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Abstract We consider an adaptive finite element method for the solution of a Fredholm integral equation of the first kind and derive a posteriori error estimates both in the Tikhonov functional and in the regularized solution of this functional. We apply nonlinear results obtained in [3–6, 12] for the case of the linear bounded operator. We formulate an adaptive algorithm and present experimental verification of our adaptive technique on the backscattered data measured in microtomography.

1 Introduction

The goal of this work is to present a posteriori error estimates for the Tikhonov functional and for the regularized solution of this functional, formulate an adaptive algorithm and apply it for the solution of a Fredholm integral equation of the first kind on the adaptively locally refined meshes.

Fredholm integral equation of the first kind arise in different applications of the mathematical physics such as image and signal processing, astronomy and geophysics, see, for example, [2, 9, 14, 17] and references therein. There exists a lot of works devoted to the solution of a Fredholm integral equations of the first kind on the finite-difference uniform grids - we refer to [16] and references therein. Since the problem of the solution of a Fredholm integral equation of the first kind is the ill-posed problem then for the solution of this equation we minimize the Tikhonov regularization functional. The main result of our work is derivation of a posteriori error estimates for the underlying Tikhonov functional and for the regularized solu-

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tion of this functional, formulation of an adaptive algorithm and application of this algorithm for the numerical solution of a Fredholm integral equation of the first kind on the locally adaptively refined meshes.

In this work we specify results of recent works [3–6, 12] for the case of a linear Fredholm integral equation. The main difference of the current work from [3–6, 12] is that we present new proofs of a posteriori error estimates for the Tikhonov functional and for the regularized solution for the case of a linear bounded operator. One more difference from the above cited works is that we consider the Tikhonov regularization term in H^1 norm. This norm is stronger than the L_2 norm which was used in proofs of [3–6, 12].

Let z_α be the regularized solution of the Tikhonov functional, and z_h be the computed finite element solution. The main goal of the adaptive error control is to find such triangulation T which has a least number of nodes such that the corresponding finite element solution z_h on this mesh satisfies to the equation

$$||z_\alpha - z_h|| \leq \varepsilon, \tag{1}$$

where ε is the desired tolerance.

To achieve criterion (1) we minimize the Tikhonov functional on a sequence of a locally adaptively refined meshes what allow us improve the resulting solution. Based on a posteriori error estimates we formulate an adaptive algorithm and apply this algorithm on the one real-life image restoration problem. Problem which we consider in our numerical examples arises in electron microscopy [14, 17]. The goal of our tests is to restore blurred images obtained by the electron microscope and find possible defects on the investigated objects. Blurred images was obtained by the microtomograph developed by professor Eduard Rau at Moscow Lomonosov State University [14, 17]. Thus, in our numerical experiments we are working with real measured data.

Our tests show that the local adaptive mesh refinement algorithm can significantly improve contrast of the blurred images using optimized number of nodes in the computational mesh.

2 Statement of the problem

Let H be the Hilbert space H^1 and let $\Omega \subset \mathbb{R}^m, m = 2,3$, be a convex bounded domain. Our goal is to solve a Fredholm integral equation of the first kind for $x \in \Omega$

$$\int_\Omega K(x-y)z(x)dx = u(y), \tag{2}$$

where $u(y) \in L_2(\overline{\Omega})$, $z(x) \in H$, $K(x-y) \in C^k\left(\overline{\Omega}\right), k \geq 0$ be the kernel of the integral equation.

Let us rewrite (2) in an operator form as

$$A(z) = u \tag{3}$$

with an operator $A : H \to L_2(\overline{\Omega})$ defined as

$$A(z) := \int_{\Omega} K(x-y)z(x)dx. \tag{4}$$

Ill-posed problem.

Let the function $z(x) \in H^1$ of the equation (2) be unknown in the domain Ω. Determine the function $z(x)$ for $x \in \Omega$ assuming the functions $K(x-y) \in C^k\left(\overline{\Omega}\right), k \geq 0$ and $u(x) \in L_2(\Omega)$ in (2) are known.

Let $\delta > 0$ be the error in the right-hand side of the equation (2):

$$A\left(z^*\right) = u^*, \quad \|u - u^*\|_{L_2(\sigma)} \leq \delta. \tag{5}$$

where u^* is the exact right-hand side corresponding to the exact solution z^*.

To find the approximate solution of the equation (2) in our numerical tests of Section 9 we will minimize the functional

$$M_\alpha(z) = \|Az - u\|^2_{L_2(\Omega)} + \alpha\|z\|^2_{H_1(\Omega)}, \tag{6}$$

$$M_\alpha : H^1 \to \mathbb{R},$$

where $\alpha = \alpha(\delta) > 0$ is the small regularization parameter.

Our goal is to solve the equation (2) on the rather coarse mesh with some regularization parameter α and then construct the sequence of the approximated solutions z_k on the refined meshes T_k with the same regularization parameter α. The regularization parameter α can be chosen using one of the methods, described in [16] (for example, the method of generalized discrepancy).

We consider now more general form of the Tikhonov functional (6). Let W_1, W_2, Q be three Hilbert spaces, $Q \subseteq W_1$ as a set, the norm in Q is stronger than the norm in W_1 and $\overline{Q} = W_1$, where the closure is understood in the norm of W_1. We denote scalar products and norms in these spaces as

$$(\cdot,\cdot), \|\cdot\| \text{ for } W_1,$$

$$(\cdot,\cdot)_2, \|\cdot\|_2 \text{ for } W_2$$

$$\text{and } [\cdot,\cdot], [\cdot] \text{ for } Q.$$

Let $A : W_1 \to W_2$ be a bounded linear operator. Our goal is to find the function $z(x) \in Q$ which minimizes the Tikhonov functional

$$E_\alpha(z) : Q \to \mathbb{R}, \tag{7}$$

$$E_\alpha(z) = \frac{1}{2}\|Az - u\|^2_2 + \frac{\alpha}{2}[z - z_0]^2, u \in W_2; z, z_0 \in Q, \tag{8}$$

where $\alpha \in (0,1)$ is the regularization parameter. To do that we search for a stationary point of the above functional with respect to z satisfying $\forall b \in Q$

$$E'_\alpha(z)(b) = 0. \tag{9}$$

The following lemma is well known [1] for the case $W_1 = W_2 = L_2$.

