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R. PETTERSSON

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
UNIVERSITY OF GOTHENBURG
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R. Pettersson

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Gothenburg, Sweden
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ON BOUNDEDNESS OF HIGHER VELOCITY MOMENTS FOR THE LINEAR BOLTZMANN EQUATION WITH DIFFUSE BOUNDARY CONDITIONS

R. PETTERSSON

Abstract. This paper considers the time and space dependent linear Boltzmann equation for elastic or inelastic (granular) collisions. First, in the angular cut-off case or with hard sphere collisions, mild L^1-solutions are constructed as limits of iterate functions. Then, in the case of hard potentials together with diffuse boundary conditions, global boundedness in time of higher velocity moments is proved, using our old collision velocity estimates together with a Jensen inequality.

Keywords: linear Boltzmann equation, hard potential collisions, higher velocity moments.

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1. Introduction

The linear Boltzmann equation is frequently used for mathematical modelling in physics (e.g. for describing the neutron distribution in reactor physics). In our papers (Pettersson, 1987, 1990, 1993, 2004, 2010) we have studied the linear Boltzmann equation, both in the elastic and the inelastic (granular) case for a function $f(\mathbf{x}, \mathbf{v}, t)$ representing the distribution of particles with mass m colliding binary with other particles of mass m_*, which have a given (known) distribution function $Y(\mathbf{x}, \mathbf{v}_*)$. Thereby we have also got results on boundedness (in time) of higher velocity moments for hard inverse collision forces with e.g. specular reflection boundary. The purpose of this paper is to get similar results for hard potentials or hard sphere collisions in the time and space dependent case with diffuse Maxwell boundary conditions; cf. (Lods and Toscani, 2004) and (Villani, 2006) for the space-homogeneous case, and also (Bobylev, 1997), with further references, for a thorough study of moment estimates in the homogeneous case.

So we will study collisions between particles with mass m and particles with mass m_*, such that momentum is conserved,

$$m\mathbf{v} + m_*\mathbf{v}_* = m\mathbf{v}' + m_*\mathbf{v}'_*, \tag{1}$$

where $\mathbf{v}, \mathbf{v}_*$ are velocities before and $\mathbf{v}', \mathbf{v}'_*$ are velocities after a collision.

In the elastic case, where also kinetic energy is conserved, one finds that the velocities after a binary collision terminate on two concentric spheres, so all velocities $\mathbf{v}'$ lie on a sphere around the center of mass, $\bar{\mathbf{v}} = (m\mathbf{v} + m_*\mathbf{v}_*)/(m + m_*)$, with radius $\frac{m_* w}{m+m_*}$, where $w = |\mathbf{v} - \mathbf{v}_*|$, and all velocities $\mathbf{v}'_*$ lie on a sphere with the same center $\bar{\mathbf{v}}$ and with radius $\frac{mw}{m+m_*}$, cf Figure 1 in (Pettersson, 1987).

In the granular, inelastic case we assume the following relation between the relative velocity components normal to the plane of contact of the two particles,

$$\mathbf{w}' \cdot \mathbf{u} = -a(\mathbf{w} \cdot \mathbf{u}), \tag{2}$$

where a is a constant, $0 < a \leq 1$, and $\mathbf{w} = \mathbf{v} - \mathbf{v}_*, \mathbf{w}' = \mathbf{v}' - \mathbf{v}'_*$ are the relative velocities before and after the collision, and $\mathbf{u}$ is a unit vector in the direction of impact, $\mathbf{u} = (\mathbf{v} - \mathbf{v}')/|\mathbf{v} - \mathbf{v}'|$. Then we find that $\mathbf{v}' = \mathbf{v}'_a$ lies on the line between $\mathbf{v}$ and $\mathbf{v}'_1$, where $\mathbf{v}'_1$ is the postvelocity in the case of elastic collision, i.e. with $a = 1$, and $\mathbf{v}'_{*a}$ lies on the (parallel) line between $\mathbf{v}_*$ and $\mathbf{v}'_{*1}$.

