CHALMERS | UNIVERSITY OF GOTHENBURG

PREPRINT 2011:5

Lipschitz continuity of the Scattering operator for nonlinear Klein-Gordon equations

PHILIP BRENNER

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
UNIVERSITY OF GOTHENBURG
Gothenburg Sweden 2011

Preprint 2011:5

Lipschitz continuity of the Scattering operator for nonlinear Klein-Gordon equations

Philip Brenner

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Gothenburg, Sweden
Gothenburg, January 2011

Preprint 2011:5
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2011

Lipschitz continuity of the Scattering operator for nonlinear Klein-Gordon equations

Philip Brenner
Department of Applied Information Technology
Chalmers|University of Gothenburg
SE-412 96 Göteborg, Sweden
email:*philip.brenner@ituniv.se*

January, 2011

Abstract

We will give an overview of the Strichartz and Space Time Integral estimates for the Klein-Gordon and subcritical nonlinear Klein-Gordon equations, respectively. In this framework, the regularity of the solution and scattering operators for the nonlinear subcritical Klein-Gordon is studied, mainly using these tools and estimates of the nonlinearity in Besov spaces. We prove that these operators are (uniformly) Hölder continuous on the energy space for space dimension $n \geq 3$, and Lipschitz continuous for $n \leq 8$.

1 Introduction

We will give an overview of the Strichartz and Space Time Integral estimates for the Klein-Gordon and subcritical nonlinear Klein-Gordon equations, respectively. We use this to study the regularity properties of the solution operator $\mathcal{E}_t$ to

$$\partial_t^2 u - \Delta u + m^2 u + f(u) = 0,\ t > 0,\ x \in \mathbf{R}^n,\ u|_0 = \phi, \partial_t u|_0 = \psi, \tag{1.1}$$

where $m > 0$ and $f(u) \approx |u|^{\rho-1}u,\ \rho > 1$ (we will give precise assumptions below) and the data ϕ, ψ belong to $(H_2^1 \cap L_{\rho+1}) \times L_2 = X_e$, the energy space for the nonlinear equation.

(A) *assumptions on f*

$$
\begin{aligned}
&f \in C^1,\ f(\mathbf{R}) \subseteq \mathbf{R},\ F(u) = \int_0^u f(v)dv \geq 0,\ u \in \mathbf{R} \\
&|f^{(j)}(u)| \leq C|u|^{\rho-j},\ j = 0,1 \\
&|f'(u) - f'(v)| \leq C|u-v|^{(\rho-1)_-}(|u|+|v|)^{(\rho-2)_+}, \\
&(\rho-1)_- = \min(\rho-1,1),\ (\rho-2)_+ = \max(\rho-2,0)
\end{aligned}
$$

(B) *Additional assumptions on f for uniform estimates and scattering*

$$
\begin{aligned}
&uf(u) - 2F(u) \geq \epsilon F(u),\ \textit{some}\ \epsilon > 0 \\
&uf(u) - 2F(u) \text{ not flat at } 0 \text{ or } \infty
\end{aligned}
$$

We also assume that $1 < \rho \leq \rho^* = \rho^*(n) = \frac{n+2}{n-2}$, $n \geq 3$.
An important, often used, example of a function f satisfying (A) and (B) is

$$
f(u) = |u|^{\rho-1}u \tag{1.2}
$$

The energy

$$
E(u) = \frac{1}{2}\int (|\partial_x u|^2 + m^2|u|^2 + |\partial_t u|^2)dx + \int F(u)dx
$$

is under our assumptions on f uniformly bounded by the energy of $u(0)$.We write $E_0(u_0)$ for the energy of the solution u_0 of the Klein-Gordon equation, i.e. with $F = 0$.
For the subcritical case $\rho < \rho^*$, any weak solution u with data in the energy space $(H_2^1 \cap L_{\rho+1}) \times L_2 = X_e$, is unique (see [14] ; an alternative proof of uniqueness can be found in Section 4.2, Theorem 3 - cf.[8]) and the energy is (actually as a consequence of the uniqueness, cf. [40]) conserved for $\rho < \rho^*$, i.e. the energy $E(u(t))$ is constant in time, again see [14].

(C) *In the critical case, i.e.* $\rho = \rho^*$, *we assume that f is of the form (1.2).*

In this cas the finite energy solutions constructed by Shatah and Struwe [39] (cf. also Ch. 6 in [40]), for short denoted Shatah - Struwe solutions, belong to and are unique in $L_\infty(H_2^1 \times L_2) \cap L_{q'}^{loc}(B_{q'}^{\frac{1}{2},2})$, $q' = 2\frac{n+1}{n-1}$. These solutions also have conserved energy. We will only treat the critical case marginally, and for completeness - leaving most of the problems for the critical case outside the scoop of this paper.
Here and in the following we use the concept and notation of Besov and Sobolev spaces as presented in [4] and [3]. We also use the notation $L_r(I; B)$ for the space of functions on $\mathbf{R} \times \mathbf{R}^n$ whose B-norm over $\mathbf{R}^n$ is L_r-integrable over I. If this

holds for arbitrary bounded intervals I, or for $I = \mathbf{R}_+$, we simply write $L_r^{loc}(B)$ and $L_r(B)$, respectively.
For $\rho = \rho^*$, let in the following $\mathcal{E}_t(\phi, \psi)$ denote the Shatah - Struwe solution, and for $\rho < \rho^*$ the unique weak finite energy solution, of (1.1).

Let $3 \leq n$, $1 + \frac{4}{n} < \rho < \rho^*(n)$. Then method of the proof of the uniquness of weak solutions of the NLKG given in [8] easily extends in a strightforward way to prove the local Lipschitz continuity of the mapping $\mathcal{E}_t : H_2^1 \times L_2 \mapsto L_{q'}(B_{q'}^{s'})$, where $B_{q'}^{s'}$ denotes the Besov space $B_{q'}^{s',2}$ with $s' \leq \frac{1}{2}\min(\rho - 1, 1)$ and with q' as above.

Using the Strichartz estimates and the Space-Time Integral (STI) estimates for the NLKG this implies in particular that $\mathcal{E}_t$ is locally Lipschitz continuous on the energy space for the whole permitted range of ρ for $n = 3,\ n = 4$.
A more systematic use of the Strichartz and uniform STI-estimates (specific) for the Klein-Gordon and NLKG, give rise to more general results on the global uniform Lipschitz continuity for subcritical NLKG. A review of STI estimates is given in Section 2, estimates originating in [42],[13], [8]; The proof of the important case (used exclusively in this paper) when there is a uniform energy bound in the STI estimates, is scetched in an appendix.
Here we are content to give the following consequences, which are proved and presented in more detail in Section 5.

Theorem 1. *Let $n \geq 3$ and $1 + \frac{4}{n} < \rho < \rho^*(n)$. Assume that f satisfies (A) and (B). Then*
(i) for $n \leq 8$, $\mathcal{E}_t : H_2^1 \times L_2 \mapsto L_\infty(H_2^1)$ is Lipschitz continuous for $\rho > 1 + \frac{4}{n}$, except for $n = 8$ and $n = 7$, in which case the lower bound becomes $1 + \frac{4.5}{n-1}$ and $1 + \frac{4}{n-1}$, respectively.
(ii) For $n \geq 6$, $\mathcal{E}_t : H_2^1 \times L_2 \mapsto L_\infty(H_2^1)$ is Hölder continuous of order α, for some $0 < \alpha = \alpha(\rho, n) \leq 1$,where $\alpha(\rho, n) = O(\frac{1}{n})$.
(iii) For all $n \geq 6$, $\mathcal{E}_t : H_2^1 \times L_2 \mapsto L_\infty(H_2^{\frac{1}{2}})$ is Lipschitz continuous.

A sharper form of (iii) is given in Theorem 4 in Section 5.2 in connection with the proof of Theorem 1.

For $n = 3$ Lipschitz estimates for the critical case have been proved by Bahouri and Gerard [1]. The methods used in the present paper break down in the critical case.

In the supercritical case, $\rho > \rho^*$, local Lipschitz estimates (on arbitrarily small time intervals containing t=0) will not hold for integer values of ρ, [9] and in case $3 \leq n \leq 6$, not for any $\rho > \rho^*$ (work in progress). For related work, see Lebeau

[30], [31].
Notice also that Lipschitz estimates for $\Phi \ni H_2^1 \times L_2 \mapsto f(\mathcal{E}_t\Phi) \in L_1(L_2)$ will only hold in low dimensions and Hölder estimates of order α will require $\alpha(\rho, n) = O(\frac{1}{n})$ (see [10] and also the comments in e.g. [16], p. 3). Since Lipschitz (or Hölder) estimates of $\mathcal{E}_t$ imply the same type of estimates for $f(\mathcal{E}_t)$, with natural changes of the spaces involved, this means that the estimates in Theorem 1 are at least in this sense best possible.

The *scattering operator* for the NLKG is the mapping $\mathcal{S} : (u_-, \partial_t u_-) \mapsto (u_+, \partial_t u_+)$ where if u is a solution of the NLKG, there exist solutions of the Klein-Gordon equation $u_\pm$ such that

$$\|u(t) - u_\pm\|_e \to 0 \text{ } as \text{ } t \to \pm\infty$$

where $\| \cdot \|_e$ denotes the energy norm (i.e. the square root of the energy - in the subcritical case the the linear and nonlinear energy norms are equivalent by Sobolev's inequality). $\mathcal{S}$ is defined on (all of) $H_2^1 \times L_2$ (Brenner [8], [7] and Ginibre and Velo [14]) for $1 + \frac{4}{n} < \rho < \rho^*$, $n \geq 3$ and assuming conditions (A) and (B).

For $n = 3$ and $\rho = 3$ the analyticity of $\mathcal{S}$ in case f is analytic was proved by Kumlin [27]. Here we prove the Lipschitz continuity of $\mathcal{S}$ as a counter part of part (i) of Theorem 1. The results corresponding to part (ii) of Theorem 1, including the comments, with obvious changes, also hold for the scattering operator.

Theorem 2. *Assume that f satisfies (A) and (B), and that* $1 + \frac{4}{n} < \rho < \rho^*(n)$.
(i) Let $3 \leq n \leq 8$ *Then the scattering operator* $\mathcal{S} : H_2^1 \times L_2 \mapsto H_2^1 \times L_2$ *for the NLKG is Lipschitz continuous for* $\rho > 1 + \frac{4}{n}$, *except for* $n = 8$ *and* $n = 7$, *in which case the lower bound becomes* $1 + \frac{4.5}{n-1}$ *and* $1 + \frac{4}{n-1}$, *respectively.*
(ii) For $n \geq 6$, *there is an* $\alpha = \alpha(\rho, n)$, $0 < \alpha \leq 1$ *such that* $\mathcal{S} : H_2^1 \times L_2 \mapsto H_2^1 \times L_2$ *is Hölder continuous of order* α .

In order to keep the size of this paper within reasonable bounds, we have omitted proofs of the basic Strichartz and space time integral (STI) estimates for the wave- and Klein-Gordon equation, respectively.We have tried to supply proper references for these. In addition, we scetch a proof of the uniform STI estimates in an appendix, following [8] and [5].
As mentioned above, we assume a basic knowledge of Besov- and Sobolev spaces, as presented e.g. in [4].
I take the opportunity to thank Peter Kumlin for his comments and suggestions during the work on this paper.

2 Strichartz and Space Time Integral (STI) estimates

2.1 Notation, solution operators and basic kernel estimates

For the remainder of this paper we fix the following notation: $p,\ p'$ will be dual exponents, $\frac{1}{p}+\frac{1}{p'}=1,\ 1\le p\le 2\le p'$. Primed exponents in L_r-spaces are assumed ≥ 2. We define $\delta=\delta_{p'}=\frac{1}{2}-\frac{1}{p'}$. We write for short B_p^s for the Besov space $B_p^{s,2}$.
Let u_0 be the solution of the Klein-Gordon equation

$$\partial_t^2 u-\Delta u+m^2u=0,\ t>0,\ x\in\mathbf{R}^n,\ u|_0=\phi,\partial_t u|_0=\psi, \tag{2.1}$$

with $m>0$ and (ϕ,ψ) *in* $H_2^1\times L_2$. Then we can write

$$u_0(t)=E_0(t)\phi+E_1(t)\psi$$

and ([6], [8])

$$\|E_\mu(t)v\|_{B_{p'}^{s'}}\le K_\mu(t)\|v\|_{B_p^s},\ \mu=0,1\ , \tag{2.2}$$

where $0\le\theta\le 1$ and

$$0\le\mu-(n+1+\theta)\delta+s-s' \tag{2.3}$$

and where $K_\mu(t)$ satisfies

$$K_\mu(t)\le C\begin{cases} t^{\mu-2n\delta+s-s'}\le t^{-(n-1-\theta)\delta} & 0<t<1,\\ (1+t)^{-(n-1+\theta)\delta} & 1\le t\end{cases} \tag{2.4}$$

Let $K(t)$ denote the upper bound in (2.4). Notice that with the assumption (2.3) K is independent of μ. If

$$(n-1+\theta)\delta>1>(n-1-\theta)\delta \tag{2.5}$$

then $K_\mu\le K\in L_r$ for $1\le r\le 1+\epsilon$, some $\epsilon=\epsilon(\delta,n)>0$ for $0<\theta\le 1$, $\frac{1}{n}<\delta\le\frac{1}{n-1}$. In addition $K\in L_r^{loc}$ if $\theta=0$ and $(n-1)\delta<1$, in which case the upper bound in (2.5) is not needed.
At this point it is convenient to notice that the finite energy solution of (1.1) is the (weak) finite energy solution of integral equation

$$u(t)=u_0(t)+\int_0^t E_1(t-\tau)f(u(\tau)d\tau \tag{2.6}$$

where u_0 is the solution of the linear Klein-Gordon equation (2.1) with the same initial data as u.

2.2 Strichartz type estimates

Using (2.4), a duality argument (a clear exposition of the duality argument can be found in Ginibre and Velo [15]) and Young's (or the Hardy-Littlewood) inequality, we obtain the following space-time estimate for the Klein - Gordon equation in the form proved in [7].

Proposition 2.1. *Let $n \geq 3$, $s' \in \mathbf{R}^n$ and $r' \geq 2$. Then if u_0 is a finite energy solution of the Klein-Gordon equation,*

$$u_0 \in L_{r'}(I; B^{s'}_{p'}) \equiv A(I), \ \textit{any interval } I \subseteq \mathbf{R}_+$$

and

$$\|u_0\|_{A(I)} \leq C\|u_0\|_{L_\infty(I;H^1_2)}$$

with C independent of I and u_0, provided (r', p', s') satisfy

$$1 - s' - \delta \geq \frac{1}{r'} \geq -1 + s' + n\delta, \tag{2.7}$$

with equality only if $r' \neq 2$, or $(n-1)\delta = 1$ and $n \geq 4$.

