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GEOMETRIC CALCULUS AND FINITE ELEMENT METHODS FOR VLASOV–MAXWELL SYSTEM

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Abstract. Geometric calculus unify, simplify and generalize many fields of mathematics that involve geometric ideas. This work is a swift introduction to a few concepts in geometric calculus leading to simple representation of Maxwell equations. We then approximate the Vlasov-Maxwell system using streamline diffusion finite element method. In this part we derive optimal convergence rates due to the maximal available regularity of the exact solution.

1. Introduction

In geometric calculus, geometric objects: *points, lines, planes, circles, ...* are represented by members of an algebra, a *geometric algebra*, rather than by equations relating coordinates. Geometric relations on objects: *rotate, translate, intersect, project, construct a circle through three points, ...* are represented by the algebraic operations on the objects. Thus, *geometric algebra is coordinate free.*

Below we introduce some basic concepts in geometric calculus used for a novel approach to a simple and general formulation for the Maxwell equation unifying its approximation strategies. The short survey on *geometric calculus* in here is based on studies by Dorst and Lasenby [3] Hestenes [5], and Macdonald [7]. Our main concern will be approximation of the Vlasov-Maxwell (VM) system by a semi-classical finite element approach: *the streamline diffusion (SD) method.* The standard finite element method for hyperbolic equation (e.g. VM), with the exact solution in the Sobolev space H^{r+1}, has an L_2-optimal convergence rate of order $\mathcal{O}(h^r)$. Whereas, with the same regularity (H^{r+1}) the corresponding optimal convergence rate for the elliptic and parabolic

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problems is: $\mathcal{O}(h^{r+1})$. SD method is, roughly, a variational formulation with the test functions possessing a multiple of convection term in the equation. This corresponds to add of extra diffusion to the continuous problem which enhance the regularity. Using the SD strategy would improve the convergence rate of the hyperbolic problems by an order of 1/2: i.e. $\mathcal{O}(h^{r+1/2})$. Then, by interpolation space techniques, one can show that, for the hyperbolic problems, this rate is optimal.

Throughout this note C will denote a generic constant, not necessarily the same at each occurrence, and independent of the parameters in the equations, unless otherwise explicitly specified.

2. The Geometric Algebra $\mathbb{G}^n$

The Geometric Algebra $\mathbb{G}^n$ is an extension of the inner product space $\mathbb{R}^n$. Every vector in $\mathbb{R}^n$ is also in $\mathbb{G}^n$. It is an associative algebra with one. i.e. a vector space with a product, called *geometric product*. In addition to being an algebra, it satisfies also the properties $G1 - G2$:

$G1$. The geometric product of $\mathbb{G}^n$ is linked to inner product of $\mathbb{R}^n$ by

$$\mathbf{vv} = \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^{\mathbf{2}}, \qquad \forall\ \mathbf{v} \in \mathbb{R}^{\mathbf{n}}.$$

$G2$. Every orthogonal basis for $\mathbb{R}^n$ defines a *canonical basis* for $\mathbb{G}^n$.

Canonical basis. We illustrate this by an example: for an orthonormal basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ for $\mathbb{R}^3$, the canonical basis for $\mathbb{G}^3$ is given by

1	basis for 0-vectors (scalars)
$\mathbf{e_1}\quad \mathbf{e_2}\quad \mathbf{e_3}$	basis for 1-vectors (vectors)
$\mathbf{e_1e_2}\quad \mathbf{e_1e_3}\quad \mathbf{e_2e_3}$	basis for 2-vectors (bivectors)
$\mathbf{e_1e_2e_3}$	basis for 3-vectors (trivectors).

