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On Lipschitz continuity of solution operators to nonlinear wave equations

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Abstract

We will study regularity properties of the solution operator $\mathcal{E}_t$ to the subcritical semi-linear wave equation, and in particular the Lipschitz continuity in the (non-linear) energy space in low dimensions.

1 Introduction

We will discuss regularity properties of the solution operator $\mathcal{E}_t$ to the semi linear hyperbolic equation

$$\partial_t^2 u - \Delta u + m^2 u + f(u) = 0,\ t > 0,\ x \in \mathbf{R}^n,\ u|_0 = \phi, \partial_t u|_0 = \psi, \tag{1.1}$$

in case $m = 0$, $f(u) = |u|^{\rho-1}u$ (we could use more assumptions on f, e.g. those in [11]) and where the data ϕ, ψ belong to $(\dot{H}_2^1 \cap L_{\rho+1}) \times L_2 = X_e$, the energy space for the non linear equation. If $m \neq 0$, the space $\dot{H}_2^1$ is replaced by the corresponding inhomogeneous space $H_2^1 = \dot{H}_2^1 \cap L_2$. If $m > 0$, we denote the equation (1.1) by NLKG (the non linear Klein-Gordon equation), and if $m = 0$, we denote it by NLWE (the non linear wave equation). Regularity properties of $\mathcal{E}_t$ for the NLKG were discussed in [11].
For any solution u with data in the energy space X_e the energy

$$E(u) = \frac{1}{2}\int(|\partial_x u|^2 + m^2|u|^2 + |\partial_t u|^2)dx + \frac{1}{\rho+1}\int |u|^{\rho+1}dx$$

is conserved for $\rho < \rho^*(n) = \frac{n+2}{n-2}$, and is for $\rho \geq \rho^*(n)$ uniformly bounded by the energy of $u(0)$ (cf. Strauss [37]).

Above $\dot{H}^s_p$ denotes the homogeneous Sobolev space of order s, based on L_p, where for $p = 2$ we may drop the reference to the L_2-space. The corresponding inhomogeneous space is then $H^s_p = \dot{H}^s_p \cap L_p$.

For the linear and non-linear Klein-Gordon equations, the energy spaces on which energy is conserved in the sub-critical case are the same (by Sobolev's inequality), that is $H^1_2 \times L_2$.
This is not the case for the linear and non-linear wave equations, however. Here the energy space for the linear equation is, leaving out non-zero constant solutions, $E = \dot{H}^1_2 \cap L_{\rho^*+1} \times L_2$. with $n\delta_{\rho^*+1} = 1$ for (corresponding to the critical case $\rho = \rho^*$ for the non-linear equation), while the NLWE has as energy space $X_e = \dot{H}^1_2 \cap L_{\rho+1} \times L_2$. We will write $e(I)$ for $L_\infty(I; X_e)$ and $E(I)$ for $L_\infty(I; E)$. Solutions with finite energy data are unique in the sub-critical case (see Ginibre and Velo [16],[18] and also [10]). We give a short proof for the NLWE below.
The following result is partly known and implicit in these uniqueness and existence results. For a proof, see e.g. [11] in the case $m > 0$, and for earlier related work for $m = 0$ in [18]. The differences in results between this paper and that of Ginibre and Velo [18] is mainly a result of a different choice of the basic energy space X_e, and the focus on slighly different problems for the NLWE.

Theorem 1.1. *Let $3 \leq n \leq 7$, and $\rho^*(n) > \rho > \rho_* = 1 + \frac{4}{n-1}$. Then $\mathcal{E}_t$ is Lipschitz continuous on the energy space X_e for $t > 0$.*

The following comments follow from the proof of Theorem 1.1.

(1) With the lower bound $\rho > 1 + \frac{4.5}{n-1}$ Theorem 1.1 holds also for n=8.

(2) Let $3 \leq n \leq 7$, and $\rho^*(n) > \rho > 1$. Then $\mathcal{E}_t : X_e \mapsto E$ is Lipschitz continuous for $t > 0$.

The question of uniform estimates in the time variable will be treated in a sequel to this paper.
As for $m > 0$ (although not proved here) the solution operator for the NLWE in the subcritical case is also Hölder continuous of order $\alpha(n) > 0$ for $n \geq 3$ on X_e. ([11]).
The restriction to low dimensions in Theorem 1.1 is natural (and the given bounds of the space dimension probably necessary) in view of the results on Lipschitz and Hölder continuous opertors in [13].
We will supply a proof of Theorem 1.1 based on the methods used in [11].
For basic properties of Besov and Sobolev spaces we refer to Bergh and Löfström [3] and also [2],[14]. For a discussion of the homogeneous Besov and Sobolev

spaces in the present context, e.g. representation of the elements in the homogeneous spaces as distributions, see Bourdaud, [5], [6] and also the Appendix in [18].

2 Energy spaces

For the linear and non-linear Klein-Gordon equations, the energy spaces on which energy is conserved in the sub-critical case are the same (by Sobolev's inequality), that is $H_2^1 \times L_2$.
As mentioned in the introduction, this is not the case for the linear and non-linear wave equations, however. Here the energy space for the linear equation is, leaving out non-zero constant solutions, $\dot{H}_2^1 \cap L_{\rho^*+1} \times L_2$. with $n\delta_{\rho^*} = 1$ (corresponding to the critical case for the non-linear equation), while the NLWE has as energy space $X_e = \dot{H}_2^1 \cap L_{\rho+1} \times L_2$. We will write X_e^1 for the first component.
Let us point out a property of finite energy solutions u(t) of the linear and non-linear wave equation (cf. [39]): Assume that additionally $u(0) = \phi \in L_2$. For such u(t), $t \mapsto \partial_t u(t)$ is weakly continuous in H_2^1 and so, again weakly, $u(t) = \int_0^t \partial_t u(\tau) d\tau + u(0)$. Thus the L_2-norm of u(t) is by the energy inequality estimated by

$$\|u(t)\|_{L_2}^2 \leq 4T^2 E(u(0)) + 2\|u(0)\|_{L_2}^2 \ , \ 0 < t \leq T$$

for any $T > 0$. If we apply (linear) interpolation between H_2^1 and $\dot{H}_2^1 \cap L_{\rho^*+1}$ we find that:

> if u_0 is a solution of the wave equation, with initial data in $X_e = \dot{H}_2^1 \cap L_{\rho+1} \times L_2$, then also $u_0 \in L_\infty^{loc}(X_e)$, and in particular, $u_0(t) \in L_{p'}$ for $\rho + 1 \leq p' \leq \rho^* + 1$, $t \in \mathbf{R}_+$, with growth in t bounded by $C(1+t)^{1-n\delta_{p'}}$.

By the energy inequality, the same statement holds for the solution u(t) of the non-linear equation, now with *uniform* estimates i time, however.

3 Solution operators and basic kernel estimates

Let u_0 be the solution of the wave equation

$$\partial_t^2 u - \Delta u = 0, \ t > 0, \ x \in \mathbf{R}^n, \ u|_0 = \phi, \partial_t u|_0 = \psi, \tag{3.1}$$

with (ϕ, ψ) *in* $X_e = \dot{H}^1_2 \cap L_{\rho+1} \times L_2$, with $\rho < \rho^*$. Then we can write

$$u_0(t) = E_0(t)\phi + E_1(t)\psi$$

and by e.g, [7]

$$\|E_\mu(t)v\|_{\dot{B}^{s'}_{p'}} \leq K_\mu(t)\|v\|_{\dot{B}^s_p},\ \mu = 0, 1\ , \tag{3.2}$$

where for $0 \leq \theta \leq 1$

$$0 = \mu - (n + 1 + \theta)\delta + s - s' \tag{3.3}$$

and where $K_\mu(t)$ satisfies

$$K_\mu(t) \leq Ct^{\mu - 2n\delta + s - s'} \leq Ct^{-(n-1-\theta)\delta} \tag{3.4}$$

Let $K(t)$ denote the upper bound in (3.4). Notice that with the assumption (3.3) K is independent of μ. If

$$(n - 1 - \theta)\delta < 1 \tag{3.5}$$

then $K \in L^{loc}_r$ with $1 \leq r < \frac{1}{(n-1-\theta)\delta}$.
If we replace the Besov spaces in (3.2) with their inhomogeneous counterparts, then we may use inequality ($\leq$) in (3.3).
At this point it is convenient to notice that the finite energy solution of (1.1) is the (weak) finite energy solution of integral equation

$$u(t) = u_0(t) + \int_0^t E_1(t - \tau)f(u(\tau)d\tau \tag{3.6}$$

where u_0 is the solution of the linear Klein-Gordon equation (3.1) with the same initial data (in X_e) as u.
Remark We have (cf. [19]),

$$\dot{B}^s_p = \dot{B}^{s,2}_p \supseteq \dot{H}^s_p,\ p \leq 2\quad ,\quad \dot{B}^{s'}_{p'} = \dot{B}^{s',2}_{p'} \subseteq \dot{H}^{s'}_{p'}\ ,\ p' \geq 2 \tag{3.7}$$

