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*PREPRINT 2013:12*

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Gothenburg, August 2013

Preprint 2013:12  
ISSN 1652-9715

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Matematiska vetenskaper  
Göteborg 2013

# RECONSTRUCTION OF THE REFRACTIVE INDEX FROM EXPERIMENTAL BACKSCATTERING DATA USING A GLOBALLY CONVERGENT INVERSE METHOD\*

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**Abstract.** The problem to be studied in this work is within the context of coefficient identification problems for the wave equation. More precisely, we consider the problem of reconstruction of the refractive index (or equivalently, the dielectric constant) of an inhomogeneous medium using one backscattering boundary measurement. The goal of this paper is to analyze the performance of a globally convergent algorithm of Beilina and Klibanov on experimental data acquired in the Microwave Laboratory at University of North Carolina at Charlotte. The main challenge working with experimental data is the the huge misfit between these data and computationally simulated data. We present data pre-processing steps to make the former somehow look similar to the latter. Results of both non-blind and *blind* targets are shown indicating good reconstructions even for high contrasts between the targets and the background medium.

**Key words.** Coefficient identification, wave equation, globally convergent algorithm, experimental data, data pre-processing.

**AMS subject classifications.** 35R30, 35L05, 78A46

**1. Introduction.** In this paper, we consider the problem of the reconstruction of the refractive indices (equivalently, the relative permittivities, or dielectric constants) of unknown targets placed in a homogeneous medium using experimental measurements of back-scattered electromagnetic waves in time domain. Mathematical speaking, this is a coefficient inverse problem (CIP) for the time-dependent wave-like equation: we reconstruct a spatially varying coefficient of this equation using measurements on a part of the boundary of the domain of interest. Potential applications of this problem are in the detection and characterization of explosives, including improvised explosive devices (IEDs). Note that IEDs are often located above the ground surface [16], which is somewhat close to our case of targets located in the air. The case when targets are buried in the ground will be reported in a future publication.

Different imaging methods have been applied to this type of measurements to obtain geometrical information such as shapes, sizes and locations of the targets, see e.g. [14, 19]. However, the refractive indices, which characterize the targets in terms of their constituent materials, are much more difficult to estimate. This is a motivation of the current publication.

For conventional gradient-based optimization approaches, there is a huge literature, see e.g. [3, 11, 12] and the references therein. It is well-known that their

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\*This research was supported by US Army Research Laboratory and US Army Research Office grants W911NF-11-1-0325 and W911NF-11-1-0399, the Swedish Research Council, the Swedish Foundation for Strategic Research (SSF) in Gothenburg Mathematical Modelling Centre (GMMC) and by the Swedish Institute, Visby Program.

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convergence is guaranteed only if the starting point of iterations is chosen to be sufficiently close to the correct solution. This means that they require some *a priori* information about the targets being found, which is not always available in many practical situations. Unlike these, a different method was proposed in [6, 7, 15, 16] and results were summarized in the book [5]. This method provides a good approximation for the exact coefficient without *a priori* knowledge of a small neighborhood of this coefficient. Its global convergence has been rigorously proved for an approximate mathematical model, see Theorem 2.9.4 in [5] and Theorem 5.1 in [7]. Due to this model, it is referred to as an *approximately globally convergent method* (globally convergent method in short). In [15] the authors demonstrated good reconstruction results for a transmitted experimental data set using this method, whereas a gradient based method with Tikhonov regularization, starting from the homogeneous medium as the first guess, failed.

The goal of this paper is to show how this globally convergent method performs on a backscattering experimental data. While previously it was demonstrated how this method works on transmitted real data [5, 6, 15], the case of backscattering data is much more complicated, because backscattered signals are much weaker than transmitted ones. In addition, a number of unwanted scattering signals caused by objects present in the room of experiments (e.g. furniture) occur in the backscattering case. Although in [16] backscattering data were treated, they were 1-d data only, while we work here with the 3-d data. To collect these data, an experimental apparatus was built in the Microwave Laboratory of University of North Carolina at Charlotte, using support of US Army Research Office.

The main challenge working with our experimental data is a *huge misfit* between these data and computationally simulated ones, also see [5, 6, 15, 16] for the same conclusion. From the Functional Analysis standpoint this means that the function expressing experimental data is far away in any reasonable norm from the range of the operator of our forward problem. And this operator should be inverted to solve the inverse problem. Hence, any inversion algorithm would fail to produce satisfactory results, if being applied to the raw data.

Therefore, the *central procedure* before applying the globally convergent algorithm is a heuristic data pre-processing procedure. This procedure makes the data look somewhat similar to the data provided by computational simulations. In other words, it moves the data closer to the range of that operator. The pre-processing of the current paper is substantially different from pre-processing procedures of [5, 6, 15, 16] because the data are different. We describe our data pre-processing procedure in section 3.

The pre-processed data are used as the input for the globally convergent method. Our goal is to image refractive indices and locations of the targets. In addition, we want to estimate sizes of those targets. We should mention that results of this method can be used as initial guesses for locally convergent methods in order to refine images, especially the targets' shapes. For example, it was shown in [5, 6, 8] that the adaptive finite element method has significantly improved images of shapes for transmitted experimental data in the case when its starting point was the solution obtained by the globally convergent method.

The experimental data sets of this paper include both non-blind and blind cases. "Blind" means that the targets were unknown for the computational team (NTT, LB, MVK) but known to MAF, who was leading the data collection process. Moreover, refractive indices of these targets were measured after the reconstruction results

were obtained. Then computational results were compared with directly measured ones. Our results indicate that we not only reconstruct accurately refractive indices and locations of targets, but also can differentiate between metallic and non-metallic targets.

**2. Problem statement and the globally convergent method.** In this section we state the forward and inverse problems under consideration as well briefly outline the globally convergent method of [5] for reader's convenience. We refer to [5] and [7] for more details.

**2.1. Forward and inverse problems.** As the forward problem, we consider the propagation of the electromagnetic wave generated by a point source in  $\mathbf{R}^3$ . Below,  $\mathbf{x} = (x, y, z)$  denotes a point in  $\mathbf{R}^3$ . Since in our experiments only one component of the electric wave field  $E$  is generated from the transmitting horn antenna (source) and the detector measures only that component of the scattered electric field, we model the wave propagation by the following Cauchy problem for the scalar wave equation

$$\epsilon(\mathbf{x})u_{tt}(\mathbf{x}, t) = \Delta u(\mathbf{x}, t), (\mathbf{x}, t) \in \mathbf{R}^3 \times (0, \infty), \quad (2.1)$$

$$u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0), \quad (2.2)$$

where  $u$  is the total wave which is equal to the sum of the incident wave  $u^i$  and the scattered wave  $u^s$  caused by the scattering from the inhomogeneous medium. To further justify our use of the single equation (2.1) instead of the full Maxwell' system, we refer to [4], where it was shown numerically that the component of  $E$  which was initially incident upon the medium, dominates two other components, at least for a rather simple medium, such as we work with, see section 7.2.2 of [4]. Besides, equation (2.1) was previously successfully used to work with transmitted experimental data [5, 6, 8]. In (2.1),  $\epsilon(\mathbf{x})$  represents the spatially distributed dielectric constant, while  $n(\mathbf{x}) := \sqrt{\epsilon(\mathbf{x})}$  is referred to as the refractive index of the medium in which the wave propagates. In our analysis, we assume that the coefficient  $\epsilon(\mathbf{x})$  is unknown inside of a bounded convex domain  $\Omega \subset \mathbf{R}^3$  with  $\partial\Omega \in C^3$ . We also assume the existence of a positive constant  $d$  such that

$$\epsilon(\mathbf{x}) \in [1, 1 + d], \quad \forall \mathbf{x} \in \mathbf{R}^3, \quad \epsilon(\mathbf{x}) \equiv 1, \quad \forall \mathbf{x} \notin \Omega. \quad (2.3)$$

In other words, the medium is assumed to be homogeneous outside of the domain  $\Omega$ . Moreover, for theoretical analysis, we assume that the coefficient  $\epsilon \in C^3(\mathbf{R}^3)$ . We also assume that the source is placed outside of  $\Omega$ , i.e.  $\mathbf{x}_0 \notin \bar{\Omega}$ .

For the convenience of the theoretical analysis, we state the inverse problem for the case when the data at the entire boundary are given. However, in our experiments, only back-scattering data are measured. In section 4, we explain how do we work with this type of data.

**CIP.** *Reconstruct the coefficient  $\epsilon(\mathbf{x})$  (or equivalently, the refractive index  $n(\mathbf{x})$ ) for  $\mathbf{x} \in \Omega$ , given the measured data at  $\partial\Omega$  for a single source position  $\mathbf{x}_0 \notin \bar{\Omega}$ .*

$$u(\mathbf{x}, t) = g(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega, \quad t \in (0, \infty). \quad (2.4)$$

The function  $g(\mathbf{x}, t)$  represents the time dependent measured data at the boundary of the domain of interest. The assumption of the infinite time interval in (2.4) is not a restrictive one, because in our method we apply the Laplace transform to  $g(\mathbf{x}, t)$  with respect to  $t$ . Since the kernel of this transform decays exponentially with respect to

$t$ , then the Laplace transform effectively cuts off to zero values of the function  $g(\mathbf{x}, t)$  for  $t$  larger than some value  $T$ .