Lemma 1. *Let $A : L_2 \to L_2$ be a bounded linear operator. Then the Fréchet derivative of the functional (6) is*

$$E'_\alpha (z) (b) = (A^*Az - A^*u, b) + \alpha [z - z_0, b], \forall b \in Q. \tag{10}$$

In particular, for the integral operator (2) we have

$$E'_\alpha (z) (b) = \int\limits_\Omega b(s) \left[\int\limits_\Omega z(y) \left(\int\limits_\Omega K(x-y)K(x-s)dx \right) dy - \int\limits_\Omega K(x-s)u(x)\,dx \right] ds \\ +\alpha [z - z_0, b], \forall b \in Q. \tag{11}$$

Lemma 2 is also well known, since $A : W_1 \to W_2$ is a bounded linear operator. We formulate this lemma only for our specific case and refer to [16] for a more general case. For the case of a nonlinear operator we refer to [3].

Lemma 2. *Let the operator $A : W_1 \to W_2$ satisfies conditions of Lemma 1. Then the functional $E_\alpha(z)$ is strongly convex on the space Q with the convexity parameter* κ such that

$$\left(E'_\alpha (x) - E'_\alpha (z), x - z\right) \geq \kappa [x - z]^2, \forall x, z \in Q. \tag{12}$$

Similarly, the functional $M_\alpha(z)$ is also strongly convex on the Sobolev space H_1:

$$\left(M'_\alpha (x) - M'_\alpha (z), x - z\right)_{H_1} \geq \kappa ||x - z||^2_{H_1}, \forall x, z \in H_1, \tag{13}$$

Remark.

We assume in (2) that $u \in L_2\left(\overline{\Omega}\right)$ since this function can be given with a noise. This is done despite to that $A(z) \in C^k\left(\overline{\Omega}\right), k \geq 0$.

3 The finite element spaces

Let $\Omega \subset \mathbb{R}^m, m = 2,3,$ be a bounded domain with a piecewise-smooth boundary $\partial\Omega$. Following [11] we discretize the domain Ω by an unstructured mesh T using non-overlapping tetrahedral elements in $\mathbb{R}^3$ and triangles in $\mathbb{R}^2$ such that $T = K_1, ..., K_l$, where l is the number of elements in Ω, and

$$\Omega = \cup_{K \in T} K = K_1 \cup K_2 ... \cup K_l.$$

We associate with the triangulation T the mesh function $h = h(x)$ which is a piecewise-constant function such that

$$h(x) = h_K \quad \forall K \in T,$$

where h_K is the diameter of K which we define as the longest side of K.

Let r' be the radius of the maximal circle/sphere contained in the element K. We make the following shape regularity assumption for every element $K \in T$

$$a_1 \leq h_K \leq r' a_2; \quad a_1, a_2 = const. > 0. \tag{14}$$

We introduce now the finite element space V_h as

$$V_h = \left\{ v(x) \in H^1(\Omega) : v \in C(\Omega),\ v|_K \in P_1(K)\ \forall K \in T \right\}, \tag{15}$$

where $P_1(K)$ denote the set of piecewise-linear functions on K. The finite dimensional finite element space V_h is constructed such that $V_h \subset V$. The finite element method which uses piecewise-linear test functions we call CG(1) method. CG(1) can be applied on the conforming meshes.

In a general case we allow also meshes in space with hanging nodes and assume that the local mesh size has bounded variation in such meshes. This means that there exists a constant $\gamma > 0$ such that $\gamma h_{K^+} \leq h_K \leq \gamma^{-1} h_{K^+}$ for all the neighboring elements K^- and K^+. Let S be the internal face of the non-empty intersection of the boundaries of two neighboring elements K^+ and K^-. We denote the jump of the function v_h computed from the two neighboring elements K^+ and K^- sharing the common side S as

$$[v_h] = v_h^+ - v_h^-. \tag{16}$$

We introduce the discontinuous finite element space W_h on such meshes as

$$W_h = \left\{ v(x) \in V :\ v|_K \in DP_1(K)\ \forall K \in T \right\}, \tag{17}$$

where $DP_1(K)$ denote the set of discontinuous linear functions on K. The finite element space W_h is constructed such that $W_h \subset V$. The finite element method which uses discontinuous linear functions we call DG(1) method.

Let $P_k : V \to M$ for $\forall M \subset V$, be the operator of the orthogonal projection. Let the function $f \in H^1(\Omega) \cap C(\Omega)$ and $\partial_{x_i} f_{x_i} \in L_\infty(\Omega)$. We define by f_k^I the standard interpolant [8] on triangles/tetrahedra of the function $f \in H$. Then by one of properties of the orthogonal projection

$$\| f - P_k f \|_{L_2(\Omega)} \leq \left\| f - f_k^I \right\|_{L_2(\Omega)}. \tag{18}$$

It follows from formula 76.3 of [8] that

$$\| f - P_k f \|_{L_2(\Omega)} \leq C_I \| h \nabla f \|_{L_2(\Omega)}, \forall f \in V. \tag{19}$$

where $C_I = C_I(\Omega)$ is positive constant depending only on the domain Ω.

4 A finite element method

To formulate a CG(1) for equation (9) we recall the definition of the space V_h. The CG(1) finite element method then reads: find $z_h \in V_h$ such that

$$E'_\alpha(z_h)(b) = 0 \ \ \forall b \in V_h. \tag{20}$$

Similarly, for DG(1) for equation (9) we recall the definition of the space W_h. The DG(1) finite element method then reads: find $z_h \in W_h$ such that

$$E'_\alpha(z_h)(b) = 0 \ \ \forall b \in W_h. \tag{21}$$

5 A posteriori error estimate for the regularized solution on locally refined meshes

In this section we will formulate theorems for accuracy of the regularized solution for the case of the more general functional E_α defined in (7).