The endpoint case $(n-1)\delta$, $r' = 2$ is due to Keel and Tao [25].
Notice that by (2.7) $s' \leq s'_{max} = 1 - \frac{n+1}{2}\delta$, and that $s' = s'_{max}$ in the endpoint case. If $s' = s'_{max}$, then $\frac{1}{r'} = \frac{1}{2}(n-1)\delta$. For the choice $s' = s'_{max}$ we get the classical Strichartz estimates, originated by Strichartz [42]. For general Strichartz estimates, see Ginibre and Velo [15], and the references given there.

Proposition 2.2 (Strichartz estimates)**.** *Let $n \geq 3$, $s' \in \mathbf{R}$ and $r' \geq 2$. Then if u_0 is a finite energy solution of the Klein-Gordon equation,*

$$u_0 \in L_{r'}(I; B^{s'}_{p'}) \equiv A(I), \ \textit{any interval } I \subseteq \mathbf{R}_+$$

and

$$\|u_0\|_{A(I)} \leq C\|u_0\|_{L_\infty(I;H^1_2)}$$

with C independent of I and u_0, provided (r', p', s') satisfy

$$(n-1)\delta \leq 1, \ s' = 1 - \frac{n+1}{2}\delta - \sigma, \ \textit{and } \frac{1}{r'} = \frac{1}{2}(n-1)\delta - \sigma, \tag{2.8}$$

for some $\sigma \in [0, \frac{1}{2}(n-1)\delta)$.
For $n = 3$ the value $r' = 2$, $(n-1)\delta = 1$ is not allowed.

The following are useful variations of Proposition 2.1 and the Strichartz estimates. Let w_0 be defined by

$$w_0(t) = \int_0^t E_1(t-\tau)h(\tau)d\tau \tag{2.9}$$

where $h = h(t,x) \in L^{loc}_1(L_2)$.

Proposition 2.3. *With the notation of (2.2) through (2.4), let $\mu = 1$ and assume that (2.3) holds with equality. Assume that $\epsilon = \theta\delta > 0$. Let $I = (t_0, t) \subseteq \mathbf{R}_+$ and $\bar{I} = (0, t - t_0)$, let w_0 be defined by (2.9) with the integral taken over I, and let $E(I) = L_\infty(I, H_2^1)$, $A = L_2(I; B_{p'}^{s'})$, and $B(I) = L_2(I, B_p^s)$. Then*

$$\|w_0\|_{E(I)} + \|w_0\|_{A(I)} \leq C\|K_1\|_{L_1(\bar{I})}\|h\|_{B(I)} \tag{2.10}$$

and with $L(I) = L_1(I, L_2)$,

$$\|w_0\|_{A(I)} \leq C\|h\|_{L(I)} \tag{2.11}$$

where

$$\frac{1}{n} < \delta < \frac{1}{n-1}, \;\; s > \frac{1}{2} + \delta, \;\; s' = 1 - s$$

and p' the dual exponent of p, so that p', s' satisfy (2.7) for $r' = 2$.

Proof: We first prove the estimate for the term $\|w_0\|_{E(I)}$.
By duality and since $E_0(t)^* = E_0(-t)$, we get

$$\|w_0(t)\|_{H_2^1} = \|\int_0^t E_1(t-\tau)h(\tau)d\tau\|_{H_2^1} = \|\int_0^t E_0(t-\tau)h(\tau)d\tau\|_{L_2}$$

and, with p', p dual exponents,

$$\|w_0\|^2_{L_\infty(I,H_2^1)} \leq (\int_I (\|\int_0^t E_0(t-\tau)h(\tau)d\tau\|_{B_{p'}^{\bar{s}}})^2 dt)^{\frac{1}{2}}\|h\|_{L_2(B_p^{\bar{s}})} \tag{2.12}$$

If

$$\bar{s} = \frac{1}{2}(n+1+\theta)\delta = n\delta - \frac{1}{2} + \frac{1}{2}(1-(n-1)\delta) + \frac{1}{2}\theta\delta$$

where by (2.5), $\theta\delta > 1-(n-1)\delta$ and hence $K \in L_1$, we get by Young's inequality and (2.12)

$$\begin{aligned}
&\|w_0()\|^2_{L_\infty(I,H_2^1)} \leq \\
&(\int_I(\int_0^t K(t-\tau)\|h(\tau)\|_{B_p^{\bar{s}}}d\tau)^2 dt)^{\frac{1}{2}}\|h\|_{L_2(B_p^{\bar{s}})} \leq \\
&\|K\|_{L_1(\bar{I})}\|h\|^2_{L_2(B_p^{\bar{s}})}
\end{aligned}$$

where $\bar{s} > n\delta - \frac{1}{2} + (1-(n-1)\delta) = \frac{1}{2} + \delta$.
The estimate of the second term follows from a slight variation of the inequalities above and Proposition 2.1:

$$\begin{aligned}
\|w_0\|_{A(I)} &= (\int_I \|\int_0^t E_1(t-\tau)h(\tau)d\tau\|^2_{B_{p'}^{s'}}dt)^{\frac{1}{2}} \\
&\leq (\int_I(\int_0^t K(t-\tau)\|h(\tau)\|_{B_p^{\bar{s}}}d\tau)^2 dt)^{\frac{1}{2}} \\
&\leq (\int_I \|h\|^2_{B_p^{\bar{s}}})^{\frac{1}{2}} = \|h\|_{B(I)}
\end{aligned}$$

In the same way,

$$\begin{aligned}\|w_0\|_{A(I)} &= (\int_I \| \int_0^t E_1(t-\tau)h(\tau)d\tau\|^2_{B^{s'}_{p'}} dt)^{\frac{1}{2}} \\ &\leq \int_I \|E_1(\cdot-\tau)h(\tau)\|_{A(I)} d\tau \\ &\leq C\int_I \|h(\tau)\|_{L_2} d\tau = C\|h\|_{L_1(I,L_2)}\end{aligned}$$

which completes the proof. $\square$

Remark In (2.11) we may obviously let $A(I) = L_{r'}(I, B^{p'}_{p'})$ for any tripple (r', p', s') satisfying(2.7) or (2.8).

In contrast to Proposition 2.5 below, Proposition 2.3 is, with the exception of (2.11), only valid for the Klein-Gordon equation (i.e. $m > 0$), since (2.4), and so (2.5), is not valid for the wave equation (i.e. m=0 in (1.1)) with $\theta > 0$.

Proposition 2.4. *With the assumptions of Proposition 2.3,define q by $\frac{1}{q} = (n-1)\delta \leq 1$, so that*

$$(n-1+\theta)\delta q > 1 > (n-1-\theta)\delta q,$$

by which $K \in L_q$. Then

$$\|w_0\|_{A(I))} \leq C\|K\|_{L_q(\bar{I})}\|h\|_{B(I)} \tag{2.13}$$

where $A = L_{r'}(I; B^{s'}_{p'})$ and $B = L_r(I; B^{s+\epsilon}_p)$ and with $s = 1 - s'$ and (r', p', s') satisfies the assumptions of the Strichartz inequality (2.8). As above, r, p are dual exponents to r', p'.

Proof: Since $K \in L_q$, the proposition follows immediatly from

$$w_0(t) = \int_0^t E_1(t-\tau)h(\tau)d\tau \ ,$$

the estimate (2.2) and Young's inequality with

$$\frac{1}{r} = \frac{1}{r'} - (n-1)\delta + 1 \ ,$$

and

$$s = (n+1+\theta)\delta - 1 + s' = \frac{1}{2}(n+1)\delta + \theta\delta = 1 - s' + \epsilon$$

since by (2.8), $\frac{1}{r'} = \frac{1}{2}(n-1)\delta$ and $s' = 1 - \frac{1}{2}(n+1)\delta$. $\square$

If we allow $\theta = 0$ in (2.3), (2.5) we get the following estimate, which, in particular that of the H^1_2-norm, originates from Kapitanski [24].

Proposition 2.5. *Let w_0 be defined by (2.9). Then*

$$\|w_0\|_e + \|w_0\|_A \le C\|h\|_B \tag{2.14}$$

where $A = L_{r'}(B^s_{p'})$ and $B = L_{\bar{r}}(B^{1-\bar{s'}}_{\bar{p}})$, and where (r', p', s') and $(\bar{r}', \bar{p}', \bar{s'})$ satisfies the assumptions of the Strichartz inequality.
The same inequality holds with integrals over intervals $I \subseteq \mathbf{R}_+$ in time.

PROOF: [after Pecher [34]] In the proof of Proposition 2.3 we let $\theta = 0$, we may take (with equality) $\bar{s} = \frac{1}{2}(n+1)\delta = n\delta - \frac{1}{2}(1-(n-1)\delta)$, $\frac{1}{r'} = \frac{1}{2}(n-1)\delta = \frac{1}{2} - \frac{1}{2}(1-(n-1)\delta)$ by (2.7). Then substitute Young's inequality by Hardy's inequality for $(n-1)\delta < 1$.
For the second term (with (r', p', s') in A independent of the corresponding parameters $(\bar{r}', \bar{p}', \bar{s'})$ for B), interpolate with the estimate

$$\begin{aligned}\|w_0\|_{A(I)} &\le \int_I \|E_1(\cdot - \tau)h(\tau)\|_{A(I)} d\tau \\ &\le C\int_I \|h(\tau)\|_{L_2} d\tau = C\|h\|_{L_1(I,L_2)}\end{aligned}$$

as follows from Proposition 2.2, in analogue to the proof of the estimate of the second term in Proposition 2.3.
We thus arrive at a proof for the non-endpoint case. The the proof of the endpoint case is more complex, and we have to refer Keel and Tao [25]. □

For a complete proof (in homogeneous norms for the wave equation), see again Keel and Tao [25] (cf. also Ginibre and Velo [15] and Pecher [34]).
The following consequence of Proposition 2.5, as well as the obvious analogue for Proposition 2.4, will be useful:

Corollary 2.1. *Let $n \ge 3$ and let u and v be solutions of the NLKG, and u_0, and v_0 the corresponding solutions of the linear equation. We use the following notation and assumptions*

$$\delta = \frac{1}{2} - \frac{1}{p'},\ 1 = \frac{1}{p'} + \frac{1}{p},\ p' \ge 2,\ s = \frac{1}{2}(n+1)\delta,\ \gamma = s \text{ and } \bar{\gamma} = 1-s,\ \frac{1}{r'} = 1 - \frac{1}{r} = \frac{1}{2}(n-1)\delta$$

It is also assumed that $(n-1)\delta \le 1$, with strict inequality for $n = 3$.
Let $0 \le \theta \le 1$. Then

$$\begin{aligned}\|u-v\|_{L_\infty(H^1_2)} + \|u-v\|_{L_{r'}(B^\gamma_{p'})} &\le C\|u_0 - v_0\|_{L_\infty(H^1_2)} \\ &\quad + C\|f(u) - f(v)\|_{L_{r(\theta)}(B^{\theta\bar{\gamma}}_{p(\theta)})}\end{aligned} \tag{2.15}$$

where
$$\frac{1}{r(\theta)} = \theta\frac{1}{r} + (1-\theta),\ \delta_{p(\theta)} = \theta\delta,\ p(\theta) \le 2$$
and, in the previous notation, with $\sigma \in \mathbf{R}$,
$$\|u-v\|_{L_{r'}(B_{p'}^{\gamma-\sigma})} \le C(\|u_0-v_0\|_{L_\infty(H_2^{1-\sigma})} + \|f(u)-f(v)\|_{L_{r(\theta)}(B_{p(\theta)}^{\theta\bar{\gamma}-\sigma})}) \qquad (2.16)$$
The same inequality holds with integrals over intervals $I \subseteq \mathbf{R}_+$ in time.

PROOF: Notice that although u and v are solutions of the NLKG, the estimates are in fact linear estimates obtained from Proposition 2.5.
In view of (2.6), the first estimates is a direct interpolation of the estimates in Proposition 2.5 between $B = L_1(L_2)$ and $B = L_2(B_p^{\bar{\gamma}})$.
In the same way, the second estimate, is obtained using the isometry of the solution operator of the Klein-Gordon equation on H_2^s in (2.15). $\square$

Remark If we instead use the estimate in Proposition 2.1, we get an additional factor $\|K\|_{L_1}^{\theta}$ in front of $\|f(u)-f(v)\|$

2.3 Space time integral (STI) estimates

Using different spaces and methods, the author [8] and Ginibre and Velo [13] obtained space-time integral inequalites (STI), originally in order to prove the existence of scattering operators on the energy space for the NLKG (cf. Section 5.3 below), that implied results for the NLKG which are very similar to the Strichartz estimates.
The original STI estimates required $K(t)$ to be integrable, i.e. that (2.5) holds for some $\theta > 0$. The estimates for $\delta \le \frac{1}{n}$ follows from interpolation of the estimate for $\delta > \frac{1}{n}$ with the energy estimate (using the convexity of the L_p-norm).
The end point case $(n-1)\delta = 1,\ r' = 2$ follows from the corresponding result for the Klein-Gordon equation and Theorem 1 of [8]. For convenience we will give a proof in Section 4 as a consequence of the corresponding case for the Klein-Gordon equation, an end point STI estimate for the NLKG in $L_2^{loc}(B_{p'}^{\gamma})$ -which we will prove here, and which can also be found in [8] - and the "close to the end point" uniform estimate in the proposition.
Since the uniform bound in the STI estimates in terms of the energy of u (or, equivalently, of u_0) wasn't explicit in [8], we scetch a proof of the STI estimate with stress on the uniform energy bound in an appendix, following the proof in [8].

Proposition 2.6 (STI)**.** *Let $n \ge 3$, $s' \in \mathbf{R}^n$ and $r' \ge 2$. Let u_0 be a finite energy solution of the Klein-Gordon equation, and u the corresponding solution* *of the NLKG (1.1) with the same finite energy initial data, and where f satisfies conditions (A) and (B). Let* $\delta_+ = \max(\delta, \frac{1}{n})$*. Then if* $1 + 4\delta_+ < \rho < \rho^* = \frac{n+2}{n-2}$*,*

$$u \in L_{r'}(I; B^{s'}_{p'}) \equiv A(I), \text{ any interval } I \subseteq \mathbf{R}_+$$

and

$$\|u\|_{A(I)} \leq C(\|u_0\|_e)$$

with C independent of I, depending continuously on the energy norm of u_0 *, provided* (r', p', s') *satisfy the conditions (2.7) in Proposition 2.1, and in particular, the conditions (2.8) in the Strichartz estimates.*

The next result is an STI for critical case $\rho = \rho^*$ and the Shatah - Struwe solutions introduced in Section 1.
Part (i) of the Proposition is due to Bahouri and Shatah [2], and the uniform energy bound for $n = 3$, as well as the statement (iii) for $n = 3$, is due to Bahouri and Gerard [1].

