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Necessary and sufficient compatibility conditions for geometric regularity for an n-patch surface with $n = 3, 4$ or 5

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Dedicated to Alma

Abstract

When considering regularity of surfaces we are primarily interested in tangent plane continuity and curvature continuity respectively. We restrict ourselves to study fundamental surfaces such as n-patch surfaces, i.e., n patches meeting at a common vertex point. The main problem here is to present necessary and sufficient compatibility conditions for such a parameterized surface to be geometrically continuous of order one and two. In this article the compatibility conditions are proved to be independent of the patch parametrization. The number of patches, n, treated here is greater or equal to 3.

Keyword: n-patch surface, geometric continuity, tangent plane continuity, curvature continuity, compatibility conditions, geometric regularity.

1 Introduction

In this paper we study a certain type of surfaces, which we call n-patch surfaces, see Figure 1. We will present general conditions for such surfaces to be regular when the the number of patches, n, is greater than or equal 3. This is a generalization of a previous paper by the author, see [8]. For $n = 3, 4, 5$, we deduce explicit necessary and sufficient compatibility condition for an n-patch surface to be regular of order 1 or 2, i.e., to be tangent continuous or curvature continuous. Since the regularity problem for 2-patch surfaces is well studied we will focus on regularity at the vertex point V, which is the intersection point of the n patches.

Nevertheless, in order to formulate necessary and sufficient compatibility conditions for an n-patch surface to be regular, the regularity results for two adjacent patches intersecting along a common boundary curve by Kahmann, see [2], is of central importance.

In this paper we consider only patches that are of parametric type, but we do not restrict us any further. Thus the results include particularly patches represented by a spline such as Bezier polynomial, B-spline or NURBS. In a previous paper, see [8], we have studied the case of a 4-patch surface. Among the many other authors that also has treated regularity problem in the 4-patch surface case, we mention Bézier [3], Sarraga [6], Ye and Nowacki [7].

2 A general perspective on an n-patch Surface

In this article we will study a general n-patch surface, for $n \geq 3$. An n-patch surface is a surface consisting of n patches connected in a way such that every pair of adjacent patches has a common boundary curve. Moreover, all the patches meet at a common vertex V. See Figure 1. Here we are going to decide necessary and sufficient (compatibility) conditions in order for such a surface to be regular of order one or two, i.e., tangent plane continuous and curvature continuous respectively.

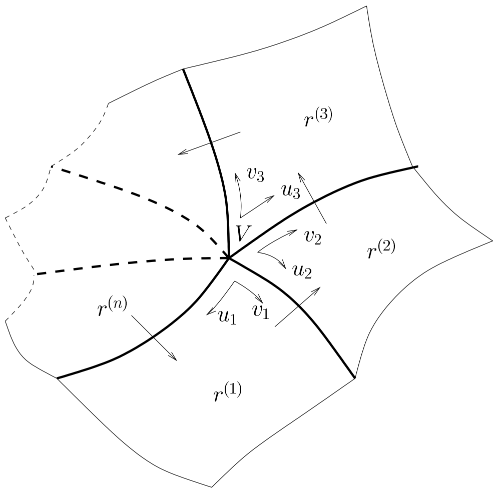

Figure 1: An n-patch surface, where the n patches intersect at a common vertex V

More precisely, let $(u_i, v_i) \mapsto \mathbf{r}^{(i)}(u_i, v_i) \in \mathbb{R}^3$ be continuous for $0 \leq u_i, v_i \leq 1$, where $i = 1, 2, \ldots, n$. An n-patch surface S of regularity G^0 is the set $S = \bigcup_{i=1}^{n} \{\mathbf{r}^{(i)}(u_i, v_i); 0 \leq u_i, v_i \leq 1\}$ such that the only intersection occurring between two of the patches, except for the vertex V, is the common boundary between two nearby patches, i.e., there exists a continuous bijective function $t_i : [0, 1] \mapsto [0, 1]$ such that for every point on the common boundary there exist parameters $u_{\tau(i)}, v_i \in [0, 1]$ satisfying

$$\mathbf{r}^{(\tau(i))}(u_{\tau(i)}, 0) = \mathbf{r}^{(i)}(0, t_i(v_i)),$$

for $\tau(i) = i \pmod{n} + 1$, where $i = 1, \ldots, n$. In order to simplify the notations we use $t_i^{-1}(u_{\tau(i)}) = v_i$, which can be done by a reparametrization. Thus, from now on we assume having the following relation

$$\mathbf{r}^{(\tau(i))}(t, 0) = \mathbf{r}^{(i)}(0, t), \tag{2.1}$$

for $t \in [0, 1]$ and $i = 1, 2, \ldots, n$.

In this section we will focus on general aspects concerning compatibility conditions for an n-patch surface to be regular. In later sections we consider such a surface for specific numbers n in order to explicitly get the compatibility conditions.

In the next subsection we consider tangent plane continuity, i.e., G^1-continuity.

2.1 Geometric continuity of order 1

The lowest order of regularity for an n-patch surface is G^0, which means that the surface is continuous across common patch boundaries as well as the included patches are continuous. The next level of regularity is tangent plane continuity, denoted by G^1, which will be defined next. For a more detailed discussion about tangent plane continuity, see Hoschek and Lasser [5] or Johansson [8].

Definition 2.1 *A continuous surface is said to be tangent plane continuous, denoted by G^1, if every point on the surface has a unique tangent plane which varies continuously on the surface. Such a surface is also said to be geometrically continuous of order one.*

The only points where G^1-continuity can fail for an n-patch surface are those at a common boundary between two patches, as long as all the patches are regularly enough, which in this case means C^1. Here we consider patches with a normal vector field, i.e., we assume that the patches are of regularity $C^1_{\#}$, which means that the representation of each patch $(u_i, v_i) \mapsto \mathbf{r}^{(i)}(u_i, v_i) \in \mathbb{R}^3$ is in $C^1([0,1]^2)$ with $\mathbf{r}^{(i)}_{u_i} \times \mathbf{r}^{(i)}_{v_i} \neq 0$ for $0 \leq u_i, v_i \leq 1$ and $i = 1, 2, \ldots, n$. Another way to guarantee tangent plane continuity under these circumstances is to require that

$$\mathbf{r}^{(\tau(i))}_{v_{\tau(i)}}(t,0) = \lambda_i(t)\mathbf{r}^{(i)}_{u_i}(0,t) + \kappa_i(t)\mathbf{r}^{(i)}_{v_i}(0,t), \tag{2.2}$$

for $t \in [0,1]$, where λ_i and κ_i are continuous functions for $i = 1, \ldots, n$, along the boundary curve, and the tangent vectors $\mathbf{r}^{(i)}_{u_i}(0,t)$ and $\mathbf{r}^{(i)}_{v_i}(t,0)$ are linearly independent for $t \in [0,1]$ and $i = 1, 2, \ldots, n$. We have here, as before, used the function $\tau(i) = i \pmod{n} + 1$.

A natural convention we follow in the rest of this article is to exclude the indices of the variables.

In the case of an n-patch surface, using the identity (2.2), we get the next system of identities along the common boundaries of two patches, i.e., for $t \in [0,1]$. We have

$$\begin{aligned}
\mathbf{r}^{(1)}_v(t,0) &= \lambda_n(t)\mathbf{r}^{(n)}_u(0,t) + \kappa_n(t)\mathbf{r}^{(n)}_v(0,t) \\
\mathbf{r}^{(2)}_v(t,0) &= \lambda_1(t)\mathbf{r}^{(1)}_u(0,t) + \kappa_1(t)\mathbf{r}^{(1)}_v(0,t) \\
\mathbf{r}^{(3)}_v(t,0) &= \lambda_2(t)\mathbf{r}^{(1)}_u(0,t) + \kappa_2(t)\mathbf{r}^{(2)}_v(0,t) \\
&\vdots \\
\mathbf{r}^{(n)}_v(t,0) &= \lambda_{n-1}(t)\mathbf{r}^{(n-2)}_u(0,t) + \kappa_{n-1}(t)\mathbf{r}^{(n-1)}_v(0,t).
\end{aligned} \tag{2.3}$$

As long as $t \neq 0$, the above equalities do not interfere with each other, but at the point $t = 0$, i.e., at the common vertex V, combining (2.1) and (2.3) it implies that the next identities have to be simultaneously satisfied. We have

$$\begin{aligned}
\mathbf{r}^{(1)}_v &= \lambda_n\mathbf{r}^{(n-1)}_v + \kappa_n\mathbf{r}^{(n)}_v \\
\mathbf{r}^{(2)}_v &= \lambda_1\mathbf{r}^{(n)}_v + \kappa_1\mathbf{r}^{(1)}_v \\
\mathbf{r}^{(3)}_v &= \lambda_2\mathbf{r}^{(1)}_v + \kappa_2\mathbf{r}^{(2)}_v \\
&\vdots \\
\mathbf{r}^{(n)}_v &= \lambda_{n-1}\mathbf{r}^{(n-2)}_v + \kappa_{n-1}\mathbf{r}^{(n-1)}_v,
\end{aligned} \tag{2.4}$$

where the notations $\lambda_i = \lambda_i(0)$, $\kappa_i = \kappa_i(0)$ and $\mathbf{r}_{v_i}^{(i)} = \mathbf{r}_{v_i}^{(i)}(0,0)$ for $i = 1, 2, \ldots, n$, are used. The compatibility conditions in the equation system (2.4) can be rephrased in such a way that we formulate necessary and sufficient conditions for the functions λ_i and κ_i, $i = 1, 2, \ldots, n$, at the intersection point V. The relations (2.4) can be written in the form

$$\begin{pmatrix} \lambda_2 & \kappa_2 & -1 & \ldots & 0 & 0 \\ 0 & \lambda_3 & \kappa_3 & \ldots & 0 & 0 \\ 0 & 0 & \lambda_4 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \ldots & \lambda_n & \kappa_n \\ \kappa_1 & -1 & 0 & \ldots & 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} \mathbf{r}_v^{(1)} \\ \mathbf{r}_v^{(2)} \\ \mathbf{r}_v^{(3)} \\ \vdots \\ \mathbf{r}_v^{(n-1)} \\ \mathbf{r}_v^{(n)} \end{pmatrix} = 0.$$