From the theory of convex optimization it is known, that Lemma 2 claims existence and uniqueness of the global minimizer of the functional E_α defined in (7) for $z_\alpha \in Q$ such that

$$E_\alpha(z_\alpha) = \inf_{z\in Q} E_\alpha(z).$$

It is well known that the operator F is Lipschitz continuous

$$\|F(z_1) - F(z_2)\| \leq ||A|| \cdot \|z_1 - z_2\| \ \ \forall z_1, z_2 \in H.$$

Because of the boundedness of the operator A there exists the constant

$$D = 2(\|A\|^2 + \alpha) = const. > 0 \tag{22}$$

such that the following inequality holds [1]

$$\left\|E'_\alpha(z_1) - E'_\alpha(z_2)\right\| \leq D\|z_1 - z_2\|, \forall z_1, z_2 \in H. \tag{23}$$

Similarly, the functional $M_\alpha(z)$ is twice Frechét differentiable [16] and the following inequality holds [1]:

$$\left\|M'_\alpha(z_1) - M'_\alpha(z_2)\right\| \leq D\|z_1 - z_2\|, \forall z_1, z_2 \in H. \tag{24}$$

Let z_k be the computed solution (minimizer) of the Tikhonov functional and $z_\alpha \in H$ be the regularized solution on the finally refined mesh. Let P_k be the operator of the orthogonal projection defined in section 3. Then the following theorem is valid for the functional (8):

Theorem 1a

Let z_k be a minimizer of the functional (8). Assume that (12) holds. Then there exists a constant D defined by (22) such that the following estimate holds

$$[z_k - z_\alpha] \leq \frac{D}{\kappa} ||P_k z_\alpha - z_\alpha||_{W^1}. \qquad (25)$$

In particular, if $P_k z_\alpha = z_\alpha$, then $z_k = z_\alpha$, which means that the regularized solution is reached after k mesh refinements.

Proof.

Through the proof the Frechét derivative E'_α and the scalar product $(\cdot,\cdot)$ are given in W^1 norm.

Since z_k is a minimizer of the functional (8) then by (12) the minimizer z_k is unique and the functional (8) is strongly convex with the strong convexity constant κ. This implies that

$$\kappa[z_k - z_\alpha]^2 \leq \left(E'_\alpha(z_k) - E'_\alpha(z_\alpha), z_k - z_\alpha\right). \qquad (26)$$

Since z_k is the minimizer of the functional (7), then

$$\left(E'_\alpha(z_k), y\right) = 0, \ \forall y \in W_1. \qquad (27)$$

Next, since z_α is the minimizer on the set Q, then

$$\left(E'_\alpha(z_\alpha), z\right) = 0, \ \forall z \in Q.$$

Using (26) with the splitting

$$z_k - z_\alpha = (z_k - P_k z_\alpha) + (P_k z_\alpha - z_\alpha),$$

together with the Galerkin orthogonality principle (27) we obtain

$$\left(E'_\alpha(z_k) - E'_\alpha(z_\alpha), z_k - P_k z_\alpha\right) = 0 \qquad (28)$$

and thus

$$\kappa[z_k - z_\alpha]^2 \leq \left(E'_\alpha(z_k) - E'_\alpha(z_\alpha), P_k z_\alpha - z_\alpha\right). \qquad (29)$$

It follows from (23) that

$$\left(E'_\alpha(z_k) - E'_\alpha(z_\alpha), P_k z_\alpha - z_\alpha\right) \leq D[z_k - z_\alpha] ||P_k z_\alpha - z_\alpha||_{W^1}.$$

Substituting above equation into (29) we obtain (25).

□

The following theorem is valid for the functional (6) when the operator $A : H^1(\Omega) \to L_2(\Omega)$:

Theorem 1b

Let z_k be a minimizer of the functional (6). Assume that (13) holds and that the regularized solution z_α is not yet coincides with the minimizer z_k after k mesh refinements. Then there exists a constant D defined by (22) such that the following *estimate holds*

$$||z_k - z_\alpha||_{H^1} \leq \frac{D}{\kappa}||P_k z_\alpha - z_\alpha||_{H^1}. \tag{30}$$

In particular, if $P_k z_\alpha = z_\alpha$, *then* $z_k = z_\alpha$, *which means that the regularized solution is reached after k mesh refinements.*

Proof.

In this proof the Frechét derivative E'_α and the scalar product $(\cdot,\cdot)$ are given in H^1 norm. Since z_k is a minimizer of the functional (6) then by (13) the minimizer z_k is unique and the functional (6) is strongly convex on the space H^1 with the strong convexity constant κ. This implies that

$$\kappa\,||z_k - z_\alpha||^2_{H^1} \leq \left(M'_\alpha(z_k) - M'_\alpha(z_\alpha), z_k - z_\alpha\right). \tag{31}$$

Since z_k is the minimizer of the functional (6), then

$$\left(M'_\alpha(z_k), y\right) = 0,\ \forall y \in H^1. \tag{32}$$

Next, since z_α is the minimizer on the set H, then

$$\left(M'_\alpha(z_\alpha), z\right) = 0,\ \forall z \in H^1.$$

Using (31) with the splitting

$$z_k - z_\alpha = (z_k - P_k z_\alpha) + (P_k z_\alpha - z_\alpha),$$

together with the Galerkin orthogonality principle (32) we obtain

$$\left(M'_\alpha(z_k) - M'_\alpha(z_\alpha), z_k - P_k z_\alpha\right) = 0 \tag{33}$$

and thus

$$\kappa\,||z_k - z_\alpha||^2_{H^1} \leq \left(M'_\alpha(z_k) - M'_\alpha(z_\alpha), P_k z_\alpha - z_\alpha\right). \tag{34}$$

It follows from (24) that

$$\left(M'_\alpha(z_k) - M'_\alpha(z_\alpha), P_k z_\alpha - z_\alpha\right) \leq D||z_k - z_\alpha||_{H^1}||P_k z_\alpha - z_\alpha||_{H^1}.$$

Substituting above equation into (34) we obtain (30).

□

In Theorem 2 we derive a posteriori error estimates for the error in the Tikhonov functional (6) on the mesh obtained after k mesh refinements.

