and graphs.** Let $\mathcal{N}$ be a poset, say $\mathcal{N} = \{1, \ldots, n\}$. A finite-dimensional representation of $\mathcal{N}$ is given by finite-dimensional $\mathbb{C}$-vector spaces $(V; (V_i)_{i\in\mathcal{N}})$ such that $V_i \subseteq V$ for all $i \in \mathcal{N}$ and $V_i \subseteq V_j$ if $i \prec j$. A morphism between two representations $(V; (V_i)_{i\in\mathcal{N}})$ and $(W; (W_i)_{i\in\mathcal{N}})$ is given by a $\mathbb{C}$-linear map $g : V \to W$ such that $g(V_i) \subseteq W_i$ for all $i \in \mathcal{N}$.

We take a weight $\chi = (\chi_0; \chi_1, \ldots, \chi_n) \in \mathbb{R}_+^{|\mathcal{N}|+1}$ and consider the $*$-algebra

$$\mathcal{A}_{\mathcal{N},\chi} = \mathbb{C}\left\langle p_1, \ldots, p_n \;\middle|\; \begin{array}{l} p_i = p_i^* = p_i^2 \\ \chi_1 p_1 + \ldots + \chi_n p_n = \chi_0 e \\ p_j p_i = p_i p_j = p_i, \quad i \prec j \end{array} \right\rangle .$$

A graph, quiver or poset is called star-shaped if each vertex (except the root) has at most two neighbours. We also consider another class of $*$-algebras which are associated to star-shaped graphs $\Gamma = (\Gamma_0, \Gamma_1)$. We identify the set of vertices Γ_0 with $(g_0; g_i^{(j)})$, where g_0 is the root vertex and $g_{i_1}^{(j)}$ and $g_{i_2}^{(j)}$ lie on the same branch of Γ. Fixing some weight $\omega \in \mathbb{R}_+^{|\Gamma_0|}$, $\omega = (\omega_0; \omega_i^{(j)})$ with $\omega_{i_1}^{(j)} > \omega_{i_2}^{(j)}$ if $i_1 > i_2$, we consider the $*$-algebra

$$\mathcal{B}_{\Gamma,\omega} = \mathbb{C}\left\langle a_1, \ldots, a_n \;\middle|\; \begin{array}{l} a_i = a_i^* \\ (a_i - \omega_1^{(i)}) \ldots (a_i - \omega_{m_i}^{(i)}) = 0 \\ a_1 + \ldots + a_n = \omega_0 e \end{array} \right\rangle .$$

Any $*$-representation of $\mathcal{B}_{\Gamma,\omega}$ in some Hilbert space is given by an n-tuple of Hermitian operators with spectra $\sigma(A_i) \in \{\omega_1^{(i)} < \ldots < \omega_{m_i}^{(i)}\}$ such that

$$A_1 + \ldots + A_n = \omega_0 I.$$

Recall that the last equation generalizes the famous problem of Hermann Weyl about the spectra of the sum of two Hermitian matrices ([5, 8, 18]). Fixing a finite-dimensional representation of $\mathcal{B}_{\Gamma,\omega}$ in some Hilbert space H, for each operator A_i we can consider its spectral decomposition

$$A_i = \omega_1^{(i)} Q_1^{(i)} + \ldots + \omega_{m_i}^{(i)} Q_{m_i}^{(i)}.$$

If the poset $\mathcal{N}$ is primitive (that means that the corresponding quiver is star-shaped), then each $*$-representation of $\mathcal{A}_{\mathcal{N},\chi}$ generates a $*$-representation of $\mathcal{B}_{\Gamma,\omega}$. More precisely, let $(P_i^{(j)})$ be a $*$-representation of $\mathcal{A}_{\mathcal{N},\chi}$, which means that $P_{i_1}^{(j)} P_{i_2}^{(j)} = P_{i_2}^{(j)} P_{i_1}^{(j)} = P_{i_1}^{(j)}$ if $i_1 < i_2$ and

$$\chi_1^{(1)} P_1^{(1)} + \ldots + \chi_{m_1}^{(1)} P_{m_1}^{(1)} + \ldots + \chi_1^{(n)} P_1^{(n)} + \ldots + \chi_{m_n}^{(n)} P_{m_n}^{(n)} = \chi_0 I.$$

Letting $Q_1^{(j)} = P_1^{(j)}$, $Q_i^{(j)} = P_i^{(j)} - P_{i-1}^{(j)}$ and taking the weight $\omega_{m_j}^{(j)} = \chi_{m_j}^{(j)}$, $\omega_i^{(j)} = \chi_i^{(j)} + \omega_{i+1}^{(j)}$, $\omega_0 = \chi_0$, we get a representation of $\mathcal{B}_{\Gamma,\omega}$. Note that one can even prove that $\mathcal{A}_{\mathcal{N},\chi}$ and $\mathcal{B}_{\Gamma,\omega}$ are isomorphic using the same transformation between the projections and the weights.

We will be interested in representations of $\mathcal{B}_{\Gamma,\omega}$ as well as in the rigidity of the corresponding matrices $A_1, \ldots, A_n$. Let us recall the definition of a rigid local system on a punctured projective line. Let $S = \{p_1, \ldots, p_n\}$ be a finite set of distinct points of $\mathbb{P}^1$ and consider the punctured projective line $\mathbb{P}^1 - S$. Take a point $x_0 \in \mathbb{P}^1 - S$. The fundamental group $\pi_1(\mathbb{P}^1 - S, x_0)$ can be understood as the free group generated by $\gamma_1, \ldots, \gamma_n$ with the relation $\gamma_1 \ldots \gamma_n = I$ where γ_i are loops around each puncture (appropriately chosen and oriented in counterclockwise direction). A rank r local system on $\mathbb{P}^1 - S$ is determined by a representation $\rho : \pi_1(\mathbb{P}^1 - S, x_0) \to Gl_r(\mathbb{C})$, and hence by some tuple $(A_1, \ldots, A_n)$ of matrices in $Gl_r(\mathbb{C})$ satisfying

$$A_1 \ldots A_n = I,$$

where A_j is the image of γ_j.

A local system given by $(A_1, \ldots, A_n)$ is said to be *physically rigid* if it is determined by local monodromies, i.e if for any tuple $(\widetilde{A}_1, \ldots, \widetilde{A}_n)$ of matrices in $Gl_r(\mathbb{C})$ satisfying $\widetilde{A}_1 \ldots \widetilde{A}_n = I$ and $\widetilde{A}_j = C_j A_j C_j^{-1}$ for $C_j \in Gl_r(\mathbb{C})$, there exists some $C \in Gl_r(\mathbb{C})$ such that $\widetilde{A}_j = C A_j C^{-1}$. The *rigidity* index $\mathrm{rig}(A_i)$ of matrices $A_i \in M_r(\mathbb{C})$ is the number

$$\mathrm{rig}(A_1, \ldots, A_n) = r^2(2 - n) + \sum_{i=1}^{n} \dim(Z(A_i)),$$

where $Z(X)$ denotes the commutator of the matrix X, i.e.

$$Z(X) = \{A \in M_r(\mathbb{C}) \mid AX = XA\}.$$

N. Katz, see [13], showed that if $(A_1, \ldots, A_n)$ is an irreducible local system then

$$\mathrm{rig}(A_1, \ldots, A_n) \in \{2m \mid m \in \mathbb{Z},\ m \leq 1\}$$

and that $\mathrm{rig}(A_1, \ldots, A_n) = 2$ if and only if the tuple is physically rigid. Following Katz we say that the set of matrices is *rigid* if its rigidity index equals 2, otherwise we say that the set of matrices is *non-rigid*.

1.2. **Strict representations of quivers. Stable representations.** Let k be an algebraically closed field. In order to study the representations of the algebra $\mathcal{A}_{\mathcal{N},\chi}$, we are going to use the notion of stable quiver representations. For an introduction to the theory of quiver representations we refer to [2].

Definition 1. *A quiver Q consists of a set of vertices Q_0 and a set of arrows Q_1 denoted by $\rho : i \to j$ for $i, j \in Q_0$. The vertex i is called tail, and the vertex j is called head of the arrow ρ.*
A vertex $q \in Q_0$ is called sink if there does not exist an arrow $\rho : q \to q' \in Q_1$.
A vertex $q \in Q_0$ is called source if there does not exist an arrow $\rho : q' \to q \in Q_1$.

In the following we only consider quivers without oriented cycles. Define the abelian group

$$\mathbb{Z}Q_0 = \bigoplus_{i \in Q_0} \mathbb{Z}i$$

and its monoid of dimension vectors $\mathbb{N}Q_0$.
A finite-dimensional k-representation of Q is given by a tuple

$$X = ((X_i)_{i \in Q_0}, (X_\rho)_{\rho \in Q_1} : X_i \to X_j)$$

of finite-dimensional k-vector spaces and k-linear maps between them. We say that X is *strict* if all maps X_ρ are injective. The dimension vector $\underline{\dim} X \in \mathbb{N}Q_0$ of X is defined by

$$\underline{\dim} X = \sum_{i \in Q_0} \dim X_i \cdot i.$$

Let $\alpha \in \mathbb{N}Q_0$ be a dimension vector. The variety $R_\alpha(Q)$ of k-representations of Q with dimension vector α is defined as the affine k-space

$$R_\alpha(Q) = \bigoplus_{\rho : i \to j} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha_j}).$$

The algebraic group $G_\alpha = \prod_{i \in Q_0} Gl_{\alpha_i}(k)$ acts on $R_\alpha(Q)$ via simultaneous base change, i.e.

$$(g_i)_{i \in Q_0} * (X_\rho)_{\rho \in Q_1} = (g_j X_\rho g_i^{-1})_{\rho : i \to j}.$$

The orbits are in bijection with the isomorphism classes of k-representations of Q with dimension vector α.