Moreover, since we want to achieve tangent plane continuity, the vector $\mathbf{r}_v = (\mathbf{r}_v^{(1)}\mathbf{r}_v^{(2)} \ldots \mathbf{r}_v^{(n)})^t$ at the point V can be represented by the linearly independent vectors $(\mathbf{r}_u^{(1)}\mathbf{r}_v^{(1)})^t$, spanning the tangent plane at the point V, as

$$\begin{pmatrix} \mathbf{r}_v^{(1)} \\ \mathbf{r}_v^{(2)} \\ \mathbf{r}_v^{(3)} \\ \vdots \\ \mathbf{r}_v^{(n-1)} \\ \mathbf{r}_v^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ \lambda_1\kappa_2 & \kappa_1\kappa_2 + \lambda_2 \\ \vdots & \vdots \\ -\kappa_n/\lambda_n & 1/\lambda_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}_u^{(1)} \\ \mathbf{r}_v^{(1)} \end{pmatrix}.$$

Finally, we have a necessary and sufficient condition for tangent plane continuity at the point V that the functions λ_i and κ_i, $i = 1, 2, \ldots, n$, satisfy the next equation system at the point V, i.e.,

$$\begin{pmatrix} \lambda_2 & \kappa_2 & -1 & \ldots & 0 & 0 \\ 0 & \lambda_3 & \kappa_3 & \ldots & 0 & 0 \\ 0 & 0 & \lambda_4 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \ldots & \lambda_n & \kappa_n \\ \kappa_1 & -1 & 0 & \ldots & 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ \lambda_1\kappa_2 & \kappa_1\kappa_2 + \lambda_2 \\ \vdots & \vdots \\ -\kappa_n/\lambda_n & 1/\lambda_n \\ 1 & 0 \end{pmatrix} = 0. \tag{2.5}$$

Thus, we have

Theorem 2.1 *Let S be a continuous n-patch surface, composed of the $C^1_{\#}$-patches $(u, v) \mapsto \mathbf{r}^{(i)}(u, v)$, $0 \le u, v \le 1$, $i = 1, 2, \ldots, n$, which intersect at the point $V = \mathbf{r}^{(i)}(0, 0)$ for $i = 1, 2, \ldots, n$. Necessary and sufficient conditions for the surface S to be G^1 are that there exist continuous functions $v \mapsto \lambda_i(v)$ and $v \mapsto \kappa_i(v)$, $0 \le v \le 1$, $i = 1, 2, \ldots, n$, satisfying the* *relations in (2.3) with the coefficients $\lambda_i = \lambda_i(0)$ and $\kappa_i = \kappa_i(0)$, $i = 1, 2, \ldots, n$, fullfilling the equation system (2.5).*

Next we continue to the case of curvature continuity, i.e., G^2-continuity.

2.2 Geometric continuity of order 2

A further level of regularity for a surface is to be curvature continuous. We next introduce the concept of curvature continuity, which also is denoted by G^2.

Definition 2.2 *A tangent plane continuous, G^1, surface is said to be curvature continuous, denoted by G^2, if every point on the surface has a unique Dupin indicatrix, which varies continuously on the surface. Such a surface is also said to be geometrically continuous of order two.*

An equivalent way to describe the notation of curvature continuity is to demand that every pair of closely connected patches in the n-patch surface, consisting of $C^2_{\#}$-patches, fulfills the next equation, together with equation (2.2), along their common boundary curve. We have

$$\begin{aligned}\mathbf{r}_{vv}^{(\tau(i))}(t,0) &= \lambda_i^2(t)\mathbf{r}_{uu}^{(i)}(0,t) + 2\lambda_i(t)\kappa_i(t)\mathbf{r}_{uv}^{(i)}(0,t) + \kappa_i^2(t)\mathbf{r}_{vv}^{(i)}(0,t) \\ &\quad + \mu_i(t)\mathbf{r}_u^{(i)}(0,t) + \nu_i(t)\mathbf{r}_v^{(i)}(0,t), \end{aligned} \tag{2.6}$$

where the functions λ_i, κ_i, μ_i and ν_i are continuous for $i = 1, 2, \ldots, n$, and $t \in [0, 1]$. Here we have used the notation $C^2_{\#} = C^2 \cap C^1_{\#}$. We assume from now on that the functions λ_i and κ_i, $i = 1, 2, \ldots, n$, are continuously differentiable. For a more detailed description of curvature continuity and its relation to the equation (2.6) we refer to Hoschek and Lasser [5]. From (2.6) it follows that for an n-patch surface the next equations must be satisfied along their common patch boundaries. Thus, we have

$$\begin{aligned}
\mathbf{r}_{vv}^{(1)}(t,0) &= \lambda_n^2\mathbf{r}_{uu}^{(n)}(0,\cdot) + 2\lambda_n\kappa_n\mathbf{r}_{uv}^{(n)}(0,\cdot) + \kappa_n^2\mathbf{r}_{vv}^{(n)}(0,\cdot) + \mu_n\mathbf{r}_u^{(n)}(0,\cdot) + \nu_n\mathbf{r}_v^{(n)}(0,\cdot)|_t \\
\mathbf{r}_{vv}^{(2)}(t,0) &= \lambda_1^2\mathbf{r}_{uu}^{(1)}(0,\cdot) + 2\lambda_1\kappa_1\mathbf{r}_{uv}^{(1)}(0,\cdot) + \kappa_1^2\mathbf{r}_{vv}^{(1)}(0,\cdot) + \mu_1\mathbf{r}_u^{(1)}(0,\cdot) + \nu_1\mathbf{r}_v^{(1)}(0,\cdot)|_t \\
\mathbf{r}_{vv}^{(3)}(t,0) &= \lambda_2^2\mathbf{r}_{uu}^{(2)}(0,\cdot) + 2\lambda_2\kappa_2\mathbf{r}_{uv}^{(2)}(0,\cdot) + \kappa_2^2\mathbf{r}_{vv}^{(2)}(0,\cdot) + \mu_2\mathbf{r}_u^{(2)}(0,\cdot) + \nu_2\mathbf{r}_v^{(2)}(0,\cdot)|_t \\
&\vdots \\
\mathbf{r}_{vv}^{(n)}(t,0) &= \lambda_{n-1}^2\mathbf{r}_{uu}^{(n-1)}(0,\cdot) + 2\lambda_{n-1}\kappa_{n-1}\mathbf{r}_{uv}^{(n-1)}(0,\cdot) + \kappa_{n-1}^2\mathbf{r}_{vv}^{(n-1)}(0,\cdot) \\
&\quad + \mu_{n-1}\mathbf{r}_u^{(n-1)}(0,\cdot) + \nu_{n-1}\mathbf{r}_v^{(n-1)}(0,\cdot)|_t
\end{aligned} \tag{2.7}$$

for $t \in [0, 1]$. We also have, as a consequence of the identities in (2.3) combined with the assumption of differentiability of λ_i and κ_i for $i = 1, 2, \ldots, n$, that the next relations must be fulfilled, i.e.,

$$\begin{aligned}
\mathbf{r}_{uv}^{(1)}(t,0) &= \lambda_n\mathbf{r}_{uv}^{(n)}(0,\cdot) + \kappa_n\mathbf{r}_{vv}^{(n)}(0,\cdot) + \lambda_n'\mathbf{r}_u^{(n)}(0,\cdot) + \kappa_n'\mathbf{r}_v^{(n)}(0,\cdot)|_t \\
\mathbf{r}_{uv}^{(2)}(t,0) &= \lambda_1\mathbf{r}_{uv}^{(1)}(0,\cdot) + \kappa_1\mathbf{r}_{vv}^{(1)}(0,\cdot) + \lambda_1'\mathbf{r}_u^{(1)}(0,\cdot) + \kappa_1'\mathbf{r}_v^{(1)}(0,\cdot)|_t \\
\mathbf{r}_{uv}^{(3)}(t,0) &= \lambda_2\mathbf{r}_{uv}^{(2)}(0,\cdot) + \kappa_2\mathbf{r}_{vv}^{(2)}(0,\cdot) + \lambda_2'\mathbf{r}_u^{(2)}(0,\cdot) + \kappa_2'\mathbf{r}_v^{(2)}(0,\cdot)|_t \\
&\vdots \\
\mathbf{r}_{uv}^{(n)}(t,0) &= \lambda_{n-1}\mathbf{r}_{uv}^{(n-1)}(0,\cdot) + \kappa_{n-1}\mathbf{r}_{vv}^{(n-1)}(0,\cdot) + \lambda_{n-1}'\mathbf{r}_u^{(n-1)}(0,\cdot) + \kappa_{n-1}'\mathbf{r}_v^{(n-1)}(0,\cdot)|_t
\end{aligned}$$

in the interval $[0,1]$. Here we use the same notations as in the previous section, complemented by $\mu_i = \mu_i(0)$, $\nu_i = \nu_i(0)$, $\lambda'_i = \lambda'_i(0)$ and $\kappa'_i = \kappa'_i(0)$ for $i = 1, 2, \dots, n$. Using this in combination with the identities (2.1) and (2.2), the two equation systems above can be rewritten at the vertex point V as follows

$$\begin{aligned}
\mathbf{r}_{vv}^{(2)} &= \lambda_1^2 \mathbf{r}_{vv}^{(n)} + 2\lambda_1\kappa_1 \mathbf{r}_{uv}^{(1)} + \kappa_1^2 \mathbf{r}_{vv}^{(1)} + \mu_1 \mathbf{r}_v^{(n)} + \nu_1 \mathbf{r}_v^{(1)} \\
\mathbf{r}_{vv}^{(3)} &= \lambda_2^2 \mathbf{r}_{vv}^{(1)} + 2\lambda_2\kappa_2 \mathbf{r}_{uv}^{(2)} + \kappa_2^2 \mathbf{r}_{vv}^{(2)} + \mu_2 \mathbf{r}_v^{(1)} + \nu_2 \mathbf{r}_v^{(2)} \\
&\vdots \\
\mathbf{r}_{vv}^{(n)} &= \lambda_{n-1}^2 \mathbf{r}_{vv}^{(n-2)} + 2\lambda_{n-1}\kappa_{n-1} \mathbf{r}_{uv}^{(n-1)} + \kappa_{n-1}^2 \mathbf{r}_{vv}^{(n-1)} + \mu_{n-1} \mathbf{r}_v^{(n-2)} + \nu_{n-1} \mathbf{r}_v^{(n-1)} \\
\mathbf{r}_{vv}^{(1)} &= \lambda_n^2 \mathbf{r}_{vv}^{(n-1)} + 2\lambda_n\kappa_n \mathbf{r}_{uv}^{(n)} + \kappa_n^2 \mathbf{r}_{vv}^{(n)} + \mu_n \mathbf{r}_v^{(n-1)} + \nu_n \mathbf{r}_v^{(n)}
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
\mathbf{r}_{uv}^{(2)} &= \lambda_1 \mathbf{r}_{uv}^{(1)} + \kappa_1 \mathbf{r}_{vv}^{(1)} + \lambda'_1 \mathbf{r}_v^{(n)} + \kappa'_1 \mathbf{r}_v^{(1)} \\
\mathbf{r}_{uv}^{(3)} &= \lambda_2 \mathbf{r}_{uv}^{(2)} + \kappa_2 \mathbf{r}_{vv}^{(2)} + \lambda'_2 \mathbf{r}_v^{(1)} + \kappa'_2 \mathbf{r}_v^{(2)} \\
&\vdots \\
\mathbf{r}_{uv}^{(n)} &= \lambda_{n-1} \mathbf{r}_{uv}^{(n-1)} + \kappa_{n-1} \mathbf{r}_{vv}^{(n-1)} + \lambda'_{n-1} \mathbf{r}_v^{(n-2)} + \kappa'_{n-1} \mathbf{r}_v^{(n-1)} \\
\mathbf{r}_{uv}^{(1)} &= \lambda_n \mathbf{r}_{uv}^{(n)} + \kappa_n \mathbf{r}_{vv}^{(n)} + \lambda'_n \mathbf{r}_v^{(n-1)} + \kappa'_n \mathbf{r}_v^{(n)}.
\end{aligned} \tag{2.9}$$