Theorem 2

Let conditions of Lemma 2 hold. Suppose that there exists minimizer $z_\alpha \in H^2(\Omega)$ *of the functional* M_α *on the set* V *and mesh* T. *Suppose also that there exists finite element approximation* z_k *of* M_α *on the set* W_h *and mesh* T. *Then the following approximate a posteriori error estimate for the error in the Tikhonov functional (6) holds*

$$|M_\alpha(z_\alpha) - M_\alpha(z_k)| \leq C_I \left\| M'_\alpha(z_k) \right\|_{H^1(\Omega)} \left(||h z_k||_{L_2(\Omega)} + ||[z_k]||_{L_2(\Omega)} + \sum_K ||[\partial_n z_k]||_{L_2(\partial K)} \right). \tag{35}$$

In the case when the finite element approximation $z_k \in V_h$ obtained by CG(1) we have

$$|M_\alpha(z_\alpha) - M_\alpha(z_k)| \leq C_I \left\| M'_\alpha(z_k) \right\|_{H^1(\Omega)} ||h z_k||_{L_2(\Omega)}. \tag{36}$$

Proof

By definition of the Frechét derivative we can write that on the mesh T we have

$$M_\alpha(z_\alpha) - M_\alpha(z_k) = M'_\alpha(z_k)(z_\alpha - z_k) + R(z_\alpha, z_k), \tag{37}$$

where by Lemma 1 $R(z_\alpha, z_k) = O((z_\alpha - z_k)^2), \ (z_\alpha - z_k) \to 0 \ \forall z_\alpha, z_k \in V$. The term $R(z_\alpha, z_k)$ is small since we assume that z_k is minimizer of the Tikhonov functional on the mesh T and this minimizer is located in a small neighborhood of the regularized solution z_α. Thus, we can neglect R in (37), see similar results for the case of a general nonlinear operator equation in $[1,3]$. Next, we use the splitting

$$z_\alpha - z_k = z_\alpha - z_\alpha^I + z_\alpha^I - z_k \tag{38}$$

and the Galerkin orthogonality [8]

$$M'_\alpha(z_k)(z_\alpha^I - z_k) = 0 \ \forall z_\alpha^I, z_k \in W_h \tag{39}$$

to get

$$M_\alpha(z_\alpha) - M_\alpha(z_k) \leq M'_\alpha(z_k)(z_\alpha - z_\alpha^I), \tag{40}$$

where z_α^I is a standard interpolant of z_α on the mesh T [8]. We have that

$$||M_\alpha(z_\alpha) - M_\alpha(z_k)||_{H^1(\Omega)} \leq ||M'_\alpha(z_k)||_{H^1(\Omega)} ||z_\alpha - z_\alpha^I||_{H^1(\Omega)}, \tag{41}$$

where the term $||z_\alpha - z_\alpha^I||_{H^1(\Omega)}$ in the right hand side of the above inequality can be estimated through the interpolation estimate with the constant C_I

$$||z_\alpha - z_\alpha^I||_{H^1(\Omega)} \leq C_I ||h\, z_\alpha||_{H^2(\Omega)}.$$

Substituting above estimate into (41) we get

$$||M_\alpha(z_\alpha) - M_\alpha(z_k)||_{L^2(\Omega)} \leq C_I \left\| M'_\alpha(z_k) \right\|_{L^2(\Omega)} ||h\, z_\alpha||_{H^2(\Omega)}. \tag{42}$$

Using the facts that [10]

$$|\nabla z_\alpha| \leq \frac{|[z_k]|}{h_K}$$

and

$$|D^2 z_\alpha| \leq \frac{|[\partial_n z_k]|}{h_K},$$

we can estimate $||h\, z_\alpha||_{H^2(\Omega)}$ in a following way:

$$
\begin{aligned}
||h\, z_\alpha||_{H^2(\Omega)} &\leq \sum_K ||h_K z_\alpha||_{H^2(K)} \leq \sum_K ||(z_\alpha + \nabla z_\alpha + D^2 z_\alpha) h_K||_{L_2(K)} \\
&\leq \sum_K \left(||z_k h_K||_{L_2(K)} + \left\| \frac{[z_k]}{h_K} h_K \right\|_{L_2(K)} + \left\| \frac{[\partial_n z_k]}{h_k} h_K \right\|_{L_2(\partial K)} \right) \qquad (43) \\
&\leq ||h z_k||_{L_2(\Omega)} + \|[z_k]\|_{L_2(\Omega)} + \sum_K \|[\partial_n z_k]\|_{L_2(\partial K)} .
\end{aligned}
$$

Here $D^2 z_\alpha$ denotes the second order derivatives of z_α. Substituting above estimate into right hand side of (42) we get estimate (35). □

6 A posteriori error estimates for the functional (44) in DG(1)

We now provide a more explicit estimate for the weaker norm $\|z_k - z_\alpha\|_{L_2(\Omega)}$ which is more efficient for practical computations since it does not involves computation of terms like $\|[\partial_n z_k]\|_{L_2(\partial K)}$ which are included in estimate (35). To do this, we replace in (6) the norm $\|z - z_0\|^2_{H^1(\Omega)}$ with the weaker norm $\|z - z_0\|^2_{L_2(\Omega)}$.

Below in Theorems 3, 4 and 5 we will consider the following Tikhonov functional

$$E_\alpha(z) : H \to \mathbb{R},$$

$$E_\alpha(z) = \frac{1}{2} \|Az - u\|_{L_2(\Omega)} + \frac{\alpha}{2} \|z - z_0\|^2_{L_2(\Omega)} . \qquad (44)$$

Theorem 3. *Let $\alpha \in (0,1)$ and $A : L_2 \to L_2$ be a bounded linear operator. Let $z_k \in W_h$ be the minimizer of the functional $E_\alpha(z)$ obtained by DG(1) on T. Assume that the regularized solution z_α is not yet reached on the mesh T and is not coincided with the minimizer z_k. Let jump of the function z_k computed from the two neighboring elements K^+ and K^- sharing the common side S on the mesh T is defined by*

$$[z_k] = z_k^+ - z_k^- . \qquad (45)$$

Then there exist constants D, C_I defined by (24),(19), correspondingly, such that the following estimate holds

$$\|z_k - z_\alpha\|_{L_2(\Omega)} \leq \frac{C_I D}{\alpha} \|[z_k]\|_{L_2(\Omega)}$$

Proof.

