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Note on modeling of fatigue damage rates for non-Gaussian stresses

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Note on modeling of fatigue damage rates for non-Gaussian stresses

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Abstract

This note reviews means for fatigue damage rates estimation using scaled Laplace distributed loads. The model is suitable for description of stresses containing transients of random amplitudes and locations. Moment method to estimate model parameters is given. Explicit formulas to compute rainflow damage rate as a function of excess kurtosis is presented. Laplace model is used to describe variability of forces measured at some location on a cultivator frame is presented. Validation of the model and uncertainty analysis in fatigue damage predictions is given.
Keywords: rainflow cycles, Laplace moving averages, cultivator loads, time scaled loads.

1 Introduction

Stochastic modeling of loads and responses is dominated by stationary Gaussian processes. Much is known about those, and well-developed numerical tools to compute probabilities of interests are available, see e.g. [5]. However, many of the environmental loads that act on ground vehicles, wind mill blades and ships in heavy seas are far from being Gaussian. Nevertheless, Gaussian models are often used, and this sometimes leads to serious underestimation of risks for fatigue or extreme responses. A popular method to compensate for non-Gaussianity is to memoryless transform a Gaussian process to achieve "correct", skewness and kurtosis, marginal distribution or level crossing intensity, see e.g. [14], [12] and [3] for more recent paper. However there are situations when transformed Gaussian models are not available for example when stresses contains transients. Transients are often caused by impulse loads, e.g. when ground vehicle passes a pothole, ship experiences slamming in rough waves or when a cultivator tine hits a stone in a field.

Durability applications of components often requires a customer or market specific load description. Hence one is interested in models that are capable of describing variability of loads with a relatively small number of parameters. These can be then used to describe the long term loading by means of a distribution of the parameters values in a given market or encountered by specific customers. For stationary non-Gaussian responses with transients Laplace models, recently introduced in [1], [7], are a possible alternative to the transformed Gaussian modeling. Laplace models are moving averages of Laplace distributed white noise and hence named as Laplace moving averages (LMA). (Note that LMA models contain Gaussian processes in the limit.)

Power spectral density (psd) is an important characteristics of stationary responses. For stationary Gaussian responses the fatigue damage rate is a function of psd alone. Even for LMA psd remains an important characteristics, however it in general does not determine the damage rate completely. In this note a very simple, yet often used, parametric model of psd is used

$$S_a(\omega) = \sigma^2 \, a \, S(a\omega) \qquad a > 0, \tag{1}$$

where $\int S(\omega)\,d\omega = 1$ and σ^2 is the variance of the response (may depend on a). LMA responses with psd given in (1) can be written as

$$X_a(t) = \sigma \, X(t/a), \tag{2}$$

where $X(t) = X_1(t)/\sigma$, having psd $S(\omega)$, is a time and scale normalized LMA. The psd (1) and process (2) have found applications in road roughness classifications, where a is a velocity a vehicle travels while $S(\omega)$ depends on the linear filter that has been used to model responses and the spectral properties of a road profile, see [6]. In numerical examples in Sections 4 an 5 LMA (2) defined by psd $S(\omega) = 0.5 \exp(-|\omega|)$, which found application to model variability of measured stresses on cultivator frame [15], is used.

The note is organized as follows. In Section 2 the damage index is introduced. In Section 3 a Gaussian process is represented as a moving average and then generalized to LMA. Further in that section estimation of parameters defining LMA is discussed and means to evaluate the damage indexes proposed. Section 4 contains example illustrating theory presented in Section 3. Finally, in Section 5, LMA is applied to predict damage indexes in measured stresses on a cultivator frame. The model is validated and uncertainty analyzes presented.

2 Damage index

Let $s(t)$, $0 \leq t \leq T$, be measured or simulated stress. In this note the fatigue damage caused by s is measured by means of the rainflow damage rate computed in the following two steps. First rainflow ranges h_i^{rfc} in $s(t)$, are found, then the rainflow damage is computed according to Palmgren-Miner rule [9], [8], viz. $D_\beta(s) = \sum (h_i^{rfc})^\beta$, see also [11] for details of this approach. The damage index is defined as the limit of damage rate as observation length T increases without bounds, viz.

$$d_\beta(s) = \lim_{T\to\infty} \frac{D_\beta(s)}{T}. \tag{3}$$

Often stresses vary in an unpredictable way and are conveniently modeled by means of random processes. Then $s(t)$ is called a realization of the process $S(t)$, say. If the stationary process S is ergodic, what is often assumed, than the damage index is independent of the particular realization of the process and

$$d_\beta(S) = \lim_{T\to\infty} \frac{D_\beta(S)}{T} = \lim_{T\to\infty} \frac{\mathsf{E}\left[D_\beta(S)\right]}{T}, \tag{4}$$

where for a random variable Z, $\mathsf{E}[Z]$ is its expected value. In other words damage index is the expected increase of damage in time unit. For any stationary process $X_a(t)$, defined by (2),

$$d_\beta(X_a) = \sigma^\beta a^{-1} d_\beta(X). \tag{5}$$

If X is Gaussian having a psd $S(\omega)$, then X_a is ergodic and $d_\beta(X)$ depends only on the psd. There are many approximations of damage indexes for Gaussian loads proposed in the literature, see e.g. [4] for comparisons of different approaches.