By $G1$, the nonzero vectors have an inverse in $\mathbb{G}^n$: $\mathbf{v}^{-1} = \mathbf{v}/|\mathbf{v}|^{\mathbf{2}}$. A polarization identity, combined with $G1$ and distributivity yields:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \frac{1}{2}((\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) - \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v}) \\ &= \frac{1}{2}((\mathbf{u}+\mathbf{v})^2 - \mathbf{u}^2 - \mathbf{v}^2) = \frac{1}{2}(\mathbf{uv}+\mathbf{vu}).\end{aligned}$$

Thus if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal then

$$\mathbf{vu} = -\mathbf{uv}, \qquad (\mathbf{u} \perp \mathbf{v}). \tag{2.1}$$

If $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u}$ and $\mathbf{v}$ are nonzero, then by $G1$: $(\mathbf{uv})^2 = \mathbf{uvuv} = -\mathbf{uuvv} = -|\mathbf{u}|^{\mathbf{2}}|\mathbf{v}|^{\mathbf{2}} < \mathbf{0}$. Thus, $\mathbf{uv}$ is not a scalar or a vector: It is a 2-*vector*, or *bivector*. For the orthonormal basis: $\mathbf{e_1e_2} = \mathbf{i}$. By (2.1) rearranging the order of the $\mathbf{e}$'s in a member of canonical basis at most changes its sign. Hence, the original product and its rearrangements are linearly dependent. Zero is a j-vector for all j. Evidently, there are no k-vectors in $\mathbb{G}^l$ with $k > l$. Finally, $dim(\mathbb{G}^l) = 2^l$.

2.1. **The Fundamental Identity.** Let $\{\mathbf{e_1}, \mathbf{e_2}\}$ be an ON-basis for a plane containing two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Let $\mathbf{u} = a\mathbf{e_1} + b\mathbf{e_2}$ and $\mathbf{v} = c\mathbf{e_1} + d\mathbf{e_2}$, ($a$, b, c and d are scalars). Then, by (G1) and (2.1)

$$\mathbf{uv} = (ac + bd) + (ad - bc)\mathbf{e_1}\mathbf{e_2} =: \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}. \tag{2.2}$$

$\mathbf{u}\cdot\mathbf{v} = (ac + bd) = |\mathbf{u}||\mathbf{v}|\cos\theta$ is the usual inner product of $\mathbf{u}$ and $\mathbf{v}$. $\mathbf{u}\wedge\mathbf{v}$ is a bivector called the *outer product*. It represents an oriented area. The identities $ad - bc = |\mathbf{u}||\mathbf{v}|\sin\theta$, and $\mathbf{e_1}\mathbf{e_2} = \mathbf{i}$ yield

$$\mathbf{u}\wedge\mathbf{v} = |\mathbf{u}||\mathbf{v}|\sin\theta\,\mathbf{i}. \tag{2.3}$$

Some important properties

- $\mathbf{u}\wedge\mathbf{v}$, unlike $\mathbf{uv}$, is always a bivector.
- $\mathbf{uv}$ is a bivector $\Leftrightarrow \mathbf{u}\cdot\mathbf{v} = 0 \Leftrightarrow \mathbf{u}\perp\mathbf{v} \Leftrightarrow \mathbf{uv} = -\mathbf{vu} \Leftrightarrow \mathbf{uv} = \mathbf{u}\wedge\mathbf{v}$. In particular, for $i \neq j$, $\mathbf{e_i}\mathbf{e_j} = \mathbf{e_i}\wedge\mathbf{e_j}$.
- $\mathbf{uv}$ is a scalar $\Leftrightarrow \mathbf{u}\wedge\mathbf{v} = 0 \Leftrightarrow \mathbf{u}\parallel\mathbf{v} \Leftrightarrow \mathbf{uv} = \mathbf{u}\cdot\mathbf{v} \Leftrightarrow \mathbf{uv} = \mathbf{vu}$.
- $\mathbf{u}\cdot\mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu})$, is the symmetric part of geometric product.
- $\mathbf{u}\wedge\mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu})$, is the antisymmetric part of geometric product.