Conditions (24) imply that

$$\left\| E'_\alpha(z_k) - E'_\alpha(z_\alpha) \right\|_{L_2(\Omega)} \leq D \|z_k - z_\alpha\|_{L_2(\Omega)} \qquad (46)$$

with a constant $D(||A||, \alpha) > 0$. By (19)

$$\|z_\alpha - P_k z_\alpha\|_{L_2(\Omega)} \leq C_I \|h\, \nabla z_\alpha\|_{L_2(\Omega)} . \qquad (47)$$

Using the Cauchy-Schwarz inequality as well as (46) and (47), we obtain from (30)

$$||z_k - z_\alpha||_{L_2(\Omega)} \leq \frac{C_I D}{\alpha} ||h \nabla z_\alpha||_{L_2(\Omega)}. \tag{48}$$

We can estimate $||h \nabla z_\alpha||$ in a following way. Using the fact that [10]

$$|\nabla z_\alpha| \leq \frac{|[z_k]|}{h_K} \tag{49}$$

we have

$$\begin{aligned} ||h \nabla z_\alpha||_{L_2(\Omega)} &\leq \sum_K ||h_K \nabla z_\alpha||_{L_2(K)} \\ &\leq \sum_K ||h_K \frac{|[z_k]|}{h_K}||_{L_2(K)} = \sum_K ||[z_k]||_{L_2(K)}. \end{aligned} \tag{50}$$

Substituting above estimate in (48) we get

$$||z_k - z_\alpha||_{L_2(\Omega)} \leq \frac{C_I D}{\alpha} \sum_K ||[z_k]||_{L_2(K)} = \frac{C_I D}{\alpha} ||[z_k]||_{L_2(\Omega)}. \tag{51}$$

□

7 A posteriori error estimate for the error in the Tikhonov functional (44)

The proof of Theorem 4 is modification of the proof given in [5]. In the proof of this Theorem we used the fact the z_k is obtained using CG(1) on T. Theorem 5 follows from the proof of Theorem 4 in the case of DG(1) method.

Theorem 4

Let conditions of Lemma 2 hold and $A : L_2 \to L_2$ be a bounded linear operator. Suppose that there exists minimizer z_α of the functional E_α on the set V and mesh T. Suppose also that there exists approximation $z_k \in V_h$ of E_α. Then the following approximate a posteriori error estimate for the error in the Tikhonov functional (6) holds

$$|E_\alpha(z_\alpha) - E_\alpha(z_k)| \leq C_I ||E'_\alpha(z_k)||_{L_2(\Omega)} \cdot ||h \nabla z_\alpha||_{L_2(\Omega)}. \tag{52}$$

Proof

By definition of the Frechét derivative we can write that on every mesh T_k

$$E_\alpha(z_\alpha) - E_\alpha(z_k) = E'_\alpha(z_k)(z_\alpha - z_k) + R(z_\alpha, z_k), \tag{53}$$

where by Lemma 1 $R(z_\alpha, z_k) = O(r^2), \;\; r \to 0 \;\; \forall z_\alpha, z_k \in V, r = |z_\alpha - z_k|$.

Now we neglect R, use the splitting

$$z_\alpha - z_k = z_\alpha - z_\alpha^I + z_\alpha^I - z_k \tag{54}$$

and the Galerkin orthogonality [8]

$$E'_\alpha(z_k)(z_\alpha^I - z_k) = 0 \ \forall z_\alpha^I, z_k \in V_h \tag{55}$$

with the space DG(1) for approximation of functions z_α, to get

$$E_\alpha(z_\alpha) - E_\alpha(z_k) \le E'_\alpha(z_k)(z_\alpha - z_\alpha^I), \tag{56}$$

where z_α^I is a standard interpolant of z_α on the mesh T [8]. Applying interpolation estimate (19) to $z_\alpha - z_\alpha^I$ we get

$$|E_\alpha(z_\alpha) - E_\alpha(z_k)| \le C_I ||E'_\alpha(z_k)||_{L_2(\Omega)} \cdot ||h \, \nabla z_\alpha||_{L_2(\Omega)}. \tag{57}$$

□

Theorem 5

Let conditions of Lemma 2 hold and $A : L_2 \to L_2$. Suppose that there exists minimizer z_α of the functional E_α on the set V and mesh T. Suppose also that there exists approximation $z_k \in W_h$ of E_α obtained by DG(1). Then the following approximate a posteriori error estimate for the error in the Tikhonov functional (6) holds

$$|E_\alpha(z_\alpha) - E_\alpha(z_k)| \le C_I ||E'_\alpha(z_k)||_{L_2(\Omega)} \cdot \|[z_k]\|_{L_2(\Omega)}. \tag{58}$$

Proof

In the case of CG(1) by Theorem 4 we have the following a posteriori error estimate for the error in the Tikhonov functional (44)

$$|E_\alpha(z_\alpha) - E_\alpha(z_k)| \le C_I ||E'_\alpha(z_k)||_{L_2(\Omega)} \cdot ||h \, \nabla z_\alpha||_{L_2(\Omega)}. \tag{59}$$

Using now for $||h \, \nabla z_\alpha||_{L_2(\Omega)}$ the estimates (49)-(50) in the case of DG(1) we get the following a posteriori error estimate

$$|E_\alpha(z_\alpha) - E_\alpha(z_k)| \le C_I ||E'_\alpha(z_k)||_{L_2(\Omega)} \cdot \|[z_k]\|_{L_2(\Omega)}. \tag{60}$$

□

Using the Theorems 2 - 5 we can now formulate our mesh refinement recommendations in CG(1) and DG(1) for a Fredholm integral equation of the first kind used in practical computations. Let us define a posteriori error indicator

$$E_h(z_k) = \int_\Omega z_k(y) \left(\int_\Omega K(x,y) K(x-s) dx \right) dy - \int_\Omega K(x-s) u(x) \, dx. \tag{61}$$

We note that a posteriori error indicator (61) is approximation of the function $|E'_\alpha(z_k)|$ which is used in the proofs of Theorems 2-5. We neglect the computation of the regularization term in the function $|E'_\alpha(z_k)|$ since this term is very small, and obtain a posteriori error indicator (61). Such approximation does not affect on the refinement of the mesh.