3 Laplace Moving Average (LMA) process

In this section we review some facts about LMA modeling. In Section 3.1 the Gaussian moving averages will be presented. These are then generalized to symmetrical LMA processes in Section3.2 and to skewed LMAs in Section 3.3. In Sections 3.4 and 3.5 some aspects of fitting LMA to data are discussed.

3.1 Gaussian Moving Average (GMA) process

A zero mean stationary Gaussian process is completely defined by its spectral density and thus any probability statement about properties of Gaussian loads can be in principle expressed by means of the spectral density. This is not always practically possible and hence Monte Carlo (MC) methods are often employed to estimate probabilities of interest. There are several ways to generate Gaussian sample paths. The algorithm proposed in [13] is often used in engineering. It is based on the spectral representation of a stationary process. Here we use an alternative way to generate Gaussian processes employing moving averages of a Gaussian white noise. The method will be extended to LMA by simply replacing the Gaussian noise by Laplace distributed white noise, see e.g. [1], [6], for more details.

Roughly speaking a moving average process is a convolution of a kernel function $g(t)$, say, with a infinitesimal "white noise" process having variance equal to the discretization step, say dt. Consider a kernel function g, which is normalized so that its square integrates to one. Then the standardized Gaussian moving average (GMA) with mean zero and variance one can be written as

$$X(t) = \int_{-\infty}^{+\infty} g(t-u)\, dB(u) \approx \sum_i g(t-t_i) Z_i \sqrt{dt}, \tag{6}$$

where $B(t)$ is a Brownian motion, Z_i's are independent standard Gaussian variables, while dt is the discretization step, i.e. $\Delta B(t_i) = B(t_i + dt) - B(t_i) = \sqrt{dt} Z_i$. A choice of appropriate length of the increment dt is related to smoothness of the kernel.

In order to get the Gaussian process with a desired spectral density one has to use an appropriate kernel g which has to satisfy the following equality

$$S(\omega) = \frac{1}{2\pi} \mathcal{F}g(\omega) \mathcal{F}g(\omega)^*, \tag{7}$$

where $\mathcal{F}g(\omega)$ stands for the Fourier transform of g, while z^* is the complex conjugate of z. Obviously for fixed spectrum $S(\omega)$ Eq. (7) does not have unique solution. However if one limits to symmetrical kernels, i.e. $g(-t) = g(t)$ then the spectrum $S(\omega)$ of $X(t)$ uniquely defines the kernel g since $\mathcal{F}g(\omega)$ is real valued and hence

$$\mathcal{F}g(\omega) = \sqrt{2\pi\, S(\omega)}. \tag{8}$$

For Gaussian processes any kernel satisfying (7) will define the same GMA. This is not true for LMA.

3.2 Symmetrical LMA

Approximately, LMA is obtained by replacing constant standard deviation of the Gaussian noise $\sqrt{dt}$ by random std $\sqrt{K_i}$, where K_i are iid. gamma distributed variables with shape parameter dt/ν and scale ν, $\nu > 0$, viz.

$$X(t) \approx \sum_i g(t-t_i) Z_i \sqrt{K_i}. \tag{9}$$

Variables $\sqrt{K_i}\, Z_i$ have the generalized symmetric Laplace distribution, see [7]. As dt tends to a zero the limiting (in distribution) process is the Laplace moving average (LMA) given by

$$X(t) = \int_{-\infty}^{+\infty} g(t-u)\, d\Lambda(u), \tag{10}$$

where $\Lambda(t)$ is the Laplace motion, see [2] for further details. Note that when ν decreases to zero LMA converges to GMA. MATLAB code to generate LMA/GMA can be found in [6].

3.3 Asymmetrical LMA

The LMA process defined above is symmetrical having skewness zero. Skewed LMA, defined in this section, will have asymmetrical (skewed) distribution, i.e. its values are asymmetricaly distributed around the mean. Further, if the kernel g satisfying (7) is asymmetrical, i.e. $g(-t) \neq g(t)$ for some t, then LMA is time asymmetrical, also called time irreversible. (Note that GMA are always value symmetrical and time reversible.) In the following skewed LMA will be introduced.