The equation system (2.9) can be written in a compact form as

$$\Lambda\, \mathbf{r}_{uv} = \kappa\, \mathbf{r}_{vv} + G\, \mathbf{r}_v, \tag{2.10}$$

where $\mathbf{r}_{uv} = (\mathbf{r}_{uv}^{(1)}\, \mathbf{r}_{uv}^{(2)}\, \mathbf{r}_{uv}^{(3)} \dots \mathbf{r}_{uv}^{(n)})^t$, i.e., the column vector of the mixed derivatives of respectively patch. Similarly for the other two column vectors $\mathbf{r}_{vv}$ and $\mathbf{r}_v$. The matrices in equation (2.10) are given by

$$\Lambda = \begin{pmatrix} -\lambda_1 & 1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & 1 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -\lambda_n \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_1 & 0 & 0 & \dots & 0 \\ 0 & \kappa_2 & 0 & \dots & 0 \\ 0 & 0 & \kappa_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \kappa_n \end{pmatrix}$$

and

$$G = \begin{pmatrix} \kappa'_1 & 0 & 0 & \dots & \lambda'_1 \\ \lambda'_2 & \kappa'_2 & 0 & \dots & 0 \\ 0 & \lambda'_3 & \kappa'_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \kappa'_n \end{pmatrix}.$$

The equalities in (2.8) can, in a similar way, also be written in a compact form. We have

$$\Gamma\,\mathbf{r}_{vv} = H\,\mathbf{r}_{uv} + F\,\mathbf{r}_v, \tag{2.11}$$

where the matrices are

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & \dots & -\kappa_n^2 \\ -\kappa_1^2 & 1 & 0 & \dots & -\lambda_1^2 \\ -\lambda_2^2 & -\kappa_2^2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & 0 & \dots & 0 & 2\lambda_n\kappa_n \\ 2\lambda_1\kappa_1 & 0 & \dots & 0 & 0 \\ 0 & 2\lambda_2\kappa_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2\lambda_{n-1}\kappa_{n-1} & 0 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 & 0 & \dots & \mu_n & \nu_n \\ \nu_1 & 0 & \dots & 0 & \mu_1 \\ \mu_2 & \nu_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \nu_{n-1} & 0 \end{pmatrix}.$$

Next, we formulate a result of the invertibility of the matrix Λ.

Lemma 2.2 *Let Λ be the matrix given above for $n = 3, 4, \dots$. Then,* $\det(\Lambda) = (-1)^n(\lambda_1 \dots \lambda_n - 1) \neq 0$. *Suppose that $\lambda_1\lambda_2 \dots \lambda_n = (-1)^n$,. Thus, for an odd integer n the matrix Λ is invertible and*

$$\Lambda^{-1} = \frac{1}{2}\begin{pmatrix} \lambda_2 \dots \lambda_n & \lambda_3 \dots \lambda_n & \lambda_4 \dots \lambda_n & \dots & 1 \\ 1 & \lambda_3 \dots \lambda_1 & \lambda_4 \dots \lambda_1 & \dots & \lambda_1 \\ \lambda_2 & 1 & \lambda_4 \dots \lambda_2 & \dots & \lambda_1\lambda_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_2 \dots \lambda_{n-1} & \lambda_3 \dots \lambda_{n-1} & \lambda_4 \dots \lambda_{n-1} & \dots & \lambda_1 \dots \lambda_{n-1} \end{pmatrix}.$$

Using the above Lemma for an odd integer n. From (2.10) we get

$$\mathbf{r}_{uv} = \Lambda^{-1}\,\kappa\,\mathbf{r}_{vv} + \Lambda^{-1}\,G\,\mathbf{r}_v,$$

which combined with (2.11) implies that

$$(\Gamma - H\,\Lambda^{-1}\,\kappa)\mathbf{r}_{vv} = (F + H\Lambda^{-1}\,G)\mathbf{r}_v. \tag{2.12}$$

In order to solve equation (2.12) we start with

$$H\,\Lambda^{-1} = \begin{pmatrix} \lambda_2\ldots\lambda_n\kappa_n & \lambda_3\ldots\lambda_n\kappa_n & \ldots & -\kappa_n \\ -\kappa_1 & \lambda_3\ldots\lambda_1\kappa_1 & \ldots & \lambda_1\kappa_1 \\ \lambda_2\kappa_2 & -\kappa_2 & \ldots & \lambda_1\lambda_2\kappa_2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_2\ldots\lambda_{n-1}\kappa_{n-1} & \lambda_3\ldots\lambda_{n-1}\kappa_{n-1} & \ldots & \lambda_1\ldots\lambda_{n-1}\kappa_{n-1} \end{pmatrix}.$$

First, a multiplication with κ and we have

$$H\,\Lambda^{-1}\,\kappa = \begin{pmatrix} \lambda_2\ldots\lambda_n\kappa_n\kappa_1 & \lambda_3\ldots\lambda_n\kappa_n\kappa_2 & \ldots & -\kappa_n^2 \\ -\kappa_1^2 & \lambda_3\ldots\lambda_1\kappa_1\kappa_2 & \ldots & \lambda_1\kappa_1\kappa_n \\ \lambda_2\kappa_2\kappa_1 & -\kappa_2^2 & \ldots & \lambda_1\lambda_2\kappa_2\kappa_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_2\ldots\lambda_{n-1}\kappa_{n-1}\kappa_1 & \lambda_3\ldots\lambda_{n-1}\kappa_{n-1}\kappa_2 & \ldots & \lambda_1\ldots\lambda_{n-1}\kappa_{n-1}\kappa_n \end{pmatrix}$$

from which it follows that

$$\Gamma - H\,\Lambda^{-1}\,\kappa = \begin{pmatrix} 1-\lambda_2\ldots\lambda_n\kappa_n\kappa_1 & -\lambda_3\ldots\lambda_n\kappa_n\kappa_2 & \ldots & 0 \\ 0 & 1-\lambda_3\ldots\lambda_1\kappa_1\kappa_2 & \ldots & -\lambda_1^2-\lambda_1\kappa_1\kappa_n \\ -\lambda_2^2-\lambda_2\kappa_2\kappa_1 & 0 & \ldots & -\lambda_1\lambda_2\kappa_2\kappa_n \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_2\ldots\lambda_{n-1}\kappa_{n-1}\kappa_1 & -\lambda_3\ldots\lambda_{n-1}\kappa_{n-1}\kappa_2 & \ldots & 1-\lambda_1\ldots\lambda_{n-1}\kappa_{n-1}\kappa_n \end{pmatrix}.$$

Second, we multiply G by $H\,\Lambda^{-1}$ to get

$$H\,\Lambda^{-1}\,G =$$

$$\begin{pmatrix} \lambda_3\ldots\lambda_n\kappa_n(\lambda_2\kappa_1'+\lambda_2') & \lambda_4\ldots\lambda_n\kappa_n(\lambda_3\kappa_2'+\lambda_3') & \ldots & \lambda_2\ldots\lambda_n\kappa_n\lambda_1'-\kappa_n\kappa_n' \\ \lambda_3\ldots\lambda_1\kappa_1\lambda_2'-\kappa_1\kappa_1' & \lambda_4\ldots\lambda_1\kappa_1(\lambda_3\kappa_2'+\lambda_3') & \ldots & \kappa_1(\lambda_1\kappa_n'-\lambda_1') \\ \kappa_2(\lambda_2\kappa_1'-\lambda_2') & \lambda_4\ldots\lambda_2\kappa_2\lambda_3'-\kappa_2\kappa_2' & \ldots & \lambda_2\kappa_2(\lambda_1\kappa_n'+\lambda_1') \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_3\ldots\lambda_{n-1}\kappa_{n-1}(\lambda_2\kappa_1'+\lambda_2') & \lambda_4\ldots\lambda_{n-1}\kappa_{n-1}(\lambda_3\kappa_2'+\lambda_3') & \ldots & \lambda_2\ldots\lambda_{n-1}\kappa_{n-1}(\lambda_1\kappa_n'+\lambda_1') \end{pmatrix}$$

and then

$$F + H\,\Lambda^{-1}\,G =$$

$$\begin{pmatrix} \lambda_3\ldots\lambda_n\kappa_n(\lambda_2\kappa_1'+\lambda_2') & \lambda_4\ldots\lambda_n\kappa_n(\lambda_3\kappa_2'+\lambda_3') & \ldots & \lambda_2\ldots\lambda_n\kappa_n\lambda_1'-\kappa_n\kappa_n'+\nu_n \\ \lambda_3\ldots\lambda_1\kappa_1\lambda_2'-\kappa_1\kappa_1'+\nu_1 & \lambda_4\ldots\lambda_1\kappa_1(\lambda_3\kappa_2'+\lambda_3') & \ldots & \kappa_1(\lambda_1\kappa_n'-\lambda_1')+\mu_1 \\ \kappa_2(\lambda_2\kappa_1'-\lambda_2')+\mu_2 & \lambda_4\ldots\lambda_2\kappa_2\lambda_3'-\kappa_2\kappa_2'+\nu_2 & \ldots & \lambda_2\kappa_2(\lambda_1\kappa_n'+\lambda_1') \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_3\ldots\lambda_{n-1}\kappa_{n-1}(\lambda_2\kappa_1'+\lambda_2') & \lambda_4\ldots\lambda_{n-1}\kappa_{n-1}(\lambda_3\kappa_2'+\lambda_3') & \ldots & \lambda_2\ldots\lambda_{n-1}\kappa_{n-1}(\lambda_1\kappa_n'+\lambda_1') \end{pmatrix}.$$