Let kQ be the path algebra of Q and let RQ be the arrow ideal. A relation in Q is a k-linear combination of paths of length at least two which have the same head and tail. For a set of relations $(r_j)_{j \in J}$ we can consider the admissible ideal I generated by these relations, that means that we have $RQ^m \subseteq I \subseteq RQ^2$ for some $m \geq 2$. Now a representation X of Q is bound by I, and thus a representation of the bound quiver (Q, I), if $X_{r_j} = 0$ for all $j \in J$. For every dimension vector this defines a closed subvariety of $R_\alpha(Q)$ denoted by $R_\alpha(Q, I)$. If R is a minimal set of relations generating I, by $r(i, j, I)$ we denote the number of relations with starting vertex i and terminating vertex j. Following [4], for the dimension of $R_\alpha(Q, I)$ we get

$$\dim R_\alpha(Q, I) \geq \dim R_\alpha(Q) - \sum_{(i,j) \in (Q_0)^2} r(i, j, I) \alpha_i \alpha_j.$$

Let $C_{(Q,I)}$ be the Cartan matrix of (Q, I), i.e. $c_{j,i} = \dim e_i(kQ/I)e_j$ where e_i denotes the primitive idempotent (resp. the trivial path) corresponding to the vertex i. On $\mathbb{Z}Q_0$ a non-symmetric bilinear form, the Euler characteristic, is defined by

$$\langle \alpha, \beta \rangle := \alpha^t (C_{(Q,I)}^{-1})^t \beta.$$

Then for two representation X and Y we have

$$\langle X, Y \rangle := \langle \underline{\dim} X, \underline{\dim} Y \rangle = \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{Ext}^i(X, Y).$$

If Q is unbound, for two representations X, Y of Q with $\underline{\dim} X = \alpha$ and $\underline{\dim} Y = \beta$ we have

$$\langle X, Y \rangle = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}(X, Y) = \sum_{q \in Q_0} \alpha_q \beta_q - \sum_{\rho : i \to j \in Q_1} \alpha_i \beta_j$$

and $\mathrm{Ext}^i(X, Y) = 0$ for $i \geq 2$.
Let X and Y be two representations of a quiver Q. Then we can consider the linear map

$$\gamma_{X,Y} : \bigoplus_{i \in Q_0} \mathrm{Hom}(X_i, Y_i) \to \bigoplus_{\rho : i \to j \in Q_1} \mathrm{Hom}(X_i, Y_j)$$

with $\gamma_{X,Y}((f_i)_{i \in Q_0}) = (Y_\rho f_i - f_j X_\rho)_{\rho : i \to j \in Q_1}$.
We have $\ker(\gamma_{X,Y}) = \mathrm{Hom}(X, Y)$ and $\mathrm{coker}(\gamma_{X,Y}) = \mathrm{Ext}(X, Y)$, see [19]. The first statement is

obvious. The second one follows because every exact sequence $E(f) \in \mathrm{Ext}(X,Y)$ is defined by a morphism $f \in \bigoplus_{\rho:i\to j\in Q_1}\mathrm{Hom}_k(X_i,Y_j)$ in the following way

$$0 \to Y \to ((Y_i \oplus X_i)_{i\in Q_0}, (\begin{pmatrix} Y_\rho & f_\rho \\ 0 & X_\rho \end{pmatrix})_{\rho\in Q_1}) \to X \to 0$$

with the canonical inclusion on the left hand side and the canonical projection on the right hand side. Now it is straightforward to check that two sequences $E(f)$ and $E(g)$ are equivalent if and only if $f-g \in \mathrm{Im}(\gamma_{X,Y})$.
As far as bound quivers are concerned, we just have consider those exact sequence such that the middle term also satisfies the relations, thus we have $\mathrm{Ext}_{(Q,I)}(X,Y) \subseteq \mathrm{Ext}_Q(X,Y)$.

In the space of $\mathbb{Z}$-linear functions $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}Q_0,\mathbb{Z})$ we consider the basis given by the elements i^* for $i \in Q_0$, i.e. $i^*(j) = \delta_{i,j}$ for $j \in Q_0$. Define $\dim := \sum_{i\in Q_0} i^*$. After choosing $\Theta \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}Q_0,\mathbb{Z})$, we define the slope function $\mu : \mathbb{N}Q_0 \to \mathbb{Q}$ via

$$\mu(\alpha) = \frac{\Theta(\alpha)}{\dim(\alpha)}.$$

The slope $\mu(\underline{\dim}X)$ of a representation X of Q is abbreviated to $\mu(X)$.

Definition 2. *A representation X of (Q,I) is semistable (resp. stable) if for all subrepresentations $U \subset X$ (resp. all proper subrepresentations $0 \neq U \subsetneq X$) the following holds:*

$$\mu(U) \leq \mu(X) \text{ (resp. } \mu(U) < \mu(X)).$$

Denote the set of semistable (resp. stable) points by $R^{ss}_\alpha(Q,I)$ (resp. $R^s_\alpha(Q,I)$).
In this situation we have the following theorem, see [14]:

Theorem 3.
(1) *The set of stable points $R^s_\alpha(Q,I)$ is an open subset of the set of semistable points $R^{ss}_\alpha(Q,I)$, which is an open subset of $R_\alpha(Q,I)$.*
(2) *There exists a categorical quotient $M^{ss}_\alpha(Q,I) := R^{ss}_\alpha(Q,I)//G_\alpha$. Moreover, $M^{ss}_\alpha(Q,I)$ is a projective variety.*
(3) *There exists a geometric quotient $M^s_\alpha(Q,I) := R^s_\alpha(Q,I)/G_\alpha$, which is a smooth subvariety of $M^{ss}_\alpha(Q,I)$.*

Remark 1.

- The moduli space $M^{ss}_\alpha(Q,I)$ does not parametrize the semistable representations, but the polystable ones. Polystable representations are such representations which can be decomposed into stable ones of the same slope, see also [14].
- For a stable representation X we have that its orbit is of maximal possible dimension. Since the scalar matrices act trivially on $R_\alpha(Q,I)$, the isotropy group is one-dimensional. Thus, if the moduli space is not empty, for the dimension of the moduli space we have the lower bound

$$\begin{aligned}\dim M^s_\alpha(Q,I) &= \dim R_\alpha(Q,I) - (\dim G_\alpha - 1)\\ &\geq 1 - \sum_{q\in Q}\alpha_q^2 + \sum_{\rho:i\to j\in Q_1}\alpha_i\alpha_j - \sum_{(i,j)\in Q_0\times Q_0} r(i,j,I)\alpha_i\alpha_j.\end{aligned}$$

  Moreover, if $I = 0$ and the moduli space is not empty, we have

$$\dim M^s_\alpha(Q) = 1 - \langle\alpha,\alpha\rangle.$$

- It is well-known that the definition of μ-stability is equivalent to that of A.King [14]. Let $\tilde{\Theta}$ be another linear form. A representation X such that $\tilde{\Theta}(\underline{\dim}X) = 0$ is semistable (resp. stable) in the sense of King if and only if

$$\tilde{\Theta}(\underline{\dim}U) \geq 0 \text{ (resp. } \tilde{\Theta}(\underline{\dim}\mathrm{U}) > 0)$$

  for all subrepresentations $U \subset X$ (resp. all proper subrepresentations $0 \neq U \subsetneq X$).

Finally, we point out some properties of (semi-)stable representations. These properties will be very useful at different points of this paper, for proofs see [10].

Lemma 4. *For a bound quiver (Q, I) let $0 \to Y \to X \to Z \to 0$ be a short exact sequence of representations.*

(1) *The following are equivalent:*
 (a) $\mu(Y) \leq \mu(X)$
 (b) $\mu(X) \leq \mu(Z)$
 (c) $\mu(Y) \leq \mu(Z)$
 The same holds when replacing $\leq$ by $<$.
(2) *The following holds:* $\min(\mu(Y), \mu(Z)) \leq \mu(X) \leq \max(\mu(Y), \mu(Z))$.
(3) *If $\mu(Y) = \mu(X) = \mu(Z)$, then X is semistable if and only if Y and Z are semistable.*

If some property is independent of the point chosen in some non-empty open subset U of $R_\alpha(Q)$, following [20], we say that this property is true for a general representation with dimension vector $\alpha \in \mathbb{N}Q_0$.
Denote by $\beta \hookrightarrow \alpha$, if a general representation of dimension α has a subrepresentation of dimension β. From [20] we get the following theorem:

Theorem 5. *Let α be a dimension vector of the quiver Q. Then α is a Schur root if and only if for all $\beta \hookrightarrow \alpha$ we have $\langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle > 0$.*

Thus, if we define $\Theta_\alpha := \langle \_, \alpha \rangle - \langle \alpha, \_ \rangle$, a general representation of dimension α is Θ_α-stable in the sense of King if and only if α is a Schur root.
Consider the n-subspace quiver $S(n)$, i.e. $S(n)_0 = \{q_0, q_1, \ldots, q_n\}$ and $S(n)_1 = \{\rho_i : q_i \to q_0 \mid i \in \{1, \ldots, n\}\}$. Define the slope μ by choosing $\Theta = (-1, 0, \ldots, 0)$. Then we have the following corollary:

Corollary 6. *A representation X with dimension vector α is μ-stable if and only if X is Θ_α-stable.*

Proof. Let U be a subrepresentation of dimension β. It is easy to check that we have

$$\frac{-\alpha_0}{\sum_{i=0}^{n} \alpha_i} > \frac{-\beta_0}{\sum_{i=0}^{n} \beta_i}$$

if and only if

$$\langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle = \sum_{i=1}^{n} \alpha_i \beta_0 - \sum_{i=1}^{n} \beta_i \alpha_0 > 0.$$

□

Remark 2.

- Note that, in general, it is not possible to choose a slope function μ once and for all such that general representations of all Schur roots are μ-stable. Actually, this is only the case if the rank of the anti-symmetrized adjacency matrix of the quiver has rank equal to two, see [21].

Finally, we will need the following lemma:

Lemma 7. *Let M and N be two representations of a bound quiver (Q, I) such that* $\mathrm{Hom}(M, N) = \mathrm{Hom}(N, M) = 0$ *and* $\mathrm{End}(N) = k$. *Let* $\dim \mathrm{Ext}(N, M) = r > 0$. *Let $e_1, \ldots, e_l \in \mathrm{Ext}(N, M)$ with $1 \leq l \leq r$ be linear independent. Consider the exact sequence*

$$e : 0 \to M \to X \to N^l \to 0$$

induced by $e_1, \ldots, e_l$. Then we have $\mathrm{End}(X) \subseteq \mathrm{End}(M)$.