2.2. **Subspaces.** A *k-blade* $\mathbf{B}$ for a k-dimensional subspace of $\mathbb{R}^n$ is a product of members of an orthogonal basis for the subspace, i.e. $\mathbf{B} = \mathbf{b_1}\mathbf{b_2}\ldots\mathbf{b_k}$. Nonzero scalars are 0-blades.

The inverse of $\mathbf{B}$ is $\mathbf{B}^{-1} = \mathbf{b_k}\ldots\mathbf{b_2}\mathbf{b_1}/|\mathbf{b_k}|^\mathbf{2}\ldots|\mathbf{b_2}|^\mathbf{2}|\mathbf{b_1}|^\mathbf{2}$.

The pseudoscalar. The *pseudoscalar* $\mathbf{I} = \mathbf{e_1}\mathbf{e_2}\ldots\mathbf{e_n}$ is the unit of $\mathbb{G}^n$. Since n-vectors form a 1-dimensional subspace of $\mathbb{G}^n$, every n-vector is a scalar multiple of $\mathbf{I}$. $\mathbf{I}^{-1} = \mathbf{e_n}\ldots\mathbf{e_2}\mathbf{e_1} = \pm\mathbf{I}$.

Duality. The *dual* of a multivector A is defined as

$$A^* = A/\mathbf{I} = A\mathbf{I}^{-1}.$$

If $\mathbf{A}$ is a j-blade, then $\mathbf{A}^*$ is an $(n-j)$-blade representing the *orthogonal complement* of $\mathbf{A}$. Extend an orthonormal basis $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_j\}$ of $\mathbf{A}$ to an orthonormal basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_j, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_n\}$ of $\mathbf{R}^n$. Then

$$\mathbf{A}^* = \mathbf{A}\mathbf{I}^{-1} = \pm\mathbf{A}\mathbf{I} = \pm(\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_j)(\mathbf{a}_1\mathbf{a}_2, \ldots, \mathbf{a}_n) = \pm\mathbf{a}_{j+1}\ldots\mathbf{a}_n.$$

2.3. **Extended Inner/Outer Products.** $\langle V\rangle_j$ denotes the j-vector part of the multivector V, e.g. $\langle\mathbf{uv}\rangle_0 = \mathbf{u}\cdot\mathbf{v}$, $\langle\mathbf{uv}\rangle_1 = 0$, $\langle\mathbf{uv}\rangle_2 = \mathbf{u}\wedge\mathbf{v}$.

Lemma 2.1. *Let $\mathbf{v}$ be a vector and $\mathbf{B}$, a k-blade. Decompose $\mathbf{v}$ with respect to $\mathbf{B}$: $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$, where $\mathbf{v}_\parallel \in \mathbf{B}$ and $\mathbf{v}_\perp \perp \mathbf{B}$. Then*

a. $\mathbf{v}_\parallel\cdot\mathbf{B} = \mathbf{v}_\parallel\mathbf{B}$, $\mathbf{v}_\parallel\wedge\mathbf{B} = 0$ *and* $\mathbf{v}_\parallel\cdot\mathbf{B}$ *is a* $(k-1)$*-blade in* $\mathbf{B}$.

b. $\mathbf{v}_\perp\wedge\mathbf{B} = \mathbf{v}_\perp\mathbf{B}$, $\mathbf{v}_\perp\cdot\mathbf{B} = 0$ *and* $\mathbf{v}_\perp\wedge\mathbf{B}$ *is a* $(k+1)$*-blade.*

Theorem 2.2 (Extended fundamental identity)**.** *Let* $\mathbf{B}$ *be a* k*-blade. Then for every vector* $\mathbf{v}$*,*

$$\mathbf{vB} = \mathbf{v} \cdot \mathbf{B} + \mathbf{v} \wedge \mathbf{B}. \tag{2.4}$$

Proof. Use Lemma 2.1. □

In general $\mathbf{AB} \neq \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}$. Example: $\mathbf{A} = \mathbf{e}_1\mathbf{e}_2$, $\mathbf{B} = \mathbf{e}_2\mathbf{e}_3$.