Thus estimates of the $\dot{B}^{s'}_{p'}$-norms (in terms of $\dot{B}^s_p$-norms) imply the corresponding estimates in the Sobolev norms $\dot{H}^{s'}_{p'}$ (in terms of the $\dot{H}^s_p$-norms). Estimates of the $\dot{H}^s_p$-norm in terms of $\dot{H}^{s'}_{p'}$-norms in the same way imply the corresponding Besov space estimates. The estimates in this section (these subsections od Section 5) will be proved in the appropriate strongest form - so that all the main estimates will be equally valid in Besov space norms as in Sobolev space norms.
We will also use the Sobolev embedding in the following form for the homogeneous Besov spaces (for a short proof, see [19]): Let $s_i \in \mathbf{R}$, $1 \leq p_1 \leq p_2$ and let $\delta_i = \delta_{p_i}$, $i = 1, 2$. Then

$$B^{s_1}_{p_1} \subseteq B^{s_2}_{p_2},\ \text{ if } \frac{n}{p_1} - s_1 = \frac{n}{p_2} - s_2, \tag{3.8}$$

and in particular if

$$n\delta_1 + s_1 = n\delta_2 + s_2, p_1 \geq 2$$

and correspondingly for $p_2 \leq 2$, changing the signs of the s_i.

4 Strichartz type estimates

Using (3.4), a duality argument (a clear exposition of the duality argument can be found in Ginibre and Velo [18]) and Young's (or the Hardy-Littlewood) inequality, we obtain the following space-time estimate the classical Strichartz estimates, originated by Strichartz [38]. For general Strichartz estimates, see Ginibre and Velo [18], and the references given there.The endpoint estimates are due to Keel and Tao [25].
First some notation: We denote $L_{r'}(I; B^{s'}_{p'})$ by $A(s', I)$ and $L_r(I; B^s_p)$ by $B(s, I)$ for any interval I in $\mathbf{R}_+$. If (r', s', p') or $(r', 1-s, p')$, where as usual r', s' are the duals of r, s, are *Strichartz triples*, i.e. satisfy (4.1) below, then we simply write $A(I)$ and $B(I)$, respectively. With $a, b, \ldots$ we denote the spaces $A, B, \ldots$ with the Besov spaces replaced by Sobolev spaces.
The corresponding spaces with homogeneous Besov or Sobolev spaces are as usual supplied with a $\cdot$ on top.
In addition we write $E = \dot{H}^1_2 \cap L_{\rho^*+1}$, $E(I) = L_\infty(I; \dot{H}^1_2 \cap L_{\rho^*+1})$ and $e(I) = L_\infty(I; X^1_e)$.

Proposition 4.1 (Strichartz estimates)**.** *Let $n \geq 3$, $s' \in \mathbf{R}$ and r', $p' \geq 2$. Then if u_0 is a finite energy solution of the wave equation,*

$$u_0 \in \dot{A}(I), \text{ any interval } I \subseteq \mathbf{R}_+$$

and

$$\|u_0\|_{\dot{A}(I)} \leq C\|u_0\|_{E(I)}$$

with C independent of I and u_0, provided (r', p', s') satisfy

$$s' = 1 - \frac{n+1}{2}\delta - s'', \text{ and } \frac{1}{r'} = \frac{1}{2}(n-1)\delta - s'' \leq \frac{1}{2}, \tag{4.1}$$

$for\ some$ $s'' \geq 0$.
For $n = 3$ the value $r' = 2$, $(n-1)\delta = 1$ is not allowed.

We call a triple (r', s', p') satisfying (4.1), with the endpoint restrictions a *Strichartz triple.*
Another way to write (4.1) is (cf. [19])

$$s' + n\delta_{p'} - \frac{1}{r'} = 1 \tag{4.2}$$

$$\frac{2}{r'} \leq \min((n-1)\delta, 1) \tag{4.3}$$

again with $r' = 2,\ (n-1)\delta = 1$ not allowed for $n = 3$.

Let as above $X_e^1 = \dot{H}_2^1 \cap L_{\rho+1}$. Then by the discussion in the end of the last subsection,

$$L_\infty^{loc}(X_e^1) \subset L_{r'}^{loc}(X_e^1) \subset L_{r'}^{loc}(L_{p'}),\ \rho+1 \leq p' \leq \rho^*+1,\ r' \leq +\infty \tag{4.4}$$

so that

$$\|u_0\|_{L_{r'}(I;L_{p'})} \leq C(I)\|u_0\|_{e(I)},\ \rho+1 \leq p' \leq \rho^*+1,\ r' \leq +\infty \tag{4.5}$$

where I is a bounded interval i $\mathbf{R}_+$. With the Strichartz estimate this proves the following estimate for solutions u_0 of the wave equation with data in X_e:

$$\|u_0\|_{A(I)} \leq C(I)\|u_0\|_{e(I)},\ I \subset\subset \mathbf{R}_+,\ \rho+1 \leq p' \leq \rho^*+1 \tag{4.6}$$

We may take this further, however. Since $u_0 \in L_{r'}(I; B_{p'}^{s'})$ for any Strichartz triple with $\rho+1 \leq p' \leq \rho^*+1$, $u_0(t) \in L_{p''}$ for $\frac{1}{p''} \geq \frac{1}{p'} - \frac{s'}{n}$ for any p' in this range: hence

$$\delta_{p''} = \frac{1}{n} + \frac{(n-1)\delta_{p'}}{2n}$$

Thus $u_0 \in L_{r''}^{loc}(B_{p''}^{s''})$, where (r'', s'', p'') is a Strichartz triple (as long as $(n-1)\delta_{p''} \leq 1$). If we bootstrap this argument, and use that

$$\frac{1}{n}\sum_0^\infty \left(\frac{(n-1)}{2n}\right)^k \geq \frac{1}{n-1},\ n \geq 3$$

with strict inequality for $n > 3$, we find that in a finite number of steps we reach $(n-1)\delta_{p'} = 1$ for $n > 3$, and come arbitrarily close to this for $n = 3$. Together with the Strichartz estimate, (4.2) and (3.8), this now proves that

$$\|u_0\|_{A(I)} \leq C(|I|)\|u_0\|_{e(I)},\ I \subset\subset \mathbf{R}_+,\ \rho+1 \leq p', \tag{4.7}$$

If we assume $\rho^*+1 \leq p'$ then $e(I)$ may be replaced by $E(I)$, with $C(|I|)$ depending on the X_e-norm of the initial data.

We now turn to some variations of Proposition 4.1 and the Strichartz estimates. Let w_0 be defined by

$$w_0(t) = \int_0^t E_1(t-\tau)h(\tau)d\tau \tag{4.8}$$

where $h = h(t,x) \in L_1^{loc}(L_2)$.

Proposition 4.2. *With the notation of (3.2) through (3.4), let $\mu = 1$. Let $I \subset \mathbf{R}_+$, let w_0 be defined by (4.8) with the integral taken over I. Let $\dot{A}(I)$ be defined as above, let $\bar{B}(I) = L_{\bar{r}}(I; \dot{B}_{\bar{p}}^{\bar{s}})$. Let (r', s', p') and $(\bar{r}', \bar{s}', \bar{p}')$ be Strichartz triples with $\bar{s} = 1 - \bar{s}'$. Then*

$$\|w_0\|_{E(I)} + \|w_0\|_{A(I)} \leq C\|h\|_{\bar{B}(I)},\ t \in I \tag{4.9}$$

with C independent of I.

For a proof, see e.g. Ginibre and Velo [18], and for the end-point estimates [25].

We will use (and prove) the following variation of Proposition 4.2.

Proposition 4.3. *With the notation of Proposition 4.2, let I be a bounded interval in $\mathbf{R}_+$, and let w_0 be defined by (4.8) with the integral taken over I. Then for $0 \leq \epsilon \leq \frac{1}{\bar{r}'}$,*

$$\|w_0\|_{\dot{A}(I)} \leq C|I|^{\epsilon}\|h\|_{\bar{B}(\bar{s}+\epsilon,I)}, \ t \in I \tag{4.10}$$

with C independent of I.