Proposition 2.7. *Let* $\rho = \rho^*$ *and let u be a Shatah - Struwe solution of the critical NLKG (with finte energy data). Then*

(i) $u \in L_{q'}(B^{\frac{1}{2}}_{q'})$, $n \geq 3$, $(n+1)\delta_{q'}$

(ii) $u \in L_2(B^{\gamma}_{p'})$, $n \geq 4$, $(n-1)\delta_{p'} = 1$, *and* $\gamma = \frac{1}{2} - \delta_{p'}$

(iii) $u \in L_{r'}(I; B^{s}_{p'})$ *where* (r', p', s) *satisfy (2.8).*

The norm in (ii) and (iii) is bounded by an increasing function of the energy of the initial data, which for $n \geq 4$*,may depend also on* $u, v \in L_{q'}(B^{\frac{1}{2}}_{q'})$*.*

We will in Section 4 give a proof of (ii), showing that it is a consequence of (i) and of the fact that the energy of the unique Shatah - Struwe solution is conserved, i.e. constant in time (cf. Shatah and Struwe [39] and also[40], in the proof of Theorem 6.6).

3 Basic estimates of $f(u)$ and $f(u) - f(v)$.

3.1 Some initial observations

By Corollary 2.1, Lipschitz and Hölder estimates of $u \mapsto f(u)$ in choosen Besov spaces should be natural tools to obtain the corresponding energy space estimates

for the solution operator $\mathcal{E}_t$ of (1.1).
First, the Besov spaces can be described in terms of integrals of moduli of continuity: Let $f_h(\cdot) = f(\cdot + h)$, and let $\omega_p(\tau, f)$ denote the modulus of continuity of f in L_p,

$$\omega_p(\tau, f) = \sup_{|h|\leq\tau} (\|f_h - f\|_{L_p}).$$

Then for $1 \leq p \leq \infty$ and $0 < s$, $B_p^{s,q}$ has for non-integers s the intrinsic norm (among many)

$$\|f\|_{B_p^{s,q}} \simeq \|f\|_{L_p} + \left(\int_0^1 (\tau^{-s}\omega_p(\tau, f))^q \frac{d\tau}{\tau}\right)^{\frac{1}{q}} \tag{3.1}$$

If s is an integer, the modulus of continuity is replaced by the second order modulus of continuity:

$$\omega_p^1(\tau, f) = \sup_{|h|\leq\tau} (\|f_h \ - 2f + f_{-h}\|_{L_p}).$$

Next,using (3.1) we will need estimates of $\omega_p(\tau, f(u))$ in tems of $\omega_{\bar{p}}(\tau, u)$, and of $\omega_p(\tau, f(u) - f(v))$ in terms of $\omega_{\bar{p}}(\tau, u - v)$, for some suitable $\bar{p}$.
This will be handled as follows: Let $f(z)$ satisfy conditions (A) , with $1 < \rho$. Defined w by

$$f(u) - f(v) = f'(w)(u - v) \tag{3.2}$$

The $w(u, u) = u$, and by the mean value theorem, $w(u, v)$ is bounded by $|u| + |v|$. If condition (A) holds, we get the desired estimate for $\omega_p(\tau, f(u))$ from (3.2) and the preceeding remarks:

$$|f(u_h) - f(u)| \leq C(|u| + |u_h|)^{\rho-1}|u_h - u|$$

For the estimate of $\omega_p(\tau, f(u) - f(v))$ we have (with $w_h = w(u_h, v_h)$ and $w = w(u, v)$)

$$\begin{aligned} |f(u_h) - f(v_h) - (f(u) - f(v))| \leq |f'(w_h) - f'(w)||u - v| \\ + |f'(w_h)||(u_h - u - (v_h - v)| \end{aligned}$$

If $|u - v| \leq |u_h - u|$ or $|u - v| \leq |v_h - v|$ we get

$$|f(u_h) - f(v_h) - (f(u) - f(v))| \leq C(|w_h|^{\rho-1} + |w|^{\rho-1})(|u_h - u| + |v_h - v|) \tag{3.3}$$

which takes care of the estimate of $\omega_p(\tau, f(u) - f(v))$ in this case. If $|u_h - u| + |v_h - v| \leq |u - v|$, $u \neq v$, we need an estimate of $f'(w_h) - f'(w)$ in terms of $|u_h - u|$ and $|v_h - v|$. Under condition (C), i.e when $f(z) = |z|^{\rho-1}z$, the estimate follows from the Lipschitz continuity of $w(u, v)$ for this function. In general the

Lipschitz continuity of $w(u,v)$ on the diagonal $u=v$ and condition (A) result in the following estimate

$$|f'(\bar{w})-f'(w)| \le C(|\bar{u}-u|^{(\rho-1)_-}+|\bar{v}-v|^{(\rho-1)_-})(|u|+|\bar{u}|+|v|+|\bar{v}|)^{(\rho-2)_+} \quad (3.4)$$

provided

$$|u-v| \ge |\bar{u}-u|+|\bar{v}-v|,\ \bar{w}=w(\bar{u},\bar{v}) \text{ and } w=w(u,v) \quad (3.5)$$

Since (3.4) will be used in the proof of the Lipschitz estimates in section 3.4, we give the details of the elementary proof.

Proof: [of (3.4)] As $w(u,u)=u$ and $w(u,v)$ is Lipschitz continuous on the diagonal $u=v$, assumption (A) proves (3.4) in case $u=v$. It is then sufficient to prove (3.4) for $u\neq v$,assuming that (3.5) holds. We also assume that $\rho\le 2$. The extension to $\rho>2$ should be obvious.We first handle the case $\bar{v}=v$.
Then

$$\begin{aligned} f(\bar{u})-f(v) &= f'(\bar{w})(\bar{u}-v) \\ f(u)-f(v) &= f'(w)(u-v) \end{aligned}$$

and so

$$f(\bar{u})-f(u) = (f'(\bar{w})-f'(w))(u-v)+f'(\bar{w})(\bar{u}-u)$$

Hence

$$\begin{aligned} (f'(\bar{w})-f'(w))(u-v) &= f(\bar{u})-f(u)-f'(\bar{w})(\bar{u}-u) \\ &= f'(\bar{u})(\bar{u}-u)+(f'(w(\bar{u},u))-f'(\bar{u}))(\bar{u}-u)-f'(\bar{w})(\bar{u}-u) \\ &= (f'(w(\bar{u},u))-f'(\bar{u}))(\bar{u}-u)-(f'(\bar{w})-f'(\bar{u}))(\bar{u}-u) \end{aligned}$$

Since $w(\cdot,\cdot)$ is Lipschitz continuous on the diagonal, we get from (A) that

$$\begin{aligned} |f'(\bar{w})-f'(w)||u-v| &\le C(|\bar{u}-u|^{1+(\rho-1)}+|\bar{u}-u||\bar{u}-v|^{\rho-1}) \\ &\le C(|\bar{u}-u|^{1+(\rho-1)}+|\bar{u}-u||u-v|^{\rho-1}) \end{aligned}$$

If then $|\bar{u}-u|\le|u-v|$, we obtain for $\bar{v}=v$:

$$|f'(\bar{u},v)-f'(u,v)| \le C|\bar{u}-u|^{\rho-1} \quad (3.6)$$

Since

$$|f'(w(\bar{u},\bar{v}))-f'(w(u,v)| \le |f'(w(\bar{u},\bar{v}))-f'(w(u,\bar{v})|+|f'(w(u,\bar{v}))-f'(w(u,v)|$$

symmetry and (3.6) proves (3.4) in case (3.5) holds. □

3.2 Besov- and Sobolev space estimates of $u \mapsto f(u)$

The following estimate of $f(u)$ is well known and is a strightforward application of Hölder's inequality, but as it contains an unusual twist in case $\rho \le 2$, we give a scetch of that part of the proof.

Lemma 3.1. *Let f satisfy (A) with $\rho < \rho^* = \frac{n+2}{n-2}$, and $\delta \le \frac{1}{n-1}$, with strict inequality for $n = 3$. Assume that $1 \ge s,\ s' > 0$ real with $s - s' = (n+1+\theta)\delta - 1$, $\theta \in [0,1]$, and $\epsilon \ge 0$. Let $1 - \rho < \eta < 1$. Then*

$$\|f(u)\|_{B_p^{s+\epsilon}} \le C\|u\|_{H_2^1}^{\rho-1+\eta}\|u\|_{B_{p'}^{s'}}^{1-\eta} \tag{3.7}$$

where for $\rho \le 2$,

$$\frac{s+\epsilon-s'}{1-s'} + 1 - \eta \le \rho \le 2 - \eta \tag{3.8}$$

provided

$$1 + 4\delta - 2\delta\eta \le \rho \le \rho(n,\delta) - 2\eta\frac{n+\delta+s'}{n-2} - 2\theta\frac{\delta}{n-2} - 2\epsilon\frac{1}{n-2} \tag{3.9}$$

$$\rho(n,\delta) = \frac{n+2(n-1)\delta}{n-2}$$

Remark For suitable choice of s, and ϵ small, any positive $s' \le s_0' = 1 - \frac{1}{2}(n+1)\delta$ is allowed: in fact, a strightforward computation shows that we may take $s' \le s_0' + \sigma$ with $\sigma < \frac{1}{2} - \delta$ for $\rho \ge 2$, i.e n=3,4 and 5 (notice that $\delta(n-1) < 1$ for $n = 3$), and with $\sigma < \frac{3}{2}\delta$, for $\rho < 2$, i.e. $n \ge 6$ (and partly for n = 5).

Proof: Assume first that $\eta = 0$ and that $\rho \le 2$.Let $\bar{s} = s + \epsilon$.
We shall use (3.1) and (3.2) to prove (3.7). With $u_h = u(\cdot + h)$, and $w_h = w(u_h, u)$,

$$|h|^{-\bar{s}}|f(u_h) - f(u)| \le |h|^{-\bar{s}}|f'(w_h)||u_h - u|$$

where

$$|f'(w_h)| \le C(|u_h| + |u|)^{\rho-1}$$

Let $a = (\rho-1)s'$, $b = (\rho-1)(1-s')$ and $c = 2 - \rho$, so that $a + b + c = 1$ and $a + b = \rho - 1$. By (3.8) we have $\bar{s} \le s' + (\rho-1)(1-s')$, and so,

$$\begin{aligned} |h|^{-\bar{s}}|u_h - u| &\le |h|^{-s'}|h|^{(\rho-1)(1-s')}|u_h - u| \\ &\le (|h|^{-1}|u_h - u|)^a(|h|^{-1}|u_h - u|)^b(|h|^{-s'}|u_h - u|)^c \end{aligned}$$

Taking L_2-norm in space of the two first factors on the right, and the $L_{p'}$-norm of the third factor, we get using (3.1) and Hölder's inequality that

$$\|f(u)\|_{B_p^{s+\epsilon}} \le C\|u\|_{H_2^1}^{a+b}\|u\|_{B_{p'}^{s'}}^{c}\|u\|_{L_r}^{\rho-1}$$

with $\frac{1}{r} \geq \frac{1}{p'} - \frac{s'}{n}$, provided (3.9) with $\eta = 0$ holds.
Next replace ρ by $\rho + \eta \leq 2$ in the above argument and we get (3.7), provided (3.8) and (3.9) hold.
The proof of (3.7) for $\rho > 2, \ \rho + \eta > 2$ is strightforward application of Hölder's inequality and (3.1). $\square$

We will also need a counterpart of Lemma 3.1 in case $0 \geq s' \geq 1 - n\delta$. In that case $(2+\theta)\delta + \epsilon \geq s \geq (1+\theta)\delta + \epsilon$, and so $s < \rho - 1$. Splitting $|h|^{-s}|u_h - u| = (|h|^{-\frac{s}{\rho-1}}|u_h - u|)^{\rho-1}|u_h - u|^{2-\rho}$ we now get:

Lemma 3.2. . *Under the assumptions of Lemma 3.1, assume now that* $0 \geq s' \geq 1 - n\delta$. *Then*

$$\|f(u)\|_{B_p^s} \leq C\|u\|_{H_2^1}^{\rho-1+\eta}\|u\|_{L_r}^{1-\eta} \tag{3.10}$$

with

$$\frac{1}{r} = \frac{1}{p'} - \frac{s'}{n}$$

Comment Since $s' \leq 0$ we have $2 < r \leq p'$, with equality only for $s' = 0$, we can't use Sobolev embedding. The following partial substitute will prove sufficient, however. Let $s' \leq 0$, $B_{r(\theta,s')}^{s'(\theta)} = (H_2^1, B_{p'}^{s'})_\theta$ be the interpolation space between $H_2^1 = B_2^1$ and $B_{p'}^{s'}$. Let $r(s') = r(\theta, s')$ for θ such that $s'(\theta) = \theta + (1-\theta)s' = 0$. Then $B_{r(s')}^0 \subseteq L_{r(s')}$, and, a strightforward computation shows that

$$\frac{1}{r(s')} = \frac{1}{p'} - \frac{s'}{n}\frac{n\delta}{1-s'}. \tag{3.11}$$

In particular, $2 < r(s') \leq r$, with equality only for $s' = 0$ and, for $n \geq 4$, also for $s' = 1 - n\delta$.
The following consequence of (3.11) will be useful later: We compute $r(s')$ in (3.11) with s' replaced by $s' + \epsilon, \ \epsilon > 0$:

$$\begin{aligned} \frac{1}{r(s')} &= \frac{1}{p'} - \frac{s' + \alpha\epsilon}{n} \\ \alpha(s') &= \frac{n\delta}{1 - s' - \epsilon}(1 + \frac{s'}{n}\frac{1}{1 - s' - \epsilon}) \end{aligned} \tag{3.12}$$

. Then $\alpha(s')$ is increasing on $[1 - n\delta, 0]$ and so $\alpha \geq \alpha(1 - n\delta) \geq 1 - \frac{1}{n}\delta \geq \frac{5}{6}$ for $n \geq 3$ *and* ϵ small enough . In addition $\alpha \geq 1$ for $s' \geq -\frac{2}{3}\delta$ and $\alpha(0) = \frac{n\delta}{1-\epsilon}$.
Summing up this discussion for future reference, we have the following lemma. .

Lemma 3.3. *Let* $n \geq 3, \ \delta \leq \frac{1}{n-1}$, *and let* $0 \geq s' \geq 1 - n\delta$ *(by which* $n\delta \geq 1$*).Let* $\epsilon = \epsilon(n)$ *be small (depending on n). Define* $r(s')$ *by* $(H_2^1, B_{p'}^{s'}) = B_{r(s')}^0$ *where* $\theta + (1-\theta)s' = 0$. *Then* $r(s')$ *is given by(3.11),* $B_{r(s')}^0 \subset L_{r(s')}$. *If* $u \in L_2^{loc}(L_{r(s')}) \cap L_2^{loc}(B_{p'}^{s'+\epsilon})$ *then* $u \in L_2^{loc}(L_{r(s'+\epsilon')})$ *for* $0 \leq \epsilon' \leq \alpha\epsilon$, *with* $\alpha \geq 1 - \frac{1}{n}\delta$ *for* $s' \geq 1 - n\delta$, *and* $\alpha \geq 1$ *for* $s' \geq -\frac{2}{3}\delta$.