Now the following relations hold for the velocities in a granular, inelastic collision

$$\mathbf{v}' = \mathbf{v} - (a+1)\frac{m_*}{m + m_*}(\mathbf{w} \cdot \mathbf{u})\mathbf{u}, \qquad \mathbf{v}'_* = \mathbf{v}_* + (a+1)\frac{m}{m + m_*}(\mathbf{w} \cdot \mathbf{u})\mathbf{u}, \tag{3}$$

where $\mathbf{w} \cdot \mathbf{u} = w\cos\theta,\ w = |\mathbf{v} - \mathbf{v}_*|$, if the unit vector $\mathbf{u}$ is given in spherical coordinates,

$$\mathbf{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad 0 \leq \theta \leq \pi/2, \ \ 0 \leq \phi < 2\pi. \tag{4}$$

Moreover, if we change notations, and let $'\mathbf{v}, '\mathbf{v}_*$ be the velocities before, and $\mathbf{v}, \mathbf{v}_*$ the velocities after a binary inelastic collision, then by (2) and (3), cf. (Pettersson, 2004; Villani, 2006),

$$'\mathbf{v} = \mathbf{v} - \frac{(a+1)m_*}{a(m + m_*)}(\mathbf{w} \cdot \mathbf{u})\mathbf{u}, \quad '\mathbf{v}_* = \mathbf{v}_* + \frac{(a+1)m}{a(m + m_*)}(\mathbf{w} \cdot \mathbf{u})\mathbf{u}. \tag{5}$$

2. Preliminaries

We consider the time-dependent transport equation for a distribution function $f(\mathbf{x}, \mathbf{v}, t)$, depending on a space variable $\mathbf{x} = (x_1, x_2, x_3)$ in a bounded convex body D with (piecewise) C^1-boundary $\Gamma = \partial D$, and depending on a velocity variable $\mathbf{v} = (v_1, v_2, v_3) \in V = \mathbb{R}^3$ and a time variable $t \in \mathbb{R}_+$. Then the linear Boltzmann equation is in the strong form

$$\frac{\partial f}{\partial t}(\mathbf{x}, \mathbf{v}, t) + \mathbf{v} \cdot \mathrm{grad}_{\mathbf{x}} f(\mathbf{x}, \mathbf{v}, t) = (Qf)(\mathbf{x}, \mathbf{v}, t), \qquad \mathbf{x} \in D, \quad \mathbf{v} \in V = \mathbb{R}^3, \quad t \in \mathbb{R}_+, \tag{6}$$

supplemented by initial data

$$f(\mathbf{x}, \mathbf{v}, 0) = f_0(\mathbf{x}, \mathbf{v}), \quad \mathbf{x} \in D, \quad \mathbf{v} \in V. \tag{7}$$

The collision term can, in the case of inelastic (granular) collision, be written, cf. (Pettersson, 2004; Villani, 2006),

$$\begin{aligned}(Qf)(\mathbf{x}, \mathbf{v}, t) &= \\ &\int_V \int_\Omega \big[J_a(\theta, w) Y(\mathbf{x}, {}'\mathbf{v}_*) f(\mathbf{x}, {}'\mathbf{v}, t) - Y(\mathbf{x}, \mathbf{v}_*) f(\mathbf{x}, \mathbf{v}, t)\big] B(\theta, w)\; d\mathbf{v}_* d\theta d\phi\end{aligned} \tag{8}$$

with $w = |\mathbf{v} - \mathbf{v}_*|$, where $Y \geq 0$ is a known distribution (with assumptions in eq. 23 below), $B \geq 0$ is given by the collision process, and finally J_a is a factor depending on the granular process (and giving mass conservation, if the gain and the loss integrals converge separately). For elastic collisions $J_a = 1$, and in the case of hard sphere collisions $J_a = a^{-2}$, cf (Lods and Toscani, 2004) and (Villani, 2006). Furthermore, ${}'\mathbf{v}$, ${}'\mathbf{v}_*$ in (8) are the velocities before and $\mathbf{v}$, $\mathbf{v}_*$ the velocities after the binary collision, cf. (5), and $\Omega = \{(\theta, \phi) : 0 \leq \theta < \hat{\theta},\ 0 \leq \phi < 2\pi\}$ represents the impact plane, where $\hat{\theta} < \frac{\pi}{2}$ in the angular cut-off case, and $\hat{\theta} = \frac{\pi}{2}$ in the infinite range case. For hard sphere collisions one can also take $\hat{\theta} = \frac{\pi}{2}$, cf (10) below. The collision function $B(\theta, w)$ is in the physically interesting case with inverse k:th power collision forces given by