Again, we want to use initial data i X_e and hence estimate also w_0 in $L_\infty^{loc}(X_e^1)$. Excluding non-zero constants (which will not be in X_e) we have by Proposition 4.3 and the invariance of $E_\mu(t)$ under (fractional) powers of $(-\Delta)$ we get for $\sigma \geq 0$

$$\|w_0(t)\|_{\dot{H}_2^{1-\sigma}} \leq C|I|^{\epsilon'}\|h\|_{\bar{B}(\bar{s}+\epsilon'-\sigma,I)}, \ t \in I \tag{4.11}$$

and since under these assumptions,

$$\dot{H}_2^{1-\sigma} \subset L_{\rho+1}, \ \sigma = 1 - n\delta_{\rho+1}$$

by Proposition 4.3 we get the following result:

Corollary 4.1. *With the above notation, let $e(I) = L_\infty(I; X_e^1)$. Then for I a bounded interval in $\mathbf{R}_+$*

$$\|w_0\|_{e(I)} \leq C|I|^{\epsilon}(\|h\|_{\bar{B}(\bar{s}+\epsilon-\sigma,I)} + \|h\|_{\bar{B}(\bar{s}+\epsilon,I)}) \tag{4.12}$$

with C independent of I, where

$$\sigma = 1 - n\delta_{\rho+1} = n(\delta_{\rho^*+1} - \delta_{\rho+1}) \tag{4.13}$$

There is another consequence of Proposition 4.3:The uniform continuity of $w(t)$ in H_2^1-norm.

Corollary 4.2. *Assume that $h \in \bar{B}(\bar{s}+\epsilon, I)$, $I \subset \mathbf{R}_+$ and $\epsilon > 0$. Then $w_0(t)$ is uniformly continuous on I in the $\dot{H}_2^1$-norm.*

Proof: Let $t, t' \in I$. We have then

$$w_0(t) - w_0(t') = \int_{t'}^{t} E_1(t-\tau)h(\tau)d\tau + (E_0(t-t') - I)w_0(t')$$

where by Proposition 4.2 and our assumptions $w_0(t')$ is uniformly bounded in E. Since $E(t'') - I$ tends to 0 uniformly on E as $t'' \to 0$, the second term is uniformly continuous on E for $t, t' \in I$. Applying Proposition 4.3, our assumptions on h imply that the first term is Hölder continuous of order $\epsilon > 0$ on E for $t, t' \in I$. This completes the proof. □

Proof: [of Proposition 4.3 based on Proposition 4.2] Let $\epsilon > 0$, Then if $(\bar{r}', 1 - \bar{s}, \bar{p}')$ is a Strichartz triple, the so is $(\bar{r}'_\epsilon, 1 - \bar{s} - \epsilon, \bar{p}')$, where $\frac{1}{\bar{r}_\epsilon} = \frac{1}{\bar{r}} + \epsilon$. Replace $\bar{r}$ by $\bar{r}_\epsilon$ and $\bar{s}$ by $\bar{s} + \epsilon$ in (4.10). Then apply Hölder's inequality to the $L_{\bar{r}_\epsilon}$-integral over I, and the statement in the Corollary follows. □

5 Besov- and Sobolev space estimates of $u \mapsto f(u)$

The following estimates of $f(u)$ are well known (cf. [8],[18]) and contains a version of Lemma 3.1 in [11] (some misprints are corrected here) for homogeneous spaces. The proof is a straightforward application of Hölder's inequality, but as it contains an unusual twist in case $\rho \leq 2$, we give a sketch of that part of the proof.
As before (motivated by our study of functions u in $\dot{H}^1_2 \cap L_{\rho+1}$) we exclude non-trivial constants from the homogeneous spaces. In applications we will usually assume (as in (3.3) through (3.5)) that $(n-1-\theta)\delta < 1$.

Lemma 5.1. *Let $f(u) = |u|^{\rho-1}u$ with $\rho < \rho^* = \frac{n+2}{n-2}$.*
Let $0 \leq \theta \leq 1$ and assume that $1 \geq s$, and that s, $s' \geq 0$, with $s - s' = (n+1+\theta)\delta - 1$. Assume in addition that ϵ and η are real.
Then

$$\|f(u)\|_{\dot{B}^{s+\epsilon}_p} \leq C \|u\|^{\rho-1+\eta}_{\dot{H}^1_2} \|u\|^{1-\eta}_{\dot{B}^{s'}_{p'}} \tag{5.1}$$

where for $\rho \leq 2 - \eta$,

$$\frac{s+\epsilon-s'}{1-s'} + 1 - \eta \leq \rho \tag{5.2}$$
$$1 - \rho \leq \eta \leq 2 - \rho$$

and where for $\rho + \eta \geq 2$, $1 - \rho \leq \eta \leq \min(1, \rho - 2)$, provided

$$\rho = \rho(n, \delta, \theta) - 2\eta \frac{n\delta - 1 + s'}{n-2} - 2\epsilon \frac{1}{n-2} \tag{5.3}$$
$$\rho(n, \delta, \theta) = \frac{n + 2(n-1-\theta)\delta}{n-2}$$

Correspondingly for non-homogeneous spaces B^s_p and $B^{s'}_{p'}$ we have

$$\|f(u)\|_{B^{s+\epsilon}_p} \leq C \|u\|^{\rho-1+\eta}_{e} \|u\|^{1-\eta}_{B^{s'}_{p'}} \tag{5.4}$$

provided that (5.3) holds with equality replaced by inequality ($\leq$), and with an additional lower bound

$$\rho \geq 1 + 4\delta - 2\eta\delta + 2\delta_{\rho+1}(\rho - 1 + \eta) \tag{5.5}$$

Remarks In particular, (5.5) holds for $\delta \leq \delta_{\rho+1}$ for any $\rho \geq 1$, and the right hand side of the inequality is then less than or equal to the right hand side of (5.3) for $\theta = 0$ and ϵ small. The maximal value of $\rho(n, \delta, \theta)$ is then (for $\theta = 0$) $\hat{\rho} = \frac{n+2-\frac{2}{n}}{n-2}$ for $\rho \leq \rho^*$, as $n\delta_{\rho^*+1} = 1$. *This implies that the non-homogeneous estimate holds for* $1 < \rho \leq \hat{\rho}$ *and* $p' = \rho + 1$.
The range of ρ in the homogeneous estimate is for $|\epsilon|$, $|\eta|$ small, and with $(n-1-\theta)\delta \leq 1$, given by $\frac{2}{n-2} \leq \rho - 1 \leq \frac{4}{n-2}$.
The value of s' may be chosen larger than in the corresponding Strichartz estimate, depending on θ.

PROOF: Assume first $\eta = 0$ and that $\rho \leq 2$.We may assume equality in (5.2), replacing B^s_p by $B^{\bar{s}}_{\bar{p}}$. Let $\bar{s} = s + n(\delta_{\bar{p}} - \delta_p) = s + \epsilon'$.
We shall use (5.2) and (5.3) to prove (5.1). With $u_h = u(\cdot + h)$, and $w_h = w(u_h, u)$,

$$|h|^{-\bar{s}}|f(u_h) - f(u)| \leq |h|^{-\bar{s}}|f'(w_h)||u_h - u|$$

where

$$|f'(w_h)| \leq C(|u_h| + |u|)^{\rho-1}$$

By (5.2) we have $\bar{s} = s' + (\rho-1)(1-s') - \theta = \rho - 1 + s'(2-\rho)$, and so,

$$|h|^{-\bar{s}}|u_h - u| \leq (|h|^{-1}|u_h - u|)^{\rho-1}(|h|^{-s'}|u_h - u|)^{2-\rho}$$

Taking L_2-norm in space of the first factor on the right, and the $L_{p'}$-norm of the second factor, we get using Hölder's inequality that

$$\|f(u)\|_{\dot{H}^{s+\epsilon}_p} \leq C\|u\|^{\rho-1}_{\dot{H}^1_2}\|u\|^{2-\rho}_{\dot{H}^{s'}_{p'}}\|u\|^{\rho-1}_{L_r}$$

with $\frac{1}{r} = \frac{1}{p'} - \frac{s'}{n}$, provided

$$\frac{1}{2} + \delta_{\bar{p}} = (\rho-1)\frac{1}{2} + (2-\rho)(\frac{1}{2} - \delta) + (\rho-1)(\frac{1}{2} - \delta - \frac{s'}{n}) = \frac{1}{2} - \delta + (\rho-1)(\frac{1}{2} - \frac{s'}{n})$$

and so

$$4n\delta + 2\epsilon' = (\rho-1)(n-2) + 2(\rho-1)(1-s')$$

Now by (5.2)

$$(\rho-1)(1-s') = s + \epsilon' - s' = \epsilon' + (n+1+\theta)\delta - 1$$

which gives the bound (5.3). If $\bar{s} = s + \epsilon' + \epsilon$, we get (5.3) with $\eta = 0$.
Next replace ρ by $\rho + \eta \leq 2$ in the above argument and we get (5.1), provided (5.2) and (5.3) hold. Estimating $f(u) = f'(w)u$ in L_p splitting $|u|$ in $|u|^{\rho-1}|u|^{2-\rho}$ and estimating the factors in $L_{\rho+1}$ and $L_{p'}$, respectively, and $|f'(w)|$ in $L_{p'}$, we obtain (5.5).
The proof of (5.1) for $\rho > 2$, $\rho + \eta > 0$ is a straightforward application of Hölder's inequality. $\square$