Proof. Consider the following long exact sequence

$$0 \longrightarrow \mathrm{Hom}(N,M)=0 \longrightarrow \mathrm{Hom}(N,X) \longrightarrow \mathrm{Hom}(N,N^l) \xrightarrow{\phi} \mathrm{Ext}(N,M)$$

induced by e. By construction ϕ is injective and, therefore, $\mathrm{Hom}(N,X)=0$. Now consider the following commutative diagram induced by e:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(N^l,M)=0 \arrow[r] \arrow[d] & \mathrm{Hom}(N^l,X) \arrow[r] \arrow[d] & \mathrm{Hom}(N^l,N^l) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(X,M) \arrow[r] \arrow[d] & \mathrm{Hom}(X,X) \arrow[r] \arrow[d, "\phi_1"] & \mathrm{Hom}(X,N^l) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(M,M) \arrow[r, "\phi_2"] & \mathrm{Hom}(M,X) \arrow[r] & \mathrm{Hom}(M,N^l)=0
\end{tikzcd}$$

Now we also have $\mathrm{Hom}(N^l,X)=0$. Thus, ϕ_1 is also injective and since ϕ_2 is an isomorphism, the claim follows. □

Note that the dual lemma dealing with sequences of the form

$$0 \to N^l \to X \to M \to 0$$

also holds and can be proven analogously.

1.3. **Quiver representations and representations of posets.** In this section we briefly recall the relation between representations of posets and representations of bound quivers. Everything presented here is well-known, see for instance [7] for a more general setup.
Let Q be a quiver without oriented cycles and multiple arrows. Moreover, we assume that all arrows are oriented to one vertex which is called the root. Let $\alpha \in \mathbb{N}Q_0$ be a dimension vector. By $S_\alpha(Q) \subset R_\alpha(Q)$ we denote the open subvariety of strict representations. For every (non-oriented) cycle $\rho_1 \dots \rho_n \tau_k^{-1} \dots \tau_1^{-1}$ with $\rho_i,\ \tau_j \in Q_1$ and $\rho_i \neq \tau_j$ we define a relation

$$r = \rho_1 \dots \rho_n - \tau_1 \dots \tau_k.$$

Let I be the ideal generated by all such relations.
Let $\mathcal{N}(Q)$ be the poset induced by the quiver Q and let V be a representation of this poset with dimension vector α. This defines a representation $F(V) \in S_\alpha(Q,I)$ satisfying the stated relations. Indeed, every inclusion $V_i \subset V_j$ defines an injective map $F(V)_{\rho_{i,j}} : V_i \to V_j$. Thus it defines a representation of $S_\alpha(Q,I)$. For two arbitrary representations V and W a morphism $g: V \to W$, defines a morphism $F(g): F(V) \to F(W)$ where $F(g)_i := g|_{V_i} : F(V)_i \to F(W)_i$.

The other way around let $X \in S_\alpha(Q,I)$. This gives rise to a representation $G(X)$ of $\mathcal{N}(Q)$ by defining $G(X)_q = X_{\rho_n^q} \circ \dots \circ X_{\rho_1^q}(X_q)$ for some path $p_q = \rho_1^q \dots \rho_n^q$ from q to q_0. This definition is independent of the chosen path. Moreover, every morphism $\varphi = (\varphi_q)_{q \in Q_0} : X \to Y$ defines a morphism $G(\varphi)$ which is induced by $\varphi_{q_0} : X_0 \to Y_0$.
Thus we get an equivalence between the categories of strict representations of Q bound by I and representations of $\mathcal{N}(Q)$. This equivalence also preserves dimension vectors. In the following, by $Q(\mathcal{N})$ we denote the quiver induced by the poset $\mathcal{N}$ and bound by all possible commutative relations, i.e. those as constructed above.

If the global dimension of $kQ(\mathcal{N})$ is at most two, for two representations X and Y with $\underline{\dim} X = \alpha$ and $\underline{\dim} Y = \beta$ we get

$$\begin{aligned}
\langle X, Y \rangle &= \dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y) + \dim \mathrm{Ext}^2(X,Y) \\
&= \sum_{q \in Q(\mathcal{N})_0} \alpha_q \beta_q - \sum_{\rho: i \to j \in Q(\mathcal{N})_1} \alpha_i \beta_j + \sum_{(i,j) \in (Q(\mathcal{N})_0)^2} r(i,j,J) \alpha_i \beta_j,
\end{aligned}$$

see also [4]. For $\alpha = \beta$ this is also known as Tits form and Drozd form respectively.

2. Unitarization

2.1. **Preliminaries.** Except for Section 2.3, in the following, we fix the base field $\mathbb{C}$. We understand a strict representation X of given a quiver $Q(\mathcal{N})$ associated to a poset $\mathcal{N}$ as a system of vector subspaces $(V;(V_i)_{i\in\mathcal{N}})$. We will use the following criteria for χ-unitarization of X (which was basically obtained by the different authors Yi Hu, A. King, A. Klyachko, T. Tao and others in different formulations).

Theorem 8. *(see for example [12]) Let $(V;(V_i)_{i\in\mathcal{N}})$ be an indecomposable strict representation. Then $(V;(V_i)_{i\in\mathcal{N}})$ is unitarizable with the weight $\chi=(\chi_0;(\chi_i)_{i\in\mathcal{N}})\in\mathbb{R}_+^{|\mathcal{N}|+1}$ if and only if for every proper subspace $0\neq U\subsetneq V$ the following holds*

$$\chi_0=\frac{1}{\dim V}\sum_{i\in\mathcal{N}}\chi_i\dim V_i,$$

$$\frac{1}{\dim U}\sum_{i\in\mathcal{N}}\chi_i\dim(V_i\cap U)<\frac{1}{\dim V}\sum_{i\in\mathcal{N}}\chi_i\dim V_i.$$

Remark 3.

- If an indecomposable strict representation X can be unitarized with the weight $\chi\in\mathbb{N}^{|\mathcal{N}|+1}$, it is obviously $\tilde\Theta$-stable in the sense of King with
$$\tilde\Theta=(\sum_{i=1}^{n}\chi_i\dim X_i,-\dim X_0\chi_1,\dots,-\dim X_0\chi_n)$$
and vice versa. Moreover, choosing the linear form $\Theta=\mu(X)\dim-\tilde\Theta$, where $\mu(X)\in\mathbb{Z}$ can be chosen arbitrarily, this representation is μ-stable. Moreover, we have
$$\chi_i=\frac{\Theta_i-\mu(X)}{\dim X_0}.$$
- It is easy to check that we can modify the linear form Θ which defines the slope μ without changing the set of stable points in the following two ways: first we can multiply it by a positive integer; second, we can add an integer multiple of the linear form dim to Θ. In particular, if we change the linear form appropriately, the weight, which it defines, can be assumed to be positive.
- Note that King [14, Proposition 6.5] gives also a connection between stability and a choice of some Hermitian structure on the corresponding vector spaces. In the primitive case it coincides with the linear relations in the algebras $\mathcal{A}_{\mathcal{N},\chi}$.

Definition 9. *Let $\alpha\in\mathbb{N}Q(\mathcal{N})_0$. If we have $\chi_i\geq 0$ for the weight induced by some linear form Θ (resp. $\tilde\Theta$), the dimension vector α is called χ-positive.*

We will also use the following lemma:

Lemma 10. *Let $X=(V;V_1,\dots,V_n)$ be a χ-unitarizable representation. Then for an arbitrary set of subspaces $V_{n+j}\subset V$, $j=1,\dots,m$, the representation $\tilde X=(V;V_1,\dots,V_n,V_{n+1},\dots,V_{n+m})$ is also unitarizable with some weight.*

Proof. We prove that $(V;V_1,\dots,V_n,V_{n+1})$ is unitarizable with some weight (the remaining part follows by induction). Let $U\subset V$ be some subspace of V such that

$$R=\frac{1}{\dim V}\sum_{i=1}^{n}\chi_i\dim V_i-\frac{1}{\dim U}\sum_{i=1}^{n}\chi_i\dim(V_i\cap U)$$

is minimal. Note that, it is clear that such a subspace exists because the right hand side only takes finitely many values. Since X is unitarizable, we have $R>0$ and there exist an $\varepsilon>0$ such that $R-\varepsilon>0$. Define $\tilde\chi$ in the following way

$$\tilde\chi_i=\chi_i,\quad i=1,\dots,n,\quad \tilde\chi_{n+1}=R-\varepsilon.$$

Our claim is that $\tilde{X}$ is $\tilde{\chi}$-unitarizable. Indeed, let $M \subset V$ be some proper subspace of V then we have

$$\begin{aligned}\frac{1}{\dim M}\sum_{i=1}^{n+1}\tilde{\chi}_i \dim(V_i\cap M) &= \frac{1}{\dim M}\sum_{i=1}^{n}\chi_i \dim(V_i\cap M) + \frac{\tilde{\chi}_{n+1}\dim(V_{n+1}\cap M)}{\dim M}\\ &\leq \frac{1}{\dim V}\sum_{i=1}^{n}\chi_i \dim V_i - R + \frac{(R-\varepsilon)\dim(V_{n+1}\cap M)}{\dim M}\\ &< \frac{1}{\dim V}\sum_{i=1}^{n}\chi_i \dim V_i < \frac{1}{\dim V}\sum_{i=1}^{n+1}\tilde{\chi}_i \dim V_i.\end{aligned}$$

Hence $(V; V_1, \dots, V_n, V_{n+1})$ is $\tilde{\chi}$-unitarizable. □

2.2. **Unitarization of general representations of unbound quivers.** Let $Q(\mathcal{N})$ corresponding to the poset $\mathcal{N}$ be unbound, i.e. for the ideal of relations I we have $I = 0$. For a vertex $q \in Q(\mathcal{N})_0$ let

$$N_q = \{q' \in Q(\mathcal{N})_0 \mid \exists \rho : q \to q' \vee \exists \rho : q' \to q\}$$

be the set of its neighbours. Moreover, define $\varphi_q : N_q \to \{\pm 1\}$ by

$$\varphi_q(q') = \begin{cases} -1 \text{ if } \rho : q' \to q, \\ 1 \text{ if } \rho : q \to q'. \end{cases}$$

and the weight χ by

$$\chi_q = \begin{cases} \sum_{q'\in N_q}\varphi_q(q')\alpha_{q'}, & q \neq q_0 \\ -\sum_{q'\in N_q}\varphi_q(q')\alpha_{q'} & q = q_0 \end{cases}.$$

Note that, if the poset $\mathcal{N}$ is primitive, then each strict dimension vector is χ-positive.