3. Geometric Calculus

3.1. **The gradient.** The gradient is defined as $\nabla = \sum_j \mathbf{e}_j \partial_j$, and acts algebraically as a vector: we can multiply it by a scalar field f, giving the vector field ∇f; dot it with a vector field $\mathbf{f}$, giving the scalar field $\nabla \cdot \mathbf{f}$; and cross it with $\mathbf{f}$, giving the vector field $\nabla \times \mathbf{f}$. In geometric calculus $\nabla \mathbf{f}$ *does* make sense:

$$\nabla \mathbf{f} = \nabla \cdot \mathbf{f} + \nabla \wedge \mathbf{f}, \qquad \text{(scalar+bivector)}. \tag{3.1}$$

3.2. **Analytic functions.** Let $f(x, y) = u(x, y) + v(x, y)\mathbf{i}$, where u and v are real valued. Then, using $\mathbf{e}_1\mathbf{e}_2 = \mathbf{i}$

$$\nabla f = \mathbf{e}_1(u_x + v_x\mathbf{i}) + \mathbf{e}_2(u_y + v_y\mathbf{i}) = \mathbf{e}_1(u_x - v_y) + \mathbf{e}_2(v_x + u_y).$$

Thus, $\nabla f = 0 \Leftrightarrow$ Cauchy-Riemann equations $\Leftrightarrow f$ is analytic.

3.3. **Generalize ∇.** *Directional derivative* of F in the "direction" A:

$$\partial_A F(X) = \lim_{\tau \to 0} \frac{F(X + \tau A) - F(X)}{\tau}.$$

If A contains grades for which F is not defined, then $\partial_A F(X) = 0$, e.g. if F is a function of a vector $\mathbf{v}$, then $\partial_{\mathbf{e}_1\mathbf{e}_e} F(\mathbf{v}) = 0$.

3.3.1. **Maxwell's equations.** Elementary electromagnetic theory is formulated in $3D$ vector calculus: The electric field $\mathbf{e}$ and the magnetic field $\mathbf{b}$, represent the electromagnetic field. The charged density scalar field ρ and the current density vector field $\mathbf{j}$ represent the distribution and motion of charges. They governed by the *Maxwell's equations*, viz

$$\nabla \cdot \mathbf{e} = 4\pi\rho, \quad \nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t}, \quad \nabla \cdot \mathbf{b} = 0, \quad \nabla \times \mathbf{b} = 4\pi\mathbf{j} + \frac{\partial \mathbf{e}}{\partial t}. \tag{3.2}$$

The spacetime $\mathbb{G}^{1,3}$ provides a more elegant formulation. A spacetime bivector field F unifying $\mathbf{e}$ and $\mathbf{b}$ represents the electromagnetic field. A spacetime vector field J unifies ρ and $\mathbf{j}$. Maxwell's four equations become a single equation:

$$\nabla F = J. \tag{3.3}$$

Multiplying $\nabla F = J$ by $\mathbf{e}_0$ and equating the $0-, 1-, 2-$, and 3-vector parts yields the standard Maxwell equation (3.2). Calculations using this formulation of Maxwell's equations are much easier than using the $\mathbb{R}^3$ formulation. This is in part due to the fact that the GA derivative ∇, unlike the divergent and *curl* in equation (3.2), is invertible.

4. Vlasov-Maxwell system in vector analysis form

The Vlasov-Maxwell (VM) system describes time evolution of collosionless plasma of particles with mass m and charge q, formulated as

$$\begin{aligned} \partial_t f + \hat{v}\cdot\nabla_x f + q(E + c^{-1}v\times B)\cdot\nabla_v f = 0,& \\ \partial_t E = c\nabla\times B - j, \qquad \nabla\cdot E = \rho,& \\ \partial_t B = -c\nabla\times E, \qquad \nabla\cdot B = 0.& \end{aligned} \tag{4.1}$$