6 Space time integral estimates of $u \mapsto f(u)$

We shall establish Space Time Integral (STI) estimates of $u \mapsto f(u)$, as they appear e.g. in Propositions 4.2 and 4.3 in terms of the norms of u given in Proposition 4.1.
We assume throughout that $f(z) = |z|^{\rho-1}z$, that $n \geq 3$, and that $1 < \rho < 1 + \frac{4}{n-2} = \rho^*$.
Then let $s = 1 - s'$, and (r', s', p') be Strichartz triple. We define

$$\frac{1}{r(\theta)} = 1 - \frac{\theta}{r'},\ \frac{1}{p(\theta)} = \frac{1}{2} + \delta\theta,\ 0 \leq \theta \leq 1 \tag{6.1}$$

so that $(r(\theta)', 1 - \theta s, p(\theta)')$ is a Strichartz triple, too.
Let $I \subseteq \mathbf{R}_+$ be an interval. We will for for short write $\tilde{A} = L_2(I; B^{s'}_{p'})$ and $B_\theta(s, I) = L_{r(\theta)}(I; B^s_{p(\theta)})$.
Define

$$\beta = \rho + \theta - 2,\ 0 \leq \theta < 1 \tag{6.2}$$

For $n = 3, 4$ we assume that β is an integer, and (in consequence) that $\beta = 0$ for $\rho \leq 2$, all $n \geq 3$. If $n = 5$ we will below choose β suitably in the range $\frac{1}{3} < \beta < 1$ for $2 < \rho \leq \rho^* = 2\frac{1}{3}$.
Notice also that $s = \frac{1}{2}(n+1)\delta$.

As before, $u_h = u(\cdot + h)$. Then

$$|f(u_h) - f(u)| \leq C(|u_h| + |u|)^{\rho-1}|u_h - u|$$

Assume that $\epsilon \geq 0$. In view of Proposition 4.3, we will eventually restrict ϵ to be $\leq \frac{1}{2}(n-1)\delta$.Let als $0 \leq \theta < 1$ and $\gamma = s'$ for $\rho \leq 2$ and $\gamma = 1$ otherwise.With $\bar{s} = \theta(s + \epsilon)$ and $\bar{\theta} = \frac{\bar{s}}{\gamma}$ we get, provided $\bar{s} = \theta s + \epsilon \leq \gamma$ (see [d], [d'] below)

$$|h|^{-\bar{s}}|u_h - u| \leq (|h|^{-\gamma}|u_h - u|)^{\bar{\theta}}|u_h - u|^{1-\bar{\theta}}$$

and so for h small

$$|h|^{-\bar{s}}|f(u_h) - f(u)| \leq C|u|^{\rho-\bar{\theta}}(|h|^{-\gamma}|u_h - u|)^{\bar{\theta}}$$

Then with a constants C and $C(I)$, where $C(I) \leq C(1 + |I|)^{(\frac{1}{2} - \frac{1}{r'})(\rho-\beta)}$ may depend on I,

$$\begin{aligned} \|f(u)\|_{B_\theta(\theta s+\epsilon, I)} &\leq C\|u\|^{\rho-\beta}_{\tilde{A}(I)}\|u\|^{\beta}_{e(I)} \\ &\leq C(I)\|u\|^{\rho-\beta}_{A(I)}\|u\|^{\beta}_{e(I)} \end{aligned} \tag{6.3}$$

and the corresponding homogeneous estimates, replacing $e(I)$ by $E(I)$, provided that, after some (straightforward) reductions of "standard" Sobolev and Hölder estimates, in particular

$$\frac{1}{2}+\delta\theta=(\rho-\bar{\theta})(\frac{1}{2}-\delta-\frac{s'}{n})+\bar{\theta}(\frac{1}{2}-\delta)$$

and for $\beta \geq 1$,

$$\frac{1}{2}+\delta\theta=(\rho-\beta)(\frac{1}{2}-\delta-\frac{s'}{n})+(\beta-\bar{\theta})(\frac{1}{2}-\frac{1}{n})+(\bar{\theta}(\frac{1}{2})$$

which give for $\beta \geq 0$

$$n \geq \rho(n-2)-(\rho-\beta)(n-1)\delta+\theta(n+1)\delta-2n\delta\theta+2\theta\epsilon=\rho(n-2)-(\rho-\beta+\theta)(n-1)\delta+2\theta\epsilon$$

If we want to use homogeneous norms in (6.3), we need in addition to the above estimates with $e(I)$ replaced by $E(I)$, and $\geq$ replaced by $=$.
In addition, for the non-homogeneous estimate we need that

$$1+2\delta\theta \leq (\rho-\beta)(1-2\delta)+\beta\delta_{\rho+1}$$

and so with (6.2)

$$\rho \geq 1+2\delta(\theta+\rho-\beta)=1+2\delta(\theta+2-\theta)+2\beta\delta_{\rho+1}=1+4\delta+2\beta\delta_{\rho+1}$$

These derivations are useful if we change (6.2) to e.g. $\beta=\rho+\theta-r''$, where $2 \leq r'' \leq r'$ to extend the global estimates from uniform L_2-estimates in time to uniform $L_{r'}$-estimates.

the following conditions hold:

[a]$\rho \leq \frac{n+2(n-1)\delta}{n-2}-\frac{2\theta\epsilon}{(n-2)}$, with $\leq$ replaced by $=$ in case of homogeneous estimates

[b] $1+4\delta+2\beta\delta_{\rho+1} \leq \rho$, left out in case of homogeneous estimates

[c] $r'+\beta-\theta \geq \rho$

[c'] $2+\beta-\theta=\rho$ (i.e. (6.2))

[d] $(2+\beta-\rho)(s+\epsilon) \leq \gamma$,
that is with [c']:

[d'] $(2-\rho)(s+\epsilon)+s \leq 1$ if $\beta=0$

Notice that [a] coincides with (5.3), and that with $\beta = \theta + \rho - 2 \le \rho - 1$ will also recover (5.5) i [b].
We will first show that for each ρ, $1 < \rho \le \rho^*$, we can find $p' = p'(\rho, n) \ge \rho + 1$ such that for $\epsilon = 0$ (or, with an obvious extension of the argument below, small) the right hand side of [a] is at least as large as the left hand side of [b], where $(n-1)\delta \le 1$:
To this end, notice that $\beta = \rho + \theta - 2 \le \rho - 1$, and hence the remark after Lemma 5.1 shows that for $1 < \rho \le \hat{\rho} = \frac{n+2-\frac{2}{n}}{n-2}$ we find that $p' = \rho + 1$ will do for all $n \ge 3$.
Next assume that $\delta_{p'} \ge \delta_{\rho^*+1}$, $p > 2$. Then we require

$$4\delta + 2\delta_{\rho+1}\beta \le \frac{2 + 2(n-1)\delta}{n-2} \tag{6.4}$$

where we have chosen p' such that the right hand side equals ρ.

n=3 and $\beta = \beta_{max} = 3$ gives

$$4\delta + 2 \le 2 + 4\delta$$

and (6.4) is proved for n=3.
n=4 and $\beta = 1$:

$$4\delta + \frac{1}{2} \le 1 + 3\delta$$

and so (6.4) holds (for $\delta \le \frac{1}{2}$).
n=5. This time take $\frac{1}{3} < \beta < \frac{5}{6}$:Then (6.4) holds for $\delta \le \frac{1}{4} = \frac{1}{n-1}$ in case n=4.
Thus (6.4) is verified for $n = 3, 4$ and 5. It remains to consider the case $n \ge 6$, when $\rho \le 2$: but then

$$4\delta \le \frac{2 + 2(n-1)\delta}{n-2}$$

implies that $(n-3)\delta \le 1$, and so all δ with $(n-1)\delta \le 1$ is allowed. This completes the verification of (6.4) for $n \ge 3$.
Thus: For $1 < \rho \le \rho^*$ and $n \ge 3$ we can find $p' = p'(\rho, n) \ge \rho + 1$ such that the right hand side of [a] is at least as large as the left hand side of [b] (with ϵ small).

To verify [d] for $\beta > 0$ (i.e $\rho > 2$) we notice that as $\theta < 1$ and $s \le 1$, and $\gamma = 1$ in this case, by which [d] follows for the above choice of p'.
If $\rho \le 2$ we have to verify [d'], which translates in $(3-\rho)s < 1$ for ϵ small. Invoking the lover bound [b] and use that $s = \frac{1}{2}(n+1)\delta$ we get the condition

$$(1 - 2\delta)\delta(n+1) < 1$$

The maximum of the left hand side is taken at $\delta = \frac{1}{4}$, which is larger than $\frac{1}{n-1}$ for $n > 5$. For $n \le 5$, the left hand side is at most $\frac{n+1}{8} \le \frac{3}{4}$. For $n > 5$ and $(n-1)\delta \le 1$ we get the upper bound $\frac{(n-3)(n+1)}{(n-1)(n-1)} = \frac{n^2-2n-3}{n^2-2n+1} < 1$ of the left hand side. *In conclusion: The conditions [a] through [d'] can be satisfied for* $1 < \rho \le \rho^*$ *for suitably chosen* $p' = p'(\rho, n) \ge \rho + 1$.