Concerning the uniqueness of this CIP, global uniqueness theorems for multi-dimensional CIPs with a single measurement are currently known only under the assumption that at least one of initial conditions does not equal zero in the entire domain  $\bar{\Omega}$ . All these theorems were proven by the method, which was originated in [10]; also, see, e.g. sections 1.10, 1.11 in [5] about this method. This technique is based on Carleman estimates. Since both initial conditions (2.2) equal zero in  $\bar{\Omega}$ , then this method is inapplicable to our case. However, since we need to solve numerically our CIP anyway, we assume that uniqueness takes place.

We remark that equation (2.1) is invalid if metallic objects are present in the domain  $\Omega$ . To deal with this type of targets, we follow the suggestion of [16]. It was established numerically in [16] that metals can be modeled as dielectrics with a high dielectric constant, which is referred to as the ‘‘appearing’’ dielectric constant of metals. It is suggested in [16] that this dielectric constant can be chosen as

$$\epsilon(\text{metals}) \in [10, 30]. \quad (2.5)$$

**2.2. The globally convergent method.** The globally convergent method of [5] works with the Laplace transformed data. However, we do not invert the Laplace transform. Let

$$w(\mathbf{x}, s) = \int_0^{\infty} u(\mathbf{x}, t) e^{-st} dt, \quad (2.6)$$

where the positive parameter  $s$  is referred to as the *pseudo frequency*. We assume that  $s \geq \underline{s} > 0$ , where the number  $\underline{s}$  is large enough, so that the Laplace transforms of  $u$  and its derivatives  $D^k u, k = 1, 2$ , converge absolutely. It follows from (2.1) that the function  $w$  satisfies the equation

$$\Delta w(\mathbf{x}, s) - s^2 \epsilon(\mathbf{x}) w(\mathbf{x}, s) = -\delta(\mathbf{x} - \mathbf{x}_0), \mathbf{x} \in \mathbf{R}^3, s \geq \underline{s}. \quad (2.7)$$

It was proved in Chapter 2 of [5] that  $w(\mathbf{x}, s) > 0$  and  $\lim_{|\mathbf{x}| \rightarrow \infty} w(\mathbf{x}, s) = 0$ . Define the function  $v$ ,

$$v := \frac{\ln w}{s^2}. \quad (2.8)$$

Substituting  $w = e^{vs^2}$  into (2.7) and keeping in mind that  $\mathbf{x}_0 \notin \bar{\Omega}$ , we obtain

$$\Delta v + s^2 |\nabla v|^2 = \epsilon(\mathbf{x}), \mathbf{x} \in \Omega. \quad (2.9)$$

Equation (2.9) shows that the coefficient  $\epsilon(\mathbf{x})$  can be computed directly via the function  $v$ . To compute  $v$ , we eliminate the unknown coefficient  $\epsilon(\mathbf{x})$  from (2.9) by taking the derivative with respect to  $s$  both sides of (2.9). Denote by  $q := \partial_s v$ . Then

$$v = - \int_s^{\infty} q d\tau = - \int_s^{\bar{s}} q d\tau + V, \quad (2.10)$$

where  $\bar{s} > \underline{s}$  is a large number and  $V(\mathbf{x}) := v(\mathbf{x}, \bar{s})$ . The number  $\bar{s}$  plays the role of a regularization parameter in the globally convergent method and it is chosen



numerically in the computational practice. The function  $V(\mathbf{x}) := v(x, \bar{s})$  is called the “tail function”. It follows from (2.8) and (2.10) that

$$V(\mathbf{x}) = \frac{\ln w(\mathbf{x}, \bar{s})}{\bar{s}^2}. \quad (2.11)$$

From (2.9) we obtain the following differential integral equation involving  $q$  and  $V$

$$\begin{aligned} \Delta q - 2s^2 \nabla q \cdot \int_s^{\bar{s}} \nabla q(\mathbf{x}, \tau) d\tau + 2s^2 \nabla V \cdot \nabla q + 2s \left| \int_s^{\bar{s}} \nabla q(\mathbf{x}, \tau) d\tau \right|^2 \\ - 4s \nabla V \cdot \int_s^{\bar{s}} \nabla q(\mathbf{x}, \tau) d\tau + 2s |\nabla V|^2 = 0, \mathbf{x} \in \Omega. \end{aligned} \quad (2.12)$$

Equation (2.12) is coupled with the following Dirichlet boundary condition for  $q$ , which follows from (2.4),

$$q(\mathbf{x}, s) = \psi(\mathbf{x}, s), \mathbf{x} \in \partial\Omega, \quad (2.13)$$

where

$$\psi(\mathbf{x}, s) = \frac{\partial}{\partial s} \left[ \frac{\ln(\varphi)}{s^2} \right] = \frac{\partial_s \varphi}{s^2 \varphi} - \frac{2 \ln(\varphi)}{s^3}. \quad (2.14)$$

Here  $\varphi$  is the Laplace transform of the measured data, i.e.  $\varphi(\mathbf{x}, s) = \int_0^\infty g(\mathbf{x}, t) e^{-st} dt$ .

Note that equation (2.12) has two unknown functions  $q$  and  $V$ . In order to approximate both of them we treat them differently. In particular, we use an iterative procedure with respect to the tail function  $V$  as described below.

**2.3. Discretization and description of the algorithm.** Divide the pseudo frequency interval  $[\underline{s}, \bar{s}]$  into  $N$  uniform sub-intervals by  $\bar{s} = s_0 > s_1 > \dots > s_N = \underline{s}$ ,  $s_n - s_{n+1} = h$ . We approximate  $q$  by  $q(\mathbf{x}, s) \approx q_n(\mathbf{x})$ ,  $s \in (s_n, s_{n-1}]$ ,  $n = 1, \dots, N$ . We also set  $q_0 \equiv 0$ . Then after some manipulations, a system of elliptic equations for functions  $q_n(x)$  is derived from (2.12) using the so-called “Carleman Weight Function”  $\exp[\lambda(s - s_{n-1})]$ ,  $s \in (s_n, s_{n-1})$ , where  $\lambda \gg 1$  is a certain parameter. We take  $\lambda = 20$  in all our computations. This system is

$$\begin{aligned} \Delta q_n + A_{1,n} \nabla q_n \cdot (\nabla V_n - \nabla \overline{q_{n-1}}) \\ = A_{2,n} |\nabla q_n|^2 + A_{3,n} (|\nabla \overline{q_{n-1}}|^2 + |\nabla V_n|^2 - 2 \nabla V_n \cdot \nabla \overline{q_{n-1}}), \end{aligned} \quad (2.15)$$

where  $A_{i,n}$ ,  $i = 1, 2, 3$ , are some coefficients, depending on  $s_n$  and  $\lambda$ , which are analytically computed, and  $\nabla \overline{q_{n-1}} = h \sum_{j=0}^{n-1} \nabla q_j$ . Here we indicate the dependence of the tail function  $V := V_n$  on the number  $n$ , because we approximate  $V$  iteratively. The discretized version of the boundary condition (2.13) is given by

$$q_n(\mathbf{x}) = \psi_n(\mathbf{x}) := \frac{1}{h} \int_{s_n}^{s_{n-1}} \psi(\mathbf{x}, s) ds \approx \frac{1}{2} [\psi(\mathbf{x}, s_n) + \psi(\mathbf{x}, s_{n-1})], \mathbf{x} \in \partial\Omega. \quad (2.16)$$

One can prove that  $|A_{2,n}| \leq C/\lambda$  for sufficiently large  $\lambda$ , where  $C > 0$  is a certain constant. Hence, the first term on the right hand side of (2.15) is dominated by the

other terms. Therefore, in the following we set  $A_{2,n}|\nabla q_n|^2 := 0$  and ignore this term in our computations. The system of elliptic equations (2.15) with boundary conditions (2.16) can be solved sequentially starting from  $n = 1$ . To solve it, we make use of the iterative process: For a given  $n$  and some approximation  $q_{n,i-1}$  of  $q_n$ , we find the next approximation  $q_{n,i}$  of  $q_n$  by solving (2.15)–(2.16). Denote by  $m_n$  the number of these iterations. The full algorithm is described as follows.

**Globally convergent algorithm.**

- Given the first tail  $V_0$ . Set  $q_0 \equiv 0$ .
- For  $n = 1, 2, \dots, N$ 
  1. Set  $q_{n,0} = q_{n-1}$ ,  $V_{n,1} = V_{n-1}$
  2. For  $i = 1, 2, \dots, m_n$ 
    - Find  $q_{n,i}$  by solving (2.15)–(2.16) with  $V_n := V_{n,i}$ .
    - Compute  $v_{n,i} = -hq_{n,i} - \overline{q_{n-1}} + V_{n,i}$ ,  $\mathbf{x} \in \Omega$ .
    - Compute  $\epsilon_{n,i}$  via (2.9). Then solve the forward problem (2.1)–(2.2) with the new computed coefficient  $\epsilon := \epsilon_{n,i}$ , compute  $w := w_{n,i}$  by (2.6) and update the tail  $V_{n,i+1}$  by (2.11).
  - End (for  $i$ )
  3. Set  $q_n = q_{n,m_n}$ ,  $\epsilon_n = \epsilon_{n,m_n}$ ,  $V_n = V_{n,m_n+1}$  and go to the next frequency interval  $[s_{n+1}, s_n]$  if  $n < N$ . If  $n = N$ , then stop.