$$B(\theta, w) = b(\theta) w^\gamma, \quad \gamma = \frac{k-5}{k-1}, \quad w = |\mathbf{v} - \mathbf{v}_*|, \tag{9}$$

with hard forces for $k > 5$, Maxwellian for $k = 5$, and soft forces for $3 < k < 5$, where $b(\theta)$ has a non-integrable singularity for $\theta = \frac{\pi}{2}$. But in the case of hard sphere collisions, then (for $\gamma = 1$) the collision function is given by

$$B(\theta, w) = \text{const} \cdot w \sin\theta \cos\theta. \tag{10}$$

So in the angular cut-off case and also in the hard sphere case, the gain and the loss terms in (8) can be separated

$$(Qf)(\mathbf{x}, \mathbf{v}, t) = (Q^{+}f)(\mathbf{x}, \mathbf{v}, t) - (Q^{-}f)(\mathbf{x}, \mathbf{v}, t), \tag{11}$$

where the gain term can be written (with a kernel K_a)

$$(Q^{+}f)(\mathbf{x}, \mathbf{v}, t) = \int_V K_a(\mathbf{x},\ {}'\mathbf{v} \to \mathbf{v}) f(\mathbf{x}, {}'\mathbf{v}, t)\ d\,{}'\mathbf{v}, \tag{12}$$

and the loss term is written with the collision frequency $L(\mathbf{x}, \mathbf{v})$ as

$$(Q^{-}f)(\mathbf{x}, \mathbf{v}, t) = L(\mathbf{x}, \mathbf{v}) f(\mathbf{x}, \mathbf{v}, t). \tag{13}$$

In the case of non-absorbing body we have that

$$L(\mathbf{x}, \mathbf{v}) = \int_V K_a(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\ d\mathbf{v}'. \tag{14}$$

Furthermore, equations (6)-(8) are in the space-dependent case supplemented by (general) boundary conditions

$$f_{-}(\mathbf{x}, \mathbf{v}, t) = \int \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) f_{+}(\mathbf{x}, \tilde{\mathbf{v}}, t) d\tilde{\mathbf{v}}, \tag{15}$$
$$\mathbf{n}\mathbf{v} < 0,\ \mathbf{n}\tilde{\mathbf{v}} > 0,\ \mathbf{x} \in \Gamma = \partial D,\ t \in \mathbb{R}_{+},$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit outward normal at $\mathbf{x} \in \Gamma = \partial D$. The function $R \geq 0$ satisfies (in the non-absorbing boundary case)

$$\int_V R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) d\mathbf{v} \equiv 1, \tag{16}$$

and f_{-} and f_{+} represent the ingoing and outgoing trace functions corresponding to f. In the specular reflection case the function R is represented by a Dirac measure $R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = \delta(\mathbf{v} - \tilde{\mathbf{v}} + 2(\mathbf{n}\tilde{\mathbf{v}})\mathbf{n})$, and in the diffuse reflection case

$R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = |\mathbf{n}\mathbf{v}| W(\mathbf{x}, \mathbf{v})$ with some given function $W \geq 0$, (e.g. Maxwellian function).

Let $t_b \equiv t_b(\mathbf{x}, \mathbf{v}) = \inf_{\tau \in \mathbb{R}_+}\{\tau : \mathbf{x} - \tau\mathbf{v} \notin D\}$, and $\mathbf{x}_b \equiv \mathbf{x}_b(\mathbf{x}, \mathbf{v}) = \mathbf{x} - t_b\mathbf{v}$, where t_b represents the time for a particle going with velocity $\mathbf{v}$ from the boundary point $\mathbf{x}_b$ to the point $\mathbf{x}$.

Then, using differentiation along the characteristics, equation (6) can formally be transformed to a *mild* equation, and also to an *exponential* form of equation in the angular cut-off case or in the hard sphere case, cf. (Pettersson, 1987, 1990, 1993).