Also notice that if $\beta > 1$ we may replace $\beta - 1$ of the factors $\|u\|_{e(I)}$ by the corresponding $L_\infty(I; L_{\bar{r}})$ for any $\bar{r}$ such that $\bar{H}_2^1 \cap L_{\rho+1} \subseteq L_{\bar{r}}$.

If I is a compact subinterval of $\mathbf{R}_+$, formally [c'] can be relaxed. If we want estimates for ρ close to ρ^*, however, we not only have to take $(n-1)\delta$ close to 1 and ϵ close to 0, but we will be forced by [a] (most easily seen before simplification) to have essentially [c'] satisfied. The restrictions in case ρ close to ρ^* will thus be the same as in the uniform case, i.e under assumption [c'].

7 Lipschitz estimates of $u \mapsto f(u)$

We keep the notation of the preceding section. Let

$$(\rho-1)_- = \min(\rho-1, 1) \text{ and } (\rho-2)_+ = \max(\rho-2, 0)$$

With $w_h = w(u_h, v_h)$ and $w = w(u, v)$ we have by

$$(f(u_h)-f(v_h)-(f(u)-f(v))) = (f'(w_h)-f'(w))(u-v)+f'(w_h)(u_h-v_h-(u-v))$$

where

$$\begin{aligned} |f'(w_h)| &\le C(|u_h|^{\rho-1} + |v_h|^{\rho-1}) \\ |f'(w_h) - f'(w)| &\le C(|u_h - u|^{(\rho-1)_-} + |v_h - v|^{(\rho-1)_-})(|u| + |u_h| + |v| + |v_h|)^{(\rho-2)_+} \end{aligned} \tag{7.1}$$

Let $\bar{s} = \theta s - \sigma + \epsilon$, $\epsilon \ge 0$ and $\max(s', \theta s) > \sigma \ge 0$. With γ, $\bar{\theta} = \frac{\bar{s}}{\gamma}$ as in the previous section, write

$$|h|^{-\bar{s}}|u_h - u|^{(\rho-1)_-} = (|h|^{-\gamma}|u_h - u|)^{\bar{\theta}}|u - u_h|^{(\rho-1)_- - \bar{\theta}}$$

and correspondingly for $v_h - v$. We then get

$$\begin{aligned} &|h|^{-\bar{s}}|f(u_h) - f(v_h) - (f(u) - f(v))| \\ &\le C(|h|^{-\gamma}(|u_h - u| + |v_h - v|))^{\bar{\theta}}(|u_h - u| + |v_h - v|)^{(\rho-1)_- - \bar{\theta}}(|u| + |u_h| + |v_h| + |v|)^{(\rho-2)_+}|u - v| \\ &+ \ C(|u_h| + |v_h|)^{\rho-1}|h|^{-\bar{s}}|(u_h - u) - (v_h - v)| \end{aligned}$$

Straightforward use of Hölder's and Sobolev's inequalities give, with C and $C(I)$, where $C(I) \leq C(1+|I|)^{(\frac{1}{2}-\frac{1}{r'})(\rho-\beta)}$ may depend on I,

$$\begin{aligned} \|f(u)-f(v)\|_{B_\theta(\theta s-\sigma+\epsilon,I)} &\leq C(\|u\|^{\rho-\beta-1}_{\tilde{A}(I)}\|u\|^{\beta}_{e(I)}+\|v\|^{\rho-\beta-1}_{\tilde{A}(I)}\|v\|^{\beta}_{e(I)})\|u-v\|_{\tilde{A}(s'-\sigma,I)} \\ &\leq C(I)(\|u\|^{\rho-\beta-1}_{A(I)}\|u\|^{\beta}_{e(I)}+\|v\|^{\rho-\beta-1}_{A(I)}\|v\|^{\beta}_{e(I)})\|u-v\|_{A(s'-\sigma,I)} \end{aligned} \tag{7.2}$$

provided, as in Section 6, conditions [a] through [d] there hold with , and in addition (as before with $\gamma = s'$ for $\rho < 2$, and equal to 1 otherwise)

[e] $\theta s - \sigma + \epsilon \leq \gamma(\rho-1)_-$, and so, with [c'],

[e'] $\sigma \geq s-(\rho-1)+\epsilon$, if $\beta = 0$

The homogeneous estimates (replacing B and A by $\dot{B}$ and $\dot{A}$, respectively) also holds, now with equality in [a], and disregarding [b]. By section 7, conditions [a] through [d'] hold under our assumptions for $\epsilon \geq 0$ small enough.
Hence for $\beta = 0$ (7.2) holds if also [e], [e'] are satisfied for sufficiently small $\epsilon > 0$.
We first consider the case $\sigma = 0$: If $\beta > 0$ we need t verify [e], i.e. $\theta s - 1 + \epsilon \leq 0$, which holds for $\epsilon > 0$ sufficiently small.
Thus for $n = 3, 4$ and 5 and for $2 < \rho < 1 + \frac{4}{n-2} = \rho^*$ we have that
$A(I) \ni u \mapsto f(u) \in B_\theta(\theta s + \epsilon, I)$ is Lipschitz continuous for suitable choice of $p' = p'(\rho, n) \geq \rho + 1$.
This is also the case for $\rho \leq 2$ if $0 > s-(\rho-1)$: Using [b] and that $s = \frac{1}{2}(n+1)\delta$, we have to verify $\frac{1}{2}(n+1)\delta \leq 4\delta$, with strict inequality for $\epsilon > 0$. Hence *the Lipschitz continuity also holds for $n = 6$ for $\epsilon > 0$ small enough, and for $n = 7$ if $\epsilon = 0$.* In addition, if we only want Lipschitz continuity for ρ close to ρ^*, then the upper limit for the dimension becomes 8, also for small ϵ (cf.[11]).
On $f(z)$: The slightly complex form of (7.1) makes it possible to extend the estimates on the NLWE to more general non-linearities f satisfying condition (A) in [11]. This is useful if we want uniform bounds (in time) in e.g Proposition 8.1, as this seems to require different behaviour at 0 and $+\infty$ of $f(z)$ (cf. [18], and [11]).

8 Space-time integral estimates (STI) of solutions of the subcritical NLWE

Proposition 8.1. *Let $n \geq 3$, let (r', s', p') be a Strichartz triple, and let I a bounded interval in $\mathbf{R}_+$. Assume that u is a solution of NLWE with finite energy data (i.e in X_e) and let u_0 be the corresponding solution of the wave equation with the same initial data as u.*

i) *[Ginibre and Velo [18], Proposition 4.1 (1)]: Let* $1 + \frac{2}{n-2} \leq \rho < \rho^*$. *Then* $u \in \dot{A}(I)$ *and*

$$\|u\|_{\dot{A}(I)} \leq C(|I|, \|u_0\|_{E(I)}, \|u\|_{E(I)}) \tag{8.1}$$

where $C(\cdot,\cdot,\cdot)$ *is of at most polynomial growth in each of the variables.*

ii) *Let* $1 < \rho \leq \rho^*$ *and* $p' \geq \rho + 1$. *Then* $u \in A(I)$ *and*

$$\|u\|_{A(I)} \leq C(|I|, \|u_0\|_{e(I)}, \|u\|_{e(I)}) \tag{8.2}$$

where $C(\cdot, \cdot\cdot)$ *as above is of at most polynomial growth in each of the variables.*

iii) *Let* $1 < \rho < \rho^*$. *Then* $u \in \dot{A}(I)$ *and* (8.1) *holds with* $E(I)$*-norms replaced by* $e(I)$*-norms.*

Proof: We first prove

(ii)* Let $1 < \rho \leq \rho^*$. Then there is a $p'(\rho, n) \geq \rho + 1$ such that for any Strichartz tripple $(r', s', p'(\rho, n)$, (6.3) holds, $u \in A(I)$ and (8.2) holds.