Theorem 11. (1) *Let α be a χ-positive Schur root of an unbound quiver $Q(\mathcal{N})$ induced by a poset $\mathcal{N}$. Let q_0 be the unique source. Then a general representation of $Q(\mathcal{N})$ with dimension vector α can be unitarized with the weight χ.*

(2) *Let α be a Schur root of an unbound quiver $Q(\mathcal{N})$ induced by a poset $\mathcal{N}$. Then a general representation of $Q(\mathcal{N})$ with dimension vector α can be unitarized with a weight χ which is induced by the above one.*

Proof. Let α be a Schur root. Let X be a general representation with dimension vector α and let β be the dimension vector of a subrepresentation of X. By Theorem 5 we have

$$\tilde{\Theta}_\alpha := \langle\beta,\alpha\rangle - \langle\alpha,\beta\rangle > 0.$$

Then it is easy to check that we have

$$\begin{aligned}\langle\beta,\alpha\rangle - \langle\alpha,\beta\rangle &= -\sum_{q\in Q(\mathcal{N})_0}\beta_q \sum_{\substack{q'\in N_q\\ \varphi_q(q')=1}}\alpha_{q'} + \sum_{q\in Q(\mathcal{N})_0}\beta_q \sum_{\substack{q'\in N_q\\ \varphi_q(q')=-1}}\alpha_{q'}\\ &= -\sum_{q\in Q(\mathcal{N})_0}\beta_q \sum_{q'\in N_q}\varphi_q(q')\alpha_{q'}.\end{aligned}$$

Now let $\chi_q = \sum_{q'\in N_q}\varphi_q(q')\alpha_{q'}$.
By Theorem 8, a representation can be unitarized with the weight χ if and only if

$$\begin{aligned}\frac{1}{\beta_{q_0}}\sum_{q\in Q(\mathcal{N})_0\setminus\{q_0\}}\beta_q\sum_{q'\in N_q}\varphi_q(q')\alpha_{q'} &< \frac{1}{\alpha_{q_0}}\sum_{q\in Q(\mathcal{N})_0\setminus\{q_0\}}\alpha_q\sum_{q'\in N_q}\varphi_q(q')\alpha_{q'}\\ &= \frac{1}{\alpha_{q_0}}\sum_{q\in N_{q_0}}\varphi_q(q_0)\alpha_{q_0}\alpha_q = -\sum_{q\in N_{q_0}}\varphi_{q_0}(q)\alpha_q\end{aligned}$$

for all subrepresentations U of dimension vector β. But this is obviously the same.

Taking into account Remark 3, the second part of the Theorem follows when changing the linear form $\tilde{\Theta}_\alpha$ appropriately. □

Corollary 12. *Let the $Q(\mathcal{N})$ induced by the poset $\mathcal{N}$ be unbound. Then the unique indecomposable representation of a real root α can be unitarized if and only if α is a real Schur root.*

Proof. If α is no Schur root, we have $\dim \mathrm{End} X_\alpha > 1$ for the unique indecomposable representation with dimension vector α. In particular, X_α cannot be stable, and thus cannot be unitarized. If α is a Schur root, X_α has a dense orbit in the affine variety $R_\alpha(Q)$. Thus it is a general representation, and we can apply the preceding theorem. □

2.3. **Unitarization of general representations of bound quivers.** In this section we state a recipe which can be used to construct unitarizable representations of bound quivers. Therefore, let k be an algebraically closed field. Moreover, let $\mathcal{N}$ be a poset and $Q(\mathcal{N})$ be the corresponding (bound) quiver as defined in Section 1.3. Note that, in general, we do not have $\mathrm{Ext}^i_{Q(\mathcal{N})}(X,Y) = 0$ if $i \geq 2$ for two arbitrary representations X and Y of the quiver $Q(\mathcal{N})$. Thus, in order to obtain some result similar to Theorem 11, the basic idea is the following: we glue polystable representations of an unbound quiver, which is a subquiver of $Q(\mathcal{N})$, with a direct sum of a simple module in order to obtain stable representations of $Q(\mathcal{N})$. Note that the global dimension of the corresponding path algebra of the unbound quiver is one.

As already mentioned, we say that a general representation of dimension α satisfies some property if there exists an open subset U of $R_\alpha(Q)$ such that every representation X_u, $u \in U$, satisfies this property. In abuse of notation, we will skip the index u in what follows. Moreover, if there is more than one property requested, we always consider elements lying in the intersection of the corresponding open subsets. In addition, when considering general representations, we restrict to dimension vectors which support can be understood as a quiver without relations. Recall that otherwise the variety of representations may not be irreducible, see [14].

Let $\mathcal{N}$ be a poset corresponding to an unbound quiver and $\mathcal{M} = \{i_1, \dots, i_l\} \subset \mathcal{N}$ be a subset of elements such that

$$t(\mathcal{M}) := \min\{q \in \mathcal{N} \cup \{0\} \mid i_j \preceq q \ \forall\, i_j \in \mathcal{M}\}$$

is unique. For this definition we assume that $i \prec 0$ for all $i \in \mathcal{N}$. Moreover, let $\mathcal{M} \subset \mathcal{N}$ such that for every two vertices $i, j \in \mathcal{M}$, we have $t(\{i,j\}) = t(\mathcal{M})$. In this setup, by induction on the number of elements of $\mathcal{M}$ we obtain that for a general representation X of $Q(\mathcal{N})$ we have

$$\dim \bigcap_{i \in \mathcal{M}} X_i = \max\{0, \sum_{i \in \mathcal{M}} \dim X_i - (|\mathcal{M}| - 1) \dim X_{t(\mathcal{M})}\}.$$

Indeed, it is straightforward to check that this condition is equivalent to the non-vanishing of certain minors of matrices which is an open condition. This formula can be extended to other subsets of $\mathcal{N}$ as well.

Let $\mathcal{N}$ be a poset such that there exists a poset $\mathcal{N}'$ corresponding to an unbound quiver obtained from $\mathcal{N}$ by deleting elements $q \in \mathcal{N}$ such that there does not exist a $j \in \mathcal{N}$ with $j \prec q$ and the corresponding relations. Moreover, we assume that for all $i, j \in N_q$ we have $t(\{i,j\}) = t(N_q)$. This corresponds to a quiver $Q(\mathcal{N}')$ which is a proper subquiver of $Q(\mathcal{N})$ obtained by deleting some sources and the corresponding arrows. E.g. starting with the non-primitive poset $(N, 5)$, we get the primitive poset $(2, 1, 5)$, see Section 4 for the notation. We call such a tuple of posets (resp. quivers) related.

In the following we assume that $\mathcal{N}'$ and $\mathcal{N} = \mathcal{N}' \cup \{q\}$ are related. This is no restriction because we will see that the case $\mathcal{N} = \mathcal{N}' \cup \{q_1, \dots, q_n\}$ can be treated by applying Lemma 10. Note that, in these cases the global dimension is at most two.
Obviously, every representation of $Q(\mathcal{N}')$ can be understood as a representation of $Q(\mathcal{N})$. Let α' be a dimension vector of $Q(\mathcal{N}')$ such that a general representation is polystable with respect to the Euler characteristic, i.e. the canonical decomposition only consists of Schur roots of

the same slope. Note that, obviously, every representation of $Q(\mathcal{N}')$ satisfies the commutative relation of $Q(\mathcal{N})$. In particular, the varieties of representations corresponding to dimension vectors $\alpha \in \mathbb{N}Q(\mathcal{N})_0$ with $\alpha_q = 0$ are irreducible. Let $X' = \oplus_{i=1}^m (X_i')^{t_i}$ with $\dim X' = \alpha'$ and $X_i' \not\cong X_j'$ for $i \neq j$ be a general polystable representation of $Q(\mathcal{N})$ and S_q be the simple module corresponding to q. Since we have $\dim X_q' = 0$, it is straightforward that we have

$$\dim \operatorname{Ext}_{Q(\mathcal{N})}(S_q, X') = \dim \bigcap_{l \in N_q} X_l' = \max\{0, \sum_{l \in N_q} \dim X_l' - (|N_q| - 1) \dim X'_{t(N_q)}\}.$$

Note that $t(N_q)$ is the vertex of the quiver $Q(\mathcal{N})$ where the relations starting at q terminate. Moreover, for two representations X' and Y' of $Q(\mathcal{N}')$ we obviously have $\operatorname{Ext}_{Q(\mathcal{N})}(X', Y') = \operatorname{Ext}_{Q(\mathcal{N}')}(X', Y')$ and $\operatorname{Hom}_{Q(\mathcal{N})}(X', Y') = \operatorname{Hom}_{Q(\mathcal{N}')}(X', Y')$. In the following we will skip the index $Q(\mathcal{N})$, and we will only use indices if we consider the quiver $Q(\mathcal{N}')$.
We should mention that, fixing a dimension vector and using the dimension formula, for a general representation of the poset $\mathcal{N}$, it is often straightforward to write down a projective and injective resolution of minimal length.
If $\dim \bigcap_{l \in N_q} X_l' \neq \{0\}$, we generally have $\operatorname{Ext}^2(S_q, X') = 0$ because

$$-\langle S_q, X' \rangle = \sum_{i \in N_q} \dim X_i' - (|N_q| - 1) \dim X'_{t(N_q)} = \dim \operatorname{Ext}(S_q, X').$$

In particular, there exists an indecomposable strict representation of $Q(\mathcal{N})$ satisfying the commutative relations.