This concludes the odd case. For even integers n, $n \geq 4$, we instead consider the equation (2.13) below. Supposing κ is invertible, it follows from (2.10) that

$$\kappa^{-1}\,\Lambda\,\mathbf{r}_{uv} = \mathbf{r}_{vv} + \kappa^{-1}\,G\,\mathbf{r}_v.$$

Combined with (2.11), we get

$$\Gamma\,\kappa^{-1}\,\Lambda\,\mathbf{r}_{uv} = \Gamma\,\mathbf{r}_{vv} + \Gamma\,\kappa^{-1}\,G\,\mathbf{r}_v = H\,\mathbf{r}_{uv} + F\,\mathbf{r}_v + \Gamma\,\kappa^{-1}\,G\,\mathbf{r}_v,$$

or more specifically

$$(\Gamma\kappa^{-1}\Lambda - H)\,\mathbf{r}_{uv} = (\Gamma\kappa^{-1}G + F)\,\mathbf{r}_v. \tag{2.13}$$

We have

$$\Gamma\kappa^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & -\lambda_n^2 & -\kappa_n^2 \\ -\kappa_1^2 & 1 & 0 & \dots & 0 & -\lambda_1^2 \\ -\lambda_2^2 & -\kappa_2^2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & -\kappa_{n-1}^2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\kappa_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\kappa_2} & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{\kappa_3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\kappa_{n-1}} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{\kappa_n} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\kappa_1} & 0 & 0 & \dots & -\frac{\lambda_n^2}{\kappa_{n-1}} & -\kappa_n \\ -\kappa_1 & \frac{1}{\kappa_2} & 0 & \dots & 0 & -\frac{\lambda_1^2}{\kappa_n} \\ -\frac{\lambda_2^2}{\kappa_1} & -\kappa_2 & \frac{1}{\kappa_3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\kappa_{n-1}} & 0 \\ 0 & 0 & 0 & \dots & -\kappa_{n-1} & \frac{1}{\kappa_n} \end{pmatrix}$$

and

$$\Gamma\kappa^{-1}\Lambda = \begin{pmatrix} \frac{1}{\kappa_1} & 0 & 0 & \dots & -\kappa_n \\ -\kappa_1 & \frac{1}{\kappa_2} & 0 & \dots & -\frac{\lambda_1^2}{\kappa_n} \\ -\frac{\lambda_2^2}{\kappa_1} & -\kappa_2 & \frac{1}{\kappa_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\kappa_n} \end{pmatrix} \begin{pmatrix} -\lambda_1 & 1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & 1 & \dots & 0 \\ 0 & 0 & -\lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -\lambda_n \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\lambda_1}{\kappa_1} - \kappa_n & \frac{1}{\kappa_1} & 0 & \dots & \lambda_n\kappa_n - \frac{\lambda_n^2}{\kappa_{n-1}} \\ \lambda_1\kappa_1 - \frac{\lambda_1^2}{\kappa_n} & -\frac{\lambda_2}{\kappa_2} - \kappa_1 & \frac{1}{\kappa_2} & \dots & \frac{\lambda_n\lambda_1^2}{\kappa_n} \\ \frac{\lambda_1\lambda_2^2}{\kappa_1} & \lambda_2\kappa_2 - \frac{\lambda_2^2}{\kappa_1} & -\frac{\lambda_3}{\kappa_3} - \kappa_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\kappa_n} & 0 & 0 & \dots & -\frac{\lambda_n}{\kappa_n} - \kappa_{n-1} \end{pmatrix}.$$

Thus, we easily se that

$$\Gamma\kappa^{-1}\Lambda - H = \begin{pmatrix} -\frac{\lambda_1}{\kappa_1} - \kappa_n & \frac{1}{\kappa_1} & \dots & \frac{\lambda_{n-1}\lambda_n^2}{\kappa_{n-1}} & -\lambda_n\kappa_n - \frac{\lambda_n^2}{\kappa_{n-1}} \\ -\lambda_1\kappa_1 - \frac{\lambda_1^2}{\kappa_n} & -\frac{\lambda_2}{\kappa_2} - \kappa_1 & \dots & 0 & \frac{\lambda_n\lambda_1^2}{\kappa_n} \\ \frac{\lambda_1\lambda_2^2}{\kappa_1} & -\lambda_2\kappa_2 - \frac{\lambda_2^2}{\kappa_1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\kappa_n} & 0 & \dots & -\lambda_{n-1}\kappa_{n-1} - \frac{\lambda_{n-1}^2}{\kappa_{n-2}} & -\frac{\lambda_n}{\kappa_n} - \kappa_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\lambda_1+\kappa_n\kappa_1}{\kappa_1} & \frac{1}{\kappa_1} & \dots & \frac{\lambda_{n-1}\lambda_n^2}{\kappa_{n-1}} & -\frac{\lambda_n(\lambda_n+\kappa_{n-1}\kappa_n)}{\kappa_{n-1}} \\ -\frac{\lambda_1(\lambda_1+\kappa_n\kappa_1)}{\kappa_n} & -\frac{\lambda_2+\kappa_1\kappa_2}{\kappa_2} & \dots & 0 & \frac{\lambda_n\lambda_1^2}{\kappa_n} \\ \frac{\lambda_1\lambda_2^2}{\kappa_1} & -\frac{\lambda_2(\lambda_2+\kappa_1\kappa_2)}{\kappa_1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\kappa_n} & 0 & \dots & -\frac{\lambda_{n-1}(\lambda_{n-1}+\kappa_{n-2}\kappa_{n-1})}{\kappa_{n-2}} & -\frac{\lambda_n+\kappa_{n-1}\kappa_n}{\kappa_n} \end{pmatrix}.$$

Also, we have

$$\Gamma\kappa^{-1}G = \begin{pmatrix} \frac{1}{\kappa_1} & 0 & \dots & -\frac{\lambda_n^2}{\kappa_{n-1}} & -\kappa_n \\ -\kappa_1 & \frac{1}{\kappa_2} & \dots & 0 & -\frac{\lambda_1^2}{\kappa_n} \\ -\frac{\lambda_2^2}{\kappa_1} & -\kappa_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\kappa_{n-1} & \frac{1}{\kappa_n} \end{pmatrix} \begin{pmatrix} \kappa_1' & 0 & \dots & 0 & \lambda_1' \\ \lambda_2' & \kappa_2' & \dots & 0 & 0 \\ 0 & \lambda_3' & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_n' & \kappa_n' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\kappa_1'}{\kappa_1} & 0 & \dots & -\frac{\lambda_n^2\kappa_{n-1}'}{\kappa_{n-1}} - \kappa_n\lambda_n' & \frac{\lambda_1'}{\kappa_1} - \kappa_n\kappa_n' \\ \frac{\lambda_2'}{\kappa_2} - \kappa_1\kappa_1' & \frac{\kappa_2'}{\kappa_2} & \dots & -\frac{\lambda_1^2\lambda_n'}{\kappa_n} & -\frac{\lambda_1^2\kappa_n'}{\kappa_n} - \kappa_1\lambda_1' \\ -\frac{\lambda_2^2\kappa_1'}{\kappa_1} - \kappa_2\lambda_2' & \frac{\lambda_3'}{\kappa_3} - \kappa_2\kappa_2' & \dots & 0 & -\frac{\lambda_2^2\lambda_1'}{\kappa_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{\lambda_n'}{\kappa_n} - \kappa_{n-1}\kappa_{n-1}' & \frac{\kappa_n'}{\kappa_n} \end{pmatrix}$$

and

$$\Gamma\kappa^{-1}G+F = \begin{pmatrix} \frac{\kappa_1'}{\kappa_1} & 0 & \dots & -\frac{\lambda_n^2\kappa_{n-1}'}{\kappa_{n-1}} - \kappa_n\lambda_n' & \frac{\lambda_1'}{\kappa_1} - \kappa_n\kappa_n' \\ \frac{\lambda_2'}{\kappa_2} - \kappa_1\kappa_1' & \frac{\kappa_2'}{\kappa_2} & \dots & -\frac{\lambda_1^2\lambda_n'}{\kappa_n} & -\frac{\lambda_1^2\kappa_n'}{\kappa_n} - \kappa_1\lambda_1' \\ -\frac{\lambda_2^2\kappa_1'}{\kappa_1} - \kappa_2\lambda_2' & \frac{\lambda_3'}{\kappa_3} - \kappa_2\kappa_2' & \dots & 0 & -\frac{\lambda_2^2\lambda_1'}{\kappa_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{\lambda_n'}{\kappa_n} - \kappa_{n-1}\kappa_{n-1}' & \frac{\kappa_n'}{\kappa_n} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & \dots & \mu_n & \nu_n \\ \nu_1 & 0 & \dots & 0 & \mu_1 \\ \mu_2 & \nu_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \nu_{n-1} & 0 \end{pmatrix}.$$

Thus

$$\Gamma\kappa^{-1}G+F =$$
$$\begin{pmatrix} \frac{\kappa_1'}{\kappa_1} & 0 & \dots & -\frac{\lambda_n^2\kappa_{n-1}'}{\kappa_{n-1}} - \kappa_n\lambda_n' + \mu_n & \frac{\lambda_1'}{\kappa_1} - \kappa_n\kappa_n' + \nu_n \\ \frac{\lambda_2'}{\kappa_2} - \kappa_1\kappa_1' + \nu_1 & \frac{\kappa_2'}{\kappa_2} & \dots & -\frac{\lambda_1^2\lambda_n'}{\kappa_n} & -\frac{\lambda_1^2\kappa_n'}{\kappa_n} - \kappa_1\lambda_1' + \mu_1 \\ -\frac{\lambda_2^2\kappa_1'}{\kappa_1} - \kappa_2\lambda_2' + \mu_2 & \frac{\lambda_3'}{\kappa_3} - \kappa_2\kappa_2' + \nu_2 & \dots & 0 & -\frac{\lambda_2^2\lambda_1'}{\kappa_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{\kappa_{n-1}'}{\kappa_{n-1}} & 0 \\ 0 & 0 & \dots & \frac{\lambda_n'}{\kappa_n} - \kappa_{n-1}\kappa_{n-1}' + \nu_{n-1} & \frac{\kappa_n'}{\kappa_n} \end{pmatrix}.$$

In order to formalize what have been discussed above, we formulate it as

Theorem 2.3 *Consider a G^1-continuous n-patch surface S, composed of the $C^2_{\#}$-patches $(u,v) \mapsto \mathbf{r}^{(i)}(u,v)$, $0 \le u,v \le 1$ for $i = 1,2,\dots,n$, which intersect at the point $V = \mathbf{r}^{(i)}(0,0)$, where $i = 1,2,\dots,n$. Necessary and sufficient conditions for the surface S to be G^2 are that there exist continuously differentiable functions $v \mapsto \lambda_i(v)$, $v \mapsto \kappa_i(v)$ and continuous functions $v \mapsto \mu_i(v)$, $v \mapsto \nu_i(v)$ for $0 \le v \le 1$ and $i = 1,2,\dots,n$, satisfying the relations (2.7), with the coefficients $\lambda_i = \lambda_i(0)$, $\kappa_i = \kappa_i(0)$, $\mu_i = \mu_i(0)$, $\nu_i = \nu_i(0)$, $\lambda_i' = \lambda_i'(0)$ and $\kappa_i' = \kappa_i'(0)$ fullfilling equation (2.12) in the case of odd integers n and equation (2.13) in the case of even integers n and invertible κ respectively.*

The cases where κ is not invertible will be handled separately in its context.