3. Construction of Solutions

We construct L^1-solutions to our problems as limits of iterate functions f^n, when $n \to \infty$. Let first $f^{-1}(\mathbf{x}, \mathbf{v}, t) \equiv 0$. Then define for given f^{n-1} the next iterate f^n, first at the ingoing boundary (using the appropriate boundary condition), and then inside D and at the outgoing boundary (using the exponential form of the equation),

(17)

$$f^n_-(\mathbf{x}, \mathbf{v}, t) = \int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) f^{n-1}_+(\mathbf{x}, \tilde{\mathbf{v}}, t) d\tilde{\mathbf{v}},$$

(18)

$$\begin{aligned} f^n(\mathbf{x}, \mathbf{v}, t) &= \bar{f}^n(\mathbf{x}, \mathbf{v}, t) \exp\Big[-\int_0^t L(\mathbf{x} - s\mathbf{v}, \mathbf{v}) ds\Big] \\ &\quad + \int_0^t \exp\Big[-\int_0^\tau L(\mathbf{x} - s\mathbf{v}, \mathbf{v}) ds\Big] \int_V K_a(\mathbf{x} - \tau\mathbf{v}, \ '\mathbf{v} \to \mathbf{v}) f^{n-1}(\mathbf{x} - \tau\mathbf{v}, \ '\mathbf{v}, t - \tau) \ d'\mathbf{v} d\tau, \end{aligned}$$

where

$$\bar{f}^n(\mathbf{x}, \mathbf{v}, t) = \begin{cases} f_0(\mathbf{x} - t\mathbf{v}, \mathbf{v}), & 0 \leq t \leq t_b, \\ f^n_-(\mathbf{x}_b, \mathbf{v}, t - t_b), & t > t_b. \end{cases} \qquad (19)$$

Let also $f^n(\mathbf{x}, \mathbf{v}, t) \equiv 0$ for $\mathbf{x} \in \mathbb{R}^3 \setminus D$. Now we get a monotonicity lemma, $f^n(\mathbf{x}, \mathbf{v}, t) \geq f^{n-1}(\mathbf{x}, \mathbf{v}, t)$, which is essential and can be proved by induction.

Then, by differentiation along the characteristics and integration (with Green's formula), we find (using the equations above, cf. (Pettersson, 2004)), that

$$\int_D\int_V f^n(\mathbf{x}, \mathbf{v}, t) \ d\mathbf{x}d\mathbf{v} \leq \int_D\int_V f_0(\mathbf{x}, \mathbf{v}) \ d\mathbf{x}d\mathbf{v}, \qquad (20)$$

so Levi's theorem (on monotone convergence) gives existence of (mild) L^1-solutions

$$f(\mathbf{x},\mathbf{v},t) = \lim_{n\to\infty} f^n(\mathbf{x},\mathbf{v},t). \tag{21}$$

Proposition (Existence). Assume (for inelastic or elastic collisions), that the function B is given by (10), or (9) with angular cut off, and that K_a, L and R are non-negative, measurable functions, such that (14) and (16) hold, and $L(\mathbf{x},\mathbf{v}) \in L^1_{loc}(D \times V)$.

Then for every $f_0 \in L^1(D \times V)$ there exists a mild L^1-solution $f(\mathbf{x},\mathbf{v},t)$ to the problem (6)-(8) with (15), satisfying the corresponding inequality in (20). Furthermore, if $L(\mathbf{x},\mathbf{v})f(\mathbf{x},\mathbf{v},t) \in L^1(D \times V)$, then equality in (20) for the limit function f holds, giving mass conservation together with uniqueness in the relevant function space (cf. (Pettersson, 1987, 1990, 1993) and also Proposition 3.3, chapter 11 in (Greenberg et al., 1987)).

Remark 1 The assumption $Lf \in L^1(D \times V)$ is, for instance, satisfied for the solution f in the case of inverse power collision forces, cf. (9), together with e.g. specular boundary reflections. This follows from a statement on global boundedness (in time) of higher velocity moments, (cf. Theorem 4.1 and Corollary 4.1 in (Pettersson, 2004)). Compare also the results for diffuse boundary reflections in the next section.