Using iid. variables K_i, introduced in the previous section, a skewed (zero mean variance one) LMA process $X(t)$ is approximated by

$$X(t) \approx \sum_i g(t-t_i) \left(\sqrt{(1-q^2)K_i}\, Z_i + \frac{q}{\sqrt{\nu}} (K_i - dt) \right), \quad -1 \le q \le 1. \tag{11}$$

Obviously if relative skewness parameter $q = 0$ then the distribution of $X(t)$ is symmetric and otherwise it is skewed. The process $X(t)$ given in (11) has zero mean and variance one, and, as dt tends to zero, the skewness

$$sk = \sqrt{\nu}\, q(3 - q^2) \int_{-\infty}^{+\infty} g(t)^3 \, dt, \tag{12}$$

and the excess kurtosis

$$\kappa_e = 3\nu(2 - (1 - q^2)^2) \int_{-\infty}^{+\infty} g(t)^4 \, dt. \tag{13}$$

3.4 Fitting LMA to data - moments method

Let consider standardized (zero mean variance one) LMA process $X(t)$. Suppose that psd of $X(t)$ and a kernel $g(t)$ satisfying (7) has been estimated by some method. (Selection of g is a difficult problem which will not be discussed in this note.) Next, in order to define X one also needs to estimate relative skewness q and scale ν. This can be done if skewness and excess kurtosis of X are available. The two parameters are functions of q, ν and the kernel g given in (12-13). The function can be inverted as will be shown next.

First consider a function

$$F(x) = \frac{x^2(3 - x^2)^2}{3(2 - (1 - x^2)^2)} \frac{\left(\int_{-\infty}^{+\infty} g(x)^3 \, dx\right)^2}{\int_{-\infty}^{+\infty} g(x)^4 \, dx}, \qquad 0 \le x \le 1.$$

The function increases from zero to

$$F_{\max} = \frac{2}{3} \frac{\left(\int_{-\infty}^{+\infty} g(t)^3 \, dt\right)^2}{\int_{-\infty}^{+\infty} g(t)^4 \, dt}.$$

The parameter q is estimated by solving the following equation

$$\frac{sk^2}{\kappa_e} = F(|q|).$$

The unique solution exists only if $\kappa_e > 0$ and $\frac{sk^2}{\kappa_e} \le F_{max}$.

Then, the scale parameter ν can be estimated using (13), viz.

$$\nu = \frac{\kappa_e}{3(2 - (1 - q^2)^2)} \frac{1}{\int_{-\infty}^{+\infty} g(t)^4 \, dt}. \tag{14}$$

3.5 Estimating X_a defined by (2).

In the previous subsection means to estimate parameters of LMA process $X(t)$ were presented. The process X has variance one and psd $S(\omega)$. It has a kernel $g(t)$, which solves (7), the scale ν and the relative skewness q chosen in such a way that skewness and kurtosis of X is equal to observed values of these parameters in the response, see (12-13). Since $X_a(t) = \sigma X(t/a)$ hence in order to define X_a one needs to estimate variance σ^2 and the scale factor a . In this note σ^2 is estimated by the observed variance while the scale factor a is estimated by requiring that, in average, X_a crosses the mean level as frequently as it is observed in data.

More precisely, let μ^{obs} be the observed mean stress crossing rate while $\mu(\nu, q)$ be the mean level crossings rate by X. Since the intensity of mean level crossings by X_a process is equal to $\mu(\nu, q)/a$ hence a is estimated by

$$a = \mu(\nu, q)/\mu^{obs}. \tag{15}$$

The intensity $\mu(\nu, q)$ could be estimated by evaluating Rice's formula [10] or by means of MC methods.

Next some simple properties of X_a will be given. The process ihas variance σ^2 and a kernel

$$g_a(t) = a^{-1/2} g(t/a). \tag{16}$$

Since $\mathcal{F}g_a(\omega) = a^{1/2}\mathcal{F}g(\omega a)$, $X_a(t)$ has psd $S_a(\omega)$ given in (1). It is easy to see that X_a has the same relative skewness q as the normalized process X while the scale ν_a, say, of X_a satisfies $\nu_a = a\,\nu$. Obviously X and X_a have the same skewness and kurtosis.