3 A 3-patch Surface

In this section we restrict ourselves to consider a 3-patch surface and formulate necessary and sufficient conditions for such a surface to be tangent plane continuous and curvature continuous respectively. Thus, we reformulate Theorem 2.1 and Theorem 2.3 for this particular case. We start by considering G^1-regularity.

3.1 Tangent plane continuity – G^1

The system of identities in (2.3) is in this case reduced to the next three equations which have to be simultaneously satisfied for $t \in [0,1]$, i.e.,

$$\begin{aligned} \mathbf{r}_v^{(1)}(t,0) &= \lambda_3(t)\mathbf{r}_u^{(3)}(0,t) + \kappa_3(t)\mathbf{r}_v^{(3)}(0,t) \\ \mathbf{r}_v^{(2)}(t,0) &= \lambda_1(t)\mathbf{r}_u^{(1)}(0,t) + \kappa_1(t)\mathbf{r}_v^{(1)}(0,t) \\ \mathbf{r}_v^{(3)}(t,0) &= \lambda_2(t)\mathbf{r}_u^{(2)}(0,t) + \kappa_2(t)\mathbf{r}_v^{(2)}(0,t). \end{aligned} \tag{3.1}$$

In order to restate Theorem 2.1 in the case $n = 3$, we have to solve the equation system (2.5), which in this case is

$$\begin{pmatrix} \lambda_2 & \kappa_2 & -1 \\ -1 & \lambda_3 & \kappa_3 \\ \kappa_1 & -1 & \lambda_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ 1 & 0 \end{pmatrix} = 0.$$

This is equivalent to the following equation systems

$$\lambda_1\kappa_2 - 1 = 0$$
$$\lambda_2 + \kappa_1\kappa_2 = 0,$$

and

$$\lambda_1\lambda_3 + \kappa_3 = 0$$
$$\lambda_3\kappa_1 - 1 = 0.$$

Thus, we can state our first result in this section as

Theorem 3.1 *Let S be a continuous 3-patch surface, composed of the three $C^1_\#$-patches $\mathbf{r}^{(1)}$, $\mathbf{r}^{(2)}$ and $\mathbf{r}^{(3)}$ which meet at the vertex point V. Necessary and sufficient conditions for the surface S to be G^1 are that there exist continuous functions λ_i and κ_i, $i = 1,2,3$, fulfilling the relations (3.1), with the coefficients $\lambda_i = \lambda_i(0)$ and $\kappa_i = \kappa_i(0)$, $i = 1,2,3$, satisfying the equations*

$$\begin{aligned} \lambda_3\kappa_1 &= 1 \\ \lambda_1\kappa_2 &= 1 \end{aligned} \quad \textit{and} \quad \begin{aligned} \lambda_2 + \kappa_1\kappa_2 &= 0 \\ \lambda_3\lambda_1 + \kappa_3 &= 0. \end{aligned} \tag{3.2}$$

Trivially consequences of Theorem 3.1 are the next two results

Corollary 3.2 *The equations in (3.2) implies*

$$\lambda_1\lambda_2\lambda_3 = -1 \quad \textit{and} \quad \kappa_1\kappa_2\kappa_3 = -1.$$

Corollary 3.3 *The equations in (3.2) also implies the natural symmetry conditions, i.e.,*

$$\lambda_2\kappa_3 = 1$$
$$\lambda_1 + \kappa_3\kappa_1 = 0$$
$$\lambda_3 + \kappa_2\kappa_3 = 0$$

and

$$\lambda_2\lambda_3 + \kappa_2 = 0$$
$$\lambda_1\lambda_2 + \kappa_1 = 0.$$

In fact, we can substitute any of the equalities in Theorem 3.1 against any other condition in Corollary 3.3. This can easily be proved, but can also be seen as a result of symmetry. In fact, any four of the above nine symmetry conditions can be used in Theorem 3.1.

Let us now continue to the next level of regularity.

3.2 Curvature continuity – G^2

In the case of a 3-patch surface to be G^2, compare (2.7), the next equation system has to be fulfilled. We have

$$\begin{aligned}
\mathbf{r}^{(1)}_{vv}(t,0) &= \lambda_3^2\mathbf{r}^{(3)}_{uu}(0,\cdot) + 2\lambda_3\kappa_3\mathbf{r}^{(3)}_{uv}(0,\cdot) + \kappa_3^2\mathbf{r}^{(3)}_{vv}(0,\cdot) + \mu_3\mathbf{r}^{(3)}_{u}(0,\cdot) + \nu_3\mathbf{r}^{(3)}_{v}(0,\cdot)|_t \\
\mathbf{r}^{(2)}_{vv}(t,0) &= \lambda_1^2\mathbf{r}^{(1)}_{uu}(0,\cdot) + 2\lambda_1\kappa_1\mathbf{r}^{(1)}_{uv}(0,\cdot) + \kappa_1^2\mathbf{r}^{(1)}_{vv}(0,\cdot) + \mu_1\mathbf{r}^{(1)}_{u}(0,\cdot) + \nu_1\mathbf{r}^{(1)}_{v}(0,\cdot)|_t \\
\mathbf{r}^{(3)}_{vv}(t,0) &= \lambda_2^2\mathbf{r}^{(2)}_{uu}(0,\cdot) + 2\lambda_2\kappa_2\mathbf{r}^{(2)}_{uv}(0,\cdot) + \kappa_2^2\mathbf{r}^{(2)}_{vv}(0,\cdot) + \mu_2\mathbf{r}^{(2)}_{u}(0,\cdot) + \nu_2\mathbf{r}^{(2)}_{v}(0,\cdot)|_t
\end{aligned} \tag{3.3}$$

for $t \in [0,1]$.

In order to formulate Theorem 2.3 in the case of $n = 3$, we start by stating that n is an odd integer and therefor we want to solve equation (2.12). From Theorem 3.1 in the previous subsection, combined with its corollaries, we get

$$\Gamma - H\,\Lambda^{-1}\,\kappa = \begin{pmatrix} 1-\lambda_2\lambda_3\kappa_3\kappa_1 & -\lambda_3^2-\lambda_3\kappa_3\kappa_2 & 0 \\ 0 & 1-\lambda_3\lambda_1\kappa_1\kappa_2 & -\lambda_1^2-\lambda_1\kappa_1\kappa_3 \\ -\lambda_2^2-\lambda_2\kappa_2\kappa_1 & 0 & 1-\lambda_1\lambda_2\kappa_2\kappa_3 \end{pmatrix} = 0$$

and

$$f + H\,\Lambda^{-1}\,g = \begin{pmatrix} \lambda_3\kappa_3(\lambda_2\kappa_1'+\lambda_2') & \kappa_3(\lambda_3\kappa_2'-\lambda_3')+\mu_3 & \lambda_2\lambda_3\kappa_3\lambda_1'-\kappa_3\kappa_3'+\nu_3 \\ \lambda_3\lambda_1\kappa_1\lambda_2'-\kappa_1\kappa_1'+\nu_1 & \lambda_1\kappa_1(\lambda_3\kappa_2'+\lambda_3') & \kappa_1(\lambda_1\kappa_3'-\lambda_1')+\mu_1 \\ \kappa_2(\lambda_2\kappa_1'-\lambda_2')+\mu_2 & \lambda_1\lambda_2\kappa_2\lambda_3'-\kappa_2\kappa_2'+\nu_2 & \lambda_2\kappa_2(\lambda_1\kappa_3'+\lambda_1') \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_3\kappa_3(\lambda_2\kappa_1'+\lambda_2') & \kappa_3(\lambda_3\kappa_2'-\lambda_3')+\mu_3 & \lambda_3(\lambda_1\kappa_3'+\lambda_1')+\nu_3 \\ \lambda_1(\lambda_2\kappa_1'+\lambda_2')+\nu_1 & \lambda_1\kappa_1(\lambda_3\kappa_2'+\lambda_3') & \kappa_1(\lambda_1\kappa_3'-\lambda_1')+\mu_1 \\ \kappa_2(\lambda_2\kappa_1'-\lambda_2')+\mu_2 & \lambda_2(\lambda_3\kappa_2'+\lambda_3')+\nu_2 & \lambda_2\kappa_2(\lambda_1\kappa_3'+\lambda_1') \end{pmatrix}.$$

Thus, in this case equation (2.12) becomes

$$\begin{pmatrix} \lambda_3\kappa_3(\lambda_2\kappa_1' + \lambda_2') & \kappa_3(\lambda_3\kappa_2' - \lambda_3') + \mu_3 & \lambda_3(\lambda_1\kappa_3' + \lambda_1') + \nu_3 \\ \lambda_1(\lambda_2\kappa_1' + \lambda_2') + \nu_1 & \lambda_1\kappa_1(\lambda_3\kappa_2' + \lambda_3') & \kappa_1(\lambda_1\kappa_3' - \lambda_1') + \mu_1 \\ \kappa_2(\lambda_2\kappa_1' - \lambda_2') + \mu_2 & \lambda_2(\lambda_3\kappa_2' + \lambda_3') + \nu_2 & \lambda_2\kappa_2(\lambda_1\kappa_3' + \lambda_1') \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ 1 & 0 \end{pmatrix} = 0.$$