Remark 2 There holds also both in the elastic and inelastic cases an *H-theorem* for a general relative entropy functional

$$H^{\Phi}_F(f)(t) = \int_D\int_V \Phi\Big(\frac{f(\mathbf{x},\mathbf{v},t)}{F(\mathbf{x},\mathbf{v})}\Big) F(\mathbf{x},\mathbf{v})\; d\mathbf{x}d\mathbf{v}, \tag{22}$$

giving that this H-functional is nonincreasing in time, if $\Phi = \Phi(z)$, $\mathbb{R}_+ \to \mathbb{R}$, is a convex C^1-function, and if there exists a corresponding stationary solution $F(\mathbf{x},\mathbf{v})$ with the same total mass as the initial data $f_0(\mathbf{x},\mathbf{v})$ for the time-dependent solution $f(\mathbf{x},\mathbf{v},t)$; cf. Theorem 5.1 in (Pettersson, 2004). By using this H-functional one can prove that every time-dependent solution $f(\mathbf{x},\mathbf{v},t)$ converges to the corresponding stationary solution $F(\mathbf{x},\mathbf{v})$ as time goes to infinity, cf. Remark 5.1 in (Pettersson, 2004) and further references.

4. Higher velocity moment estimates for a diffuse boundary

In this section we will generalize a result on global boundedness of higher velocity moments to the case of hard potentials or hard sphere collisions. Then we start with some (old) velocity estimates for a binary collision, and also give the corresponding moment estimates, cf. Propositions 1.1 and 1.2 in (Pettersson, 1987).

Proposition A. If $\mathbf{v}$ and $\mathbf{v}'_a(\theta, \phi)$ are the velocities before and after a (granular) binary collision, then, with $w = |\mathbf{v} - \mathbf{v}_*|$,

$$|\mathbf{v}'_a(\theta,\phi)|^2 - |\mathbf{v}|^2 \;\leq\; 2\,(a+1)\frac{m_*}{m+m_*}w\cos\theta\Big[\,3\,|\,\mathbf{v}_*| - \frac{m_*}{m+m_*}\,|\mathbf{v}|\cos\theta\,\Big].$$

Proposition B. If $\sigma > 0$, there exist constants $c_1 > 0,\ c_2 > 0$ (depending on $\sigma,\ m,\ m_*$ and a) such that

$$\begin{aligned}&\big(1+|\mathbf{v}'_a(\theta,\phi)|^2\big)^{\frac{\sigma}{2}} - \big(1+|\mathbf{v}|^2\big)^{\frac{\sigma}{2}}\\ \leq\ & c_1 w\cos\theta\big(1+|\mathbf{v}_*|\big)^{\max(1,\sigma-1)}\big(1+|\mathbf{v}|^2\big)^{\frac{\sigma-2}{2}} - c_2 w\cos^2\theta\big(1+|\mathbf{v}|^2\big)^{\frac{\sigma-1}{2}}.\end{aligned}$$

By using these propositions we have earlier got results on boundedness of higher velocity moments for hard inverse collision forces, $0 \leq \gamma < 1$, and also, by using a Jensen inequality to get the analogous results for hard sphere collisions, $\gamma = 1$, in the space-dependent case with e.g. specular reflection boundary. Here, we will now study the case with diffuse boundary conditions.

We start with an elementary lemma (used in the theorem below) for the velocities in a binary collision, where $v = |\mathbf{v}|,\ v_* = |\mathbf{v}_*|$ and $w = |\mathbf{w}|$, cf. (Pettersson, 1987).

Lemma For $\gamma \geq 0$ it holds that

$$-w^{\gamma+1} \leq \big(1+v_*\big)^{\gamma+1} - 2^{-\gamma}\big(1+v^2\big)^{\frac{\gamma+1}{2}}, \quad \text{where} \quad \mathbf{w} = \mathbf{v}-\mathbf{v}_* \quad \text{is the relative velocity.}$$

Now we can formulate our main result on global boundedness (in time) for hard potentials or hard sphere collisions, i.e. with $0 \leq \gamma \leq 1$, together with diffuse boundary conditions, in the following theorem. Compare Theorem 4.1 in (Pettersson, 2004) for the case of hard inverse forces.