Definition 13. *We call a dimension vector α' of $Q(\mathcal{N}')$ strongly strict if for a general representation X' with $\underline{\dim} X' = \alpha'$, we have $\operatorname{Ext}(S_q, X') \neq 0$.*

For instance, in the case of the poset $(N, 5)$ we may consider the related poset $(2, 1, 5)$ and the unique imaginary Schur root $\alpha' = (6; 2, 4; 3; 1, 2, 3, 4, 5)$. This root is strongly strict and we get a representation of the poset $(N, 5)$ with dimension vector $\alpha = (6; 2, 4; 1, 3; 1, 2, 3, 4, 5)$ by an extension with the simple module corresponding to the additional source.
We will need the following lemma:

Lemma 14. *Let $X' = \bigoplus_{i=1}^m (X_i')^{t_i}$ with $X_i' \not\cong X_j'$ for $i \neq j$ be a polystable representation. If Y' is an indecomposable subrepresentation such that $\operatorname{Hom}(X', Y') \neq 0$, it follows that $Y' \cong X_i'$ for some $i \in \{1, \ldots, m\}$.*

Proof. Since the canonical composition $\tau : X_j' \hookrightarrow X' \to Y'$ is not zero for some j, this defines a factor representation $\operatorname{Im}(\tau) = U$ of X_j' which is a subrepresentation of X'. Thus we have $\mu(X_j') \leq \mu(U) \leq \mu(X')$ and thus $\mu(X_j') = \mu(U) = \mu(X')$. It follows that $U \cong X_j'$.
Moreover, Y defines a subrepresentation of some X_i' via the canonical composition $\phi : Y' \hookrightarrow X' \twoheadrightarrow X_i'$. Since this defines a non-zero homomorphism $\phi \circ \tau : X_j' \to X_i'$, we have $i = j$. Moreover, $\phi \circ \tau$ is forced to be an isomorphism and, therefore, X_j' is a direct summand of Y'. Since Y' is indecomposable, we have $Y' \cong X_j'$. □

Define $n_j := \dim \operatorname{Ext}(S_q, X_j')$. Consider the quiver $\tilde{Q}$ with vertices $\tilde{Q}_0 = \{i_0, i_1, \ldots, i_m\}$ and arrows $\tilde{Q}_1 = \{\rho_{j,l_j} : i_0 \to i_j \mid j \in \{1, \ldots, m\}, l_j \in \{1, \ldots, n_j\}\}$. Then every representation of this quiver with dimension vector $t = (t_0, t_1, \ldots, t_m)$ induces an exact sequence $e \in \operatorname{Ext}(S_q^{t_0}, X')$. To see this, we keep in mind the description of exact sequences given in Section 1.2 and that the given Schur roots (including the simple one) are pairwise orthogonal, i.e. there exist no homomorphisms between them. Thus, fixing representations of the respective dimension vectors, they correspond to the simple modules of the category containing these modules and being closed under extensions, see [19, Theorem 1.2].
We call the dimension vector t stable if it is a Schur root and polystable if it has the canonical decomposition $t = \oplus_{l=1}^m \alpha_l^{t_l}$ with $\alpha_l = i_0 + n_l i_l$, see [20] for the general theory and [6] for an algorithm determining the canonical decomposition. Moreover, we call the extension e stable (resp. polystable) if the corresponding representation of $\tilde{Q}$ is stable (resp. polystable).

In this setup, let X be a stable extension of some general representation X' with $\underline{\dim}X' = \alpha'$ where α' is strongly strict. Every subrepresentation Y of X induces a subrepresentation Y' of X'. In particular, we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] & X \arrow[r] & S_q^r \arrow[r] & 0 \\ 0 \arrow[r] & Y' \arrow[r] \arrow[u, hook] & Y \arrow[r] \arrow[u, hook] & S_q^l \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

Since α' is strongly strict, we have $\mathrm{Ext}^2(Y, X) \cong \mathrm{Ext}^2(Y', X) = 0$. Moreover, since $\mathrm{Hom}(Y', S_q) = \mathrm{Ext}(Y', S_q) = 0$, we get $\mathrm{Hom}(Y', X') \cong \mathrm{Hom}(Y', X)$ and $\mathrm{Ext}(Y', X') \cong \mathrm{Ext}(Y', X)$. Moreover, we have $\mathrm{Hom}(S_q, X) = 0$. Since we have $\mathrm{Ext}^2(S_q, X') = \mathrm{Ext}^2(S_q, X) = 0$, from the long exact sequence

$$0 \to \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Y', X) \to \mathrm{Ext}(S_q^l, X) \to \mathrm{Ext}(Y, X) \to \mathrm{Ext}(Y', X) \to 0$$

we get

$$\dim \mathrm{Hom}(Y, X) - \dim \mathrm{Ext}(Y, X) = \dim \mathrm{Hom}(Y', X') - \dim \mathrm{Ext}(Y', X') - \dim \mathrm{Ext}(S_q^l, X).$$

First assume that $l > 0$ and that no direct summand of Y' is a direct summand of Y. Then, we generally have $\mathrm{Ext}^2(S_q, Y') = 0$. Indeed, we again generally have

$$-\langle S_q, Y'\rangle = \sum_{i \in N_q} \dim Y_i' - (|N_q| - 1) \dim Y'_{t(N_q)} = \dim \mathrm{Ext}(S_q, Y') > 0.$$

Thus we get long exact sequences

$$0 \to \mathrm{Hom}(X, Y') \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, S_q^l) \to \mathrm{Ext}(X, Y') \to \mathrm{Ext}(X, Y) \to 0$$

and

$$0 \to \mathrm{Hom}(X, Y') \to \mathrm{Hom}(X', Y') \to \mathrm{Ext}(S_q^r, Y') \to \mathrm{Ext}(X, Y') \to \mathrm{Ext}(X', Y') \to 0.$$

Thus we get

$$\begin{aligned} \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}(X, Y) &= \dim \mathrm{Hom}(X', Y') - \dim \mathrm{Ext}(X', Y') \\ &\quad + \dim \mathrm{Hom}(X, S_q^l) - \dim \mathrm{Ext}(S_q^r, Y'). \end{aligned}$$

Thus in summary we get

$$\langle Y, X\rangle - \langle X, Y\rangle = \langle Y', X'\rangle - \langle X', Y'\rangle - \dim \mathrm{Ext}(S_q^l, X) - \dim \mathrm{Hom}(X, S_q^l) + \dim \mathrm{Ext}(S_q^r, Y').$$

Define

$$\Psi(X, Y) = \dim \mathrm{Ext}(S_q^l, X) + \dim \mathrm{Hom}(X, S_q^l) - \dim \mathrm{Ext}(S_q^r, Y').$$

Then we have

$$\begin{aligned} \Psi(X, Y) &= l(\dim \bigcap_{i \in N_q} X_i - \dim X_q) + l \dim X_q - r \dim \bigcap_{i \in N_q} Y_i' \\ &= l \dim \bigcap_{i \in N_q} X_i - r \dim \bigcap_{i \in N_q} Y_i'. \end{aligned}$$

For a strongly strict dimension vector α of $Q(\mathcal{N})$ we fix the linear form $\Theta_\alpha : \mathbb{Z}Q_0 \to \mathbb{Z}$ given by

$$\begin{aligned} \Theta_\alpha(\beta) &= \langle \beta, \alpha\rangle - \langle \alpha, \beta\rangle \\ &= -\sum_{\rho: i \to j \in Q(\mathcal{N})_1} \beta_i \alpha_j + \sum_{(i,j) \in (Q_0)^2} r(i, j, I) \beta_i \alpha_j \\ &\quad + \sum_{\rho: i \to j \in Q(\mathcal{N})_1} \alpha_i \beta_j - \sum_{(i,j) \in (Q_0)^2} r(i, j, I) \alpha_i \beta_j. \end{aligned}$$

Remark 4.

- Let $\mathcal{N} = \mathcal{N}' \cup \{q\}$ be a poset related to a poset $\mathcal{N}'$ without cycles. Let α and β be two dimension vectors of $\mathcal{N}$ and let α' and β' be the corresponding dimension vectors of $\mathcal{N}'$ such that $\beta' \hookrightarrow \alpha'$. Consider the linear form induced by the considerations from above:

$$\begin{aligned}\tilde{\Theta}_\alpha(\beta) &= \langle \beta', \alpha' \rangle - \langle \alpha', \beta' \rangle - \beta_q(\sum_{i \in N_q} \alpha_i - (|N_q| - 1)\alpha_{t(N_q)}) \\ &\quad + \alpha_q(\sum_{i \in N_q} \beta_i - (|N_q| - 1)\beta_{t(N_q)}).\end{aligned}$$

  It is straightforward that we have $\Theta_\alpha = \tilde{\Theta}_\alpha$. Let χ' be the weight as given in Theorem 11. Let χ be the weight such $\chi_q = \sum_{i \in N_q} \alpha_i - (|N_q| - 1)\alpha_{t(N_q)}$, $\chi_i = \chi'_i - \alpha_q$ for all $i \in N_q$, $\chi_{t(N_q)} = \chi'_{t(N_q)} + \alpha_q(|N_q| - 1)$ and $\chi_i = \chi'_i$ for the remaining vertices. Like in the proof of Theorem 11 one checks that a representation X of dimension α can be unitarized if and only if we have $\tilde{\Theta}_\alpha(\underline{\dim} Y) > 0$ for all subrepresentations Y. If there is at least one such representation, there exists an open, not necessary dense, subset of unitarizable representations.

By $\hom(\alpha, \beta)$ (resp. $\operatorname{ext}(\alpha, \beta)$) we denote the general dimension of the vector space of homomorphisms (resp. extensions) between general representations of dimension α and β of some unbound quiver. These values are well-defined, see for instance [20].

Proposition 15. *Let a general representation with dimension vector α' be polystable and let*

$$0 \to X' \to X \to S_q^r \to 0$$

be some stable extension of some general representation X' with $\underline{\dim} X' = \alpha'$. Moreover, assume $\beta' \hookrightarrow \alpha'$. Let

$$0 \to Y' \to Y \to S_q^l \to 0$$

with $1 \leq l \leq r$ and $\underline{\dim} Y' = \beta'$ such that Y is a subrepresentation of X. Then we have

$$\dim \operatorname{Hom}(Y', X') > \Psi(X, Y)$$

and Y has no direct summand isomorphic to S_q.

Proof. The second statement is obvious since the first sequence is stable. Let s be maximal such that $\phi : \oplus_{i=1}^s Y' \hookrightarrow X'$. Then we obviously have $s \leq \dim \operatorname{Hom}(Y', X')$. Let $d := \dim \bigcap_{i \in N_q} Y'_i$ and $n := \dim \bigcap_{i \in N_q} X'_i$. Every injection φ_i, $i = 1, \ldots, s$, defines an d-dimensional subspace of $\bigcap_{i \in N_q} X'_i$. Thus we have a decomposition $\bigcap_{i \in N_q} X'_i = \oplus_{i=1}^s \varphi_i(\bigcap_{i \in N_q} Y'_i) \oplus U$ with $\dim U = n - sd$. Every representation M of the quiver

$$\begin{tikzcd} 1 \arrow[rr, bend left=40, "\alpha_1"] \arrow[rr, bend left=15, "\alpha_2"] \arrow[rr, bend right=30, "\alpha_d"'] \arrow[d] & \vdots & 2 \\ 3 \end{tikzcd}$$

with $\dim(M_1) = r$, $\dim(M_2) = s$ and $\dim(M_3) = n - sd$ defines an extension $\operatorname{Ext}(S_q^r, X')$. In order to obtain a condition as mentioned we have to investigate whether a general representation with this kind of dimension vector has a subrepresentation of dimension vector $(l, 1, 0)$ because such representations correspond to extensions $0 \to Y' \to Y \to S_q^l \to 0$. If $r \leq n - sd$, we obviously have no such subrepresentation. Thus let $r > n - sd$. Considering the canonical decomposition of the dimension vector $(r, s, n - sd)$, it is easy to check that $(l, 1, 0) \hookrightarrow (r, s, n - sd)$ if and only if $(l, 1) \hookrightarrow (r - n + sd, s)$ considering the quiver without oriented cycles having two vertices and d arrows. Following [20, Theorem 5.4] we have $\operatorname{ext}(\alpha, \beta) = \max_{\alpha' \hookrightarrow \alpha}\{-\langle \alpha', \beta \rangle\}$. Since the only possible subrepresentations of general representations of dimension vector $(l,1)$ are of dimension $(l',1)$ with $0 \le l' \le l$ we get the condition

$$\hom((l,1),((r-n+sd),s)) = \langle (l,1),((r-n+sd),s)\rangle = l(r-n+sd)+s-lds > 0.$$

But this is equivalent to $s > ln - lr \ge ln - dr$. $\square$

Next we will show that every subrepresentation of X' also does not contradict the stability condition.