Finally, since LMA is ergodic, see [2], the damage index of X is a function of kernel kernel g, relative skewness q and scale ν and damage exponent β, viz.

$$d_\beta(X) = f_\beta(\nu, q), \tag{17}$$

and hence the damage index of $X_a(t)$ is given by

$$d_\beta(X_a) = \sigma^\beta\, a^{-1} d_\beta(X) = \sigma^\beta\, a^{-1} f_\beta(\nu, q). \tag{18}$$

4 Example - LMA with exponential psd

In this section formulas given in Sections 3.4 and 3.5 by means of LMA having the following psd

$$S_a(\omega) = 0.5\sigma^2\, a \exp(-a|\omega|), \qquad a > 0. \tag{19}$$

4.1 Kernel $g_a(t)$

For an exponential psd there is a single parameter family of kernels for which moving average process will have the spectrum (19). The kernels are defined by their Fourier transforms

$$\mathcal{F}g_a(\omega; b) = \sqrt{0.5a}\, e^{-0.5a|w|\cdot(1+i\,b\,\mathrm{sgn}(w)\ln(|w|)2/\pi)}, \qquad -1 \le b \le 1, \tag{20}$$

where $i = \sqrt{-1}$ and $\mathrm{sgn}(x)$ is the sign of x. For $b = 0$ the kernel is symmetrical and given by

$$g_a(t) = \frac{2}{\sqrt{\pi}} \frac{a^{-1/2}}{1 + (2t/a)^2}. \tag{21}$$

For $b \neq 0$ kernels are asymmetrical. In the following two kernels will be used; symmetrical g ($a = 1$, $b = 0$ in (20)) and the "most" asymmetrical g defined by parameters $a = 1$, $b = -1$. The kernels are shown in Figure 1 as solid, dashed lines, respectively.

4.2 The parameters q, ν

The relative skewness q and the scale ν are evaluated by inverting the relations (12-13). For LMA model with symmetrical kernel (21) the equation system reads

$$\begin{aligned} \frac{sk^2}{\kappa_e} &= \frac{0.3q^2(3-q^2)^2}{(2-(1-q^2)^2)}, \qquad -1 \le q \le 1, \\ \kappa_e &= \nu\,(2-(1-q^2)^2)\frac{7.5}{\pi}. \end{aligned} \tag{22}$$

Now for the asymmetric kernel, $a = 1$, $b = -1$, (12-13) become

$$\begin{aligned} \frac{sk^2}{\kappa_e} &= \frac{0.3q^2(3-q^2)^2}{(2-(1-q^2)^2)}, \qquad -1 \le q \le 1, \\ \kappa_e &= 2\,\nu\,(2-(1-q^2)^2). \end{aligned} \tag{23}$$

One can see that q, ν estimates depend on the chosen kernel g. For example consider a stress having excess kurtosis $\kappa_e = 1$ and skewness $sk = 0$, and hence $q = 0$. Then, time symmetrical LMA ($b = 0$) will have the scale $\nu = \pi/7.5$ while the time asymmetrical LMA, having kernel (20) with $a = 1, b = -1$, will have the scale $\nu = 1/2$.

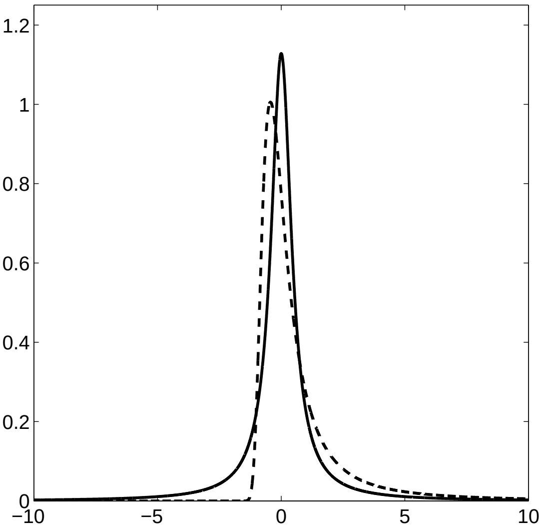

Figure 1: Two kernels having Fourier transform (20); symmetrical kernel $g(t)$, $a = 1, b = 0$ (solid line); asymmetrical kernel, $a = 1, b = -1$ (dashed line).

4.3 The damage index $d_\beta(X)$

In this section dependence of the damage index on the choice of kernel will be illuminated. As mentioned before LMA is an ergodic process and hence the damage index is a deterministic function of model parameters and the kernel, see (18). Here MC study and non linear regression is used to estimate the function. Only the case $\beta = 3$ is considered here.