Here f is density in phase space, c is speed of light and $\hat{v}$ the velocity

$$\hat{v} = (m^2 + c^{-2}|v|^2)^{-1/2}v \qquad (v \text{ is momentum}).$$

Further, the charge and current densities are given by

$$\rho(t,x) = 4\pi\int qf\,dv, \quad j(t,x) = 4\pi\int qf\hat{v}\,dv. \tag{4.2}$$

A proof for the existence (and uniqueness) of the solution to VM system can be obtained using the Schauder fixed point theorem: Insert an assumed and given g for f in (4.2). Compute ρ_g, j_g and insert the results in Maxwell equations to get E_g, B_g. Then insert, such obtained, E_g and B_g in the Vlasov equation to get f_g via an operator Λ: $f_g = \Lambda g$. A fixed point of Λ is the solution of the VM system. For the discretized version one should, instead, use the Brouwver fixed point theorem. Both these proofs are rather involved and non-trivial.

The most convenient VM system to discretize is the *relativistic model* (RVM) in *one and one-half dimensional geometry* ($x\in\mathbb{R}, v\in\mathbb{R}^2$) and $E = (E_1, E_2)$, which then can be generalized to higher dimensions:

$$\left.\begin{aligned} &\partial_t f + \hat{v}_1\cdot\partial_x f + q(E + BM_0\hat{v})\cdot\nabla_v f = 0, \\ &\partial_t E_1 = -4\pi j_1, \qquad\qquad \partial_x E_1 = 4\pi\rho, \\ &\partial_t E_2 = -\partial_x B - 4\pi j_2, \qquad \partial_t B = -\partial_x E_2. \end{aligned}\right\} \qquad M_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To carry out discrete analysis, we need global existence of classical solution. To this end we need to make the following some assumptions:

The background density $n(x)$ is smooth, has compact support and is neutralizing. This yields, for

$$\rho(t,x) = q\int f\,dv - n(x); \qquad \text{that} \int_{-\infty}^{\infty}\rho(0,x)\,dx = 0. \tag{4.3}$$

We also assume that $f^0(x,v) := f(0,x,v) \geq 0$. Then choosing

$$E_1(0,x) = 4\pi\Big(\int_{-\infty}^{x} f^0(y,v)\,dv - n(y)\Big)\,dy, \tag{4.4}$$

$\partial_x E_1 = 4\pi\rho$ is the only possibility that lead to finite energy solution.

Theorem 4.1 (Glassy, Schaeffer, [4]). *Assume that n is naturalizing,*

$$(i)\ 0 \leq f^0(x,v) \in C_0^1(\mathbb{R}^3), \qquad (ii)\ E_2^0,\ \ B^0 \in C_0^2(\mathbb{R}^1).$$

Then, there exists global C^1 solutions of RVM. If $0 \leq f^0 \in C_0^r(\mathbb{R}^3)$ and $E_2^0,\ B^0 \in C_0^{r+1}(\mathbb{R}^1)$, then (f,E,B) is of class C^r over $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$.

The theorem is an existence result. For $r = 2$ we differentiate to get

$$\begin{cases} \partial_t E_2 = -\partial_x B - 4\pi j_2 & \text{w.r.t } x(t) \Rightarrow \quad \partial_x\partial_t E_2 = -\partial^2_{xx} B - \partial_x 4\pi j_2 \\ \partial_t B = -\partial_x E_2 & \text{w.r.t } t(x) \Rightarrow \quad \partial^2_{tt} B = -\partial_t\partial_x E_2. \end{cases}$$

Subtracting the resulting equations, both E_2 and B, satisfying the wave equation, have solutions of *d' Alembert* type. The closed form solution for E_1 is yet simpler. Hence, (by uniqueness of the solution for the wave equation), we have now both existence and uniqueness.