Stopping criteria of this algorithm with respect to  $i, n$  should be addressed computationally, see details in section 4.

**2.4. The initial tail function and the global convergence.** We remark that the convergence of this algorithm depends on the choice of the initial tail function  $V_0$ . In [5], see also [7], the global convergence of this algorithm was proved within the context of an approximate mathematical model. First, it has been proved in [5] that under some conditions, there exists a function  $p(\mathbf{x}) \in C^{2+\alpha}(\overline{\Omega})$  such that

$$V(\mathbf{x}, s) = \frac{p(\mathbf{x})}{s} + O\left(\frac{1}{s^2}\right), s \rightarrow \infty, \quad (2.17)$$

where  $C^{2+\alpha}(\overline{\Omega})$  is the Hölder space. Due to this asymptotic behavior, we assume that the exact tail is given by

$$V(\mathbf{x}, s) = \frac{p(\mathbf{x})}{s} = \frac{\ln w(\mathbf{x}, s)}{s^2}, \forall s \geq \bar{s}. \quad (2.18)$$

We use this assumption only on the initializing iteration to obtain  $V_0(\mathbf{x})$ . That is, we make use of only the first order term in the asymptotic expansion (2.17). Under this assumption, it follows from (2.12), (2.13) that  $p(\mathbf{x})$  satisfies

$$\Delta p(\mathbf{x}) = 0, \mathbf{x} \in \Omega, p \in C^{2+\alpha}(\overline{\Omega}), \quad (2.19)$$

$$p|_{\partial\Omega} = -\bar{s}^2\psi(\mathbf{x}, \bar{s}), \quad (2.20)$$

As the first guess for the tail function we take

$$V_0(\mathbf{x}) := \frac{p(\mathbf{x})}{\bar{s}}, \mathbf{x} \in \Omega. \quad (2.21)$$

With this choice of the initial tail function, it was proved in Chapter 2 of [5], Theorem 2.9.4 (see also Theorem 5.1 in [7]) that the proposed algorithm is convergent in the

following sense: it delivers some points in a sufficiently small neighborhood of the exact coefficient  $\epsilon(\mathbf{x})$ . The latter is sufficient for computational purposes. The size of this neighborhood depends on the noise in the data  $\psi$ , the discretization step size  $h$  of the interval  $[\underline{s}, \bar{s}]$  and the domain  $\Omega$ . We note that no *a priori* information about the unknown coefficient is used here. Therefore, we say that the algorithm is globally convergent within the framework of the approximation (2.18).

In our computations, the bound constraints (2.3) is used to truncate the coefficient  $\epsilon_{n,i}(\mathbf{x})$  on each iteration.

**2.5. Numerical solution of the forward problem.** Although a point source is used in the forward model (2.1)–(2.2) for theoretical analysis, we make use of an incident plane wave in our numerical implementation. Moreover, since it is impossible to solve the forward problem numerically in the infinite space  $\mathbf{R}^3$ , we consider the wave equation in a bounded domain  $G \subset \mathbf{R}^3$  such that  $\Omega \subset G$ . For simplicity, we choose  $G$  as the prism

$$G := \{\mathbf{x} \in \mathbf{R}^3 : X_l \leq x \leq X_u, Y_l \leq y \leq Y_u, Z_l \leq z \leq Z_u\}.$$

We denote by  $\partial G_z^l := \{z = Z_l\}$ ,  $\partial G_z^u := \{z = Z_u\}$  and  $\partial G_{xy} = \partial G \setminus (\partial G_z^l \cup \partial G_z^u)$ . An incident plane wave of a short time period is excited at  $\partial G_z^u$  and propagates in the negative  $z$  direction. At the plane  $\partial G_z^l$  we assume that the absorbing boundary condition is satisfied, and at  $\partial G_{xy}$  we assign the homogeneous Neumann boundary condition. More precisely, we solve the following problem

$$\epsilon(\mathbf{x})u_{tt}(\mathbf{x}, t) = \Delta u(\mathbf{x}, t), (\mathbf{x}, t) \in G \times (0, T), \quad (2.22)$$

$$u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = 0, \mathbf{x} \in G, \quad (2.23)$$

$$\partial_\nu u = f(t), (\mathbf{x}, t) \in \partial G_z^u \times (0, t_1), \quad (2.24)$$

$$\partial_\nu u = -u_t, (\mathbf{x}, t) \in \partial G_z^l \times (t_1, T), \quad (2.25)$$

$$\partial_\nu u = -u_t, (\mathbf{x}, t) \in \partial G_z^l \times (0, T), \quad (2.26)$$

$$\partial_\nu u = 0, (\mathbf{x}, t) \in \partial G_{xy} \times (0, T), \quad (2.27)$$

where  $\nu$  is the outward normal vector of  $\partial G$  and  $t_1 := 2\pi/\omega$  is the duration of the excitation of the incident plane wave. Function  $f$  is the incident waveform chosen by

$$f(t) = \sin(\omega t), 0 \leq t \leq t_1 = 2\pi/\omega.$$

Here  $\omega$  represents the angular frequency of the incident plane wave. The forward problem (2.22)–(2.27) is solved using the software package WavES [20] via the hybrid FEM/FDM method described in [9].

### 3. The experimental setup and data pre-processing.

**3.1. Data acquisition.** Our experimental configuration is shown in Figure 3.1. The setup of our measurements included a horn antenna (transmitter) fixed at a given position and a detector scanned in a square of a vertical plane, which we refer to as the measurement plane. Consider the Cartesian coordinate system  $Oxyz$  as shown in Figure 3.1(b). The scanning area was 1 m by 1 m with the step size of 0.02 m, starting at  $(x, y) = (-0.5, -0.5)$ , and ends at  $(x, y) = (0.5, 0.5)$ .

In our model (2.1)–(2.2), we assumed that the source point  $\mathbf{x}_0$  is in  $\mathbf{R}^3 \setminus \bar{\Omega}$ . However, due to some technical difficulties with the mechanical scanning system, the horn antenna was not placed behind but in front of the measurement plane (between

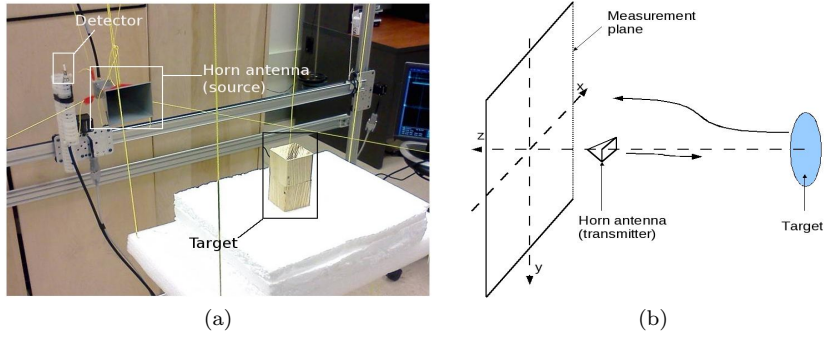


Fig. 3.1: (a): A picture of our experiment setup; (b) Diagram of our setup.

the measurement plane and the targets). Therefore a small area in the center of the scanning area on the measurement plane was shaded by the horn. The horn was placed at the distance of about 0.2-0.25 m from the measurement plane and the distances from the targets to the measurement plane are about 0.8 m.

At each position of the detector, a number of electric pulses were emitted by the horn. The detector received two types of signals: the direct signals from the source and the backscattered signals by the targets. The direct signals are used for time reference in data pre-processing. There were also other unwanted signals due to scattering by some objects in the room. To reduce the instability of the recorded signals in terms of magnitude, the measurements were repeated 800 times at each detector position and the recorded signals were averaged. By scanning the detector and repeating the measurements, we obtained essentially the same signals as using one incident signal and multiple detectors at the same time.

Pulses were generated by the picosecond pulse generator 10070A. The scattered signals were measured by a Tektronix DSA70000 series real-time oscilloscope. The emitted pulses were of 300 picoseconds duration. The wavelength of the incident pulses was about 0.03 m. The sampling rate (the step size in time between two consecutive records of captured signals) was  $\Delta t = 10$  picoseconds. Each signal was recorded for 10 nanoseconds.

**3.2. Data pre-processing.** This biggest challenge in working with these experimental data is to deal with the *huge misfit* between these data and the data produced via computational simulations, also see [5, 6, 15, 16] for the same conclusion. There are several causes of this misfit such as (i) the instability of the amplitude of the emitted signals (incident waves) which causes the instability of the received signals, (ii) unwanted scattered waves by several existing objects around our devices (see Figure 3.2(a)), (iii) the shadow on the measurement plane caused by the transmitting horn antenna, and (iv) the difference between the experimental and simulated incident waves. Figure 3.3 compares the Laplace transform of an experimental scattered wave and the corresponding simulated one, which shows a huge misfit between them. Note that the Laplace transform of the experimental wave was carried out after removing the incident wave and unwanted parts, as shown in Figure 3.2(c).

Therefore, the *central procedure* before applying inversion methods is the data pre-processing. This procedure is usually heuristic and cannot be rigorously justified. Our data pre-processing consists of six steps described below. We do not describe

steps 1-3 in detail here, since they are straightforward.