For the exponential spectrum the estimated damage indexes $d_3(X) = f_3(\nu, q)$ for $q = 0, 0.3, 0.4$ and $b = 0, -1$, are presented in Figure 2. The fitted regressions for symmetrical kernel ($b = 0$) are

$$\begin{aligned} f_3(\nu, 0) &\approx 4.84 + 0.06\nu + 8.32\nu^{1/2} - 5.15\nu^{1/3}, && (24)\\ f_3(\nu, 0.3) &\approx 4.84 + 0.03\nu + 9.28\nu^{1/2} - 6.58\nu^{1/3}, && (25)\\ f_3(\nu, 0.4) &\approx 4.84 + 0.93\nu + 1.63\nu^{1/2} + 0.16\nu^{1/3}, && (26)\end{aligned}$$

while for asymmetrical kernel ($b = -1$) the damage indexes are

$$\begin{aligned} f_3(\nu, 0) &\approx 4.84 + 0.37\nu + 3.80\nu^{1/2} - 2.23\nu^{1/3}, && (27)\\ f_3(\nu, 0.3) &\approx 4.84 - 0.02\nu + 6.46\nu^{1/2} - 4.88\nu^{1/3}, && (28)\\ f_3(\nu, 0.4) &\approx 4.84 + 0.14\nu + 4.49\nu^{1/2} - 3.10\nu^{1/3}. && (29)\end{aligned}$$

The regression lines are shown in Figure 2. One can see that LMA with symmetrical kernel is more damaging than the LMA with asymmetrical kernel.

5 Modeling cultivator loads

In this section LMA with exponential spectrum is used to model damage accumulation in a frame of a cultivator working in a sandy soil. It is assumed that the model is sufficiently accurate if the observed damage index (estimated from measured stresses) differs by less than factor two from the "theoretical" damage index (estimated from the model).

The data set consists of 12 measured multi-axial loads. Measurements are about 70 seconds long, sampled with 500 Hz frequency. Loads are six dimensional, three forces and three moments. Here only force F_z will be modeled. In Figure 3 (left plot) fitted exponential psd (19) is compared with the

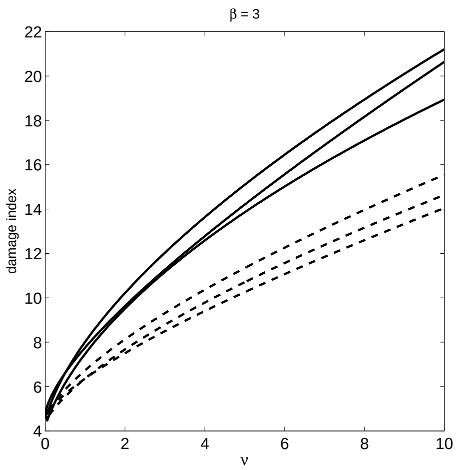

Figure 2: Examples of the regression fits for $f_\beta(\nu, q)$, $\beta = 3$, as function of ν, for the symmetrical kernel (24-26) solid lines and dashed lines for time asymmetrical kernel (27-29). The relative skewness $q = 0, 0.3, 0.4$.

"empirical" psd. The empirical psd differs significantly from the fitted exponential psd, which basically reflects a general decrease of signals energy with increase of frequency. In Figure 3 (right plot) a part of measured force $F_z(t)$ is compared with corresponding simulation of fitted LMA. The simulated force seams to vary in a similar manner as the measured one. Accuracy of prediction of damage indexes by means of LMA is investigated next.

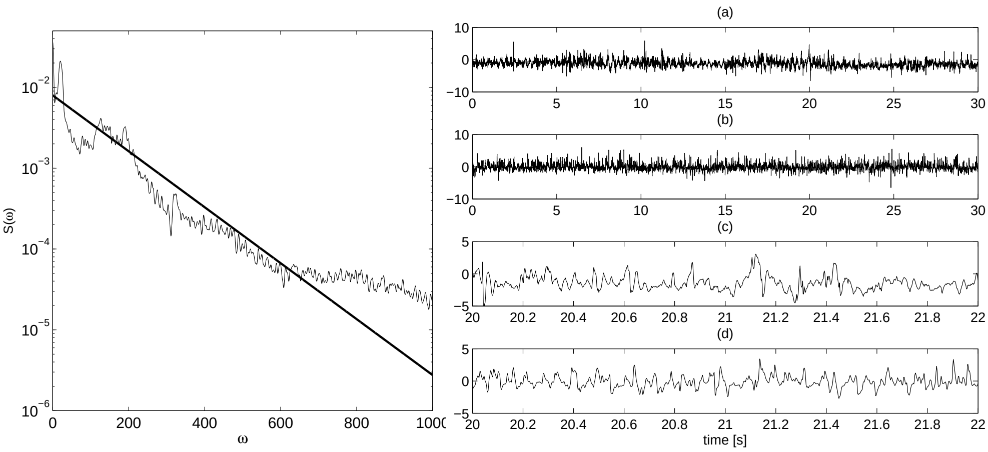

Figure 3: *Left:* Logarithms of empirical psd of $F_z(t)$ (irregular line) and fitted exponential psd (solid line). *Right:* Plot (a) - one minute of measured force on a cultivator frame. Plot (b) - one minute of simulated LMA model of the force. Plot (c) - zoomed plot (a). Plot (d) zoomed plot (b).