Theorem Assume for hard potentials or hard sphere collisions with $0 \leq \gamma \leq 1$ that the function $B(\theta, w)$ is given by equation (9) or (10), and suppose that the function $Y(\mathbf{x}, \mathbf{v}_*)$ satisfies the following conditions:

$$(23) \quad \int_V \big(1+v_*\big)^{\gamma+\max(2,\sigma)} \sup_{\mathbf{x}\in D} Y(\mathbf{x}, \mathbf{v}_*)\, d\mathbf{v}_* < \infty \text{ and } \int_V \inf_{\mathbf{x}\in D} Y(\mathbf{x}, \mathbf{v}_*)\, d\mathbf{v}_* > 0.$$

Let the boundary conditions (15) be given by diffuse Maxwell reflections,

$$R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = M(\mathbf{x}, \mathbf{v})|\mathbf{n}\mathbf{v}|,$$

where

$$(24) \qquad M(\mathbf{x}, \mathbf{v}) = \frac{1}{2\pi T^2} e^{-\frac{v^2}{2T}}$$

with $0 < T_1 \leq T = T(\mathbf{x}) \leq T_2 < \infty$. Then the higher velocity moments belonging to the mild solution $f(\mathbf{x}, \mathbf{v}, t)$ given by (21) are all bounded (globally in time),

$$\int_D\int_V \big(1+v^2\big)^{\sigma/2} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x} d\mathbf{v} \leq C_\sigma < \infty,\ \sigma > 0,\ t > 0,\ 0 < a \leq 1,$$
$$\text{if } \big(1+v^2\big)^{\sigma/2} f_0(\mathbf{x}, \mathbf{v}) \in L^1(D \times V).$$

Proof: Start from the definition of the iterate function $f^n(\mathbf{x}, \mathbf{v}, t)$ in equations (17)-(19), and differentiate along the characteristics, using the corresponding mild form of the equation, and then multiply by $\left(1+\frac{v^2}{2T}\right)^{\sigma/2}$, where $v = |\mathbf{v}|$, $\sigma > 0$ and $0 < T_1 \leq T \leq T_2 < \infty$ with (for simplicity) constant temperature. Then

$$\frac{d}{dt}\Big[\left(1+\frac{v^2}{2T}\right)^{\sigma/2} f^n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t)\Big] =$$
$$\int_V K_a(\mathbf{x}+t\mathbf{v}, {}'\mathbf{v} \to \mathbf{v}) \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f^{n-1}(\mathbf{x}+t\mathbf{v}, {}'\mathbf{v}, t) d'\mathbf{v}$$
$$- L(\mathbf{x}+t\mathbf{v}, \mathbf{v}) \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f^n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t)$$

Integrating $\iiint \dots d\mathbf{x}d\mathbf{v}d\tau$ (with Green's formula) gives

(25)

$$\begin{aligned}
&\int_D\int_V \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f^n(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} \\
&\quad = \int_D\int_V \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f_0(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} \\
&\qquad + \int_0^t\int_\Gamma\Big[\int_V \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f_-^n(\mathbf{x},\mathbf{v},\tau)|\mathbf{n}\mathbf{v}|d\mathbf{v} \\
&\qquad - \int_V \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f_+^n(\mathbf{x},\mathbf{v},\tau)|\mathbf{n}\mathbf{v}|d\mathbf{v}\Big]d\Gamma d\tau \\
&\qquad + \int_0^t\int_D\int_V\int_V K_a(\mathbf{x},{'\mathbf{v}}\to\mathbf{v}) \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f^{n-1}(\mathbf{x},{'\mathbf{v}},\tau)d\mathbf{x}d\mathbf{v}d{'\mathbf{v}}d\tau \\
&\qquad - \int_0^t\int_D\int_V L(\mathbf{x},\mathbf{v}) \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f^n(\mathbf{x},\mathbf{v},\tau)d\mathbf{x}d\mathbf{v}d\tau,
\end{aligned}$$

where all integrals exist inductively.

Let the velocity moment be defined by

$$M_\sigma(t) \equiv M^n_{\sigma,T}(t) = \int_D\int_V \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f^n(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} \tag{26}$$

and differentiate equation (25). For an estimate of the gain and loss terms in (26), use Proposition B and the Lemma above together with the assumptions (23) to get (for general $\gamma \geq 0$), cf. (Pettersson, 2004), that

$$\begin{aligned}
&\int_D\int_V \Big[\int_V K_a(\mathbf{x},{'\mathbf{v}}\to\mathbf{v}) \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f^{n-1}(\mathbf{x},{'\mathbf{v}},t)d{'\mathbf{v}} \\
&\qquad - \int_V K_a(\mathbf{x},\mathbf{v}\to\mathbf{v}') \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f^n(\mathbf{x},\mathbf{v},t)d\mathbf{v}'\Big]d\mathbf{x}d\mathbf{v} \\
&\quad \leq C_1 M_{\sigma+\gamma-1}(t) + C_2 M_{\sigma-1}(t) - C_0 M_{\sigma+\gamma}(t),
\end{aligned} \tag{27}$$

with constants $C_1 > 0$, $C_2 > 0$, $C_0 > 0$.