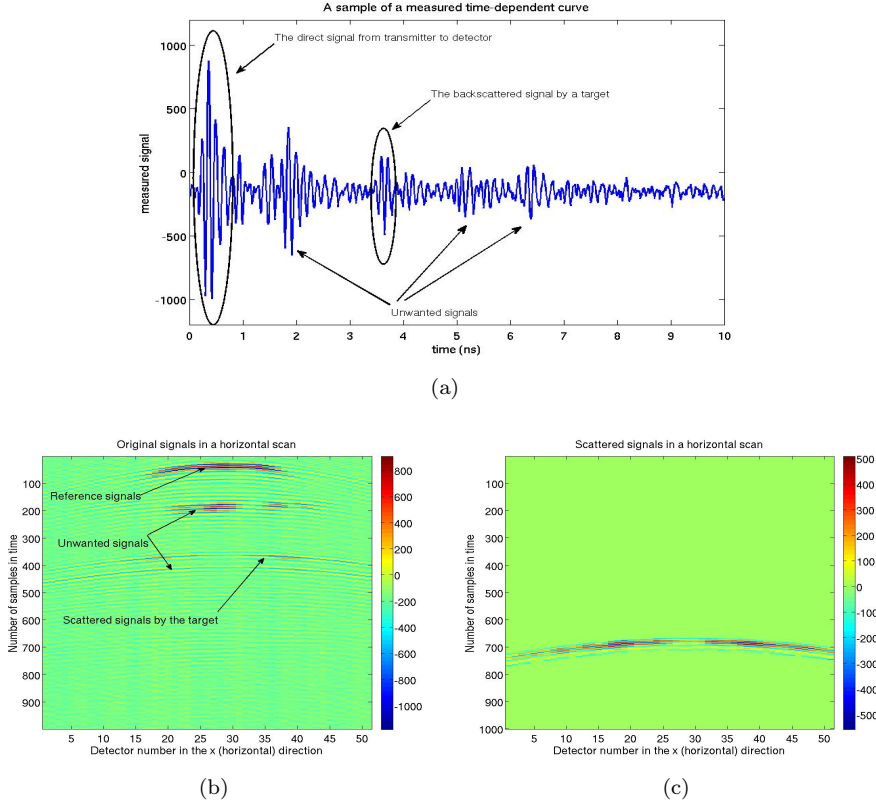


Fig. 3.2: An experimental data set: (a) a 1D curve; (b) a 2D data on a horizontal scan; (c) After steps 1-4 of the data pre-processing.

*Step 1. Off-set correction.* The acquired signals are usually shifted from the zero mean value. This can be corrected by subtracting the mean value from them.

*Step 2. Time-zero correction.* Time-zero refers to the moment at which the signal is emitted from the transmitter. The recorded signals may be shifted in time. We use the direct signals from the transmitter to the detector to correct the time-zero.

*Step 3. Source shift.* As mentioned above, the horn antenna in our experiments is placed between the targets and the measurement plane. To make the data consistent with our assumption that the source is out side of  $\Omega$ , we artificially “shift” the horn in the positive  $z$ -direction such that it is 0.4 m further than the measurement plane from the targets. It is done by shifting the whole time-dependent data by a number of samples which corresponds to the shifted distance.

*Step 4. Extraction of scattered signals.* Apart from the signals backscattered by the targets, our measured data also contain various types of signals as mentioned above. What we need, however, is the scattered signals by the targets only. The extraction of these scattered signals for each target is done as follows. First, we exclude the direct signals and the unwanted signals, which come earlier than the scattered signals by the target (see Figure 3.2(a)-(b)), by calculating the time of arrival. These unwanted signals are due to the reflection of the direct signals by the metallic structure behind

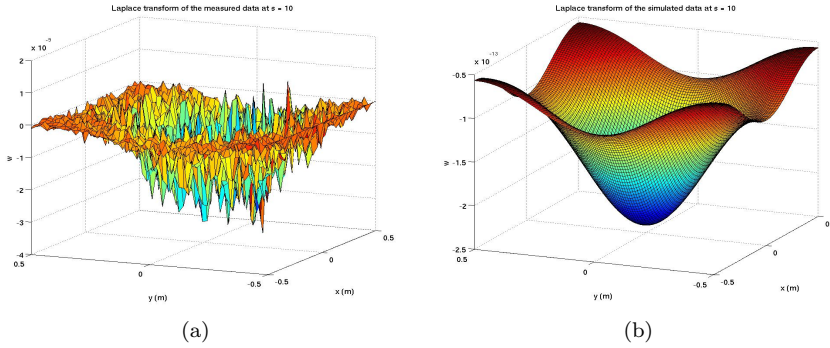


Fig. 3.3: Experimental vs. simulated scattered waves on the measurement plane after Laplace transform: (a) measured data; (b) simulated data.

the measurement plane, so we can estimate their times of arrival as we know the distance from the measurement plane to this structure. Then, as we observed that the scattered signals by the target are the strongest peaks of the remaining, we first detect, for each detector position, the strongest negative peak after removing the aforementioned signals. Then, the scattered signals by the target are taken as 7 peaks (4 negative peaks and 3 positive peaks) starting from the first negative peak prior to the strongest one, if its amplitude is less than 80% of the strongest one (see Figure 3.2(a)). Otherwise, we start from the second negative peak prior to the strongest one. The reason for choosing 7 peaks for the scattered signals is due to the fact that the incident pulses also contain 7 strong peaks. We note that having the scattered signals by the target, we can easily determine the distance from its front side to the measurement plane by calculating the time of arrival.

*Step 5. Data propagation.* After getting the scattered signals, the next step of data pre-processing is to propagate the data closer to the targets, i.e. to approximate the scattered waves on a plane closer to the targets, compared to the measurement plane. There are two reasons for doing this. The first one is that since the Laplace transform decays exponentially in terms of the time delay, which is proportional to the distance from the targets to the measurement plane, then the amplitude of the data after the Laplace transform on the measurement plane is very small and can be dominated by the computational error. The second reason is that this propagation procedure helps to reduce the computational cost substantially as the computational domain  $\Omega$  is reduced. We have also observed that the data propagation helps to reduce the noise in the measured data.

*Step 6. Data calibration.* Finally, since the amplitude of the experimental incident and scattered waves are usually quite different from simulations, we need to bring the former to the same level of the amplitude as the latter. This is done using a known target referred to as *calibrating object*.

Figure 3.2 shows an original data set and data after steps 1-4 for a horizontal scan of the detector. One can see that steps 1-4 help to remove the unwanted signals. In the following, we present our methods for steps 5 and 6.

**3.2.1. Data propagation.** Denote by  $P_m$  the measurement plane and by  $P_p$  the propagation plane, which is closer to the targets than  $P_m$ . Without a loss of

the generality, we denote by  $P_m = \{z = a\}$  and  $P_p = \{z = 0\}$ , where the number  $a > 0$ . Moreover, denote by  $\Gamma = (-0.5, 0.5) \times (-0.5, 0.5) \subset \mathbf{R}^2$  the scanning area of the detector on the plane  $P_m$ . Let  $\Gamma_m = \{(x, y, a) \in \mathbf{R}^3 : (x, y) \in \Gamma\}$  and  $\Gamma_p = \{(x, y, 0) \in \mathbf{R}^3 : (x, y) \in \Gamma\}$ . We also denote by  $u^s(\mathbf{x}, t)$  the scattered wave. Note that the medium between  $P_m$  and  $P_p$  is homogeneous with  $\epsilon = 1$  and the scattered wave  $u^s$  propagates in the direction from  $P_p$  to  $P_m$ . The aim of the data propagation is to approximate  $u^s|_{\Gamma_p \times (0, \infty)}$  from the measured data  $\tilde{g}(x, y, t) := u^s|_{\Gamma_m \times (0, \infty)}$ .

To do this, we make use of a time reversal method. Its idea is to reverse the scattered wave in time via solution of an initial boundary value problem for the time-reversed wave function. We proceed as follows.