The First Mesh Refinement Recommendation. *Using the Theorems 4 and 5 we can conclude that we should refine the mesh in neighborhoods of those points in* Ω *where the function* $|E'_\alpha(z_k)|$ *or the function* $|E_h(z_k)|$ *attains its maximal values. More precisely, let* $\varkappa \in (0,1)$ *be the tolerance number which should be chosen in computational experiments. Refine the mesh in such subdomains of* Ω *where*

$$\left|E'_\alpha(z_k)\right| \geq \varkappa \max_\Omega \left|E'_\alpha(z_k)\right|$$

or

$$|E_h(z_k)| \geq \varkappa \max_\Omega |E_h(z_k)| \,.$$

The Second Mesh Refinement Recommendation. *Using the Theorem 3 we can conclude that we should refine the mesh in neighborhoods of those points in* Ω *where the function* $|z_k|$ *attains its maximal values. More, precisely in such subdomains of* Ω *where*

$$|z_k| \geq \widetilde{\varkappa} \max_\Omega |z_k|$$

where $\widetilde{\varkappa} \in (0,1)$ *is the number which should be chosen computationally*.

8 The Adaptive Algorithm

In this section for solution of a Fredholm integral equation of the first kind (2) we present adaptive algorithms which we apply in numerical examples of section 9. Our algorithms use mesh refinement recommendations of section 7. In these algorithms we also assume that the kernel in (2) is such that $K(x-y) = \rho(y-x)$. Next, using the convolution theorem we can determine the functions $z(x)$ in (2) and the regularized solution z_α of (6), correspondingly. For example, for calculation of the function $z_\alpha(x)$ in numerical examples of section 9 we use (69). In our algorithms we define the minimizer and its approximation by z_α and z_k, correspondingly.

In Algorithm 1 we apply the first mesh refinement recommendation of Section 7, while in Algorithm 2 are used both mesh refinement recommendations of section 7. These algorithms are successfully tested by numerical examples of section 9.

Algorithm 1

- Step 0. Choose an initial mesh T_0 in Ω and obtain the numerical solution z_0 of (6) on T_0 using the finite element discretization of (20) for CG(1) or (21) for DG(1) and discretization of the convolution theorem (69). Compute the sequence $z_k, k > 0$, via following steps:

- Step 1. Interpolate the given right hand side of (2) and the solution z_{k-1} from the mesh T_{k-1} to the mesh T_k and obtain the numerical solution z_k of (6) on T_k using the finite element discretization of (20) for CG(1) or (21) for DG(1) and discretization of (69).
- Step 2. Refine the mesh T_k at all points where

$$|B_h(z_k)| \geq \beta_k \max_{\Omega} |B_h(z_k)|, \tag{62}$$

with

$$B_h(z_k) = \int_{\Omega} z_k(y) \left(\int_{\Omega} \rho(x,y)\rho(x,s)dx \right) dy - \int_{\Omega} \rho(x,s)\, u(x)\; dx. \tag{63}$$

Here the tolerance number $\beta_k \in (0,1)$ is chosen by the user.
- Step 3. Construct a new mesh T_{k+1} in Ω and perform steps 1-3 on the new mesh. Stop mesh refinements when $||z_k - z_{k-1}|| < \varepsilon$ or $||B_h(z_k)|| < \varepsilon$, where ε is tolerance chosen by the user.

Algorithm 2

- Step 0. Choose an initial mesh T_0 in Ω and obtain the numerical solution z_0 of (6) on T_0 using the finite element discretization of (20) for CG(1) or (21) for DG(1) and discretization of the convolution theorem (69). Compute the sequence $z_k, k > 0$, via following steps:
- Step 1. Interpolate the given right hand side of (2) and the solution z_{k-1} from the mesh T_{k-1} to the mesh T_k and obtain the numerical solution z_k of (6) on T_k using the finite element discretization of (20) for CG(1) or (21) for DG(1) and discretization of (69).
- Step 2. Refine the mesh T_k at all points where

$$|B_h(z_k)| \geq \beta_k \max_{\Omega} |B_h(z_k)| \tag{64}$$

with $B_h(z_k)$ defined by (63), and where

$$|z_k(x)| \geq \widetilde{\varkappa}_k \max_{\Omega} |z_k(x)|. \tag{65}$$

Here the tolerance numbers $\beta_k, \widetilde{\varkappa}_k \in (0,1)$ are chosen by the user.
- Step 3. Construct a new mesh T_{k+1} in Ω and perform steps 1-3 on the new mesh. Stop mesh refinements when $||z_k - z_{k-1}|| < \varepsilon$ or $||B_h(z_k)|| < \varepsilon$, where ε is tolerance chosen by the user.

Remarks

- 1. We note that the choice of the tolerance numbers $\beta_k, \widetilde{\varkappa_k}$ in (63), (65) depends on the concrete values of $\max_\Omega |B_h(z_k)|$ and $\max_\Omega |z_k(x)|$, correspondingly. If we would choose $\beta_k, \widetilde{\varkappa_k}$ very close to 1 then we would refine the mesh in very narrow region of the computational domain Ω, and if we will choose $\beta_k, \widetilde{\varkappa_k} \approx 0$ then almost all mesh of the domain Ω will be refined what is unsatisfactory. Thus, the values of the numbers $\beta_k, \widetilde{\varkappa_k}$ should be chosen in optimal way. Our numerical tests show that the choice of $\beta_k, \widetilde{\varkappa_k} = 0.5$ is almost optimal one, however, it can be changed during the iterations in adaptive algorithms from one mesh to other.
- 2. We also note that we neglect the computation of the regularization term in a posteriori error indicator (63) since this term is very small and does not affects on the refinement procedure. However, this term is included in the minimization procedure of the Tikhonov's functional (6).

9 Numerical studies of the adaptivity technique in microtomography

Microtomography in the backscattering electron mode allows obtain the picture of a plain layer which is located at a some depth below the surface of the investigated object. Due to the fact that the electron probe (monokinetic electron beam) has the finite radius the measured signal of this layer is distorted. It was shown in [13] and [14] that there exists connection between the measured signal obtained by the electron microscope and the real scattering coefficient of the object under investigation. The measured signal $u(\xi,\eta)$ can be described by a Fredholm integral equation of the first kind

$$u(\xi,\eta) = \int_\Omega z(x,y)\rho(x-\xi,y-\eta)dxdy. \tag{66}$$

Here, the kernel is given by the relation

$$\rho(x,y) = \frac{1}{2\pi r^2}\exp(-\frac{x^2+y^2}{2r^2}) \tag{67}$$

with the variance function $r = r(t)$ depended on the depth t of the layer under investigation.