We begin by proving (ii)* for $1 < \rho \leq \hat{\rho} = 1 + \frac{4-\frac{2}{n}}{n-2}$ by choosing $\rho + 1 = p'$: Let $s' \geq 0,\ \epsilon > 0$ as in Lemma 5.1. Then by (3.6), Proposition 4.3, the remark after that proposition, and Lemma 5.1, with our choice of p':

$$\|u\|_{B^{s'+\epsilon}_{p'}} \leq \|u_0\|_{B^{s'+\epsilon}_{p'}} + C\|f(u)\|_{B^{s}_{p}+\epsilon} \leq \|u_0\|_{B^{s'+\epsilon}_{p'}} + C\|u\|_e\|u\|_{B^{s'}_{p'}}$$

which for $s' = 0$ have a right hand side that (by definition of X_e and (4.7)) belongs to $L^{loc}_{r'}$ for $\rho + 1 \leq p' \leq \rho^* + 1$. Since u belongs to $L^{loc}_r(L'_p)$, u also belongs to $L^{loc}_{r'}(B^{\epsilon}_{p'})$. After a finite number of steps we then reach $s' + \epsilon = 1 - \frac{1}{2}(n+1)\delta_{p'} = s'_{\rho+1}$. This extends to all $p' \geq \rho + 1$ with $n\delta_{p'} + s' = n\delta_{\rho+1} + s'_{\rho+1} = 1 + \frac{1}{r'}$ by (4.2).
Let $\hat{\rho} \leq \rho \leq \rho^*$. Choose $p' = p'(\rho, n)$ so that (5.2) holds with $\theta = \eta = \epsilon = 0$. By (i) then $u \in L^{loc}_{r'}(\dot{B}^{s'}_{p'})$, which implies that $u \in L^{loc}_{r'}(L_{\hat{p}'})$, where $\hat{p}' > p'(\rho, n)$. As $u \in L_{infty}(L_{\rho+1})$ interpolation (convexity) gives that $u \in L_{r'}(L'_p)$, and hence $u \in A(I)$, with $p' = p'(\rho, n) \geq \rho + 1$ also for $\hat{\rho} \leq \rho \leq \rho^*$. The above choice of p' is the same as that in the verification of (6.3). This completes the proof of (ii)*.
We now invoke (6.3), the comments on that inequality, and Proposition 4.2 to prove the first part of (ii): Let (r', s', p') be a Strichartz triple, let $\bar{p}' \geq \rho + 1$ be determined as in (ii)*, and let $\bar{B}(I) = L_r(I : B^{\bar{s}}_{\bar{p}}),\ \bar{A}(I) = L_{r'}(I : B^{\bar{s}'}_{\bar{p}'})$. Then

$$\begin{aligned}\|u\|_{A(I)} &\leq \|u_0\|_{A(I)} + \|f(u)\|_{\bar{B}(I)} \\ &\leq \|u_0\|_{A(I)} + \|u\|^{\rho-\beta}_{\bar{A}(I)}\|u\|^{\beta}_{e} \\ &\leq \|u_0\|_{A(I)} + C(|I|, \|u_0\|_{e(I)}, \|u\|_{e(I)})\end{aligned}$$

which proves (ii). In the proof of (iii) we only have to consider $1 < \rho \leq 1 + \frac{2}{n-2}$. Let (as in (ii)*) $p' = \rho + 1$. Then as above

$$\begin{aligned}\|u\|_{\dot{A}(I)} &\leq \|u_0\|_{\dot{A}(I)} + \|f(u)\|_{\bar{B}(I)} \\ &\leq \|u_0\|_{\dot{A}(I)} + \|u\|^{\rho-\beta}_{\bar{A}(I)}\|u\|^{\beta}_{e} \\ &\leq \|u_0\|_{\dot{A}(I)} + C(|I|, \|u_0\|_{e(I)}, \|u\|_{e(I)})\end{aligned}$$

With (3.6) this completes the proof of (iii). □

Remark If we let $p' = \rho^* + 1$, then X_e may be replaced by $\dot{H}^1_2 \times L_2$ in the above argument.
Remark Even if the norms of u_0 and u are uniformly bounded , as they are in case of $E(I)$-norms, the dependence on I is in view of (4.4) is unavoidable with the present method of proof.
The estimate of the norm of u in E follows from the energy inequality.

9 Lipschitz etimates of $\mathcal{E}_t$

Let $u,\ v$ be solutions of the NLWE and $u_0,\ v_0$ the corresponding solutions with the same finite energy data (in X_e) as $u,\ v$. Let e_0 denote the energy norm of the data, i.e the norm in X_e. Then, by the energy inequality, and (4.8), u_0 and u are bounded by in $e(I) = L_\infty(I; \dot{H}^1_2 \cap L_{\rho+1} \times L_2) \subset E(I) = L_\infty(I; \dot{H}^1_2 \times L_2)$ by e_0, $I \subset\subset \mathbf{R}_+$.

We let (r', s', p') be a Strichartz triple and $0 \leq \sigma \leq \theta(1 - s')$ as in (7.2), where then $\sigma > 0$ is a possible (and for $n < 8$ necessary) choice for $1 < \rho < \rho^*$ and $n \geq 6$.

Fix $\bar{I} \subset\subset \mathbf{R}_+$. Let $I \subset \bar{I}$. By Proposition 4.3 and the Lipschitz estimate (7.2) for $f(u)$, using Proposition 8.1, we then get

$$\| \int_I E_1(\cdot - \tau)(f(u(\tau)) - f(v(\tau)))d\tau \|_{A(s'-\sigma;I)} \leq J(I)\|u - v\|_{A(s'-\sigma;I)} \tag{9.1}$$

$$\|u - v\|_{A(s'-\sigma;I)} \leq \|u_0 - v_0\|_{A(s'-\sigma;I)} + J(|I|)\|u - v\|_{A(s'-\sigma;I)} \tag{9.2}$$

$$J(|I|) = |I|^\epsilon C(e_0, \bar{I}),$$

for $\epsilon = \epsilon(\rho, n) > 0$ small enough, and thus for $|I| > 0$ small enough, depending only on the energy e_0 (and on the constants $\epsilon,\ \bar{I}$),

$$\|u - v\|_{A(s'-\sigma;I)} \leq 2\|u_0 - v_0\|_{A(s'-\sigma;I)}\ ,\ \ |I| \leq \epsilon(e_0).$$

Let us make a small diversion in order to prove

Proposition 9.1 (Uniqueness of finite energy solutions of NLWE). *Let u be a finite energy solution with finite energy data (i.e. in X_e). Then u is uniquely determined by the initial data.*

PROOF: Since the solution operator of the wave equation is Lipschitz continuous as a map from E to $\dot{A}$ (which follows from the linearity and the Strichartz estimate (Proposition 4.1, if u and v have the same initial data then so has u_0 and v_0, and $u - v = 0$ in $\dot{A}(I)$. Hence $u = v$ a.e. on $\mathbf{R}_+ \times \mathbf{R}^n$. By the continuity of u, v in E, $u(t) = v(t)$ a.e on $\mathbf{R}^n$ for $t \in I$, and so are equal in E and X_e in a neighbourhood of 0. The translation invariance of the NLWE completes the proof. $\square$

Let us now go back to the Lipschitz estimates:
For $t_0 \geq 0$ we have

$$u(t+t_0)-v(t+t_0) = u_0(t+t_0)-v_0(t+t_0)+\int_0^{t+t_0} E_1(t+t_0-\tau)(f(u(\tau))-f(v(\tau)))d\tau$$

where we split the integral term (for $t_0 > 0$) in two parts, with integration over $I_0 = (0, t_0)$ and $(t_0, t_0 + t)$, respectively.
Assume now that

$$\|u - v\|_{A(s'-\sigma;I_0)} \leq C(I_0, e_0)\|u_0 - v_0\|_{A(s'-\sigma;I_0)} \tag{9.3}$$

Then (9.1) implies that for $I = (0, t_0 + t)$ with $t \leq \epsilon(e_0)$

$$\begin{aligned}\|u - v\|_{A(s'-\sigma;I)} &\leq (C(I_0, e_0) + 1)\|u_0 - v_0\|_{A(s'-\sigma;I)} + \frac{1}{2}\|u - v\|_{A(s'-\sigma;I\setminus I_0)} \\ &\quad + J(|I_0|)\|u - v\|_{A(s'-\sigma;I_0)} \\ &\leq \|u_0 - v_0\|_{A(s'-\sigma;I)} + \frac{1}{2}\|u - v\|_{A(s'-\sigma;I)} + C(I_0, e_0)\|u_0 - v_0\|_{A(s'-\sigma;I_0)}\end{aligned}$$

By induction then (starting with $t_0 = \epsilon(e_0)$) for each bounded interval $I \subset \bar{I}$ there is a continuous function $C(I, e_0)$ such that

$$\|u - v\|_{A(s'-\sigma;I)} \leq C(I, e_0)\|u_0 - v_0\|_{A(s'-\sigma;I)} \tag{9.4}$$