5. The streamline diffusion method

We shall assume $(x,v) \in \Omega_x \times \Omega_v \subset \mathbb{R} \times \mathbb{R}^2$, that f , E_2, B and n have compact support in Ω_x and that f has compact support in Ω_v. Since $\int \rho(0,x)dx = 0$, it follows that E_1 has compact support in Ω_x. Let now $\Omega := \Omega_x \times \Omega_v$, $t \in [0,T]$ and $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ be a partition of $[0,T]$. For $k = 0, 1, \ldots$ we define

$$V_h = \{w \in \mathcal{H} : w|_K \in P_k(\tau) \times P_k(I_m); \forall K = \tau \times I_m\},$$

where $S_m = I_m \times \Omega = [t_{m-1}, t_m] \times \Omega$ and $\mathcal{H} := \Pi_{m=1}^M H^1(S_m)$. We write

$$(f,g)_m := (f,g)_{S_m}, \qquad (f,g)_D := \int_D fg,\ (D \text{ any domain}).$$

Further $\|g\|_m := (g,g)_m^{1/2}$ and

$$\langle f,g\rangle_m = (f(t_m,\cdot), g(t_m,\cdot))_\Omega, \quad |g|_m = \langle g,g\rangle_m^{1/2}.$$

6. SD for Maxwell equations in vector analysis form

In this section, $\Omega = \Omega_x$, we state the main SD results for the Maxwell equations. Detailed proofs follow the path of analysis in [1]-[2] and therefore are omitted. We set

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and let $W = (E_1, E_2, B)^T$, $W^0 = (E_1^0, E_2^0, B^0)^T$ and $b = (\rho, j_1, j_2, 0)^T$. The Maxwell equations can then be written, in the concise form, as

$$\begin{cases} M_1 W_t + M_2 W_x = b \\ W(0,x) = W^0(x). \end{cases} \tag{6.1}$$

The SD method for the Maxwell equations can now be formulated as: Find $W^h \in V_h$ such that for $m = 1, \ldots, M$,

$$\begin{aligned}(M_1 W_t^h + M_2 W_x^h, g + \delta(M_1 g_t + M_2 g_x))_m + \langle W_+^h, g_+\rangle_m = \\ = (b, g + \delta(M_1 g_t + M_2 g_x))_m + \langle W_-^h, g_+\rangle \quad \forall\, g \in V_h.\end{aligned}$$

Here, $g = (g_1, g_1, g_2, g_3)^T$, δ is a multiple of h, and we have a variational formulation with the test function $g + \delta(M_1 g_t + M_2 g_x)$. We introduce

$$\begin{aligned}\tilde{\mathcal{B}}(W, g) = &\sum_{m=0}^{M-1} (M_1 W_t + M_2 W_x, g + \delta(M_1 g_t + M_2 g_x))_m \\ &+ \sum_{m=1}^{M-1} \langle [W], g_+\rangle_m + \langle W_+, g_+\rangle_0,\end{aligned}$$

which is a bilinear form, and define the linear form

$$\tilde{\mathcal{L}}(b, g) = \sum_{m=0}^{M-1} (b, g + \delta(M_1 g_t + M_2 g_x))_m + \langle W^0, g_+\rangle_0.$$

Thus, in short, we have the SD problem: Find $W^h \in V_h$ such that

$$\tilde{\mathcal{B}}(W^h, g) = \tilde{\mathcal{L}}(b, g) \quad \forall\, g \in V_h. \tag{6.2}$$

Then, the triplenorm below will be an adequate measuring instrument:

$$|||g|||^2 = \frac{1}{2}\Big(|g_+|_0^2 + |g_-|_M^2 + \sum_{m=1}^{M-1} |[g]|_m^2 + 2\delta \sum_{m=0}^{M-1} \|M_1 g_t + M_2 g_x\|_m^2\Big),$$

where $[g] = g_+ - g_-$ is the jump with $g_\pm = \lim_{s\to 0+} g(x \pm s)$.

It is the spirit of Reisz representation and Lax-Milgram theorems to guarantee existence of a unique solution for (6.2) via Lemmas 6.1-6.2.