Furthermore, the (total) ingoing moment flow is, for the diffuse boundary, given by

$$\begin{aligned}
\Lambda^-(f^n)_{\sigma,T}(t) &= \int_\Gamma\int_V \left(1+\frac{v^2}{2T}\right)^{\sigma/2} f_-^n(\mathbf{x},\mathbf{v},t)|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma \\
&= \int_\Gamma\int_V \left(1+\frac{v^2}{2T}\right)^{\sigma/2} \cdot \frac{1}{2\pi T^2} e^{-\frac{v^2}{2T}}|\mathbf{n}\mathbf{v}|d\mathbf{v}\left(\int_V f_+^{n-1}(\mathbf{x},\tilde{\mathbf{v}},t)|\mathbf{n}\tilde{\mathbf{v}}|d\tilde{\mathbf{v}}\right)d\Gamma,
\end{aligned}$$

where (by a substitution $\xi = v^2/2T$), the integral

$$I_\sigma(\mathbf{x}) = \frac{1}{2\pi T^2}\int_0^\infty \int_0^{\pi/2}\int_0^{2\pi}\left(1+\frac{v^2}{2T}\right)^{\sigma/2} e^{-\frac{v^2}{2T}}\cdot v\cos\theta\cdot v^2\sin\theta d\theta d\phi dv$$
$$= \int_0^\infty (1+\xi)^{\sigma/2}\xi e^{-\xi}d\xi = I_\sigma$$

is independent of $\mathbf{x}$ and $T = T(\mathbf{x})$, and

$$\Lambda^+(f^{n-1})(t) = \int_\Gamma\int_V f_+^{n-1}(\mathbf{x},\tilde{\mathbf{v}},t)|\mathbf{n}\tilde{\mathbf{v}}|d\tilde{\mathbf{v}}d\Gamma$$

is the (total) outgoing mass flow (for large velocities $|\mathbf{v}| \geq \eta > 0$, and large times $t \geq \text{diam }(D)/\eta$).

By a substitution, cf. (Falk, 2003),

$$d\mathbf{x} = \frac{|\mathbf{n_x}\cdot\mathbf{v}|}{|\mathbf{v}|}d\Gamma_{\mathbf{x}}\cdot|\mathbf{v}|d\tau,$$

then we have for large velocities, $|\mathbf{v}| \geq \eta > 0$, that the (outgoing and ingoing) mass flow is

$$m_\eta(t) = \int_D\int_{V_\eta} f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} = \int_{V_\eta}d\mathbf{v}\int_{\Gamma_-}\frac{|\mathbf{n_x}\cdot\mathbf{v}|}{|\mathbf{v}|}d\Gamma_{\mathbf{x}}\int_0^{t_b(\mathbf{x},\mathbf{v})} f(\mathbf{x}-\tau\mathbf{v},\mathbf{v},t-\tau)|\mathbf{v}|d\tau,$$

and the (total) mass flow out (for large times) is then

$$\Lambda_\eta^+(f^{n-1})(t) = \int_\Gamma\int_{V_n} f^{n-1}(\mathbf{x},\mathbf{v},t)|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma$$
$$\leq m_\eta(t) \leq m(t) \leq \int_D\int_V f^n(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}$$
$$\leq M_0 = \int_D\int_V f_0(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} < \infty.$$

So the ingoing moment flow is estimated by

$$\Lambda^-(f^n)_{\sigma,T}(t) \leq I_\sigma M_0.$$

Now the velocity moments $M_\sigma(t) = M^n_{\sigma,T}(t)$ satisfy

$$\frac{d}{dt}\left(M_\sigma(t)\right) \leq I_\sigma M_0 + C_1 M_{\sigma+\gamma-1}(t) + C_2 M_{\sigma-1}(t) - C_0 M_{\sigma+\gamma}(t) \tag{28}$$

with positive constants $C_1, C_2, C_0 > 0$ for $0 \leq \gamma \leq 1$.