3.3 Spacetime integral estimates of $u \mapsto f(u)$

We shall establish Time Space Integral (STI) estimates of $u \mapsto f(u)$, as they appear e.g. in Propositions 2.3 and 2.4 and their corollaries, in terms of the norms of u given in Proposition 2.1.
We assume throughout that (A) holds, that $n \geq 3$, and that $1+\frac{4}{n} < \rho < 1+\frac{4}{n-2} = \rho^*$.
We use the standard notation of this paper. Then let $\bar{\gamma} = 1-\gamma$, with $\gamma \leq \frac{1}{2} - \delta$, with equality for $(n-1)\delta = 1$ and with $\gamma < \frac{1}{2} - \delta$ otherwise, to be choosen close to this upper limit. We also define

$$\frac{1}{r(\theta)} = 1 - \frac{1}{2}\theta, \ \frac{1}{p(\theta)} = 1 + \delta\theta, \ 0 \leq \theta \leq 1$$

Define the integer $\beta = \rho + \theta - 2, \ 0 \leq \theta < 1$. Then $\beta = 0$ for $\rho \leq 2$, and in particular for $n \geq 6$ (when $\rho^* \leq 2$).
Notice that $\bar{\gamma} - \gamma \approx 2\delta$, with equality for $(n-1)\delta = 1$.
Let $I \subseteq \mathbf{R}_+$ be an interval, and let s and s' be real. We will for short write

$$B(s; I) = L_{r(\theta)}(I; B^s_{p(\theta)}),$$

and

$$A(s'; I) = L_2(I; B^{s'}_{p'})$$

and also $X^1_e(I) = L_\infty(I; H^1_2)$ for the first component of the energy space $X_e(I) = L_\infty(I; H^1_2 \times L_2)$ over I. We let $\|\cdot\|_{e(I)}$ and $\|\cdot\|_{e_1(I)}$ denote the norm in $X_e(I)$ and $X^1_e(I)$, respectively.
If $s = \theta\bar{\gamma}$ and $s' = \gamma$, we write $B(s; I) = B(I)$ and $A(s'; I) = A(I)$, respectively.
If $I = \mathbf{R}_+$, we drop I in the notation.
As in the previous section we use (3.1) and (3.2) to estimate $f(u)$ in $B^s_{p(\theta}$. As before, let $h > 0$, $u_h = u(\cdot + h)$ and $w_h = w(u_h, u)$. Then by (3.2) and condition (A),

$$|f(u_h) - f(u)| \leq |f'(w_h)||u_h - u| \leq C(|u_h| + |u|)^{\rho-1}|u_h - u|$$

Assume that $\epsilon \geq 0$ and $0 \leq \theta < 1$.With $s = \theta\bar{\gamma} + \epsilon$ and $\tilde{\gamma} = \gamma$ for $\rho < 2$, and $\tilde{\gamma} = 1$ otherwise, we get, provided $s \leq \tilde{\gamma}$ (see [d], [d'] below)

$$|h|^{-s}|u_h - u| \leq (|h|^{-\tilde{\gamma}}|u_h - u|)^{\frac{s}{\tilde{\gamma}}}|u_h - h|^{1-\frac{s}{\tilde{\gamma}}}$$

and so for h small

$$|h|^{-s}|f(u_h) - f(u)| \leq C|u|^{\rho - \frac{s}{\tilde{\gamma}}}(|h|^{-\tilde{\gamma}}|u_h - u|)^{\frac{s}{\tilde{\gamma}}}$$

Then by (3.1), with a constant C independent of u and I,

$$\|f(u)\|_{B(\theta\bar{\gamma}+\epsilon; I)} \leq C\|u\|^{\rho-\beta}_{A(\gamma; I)}\|u\|^{\beta}_{e_1(I)} \tag{3.13}$$

provided, efter some (strightforward) reductions of “standard” Sobolev and Hölder estimates, that

[a] $n+2 \geq \rho(n-2)+2\epsilon+4(1-(n-1)\delta)$

[b] $1+4\delta \leq \rho$

[c] $2+\beta-\theta=\rho$

[d] $\epsilon+(2-\rho)\bar{\gamma} < \tilde{\gamma}$, that is with [c]

[d'] $\epsilon < (\rho-1)\bar{\gamma}-2\delta$ if $\beta=0$

If we let $\delta=\frac{\rho-1}{4}$ for $\frac{1}{n}<\delta<\frac{1}{n-1}$, and take $(n-1)\delta=1$ for $\rho \geq 1+\frac{4}{n-1}$ for $n>3$, while for $n=3$ take $(n-1)\delta$ less than and sufficiently close to 1 for $\rho \geq 1+\frac{4}{n-1}$, then [a] through [c] can be satisfied for $\epsilon=\epsilon(n,\rho)>0$ small enough, where in case $\beta=0$, we assume that [d'] holds. Notice that the right hand side of [d'] is positive under our assumptions and [b]. The conditon [d] for $\beta \geq 1$ is $\theta\bar{\gamma}+\epsilon \leq 1$, which holds for all $\epsilon \leq 1-\bar{\gamma}=\gamma$.
Also notice that if $\beta>1$ we may replace $\beta-1$ of the factors $\|u\|_{e_1(I)}$ by the corresponding $L_\infty(I;L_{\bar{r}})$ for any $\bar{r}$ such that $H_2^1 \subseteq L_{\bar{r}}$.
The estimate (3.13) is known in various forms - the one presented here will suit our purposes: The choice of γ and $\bar{\gamma}$ according to Proposition 2.1 rather than the Strichartz estimate Proposition 2.2, is caused by the use of the L_2-norm in time - which is not possible for $(n-1)\delta<1$ in the clasical Strichartz estimates. The choice made here gives slightly better estimates for $\frac{1}{n}<\delta<\frac{1}{n-1}$ then the use of the classical Strichartz estimate would give (the lover bound for $n=6$, for example). For global non-uniform estimates (i.e estimates over compact time intervals), the choice of (r',p',γ) as in the Strichartz estimate is optimal, using that $L_{r'}^{loc} \subseteq L_2^{loc}$, however.
If I is a compact subinterval of $\mathbf{R}_+$, formally [c] can be relaxed. If we want estimates for ρ close to ρ^*, however, we not only have to take $(n-1)\delta$ close to 1 and ϵ close to 0, but we will be forced by [a] (most easily seen before simplification) to have essentially [c] satisfied. The restrictions in case ρ close to ρ^* will thus be the same as in the uniform case.

3.4 Lipschitz estimates of $u \mapsto f(u)$

We keep the notation of the preceeding section. This time we use (3.1) and (3.4) to estimate the $B^s_{p(\theta)}$-norm of $f(u)-f(v)$. With $w_h=w(u_h,w_h)$ and $w=w(u,v)$ we have by (3.2)

$$(f(u_h)-f(v_h)-(f(u)-f(v)))=(f'(w_h)-f'(w))(u-v)+f'(w_h)(u_h-v_h-(u-v))$$

Let $s \geq \sigma \geq 0$. Write

$$|h|^{-(s-\sigma)}|u_h-u|^{(\rho-1)_-}=(|h|^{-\frac{s-\sigma}{(\rho-1)_-}}|u_h-u|)^{(\rho-1)_-}$$

and correspondingly for $vh - v$. By (3.3) and (3.4) we then get

$$|h|^{-(s-\sigma)}|f(u_h) - f(v_h) - (f(u) - f(v))|$$
$$\leq C(|h|^{-\frac{s-\sigma}{(\rho-1)_-}}(|u_h - u| + |v_h - v|))^{(\rho-1)_-}(|u| + |u_h| + |v_h| + |v|)^{(\rho-2)_+}|u - v|$$
$$+ \ C(|u_h| + |v_h|)^{\rho-1}|h|^{-(s-\sigma)}|(u_h - u) - (v_h - v)|$$

Let $s = \theta\gamma + \epsilon'$, $\epsilon' > 0$. Application of (3.1) and strightforward use of Hölder's and Sobolev's inequalities give,

$$\|f(u) - f(v)\|_{B(s-\sigma,I)} \leq C(\|u\|^{\rho-\beta-1}_{A(\gamma,I)}\|u\|^{\beta}_{e_1(I)} + \|v\|^{\rho-\beta-1}_{A(\gamma,I)}\|v\|^{\beta}_{e_1(I)})\|u - v\|_{A(\gamma-\sigma,I)} \quad (3.14)$$

provided, as in Section 3.3, conditions [a] through [d] there hold with ϵ replaced by ϵ', and in addition (as before with $\tilde{\gamma} = \gamma$ for $\rho < 2$, and equal to 1 otherwise)

[e] $\theta\bar{\gamma} - \sigma + \epsilon' \leq \tilde{\gamma}(\rho - 1)_-$, and so if $\beta = 0$, with [c],

[e'] $\sigma \geq \bar{\gamma} - (\rho - 1) + \epsilon'$

By the discussion in section 3.3, conditions [a] through [d'] hold under our assumptions on ρ, δ, γ and $\bar{\gamma}$ for $\epsilon' > 0$ small enough.
Hence for $\beta = 0$ (3.14) holds if $\sigma \geq 0$, such that also [e'] is satisfied for sufficiently small $\epsilon' > 0$. If $\beta \geq 1$ we get instead of [e'] that $\theta\bar{\gamma} - 1 + \epsilon' \leq \sigma$, which holds for $\sigma = 0$, again with $\epsilon' > 0$ sufficiently small.
Thus for $n = 3, 4$ and 5 and for $\rho > 2$ and $1 + \frac{4}{n} < \rho < 1 + \frac{4}{n-2} = \rho^*$ we have that $\sigma = 0$, i.e. $A(I) \ni u \mapsto f(u) \in B(I)$ is Lipschitz continuous. This is also the case if $0 > \bar{\gamma} - (\rho - 1)$, i.e. if $n \leq 6$. In addition $\sigma = 0$ also for $n = 7$ and 8 with lower bounds of ρ equal to $1 + \frac{4}{n-1}$ and $1 + \frac{4.5}{n-1}$, respectively, and so we have Lipschitz continuity also in this case.
Let us point out that (even with additional regularity of f at 0) a limitation of the dimension n for which the mapping $A(I) \ni u \mapsto f(u) \in B(I)$ is Lipschitz continuous is expected in a qualitative sense from the results in [10]. See also the discussion in [17] on p. 549, related to Lemma 2.3 in that paper.
In [10] it was also proved that for large dimensions n, in case of Hölder continuity of the mapping of order α, one should expect $\alpha = \mathcal{O}(\frac{1}{n})$. A step towards this result is the following crude but useful estimate, which is an immediate consequence of (3.13) and (3.14). We again give the uniform global estimate in time; the corresponding estimates for subintervals I of $\mathbf{R}_+$ are the same (with constants independent of I). Since we have Lipschitz estimates for $n \leq 6$, we assume that $n > 6$ and so in particular the integer $\beta = 0$.

$$\|f(u) - f(v)\|_{B(\theta\bar{\gamma}+\epsilon')} \leq C(\|u\|^{\rho-\alpha}_{A} + \|v\|^{\rho-\alpha}_{A})\|u - v\|_{A(\gamma-\sigma)} \quad (3.15)$$

where $\alpha = \frac{\epsilon}{\sigma+\epsilon}$, with $\epsilon > 0$, with $\epsilon' > 0$ sufficiently small satisfying the assumptions of (3.13) and σ those of (3.14), that is, spelling out the choice of γ and

$\bar{\gamma}$,

$$\epsilon = \min(\frac{(n-2)(\rho^* - \rho)}{2} - 2(1-(n-1)\delta), (\rho - 1)(\frac{1}{2} + \delta) - 2\delta)$$

and

$$\sigma = \max(\frac{1}{2} + \delta - (\rho - 1) + \epsilon', 0)$$

In particular, $\sigma = 0$ implies $\alpha = 1$ in (3.15), which in view (3.14) of is at least one redeeming fact in the choice of α.
The derivation of (3.15) is short: To simplify the notation, write $s = \theta\bar{\gamma} + \epsilon'$. By convexity

$$\|f(u) - f(v)\|_{B(s)} \leq \|f(u) - f(v)\|_{B(s+\epsilon)}^{1-\alpha} \|f(u) - f(v)\|_{B(s-\sigma)}^{\alpha}$$

with $\alpha = \frac{\epsilon}{\sigma+\epsilon}$. Next estimate the first factor by (3.13) and the second by (3.14). This proves (3.15) and gives the conditions on ϵ and σ as stated above.
Notice that, as expected, $\alpha = \mathcal{O}(\frac{1}{n})$ as n tends to ∞.

3.5 Comments on the critical case $\rho = \rho^*$

In the critical case $\rho = \rho^*$ the estimates (3.7), (3.13), and (3.14) hold with some restrictions: The parameters θ, ϵ in (3.7) have to be 0, and η has to be ≤ 0. In the estimate (3.13)and (3.14,) necessarily $\epsilon = 0$, $\epsilon' = 0$ and $\beta(1-(n-1)\delta) = 0$. The Hölder estimate (3.15) is then of no interest, since α becomes 0 for $\sigma > 0$. In addition, in the Strichartz estimates the endpoint case $n = 3$ and $(n-1)\delta = 1$ is excluded, and so (3.13) is of little interest for the critical case for $n = 3$.
To obtain Lipschitz estimates for the solution operator of the NLKG/NLWE for $n = 3$ in the critical case $\rho = 5$ more refined estimates are necessary (see Bahouri and Gerard [1]).
The following are simple inequalities which can be used to obtain global (not necessarily uniform) estimates of the solution for the NLKG/NLWE.
Let $\frac{1}{q'} = \frac{1}{2} - \frac{1}{n+1}$, $n = 3$ and $1 < \rho \leq 5$. I will be a compact interval in $\mathbf{R}_+$, $C(\cdot)$ a continuous function of $|I|$. Then

$$\|f(u) - f(v)\|_{L_1(I;L_2)} \leq C(I) \|u\|^{\rho-1}_{L_{q'}(I;B^{\frac{1}{2}}_{q'})} \|u - v\|_{L_\infty(I;H^1_2)}$$

If $n = 4$, $1 < \rho \leq 3$ and we use A(I) in the notation above,we get

$$\|f(u) - f(v)\|_{L_1(I;L_2)} \leq C(I) \|u\|^{\rho-1}_{A(I)} \|u - v\|_{L_\infty(I;H^1_2)}$$

We will not pursue these questions further in this paper.

4 STI estimates, uniqueness and endpoint estimates

4.1 Global non-uniform STI estimates

As pointed out in Section 2, the endpoint estimates in the STI estimates in the subcritical case follow directly from Theorem 1 in [8]. We will, however, as promised, include a proof, partly following the exposition in [8], but using the non-endpoint STI estimates of [8] and Ginibre and Velo, [13], [14].
We will to this end first prove a global non-uniform STI estimate, which includes (by the endpoint Strichartz estimates) the endpoint estimates in this setting (i.e in L_2^{loc} in time).
We introduce the condition

$$(*)\ (n-1+\theta)\delta > 1 > (n-1-\theta)\delta,, \quad some\ \theta \in (0,1],$$
$$s - s' \geq (n+1+\theta)\delta - 1,\ s,\ s'\ real,\ s \leq 1$$

We denote by (*)' condition (*) without the upper bound $(n-1+\theta)\delta > 1$.
We then prove the following special case of Corollary 1 in [8].