If we multiply the above equation with the invertible matrix

$$\begin{pmatrix} 1 & 0 & -\lambda_3/\lambda_2\kappa_2 \\ -\lambda_1/\lambda_3\kappa_3 & 1 & 0 \\ 0 & -\lambda_2/\lambda_1\kappa_1 & 1 \end{pmatrix},$$

we get

$$\begin{pmatrix} 2\lambda_3\kappa_3\lambda_2' - \frac{\lambda_3\mu_2}{\lambda_2\kappa_2} & 2\lambda_3\kappa_3\kappa_2' + \mu_3 - \frac{\lambda_3\nu_2}{\lambda_2\kappa_2} & \nu_3 \\ \nu_1 & 2\lambda_1\kappa_1\lambda_3' - \frac{\lambda_1\mu_3}{\lambda_3\kappa_3} & 2\lambda_1\kappa_1\kappa_3' + \mu_1 - \frac{\lambda_1\nu_3}{\lambda_3\kappa_3} \\ 2\lambda_2\kappa_2\kappa_1' + \mu_2 - \frac{\lambda_2\nu_1}{\lambda_1\kappa_1} & \nu_2 & 2\lambda_2\kappa_2\lambda_1' - \frac{\lambda_2\mu_1}{\lambda_1\kappa_1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ 1 & 0 \end{pmatrix} = 0,$$

which is equivalent to the next equation system

$$\begin{aligned}
&\lambda_1(2\lambda_3\kappa_3\kappa_2' + \mu_3 - \tfrac{\lambda_3\nu_2}{\lambda_2\kappa_2}) + \nu_3 = 0 \\
&2\lambda_3\kappa_3\lambda_2' - \tfrac{\lambda_3\mu_2}{\lambda_2\kappa_2} + \kappa_1(2\lambda_3\kappa_3\kappa_2' + \mu_3 - \tfrac{\lambda_3\nu_2}{\lambda_2\kappa_2}) = 0 \\
&\lambda_1(2\lambda_1\kappa_1\lambda_3' - \tfrac{\lambda_1\mu_3}{\lambda_3\kappa_3}) + 2\lambda_1\kappa_1\kappa_3' + \mu_1 - \tfrac{\lambda_1\nu_3}{\lambda_3\kappa_3} = 0 \\
&\nu_1 + \kappa_1(2\lambda_1\kappa_1\lambda_3' - \tfrac{\lambda_1\mu_3}{\lambda_3\kappa_3}) = 0 \\
&\lambda_1\nu_2 + 2\lambda_2\kappa_2\lambda_1' - \tfrac{\lambda_2\mu_1}{\lambda_1\kappa_1} = 0 \\
&2\lambda_2\kappa_2\kappa_1' + \mu_2 - \tfrac{\lambda_2\nu_1}{\lambda_1\kappa_1} + \kappa_1\nu_2 = 0.
\end{aligned}$$

A further simplification using Theorem 3.1 and its Corollaries combined with multiplying the above equation system by the following invertible matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ \lambda_2^2 & \lambda_2 & 0 & 0 & 0 & 0 \\ \lambda_2\kappa_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3^2 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_3\kappa_3 & 0 & 0 \end{pmatrix}$$

gives us the next result, which states

Theorem 3.4 *Let S be a G^1-continuous 3-patch surface, composed of the three $C^2_\#$-patches $\mathbf{r}^{(1)}$, $\mathbf{r}^{(2)}$ and $\mathbf{r}^{(3)}$ which meet at the vertex point V. Necessary and sufficient conditions for the surface S to be G^2 are that there exist continuously differentiable functions λ_i, κ_i and continuous functions μ_i, ν_i for $i = 1,2,3$, fulfilling the relations (3.3), with the coefficients $\lambda_i = \lambda_i(0)$, $\kappa_i = \kappa_i(0)$, $\mu_i = \mu_i(0)$, $\nu_i = \nu_i(0)$, $\lambda'_i = \lambda'_i(0)$ and $\kappa'_i = \kappa'_i(0)$ for $i = 1,2,3$, satisfying the equations*

$$\begin{aligned}
&2\lambda_2\lambda'_1 + \kappa_2\mu_1 + \lambda_1^2\nu_2 = 0\\
&2\lambda_2\kappa'_1 + \lambda_1\mu_2 + \kappa_2\nu_1 + \lambda_1\kappa_1\nu_2 = 0\\
&2\lambda_3\lambda'_2 + \kappa_3\mu_2 + \lambda_2^2\nu_3 = 0\\
&2\lambda_3\kappa'_2 + \lambda_2\mu_3 + \kappa_3\nu_2 + \lambda_2\kappa_2\nu_3 = 0\\
&2\lambda_1\lambda'_3 + \kappa_1\mu_3 + \lambda_3^2\nu_1 = 0\\
&2\lambda_1\kappa'_3 + \lambda_3\mu_1 + \kappa_1\nu_3 + \lambda_3\kappa_3\nu_1 = 0.
\end{aligned}$$

It can be worth noticing that the case $\mu_i = \nu_i = 0$ for $i = 1,2,3$, implies that $\lambda'_i = \kappa'_i = 0$ for $i = 1,2,3$. On the other hand, if $\lambda'_i = \kappa'_i = 0$ for $i = 1,2,3$, then $\mu_i = \nu_i = 0$ for $i = 1,2,3$.

4 A 4-patch Surface

In this section we study the most common case, i.e., a 4-patch surface. Just as in the previous section, we use the results in the second section applied to the case where $n = 4$ in order to formulate necessary and sufficient conditions for the surface to be tangent plane continuous as well as curvature continuous.

4.1 Tangent plane continuity – G^1

In the case of a 4-patch surface we have, as a consequence from (2.3), the next equation system for $t \in [0,1]$

$$\begin{aligned}
\mathbf{r}_v^{(1)}(t,0) &= \lambda_4(t)\mathbf{r}_u^{(4)}(0,t) + \kappa_4(t)\mathbf{r}_v^{(4)}(0,t)\\
\mathbf{r}_v^{(2)}(t,0) &= \lambda_1(t)\mathbf{r}_u^{(1)}(0,t) + \kappa_1(t)\mathbf{r}_v^{(1)}(0,t)\\
\mathbf{r}_v^{(3)}(t,0) &= \lambda_2(t)\mathbf{r}_u^{(2)}(0,t) + \kappa_2(t)\mathbf{r}_v^{(2)}(0,t)\\
\mathbf{r}_v^{(4)}(t,0) &= \lambda_3(t)\mathbf{r}_u^{(3)}(0,t) + \kappa_3(t)\mathbf{r}_v^{(3)}(0,t).
\end{aligned} \tag{4.1}$$

In this case equation (2.5) reads

$$\begin{pmatrix} \lambda_2 & \kappa_2 & -1 & 0\\ 0 & \lambda_3 & \kappa_3 & -1\\ -1 & 0 & \lambda_4 & \kappa_4\\ \kappa_1 & -1 & 0 & \lambda_1 \end{pmatrix}\begin{pmatrix} 0 & 1\\ \lambda_1 & \kappa_1\\ \lambda_1\kappa_2 & \kappa_1\kappa_2+\lambda_2\\ 1 & 0 \end{pmatrix} = 0,$$

from which we get

$$\lambda_1\lambda_3 + \lambda_1\kappa_2\kappa_3 - 1 = 0$$
$$\kappa_1\lambda_3 + \kappa_3(\kappa_1\kappa_2 + \lambda_2) = 0$$
$$\lambda_4\lambda_1\kappa_2 + \kappa_4 = 0$$
$$-1 + \lambda_4(\kappa_1\kappa_2 + \lambda_2) = 0.$$

Thus, we have the following result

Theorem 4.1 *Let S be a continuous 4-patch surface, composed of the $C^1_{\#}$-patches $\mathbf{r}^{(i)}$, for $i = 1, \ldots, 4$, which meet at the vertex point V. Necessary and sufficient conditions for the surface S to be G^1 are that there exist continuous functions λ_i and κ_i, $i = 1, \ldots, 4$, fulfilling the relations (4.1) and the coefficients $\lambda_i = \lambda_i(0)$ and $\kappa_i = \kappa_i(0)$, $i = 1, \ldots, 4$, satisfy the equations*

$$\lambda_1(\lambda_3 + \kappa_2\kappa_3) = 1$$
$$\lambda_3\lambda_4\kappa_1 + \kappa_3 = 0$$

and

$$\lambda_4(\lambda_2 + \kappa_1\kappa_2) = 1$$
$$\lambda_4\lambda_1\kappa_2 + \kappa_4 = 0.$$

With the same assumptions as in Theorem 4.1, we can prove the next three Corollaries.

Corollary 4.2

$$\lambda_1\lambda_2\lambda_3\lambda_4 = 1.$$

Proof

$$\begin{aligned} 1 &= \lambda_1(\lambda_3 + \kappa_2\kappa_3) = \lambda_1(\lambda_3 - \kappa_2\lambda_3\lambda_4\kappa_1) \\ &= \lambda_1\lambda_3(1 - \lambda_4\kappa_1\kappa_2) = \lambda_1\lambda_3(1 - (1 - \lambda_2\lambda_4)) \\ &= \lambda_1\lambda_2\lambda_3\lambda_4. \end{aligned}$$

□

We can easily see that the symmetry conditions also hold. We have

$$\begin{aligned} \lambda_2\lambda_3\kappa_4 + \kappa_2 &= \lambda_2\lambda_3\kappa_4 + \lambda_1\lambda_2\lambda_3\lambda_4\kappa_2 = \lambda_2\lambda_3(\kappa_4 + \lambda_4\lambda_1\kappa_2) = 0 \\ \lambda_1\lambda_2\kappa_3 + \kappa_1 &= \lambda_1\lambda_2\kappa_3 + \lambda_1\lambda_2\lambda_3\lambda_4\kappa_1 = \lambda_1\lambda_2(\kappa_3 + \lambda_3\lambda_4\kappa_1) = 0 \end{aligned}$$

and

$$\begin{aligned} \lambda_2(\lambda_4 + \kappa_3\kappa_4) &= \lambda_2(\lambda_4 + \lambda_3\lambda_4\kappa_1\lambda_4\lambda_1\kappa_2) = \lambda_4(\lambda_2 + \kappa_1\kappa_2) = 1 \\ \lambda_3(\lambda_1 + \kappa_4\kappa_1) &= \lambda_3(\lambda_1 + \lambda_4\lambda_1\kappa_2\lambda_1\lambda_2\kappa_3) = \lambda_1(\lambda_3 + \kappa_2\kappa_3) = 1. \end{aligned}$$

Corollary 4.3 *It holds that*

$$\kappa_1\kappa_2 = \lambda_2(\lambda_1\lambda_3 - 1)$$
$$\kappa_3\kappa_4 = \lambda_4(\lambda_1\lambda_3 - 1)$$

and

$$\kappa_2\kappa_3 = \lambda_3(\lambda_2\lambda_4 - 1)$$
$$\kappa_4\kappa_1 = \lambda_1(\lambda_2\lambda_4 - 1).$$

Corollary 4.4 *It also holds that*

$$\kappa_1\kappa_2\kappa_3\kappa_4 = \frac{(1-\lambda_1\lambda_3)^2}{\lambda_1\lambda_3} = \frac{(1-\lambda_2\lambda_4)^2}{\lambda_2\lambda_4}.$$