Since short pulses are used as incident waves, it is reasonable to assume that the scattered wave  $u^s$  in the domain between  $P_m$  and  $P_p$  vanishes along with its time derivative  $u_t^s$  after some time  $T$ . Therefore in the following we only consider the finite time interval  $(0, T)$ . Denote  $\tau := T - t$ . Then the time-reversed wave function  $u^r(\mathbf{x}, \tau) := u^s(\mathbf{x}, t)$  satisfies the homogeneous wave equation. Moreover, it propagates in the negative  $z$  direction, i.e. from  $P_m$  to  $P_p$ . To find  $u^r|_{\Gamma_p}$ , we consider the domain  $D := \{\mathbf{x} \in \mathbf{R}^3 : (x, y) \in \Gamma, b < z < a\}$  with  $b < 0$ . Note that  $\Gamma_p \subset D$ . The reason for choosing this larger domain is that we need to assign boundary conditions at  $\partial D$ . Indeed, we assume that  $u^r$  satisfies the absorbing boundary condition at  $\Gamma_b := \{(x, y, b) : (x, y) \in \Gamma\}$ . On  $\Gamma_b$ , far from our propagation plane, this boundary condition means, heuristically, that we “send back” the original scattered wave  $u^s$  recorded at  $P_m$ . On the other hand, we impose the zero Neumann boundary condition at the rest of the boundary of  $D$ , except of  $\Gamma_m$ . Denote  $Q_T = D \times (0, T)$  and  $\Gamma_3 := \partial D \setminus (\Gamma_m \cup \Gamma_b)$ . We obtain the following problem for the function  $u^r(\mathbf{x}, \tau)$

$$u_{\tau\tau}^r = \Delta u^r, (\mathbf{x}, \tau) \in Q_T, \quad (3.1)$$

$$u^r(\mathbf{x}, 0) = u_\tau^r(\mathbf{x}, 0) = 0, \mathbf{x} \in D, \quad (3.2)$$

$$u^r|_{\Gamma_m \times (0, T)} = \tilde{g}(x, y, T - \tau), \quad (3.3)$$

$$(\partial_\nu u^r + \partial_\tau u^r)|_{\Gamma_b \times (0, T)} = 0, \quad (3.4)$$

$$\partial_\nu u^r|_{\Gamma_3 \times (0, T)} = 0. \quad (3.5)$$

A similar procedure was proposed and numerically implemented for computationally simulated data in [1]. However, the absorbing boundary condition for the original scattered wave  $u^s$  was applied in [1, 2]. Since our time-reversed wave function  $u^r$  propagates from  $\Gamma_m$  to  $\Gamma_b$ , we believe that it is more natural to apply the absorbing boundary condition on  $\Gamma_b$  to  $u^r$ . Note that we do not locate scatterers here, since we know from our previous pre-processing steps an approximate distance to the targets. We refer the reader to [13, 18] and the references therein for experimental time reversal mirror method, a similar procedure but using a physical mirror to reverse the received signals in time and send back to the targets.

Theorem 3.1 below shows the stability of the problem (3.1)–(3.5). We do not prove this theorem here due to the space limit. This theorem can be extended to more general domains. We note that its proof is similar with the proof of Theorem 4.1 of [4] for the Maxwell’s system. For brevity we are not concerned here with minimal smoothness assumptions and leave aside the question of existence. We conjecture that it can be addressed via the technique of chapter 4 of [17].

**THEOREM 3.1.** *Assume that there exists a solution  $u^r \in H^2(Q_T)$  of the problem (3.1)–(3.5). Also, assume that the function  $\tilde{g} \in H^2(\Gamma_m \times (0, T))$  and there exists such*

a function  $F \in H^2(Q_T)$  that

$$\begin{aligned} F(\mathbf{x}, 0) = F_\tau(\mathbf{x}, 0) = 0, (\partial_\nu F + F_\tau) |_{\Gamma_b \times (0, T)} = 0, \partial_\nu F |_{\Gamma_3 \times (0, T)} = 0, \\ F |_{\Gamma_m \times (0, T)} = \tilde{g}(\mathbf{x}, t), \quad \|F\|_{H^2(Q_T)} \leq C \|\tilde{g}\|_{H^2(\Gamma_m \times (0, T))}, \end{aligned}$$

where  $C > 0$  is a certain number. Then that solution  $u^r$  is unique and the following stability estimate holds with a constant  $C_1 = C_1(C, Q_T) > 0$  depending only on the listed parameters

$$\|u^r\|_{H^1(Q_T)} \leq C_1 \|\tilde{g}\|_{H^2(\Gamma_m \times (0, T))}.$$

By solving (3.1)–(3.5), we obtain an approximation of  $u^r(\mathbf{x}, \tau)$  and then obtain an approximation of  $u^s$  for  $\mathbf{x} \in \Gamma_p$ . In this work, we use the finite difference method to solve this problem. Since the formulation is very standard, we omit the details here. We note that other methods can also be used to solve this data propagation problem such as the Fourier transform method or the quasi-reversibility method. We will discuss these two methods in future works.

**3.2.2. Data calibration.** Usually the experimental data have quite different amplitudes compared to the simulations, see Figure 3.4 which shows that the minimum of the Laplace transform of the propagated measured data is approximately  $-2 \times 10^{-5}$ , whereas the minimum of the simulated data is about  $-5 \times 10^{-9}$ . We choose a number, which is called *calibration factor*, to scale the measured data to the same scaling as in our simulations. To do this, we make use of the measured data of a single calibrating object whose location, shape, size and material are known. The word “single” is important here to ensure that the calibration procedure is unbiased, i.e. it remains the same for all targets.

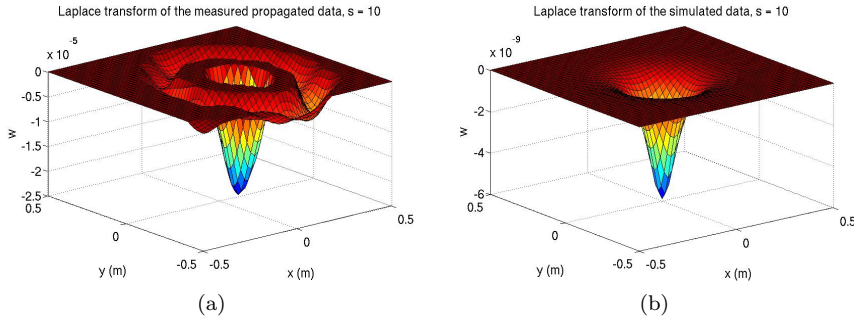


Fig. 3.4: Laplace transform of the scattered wave on the propagation plane  $P_p$ : (a) measured data; (b) simulated data.

First, we computationally simulate the data on  $\Gamma_p$  for the calibrating object by solving problem the (2.22)–(2.27). Next, we compute the Laplace transform (2.6) of this computationally simulated solution. Just as in [5], in studies below we work with  $s \in [\underline{s}, \bar{s}]$ . Numbers  $\underline{s}, \bar{s}$  are chosen numerically, see section 4. Denote by  $w_{sim}^t(\mathbf{x}, s)$ ,  $w_{sim}^s(\mathbf{x}, s)$  and  $w_{sim}^i(\mathbf{x}, s)$  respectively the Laplace transforms of the total wave, the scattered wave and the incident wave of the simulated solution for the calibrating object. Clearly,  $w_{sim}^s(\mathbf{x}, s) = w_{sim}^t(\mathbf{x}, s) - w_{sim}^i(\mathbf{x}, s)$ . It can be proved that



$w_{sim}^s(\mathbf{x}, s) \leq 0$ , see Figure 3.4(b). Figure 3.4(b) displays the function  $w_{sim}^s(\mathbf{x}, s)$  for  $\mathbf{x} \in \Gamma_p$  and qualitatively this is a typical behavior for all targets. Let

$$d_{sim,s} = \min_{\Gamma_p} w_{sim}^s(\mathbf{x}, s).$$

Next, for  $\mathbf{x} \in \Gamma_p$  let  $w_{exp}^s(\mathbf{x}, s)$  be the Laplace transform of the propagated experimental data for the calibrating object, see Figure 3.4(a). Denote by

$$d_{exp,s} = \min_{\Gamma_p} w_{exp}^s(\mathbf{x}, s).$$

The number  $d_{sim,s}/d_{exp,s}$  is used as the calibration factor for all targets at pseudo-frequency  $s$ . That means, the propagated measured data of all targets are multiplied by this calibration factor before being used in the inversion algorithm.

For the data sets used in this paper, we have two types of targets: dielectric and metallic targets. We have observed that two different calibration factors should be used for these two types of targets, because the signals from them have different levels of amplitude. First of all, we differentiated these two types of targets by comparing the amplitudes of the recorded signals. Indeed, we have consistently observed that the maximal values of amplitudes of measured signals are at least two times larger for metallic targets than for dielectric ones on those positions of detectors which are most sensitive to the presence of targets. Next, we chose in each type a known object as the calibrating object. In other words, we should use a dielectric calibrating object for all dielectric targets and another metal calibrating object for all metallic targets.

The value of  $\epsilon(\mathbf{x})$  for the dielectric calibrating object was taken as  $\epsilon(\mathbf{x}) = 4.28$  inside that target and  $\epsilon(\mathbf{x}) = 1$  outside of it. For the metallic calibrating object, as suggested by (2.5), we took  $\epsilon(\mathbf{x}) = 12$  inside and  $\epsilon(\mathbf{x}) = 1$  outside of it.

**4. Numerical implementation and reconstruction results.** Now we describe some details of the numerical implementation and present reconstruction results for our experimental data using the globally convergent algorithm of section 2.2. In our computations, the frequency of the simulated incident wave was chosen as  $\omega = 30$ .

There were ten data sets tested in this paper. Each data set consisted of only one target numbered from 1 to 10. Four of them (targets 1 - 4) were non-blind and six of them (targets 5 - 10) were blind. In our data pre-processing, we chose target 1 (a piece of wood) and target 3 (a metallic sphere) as our calibrating objects.

**4.1. Dimensionless variables.** The spatial dimensions in our experiment were given in meters. Since the scanning step in our measured data was 0.02 m in both  $x$  and  $y$  directions, we chose the dimensionless spatial variable  $\mathbf{x}'$  to be  $\mathbf{x}' = \mathbf{x}/1(m)$ . Next, to scale the wave speed to be 1 in the homogeneous medium, as in our simulations, we chose the dimensionless time  $t'$  by  $t' = 0.3t$  where  $t$  is the time given in nanoseconds (ns). The factor 0.3 is the speed of light in meters per nanosecond in the free space. For the simplicity of notations, we use  $\mathbf{x}$  and  $t$  again to denote the dimensionless variables.