Proposition 16. *Let α be χ-positive where the weight is given as in Remark 4. If $l = 0$, i.e. $Y \cong Y'$, and Y' has no direct summand isomorphic to X_i' for some $i = 1,\dots m$, we have*

$$\langle Y', X\rangle - \langle X, Y'\rangle > 0.$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & S_q^r \arrow[u] \arrow[r, equal] & S_q^r \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & Y' \arrow[u] \arrow[r] & X \arrow[u] \arrow[r] & X/Y' \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & Y' \arrow[u, equal] \arrow[r] & X' \arrow[u] \arrow[r] & X'/Y' \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since $\mathrm{Ext}(Y', X/Y') \cong \mathrm{Ext}(Y', X'/Y')$ we have $\mathrm{Ext}(Y', X/Y') = 0$ by [20, Theorem 3.3]. Moreover, since X' is polystable, by Lemma 14 we get $\mathrm{Hom}(X/Y', Y') \cong \mathrm{Hom}(X'/Y', Y') = 0$. Therefore, we have

$$\begin{aligned}
\langle Y', X\rangle - \langle X, Y'\rangle &= \langle Y', X/Y'\rangle - \langle X/Y', Y'\rangle \\
&= \dim \mathrm{Hom}(Y', X/Y') + \dim \mathrm{Ext}(X/Y', Y') - \dim \mathrm{Ext}^2(X/Y', Y').
\end{aligned}$$

Define $X_\cap := \bigcap_{i\in N_q} X_i$. First assume that $Y_0' \cap X_\cap \neq 0$. For $i \in N_q$ let τ_i be the unique path from i to $t(N_q)$ and $\tau_{t(N_q)}$ be the unique path from $t(N_q)$ to the root. Let T_i and $T_{t(N_q)}$ respectively be the representations such that $(T_i)_j = k$ for all j such that j is tail of some arrow in τ_i and $(T_i)_j = 0$ otherwise. Moreover, we assume $(T_i)_\rho = \mathrm{id}$ where it makes sense. In the same way, we define $T_{t(N_q)}$. Define $k_0 := \dim Y_0' \cap X_\cap - \dim Y'_{t(N_q)} \cap X_\cap$. Then we obtain some exact sequence

$$0 \to Y' \to Y'' \to T_{t(N_q)}^{k_0} \to 0$$

where we just glue $T_{t(N_q)}^{k_0}$ to $Y_0' \cap X_\cap \neq 0$. Moreover, if we define $k_i := \dim Y_0'' \cap X_\cap - \dim Y_i'' \cap X_\cap$ for all $i \in N_q$, in the same manner we get

$$0 \to Y'' \to \tilde{Y} \to \oplus_{i\in N_q} T_i^{k_i} \to 0.$$

Now by construction we have $\dim \mathrm{Ext}^2(S_q, \tilde{Y}) = 0$ and, therefore, $\dim \mathrm{Ext}^2(X/\tilde{Y}, \tilde{Y}) = 0$. Note that, we again have $\dim \mathrm{Ext}(S_q, \tilde{Y}) = \dim \cap_{i\in N_q} \tilde{Y}_i \neq 0$. Moreover, since the dimension vector is χ-positive we have

$$\langle T_i, X\rangle - \langle X, T_i\rangle \le 0$$

for $i \in N_q \cup t(N_q)$. Thus we obtain

$$0 < \langle \tilde{Y}, X\rangle - \langle X, \tilde{Y}\rangle = \langle Y', X\rangle - \langle X, Y'\rangle + \sum_{i\in N_q\cup t(N_q)} \langle T_i, X\rangle - \langle T_i, X\rangle \le \langle Y', X\rangle - \langle X, Y'\rangle.$$

Now assume $Y_0' \cap X_\cap = 0$. In particular, S_q is no direct summand of X'/Y'. Let $P(q)$ be the indecomposable projective module corresponding to the vertex q which is given by the vector spaces $P(q)_j = k$ for all $j \succeq q$ and $P(q)_0 = k$ and the identity map where it makes sense. Now it is straightforward that the injective dimension of $P(q)$ is one because the cokernel of $P(q) \hookrightarrow I(0)$ is also injective. Since $\mathrm{Hom}(P(q), Y') = 0$, there exists a short exact sequence

$$0 \to P(q)^{\dim(X/Y')_q} \to X/Y' \to \overline{X/Y'} \to 0.$$

Since $P(q)$ has injective dimension one and $\dim(\overline{X/Y'})_q = 0$, we have $\mathrm{Ext}^2(X/Y', Y') = 0$. Thus the claim follows. $\square$

Let X' be a general representation of $\mathcal{N}'$ of dimension α' and let $\beta' \hookrightarrow \alpha'$ such that a general representation of dimension β' has no direct summand which is isomorphic to X_i' for all $i = 1, \ldots, m$. Moreover, we assume that Y' is a general representation of dimension β'. Consider the exact sequence

$$0 \to Y' \to X' \to X'/Y' \to 0$$

Since $\mathrm{Hom}(X', Y') = 0$ and $\mathrm{Ext}^2(X'/Y', Y') = 0$, this induces the following long exact sequences

$$0 \to \mathrm{Hom}(Y', Y') \to \mathrm{Ext}(X'/Y', Y') \to \mathrm{Ext}(X', Y') \to \mathrm{Ext}(Y', Y') \to 0$$

and

$$0 \to \mathrm{Hom}(Y', Y') \to \mathrm{Hom}(Y', X') \to \mathrm{Hom}(Y', X'/Y') \to \mathrm{Ext}(Y', Y') \to \mathrm{Ext}(Y', X') \to 0.$$

Thereby, $\dim \mathrm{Ext}(Y', X'/Y') = 0$ by [20, Theorem 3.3]. Since we have $\dim \mathrm{Ext}(Y', X') \leq \dim \mathrm{Ext}(Y', Y') = -\langle Y', Y' \rangle + \dim \mathrm{Hom}(Y', Y')$, we get

$$\begin{aligned} \dim \mathrm{Ext}(X', Y') &= \dim \mathrm{Ext}(X'/Y', Y') - \langle Y', Y' \rangle \\ &\geq \dim \mathrm{Ext}(X'/Y', Y') + \dim \mathrm{Ext}(Y', X') - \dim \mathrm{Hom}(Y', Y'). \end{aligned}$$

Since $\dim \mathrm{Ext}(X'/Y', Y') \geq \dim \mathrm{Hom}(Y', Y')$, we obtain

$$\dim \mathrm{Ext}(X', Y') \geq \dim \mathrm{Ext}(Y', X').$$

Theorem 17. *Let a general representation with dimension vector α' be polystable with respect to $\Theta_{\alpha'} = \langle \_, \alpha' \rangle - \langle \alpha', \_ \rangle$ and let*

$$0 \to X' \to X \to S_q^r \to 0$$

be some stable extension of some general representation X' with $\underline{\dim} X' = \alpha'$ such that α is χ-positive where χ is given as in Remark 4. Then X is stable and can be unitarized with the weight χ.

Proof. We use the notations from above. Let Y be a proper subrepresentation of X and Y' be the induced representation of the primitive poset $\mathcal{N}'$.
If $l = 0$ and Y' has no direct summand isomorphic to X_i', the claim follows from Proposition 16. Next assume that Y' has no direct summand isomorphic to X_i', $\dim Y_q > 0$ and that no direct summand of Y' is a direct summand of Y. By the consideration from above we have $\dim \mathrm{Ext}(X', Y') \geq \dim \mathrm{Ext}(Y', X')$. Moreover, by Proposition 15 we have $\dim \mathrm{Hom}(Y', X') > \Psi(X, Y)$ and by Lemma 14 we have $\mathrm{Hom}(X', Y') = 0$. Thus we get $\tilde{\Theta}_\alpha(\underline{\dim} Y) > 0$.
Now assume that $Y' \cong \oplus (X_i')^{b_i}$ with $b_i \leq t_i$. Consider

$$-l_q \dim \bigcap_{i \in N_q} X_i' + r_q \dim \bigcap_{i \in N_q} Y_i'.$$

Since the extension is stable, we have $t := (r_q, t_1, \ldots, t_m)$ is a Schur root of the quiver $\tilde{Q}$ considered in this section. Moreover, we have $b := (l_q, b_1, \ldots, b_m) \hookrightarrow t$. In particular, we have $\langle b, t \rangle - \langle t, b \rangle > 0$. But, this means

$$-l_q \sum_{i=1}^{m} t_i n_i + r_q \sum_{i=1}^{m} n_i b_i > 0.$$

Combining these three cases, the claim follows. $\square$

We have the following Corollary:

Corollary 18. *Let a general representation with dimension vector α' be polystable with respect to $\Theta_{\alpha'}$ and let*

$$0 \to X' \to X \to \bigoplus_{q \in \mathcal{N} \setminus \mathcal{N}'} S_q^{r_q} \to 0$$

be an extension of some general representation X' with $\underline{\dim} X' = \alpha'$. Moreover, let the induced extensions $e_q \in \mathrm{Ext}(S_q^{r_q}, X')$ be polystable such that at least one extension is stable and such that the dimension vector induced by the stable extension is χ-positive. Then X is stable and can be unitarized with some weight χ.