Proposition 4.1. *Assume that (*)' holds and that* $\frac{s-s'}{1-s'} < \rho - 1$ *if* $\rho \leq 2$, *and let* r', p', s' *satisfy the conditions (2.8) in the Strichartz estimate, where* $r' = 2$, $(n-1)\delta = 1$ *is not allowed for* $n = 3$. *Let* f *satsify condition (A) with* $1 + 4\delta < \rho < \rho^* = \frac{n+2}{n-2}$ *and let* u *be a solution of the NLKG with finite energy data. Then* $u \in L_{r'}(I; B_{p'}^{s'})$, *with a bound continuously depending on* $I \subset\subset \mathbf{R}_+$ *and the energy of the data.*

Let $\mathcal{K}_M$ denote the M-fold konvolution with the kernel K_1 of Section 2, and K_M the kernel of $\mathcal{K}_M$. If we assume that (*) holds,then $K_1 \in L_1 \cap L_{1+\epsilon}$, some $\epsilon > 0$ by (2.4). If instead only (*)' holds, then $K_1 \in L_1^{loc} \cap L_{1+\epsilon}^{loc}$.
Let u_0 be a solution of the Klein-Gordon equation with finite energy data. Then by the Strichartz estimate $u_0 \in L_{r'}(B_{p'}^{s'})$. For M large enough, if (*) or (*)' holds, then $K_M * \|u\|_{B_{p'}^{s'}} \in L_\infty$ or $K_M * \|u\|_{B_{p'}^{s'}} \in L_\infty^{loc}$, respectively.
We will also use an elementary estimate:

Lemma 4.1. *Let* $U(t)$ *be an increasing non-negative function for* $t \geq 0$, *with* $U(0) = 0$, *and let* A *and* B *be non-negative. Assume that*

$$U(t) \leq A + BU(t)^\alpha,\ t \geq 0,\ \alpha < 1.$$

Then

$$U(t) \leq \frac{1}{1-\alpha}A + B^{\frac{1}{1-\alpha}}$$

Using these observations, we prove the following version of Lemma 1.1 in [8]:

Lemma 4.2. *Assume that the conditions of Proposition 1 hold. Then there is an integer M_0 such that for $M \geq M_0$*

$$\mathcal{K}_M \|u(t)\|_{B^{s'}_{p'}} = \int_0^t K_M(t-\tau)\|u(\tau)\|_{B^{s'}_{p'}} d\tau \in L^{loc}_\infty(\mathbf{R}_+)$$

with a bound depending continuously on t and on the energy norm of the initial data. If (*) holds, and if $1+4\delta_+ < \rho < \rho^*$, then

$$\mathcal{K}_M \|u(t)\|_{B^{s'}_{p'}} \in L_\infty(\mathbf{R}_+)$$

with a bound depending continuously on the energy of the initial data.

Remark In view of Lemma 3.1, the norm of u could be taken in $L_{\bar{r}}$ instead, with $\frac{1}{\bar{r}} = \frac{1}{p'} - \frac{s'}{n}$.
To see that the quantities used in the proof of Lemma 4.2 make sense, we first prove a crude version of Proposition 4.1. In the proof we use a bootstrapping device introduced in a more precise and complex setting by Ginibre and Velo [13] used by them to prove a result corresponding to Proposition 4.1. Since we want to include the global result in Lemma 4.2 we are satisified with the following result:

Lemma 4.3. *Let u satisfy the conditions of Lemma 4.2. Then $u \in L^{loc}_2(B^{s'}_{p'})$.*

Proof: We assume for simplicity that $\sigma = 0$ in (2.8).The changes for $\sigma > 0$ should be obvious.
Let $0 < \delta \leq \frac{1}{n-1}$, with sharp upper inequality for $n = 3$. Define $Z^{s'}_{p'}$ by

$$Z^{s'}_{p'} = \begin{cases} L_{r(s')} \ , \ \frac{1}{r(s')} = \frac{1}{p'} - \frac{s'}{n}, & s' \leq 0 \\ B^{s'}_{p'}, & s' > 0 \end{cases} \tag{4.1}$$

Since

$$u(t) = u_0(t) + \int_0^t E_1(t-\tau) f(u(\tau)) d\tau$$

we get by (2.2), (2.4), Lemma 3.1 and Lemma 3.2,

$$\|u\|_{B^{s'+\epsilon}_{p'}} \leq \|u_0\|_{B^{s'+\epsilon}_{p'}} + C(t, \|u\|_{H^1_2}) \|u\|_{Z^{s'}_{p'}} \tag{4.2}$$

Here $C(\cdot, H^1_2) \in L^{loc}_\infty$. Assume first that $n\delta > 1$. Here $u_0 \in L_{r'}(B^{s'+\epsilon}_{p'})$ for some $r' > 2$ if $s' + \epsilon \leq 1 - \frac{1}{2}(n+1)\delta$, and so to $L^{loc}_2(B^{s'+\epsilon}_{p'})$ by the Strichartz estimates, and $Z^{s'}_{p'} = L_{r(s')} \supseteq H^1_2$ for $s' = 1 - n\delta < 0$, and so $u \in L_\infty(H^1_2) \subset L^{loc}_2(Z^{s'}_{p'})$, and

thus for $\epsilon > 0$ small enough, (4.2) shows that $u \in L_2^{loc}(B_{p'}^{s'+\epsilon})$. Invoking Lemma 3.3 we find that $u \in L_2^{loc}(L_{r(s'')})$ with $s'' = s' + \alpha\epsilon$. Thus, again by Lemma 3.3, we find that $u \in L_2^{loc}(B_{p'}^{s''})$. If $s'' < 0$, we repeat this argument. Since $\alpha \geq \frac{5}{6} > \frac{1}{2}$ we reach either a value of $s'' \geq 0$ or else $s' + \epsilon$. In the latter case we restart the process replacing s' by $s' + \epsilon$.
In a finite number of steps we reach that $u \in L_2^{loc}(L_{p'})$. If $n\delta \leq 1$, then $s' = 1 - n\delta \geq 0$, and in that case $u \in L_2^{loc}(L_{p'})$. From then on $Z_{p'}^{s'} = B_{p'}^{s'}$, and with suitable choice of ϵ we reach by (4.2) in a finte number of steps that $s' + \epsilon = 1 - \frac{1}{2}(n+1)\delta$, which proves the lemma. $\square$

PROOF: [of Lemma 4.2] Let $\|\cdot\|$ denote the norm in $B_{p'}^{s'}$. Let u_0 be the solution of the Klein-Gordon equation with the same initial data as u. The by the Strichartz estimates $u_0 \in L_{r'}(B_{p'}^{s'})$ with $\frac{2}{r'} = (n-1)\delta$. Then as in (4.2) using Lemma 3.1 with $\eta > 0$, we get

$$\|u(t)\| \leq \|u_0\| + C\int_0^t K_1(t-\tau)\|u(\tau)\|^{1-\eta} d\tau$$

Multiplying with $K_M(\sigma - \tau)$ and integrate over $(0, \sigma)$:

$$\int_0^\sigma K_M(\sigma - t)\|u(t)\| dt \leq$$
$$\int_0^\sigma K_M(\sigma - t)\|u_0(t)\| dt + \sup_{t \leq \sigma} C(\int_0^t K_M(t-\tau)\|u(\tau)\| d\tau)^{1-\eta} \tag{4.3}$$

since

$$\int_0^\sigma K_M(\sigma - t)\int_0^t K_M(t-\tau)\|u(\tau)\|^{1-\eta} d\tau dt \leq$$
$$\int_0^\sigma K_M(\sigma - t)(\int_0^t K_M(t-\tau)\|u(\tau)\| d\tau)^{1-\eta}(\int_0^t K_M(\tau)d\tau)^\eta dt$$

Define

$$g(t) = \int_0^t K_M(t-\tau)\|u(\tau)\| d\tau$$
$$g_0(t) = \int_0^t K_M(t-\tau)\|u_0(\tau)\| d\tau$$

Then by (4.3)

$$g(t) \leq g_0(t) + C(t, \|u\|_{L_\infty(H_2^1)}) \sup_{\tau \leq t} g(\tau)^{1-\eta}$$

where $C(\cdot, \cdot)$ is an increasing continuous function in each of the variables By assumption, $g_0(t)$ is uniformly bounded on $\mathbf{R}_+$ by the energy norm of the initial data, and hence the energy bound for u and Lemma 4.1 completes the proof of Lemma 4.2. If (*) holds, and if $1 + 4\delta_+ < \rho < \rho^*$ then $C()$ can be taken independent of $t \in \mathbf{R}_+$, and the uniform version of Lemma 4.2 follows. $\square$

We are now in position to prove Proposition 4.1

PROOF: [of Proposition 4.1] Again, let $\|\cdot\|$ denote the norm in $B^{s'}_{p'}$. By the Strichartz estimate

$$\|u_0\| \in L_{r'},\ \frac{2}{r'} = (n-1)\delta$$

and by Lemma 4.2

$$K_M * \|u\| \in L_\infty \subseteq L^{loc}_{r'},\ M \geq M_0.$$

Thus

$$K_{j-1} * \|u\| \leq K_{j-1} * \|u_0\| + CK_j * \|u\| \tag{4.4}$$

If $K_j * \|u\| \in L^{loc}_{r'}$ then by (4.4) $K_{j-1} * \|u\| \in L^{loc}_{r'}$. Recursion over j now proves Proposition 4.1. □

4.2 Uniqueness

Neither uniqueness nor energy conservation was assumed in the proof of Proposition 4.1, only the energy inequality providing uniform energy bound on $\mathbf{R}_+$. Uniquness is however an important ingredience in the proof of the uniform STI estimates (originally in the non-endpoint estimates) and implies energy conservation. We will here give a proof of the uniqueness of (weak) finite energy solutions of the NLKG under the assumption (A) and for $1 \leq \rho < \rho^* = \frac{n+2}{n-2},\ n \geq 3$, i.e. the subcritical case, based on Proposition 4.1.
The result as such is due to Ginibre and Velo [13] (for earlier results, see Glassey and Tsusumi [19]).

Theorem 3 (Uniquness)**.** *Let $1 \leq \rho < \rho^*$ and assume that condition (A) holds. Then for any finite energy initila data, the NLKG has a unique weak finite energy solution.*

PROOF: Existence (without limitations on $\rho \geq 1$) for finite energy solutions was proved by Segal already 1962-63 ([37], [38]). The argument (and later modifications) is a compactness argument that doesn't imply uniqueness.
Let u and v be finite energy solutions of the NLKG with the same initial data. For the validity of the inequalities which will follow, notice that for $u, v \in L_\infty(H^1_2) \subseteq L_\infty(L_{p'})$.
For $1 + \frac{4}{n-1} \leq \rho \leq \rho^* = 1 + \frac{4}{n-2}$,

$$\|f(u) - f(v)\|_{L_p} \leq h(u,v)\|u - v\|_{L_{p'}}$$

where

$$h(u,v) \leq C(\|u\|_B^{\rho-1} + \|v\|_B^{\rho-1}),\ C = C(n,p')\ constant$$
$$B = B_{p'}^{s'},\ \delta = \frac{1}{n+1},\ s' = \frac{1}{2} - \frac{1}{2}\delta$$

which is easily verified by strightforward application of Hölder's inequality and Sobolev embedding. The Strichartz estimates and Proposition 4.1 show that $u \in L_{r'}(B)$, $\frac{1}{r'} = \frac{1}{p'} - \frac{1}{2}\delta$. This part of Proposition 4.1 can be proved without the use of Lemma 3.2 and Lemma 3.3. Hence for $t \leq 1$

$$\|u(t) - v(t)\|_{L_{p'}} \leq \int_0^t K(t-\tau)h(u(\tau), v(\tau))\|u(\tau) - v(\tau)\|_{L_{p'}} d\tau$$

where

$$K(t) \leq Ct^{-\frac{n-1}{n+1}},\ 0 < t \leq 1$$

Since

$$\frac{n-1}{n+1} + \frac{\rho-1}{r'} < 1\ for\ \rho - 1 < \frac{4}{n-2}$$

Young's inequality shows that

$$I(t) = \int_0^t K(t-\tau)h(u(\tau), v(\tau))d\tau$$

is continuous and I(0) = 0. It follows that

$$\|u(t) - v(t)\|_{L_{p'}} \leq I(t) \sup_{0 \leq \tau \leq t} \|u(\tau) - v(\tau)\|_{L_{p'}} \tag{4.5}$$

If $\rho \leq 1 + \frac{4}{n-1}$, we apply Lemma 3.2 with $\eta = 0,\ \theta = 0,\ and\ \epsilon = 0$. Then first take $n\delta = 1$ and $s = \delta,\ s' = 0$; the estimate (4.5) follows as above for $1 + \frac{4}{n} \leq \rho \leq 1 + \frac{4}{n-1}$. If $\rho \leq 1 + \frac{2}{n-2}$, then $\delta = 0$, i.e. $p' = 2$ is allowed, and we get the lower bound $1 \leq \rho$ in (4.5). Hence if we have (4.5) for $1 + 4\delta' \leq \rho < \rho^*$, and if $4\delta' \leq \frac{2}{n-2}$, we may take $\delta = 0$ in the next step, and (4.5) follows for the whole range $1 \leq \rho < \rho^*$. If $4\delta' > \frac{2}{n-2}$, we have $\delta = \alpha\delta'$, where by Lemma 3.2

$$4\delta' \leq \rho - 1 < \frac{2(n-2)\delta + 2}{n-2}$$

and so a strightforward computation gives

$$\alpha = \frac{2(n-2)\delta' - 1}{(1+\delta')(n-1)\delta'}$$

by which $\frac{1}{2} > \alpha > 0$. After a finite number of steps we then reach $4\delta \leq \frac{2}{n-2}$. Thus (4.5) is proved with $I(0) = 0$, I(t) continous and increasing. It follows that for $0 < t \leq 1$,

$$\sup_{0 \leq \tau \leq t} \|u(\tau) - v(\tau)\|_{L_{p'}} \leq I(t) \sup_{0 \leq \tau \leq t} \|u(\tau) - v(\tau)\|_{L_{p'}}$$

and letting $t \to 0$ the Proposition is proved. □

4.3 Endpoint STI estimates for the NLKG

Proof: [of Proposition 2.7] As mentioned the proof of (i) can be found in [39], [40] and for the estimate over all of $\mathbf{R}_+$ in time, in Bahouri and Shatah [2], formally stated only for $n = 3$, but as commented in the end of that paper, easily extended to all $n \geq 3$. The uniform energy bound for n= 3 in (i) is due to Bahouri and Gerard [1]. It remains to prove (ii):

The Shatah - Struwe solution of the critical NLKG:

Let u_0 be a solution of the Klein-Gordon equation. Then there is a solution u^k of the NLKG (1.1) with critical nonlinearity (i.e. $\rho = \rho^*$) with the same initial data as u_0, but with f satisfying (C) replaced by f_k:

$$f_k(z) = z \min(|z|, k)^{\rho^* - 1}$$

Notice that f_k satisfy conditions (A) and (B) for $\rho < \rho^*$,and in particular u_k satisfies the endpoint STI of Proposition 2.6, though of course not uniformly in k. A (subsequence of) u^k converges weakly to a finite energy solution u - *the Shatah - Struwe solution* - of the NLKG (1.1): This solution u is unique in $C(H_2^1 \times L_2) \cap L_{q'}^{loc}(B_{q'}^{\frac{1}{2}})$.The (given sub-) sequence u^k can be shown to converge to u in H_2^1 and has the same (conserved) energy as u (and u_0). We first treat the case $n \geq 6$: Let $\rho = \rho^*$ in the remainder of this proof.