Lemma 6.1. *In the above setting, we have the coercivity relation*

$$\tilde{\mathcal{B}}(g,g) = |||g|||^2 \quad \forall g \in \mathcal{H}, \qquad \mathcal{H} := \Pi_{m=1}^{M} H^1(S_m).$$

Lemma 6.2. *For any constant C we have for $g \in \mathcal{H}$,*

$$\|g\|_m^2 \le \left(|g_-|_{m+1}^2 + \frac{1}{C}\|M_1 g_t + M_2 g_x\|_m^2 \right) h e^{2Ch}.$$

Lemma 6.3. *For any $h > 0$ the problem (6.2) has a solution and if h is small enough the solution is unique.*

Let now $F = E_2 + B$ and $G = E_2 - B$. By adding and subtraction the equations for E_2 and B, we get the following equations for F and G:

$$\begin{cases} \partial_t F + \partial_x F = j_2(t,x), & F(0,x) = E_2^0(x) + B^0(x) \\ \partial_t G - \partial_x G = j_2(t,x), & G(0,x) = E_2^0(x) - B^0(x). \end{cases}$$

Using the facts that $E_2 = \frac{1}{2}(F+G)$ and $B = \frac{1}{2}(F-G)$, we have

$$E_2(t,x) = \frac{1}{2}\big(E_2^0(x-t) + E_2^0(x+t) + B^0(x-t) - B^0(x+t)\big) + \\ + \frac{1}{2}\int_0^t j_2(\tau, x+\tau-t) + j_2(\tau, x+t-\tau)d\tau,$$

$$B(t,x) = \frac{1}{2}\big(E_2^0(x-t) - E_2^0(x+t) + B^0(x-t) + B^0(x+t)\big) + \\ + \frac{1}{2}\int_0^t j_2(\tau, x+\tau-t) - j_2(\tau, x+t-\tau)d\tau.$$

Further, for $x \in [x_0, x_1]$, the equations for E_1 yields

$$E_1(t,x) = \int_{x_0}^{x} \left(\int f(t,y,v)dv - n(y) \right) dy.$$

We set $Q_T = [0,T] \times \Omega_x$, by some standard inequalities we can derive

$$\|E_1\|_{Q_T}^2 \le C\left(\|f\|_{Q_T}^2 + T\int_{\Omega_x} |n(x)|^2 dx \right),$$

and

$$\|E_2\|_{Q_T}^2 \le CT\left(\int_{\Omega_x} |E_2^0(x)|^2 dx + \int_{\Omega_x} |B^0(x)|^2 dx + \|\hat{v}_2 f\|_{Q_T}^2 \right).$$

In a similar way we get, for $i = 1, 2$, that

$$\|\hat{v}_i B\|_{Q_T}^2 \le CT\left(\int_{\Omega_x} |B^0(x)|^2 dx + \int_{\Omega_x} |E_2^0(x)|^2 dx + \|\hat{v}_2 f\|_{Q_T}^2 \right).$$

These inequlities are the main ingredients to perform error analysis. Then, the proof of the convergence theorem below relay on the interpolation (η) and projection (ξ) error estimates in the following split: let W^h be the SD solution for (6.2) and $\tilde{W}$ an interpolant of W, set

$$W - W^h = (W - \tilde{W}) - (W^h - \tilde{W}) := \eta - \xi.$$

Theorem 6.4. *If W^h is a solution to (6.2) and the exact solution W of (6.1) satisfies $\|W\|_{k+1} \le C$, then there exists a constant C such that*

$$|||W - W^h||| \le Ch^{k+1/2}, \qquad W \in H^{k+1}(\Omega_x).$$

7. SD for the Vlasov-Maxwell system

As for the Vlasov-Maxwell system we relay on Schauder fixed point approach and let $\Omega_x \subset \mathbb{R}^d,\ \Omega_v \subset \mathbb{R}^d$ and $\Omega := \Omega_x \times \Omega_v,\ d = 1, 2, 3.$ We consider the divergent-free form of the Vlasov-Maxwell system, viz

$$\begin{cases} f_t + G \cdot \nabla f = 0, & \text{in } \Omega_T := \Omega \times (0,T) := \Omega_x \times \Omega_x \times (0,T), \\ f(x,v,0) = f_0(x,v), & \text{in } \Omega_0 := \Omega \times \{0\} \qquad f = f_b, \quad \text{on } \partial\Omega \times (0,T). \end{cases}$$