Proof. We first consider the stable extension

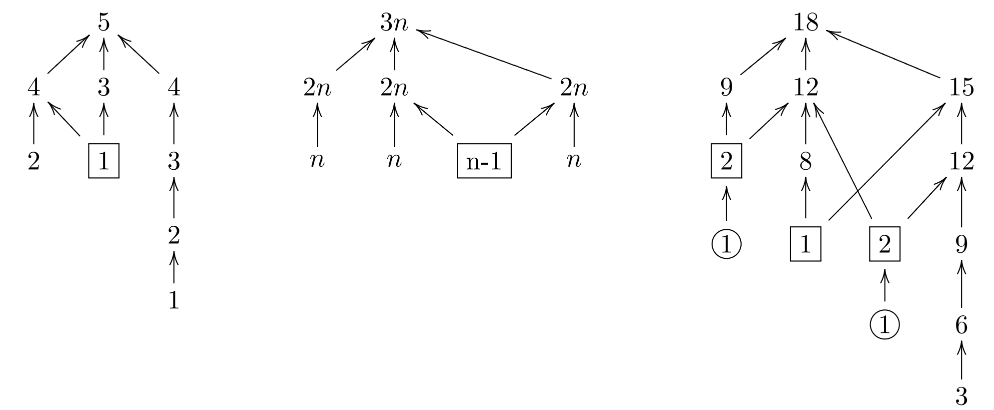
$$0 \to X' \to X'' \to S_q^{r_q} \to 0.$$

By Theorem 17 we have that X'' can be unitarized with the weight as given in Remark 4. Now we can apply Lemma 10 in order to obtain the result. □

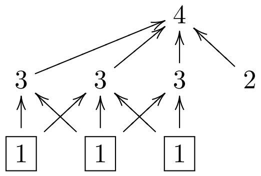
Note that, having constructed a stable representation, by Remark 1, we know a lower bound of the dimension of the moduli space of stable points.

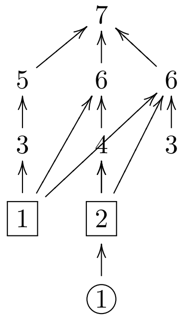
3. Unitarizable and non-unitarizable representations of posets

3.1. **Some examples of unitarizable representations.** Using the algorithm provided in Section 2.3, for instance in tame cases, fixing a dimension vector one can build families of unitarizable Schurian representations of non-primitive posets that depend on several complex parameters. Below we provide a few examples of such posets and dimension vectors. We start with polystable representations of primitive posets. Then we glue some subspaces using Corollary 18 in order to construct Schurian representations, and afterwards we glue an extra subspaces as described in Lemma 10.



Here by $\boxed{i}$ we denote those elements that are glued to the primitive posets as in Corollary 18, and by j we denote elements glued to stable representation using Lemma 10. It is clear that one can produce many of such examples. Notice that for some posets and their dimension vectors the provided technique is not applicable, for example in the following cases





In these cases the corresponding representations of the primitive posets are not polystable because the canonical decompositions of the dimension vectors are $(4;3;3;3;2) = (3;2;2;2;2) \oplus (1;1;1;1;0)$ and $(7;3,5;4,6;3,6) = (2;1,1;1,2;1,2) \oplus (1;0,1;1,1;0,1) \oplus (4;2,3;2,3;2,3)$.

3.2. **Unitarization of rigid modules and rigid local systems.** We will need the following lemma.

Lemma 19. *Let $A \in M_n(\mathbb{C})$ be an arbitrary Hermitian matrix with eigenvalues $\{\lambda_i\}_{i=1}^{j}$. Let the multiplicity of each λ_i be d_i. Then*

$$\dim Z(A) = d_1^2 + \ldots + d_j^2.$$

Proof. It is clear that the dimension of the commutator of the matrix A does not depend on the conjugacy class of A. Hence we can assume that

$$A = \operatorname{diag}\{\lambda_1, \ldots, \lambda_1, \ldots, \lambda_j, \ldots, \lambda_j\}.$$

Then $Z(A) = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_j}(\mathbb{C})$. Now the statement is obvious. □

Recall that a module X is called *rigid* if $\operatorname{Ext}(X, X) = 0$.

Assume that Q is a star-shaped quiver, i.e. induced by a primitive poset, such that the arrows are oriented towards the unique source q_0. Recall that a module X is called *rigid* if $\operatorname{Ext}(X, X) = 0$.

Proposition 20. *Assume that X is an indecomposable rigid strict representation of Q. Then it is unitarizable and the corresponding system of Hermitian matrices $A_1, \ldots, A_n$ is rigid.*

Proof. By Corollary 12 the unique indecomposable representation X of a real Schur root can be unitarized. This representation is *rigid* due to $\dim \operatorname{End}(X) - \dim \operatorname{Ext}(X, X) = 1$. In this case Schofield's Theorem 5 can be checked easily, see also [11, Lemma 5.1]. We take the Euler characteristic for X, i.e.

$$\langle X, X \rangle = \dim \operatorname{End}(X) - \dim \operatorname{Ext}(X, X) = \sum_{i \in Q_0} \dim X_i \dim X_i - \sum_{\rho: i \to j} \dim X_i \dim X_j = 1.$$

Using Lemma 19 we have that

$$\operatorname{rig}(A_1, \ldots, A_n) = r^2(2-n) + \sum_{i=1}^{n} \dim(Z(A_i)) = r^2(2-n) + \sum_{i=1}^{n} \sum_{j=1}^{m_i+1} {d_j^{(i)}}^2,$$

where $r = \dim X_0$ and $d_j^{(i)}$ is the dimensions of the j-th eigenspace of the corresponding Hermitian matrix A_i, which are given by

$$d_1^{(i)} = \dim X_1^{(i)}, \quad d_j^{(i)} = \dim X_j^{(i)} - \dim X_{j-1}^{(i)}, \quad 2 \leq j \leq m_i,$$
$$d_{m_i+1}^{(i)} = \dim X_0 - \dim X_{m_i}^{(i)}.$$

Then taking $\langle X, X\rangle$ we get

$$\begin{aligned}\langle X, X\rangle &= (\dim X_0)^2 + \sum_{i=1}^{n}\sum_{j=1}^{m_i}(\dim X_j^{(i)})^2 \\ &- \sum_{i=1}^{n}\sum_{j=1}^{m_i-1}(\dim X_j^{(i)})(\dim X_{j+1}^{(i)}) - \sum_{i=1}^{n}(\dim X_0)(\dim X_{m_i}^{(i)}) \\ &= (\dim X_0)^2 + \frac{1}{2}\sum_{i=1}^{n}\left((\dim X_1^{(i)})^2 + \sum_{j=2}^{m_i}((\dim X_j^{(i)}) - (\dim X_{j-1}^{(i)}))^2\right) \\ &+ \frac{1}{2}\sum_{i=1}^{n}((\dim X_0) - (\dim X_{m_i}^{(i)}))^2 - \frac{n}{2}(\dim X_0)^2 \\ &= \frac{1}{2}(r^2(2-n) + \sum_{i=1}^{n}\sum_{j=1}^{m_i+1} {d_j^{(i)}}^2) = \frac{1}{2}\mathrm{rig}(A_1, \dots, A_n).\end{aligned}$$

Since $\langle X, X\rangle = 1$ because X is rigid, the corresponding set of the matrices is also rigid. □

Let $\mathcal{N}$ be non-primitive poset, and let $\mathcal{N}'$ be a related primitive poset, i.e. $\mathcal{N}' \cup \{q_1, \dots, q_n\}$. The following Corollary is straightforward:

Corollary 21. *Assume that X is a rigid Schurian representation of $\mathcal{N}$ such that the following condition holds*

$$\dim X_{q_i} \leq \sum_{l \in N_{q_i}} \dim X_l - (|N_{q_i}| - 1)\dim X_{t(N_{q_i})}$$

for all q_i and that the corresponding representation of the related primitive poset is Schurian. Then X can be unitarized with some weight.

Proof. It is straightforward to see that the corresponding representation X' of $\mathcal{N}'$ is rigid. Then we can apply Proposition 20 and Corollary 18 to obtain the statement. □

Remark 5.

- An interesting question is whether the stability condition $\Theta_\alpha = \langle \_, \alpha\rangle - \langle \alpha, \_\rangle$ determines a dense subset of Schurian representations of the poset $\mathcal{N}$ with dimension α (an analogue of Schofield's Theorem 5). Notice that it is straightforward to check that each indecomposable quite sincere representation of posets of finite type (see [9]) is stable with the weight $\Theta_{\underline{\dim} X}$. In some cases, i.e. if the canonical decomposition of the dimension vector of the related primitive poset consists of Schurian roots of the same slope, they can be constructed by applying Theorem 17.

3.3. **ADE classification of unitarizable representations.**

Theorem 22. *Let Q be an unbound quiver induced by a primitive poset. Then we have:*

1. *Every indecomposable strict representation of Q is unitarizable if and only if Q is a Dynkin quiver.*
2. *Every Schurian strict representation of Q is unitarizable if and only if Q is a subquiver of an extended Dynkin quiver.*
3. *There is a family of non-isomorphic non-unitarizable Schurian strict representations that depends on arbitrary many continuous parameters if and only if Q contains an extended Dynkin quiver as a proper subquiver.*

Proof. The first part trivially follows from the previous section observing that in this case all indecomposable representations are Schurian and rigid. Moreover, if the underlying quiver is not of Dynkin type, there always exist non Schurian roots. Indeed, we may consider an isotropic root α, i.e. $\langle\alpha,\alpha\rangle = 0$. Now it is easy to check that 2α is no Schur root, but a root, since the canonical decomposition of 2α is $\alpha \oplus \alpha$, see also [20].

Second part. The representations that correspond to real roots are obviously rigid and hence unitarizable. In general, by [11, Proposition 5.2] any Schur representation is stable for some linear form Θ. Thus following Remark 3 it can be unitarized with some weight. Let us notice that this result (together with the description of possible weights) was alternatively obtained in the series of D.Yakimenko's papers.

Third part. Let α be an indivisible isotropic Schur root of an extended Dynkin quiver. Thus a general representation X with dimension vector α is Schurian and can be unitarized by Theorem 11. By adding an extra vertex with fixed dimension d to a vertex q with $\dim X_q > d \geq 1$ to the extended Dynkin quiver we again get a Schurian representation, say with dimension vector $\tilde{\alpha}$. In particular, $\tilde{\alpha}$ is a Schur root. Indeed, we may for instance apply Lemma 7 in order to see that the new representation is a Schurian representation.