5.1 Accuracy of LMA model - simulation study

This section is devoted to investigation of availability of LMA with exponential psd to predict amount of fatigue damage accumulated in cultivator frame caused by experienced environmental loads. The twelve measured responses are modeled separately. For each of measured forces three models are fitted. Two of them are LMA with symmetrical and asymmetrical kernels, i.e. $b = 0, -1$ in (20), respectively, while the third is Gaussian model. In the following simulation study one is investigating accuracy of the damage index predictions for the three models. The simulation study is performed as follows.

The total length of the measured signals is about 14 minutes. Fifty signals of that length were simulated using the fitted models and relative indexes computed. (Note that parameters in LMA, GMA models change each 70 seconds.) The relative damage index is defined as a ratio between simulated index and the observed one. Obviously, relative index exceeding one means that a model is conservative while relative index below one means that a model is underestimating the damage. Next 50 relative damage indexes were averaged and results are given in Table 1. In the table one can see that both LMA models show good accuracy. LMA with symmetrical kernel is somewhat conservative. The GMA model severely underestimate the damage for damage exponent $\beta = 5$.

Each 14 minutes long simulation consists of twelve 70 seconds long simulations of "local" LMA models. In total 600 LMA samples were generated and damage indexes evaluated. The 600 damage indexes are presented in Figures 4 and 5 for damage exponents $\beta = 3, 5$, respectively. The indexes are presented as irregular lines. The solid lines are the observed damage indexes for the corresponding measured stresses. The solid lines jumps between twelve levels since there were only 12 observed damage indexes. In the figures one can see that most damaging loads are very well modeled by LMA, while errors are larger for the less damaging loads. The models are more accurate for the higher damage exponent as can be seen by comparing the plots in the two figures. Further, one can see that LMA with asymmetrical kernel ($b = -1$) is less conservative than LMA with symmetrical kernel ($b = 0$).

Averages of relative damage indexes		
damage exponents	$\beta = 3$	$\beta = 5$
LMA, $b = 0$	1.28	1.32
LMA, $b = -1$	1.08	0.92
GMA	0.78	0.25

Table 1: Averages of 50 relative damage indexes for the proposed three type of models.

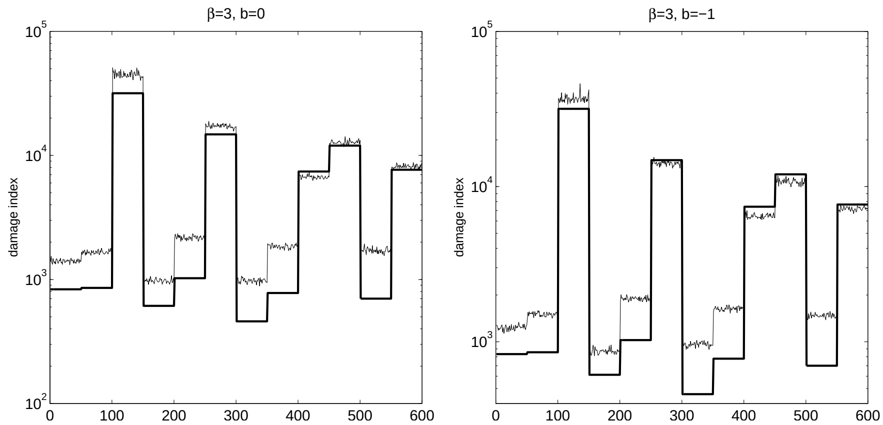

Figure 4: Six hundred simulated damage indexes with damage exponent $\beta = 3$, the irregular line. The corresponding observed damage indexes, the solid line. *Left:* LMA with symmetrical kernels $b = 0$. *Right:* LMA with asymmetrical kernels $b = -1$.