Let $I = (t_0, t_0+t) \subseteq \mathbf{R}_+$, $t > 0$ finite. Then by Corollary 2.1, and the STI applied to u^k (which, together with the above remark on f_k, ensures the boundedness of the norms)

$$\|u^k\|_{L_2(I;B_{p'}^{\gamma})} \leq C\|u_0\|_{L_\infty(I;H_2^1)} + \|f_k(u^k)\|_{L_{r(\theta)}(I;B_{p(\theta)}^{\theta\bar{\gamma}})}$$

with

$$\frac{1}{r(\theta)} = \frac{1}{2} + (1-\theta)\frac{1}{2},\ \delta_{p(\theta)} = \theta\delta_{p'},\ p(\theta) \leq 2,\ 0 \leq \theta \leq 1.$$

By a variation of (2.1) we have (with $\rho = \rho^*$),for $\epsilon > 0$ choosen such that $\rho - \epsilon < 1$,

$$\|f_k(u^k)\|_{L_{r(\theta)}(I;B_{p(\theta)}^{\theta\bar{\gamma}})} \leq C_0\|u^k\|^{\epsilon}_{L_{q'}(I;B_{q'}^{\frac{1}{2}})}\|u^k\|^{\rho-\epsilon}_{L_2(I;B_{p'}^{\gamma})} \tag{4.6}$$

where

$$\frac{1}{2} + \theta\delta_{p'} \geq (\rho - \epsilon)\frac{n-3}{2n} + \epsilon(\frac{n-3}{2n} + \frac{1}{n(n+1)}) + \theta\frac{\bar{\gamma}}{n}$$

and

$$\frac{1}{2} + \theta\delta_{p'} \leq (\rho - \epsilon)(\frac{1}{2} - \delta_{p'}) + \epsilon(\frac{1}{2} - \delta_{q'})$$

and additionally (for the time integrals),

$$\frac{1}{r(\theta)} = 1 - \theta\frac{1}{2} = \frac{\rho - \epsilon}{2} + \frac{\epsilon}{q'}$$

which are satisfied if with $\bar\theta = \theta - \epsilon\frac{2}{n+1}$,

$$\begin{aligned} &2 - \bar\theta = \rho \\ &n + \bar\theta \geq \rho(n-3) \\ &n - 1 + 2\bar\theta \leq \rho(n-3) \end{aligned} \tag{4.7}$$

with the restrictions

$$0 \leq \theta \leq \frac{\gamma}{\bar\gamma} = \frac{n-3}{n+1}$$

The last two inequalities in (4.7) become

$$\frac{n+2}{n-2} \geq \rho \geq 1 + \frac{4}{n-1}$$

which is obviously satisfied for $\rho = \rho^* = \frac{n+2}{n-2}$. For $\epsilon > \rho - 1$ sufficiently close to $\rho - 1$ we may choose θ in the allowed range $0 \leq \theta \leq \frac{\gamma}{\bar\gamma} = \frac{n-3}{n+1}$, with $\bar\theta$ satisfying (4.7).

Let

$$U_k(t) = \|u^k\|_{L_2(I;B^{\gamma}_{p'})}$$

and let

$$D_k = C_0\|u^k\|_{L_{q'}(\mathbf{R}_+;B^{\frac{1}{2}}_{q'})}$$

Then U(0) = 0 and by (4.6)

$$U_k(t) \leq C\|u_0\|_{L_\infty(\mathbf{R}_+;H^1_2)} + D_k^\epsilon U_k(t)^{\rho-\epsilon}$$

and since by our choice of ϵ, $\alpha = \rho - \epsilon < 1$ close to 1, it follows using Lemma 4.1 above and the uniqueness of the Shatah - Struwe solution, and the convergence properies of u^k,

$$U_k(t) \leq \frac{1}{1-\alpha}C\|u_0\|_{L_\infty(\mathbf{R}_+;H^1_2)} + (C_0\|u^k\|_{L_{q'}(\mathbf{R}_+;B^{\frac{1}{2}}_{q'})})^{\frac{1}{1-\alpha}}$$

Thus

$$\begin{aligned} \|u\|_{L_2(I;B^{\gamma}_{p'})} &\leq \lim_k \|u^k\|_{L_2(I;B^{\gamma}_{p'})} \\ &\leq \frac{1}{1-\alpha}C\|u_0\|_{L_\infty(\mathbf{R}_+;H^1_2)} + (C_0\|u\|_{L_{q'}(\mathbf{R}_+;B^{\frac{1}{2}}_{q'})})^{\frac{1}{1-\alpha}}, \end{aligned} \tag{4.8}$$

which proves (ii) for $n \geq 6$. The proof of (4.8) for $n = 4$ *and* $n = 5$ are similar, now using a variant replacing ρ by $\rho - 1$ in the above argument for $n \geq 6$ in the following way: Again taking the time integrals over I, we this time estimate f(u) by

$$\|f_k(u^k)\|_{L_{r(\theta)}(B^{\theta\bar\gamma}_{p(\theta)})} \leq C_0\|u^k\|^\epsilon_{L_{q'}(B^{\frac{1}{2}}_{q'})}\|u^k\|^{\rho-1-\epsilon}_{L_2(B^{\gamma}_{p'})}\|u^k\|_{L_\infty(H^1_2)}$$

where now

$$\frac{1}{2}+\theta\delta_{p'} \geq (\rho-1-\epsilon)\frac{n-3}{2n}+\epsilon(\frac{n-3}{2n}+\frac{1}{n(n+1)})+\frac{1}{2}-\frac{1}{n}+\theta\frac{\bar{\gamma}}{n}$$

and

$$\frac{1}{2}+\theta\delta_{p'} \leq (\rho-1)\frac{n-3}{2(n-1)}+\epsilon(\frac{1}{n-1}-\frac{1}{n+1})+\frac{1}{2}$$

and additionally for the time integrals,

$$\frac{1}{r(\theta)} = 1-\theta\frac{1}{2} = \frac{\rho-1-\epsilon}{2}+\frac{\epsilon}{q'}$$

which are satisfied (again with $\bar{\theta}=\theta-\epsilon\frac{2}{n+1}$) if

$$\begin{aligned} 3-\bar{\theta} &= \rho, \\ 2+\bar{\theta} &\geq (\rho-1)(n-3), \\ 2+2\bar{\theta} &\leq (\rho-1)(n-1), \end{aligned} \tag{4.9}$$

now with the restriction $0 \leq \theta \leq 1$, the differentiation this time included in the H_2^1-term.
Clearly the first equation in(4.9)can be satisfied for ϵ with $\rho-1-\epsilon < 1$ close to 1, i.e $\epsilon > \rho-2$ close to $\rho-2$. Substituting the first equation in the last two equations, we get

$$\begin{aligned} 4 &\geq (\rho-1)(n-2), \\ 4 &\leq (\rho-1)(n-1), \end{aligned}$$

which are both satisfied for $\rho-1=\rho^*-1=\frac{4}{n-2}$. With $\alpha=\rho-1-\epsilon$ the proof of (4.8) is then completed as for $n \geq 6$.
Part (iii) follows by interpolation between (ii) and the energy estimate. $\square$

The following is a slight, but natural, extension of the original STI-estimates (which are assumed - i.e the non-endpoint estimates) in Proposition 2.6, allowing endpoint estimates as in the Strichartz estimate for the Klein-Gordon equation.

Proposition 4.2. *Let $n \geq 4$ and u a solution of the NLKG with finite energy data, and let $(n-1)\delta_{p'}=1,\ p'>2,$ and $\gamma=\frac{1}{2}-\delta_{p'}$. Then*

$$u \in L_2(B^{\gamma}_{p'}),\ 1+\frac{4}{n-1} < \rho < \rho^*$$

The norms are bounded by an increasing function of the energy of u(0), i.e of the initial data.

PROOF: Let $\tilde{p'} > 2$, $\tilde{\delta} = \frac{1}{2} - \frac{1}{\tilde{p'}}$ with $(n-1)\tilde{\delta} < 1$ close to 1, to be choosen below. Let $(\tilde{r'}, \tilde{p'}, \tilde{s'}$ be the corresponding tripple in the Strichartz estimates. We also let $\delta = \frac{1}{2} - \frac{1}{p'} = \frac{1}{n-1}$ with $\bar{\gamma} = \frac{1}{2}(n+1)\delta = \frac{1}{2} + \delta$ and $\gamma = 1 - \bar{\gamma}$; as before p denotes the dual of p', $\frac{1}{p} + \frac{1}{p'} = 1$. Let u be a solution of the NLKG with initial data in $H_2^1 \times L_2$, and let u_0 be the corresponding solution of the Klein-Gordon equation with the same initial data.Then (by Theorem 3) u is the unique finite energy solution of NLKG, and if $\tilde{B} = L_{\tilde{r'}}(B_{\tilde{p'}}^{\tilde{s'}})$, then by the STI estimate in the non-endpoint case, $u \in \tilde{B}$ and $\|u\|_{\tilde{B}}$ is bounded by an increasing function of the energy of u (which is constant in $t \in \mathbf{R}$). Let

$$\frac{1}{r(\theta)} = 1 - \frac{1}{2}\theta,\ \frac{1}{2} - \frac{1}{p(\theta)} = \delta_{p(\theta)} = \theta\delta,\ 0 \leq \theta < 1$$

and define

$$A_\theta(T) = L_{r(\theta)}(0, T; B_{p(\theta)}^{\bar{\gamma}\theta}),\ A_\theta = A_\theta(+\infty)$$

Then

$$\|f(u)\|_{A_\theta} \leq C\|u\|_{\tilde{B}}^\rho \tag{4.10}$$

provided

$$n + \theta \geq \rho(n-3) + \rho(1 - (n-1)\tilde{\delta}) \tag{4.11}$$

$$\begin{aligned} 2 - \theta &= \rho - \rho(1 - (n-1)\tilde{\delta}) \\ n - 1 + 2\theta &\leq \rho(n-3) + 2\rho(1 - (n-1)\tilde{\delta}) \end{aligned} \tag{4.12}$$

Let $\bar{\theta} = \theta - \rho(1 - (n-1)\tilde{\delta})$. Then (4.11) is the same as (4.7) and holds for $\theta\bar{\gamma} \leq \gamma$ if $\epsilon = \rho(1 - (n-1)\tilde{\delta})$ is sufficiently small, i.e. $(n-1)\tilde{\delta} < 1$ is sufficiently close to 1. This is verified in the same way as (4.7).
If $n = 4$ *or* 5, then replace (4.10) by

$$\|f(u)\|_{A_\theta} \leq C\|u\|_{\tilde{B}}^{\rho-1}\|u\|_{L_\infty(H_2^1)} \tag{4.13}$$

and, again with $\bar{\theta} = \theta - \rho(1 - (n-1)\tilde{\delta}) = \theta - \epsilon$, we this time see that (4.13) holds if (4.9) is valid for ϵ small enough, which was verified in the previous proof.
Since, by the energy inequality and the STI estimate Proposition 2.6, in the non-endpoint case, the norms on the right hand side of (4.10) and (4.13) are all bounded by an increasing function $C_0(\cdot)$ of the energy of u, i.e. the $H_2^1 \times L_2$ of the initial data, we have

$$\|f(u)\|_{A_\theta} \leq C_0(\|u(0)\|_{L_\infty(H_2^1)}) \tag{4.14}$$

Next, by Proposition 4.1, $u \in L_2^{loc}(B_{p'}^\gamma)$. Thus with $B(T) = L_2(0, T; B_{p'}^\gamma)$ we get from (4.14) and Proposition 2.5 that for any finte $T > 0$,

$$\begin{aligned}\|u\|_{B(T)} &\leq C_1\|u_0\|_{L_\infty(H_2^1)} + C\|f(u)\|_{A_\theta(T)} \\ &\leq C_1\|u_0\|_{L_\infty(H_2^1)} + C_0(\|u(0)\|_{L_\infty(H_2^1)}) \\ &\leq C_2(\|u_0\|_{L_\infty(H_2^1)})\end{aligned}$$

. Hence $u \in B$, and the endpoint estimate for the (unique) finite energy solution of the NLKG is proved. $\square$

5 Proof of the main results

In this section we will prove Theorems 1 and 2. In order to do this we need a uniform decay result which we state and prove in the first part of this section.
As before u and v will denote finite energy solutions of the NLKG, u_0, v_0 the corresponding solutions of the Klein - Gordon equation (with the same initial data as u and v, respectively). We assume in this section, if nothing else is said, that $n \geq 3$ and $1 + \frac{4}{n} < \rho < \rho^* = 1 + \frac{4}{n-2}$, and that f satisfies conditions (A) and (B). A and A(I) will denote any of the spaces used in the formulation of the space time estimates in Proposition 2.6.
Let $E_0 = E_0(u_0)$. Then conservation of energy for the NLKG implies that (for $\rho < \rho^*$) $E(u(t)) = E(u)$ and $E_0(u_0)$ are equibounded. As before, X_e will denote the energy space with norm $\|\cdot\|_e = E_0(.)^{\frac{1}{2}}$. The norm of u_0 in $A(I)$ is continuous on the energy space, since

$$\|u_0 - v_0\|_{A(I)} \leq C\|u_0 - v_0\|_e$$

by the Strichartz estimates, Propositions 2.1 and 2.2.