Thus by (4.11) and (7.2) (and Proposition 8.1) for any interval $I \subset \bar{I}$, using that by linearity $u_0 - v_0$ is solution of the wave equation,

$$\begin{aligned}\|u - v\|_{L_\infty(I,\dot{H}_2^1)} &\leq \|u_0 - v_0\|_{E(I)} + C(e_0, \bar{I})\|u - v\|_{A(s',I)} \\ &\leq C'(e_0, \bar{I})\|u_0 - v_0\|_{e(I)} \end{aligned} \tag{9.5}$$

since $\sigma = 0$ is allowed in (7.2) for $n \leq 7$ (and for $n = 8$ if $\rho < \rho^*$ is sufficintly close to ρ^*) by the analysis following that equation, and $p' \geq \rho + 1$. This completes that proof of the Lipschitz continuity of $\mathcal{E}_t$ as a map from e_0 to $E(I)$, $I \subset\subset \mathbf{R}_+$. If we may choose $\sigma = 1 - n\delta_{\rho+1} = n(\delta_{\rho^*+1} - \delta_{\rho+1})$ in (9.4), it will follow that

Proposition 9.2. *Let $3 \leq n \leq 8$. Then $\mathcal{E}_t$ is Lipschitz continuous on X_e for $t > 0$, and $\rho < \rho^*$ sufficiently close to ρ^*.*

PROOF: We first have to verify that $\sigma \leq \theta s$. With $s = \frac{1}{2}(n+1)\delta$, $\theta = 2 + \beta - \rho$, chosing $\beta = 2 + \rho^*$ we get after dividing out the common factor $\rho^* - \rho$ the condition

$$\frac{n-2}{2}\frac{1}{1+\rho} \leq \frac{1}{2}(n+1)\delta \tag{9.6}$$

For $\rho = \rho^*$ and $(n-1)\delta = 1$, we get strict inequality in (9.6), since

$$\frac{(n-2)^2}{2n} < \frac{n+1}{n-1} \text{ for } 3 \leq n \leq 6$$

If $n > 6$, $\rho^* < 2$, and so

$$\frac{n-2}{2}\frac{1}{1+\rho}(\rho^* - \rho) \leq \frac{1}{2}(n+1)\delta(2-\rho), \ n \geq 6 \tag{9.7}$$

holds for $\rho < \rho^*$ sufficiently close to ρ^*. Thus $\sigma = 1 - n\delta_{\rho+1l} \leq \theta s$, $3 \leq n \leq 8$ and $\rho < \rho^*$ sufficiently close to ρ^*. We have

$$\|u - v\|_{e(I)} \leq \|u_0 - v_0\|_{e(I)} + \| \int_I E(t-\tau)(f(u(\tau)) - f(v(\tau))d\tau\|_{e(I)}$$

and so by Corollary 4.1, with $\sigma = 1 - n\delta_{\rho+1} \leq \theta s$,

$$\|u - v\|_{e(I)} \leq \|u_0 - v_0\|_{e(I)} + C(I)\|f(u) - f(v)\|_{\dot{B}(\theta s - \sigma, I)}$$

By our assumption (9.4) holds. Thus

$$\begin{aligned} \|u - v\|_{e(I)} &\leq \|u_0 - v_0\|_{e(I)} + C\|u - v\|_{A(s'-\sigma,I)} \\ &\leq \|u_0 - v_0\|_{e(I)} + C\|u_0 - v_0\|_{A(s'-\sigma,I)} \end{aligned} \tag{9.8}$$

We now by obtain (again using that by linearity $u_0 - v_0$ is solution of the wave equation, and the invariance of the $\dot{A}$-norm of these under powers of $(-\Delta)$):

$$\|u_0 - v_0\|_{A(s'-\sigma;I)} \leq \|u_0 - v_0\|_{A(I)} \leq C(I)\|u_0 - v_0\|_{e(I)}$$

and so by (9.8)

$$\|u - v\|_{e(I)} \leq C(|I|, e_0(u), e_0(v))\|u_0 - v_0\|_{e(I)} \tag{9.9}$$

where $e_0(u)$, $e_0(v)$ denotes the (non-linear) energy of the initial data of u, and v, respectively. As the solution operator of the wave equation is (by linearity) Lipschitz continuous on X_e on bounded intervals I in $\mathbf{R}_+$, this completes the proof of the Lipschitz continuity in X_e on bounded subintervals of $\mathbf{R}_+$ for $\rho < \rho^*$ close to ρ^*. □

In the next section we will determine a lower bound for ρ for which $\mathcal{E}_t$ is Lipschitz continuous on X_e by establishing a variation of the Lipschitz estimates in Section 7.

10 On the Lipschitz continuity of $\mathcal{E}_t$ and uniform continuity in time.

As in the previous sections, let $u,\ v$ be solutions of the NLWE and $u_0,\ v_0$ the corresponding solutions with the same finite energy data (in X_e) as $u,\ v$. Let us begin with the following result.

Proposition 10.1. *Let $3 \le n \le 8$, and let $I \subset\subset \mathbf{R}_+$. Assume that $1+\frac{4}{n-1} = \rho_* \le \rho < \rho^*$ for $n \le 7$, with a lower bound $1+\frac{4.5}{n-1} \le \rho$ for $n=8$. Then*

$$\|f(u)-f(v)\|_{\dot{B}(\theta s-\sigma+\epsilon,I)} \le C(|I|,u,v)\|u-v\|_{A(s'-\sigma,I)} \tag{10.1}$$

holds for $\sigma = 1-n\delta_{\rho+1} = n(\delta_{\rho^*+1}-\delta_{\rho+1})$ with (r',s',p') a Strichartz triple with $\frac{1}{r'} = \frac{(n-1)\delta_{p'}}{2}$ and $p' \ge \rho+1$. Here

$$C(|I|,u,v) \le C(\|u\|_{A(I)}^{\rho-\beta-1}\|u\|_E^{\beta} + \|v\|_{A(I)}^{\rho-\beta-1}\|v\|_E^{\beta}) \tag{10.2}$$

where C is independent of $I,\ u$, and v.

Proof: We let $\beta = \beta(\rho,n)$ be a linear function of ρ for $n<6$, and piecewise linear for $n \ge 6$, with $1 < \rho-\beta \le 2$, so that

$$r'-(\rho-\beta) = \theta, 0<\theta<1. \tag{10.3}$$

If $\beta = 0$ (i.e $\rho<2$, which in view of [b"] below means $n \ge 6$), we let $r' = r'(\rho,n)$ be piecewise constant, and for $\beta>0$ we assume that $r' = r(\rho,n)$ is linear in ρ. As in the proof of (6.3) and (7.2) we get the following conditions under which (10.1) holds (with $\epsilon = 0$), keeping the definitions of γ and $(\rho-1)_-$:

[a" $]n+2 \ge \rho(n-2)+2-r'(n-1)\delta = \rho(n-2)$, i.e. $\rho^* \ge \rho$

[b" $]\rho \ge 1+2r'\delta = 1+\frac{4}{n-1}+2\beta\delta_{\rho+1}$

[c" $]r'-(\rho-\beta) = \theta$

[e" $](r'-(\rho-\beta))s \le \sigma+\gamma(\rho-1)_-$

[f" $]\sigma \le (r'-(\rho-\beta))s(\le s')$

where the condition $p' \ge \rho+1$ implies that

$$2 \le r' = \frac{2}{(n-1)\delta} \le \frac{2}{(n-1)\delta_{\rho+1}} = \hat{r}(\rho,n),\ n\delta_{\rho+1} \le \delta_{\rho^*+1} \le 1 \tag{10.4}$$

and where $\sigma = 1 - n\delta_{\rho+1}$ ($\sigma = 0$ was handled in the preceding section).
By $[b'']$ we have $\rho \geq 1 + \frac{4}{n-1}$, which means that $\rho \geq 2$ for $n < 6$, and by $[a'']$, that $\rho < 2$ for $n \geq 6$. Further $\gamma(\rho-1)_{-}$ equals 1 for $n < 6$ and for $n \geq 6$ equals $s'(\rho-1) = (1-s)(\rho-1)$. *Since $\theta s \leq 1 \leq \sigma + 1$ for $n < 6$, [e"] holds , and will be disregarded for these dimensions.*
Also notice that $s = \frac{1}{2}(n+1)\delta_{p'} = \frac{n+1}{n-1}\frac{1}{r'}$. In addition

$$\sigma = 1 - n\delta_{\rho+1} = n(\delta_{\rho^*+1} - \delta_{\rho+1)} = \frac{n-2}{2}\frac{\rho^*-\rho}{1+\rho}$$

The equations $[e'']$ and $[f'']$ become after some simplification and reshuffling the following form, which will be useful in the analysis of [e"] (for $n \geq 6$) and [f"]:

$$\frac{n+1}{n-1} \leq \sigma + \frac{n+1}{n-1}\frac{1}{r'} + \rho - 1, \; [e''] \; n \geq 6 \tag{10.5}$$

$$\frac{n+1}{n-1} \geq \sigma + (\rho - \beta)\frac{n+1}{n-1}\frac{1}{r'}, \; [f''] \tag{10.6}$$

$$\sigma = \frac{n-2}{2}\frac{\rho^*-\rho}{1+\rho}$$

The first inequality is already verified for $r' = 2$ and $\sigma = 0$ for the range $\rho_*(n) \leq \rho < \rho^*(n)$, where $\rho_* = 1 + \frac{4}{n-1}$ for $3 \leq n \leq 7$ and $\rho_* = 1 + \frac{4.5}{n-1}$ for $n = 8$. If the inequality holds for some $r' > 2$, it evidently also holds for $r' = 2$.
The equation [f"] may also be written

$$\frac{n-2}{2}\frac{\rho^*-\rho}{1+\rho} \leq (r' - (\rho-\beta))\frac{n+1}{n-1}\frac{1}{r'} \tag{10.7}$$

In order to simplify some computations, note that

$$\frac{n-2}{2}\frac{\rho^*-\rho_*}{1+\rho_*} = \frac{1}{n+1} \tag{10.8}$$

and that the right hand side of (10.7) is an increasing function of r' (and the left hand side independent of r').