5.2 Uncertainty analysis

Performing uncertainty analysis when predicting the accumulated fatigue damage is important. The analysis can help in planning measuring campaigns to limit prediction errors and biases. In this section

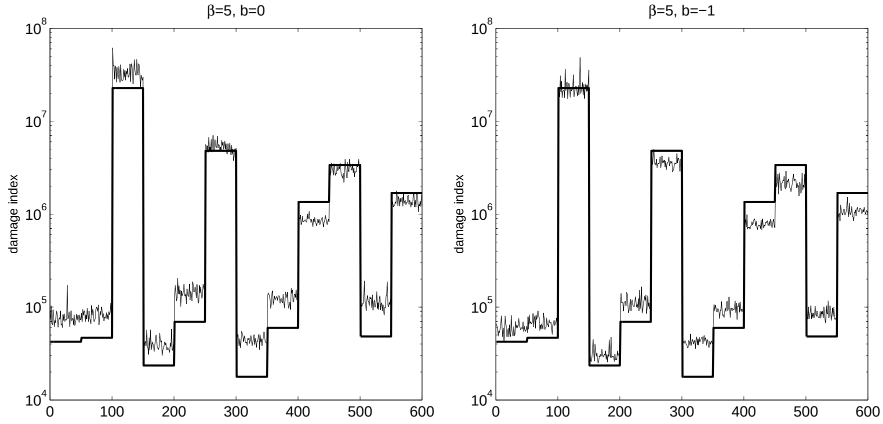

Figure 5: Six hundred simulated damage indexes with damage exponent $\beta = 5$, the irregular line. The corresponding observed damage indexes, the solid line. *Left:* LMA with symmetrical kernels $b = 0$. *Right:* LMA with asymmetrical kernels $b = -1$.

only two aspects of the analysis will be considered. Firstly main factors influencing the damage predictions will be found and their variability studied. Secondly, statistical uncertainties in damage index estimation due to limited length of measured load will be investigated. In order to make presentation short and transparent several parameters will be kept constant and their uncertainty will be neglected. These parameters will be presented next.

Assumptions: Measured loads can be modeled by means of LMA having exponential spectrum and symmetrical kernels, i.e. $b = 0$. (Results presented in Figures 4 and 5 indicates that b is not an influential factor.) It is assumed that stresses have constant skewness, and hence the average relative skewness parameter is used, viz. $q = 0.4$. (Results presented in Figure 2 indicate that skewness of response is not very influential factor on damage index.) The damage exponent β is set to be three. (This value is often used for welded structures.)

Main factors: Since LMA models with an exponential spectrum and symmetrical kernel ($b = 0$), the skewness parameter $q = 0.4$, are used to model measured response the theoretical damage index $d_3(X_a)$, given by (18), can be written in an explicit algebraic form

$$\begin{aligned} d_3(X_a) &= \sigma^3\, a^{-1}(4.84 + 0.3\kappa_e + 0.93\kappa_e^{1/2} + 0.11\kappa_e^{1/3}) \\ &= \sigma^3\, a^{-1}\, f(\kappa_e). \end{aligned} \tag{30}$$

(Equations (26) and (23) were used to derive (30). Three main factors influencing the damage index are σ^3, a^{-1} and $f(\kappa_e)$. Variability of the factors will be studied next.

Uncertainty of main factors: Using the twelve measure signals a sample of twelve "observed" values of the factors were estimated. In Table 2 the variability of the factors is presented in form of means and variances of natural logarithms of the factors. Sizes of statistical errors of estimates of the observed factors were evaluated. The statistical uncertainties were negligible. Obviously the estimates of means and variances of the logarithms of factors are statistically uncertain since these are based only on twelve observations.

The results shown in the table indicate that the uncertainty of factor $f(\kappa_e)$ is negligible and hence the parameter ν can be kept constant in all twelve LMA models. The variability of the factor a^{-1} is not negligible and contributes to the average damage index by about 7%. (Here, as it is customary in such rough uncertainty analysis, it is assumed that the factors are independent and log normally distributed.) Since LMA model is in average about 30% conservative, choosing a constant value for the factor a^{-1} would decrease the conservatism of LMA to 20%. The conclusion is that only variability of standard deviation σ should not be neglected and suitably modeled, for example by means of a log normal distribution.

Statistical uncertainties of damage index estimates: As mentioned above the statistical uncertainty in estimated values of factors in (30) is negligible. What remains is to investigate statistical uncertainties of damage index estimates caused by finite length of measured signals. This will be done as follows.

Note that $d_3(X_a)$ (30) is the expected damage index. The formula was derived by means of theoretical reasoning and a very long simulations of LMA. Now the 70 second long simulation may not be long enough for accurate estimation of the index. In order to estimate this statistical uncertainty the relative damage index, i.e. a ratio between estimated damage index from 70 seconds long sample of LMA and the theoretical damage index (30), will be used. This relative damage index k, say, is defined by

$$d_\beta(x_a) = k \cdot d_\beta(X_a), \tag{31}$$

where x_a is a simulated 70 second long sample of X_a.