5.1 Uniform decay in space-time

Lemma 5.1. *Let $a, \frac{1}{\epsilon}$ be positive (and large). Then there is a continuous and increasing function $b = b(a, \frac{1}{\epsilon}, E_0) > a$ such that*

$$\|u\|_{A(t \geq b)} \leq \|u_0\|_{A(t \geq a)} + \epsilon \tag{5.1}$$

In view of the comment on the continuous dependence of the $A(I)$ norm on the energy, we get the following uniform decay estimate in space-time:

Corollary 5.1. *Let $\epsilon > 0$. The there is a continuous function $b = b(\frac{1}{\epsilon}, u_0)$ on $\mathbf{R}_+ \times E$ such that*

$$\|u\|_{A(t \geq b)} \leq \epsilon \tag{5.2}$$

PROOF: [of Lemma 5.1] The proof consists of three basic steps:

a) $\|u\|_A \leq C(E_0)$

b) Let a, T and $\frac{1}{\epsilon}$ be positve (and large). Then there is a $b(a, T, \frac{1}{\epsilon}, E_0) > a$ increasing and continuous in each variable, and an interval $I = (t_* - 2T, t_*) \subseteq [a, b]$ such that

$$\|u\|_{A(I)} < \epsilon$$

c) Let $I_* = (t_* - 2T, \infty)$. Then, under the assumptions of b),

$$\|u\|_{A(I_*)} < \|u_0\|_{A(I_*)} + \epsilon \tag{5.3}$$

Since $a \leq t_* - 2T < t_* \leq b$, (5.1) follows from c).
Claim a) is the STI estimates in Proposition 2.6, since as mentioned, $E(u)$ *and* $E_0(u_0) = E_0$ are equibounded.
Next, a) implies b):
Let $A = L_{r'}(I; B^{s'}_{p'})$ and define

$$I_n = (a + 2t(n-1), a + 2Tn) \textit{ and } a_n = \|u\|^{r'}_{A(I_n)}$$

If $a_n \leq \epsilon^{r'}$ for some $n \leq N$, then b) follows. If this is not the case then $a_n \geq \epsilon^{r'}$ for $n = 1, .., N$, and $N\epsilon^{r'} \leq C(E_0)$, that is $N \leq \epsilon^{-r'}C(E_0)$. Thus $2TN \leq b - a - 2T$ with $b \geq 2T\epsilon^{-r'}C(E_0) + a + T$. This proves b).
We may then in a fairly standard fashion prove c):
Assume that t_{**} is the largest t for which (5.3) holds with I_* replaced by $I = (t_* - 2T, t)$. Clearly $t_{**} \geq t_*$. We may assume that t_{**} is finte, since otherwise there is nothing to prove. Let $t > t_{**}$. Then, applying (3.13), and using a)

$$\begin{aligned}
\|u\|_{A(I)} &\leq \|u_0\|_{A(I)} + (\int_T^\infty K(\tau)d\tau)C(E_0) + C(E_0)\|u\|^{(2-\theta)}_{A((t_*-2T,t_{**})} \\
&\quad + (\int_0^{t-t_{**}} K(\tau)d\tau)C(E_0)\|u\|_{A(t_{**},t)} \\
&\leq \|u_0\|_{A(I)} + C(E_0)(\int_T^\infty K(\tau)d\tau) \\
&\quad + C(E_0)\epsilon^{2-\theta} + C(E_0)(t - t_{**})^\alpha
\end{aligned} \tag{5.4}$$

By choosing $T,$ *and* ϵ large and small enough, respectively, we find that for $t - t_{**}$ small enough, we obtain the estimate (5.3) for some $t > t_{**}$. Here we used that $K(\tau) \leq C\tau^{\alpha-1}$, $\alpha > 0$, $0 < \tau < 1$, and that $2 - \theta > 1$. The estimate (5.4) contradicts the existence of a finite t_{**}, and the proof of c) is completed. □

5.2 Lipschitz and Hölder estimates for $\mathcal{E}_t$

We first provide a global, non-uniform Lipschitz estimate on $A(s'-\sigma;I)$ over compact intervals $I \subset\subset \mathbf{R}_+$ with Lipschitz constants depending on I (and the energy E). We then extend this to a uniform estimate on $A(s'-\sigma;\mathbf{R}_+)$ by use of Lemma 5.1.
Let, in the notation of Section 3, $s' \geq \sigma \geq 0$, $s' = \gamma$ and $\sigma > \bar{\gamma} - (\rho - 1)$ (sufficiently) close to this lower bound if $n > 6$ and $\sigma = 0$ if $n \leq 6$. By Proposition 2.3 and 2.6, and by the Lipschitz estimate (3.14) for $f(u)$ we then get,

$$\|u-v\|_{A(s'-\sigma;I)} \leq \|u_0 - v_0\|_{A(s'-\sigma;I)} + (\int_0^{|I|} K(\tau)d\tau)^\theta C(E_0)\|u-v\|_{A(s'-\sigma;I)}$$

Thus for $|I|$ small enough, depending only on the energy E_0,

$$\|u-v\|_{A(s'-\sigma;I)} \leq \|u_0 - v_0\|_{A(s'-\sigma;I)} \ , \ |I| < \epsilon(E_0).$$

Hence by translation invariance and energy conservation, for each bounded interval I there is a continuous function $C(I, E_0)$ such that

$$\|u-v\|_{A(s'-\sigma;I)} \leq C(I,E_0)\|u_0 - v_0\|_{A(s'-\sigma;I)}$$

Next, let $\bar{I} = \mathbf{R}_+ \setminus I = (2T, \infty)$. Write for $t \geq 2T$,

$$u(t) - v(t) = u_0(t) - v_0(t) + \int_0^T E_1(t-\tau)f(u(\tau))d\tau + \int_T^t E_1(t-\tau)f(u(\tau))d\tau$$

and then take the $A(s'-\sigma : \bar{I})$-norm of each term on the right hand side. Let C_1 be a constant only depending on A, i.e on r', s', n *and* δ. By Proposition 2.3 and the (uniform) STI estimate we get for the second term an estimate

$$C_1(\int_T^\infty K(\tau)d\tau)^\theta E_0^{\rho-1}\|u-v\|_{A(s'-\sigma;(T,\infty))} \tag{5.5}$$

and for the third term

$$C_1(\|u\|^{\rho-1}_{A(T,\infty)} + \|v\|^{\rho-1}_{A(T,\infty)})\|u-v\|_{A(s'-\sigma;(T,\infty))} \tag{5.6}$$

Let $\epsilon > 0$ such that $2\epsilon < \frac{1}{2}$.Then choose T depending on ϵ *and* E_0 such that by (5.5) and (5.6),

$$\|u-v\|_{A(s'-\sigma,\bar{I})} \leq \|u_0 - v_0\|_{A(s'-\sigma,\bar{I})} + \epsilon\|u-v\|_{A(s'-\sigma)} + \epsilon\|u-v\|_{A(s'-\sigma;(T,\infty))}$$

Thus

$$\begin{aligned}
&\|u-v\|_{A(s'-\sigma)} \leq \|u-v\|_{A(s'-\sigma;I)} + \|u-v\|_{A(s'-\sigma;\bar{I})} \\
&\leq C(I,E_0)\|u_0 - v_0\|_{A(s'-\sigma;I)} + \|u_0 - v_0\|_{A(s'-\sigma;\bar{I})} + 2\epsilon\|u-v\|_{A(s'-\sigma)}
\end{aligned}$$

and we conclude that

$$\|u - v\|_{A(s'-\sigma)} \leq C(E_0)\|u_0 - v_0\|_{A(s'-\sigma)} \tag{5.7}$$

Proposition 2.3 and Corollary 2.1, the Lipschitz estimate (3.14) and the STI estimate (Proposition 2.6), then give with our choice of $s' = \gamma$ and σ that

$$\begin{aligned}\|u - v\|_{H_2^{1-\sigma}} &\leq \|u_0 - v_0\|_{H_2^{1-\sigma}} + C_1(\|u\|_A^{\rho-1} + \|v\|_A^{\rho-1})\|u - v\|_{A(s'-\sigma)} \\ &\leq C(E)\|u_0 - v_0\|_{H_2^{1-\sigma}} \end{aligned} \tag{5.8}$$

This proves the first and, since $\bar{\gamma} - (\rho - 1) < \frac{1}{2} + \delta - 4\delta < \frac{1}{2}$, for σ sufficiently close to $\bar{\gamma} - (\rho - 1)$, also the third part of Theorem 1. To prove the Hölder estimate in Theorem 1, simply use (3.15) to replace this last estimate by

$$\begin{aligned}\|u - v\|_{H_2^1} &\leq \|u_0 - v_0\|_{H_2^1} + C_1(\|u\|_A^{\rho-1} + \|v\|_A^{\rho-\alpha})\|u - v\|_{A(s'-\sigma)}^{\alpha} \\ &\leq C(E)(\|u_0 - v_0\|_{H_2^1} + \|u_0 - v_0\|_{H_2^1}^{\alpha})\end{aligned}$$

This estimate completes the proof of Theorem 1.
Let us collect the relevant assumptions (besides the standard notation and assumptions of this paper and in particular of this section) in a theorem which using (5.8) slightly generalizes Theorem 1, parts (i) and (iii).

Theorem 4. *Let $\gamma, \bar{\gamma}$ be defined as in Section 3.3. Assume that $\gamma \geq \sigma \geq 0$, with $\sigma = 0$ for $n \leq 6$ and $\sigma > \bar{\gamma} - (\rho - 1)$. Then*

$$\|u - v\|_{H_2^{1-\sigma}} \leq C(E(u_0, v_0))\|u_0 - v_0\|_{H_2^{1-\sigma}}$$

where $C(\cdot, \cdot)$ is continuous on $\mathbf{R}_+^2$.
In particular, $\mathcal{E}_t : H_2^1 \times L_2 \mapsto H_2^{1-\sigma}$ is Lipschitz continuous.

Theorem 4 is a considerable improvment of Theorem 1 (iii) for small values of $n > 8$ and ρ close to ρ^*. For example, for $n = 9$ and $\rho \approx \rho^*$ we may take any $1 - \sigma < \frac{53}{56}$, rather than $1 - \sigma = \frac{1}{2}$.

It follows from the remarks in Section 3.4 and the above derivation that Theorem 4 can't be improved by the method used in this paper in the case of global non-uniform or local estimates when ρ is sufficiently close to ρ^* .

5.3 The existence of an everywhere defined scattering operator $\mathcal{S}$ on the energy space

Let u_- be a finite energy solution of the Klein - Gordon equation and define the mapping

$$A \cap E \ni u \mapsto \mathcal{F}u(t) = u_-(t) + \int_{-\infty}^{t} E_1(t - \tau)f(u(\tau))d\tau \in A \cap X_e$$

which is well defined by Proposition 2.3 and (3.13). Let $I = (-\infty, -T)$, $T > 0$. If u and v belong to $A \cap X_e$ the $\|u\|_{A(I)}$, $\|v\|_{A(I)}$ tend to 0 as $T \to \infty$. Now by (3.15)

$$\begin{aligned}
&\|\mathcal{F}u - \mathcal{F}v\|_{A(I)} + \|\mathcal{F}u - \mathcal{F}v\|_{E(I)} \\
&\leq \| \int_{-\infty}^{t} E_1(t-\tau)(f(u(\tau)) - f(v(\tau)))d\tau\|_{E(I)} \\
&\leq C\|f(u) - f(v)\|_{B(I)} \\
&\leq C(\|u\|_{A(I)}^{\rho-\alpha} + \|u\|_{A(I)}^{\rho-\alpha})\|u - v\|_{A(I)}^{\alpha}
\end{aligned}$$

By choosing T large enough, we can apply Tychonoff's fixed point theorem in a standard fashion for hyperbolic problems, i.e. problems with a finite speed of propagation (cf. [12] and Pecher [34]), and conclude that there is at least one $u \in A \cap X_e$ such that

$$u(t) = u_-(t) + \int_{-\infty}^{t} E_1(t-\tau)f(u(\tau))d\tau, \tag{5.9}$$

for $t \leq -T$.
Remark The method referred to can be scetched as follows: Assume that u_- has compactly supported initial data, so that u_- has support in a truncated cone. Let us consider the subset of $A \cap X_e$ of fuctions u with $supp(u) \subseteq supp(u_-)$. Then $supp(\mathcal{F}u) \subseteq supp(u_-)$, too. Tychonoff's fixpoint theorem now can be applied. Since the estimates are independent of the support of the functions involved, a compactness and strightforward limiting argument removes the assumption of compact supports, and (5.9) follows.
Now u will be a weak finite energy solution of the NLKG on I, and so unique. By uniqueness and conservation of energy, u can be extended to a solution of (5.9) on all of **R**.
Then define

$$u_+(t) = u_-(t) - \int_{-\infty}^{+\infty} E_1(t-\tau)f(u(\tau))d\tau$$

so that

$$u(t) = u_+(t) + \int_{-t}^{+\infty} E_1(t-\tau)f(u(\tau))d\tau$$

Then

$$\|u - u_+\|_{E(t \geq T)} \leq C\|u\|_{A(t \geq T}^{\rho} \to 0 \ as \ T \to \infty \tag{5.10}$$

and

$$\|u - u_-\|_{E(t \leq -T)} \leq C\|u\|_{A(t \leq -T}^{\rho} \to 0 \ as \ T \to -\infty \tag{5.11}$$

In conclusion, we have defined the scattering operator $\mathcal{S} : u_- \mapsto u \mapsto u_+$ on all of the energy space.
We also have

$$E(u) = E_0(u_+) = E_0(u_-) \tag{5.12}$$

To see this, notice that by linearity and energy conservation $\|u_\pm(t)\|_{H^1_2}$ is uniformly continuous on $\mathbf{R}$. By Sobolev embedding,

$$L_{q'}(L_{\rho+1}) \supset L_{q'}(B^{\frac{1}{2}}_{q'}),\ \delta_{q'} = \frac{1}{2} - \frac{1}{n+1}.$$

Thus $\|u_\pm(t)\|_{H^1_2} \in L_{q'}$, and the uniform continuity implies that $\|u_\pm(t)\|_{\rho+1} \to 0$ as $t \to \pm\infty$. By (5.10) and (5.11) then also $\|u(t)\|_{\rho+1} \to 0$ as $t \to \pm\infty$. Hence $E(u(t)) - E_0(u(t)) = \frac{1}{2}\|u(t)\|_{\rho+1} \to 0$ as $t \to \pm\infty$. Then (5.10) and (5.11) and energy conservation for solutions of the NLKG completes the proof of (5.12).

5.4 The Lipschitz continuity of the scattering operator

We will prove Theorem 2 using a slight variation of the proof of the Lipschitz estimates for the solution operator $\mathcal{E}_t$.We assume that $\rho < \rho^*$ and n satisfy the restrictions for the validity of the Lipschitz estimates in Theorem 2 (i).
Let $u_-,\ u,\ u_+$ and $v_-,\ v,\ v_+$ be sequences defined by the scattering operator on the energy space. Then we will prove that

$$\|u_+ - v_+\|_e \le C\|u - v\|_e \le C\|u_- - v_-\|_e \tag{5.13}$$

which proves Theorem 2 (i). The proof of the Hölder estimate is similiar to the proof of the Lipschitz estimate, and is omitted.
Proof of the second inequality in (5.13):
Let $\epsilon > 0$. Then Lemma 5.1 provides the exitstens of a $T = T(\epsilon, E(u), E(v))$ such that

$$\begin{gathered}\|u\|_{A(t\le -T)} + \|u\|_{A(t\ge T)} < \epsilon \\ \|v\|_{A(t\le -T)} + \|v\|_{A(t\ge T)} < \epsilon\end{gathered}$$

Now on any finite interval I we have (as in ...) that by (5.9)

$$\|u - v\|_{A(I)} \le C\|u_- - v_-\|_{A(I)}$$

with a continuous function $C = C(|I|, E(u), E(v))$. Let $I = (-T, T)$. Then

$$\|u - v\|_A = \|u - v\|_{A(I)} + \|u - v\|_{A(\mathbf{R}\setminus I)}.$$

The last term is, using (5.9), estimated by

$$\begin{aligned}&\|u - v\|_{A(\mathbf{R}\setminus I)} \le \\ &2\|u_- - v_-\|_{A(\mathbf{R}\setminus I)} + C(E(u), E(v))(\int_T^{infty} K(\tau)d\tau\|u - v\|_A + \epsilon\|u - v\|_A) \\ &\le 2\|u_- - v_-\|_A + 2C(E(u), E(v))\epsilon\|u - v\|_A\end{aligned}$$

taking if necessary a larger value of T. Hence

$$\|u-v\|_A \le C\|u_- - v_-\|_A + \epsilon'\|u-v\|_A$$

with $\epsilon' < 1$ for ϵ, T small enough depending possibly on $E(u)$, $E(v)$, equal to $E_0(u_-)$ and $E_0(v_-)$, respectvely, by(5.12). Thus by the Strichartz inequality,

$$\|u-v\|_A \le C\|u_- - v_-\|_e$$

with $C = C(E_0(u_-), E_0(v_-))$. But then Proposition 2.3 and the Lipschitz estimate for $f(u)$ gives via (5.9) that

$$\begin{aligned}\|u-v\|_E &\le \|u_- - v_-\|_e + C\|f(u) - f(v)\|_B \\ &\le \|u_- - v_-\|_e + C(\|u\|_A^{\rho-1} + \|v\|_A^{\rho-1})\|u-v\|_A \\ &\le \|u_- - v_-\|_e + C(E_0(u_-), E_0(v_-))\|u_- - v_-\|_e\end{aligned}$$

which proves the second inequality in (5.13).
Since,

$$u_+(t) - v_+(t) = u(t) - v(t) + \int_t^{infty} E_1(t-\tau)(f(u(\tau)) - f(v(\tau))d\tau$$

the first estimate in (5.13) follows in the same way from Proposition 2.3 and the Lipschitz estimate (3.14) for $f(u)$.
This completes the proof of Theorem 2. The Hölder estimate follows with obvious changes in the above proof.