4.2 Curvature continuity – G^2

Since we are now dealing with a 4-patch surface and its curvature continuity, the next equations have to be fulfilled, i.e., for $t \in [0,1]$ we have

$$\begin{aligned}
\mathbf{r}^{(1)}_{vv}(t,0) &= \lambda_4^2\mathbf{r}^{(4)}_{uu}(0,\cdot) + 2\lambda_4\kappa_4\mathbf{r}^{(4)}_{uv}(0,\cdot) + \kappa_4^2\mathbf{r}^{(4)}_{vv}(0,\cdot) + \mu_4\mathbf{r}^{(4)}_{u}(0,\cdot) + \nu_4\mathbf{r}^{(4)}_{v}(0,\cdot)|_t \\
\mathbf{r}^{(2)}_{vv}(t,0) &= \lambda_1^2\mathbf{r}^{(1)}_{uu}(0,\cdot) + 2\lambda_1\kappa_1\mathbf{r}^{(1)}_{uv}(0,\cdot) + \kappa_1^2\mathbf{r}^{(1)}_{vv}(0,\cdot) + \mu_1\mathbf{r}^{(1)}_{u}(0,\cdot) + \nu_1\mathbf{r}^{(1)}_{v}(0,\cdot)|_t \\
\mathbf{r}^{(3)}_{vv}(t,0) &= \lambda_2^2\mathbf{r}^{(2)}_{uu}(0,\cdot) + 2\lambda_2\kappa_2\mathbf{r}^{(2)}_{uv}(0,\cdot) + \kappa_2^2\mathbf{r}^{(2)}_{vv}(0,\cdot) + \mu_2\mathbf{r}^{(2)}_{u}(0,\cdot) + \nu_2\mathbf{r}^{(2)}_{v}(0,\cdot)|_t \\
\mathbf{r}^{(4)}_{vv}(t,0) &= \lambda_3^2\mathbf{r}^{(3)}_{uu}(0,\cdot) + 2\lambda_3\kappa_3\mathbf{r}^{(3)}_{uv}(0,\cdot) + \kappa_3^2\mathbf{r}^{(3)}_{vv}(0,\cdot) + \mu_3\mathbf{r}^{(3)}_{u}(0,\cdot) + \nu_3\mathbf{r}^{(3)}_{v}(0,\cdot)|_t.
\end{aligned} \tag{4.2}$$

As a consequence of this, we refer to Theorem 2.3 under the assumption that $\kappa_1\kappa_2 \neq 0$, the equation (2.13) must be satisfied. We have for $n = 4$ that

$$\Gamma\kappa^{-1}\Lambda - H = \begin{pmatrix}
-\frac{\lambda_1+\kappa_4\kappa_1}{\kappa_1} & \frac{1}{\kappa_1} & \frac{\lambda_3\lambda_4^2}{\kappa_3} & -\frac{\lambda_4(\lambda_4+\kappa_3\kappa_4)}{\kappa_3} \\
-\frac{\lambda_1(\lambda_1+\kappa_4\kappa_1)}{\kappa_4} & -\frac{\lambda_2+\kappa_1\kappa_2}{\kappa_2} & \frac{1}{\kappa_2} & \frac{\lambda_4\lambda_1^2}{\kappa_4} \\
\frac{\lambda_1\lambda_2^2}{\kappa_1} & -\frac{\lambda_2(\lambda_2+\kappa_1\kappa_2)}{\kappa_1} & -\frac{\lambda_3+\kappa_2\kappa_3}{\kappa_3} & \frac{1}{\kappa_3} \\
\frac{1}{\kappa_4} & \frac{\lambda_2\lambda_3^2}{\kappa_2} & -\frac{\lambda_3(\lambda_3+\kappa_2\kappa_3)}{\kappa_2} & -\frac{\lambda_4+\kappa_3\kappa_4}{\kappa_4}
\end{pmatrix} \tag{4.3}$$

and

$$\Gamma\kappa^{-1}G + F$$
$$= \begin{pmatrix}
\frac{\kappa_1'}{\kappa_1} & -\frac{\lambda_4^2\lambda_3'}{\kappa_3} & -\frac{\lambda_4^2\kappa_3'}{\kappa_3} - \kappa_4\lambda_4' + \mu_4 & \frac{\lambda_1'}{\kappa_1} - \kappa_4\kappa_4' + \nu_4 \\
\frac{\lambda_2'}{\kappa_2} - \kappa_1\kappa_1' + \nu_1 & \frac{\kappa_2'}{\kappa_2} & -\frac{\lambda_1^2\lambda_4'}{\kappa_4} & -\frac{\lambda_1^2\kappa_4'}{\kappa_4} - \kappa_1\lambda_1' + \mu_1 \\
-\frac{\lambda_2^2\kappa_1'}{\kappa_1} - \kappa_2\lambda_2' + \mu_2 & \frac{\lambda_3'}{\kappa_3} - \kappa_2\kappa_2' + \nu_2 & \frac{\kappa_3'}{\kappa_3} & -\frac{\lambda_2^2\lambda_1'}{\kappa_1} \\
-\frac{\lambda_3^2\lambda_2'}{\kappa_2} & -\frac{\lambda_3^2\kappa_2'}{\kappa_2} - \kappa_3\lambda_3' + \mu_3 & \frac{\lambda_4'}{\kappa_4} - \kappa_3\kappa_3' + \nu_3 & \frac{\kappa_4'}{\kappa_4}
\end{pmatrix}.$$

Using the identities in Theorem 4.1 and its Corollaries we can simplify the matrix in (4.3) in such a way that we get

$$\Gamma\kappa^{-1}\Lambda - H = \begin{pmatrix} -\frac{1}{\lambda_3\kappa_1} & \frac{1}{\kappa_1} & \frac{\lambda_3\lambda_4^2}{\kappa_3} & -\frac{\lambda_4}{\lambda_2\kappa_3} \\ -\frac{\lambda_1}{\lambda_3\kappa_4} & -\frac{1}{\lambda_4\kappa_2} & \frac{1}{\kappa_2} & \frac{\lambda_4\lambda_1^2}{\kappa_4} \\ \frac{\lambda_1\lambda_2^2}{\kappa_1} & -\frac{\lambda_2}{\lambda_4\kappa_1} & -\frac{1}{\lambda_1\kappa_3} & \frac{1}{\kappa_3} \\ \frac{1}{\kappa_4} & \frac{\lambda_2\lambda_3^2}{\kappa_2} & -\frac{\lambda_3}{\lambda_1\kappa_2} & -\frac{1}{\lambda_2\kappa_4}. \end{pmatrix}.$$

Introducing the notation

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \Gamma\kappa^{-1}G + F,$$

the equation (2.13) transforms into

$$\begin{pmatrix} -\frac{1}{\lambda_3\kappa_1} & \frac{1}{\kappa_1} & \frac{\lambda_3\lambda_4^2}{\kappa_3} & -\frac{\lambda_4}{\lambda_2\kappa_3} \\ -\frac{\lambda_1}{\lambda_3\kappa_4} & -\frac{1}{\lambda_4\kappa_2} & \frac{1}{\kappa_2} & \frac{\lambda_4\lambda_1^2}{\kappa_4} \\ \frac{\lambda_1\lambda_2^2}{\kappa_1} & -\frac{\lambda_2}{\lambda_4\kappa_1} & -\frac{1}{\lambda_1\kappa_3} & \frac{1}{\kappa_3} \\ \frac{1}{\kappa_4} & \frac{\lambda_2\lambda_3^2}{\kappa_2} & -\frac{\lambda_3}{\lambda_1\kappa_2} & -\frac{1}{\lambda_2\kappa_4} \end{pmatrix} \mathbf{r}_{uv} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} \mathbf{r}_v.$$

Using Theorem 4.1 and its Corollaries once more, we get

$$\begin{pmatrix} -\frac{1}{\lambda_3\kappa_1} & \frac{1}{\kappa_1} & \frac{\lambda_3\lambda_4^2}{\kappa_3} & -\frac{\lambda_4}{\lambda_2\kappa_3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{r}_{uv} = \begin{pmatrix} h_1 \\ h_2 - \frac{\lambda_1\kappa_1}{\kappa_4} h_1 \\ h_3 - \frac{\lambda_2\kappa_2}{\kappa_1} h_2 \\ h_4 - \frac{\lambda_3\kappa_3}{\kappa_2} h_3 \end{pmatrix} \mathbf{r}_v.$$

Thus, we want to solve the equation system

$$\begin{pmatrix} h_2 - \frac{\lambda_1\kappa_1}{\kappa_4} h_1 \\ h_3 - \frac{\lambda_2\kappa_2}{\kappa_1} h_2 \\ h_4 - \frac{\lambda_3\kappa_3}{\kappa_2} h_3 \end{pmatrix} \mathbf{r}_v = 0. \tag{4.4}$$

From (2.4) we have a relation to express $\mathbf{r}_v$ in the linear independent vectors $\mathbf{r}_u^{(1)}$ and $\mathbf{r}_v^{(1)}$ as

$$\mathbf{r}_v = \begin{pmatrix} \mathbf{r}_v^{(1)} \\ \lambda_1\mathbf{r}_u^{(1)} + \kappa_1\mathbf{r}_v^{(1)} \\ \lambda_1\kappa_2\mathbf{r}_u^{(1)} + (\lambda_2 + \kappa_1\kappa_2)\mathbf{r}_v^{(1)} \\ \mathbf{r}_u^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ \lambda_1\kappa_2 & \frac{1}{\lambda_4} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}_u^{(1)} \\ \mathbf{r}_v^{(1)} \end{pmatrix}. \tag{4.5}$$