Employing 600 simulations presented in the previous section the 600 values of the factosr k were evaluated. In Table 2, last row, the mean and the variance of $\ln(k)$ are given. One can see that variability of the factor is negligible, however the factors are in average smaller than one indicating a bias. Conclusion is that measuring only for 70 seconds may lead to underestimation of the observed damages by about 9%. The reason for this is that the large transients that contributes largely to the damage occur seldom and can be missed in 70 second long measured signals. This decreases conservatism of the LMA model to about 10% in average. If this is too low value one could slightly increase the conservatism of LMA by using symmetrical (skewness zero $q = 0$) LMA models. This is recommendable since the goal of the modeling is to propose sufficiently accurate models with minimal number of parameters that can change with loading environment.

Factors (30-31)	mean of log factor	variance of log factor
σ^3	1.40	1.462
a^{-1}	4.95	0.132
$f(\kappa_e)$	2.02	0.029
k	-0.18	0.005

Table 2: Estimates of mean and variance of natural logarithms of factors defined in equations (30 - 31).

6 Conclusions

Laplace moving averages has been successfully used to model variability of measured stresses containing transients. A hierarchical LMA model having random variance σ^2 and exponentially decaying spectrum has been proposed for the measured stresses on a cultivator frame. The parameters needed to define the model are; mean and variance of $\ln(\sigma^2)$; intensity of mean level crossings, giving the relative time scale a, and the excess kurtosis κ_e. The model has been validated. It was found that statistical errors in estimates of the parameters are negligible. However the relative short measuring period (70 seconds) caused systematic underestimation of damage indexes by about 9%. A possible explanation is that transients with large amplitudes are rare and can be missed in short measured records.

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References

[1] S. Åberg, K. Podgórski, and I. Rychlik, *Fatigue damage assessment for a spectral model of non-Gaussian random loads*, Prob. Eng. Mech. **24** (2009), pp. 608-617.

[2] S. Åberg and K. Podgórski, *A class of non-Gaussian second order random fields*, Extremes **14** (2010), pp. 187-222.

[3] Azais, J-M., Déjean, S., Leon, J. R., Zwolska, F. *Transformed Gaussian stationary models for ocean waves*, Probabilistic Engineering Mechanics, **26**, (2011), pp. 342-349

[4] Bengtsson, A. and Rychlik, I. *Uncertainty in fatigue life prediction of structures subject to Gaussian loads*, Probabilistic Engineering Mechanics, **24** (2009), pp. 224-235.

[5] P.A. Brodtkorb, P. Johannesson, G. Lindgren, I. Rychlik, J. Rydén, and E. Sjö, *WAFO - a Matlab toolbox for analysis of random waves and loads* in *Proc. 10th Int. Offshore and Polar Eng. Conf., Seattle*, USA, Vol III, 2000, pp. 343-350.

[6] K. Bogsjö, K. Podgórski and I. Rychlik *Models for road surface roughness* Vehicle System Dynamics, **50** (2012), pp. 725-747.

[7] Kotz, S. Kozubowski, T.J. and Podgórski, K. *The Laplace distribution and generalizations: A revisit with applications to communications, economics, engineering and finance.* Birkhaüser, Boston, 2001.

[8] Miner, M.A. *Cumulative damage in fatigue*, J. Appl. Mech. **12** (1945), pp. A159–A164.

[9] A. Palmgren, *Die Lebensdauer von Kugellagern*, VDI Zeitschrift **68** (1924), pp. 339–341.

[10] Rice, S. O. *The Mathematical Analysis of Random Noise*, Bell Syst Tech J., **23, 24** (1944, 1945), pp. 282-332 and pp. 46-156.

[11] I. Rychlik, *A new definition of the rainflow cycle counting method*, Int. J. Fatigue **9** (1987), pp. 119-121.

[12] Rychlik I., Johannesson P. and Leadbetter M. R. *Modelling and Statistical Analysis of Ocean-Wave Data using transformed Gaussian processes* Marine Structures, **10** (1997), pp. 13-47.

[13] M. Shinozuka, *Simulation of Multivariate and Multidimensional Random Processes*, J. Acoust. Soc. Am. **49** (1971), pp. 357-368.

[14] Winterstein S.R. *Nonlinear vibration models for extremes and fatigue*. American Society of Civil Engineers, Journal of Engineering Mechanics, **114**, (1988), pp. 1772-1790.

[15] Xin, Z. *Modeling and simulations of cultivator load signals with a fatigue damage perspective*, *Master Thesis at Dept. of Mathematical Sciences, Chalmers*, 2011, pp. 1-58.