6 Appendix: On uniform bounds for STI estimates in the energy space

We will make explicit the energy bounds of the STI estimates in the slightly more general situation treated in [8], mainly by referring to the proof in that paper. Since [8] is out of print, we will make references to Blomqvist thesis [5], which (intentionally,since the proofs were need to prove the decay results in [5]) gives a detailed account of the proof of the STI in [8], and is available online (see the references).This said, we turn to the estimates.
We keep the notation of section 5. In addition K denotes the kernel defined in section 2 after formula (2.4), where we assume that $\theta\delta > 0$ in (2.5), so that there is an $\alpha > 0$ such that

$$K(t) \le C \begin{cases} t^{-1+\alpha} & 0 < t < 1, \\ t^{-1-\alpha} & 1 \le t \end{cases} \tag{6.1}$$

We also denote, as in section 4, the M-fold convolution with K by $\mathcal{K}_M$ with kernel K_M. With u and u_0 the solutions of the NLKG and the Klein-Gordon equation, respectively, with the same initial data, we write for short $U(t) = \|u(t)\|_{B^{s'}_{p'}}$ and $U_0(t) = \|u_0(t)\|_{B^{s'}_{p'}}$. We also assume that$(r', p', s')$ to be a tripple satisfying (2.7) (or, slightly more restrictive, (2.8)). Then the energy bound in the STI (in the non-endpoint case) follows from the following Proposition.

Proposition 6.1. *Assume in the above notation that for some $\epsilon > 0$ and $\eta_0 > 0$ small enough,*

$$U(t) \le U_0(t) + \int_0^t K(t-\tau)U(\tau)^{1+\eta}d\tau, \ 0 \le |\eta| < \eta_0 \tag{6.2}$$

$$\int_0^t K_M(t-\tau)U(\tau)d\tau \le C(E_0(u_0)) \tag{6.3}$$

$$\sup_{t \ge t^*} \int_{t^*}^t K_M(t-\tau)U(\tau)d\tau < \epsilon, \ t^* \ge b(\epsilon, u_0) \tag{6.4}$$

where C and b are continuous functions on $\mathbf{R}_+$ and $\mathbf{R}_+ \times X_e$, where as before X_e is the energy space.
Then $U \in L_{r'}$ with a bound depending continuously on u_0 in the energy space.

Remark The estimate (6.4) is a consequence of

$$\sup_{t \ge t^*} \int_{t^*}^t K_M(t-\tau)U(\tau)d\tau < C\|u_0\|_{A(t \ge a)} + \epsilon, \ t^* \ge b_0(\epsilon, E_0(u_0), a) \tag{6.5}$$

with b_0 depending continuously on $\epsilon, E_0(u_0)$, and a. As the estimate of u_0 in $A(t \ge a)$ is continuous on the energy space (as noticed in the introduction to section 5), (6.4) follows from (6.5).
The estimates (6.2) and (6.3) follow from Lemma 3.1 and Lemma 4.2, respectively. We will come back to the estimate (6.4), or equivalently (6.5), after an indication of the proof of Proposition 6.1.

Proof: [scetched] For a detailed proof of the proposition, without the uniform energy bound, se [5], Chapter I:9, Theorem 9.1, p.33 ff. We will thus here only indicate how the uniform energy bound follows for STI estimate from the uniform bounds in estimates (6.2) through (6.5).
The proof is carried out by estimates depending on K (i.e. α), on the estimates (6.3) and (6.4), and in addition estimates of U_0 in $L_{r'}$ on $\mathbf{R}_+$ and on $\{t \ge a\}$ for a large, i.e estimates with bounds that are continuous functions of the energy of u_0.
In this way, using (6.2) it is proved step by step that

$$U \in L_{r'} + o(1)L_{\frac{r'}{j\eta}} \cap L_\infty, \ j = 1, ..J$$

where $(J-1)\eta < 1 < J\eta$, with the $o(1)$ factor is a continuouos function of the energy of u_0. This then completes the proof of the proposition. $\square$

We now turn to a scetch of the proof of (6.4) (or equivalently,(6.5)). The proof of (6.4) is based on two results.The first is Lemma 4.2, the second is an inequality due to Morawetz for solutions u of the NLKG

$$\int\int \frac{F(u(x,t))}{1+|x|} dxdt \leq C(E(u)) \tag{6.6}$$

from which Morawetz and Strauss [33] derived a crucial asymptotic result:
Let ϵ, T, $a > 0$. Then there exist a b depending continuously on ϵ, T, a and the energy $E(u)$, and an interval $I = (t^* - 2T, t^*) \subset [a, b]$ such that

$$\int\int_{R^n \times I} F(u(x,t))dxdt < \epsilon \tag{6.7}$$

(In fact, valid for u with compactly supported data, with b independent of the support of the data - this restriction can be shown to be disregarded in our application of the estimate).
Using (6.3) and the assumption (B) on f, strightforward estimates give (cf. [5] I:7, Lemma 7.2) with $I_* = (t^* - T,\ t^*)$,

$$\int_{I_*} K_M(t-\tau)\|u(\tau)\|d\tau < \epsilon\ for\ t^* \geq b \tag{6.8}$$

By a slightly complex convexity argument (see [5] I:7), deriving from estimates of

$$\int_{I_*} K * (\mathcal{K}_{M-1}U)^{1+\eta}$$

estimates of

$$\int_{I_*} K * (\mathcal{K}_M(U^{1+\eta})) = \int_{I_*} K_M * (U^{1+\eta})$$

in each step using bounds for t^* only depending, besides on ϵ, T, a and the energy $E(u)$, on η and the kernel K (or rather, α in (6.1)). Using these estimates we conclude (cf. [5] I:7, Lemma 7.7 and 7.8) that if

$$\int_{I_*} K_M * U_0 < \frac{1}{3}\epsilon,\ \int_{I_*} K_M * U < \epsilon,\ t^* \geq b$$

then

$$\int_{I_*} K_M * U^{1+\eta} < \epsilon^{1+\eta},\ t^* \geq b.$$

With a slight variation of the argument of the proof of c) in Lemma 5.1, this proves (6.4) using (6.2) (cf. [5] I:8)
This completes the scetch of the proof of the proposition.

References

[1] Bahouri, H., Gerard, P., High frequency approximation of solutions to critical nonlinear equations, Amer. J. Math. 121, (1999), 131 - 175.

[2] H. Bahouri and J. Shatah, Global estimate for the critical semilinear wave equation, Ann. Inst. H. Poincaré Anal. Non Linéaire,15 (1998), 783-789

[3] Bennet, C., Sharpley, R.: Interpolation of operators, Academic Press 1988.

[4] Bergh, J., Löfström, J.: Interpolation spaces, Berlin Heidelberg, New York: Springer 1976.

[5] Blomqvist, H.:On space-time means for solutions of nonlinear Klein-Gordon equations,Thesis 2003, Department of Mathematics, Chalmers University of Technology. Available online:http://publications.lib.chalmers.se/cpl/

[6] Brenner, P.: On $L_p - L_{p'}$ estimates for the wave equation, Math. Z. 145 (1975), 251-254.

[7] Brenner, P.:Space-time means and everywhere defined scattering operators for nonlinear Klein-Gordon equations, Math. Z. 186 (1984), 383-291.

[8] Brenner, P.: Space-time means and nonlinear Klein-Gordon equations, Report No. 1985-19, CTH/GU.

[9] Brenner,P., Kumlin, P.:On Lipschitz continuity for solution operators for wave equations with supercritical nonlinearities, Arch. Math. 74 (2000), 129 - 147.

[10] Brenner,P., Kumlin, P.:Nonlinear Maps between Besov- and Sobolev spaces, Annales de la Faculté des Sciences de la Université de Toulouse 19 (2010), 105-120

[11] Brenner, P. Thomée, V., Wahlbin, L.: Besov spaces and applications to difference methods for initial value problems. Lecture Notes in Mathematics, Vol. 434. Berlin Heidelberg, New York: Springer 1975.

[12] Brenner, P., von Wahl, W.: global classical solutions of nonlinear wave equations, Math. Z. 176 (1981), 87-121.

[13] Ginibre, J. Velo, G.: The global Cauchy problem for the non-linear Klein-Gordon equation, Math. Z. 189 (1985), 487-505.

[14] Ginibre, J. Velo, G.:Time decay of finite energy solutions of the nonlinear Klein-Gordon and Schrödinger equations, Annales de l'I.H.P.,A, 43 (1985), 399-442

[15] Ginibre, J. Velo, G.:Generalized Strichartz inequalities for the wave equaton, J. Funct. Anal. 133(1995), 50-68

[16] Ginibre, J. Velo, G.: Regularity of solutions of critical and subcritical nonlinear wave equations, Nonlinear Analysis 22 (1994), 1-19

[17] Ginibre, J. Velo, G.:Scattering Theory in the energy space for a class of nonlinear wave equations, Comm. Math. Phys. 123 (1989), 535-573

[18] Ginibre, G. Soffer, A. J. Velo,:The global Cauchy problem for the critical nonlinear wave equation. J. Funct. Anal. 110 (1992), 96130.

[19] Glassey, R. Tsusumi, M.: On uniqueness of weak solutions of solutions to semi-linear wave equations, Commun. Partial Differ. Equations 7 (1982), 153 - 195.

[20] Grillakis, M.: Regularity and asymptotic behavior of the wave equation with a critical nonlinearity, Ann. of Math. 132 (1990), 485-509.

[21] Grillakis, M.: Regularity for the wave equaiton with a critical nonlinearity, Comm. Pure Appl. Math. 45 (1992), 749-774.

[22] Heinz, E., von Wahl, W.: Zu einem Satz von F. E. Browder über nichtlineare Wellengleichungen, Math. Z. 141 (1975), 33-45.

[23] John, F.: Blow-up for quasilinear wave equation in three space dimensions, Comm. Pure Appl. Math. 33 (1980), 43-101.

[24] Kapitanski, L.: Weak and yet weaker solutions of semilinear Wave Equations, J.Functional Analysis,130(1995), 357-426

[25] Keel, M., Tao, T.: Endpoint Strichartz estimates, American J. Math. 120 (1998), 955-980

[26] Kumlin, P.: On mapping properties for some nonlinear operators related to hyperbolic problems, Göteborg 1985 (thesis).

[27] Kumlin, P.:Analyticity of the Scattering Operator for the Nonlinear Klein-Gordon Equation with Cubic Nonlinearity, Commun. Math. Phys. 144 (1992), 257-278

[28] Kumlin, P.: A Note on Maximal Rate of Decay for Solutions to Some Semilinear Wave Equations, Math. Z. 197 (1988), 465-481.

[29] Klainerman, S., Ponce, G.: Global small amplitude solutions to nonlinear evolution equations, Commun. Pure Appl. Math. 36 (1983), 144-141.

[30] Lebeau, G., Perte de regularité pour les equations d'ondes sur-critiques, *Bull. Soc. Math. France*, 133 (2005), 145 - 157.

[31] Lebeau, G., Nonlinear optics and supercritical wave equation, *Bull. Soc. Math. Liege*, 70 (2001), 267 - 306.

[32] Lindblad, H. , Sogge,C.D.: On existence and scattering with minimal regularity for semilinear wave equations,J. Functional Analysis.130 (1995), 357 - 426

[33] Morawetz,C.S., Strauss,A.W.Decay and Scatteringof solutions of nonlinear relativistic wave equation, Comm.Pure Appl. Math. 25 (1972), 1-31

[34] Pecher, H.: Local solutions of semilinear equations in H^{s+1}, Math.Methods in applied Sciences, 19 (1996), 145-170

[35] Pecher, H.: Low energy scattering for nonlinear Klein-Gordon equations, J. Functional Analysis, 63 (1985), 101-122

[36] Peetre, J.: Interpolation of Lipschitz operators and metric spaces, Mathematica (Cluj) 12:35 (1970), 325-334.

[37] Segal, I.E.: Nonlinear semigroups, Ann. of Math. 78 (1963), 339-364

[38] Segal, I.E.: The global Cauchy problem for a relativistic scalarfield with power interaction, Bull. Soc. Math. France, 92 (1963), 129-138

[39] Shatah, J., Struwe, M.: Regularity results for nonlinear wave equations, Ann. of Math. 138 (1993), 503-518.

[40] Shatah, J., Struwe, M.:Geometric Wave Equations,Courant Lecture Notes in Mathematics, 2nd ed., Courant Institute of Mathematical Sciences, New York.American Mathenmatical Society, Rhode Island, 2000.

[41] Strauss, A. W.:Decay and asymptotics for $\Box u = F(u)$, J.Funct.Anal. 2 (1968), 409-457.

[42] Strichartz, R. S.: A priori estimates for the wave equation and some applications, Journ. Funct.Anal. 5 (1970), 218-235.

[43] Struwe, M., On Uniqueness and Stability for Supercritical Nonlinear Wave and Schrödinger Equations, Int. Math. Res. Not. 2006, Art. ID 76737, 1-14

[44] Struwe, M., Globally regular solutions to the u^5-Klein-Gordon equations, Ann. Scuola Norm. Sup. Pisa, Ser. 4, 15 (1988), 495-513.

[45] Struwe, M., Semilinear wave equations, Bull. A. M. S. 261 (1992), 53-85.

[46] von Wahl, W.: Über die klassische Lösbarkeit des Cauchy-Problems für nichtlineare Wellengleichungen bei kleinen Anfangswerten und das asymptotische Verhalten der Lösungen, Math. Z. 114 (1970), 281-299.