**4.2. Choosing the domains.** Before applying the inversion algorithm, some information about the targets was in place already from the measured data due to the data pre-processing. First, we obtained the distance from the targets to the measurement plane. Second, by the data propagation, the targets' locations in the  $xy$  plane were estimated. Thirdly, we also differentiated between nonmetallic and metallic targets directly from the measured data based on signal amplitudes.

Given the estimated distances from the targets to the measurement plane, which were about 0.8 m, we propagated the measured data from the measurement plane  $P_m = \{z = 0.8\}$  to the plane  $P_p = \{z = 0.04\}$ . This means that the distance from the front sides of the targets to the backscattering boundary of our inversion domain was about 0.04 m. The reason for choosing this distance was due to good reconstruction results we have obtained for several non-blind targets. The domain  $\Omega$  was chosen by

$$\Omega = \{\mathbf{x} \in (-0.5, 0.5) \times (-0.5, 0.5) \times (-0.1, 0.04)\}. \quad (4.1)$$

For solving the forward problem (2.22)–(2.27), using the hybrid FDM/FEM method, we chose the domain  $G$  as

$$G = \{\mathbf{x} \in (-0.56, 0.56) \times (-0.56, 0.56) \times (-0.16, 0.1)\}.$$

This domain  $G$  was decomposed into two subdomain:  $G = \Omega \cup (G \setminus \Omega)$ . We recall that  $\epsilon(\mathbf{x}) = 1$  in  $G \setminus \Omega$ . Therefore, it is only necessary to solve the inverse problem in  $\Omega$ . In  $\Omega$  we use a FEM mesh with tetrahedral elements, while in  $G \setminus \Omega$  we use a FDM mesh with the mesh size of 0.02 by 0.02 by 0.02 in Test 1 and 0.01 by 0.01 by 0.01 in Test 2 below. The reason for using the FEM mesh in  $\Omega$  is that it is possible to refine the reconstruction using adaptive mesh refinement. However, we do not discuss this step in this work. We refer to [5, 8] for more details about the adaptivity.

The time interval  $(0, T)$  in the forward problem (2.22)–(2.27) was chosen equal to  $(0, 1.2)$ . Since the explicit scheme in time was used in WaveES, the time step size was chosen as  $\Delta t = 0.0015$  which satisfies the CFL stability condition.

The pseudo frequencies  $s_n$  were chosen from  $\underline{s} = 8$  to  $\bar{s} = 10$  with the step size  $h = 0.05$ . This frequency interval was chosen because it gave good reconstructions of the non-blind targets.

**4.3. Estimation of the  $xy$  projection.** During our data pre-processing for non-blind targets, we observed that the  $xy$  projection of a target can be roughly estimated directly from the propagated data. Indeed, we define  $\Gamma_T$  as

$$\Gamma_T = \{(x, y) : v_{prop}(x, y, \bar{s}) < 0.85 \min v_{prop}(x, y, \bar{s})\}, \quad (4.2)$$

where  $v_{prop}$  is the function  $v$  in (2.8) which is constructed from the propagated measured data on the propagation plane  $\Gamma_p$ . Note that the function  $v_{prop}$  has a negative peak corresponding to each target, see Figure 3.4. The truncation value 0.85 was chosen based on trial-and-error tests on some non-blind targets with known sizes. We observed that  $\Gamma_T$  provided a good approximation for the  $xy$  projection of a target. Note that the same truncation was applied to blind targets as well. Hence, it is not biased.

Figure 4.1 shows the estimated  $xy$  projections of targets 4 and 10 in our experiments, see Table 4.1. Although the corners of the targets are not well estimated, we see that their shapes and sizes are reasonably good. For target 10, since its lower part was filled with the sand and the upper part was air inside of a wooden cover (see section 4.7 for details), we can see only the lower part due to its higher refractive index compared to the upper part.

**4.4. Complementing backscattered data.** We recall that only backscattered signals were measured in our experiments. This means that after data propagation, the function  $\psi(\mathbf{x}, s)$  was known only on the side  $\Gamma_p = \{\mathbf{x} \in \partial\Omega : |x|, z = 0.04\}$  of  $\Omega$ . As in [7], we replace the missing data on the other parts of  $\partial\Omega$  by the corresponding

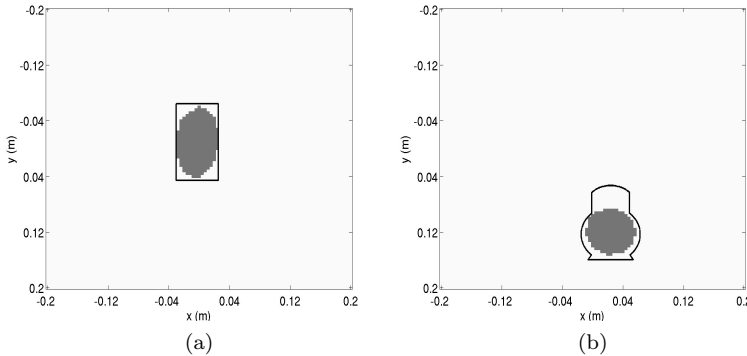


Fig. 4.1: Estimation of target's  $xy$  projection: (a): target 4 (a metallic cylinder); (b): target 10 (a wooden doll partly filled with sand). Thin lines indicate boundaries of correct  $xy$  projections.

solution of the forward problem in the homogeneous medium. In other words, in our computations, function  $\psi$  is given by

$$\psi(\mathbf{x}, s) = \begin{cases} \psi_{prop}(\mathbf{x}, s), & \mathbf{x} \in \Gamma_p, s \in (\underline{s}, \bar{s}), \\ \psi_{sim}^i(\mathbf{x}, s), & \mathbf{x} \in \partial\Omega \setminus \Gamma_p, s \in (\underline{s}, \bar{s}), \end{cases} \quad (4.3)$$

where  $\psi_{prop}$  is the function  $\psi$  of (2.14) computed from the propagated measured data at  $\Gamma_p$  and  $\psi_{sim}^i$  is computed from the simulated solution of the problem (2.22)–(2.27) with  $\epsilon(\mathbf{x}) \equiv 1$  in the wave equation (2.22). In the following, we also denote by  $V_{prop}(\mathbf{x}) := v_{prop}(x, y, \bar{s})$ ,  $\mathbf{x} \in \Gamma_p$ ,  $s \in [\underline{s}, \bar{s}]$  the tail function (2.11) of the propagated measured data.

Recall that our measured data were collected with the step size of 0.02 m in  $x$  and  $y$  directions. To obtain the data at the same grid size as in our computational domain, we applied the linear interpolation to the Laplace transform of the propagated measured data.

Below we present the reconstruction results of two different tests: **Test 1** and **Test 2**. In Test 1, we made use of the first tail function computed from the boundary value problem (2.19)–(2.20). As it was remarked in section 2.4, the global convergence with this choice of the initial tail function is rigorously guaranteed. In Test 2, we choose the first tail function from some information about the targets which we obtained in the data pre-processing. Although the convergence of the resulting algorithm has not been rigorously proved yet, our numerical results show good reconstructions. As we mentioned in section 2.3, stopping criteria of the algorithm should be addressed numerically. We will discuss this in the following.

**4.5. Test 1.** When testing the algorithm for different non-blind targets in this test, we have developed a reliable stopping criterion which includes two steps.

**Stopping criterion for inner iterations with respect to  $i$ .** The inner iterations are stopped at  $i = m_n$  such that either (4.4) or (4.5) is fulfilled,

$$E_{n,i} \geq E_{n,i-1} \text{ or } E_{n,i} \leq \eta, \quad (4.4)$$

$$D_{n,i} \geq D_{n,i-1}, \text{ or } D_{n,i} \leq \eta, \quad (4.5)$$

where  $\eta = 10^{-6}$  is a chosen tolerance and  $D_{n,i} = \|V_{n,i}|_{\Gamma_p} - V_{prop}\|_{L_2(\Gamma_p)}$ . In (4.4)  $E_{n,i}$  represents the relative error of the coefficient, which is given by

$$E_{n,i} = \frac{\|\epsilon_{n,i} - \epsilon_{n,i-1}\|_{L_2(\Omega)}}{\|\epsilon_{n,i-1}\|_{L_2(\Omega)}}.$$

**Criterion for choosing the final coefficient.** In Test 1, the inversion algorithm was run for all pseudo frequencies in the chosen interval  $s \in [8, 10]$  and then the final reconstructed coefficient was chosen as follows. On every pseudo frequency interval  $[s_n, s_{n-1})$  we took the norms  $D_{n,first} = D_{n,1}$  and  $D_{n,final} := D_{n,m_n}$  at the first and the final inner iteration respectively. We have always observed in all ten sets of our data that the first norm  $D_{n,first}$  increases first with respect to  $n$ , then decreases and attains a unique minimum with respect to  $n \in [1, N] = [1, 40]$ . On the other hand, the final norm  $D_{n,final}$  has either one or two local minima, see Figure 4.2(a). Let  $n_1$  be the number of the iteration  $n$  at which the minimum of the first norms is achieved, e.g.  $n_1 = 16$  in Figure 4.2(a). As the reconstructed coefficient  $\epsilon_{rec}(\mathbf{x})$ , we first choose  $\epsilon_{rec}(\mathbf{x}) := \epsilon_{n_1}(\mathbf{x})$ . Next, if  $\max_{\bar{\Omega}} \epsilon_{rec}(\mathbf{x}) < 5$  or  $\max_{\bar{\Omega}} \epsilon_{rec}(\mathbf{x}) > 10$ , then we take the function  $\epsilon_{rec}(\mathbf{x})$  as the final reconstruction. Suppose now that  $5 \leq \max_{\bar{\Omega}} \epsilon_{rec}(\mathbf{x}) \leq 10$ . Then we consider the iteration number  $n_2$  at which the smallest local minimum of the final norm  $D_{n,final}$  is achieved, e.g.  $n_2 = 33$  in Figure 4.2(a). Then we take the function  $\epsilon_{rec}(\mathbf{x}) := \epsilon_{n_2}(\mathbf{x})$  as the final reconstruction.