For the cases $\gamma = 0$ and $\gamma = 1$, cf. analogous results in (Pettersson, 1987) and (Pettersson, 2010). Here we will, in order to study the case $0 < \gamma < 1$, use a Jensen's inequality, $\phi\left(\int_\Omega g d\mu\right) \leq \int_\Omega \phi(g)d\mu$, where $\mu(\Omega) = 1$, $g \in L^1(\mu)$ and ϕ is a convex function. Then we take $d\mu = f^n d\mathbf{x}d\mathbf{v}/\|f^n\|$, where

$$\|f^n\| = \int_D \int_V f^n d\mathbf{x}d\mathbf{v} \leq M_0 = \int_D \int_V f_0(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v},$$

and $g(v) = \left(1 + \frac{v^2}{2T}\right)^{\sigma/2}$ together with the convex function $\phi(z) = z^\rho$, where $\rho = \frac{\sigma+\gamma}{\sigma+\gamma-1} > 1$.

Then, by Jensen's inequality,

$$\left[\int_D \int_V \left(1 + \frac{v^2}{2T}\right)^{\sigma/2} f^n d\mathbf{x}d\mathbf{v}/\|f^n\|\right]^\rho \leq \int_D \int_V \left(\left(1 + \frac{v^2}{2T}\right)^{\sigma/2}\right)^\rho f^n d\mathbf{x}d\mathbf{v}/\|f^n\|,$$

so

$$\left[\int_D \int_V \left(1 + \frac{v^2}{2T}\right)^{\sigma/2} f^n d\mathbf{x}d\mathbf{v}\right]^\rho \leq \|f^n\|^{\rho-1} \cdot \int_D \int_V \left(1 + \frac{v^2}{2T}\right)^{\frac{\rho}{2}} f^n d\mathbf{x}d\mathbf{v},$$

and

$$-M_{\sigma+\gamma}(t) \leq -C(M_\sigma(t))^\rho$$

with constant $C > 0$, and $\rho = \frac{\sigma+\gamma}{\sigma+\gamma-1} > 1$.

And, by (28) and with $C_3 = C_1 + C_2$,

$$M'_\sigma(t) \leq I_\sigma M_0 + C_3 M_{\sigma+\gamma-1}(t) - CC_0\left(M_\sigma(t)\right)^\rho,$$

so $M'_\sigma(t) \leq A_1 M_\sigma(t) - A_0\left(M_\sigma(t)\right)^\rho$, $\rho > 1$, with positive constants $A_1, A_0 > 0$, in the case when $M_\sigma(t) \geq$ Constant (and $I_\sigma M_0 \geq C_3 M_\sigma(t)$).

Let $y(t) = \left(M_\sigma(t)\right)^{1-\rho}$, $\rho = (\sigma+\gamma)/(\sigma+\gamma-1)$, integrate and use a maximum principle, cf. (Lods and Toscani, 2004; Pettersson, 2010). Then

$$M_\sigma(t) \leq \max\left[M_\sigma(0), \left(\frac{A_1}{A_0}\right)^{\frac{\sigma+\gamma}{\sigma+\gamma-1}}\right] < \infty$$

(for $0 < \gamma < 1$), so the velocity moments

$$\int_D \int_V \left(1 + \frac{v^2}{2T}\right)^{\sigma/2} f^n(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}$$

are all globally bounded in time, $M^n_{\sigma,T}(t) \leq C_\sigma < \infty$.

Finally, let $n \to \infty$, and use that the temperature is bounded, $0 < T_1 \leq T = T(\mathbf{x}) \leq T_2 < \infty$. Then the higher velocity moments are all bounded, if they exist at $t = 0$.

Remark The estimate (28) would lead to a creation of moments of arbitrary order for $t > 0$, cf. (Desvillettes, 1993; Mischler and Wennberg, 1999) and our forthcoming papers.

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Department of Mathematics, Chalmers University of Technology and Göteborg University, SE 412 96 Göteborg, Sweden, e-mail: rolfp@chalmers.se