To solve the equation (66) we minimize the following Tikhonov functional on Sobolev space H^1

$$M_\alpha(z) = ||\int_\Omega \rho(x-\xi,y-\eta)z(x,y)dxdy - u(\xi,\eta)||^2_{L_2(\Omega)} + \alpha\|z(x,y)\|^2_{H^1}. \tag{68}$$

Using the convolution theorem we can obtain the following expression for the minimizer $z_\alpha(x,y)$ (see, for example, [15]) of the functional (68)

$$z_\alpha(x,y) = \int_{R^2} e^{-i(\lambda x + \nu y)} \frac{P^*(\lambda,\nu)U(\lambda,\nu)}{|P(\lambda,\nu)|^2 + \alpha(1+\lambda^2+\nu^2)^2} d\lambda d\nu, \tag{69}$$

where functions U and P are the Fourier transforms of the functions u and ρ, respectively, and P^* denotes the complex conjugated function to the function P.

The goal of our computational test was to restore image of Figure 1-a) which represents the part of the planar microscheme obtained from the experimentally measured data by microtomograph [14]. This image was measured on the depth 0.9 μm of the microscheme with the smearing parameter $r = 0.149$ mkm in (67). Real area of the image of Figure 1-a) is $\Omega = 16.963$ mkm. For restoration of the image of Figure 1-a) we apply the adaptive algorithm of section 8.

First, we compute z_0 on the initial coarse mesh T_0 using the finite element discretization of (69) as described in section 3 with the regularization parameter $\alpha = 2e10-07$ in (68) on the coarse mesh presented in Figure 1-g). Let us define the function

$$B_h(z_k) = \int\limits_\Omega z_k(y) \left(\int\limits_\Omega \rho(x-\xi, y-\eta)\rho(x-\xi, s-\eta)\, dx \right) dy - \int\limits_\Omega \rho(x-\xi, s-\eta) u(x)\, dx, \tag{70}$$

where Ω is our two-dimensional domain. We refine the mesh in all subdomains of Ω where the gradient of the function $B_h(z_k)(x)$ attains its maximal values, or where

$$|B_h(z_k)| \geq \beta_k \max_\Omega |B_h(z_k)| \tag{71}$$

with $\beta_k = 0.5$. Next, we perform all steps of the adaptive algorithm until the desired tolerance $||z_k - z_{k-1}|| < \varepsilon$ with $\varepsilon = 10e-05$ is achieved or the computed L_2- norms of the differences $||z_k - z_{k-1}||$ are started abruptly grow.

Figures 1-b)-f) show results of the reconstruction on the adaptively refined meshes which are presented in Figures 1-h)-l). Using the Figure 1 we observe that on the fifth refined mesh corresponding to the Figure 1-l) we obtain the best restoration results. Since the computed L_2- norms $||z_k - z_{k-1}||$ are started abruptly grow after the fifth refinement of the initial mesh we conclude that the restoration image of the Figure 1-f) is the resulting one.

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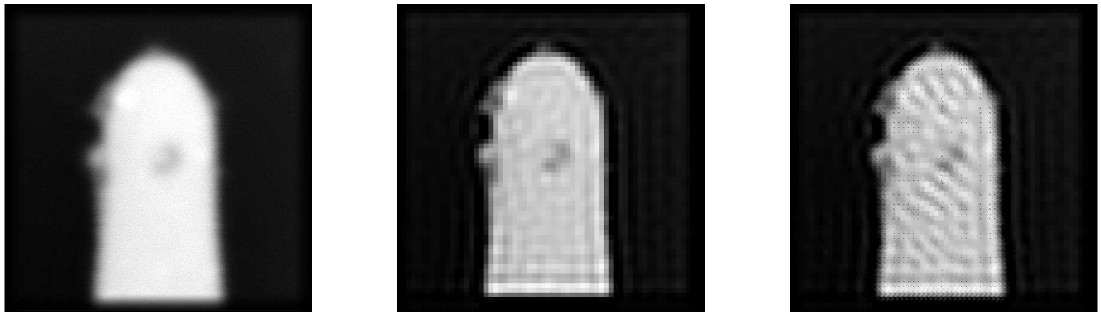
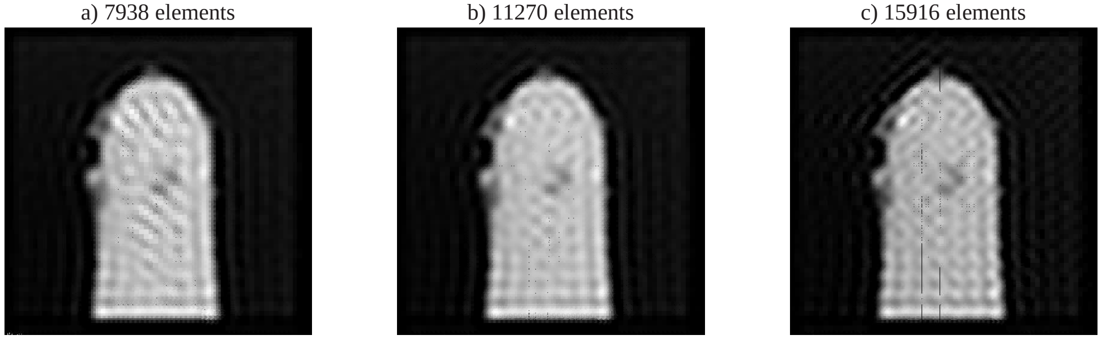
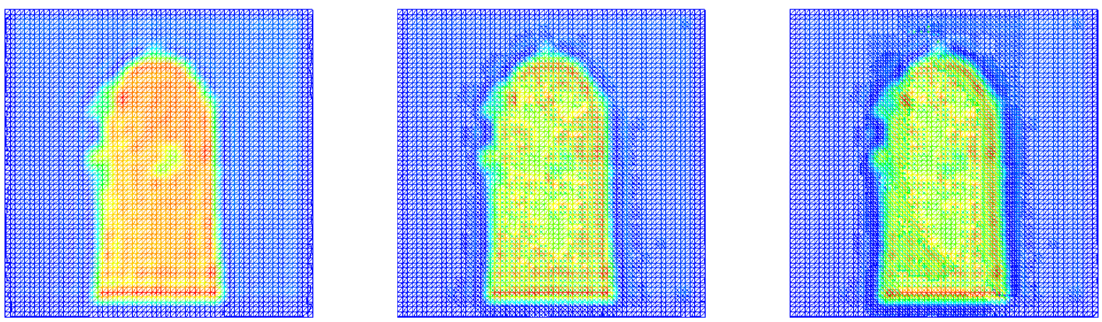