As shown in Table 4.1, the reconstructed refractive indices are quite close to the true values for all dielectric targets. Table 4.2 shows that reconstructed appearing dielectric constants of metallic targets are always in the required range (2.5). However, the shapes and sizes of the targets were not well reconstructed, in particular the “depth” in the  $z$ -direction. To improve this, we use the following post-processing procedure. Let  $P_{z_0} := \{z = z_0\}$  be the plane where the function  $\epsilon_{rec}(\mathbf{x})$  achieves its maximal value. Then we compute the truncated coefficient function  $\tilde{\epsilon}_{rec}(\mathbf{x})$  as

$$\tilde{\epsilon}_{rec}(\mathbf{x}) = \begin{cases} \epsilon_{rec}(\mathbf{x}) & \text{if } \epsilon_{rec}(x, y, z_0) > \gamma \max \epsilon_{rec}(x, y, z_0), \\ 1 & \text{otherwise,} \end{cases} \quad (4.6)$$

where  $\gamma$  is a truncation factor chosen such that the area of  $\{\tilde{\epsilon}(x, y, z_0) > 1\}$  is the same as that of  $\Gamma_T$ , see (4.2) for  $\Gamma_T$ . Finally, we approximate the depth in the  $z$  direction by truncating  $\tilde{\epsilon}_{rec}(\mathbf{x})$  by 90% of its maximal value. This truncation value is chosen based on the trial-and-error tests with non-blind targets. Figure 4.3 shows imaging results for targets 4 and 10.

**4.6. Test 2.** In this test, we chose the mesh size of 0.01 by 0.01 by 0.01 in order to represent the targets’ shapes more accurately. We use an information about the targets in our data pre-processing to restrict the estimated coefficient  $\epsilon$  in a subdomain of  $\Omega$  and to choose the first tail function. More precisely, for each target, let  $x_{t,min} = \min\{x \in \Gamma_T\}$ ,  $x_{t,max} = \max\{x \in \Gamma_T\}$ . The numbers  $y_{t,min}$  and  $y_{t,max}$  are defined similarly. Then, we define the extended projection by

$$\Gamma_{T,ext} = \{x_{t,min} - 0.03 \leq x \leq x_{t,max} + 0.03, y_{t,min} - 0.03 \leq y \leq y_{t,max} + 0.03\}.$$

Moreover, denote by  $z_{t,front}$  the estimated location of the front side of the target in the  $z$  direction. We then define the following domain  $\Omega_{T,ext}$

$$\Omega_{T,ext} := \{\mathbf{x} \in \Omega : (x, y) \in \Gamma_{T,ext}, -0.1 \leq z \leq z_{t,front}\}.$$

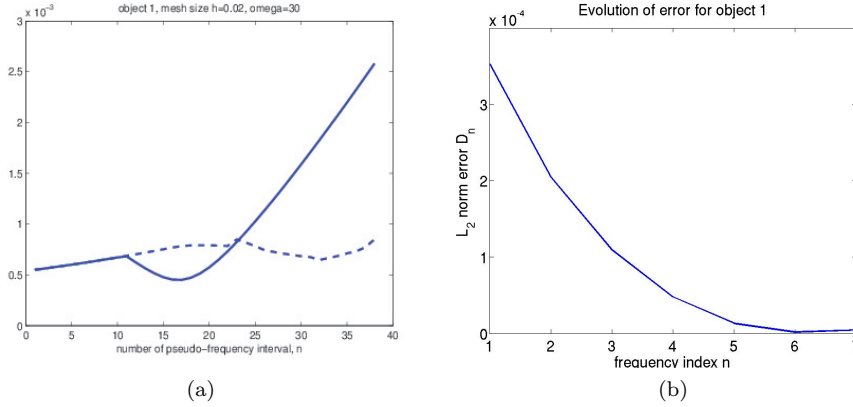


Fig. 4.2: (a) Behavior of the norms  $D_{n,first}$  (solid curve) and  $D_{n,final}$  (dash curve) of Test 1; (b) Behavior of  $D_{n,final}$  of Test 2.

Clearly,  $\Omega_{T,ext} \subset \Omega$ . Moreover, this domain should contain the unknown target we are looking for. Next, we chose the first tail function  $V_0$  as the function (2.11), where the function  $w(\mathbf{x}, \bar{s})$  is the Laplace transform (2.6) at  $s = \bar{s}$  of the solution of the forward problem (2.22)–(2.27) with the coefficient  $\epsilon = \epsilon_0$ , where

$$\epsilon_0(\mathbf{x}) = 1 + d, \text{ for } \mathbf{x} \in \Omega_{T,ext}, \quad \epsilon_0(\mathbf{x}) = 1, \text{ for } \mathbf{x} \notin \Omega_{T,ext}.$$

Moreover, the coefficient is truncated by

$$\epsilon_{n,i}(\mathbf{x}) = \begin{cases} \epsilon_{n,i}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_{T,ext} \text{ and } 1 \leq \epsilon_{n,i}(\mathbf{x}) \leq 1 + d, \\ 1, & \text{if } \epsilon_{n,i}(\mathbf{x}) < 1, \\ 1 + d, & \text{if } \epsilon_{n,i}(\mathbf{x}) > 1 + d. \end{cases} \quad (4.7)$$

In this paper, we chose  $1 + d$  to be 10 for nonmetallic targets and 20 for metallic ones.

**Stopping criterion.** In this test, the inner iterations were stopped using the same criterion as in Test 1. However, we also set the maximum number of inner iterations to be 5. That means, the inner iterations were stopped if either (4.4) or (4.5) was satisfied for  $i < 5$ , or if  $i = 5$ .

Concerning the outer iterations, we have observed that the error  $D_{n,final}$  decreased with respect to  $n$  first, and then increased after reaching a minimum, see Figure 4.2(b). At that minimum, the estimated coefficient was a good approximation of the true one for some non-blind targets. Therefore, we stopped the algorithm when this error function attained the minimum.

We have observed through our tests that the shapes of the targets are quite well reconstructed. Figure 4.4 shows the results of targets 4 and 10 using Test 2. For target 10, we again obtained the lower part, which was filled with the sand, see Figure 4.1(c) for its  $xy$  projection estimated from the data (recall that air was inside the wooden cover in the upper part of that target).

**4.7. Summary of reconstruction results.** To compare our computational results with true ones, we have directly measured *a posteriori* refractive indices  $n = \sqrt{\epsilon}$  of dielectric targets. In Tables 4.1, 4.2, the computed values of the refractive index

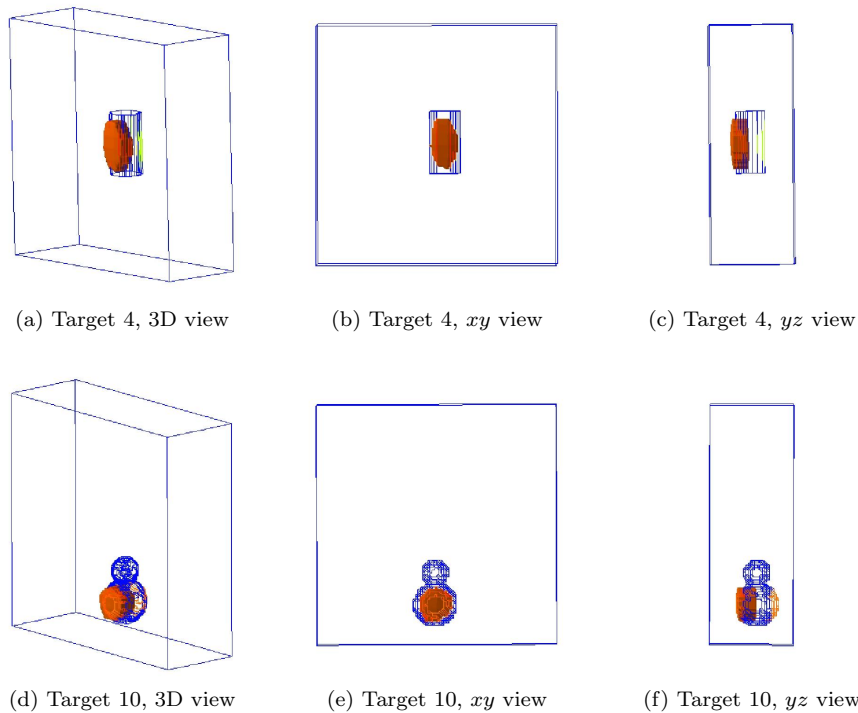


Fig. 4.3: Some reconstruction results of Test 1.  $xy$  view means the projection of the target on the  $xy$  plane.  $yz$  view means the projection of the target on the  $yz$  plane. Thin lines indicate correct shapes.