Here $\nabla f := (\nabla_x f, \nabla_v f)$, and $G := \Big(v, (E + BM_0 v)\Big) \ \Rightarrow \text{div} G(f) = 0.$

Let $\mathcal{F}$ be a certain (linear) function space, $\tilde{f} \in \mathcal{F}$ an approximation of f and $\Pi f \in \mathcal{F}$ a projection of f into $\mathcal{F}$ (e.g. an interpolant), then to estimate the approximation error, once again, we relay on an split:

$$f - \tilde{f} = (f - \Pi f) + (\Pi f - \tilde{f}) \equiv \eta + \xi; \quad \xi \in \mathcal{F}.$$

We discretize Ω_T using streamline diffusion method with test functions of the form $u + \delta\Big(u_t + G(\tilde{f}) \cdot \nabla u\Big)$, and $\delta \sim h$, the mesh size. Then we
(i) Use approximation theory to derive sharp error bounds for the interpolant η: $|||\eta||| \le |||data|||, \quad data = f_0,\ f_b.$
(ii) Establish $|||\xi||| \le C|||\eta|||. \quad (|||\cdot||| \text{ is a certain norm below}).$
Observe that (i) and (ii) work only if u_t is included inside test function!

7.1. **stability.** In the stationary problems (with no f_t), the modification test function $u + \delta(G^h \cdot \nabla u)$ together with $G^h \cdot \nabla f_h$ introduces a term of the form $\delta(G^h \cdot \nabla f_h, G^h \cdot \nabla u) := \delta(f^h_\gamma, u_\gamma),\ (\gamma := G^h, \zeta_\gamma := \gamma \cdot \nabla \zeta)$, interpreted as resulting from a diffusion $-\delta f_{\gamma\gamma}$ acting *only* in the streamline direction γ: Motivation for the use of the term *streamline diffusion*.

Lemma 7.1 (coercivity/stability)**.**

$$\forall u \in \mathcal{H}, \quad B^\delta\Big(G(f^h); u, u\Big) \ge \frac{1}{2}|||u|||^2. \quad \forall C_1 \ge 0.$$

$$||u||^2_{\Omega_T} \le \Big[\tfrac{1}{C_1}||u_t + G(f^h) \cdot \nabla u||^2 + \textstyle\sum_{m=1}^{M-1}||[g]||^2_m + \int_{\partial\Omega \times I} g^2 |G^h \cdot n|\Big]\delta e^{C_1 \delta}.$$

Lemma guarantees existence and uniqueness for the approximate VM.

The SD test function adds numerical dissipation in the vicinity of large gradients improving convergence rates: The streamline-diffusion approximation, with f_{SD} in a discrete space V_h, satisfies

$$||f - f_{SD}||_{SD} \leq Ch^{k+1/2}||f||_{H^{k+1}(\Omega_T)}.$$

where

$$\begin{aligned}||f||^2_{SD} =& \frac{1}{2}\Big[2\sigma||\nabla_v g^2_{\Omega_T} + |g|^2_M + |g|^2_0 \\ &+ \sum_{m=1}^{M-1}|[g]|^2_m + 2||g_t + G(f^h)\cdot\nabla g||^2_{\Omega_T} + \int_{\partial\Omega\times I} g^2|G^h\cdot n|\,d\nu ds\Big].\end{aligned}$$

For the analysis and proofs follow [1]-[2] and work out the details.

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