$n \geq 6$:
Here $\beta = 0$, and r' is assumed piecewise constant: In particular the derivative of (10.6) becomes

$$-\frac{n}{(1+\rho)^2} + \frac{n+1}{n-1}\frac{1}{r'} \tag{10.9}$$

If $n = 6$, then $r' = 2 (= \rho^*)$, and by (10.7)

$$\frac{n-2}{2}\frac{1}{\rho+1} \leq \frac{n+1}{n-1}\frac{1}{2} \tag{10.10}$$

for $\rho \geq 1\frac{6}{7}$. For ρ less than this value, (10.9) is negative, and hence it is in this case enough to compute a bound for r' when $\rho = \rho_*$. This gives

$$\frac{1}{n+1} \leq (r' - \rho_*)\frac{n+1}{n-1}\frac{1}{r'} \tag{10.11}$$

which requires $r' \geq 2\frac{1}{220}$ (which also satisfies [e"]). This completes the case $n = 6$. For $n \geq 7$ (10.9) is negative. Since $\sigma(\rho^*) = 0$, $n = 7$ and 8 are as in the case $\sigma = 0$ the only allowed dimensions. Equation (10.5) limits the range of ρ: As mentioned above, we only have to consider $r' = 2$, i.e.

$$\frac{1}{2}\frac{n+1}{n-1} \leq \rho - 1 + \sigma$$

With $\sigma = 0$, we get $n \leq 7$ for $\rho = \rho_*$, while $n = 8$ requires the lower bound $1 + \frac{4.5}{n-1}$. If $\sigma = \frac{n-2}{2}\frac{\rho^*-\rho}{1+\rho}$ the right hand side of the inequality is decreasing as a function of ρ (the derivate is $1 - \frac{n}{(1+\rho)^2} < 0$), and so the whole range $\rho_* \leq \rho \leq \rho^*$ is allowed also for $n = 8$.

For dimensions $n = 3$ through 5, we only have to verify [f"] and that $r'(\rho, n) \leq \hat{r}(\rho, n) = \frac{2}{(n-1)\delta_{\rho+1}} = \frac{4}{n-1}\frac{\rho+1}{\rho-1}$, $\rho_* \leq \rho \leq \rho^*$ (in order to fulfil the condition $p' \geq \rho + 1$).

n=5:
We let $r(\rho^*) = 2$, $r(\rho^*) + \beta(\rho^*) = \rho^* = 2\frac{1}{3}$, and $\beta(\rho_*) = 0$, where now $\rho_* = 2$. Thus $\beta(\rho) = \rho - \rho_*$. If we let $r'(\rho) + \beta(\rho) = \rho^*$, then $r'(\rho) = \rho^* + \rho_* - \rho$.
After dividing ou the common factor $\rho^* - \rho$, [f"] becomes

$$\frac{n+1}{n-1}\frac{1}{r'(\rho)} \geq \frac{n-2}{2}\frac{1}{1+\rho}, \quad n = 5$$

which with our choice of $r'(\rho)$ holds in the range $\rho_* = 2 \leq \rho \leq \rho^*$. This proves [f"] for n=5.
Now $r'(\rho) \leq r'(\rho_*) = 2\frac{1}{3} \leq \hat{r}(\rho^*) = 2\frac{1}{2} \leq \hat{r}(\rho)$, $\rho \leq \rho^*$. This completes the case $n = 5$.

n=4:
Here $\rho^* = 3$ and $\rho_* = 2\frac{1}{3}$.We choose $r'(\rho^*) = 2$, $r'(\rho^*) + \beta(\rho^*) = \rho^*$, and so $\beta(\rho^*) = 1$ and we (have to) let $\beta(\rho_*) = 0$. Hence $\beta(\rho) = \frac{3}{2}(\rho - \rho_*)$. We next choose $r'(\rho_*) = \hat{r}(\rho^*, 4) = \frac{8}{3}$, so that $p' \geq \rho + 1$, $\rho \leq \rho^*$ will be satisfied. Then $r'(\rho) = 5 - \rho$ and $r' + \beta - \rho = \frac{1}{2}(\rho^* - \rho)$. After dividing out the common factor $\rho^* - \rho$ in [f"] (i.e. (10.7)) we get

$$\frac{1}{2}\frac{n+1}{n-1}\frac{1}{r'(\rho)} \geq \frac{n-2}{2}\frac{1}{1+\rho}, \quad n = 4$$

which requires $r'(\rho) \leq \frac{25}{9} = 2\frac{7}{9}$, which is the case since by our choice $r'(\rho) \leq r'(\rho_*) = 2\frac{2}{3} = 2\frac{6}{9}$. This completes the case $n = 4$.

n=3:
Here $\rho^* = 5$ and $\rho_* = 3$. We let $r'(\rho^*) = 2$ and choose $\beta(\rho^*) = 3$ and $\beta(\rho_*) = 0$. If we then take $r'(\rho_*) = \hat{r}(\rho_*) = 4$, we get $r + \beta - \rho = \frac{1}{2}(\rho^* - \rho)$. Hence [f"] becomes (again dividing out the common factor) by (10.7)

$$\frac{1}{2}\frac{n+1}{n-1}\frac{1}{r'} \geq \frac{n-2}{2}\frac{1}{1+\rho}, \quad n = 3$$

which amounts to $r' \leq 8$ which clearly is the case for $\rho_* = 3 \leq \rho \leq \rho^*$. Will $r'(\rho) \leq \hat{r}(\rho)$ in this range? The tangent of the concave curve described by $\hat{r}(\rho) = 2\frac{\rho+1}{\rho-1} = 2 + \frac{4}{(1+\rho)^2}$ has the same direction -1 at $\rho = \rho_*$ as $r'(\rho) = 4 + \rho_* - \rho$; as r' and $\hat{r}$ have the same value at $\rho = \rho_*$, $r'(\rho)$ is the tangent at this point, and so is smaller than $\hat{r}(\rho)$ for $\rho \geq \rho_*$. This completes the proof for n=3.
The above give together a proof of Proposition 10.1. □

Proof of Theorem 1.1:
In the proof in Section 9 replacing (7.2) by (10.1),(10.2) we obtain (9.4) for $\sigma = 1 - n\delta_{\rho+1}$, $\rho_* \leq \rho \leq \rho^*$. We then complete the proof as in the last part of the proof of Proposition 9.2.

We may now as a final result extend Corollary 4.2 to solutions of the NLWE.

Corollary 10.1. *Let $I \subset\subset \mathbf{R}_+$, an u a finite energy solution of the NLWE with finite energy data. Assume that $\rho_* \leq \rho \leq \rho^*$. Then $u(t)$ is uniformly continuous on X_e and on E in the time variable t for $t \in I$.*

Proof: An immediate consequence of the Lipschitz continuity and the Lipschitz estimate for $n \leq 7$.
For higher dimensions we prove that (using Corollary 4.1) $\|h\|_{\bar{B}(\bar{s}+\epsilon-\sigma,I)} + \|h\|_{\bar{B}(\bar{s}+\epsilon,I)}$, with $h = f(u)$, is bounded for $\epsilon > 0$ small enough and $\sigma = 1 - n\delta_{\rho+1}$. We then follow the argument in Corollary 4.2.
In view of conditions [a] through [e] (or for $n > 6$, [e']) we then only additionally have to prove that for $n \geq 8$,

$$\sigma < (2-\rho)s < 1 - s + \sigma, \ s = \frac{n+1}{2}\delta$$

which certainly holds for $\rho \geq \rho_* = 1 + \frac{4}{n-1}$, $\delta = \frac{1}{n-1}$ and $n \geq 8$ (use that (10.9) is negative for $n \geq 7$, and apply (10.8)). □

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