Table 4.1: Computed and measured refractive indices of six dielectric targets. The average error in direct measurements is 24%. The average error of Test 1 is 8% and Test 2 is 3.4%.

Target ID	1	2	5	8	10
blind/non-blind (yes/no)	no	no	yes	yes	yes
Measured value, error	2.11, 19%	1.84, 18%	2.14, 28%	1.89, 30%	2.1, 26%
$n^{\text{comp}}$ Test 1, error:	1.92, 10%	1.80, 2%	1.83, 17%	1.86, 2%	1.92, 9%
$n^{\text{comp}}$ Test 2, error:	2.03, 4%	1.96, 7%	2.10, 2%	1.85, 2%	2.05, 2%

$n^{\text{comp}}$  of dielectrics (respectively appearing dielectric constant  $\epsilon^{\text{comp}}$  of metals) were chosen as the square root of the maximal values (respectively, the maximal values) of the reconstructed coefficient. Table 4.1 lists both computed  $n^{\text{comp}}$  and directly measured refractive indices  $n$  of dielectric targets for both Test 1 and Test 2. This table also shows the measurement error in direct measurements of  $n$ . Table 4.2 lists computed appearing dielectric constants  $\epsilon^{\text{comp}}$  of metallic targets. We see from Tables 4.1 and 4.2 that  $(n^{\text{comp}})^2 < 5$  for all dielectric targets, while  $\epsilon^{\text{comp}} > 14$  for all metallic targets in Test 1 and  $\epsilon^{\text{comp}} > 7.59$  in Test 2. Thus, our algorithm can differentiate quite well between dielectric and metallic targets.

It can be seen from Table 4.1 that both tests image refractive indices of both blind

Table 4.2: *Computed appearing dielectric constants  $\epsilon^{comp}$  of metallic targets.*

Target ID	3	4	6	7	9
blind/non-blind (yes/no)	no	no	yes	yes	yes
$\epsilon^{comp}$ Test 1:	14.37	16.93	25.0	13.61	13.56
$\epsilon^{comp}$ Test 2:	7.59	10.76	19.55	8.12	7.89

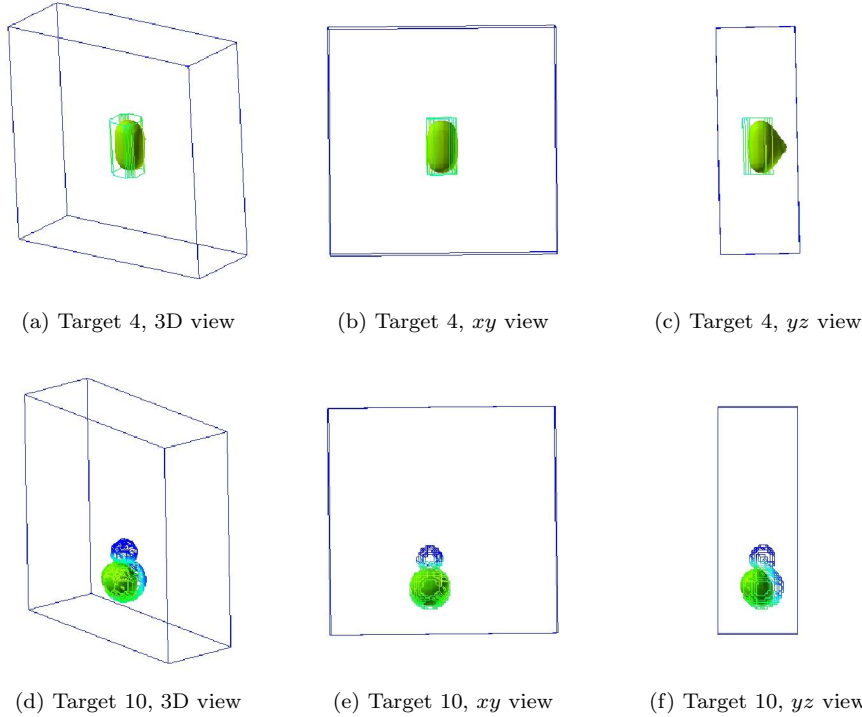


Fig. 4.4: Some reconstruction results of Test 2. Thin lines indicate correct shapes.

and non-blind dielectric targets with only a few percent of error, which is even smaller than the error of direct measurements. The average error of computed refractive indices  $n^{comp}$  in Tests 1 and 2 is respectively three and seven times less than the average error of direct measurements. Test 1 obtains higher appearing dielectric constant of some metallic targets than Test 2. However, Test 2 provides better shapes.

Unlike targets 1 - 7, which are homogeneous, targets 8, 9 and 10 are heterogeneous. Target 8 is a wooden doll with air inside. Target 9 is that doll with a piece of metal inside, i.e. this is a metal covered by a dielectric. We can see that only the metal was imaged, because its reflection is much stronger than that of the wood. Target 10 is the same doll partially filled with sand inside (except of the top), i.e. this is one dielectric covered by another one. One can see that only the part with the sand was imaged in target number 10, since its dielectric constant is higher than the air inside the top. Moreover, the reconstructed refractive index is about the average of those of the wood and the sand.

In conclusion, we can see from our tests that, with the proposed data pre-processing procedure, the globally convergent algorithm can image quite well both geometries and materials of the targets in our experimental data even though there is a huge misfit between these data and simulations. Moreover, it can image large inclusion/background contrasts, the case that is well known to be difficult for conventional least squares approaches.

**Acknowledgment.** The authors are grateful to Mr. Steven Kitchin for his excellent work on data collection.

#### REFERENCES

- [1] F. Assous, M. Kray, and F. Nataf. Time-reversed absorbing conditions in the partial aperture case. *Wave Motion*, 49(7):617–631, 2012.
- [2] F. Assous, M. Kray, F. Nataf, and E. Turkel. Time-reversed absorbing condition: application to inverse problems. *Inverse Problems*, 27(6):065003, 18, 2011.
- [3] A. B. Bakushinsky and M. Y. Kokurin. *Iterative methods for approximate solution of inverse problems*. Springer, Dordrecht, 2004.
- [4] L. Beilina. Energy estimates and numerical verification of the stabilized domain decomposition finite element/finite difference approach for the maxwell’s system in time domain. *Central European Journal of Mathematics*, 11(4):702–733, 2013.
- [5] L. Beilina and M. Klibanov. *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems*. Springer, New York, 2012.
- [6] L. Beilina and M. V. Klibanov. Reconstruction of dielectrics from experimental data via a hybrid globally convergent/adaptive inverse algorithm. *Inverse Problems*, 26(12):125009, 30, 2010.
- [7] L. Beilina and M. V. Klibanov. A new approximate mathematical model for global convergence for a coefficient inverse problem with backscattering data. *Journal of Inverse and Ill-posed Problems*, 20(4):513–565, 2012.
- [8] L. Beilina and M. V. Klibanov. Relaxation property for the adaptivity for ill-posed problems. *Applicable Analysis*, 2013. DOI:10.1080/00036811.2013.768339.
- [9] L. Beilina, K. Samuelsson, and K. Åhlander. Efficiency of a hybrid method for the wave equation. In *International Conference on Finite Element Methods*, Gakuto International Series Mathematical Sciences and Applications,. Gakkotosho CO., LTD, 2001.
- [10] A. Bukhgeim and M. Klibanov. Uniqueness in the large of a class of multidimensional inverse problems. *Soviet Math. Doklady*, 17:244–247, 1981.
- [11] G. Chavent. *Nonlinear least squares for inverse problems. Theoretical foundations and step-by-step guide for applications*. Springer, New York, 2009.
- [12] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems*, volume 375 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1996.
- [13] M. Fink. Time reversed acoustics. *Physics Today*, 50(3):34–40, 1997.
- [14] V. Isakov. Inverse obstacle problems. *Inverse Problems*, 25:123002, 2009.
- [15] M. V. Klibanov, M. A. Fiddy, L. Beilina, N. Pantong, and J. Schenk. Picosecond scale experimental verification of a globally convergent algorithm for a coefficient inverse problem. *Inverse Problems*, 26(4):045003, 30, 2010.
- [16] A. Kuzhuget, L. Beilina, M. Klibanov, A. Sullivan, L. Nguyen, and M. Fiddy. Blind backscattering experimental data collected in the field and an approximately globally convergent inverse algorithm. *Inverse Problems*, 28(9):095007, 33, 2012.
- [17] O. A. Ladyzhenskaya. *The Boundary Value Problems of Mathematical Physics*. Springer-Verlag, New York, 1985.
- [18] G. Lerosey, J. de Rosny, A. Tourin, , and M. Fink. Focusing beyond the diffraction limit with far-field time reversal. *Science*, 315:1120–1122, 2007.
- [19] M. Soumekh. *Synthetic aperture radar signal processing*. John Wiley & Son, New York, 1999.
- [20] WavES. The software package. Available online at <http://waves24.com>.