It is easy to check that $\langle\tilde{\alpha},\tilde{\alpha}\rangle = \langle\alpha,\alpha\rangle + d^2 - d\alpha_q < 0$. For two general stable representations X and Y of dimension $\tilde{\alpha}$ we have $\mathrm{Hom}(X,Y) = \mathrm{Hom}(Y,X) = 0$. Let

$$0 \to X \to Z \to Y \to 0$$

be a non-splitting exact sequence. Then by Lemma 7, we have $\mathrm{End}(Z) \subseteq \mathrm{End}(X) = \mathbb{C}$. Thus, Z is a semistable Schurian representation which is not stable. Now we can check by a direct calculation that for every weight χ we have that X is a subrepresentation which contradicts χ-stability, see Lemma 4.

If we want to glue a vertex q to some vertex q' of dimension one we proceed as follows: first we add an extra arrow $\rho : q' \to q$ and consider some non-splitting exact sequence $0 \to S_q \to Z \to X \oplus X' \to 0$ where S_q is the simple module corresponding to the vertex q and X and X' are non-isomorphic Schurian of dimension α, thus $\mathrm{Hom}(X,X') = 0$. Then, applying Lemma 7 to the induced sequences $0 \to S_q \to Z' \to X \to 0$ and $0 \to Z' \to Z \to X' \to 0$ we obtain $\mathrm{End}(Z) = \mathbb{C}$. It is easy to check that $\dim Z$ is an imaginary root which is not isotropic. Now by applying the reflection functor, see [3], corresponding to the vertex q we again get a Schurian representation $\tilde{Z}$. But $\tilde{Z}$ corresponds to some filtration and we can proceed as in the first case. □

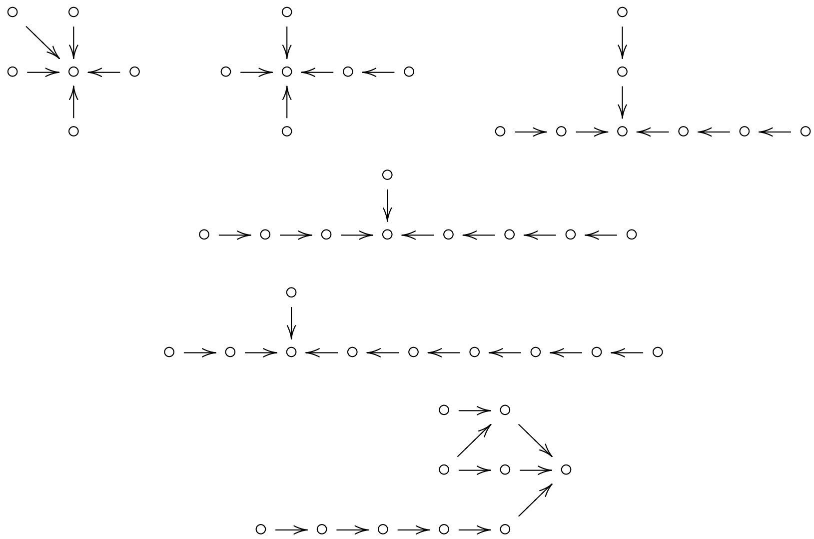
Remark 6.

- Let us remark that the first and third part of the theorem hold for posets in general.
  If the poset is of finite type, then each indecomposable representation can be unitarized with some weight (see [9] for the proof).
  If the poset contains a poset of wild type as a subposet, the same argument as for star-shaped quivers can be applied. Thus, there is a family of non-isomorphic non-unitarizable Schurian representations of the poset that depends on arbitrary many continuous parameters.
  But it is an open question whether all Schurian representations of tame posets with unoriented cycles are unitarizable. Like in Section 3.1, in many cases it is possible to construct an open subset of unitarizable representations. But as in the case without cycles the constructed weight does not apply for all Schurian representations.

4. Complexity of the description of $*$-representations of $\mathcal{A}_{\mathcal{N},\chi}$

Theorem 23. *Let $\mathcal{N}$ be a poset of wild type. Then it is possible to choose the weight $\chi_{\mathcal{N}}$ in such a way that for an arbitrary natural number n there exists a family of non-isomorphic Schurian representations of $\mathcal{N}$ depending on at least n complex parameters which can be unitarized with the weight $\chi_{\mathcal{N}}$.*

Proof. Due to [17] we only need to consider critical posets corresponding to quivers of the following type:



We denote a star-shaped quiver by $(n_1, \dots, n_k)$, where n_i is the number of vertices of the i-th branch. The only non star-shaped quiver is denoted by $(N, 5)$. Let us consider the dimension vectors $(2;1;1;1;1;1)$, $(4;2;2;2;1,2)$, $(6;2,4;2,4;1,2,4)$, $(8;4;2,4,6;1,2,4,6)$ and $(12;6;4,8;1,2,4,6,8,10)$ respectively of the quivers corresponding to $(1,1,1,1,1)$, $(1,1,1,2)$, $(2,2,3)$, $(1,3,4)$ and $(1,2,6)$ respectively. In order to see that these are Schur roots, we can easily construct a Schurian representation of these dimension vectors. For instance for $(4;2;2;2;1,2)$ we consider a non-splitting short exact sequence $0 \to X' \oplus X \to Y \to S_5 \to 0$ where X and X' are Schurian representations of dimension vector $(2;1;1;1;0,1)$ with $\mathrm{Hom}(X, X') = 0$. As in the proof of Theorem 22 we may apply Lemma 7 to the two induced sequences. The other cases behave analogously.

For the first five dimension vectors α_i we have that $\langle \alpha_i, \alpha_i \rangle = -1$. Hence, following Remark 1 it is possible to choose a two parameter family of Schurian representations. By Theorem 11, a general representation with this dimension vector can be unitarized with the weights

$$\begin{aligned}
\chi_{(1,1,1,1,1)} &= (5;2;2;2;2;2),\\
\chi_{(1,1,1,2)} &= (8;4;4;4;2,3),\\
\chi_{(2,2,3)} &= (12;4,4;4,4;2,3,4),\\
\chi_{(1,3,4)} &= (16;8;4,4,4;2,3,4,4),\\
\chi_{(1,2,6)} &= (24;12;8,8;2,3,4,4,4,4).
\end{aligned}$$

For two general unitarizable representations X and X' of dimension α we have $\mathrm{Hom}(X, X') = \mathrm{Hom}(X', X) = 0$. The middle term Z of every non-splitting exact sequence

$$0 \to X \to Z \to X' \to 0$$

has dimension vector 2α. Moreover, such representations, even if they are not stable, are Schurian by Lemma 7. Thus there exists a non-empty open subset of Schurian representations having the same dimension vector. Following Theorems 5 and 11, there exists a non-empty open subset of representations which can be unitarized with the weight $2\chi_{\mathcal{N}}$ and, therefore, with the weight $\chi_{\mathcal{N}}$, too. The dimension of the corresponding moduli space is $1 - \langle \dim Z, \dim Z \rangle = 1 - \langle 2\alpha, 2\alpha \rangle = 1 - 4\langle \alpha, \alpha \rangle = 5$, see Remark 1. Hence there exists a 5-parameters family of non-isomorphic representations having dimension vector 2α which can be unitarized with the same weight $\chi_{\mathcal{N}}$.

Then iterating the same procedure for the dimension vector 2α, we will obtain the desirable result due to the fact that $1 - \langle 2^n\alpha, 2^n\alpha\rangle = 1 + 2^{2n}$ growths when iterating.

In the case of the poset $(N, 5)$ we proceed as follows. We consider the related poset $(2, 1, 5)$ and the unique imaginary Schur root $\alpha = (6; 2, 4; 3; 1, 2, 3, 4, 5)$. This root is strongly strict. Taking a general polystable representation $X' = \oplus_{i=1}^{s}(X_i')^{t_i}$ with $\underline{\dim} X_i' = \alpha$ and $X_i' \not\cong X_j'$, we can consider stable extensions

$$0 \to X' \to X \to S_q^r \to 0.$$

Therefore, $(r, t_1, \ldots, t_s)$ has to be a Schur root of the s-subspace quiver $S(s)$. Note that the intersection of the two questioned subspaces is of dimension one. By Theorem 17 any such representation X is stable, i.e. in particular Schurian, and can be unitarized with the weight $\chi_{(N,5)} = (20; 8, 4; 4, 8; 2, 3, 4, 4, 4)$. Since α is an isotropic root we have a one-parameter family of stable representations of dimension α. In particular, we have a t-parameter family of polystable representations for every tuple $(t_1, \ldots, t_s)$ with $\sum_{i=1}^{s} t_i = t$. □

Corollary 24. *Let Γ be a star-shaped graph that contains an extended Dynkin graph as a proper subgraph. Then there exists a weight ω_Γ such that the algebra $\mathcal{B}_{\Gamma,\omega_\Gamma}$ has a family of unitary-nonequivalent irreducible $*$-representations which depends on an arbitrary number of continuous parameters.*

Proof. Using the previous theorem and the relations between unitarizable systems of subspaces and $*$-representations of $\mathcal{B}_{\Gamma,\omega_\Gamma}$ it is easy to check that letting

$$\begin{aligned}
\omega_{(1,1,1,1,1)} &= (5; 2; 2; 2; 2; 2),\\
\omega_{(1,1,1,2)} &= (8; 4; 4; 4; 2, 5),\\
\omega_{(2,2,3)} &= (12; 4, 8; 4, 8; 2, 5, 9),\\
\omega_{(1,3,4)} &= (16; 8; 4, 8, 12; 2, 5, 9, 13),\\
\omega_{(1,2,6)} &= (24; 12; 8, 16; 2, 5, 9, 13, 17);
\end{aligned}$$

we obtain desirable statement. □

Remark 7.

- It is an open question whether the corresponding algebras $\mathcal{B}_{\Gamma,\omega_\Gamma}$ are of $*$-wild type. The previous corollary gives possible candidates among all possible weights, since it is evident that for such weights the corresponding algebras are not $*$-tame. Let us also note that in the case $\Gamma = (1, 1, 1, 1, 1)$, it was proved that the algebra $\mathcal{B}_{\Gamma,(5;2;2;2;2;2)}$ is of $*$-wild type (see for example [18]) since it contains the $*$-wild subproblem of describing three reflections such that two of them anticommute.

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