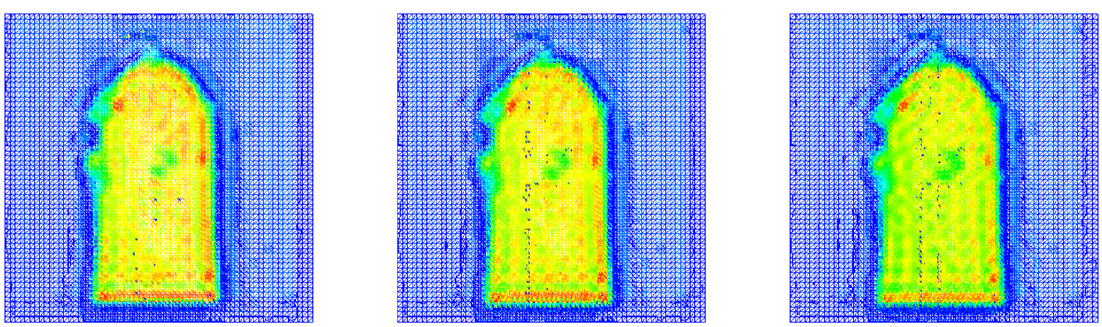
a) 7938 elements b) 11270 elements c) 15916 elements



d) 24262 elements e) 40358 elements f) 72292 elements

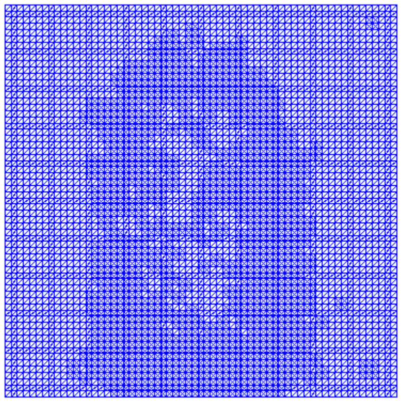


g) 7938 elements h) 11270 elements i) 15916 elements

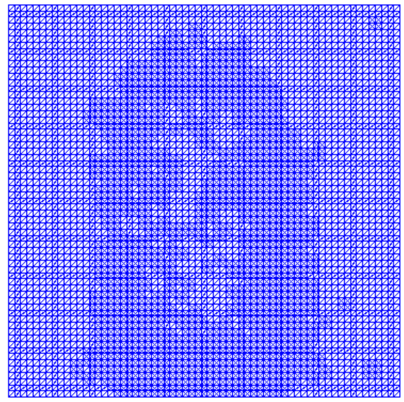


j) 24262 elements k) 40358 elements l) 72292 elements

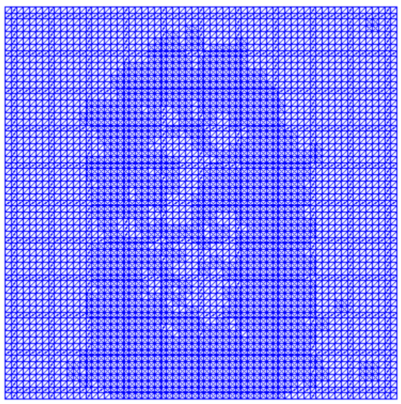
Fig. 1 Reconstructed images from the experimental backscattering data obtained by the microtomograph [14, 17]. On a) we present the real measured signal on the part of the planar microscheme obtained by microtomograph on the depth 0.9 μm. On b)-f) we show results of the image restoration presented on a) on different adaptively refined meshes using the algorithm of section 8. Reconstruction results together with correspondingly adaptively refined meshes are presented on g)-l).



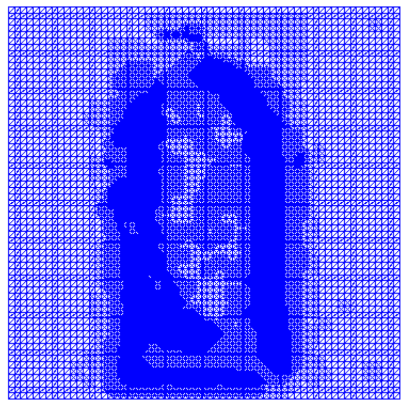
a) 7938 elements



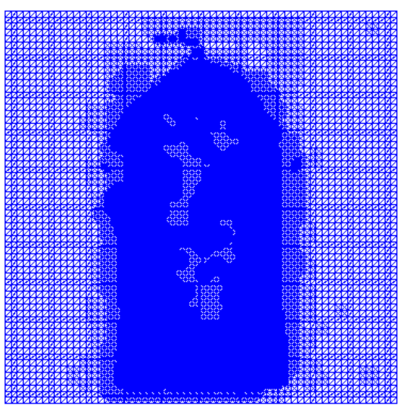
b) 11270 elements



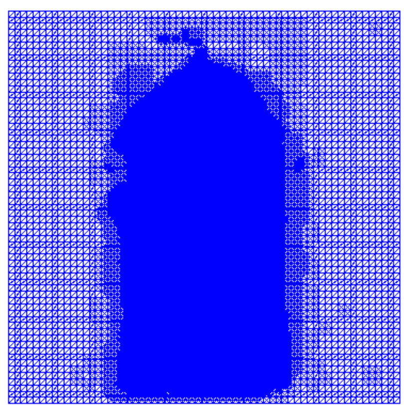
c) 7938 elements



d) 15916 elements



e) 40358 elements



f) 72292 elements

Fig. 2 Adaptively refined meshes which correspond to the images of Figures 1-g)-l) are presented on a)-f). Also, reconstructed images of Figures 1-b)-f) correspond to the adaptively refined meshes shown on b)-f).

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