Substituting the identity (4.5) into equation (4.4) we can rewrite it as

$$\begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ \lambda_1\kappa_2 & \frac{1}{\lambda_4} \\ 1 & 0 \end{pmatrix} = 0, \tag{4.6}$$

where $q_i,\ i = 1, \ldots, 4$, is naturally defined from equation (4.4). In fact, we have

$$q_1 = \begin{pmatrix} \frac{\lambda_2'}{\kappa_2} - \frac{\kappa_1'}{\lambda_3\kappa_4} + \nu_1 \\ -\frac{\lambda_2'}{\lambda_4\kappa_1} - \frac{\lambda_2\kappa_1'}{\kappa_1}(\lambda_2 - \kappa_1\kappa_2) + \mu_2 - \frac{\lambda_2\kappa_2}{\kappa_1}\nu_1 \\ -\frac{\lambda_3\lambda_2'}{\kappa_2}(\lambda_3 - \kappa_2\kappa_3) + \frac{\lambda_2^2\lambda_3\kappa_3\kappa_1'}{\kappa_1\kappa_2} - \frac{\lambda_3\kappa_3}{\kappa_2}\mu_2 \end{pmatrix}$$

$$q_2 = \begin{pmatrix} \frac{\kappa_2'}{\kappa_2} + \frac{\lambda_4^2\lambda_1\kappa_1\lambda_3'}{\kappa_3\kappa_4} \\ \frac{\lambda_3'}{\kappa_3} - \frac{\kappa_2'}{\lambda_4\kappa_1} + \nu_2 \\ -\frac{\lambda_3'}{\lambda_1\kappa_2} - \frac{\lambda_3\kappa_2'}{\kappa_2}(\lambda_3 - \kappa_2\kappa_3) + \mu_3 - \frac{\lambda_3\kappa_3}{\kappa_2}\nu_2 \end{pmatrix}$$

$$q_3 = \begin{pmatrix} -\frac{\lambda_1\lambda_4'}{\kappa_4}(\lambda_1 - \kappa_4\kappa_1) + \frac{\lambda_4^2\lambda_1\kappa_1\kappa_3'}{\kappa_3\kappa_4} - \frac{\lambda_1\kappa_1}{\kappa_4}\mu_4 \\ \frac{\kappa_3'}{\kappa_3} + \frac{\lambda_1^2\lambda_2\kappa_2\lambda_4'}{\kappa_4\kappa_1} \\ \frac{\lambda_4'}{\kappa_4} - \frac{\kappa_3'}{\lambda_1\kappa_2} + \nu_3 \end{pmatrix}$$

$$q_4 = \begin{pmatrix} -\frac{\lambda_1'}{\lambda_3\kappa_4} - \frac{\lambda_1\kappa_4'}{\kappa_4}(\lambda_1 - \kappa_4\kappa_1) + \mu_1 - \frac{\lambda_1\kappa_1}{\kappa_4}\nu_4 \\ -\frac{\lambda_2\lambda_1'}{\kappa_1}(\lambda_2 - \kappa_1\kappa_2) + \frac{\lambda_1^2\lambda_2\kappa_2\kappa_4'}{\kappa_4\kappa_1} - \frac{\lambda_2\kappa_2}{\kappa_1}\mu_1 \\ \frac{\kappa_4'}{\kappa_4} + \frac{\lambda_2^2\lambda_3\kappa_3\lambda_1'}{\kappa_1\kappa_2} \end{pmatrix}.$$

Next we multiply equation (4.6) by the invertible matrix

$$\begin{pmatrix} -\lambda_2\lambda_3\kappa_4 & \frac{\lambda_4\lambda_1}{\kappa_4} & -\frac{\lambda_1\kappa_2}{\lambda_3^2\kappa_4\kappa_1} \\ \lambda_2\lambda_3^2\kappa_4 & -\lambda_3\lambda_4\kappa_1 & 0 \\ \lambda_2\lambda_3^2\lambda_4\kappa_4 & 0 & -\lambda_4\lambda_1\kappa_2 \end{pmatrix}$$

to get

$$\begin{pmatrix} \tilde{q}_1 & \tilde{q}_2 & \tilde{q}_3 & \tilde{q}_4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ \lambda_1\kappa_2 & \frac{1}{\lambda_4} \\ 1 & 0 \end{pmatrix} = 0, \tag{4.7}$$

where

$$\tilde{q}_1 = \begin{pmatrix} \frac{\lambda_1\lambda_2\lambda_3}{\kappa_1}\nu_1 \\ 2\lambda_2^2\lambda_3^2\lambda_4\kappa_4\kappa_1\kappa_1' - \lambda_3\lambda_4\kappa_1\mu_2 - \lambda_2\lambda_3\kappa_2\kappa_3\kappa_4\nu_1 \\ -2\lambda_3^2\lambda_4\kappa_4\kappa_1\lambda_2' - \lambda_3^2\lambda_4^2\lambda_1\kappa_1\mu_2 + \lambda_2\lambda_3^2\lambda_4\kappa_4\nu_1 \end{pmatrix}$$

$$\tilde{q}_2 = \begin{pmatrix} 2\lambda_4\lambda_1\lambda_2\lambda_3' + \frac{1}{\lambda_3^2\lambda_4\kappa_1}\mu_3 \\ -\lambda_3\lambda_4\kappa_1\nu_2 \\ -2\lambda_3^2\lambda_4\kappa_4\kappa_1\kappa_2' + \kappa_4\mu_3 - \lambda_3^2\lambda_4^2\lambda_1\kappa_1\nu_2 \end{pmatrix}$$

$$\tilde{q}_3 = \begin{pmatrix} 2\lambda_1^2\lambda_2\lambda_3\lambda_4' + 2\lambda_4\lambda_1\lambda_2\kappa_3' + \lambda_1\lambda_2\lambda_3\kappa_1\mu_4 + \frac{1}{\lambda_3^2\lambda_4\kappa_1}\nu_3 \\ 2\lambda_1\lambda_2\lambda_3^2\kappa_4\kappa_1\lambda_4' - \lambda_1\lambda_2\lambda_3^2\kappa_1\mu_4 \\ 2\lambda_3\kappa_4\kappa_1\lambda_4' - \lambda_3\kappa_1\mu_4 + \kappa_4\nu_3 \end{pmatrix}$$

$$\tilde{q}_4 = \begin{pmatrix} 2\lambda_1^2\lambda_2\lambda_3\kappa_4' + \frac{1}{\lambda_4\kappa_1}\mu_1 + \lambda_1\lambda_2\lambda_3\kappa_1\nu_4 \\ 2\lambda_2^2\lambda_3^2\lambda_4\kappa_4\kappa_1\lambda_1' + 2\lambda_1\lambda_2\lambda_3^2\kappa_4\kappa_1\kappa_4' - \lambda_2\lambda_3\kappa_2\kappa_3\kappa_4\mu_1 - \lambda_1\lambda_2\lambda_3^2\kappa_1\nu_4 \\ 2\lambda_3\kappa_4\kappa_1\kappa_4' + \lambda_2\lambda_3^2\lambda_4\kappa_4\mu_1 - \lambda_3\kappa_1\nu_4 \end{pmatrix}.$$

Equation (4.7) can also be written as the following equation system

$$\begin{aligned}
&\lambda_1\big(2\lambda_4\lambda_1\lambda_2\lambda_3' + \tfrac{1}{\lambda_3^2\lambda_4\kappa_1}\mu_3\big) + \lambda_1\kappa_2\big(2\lambda_1^2\lambda_2\lambda_3\lambda_4' + 2\lambda_4\lambda_1\lambda_2\kappa_3' + \lambda_1\lambda_2\lambda_3\kappa_1\mu_4 \\
&\qquad + \tfrac{1}{\lambda_3^2\lambda_4\kappa_1}\nu_3\big) + 2\lambda_1^2\lambda_2\lambda_3\kappa_4' + \tfrac{1}{\lambda_4\kappa_1}\mu_1 + \lambda_1\lambda_2\lambda_3\kappa_1\nu_4 = 0 \\
&\tfrac{\lambda_1\lambda_2\lambda_3}{\kappa_1}\nu_1 + \kappa_1\big(2\lambda_4\lambda_1\lambda_2\lambda_3' + \tfrac{1}{\lambda_3^2\lambda_4\kappa_1}\mu_3\big) + \tfrac{1}{\lambda_4}\big(2\lambda_1^2\lambda_2\lambda_3\lambda_4' + 2\lambda_4\lambda_1\lambda_2\kappa_3' \\
&\qquad + \lambda_1\lambda_2\lambda_3\kappa_1\mu_4 + \tfrac{1}{\lambda_3^2\lambda_4\kappa_1}\nu_3\big) = 0 \\
&\lambda_1(-\lambda_3\lambda_4\kappa_1\nu_2) + \lambda_1\kappa_2\big(2\lambda_1\lambda_2\lambda_3^2\kappa_4\kappa_1\lambda_4' - \lambda_1\lambda_2\lambda_3^2\kappa_1\mu_4\big) + 2\lambda_2^2\lambda_3^2\lambda_4\kappa_4\kappa_1\lambda_1' \\
&\qquad + 2\lambda_1\lambda_2\lambda_3^2\kappa_4\kappa_1\kappa_4' - \lambda_2\lambda_3\kappa_2\kappa_3\kappa_4\mu_1 - \lambda_1\lambda_2\lambda_3^2\kappa_1\nu_4 = 0 \\
&2\lambda_2^2\lambda_3^2\lambda_4\kappa_4\kappa_1\kappa_1' - \lambda_3\lambda_4\kappa_1\mu_2 - \lambda_2\lambda_3\kappa_2\kappa_3\kappa_4\nu_1 + \kappa_1(-\lambda_3\lambda_4\kappa_1\nu_2) \\
&\qquad + \tfrac{1}{\lambda_4}\big(2\lambda_1\lambda_2\lambda_3^2\kappa_4\kappa_1\lambda_4' - \lambda_1\lambda_2\lambda_3^2\kappa_1\mu_4\big) = 0 \\
&\lambda_1\big(-2\lambda_3^2\lambda_4\kappa_4\kappa_1\kappa_2' + \kappa_4\mu_3 - \lambda_3^2\lambda_4^2\lambda_1\kappa_1\nu_2\big) + \lambda_1\kappa_2\big(2\lambda_3\kappa_4\kappa_1\lambda_4' - \lambda_3\kappa_1\mu_4 \\
&\qquad + \kappa_4\nu_3\big) + 2\lambda_3\kappa_4\kappa_1\kappa_4' + \lambda_2\lambda_3^2\lambda_4\kappa_4\mu_1 - \lambda_3\kappa_1\nu_4 = 0 \\
&-2\lambda_3^2\lambda_4\kappa_4\kappa_1\lambda_2' - \lambda_3^2\lambda_4^2\lambda_1\kappa_1\mu_2 + \lambda_2\lambda_3^2\lambda_4\kappa_4\nu_1 + \kappa_1\big(-2\lambda_3^2\lambda_4\kappa_4\kappa_1\kappa_2' + \kappa_4\mu_3 \\
&\qquad - \lambda_3^2\lambda_4^2\lambda_1\kappa_1\nu_2\big) + \tfrac{1}{\lambda_4}\big(2\lambda_3\kappa_4\kappa_1\lambda_4' - \lambda_3\kappa_1\mu_4 + \kappa_4\nu_3\big) = 0.
\end{aligned}$$

In order to simplify these equations we multiply the above equation system by the invertible matrix

$$\begin{pmatrix} \frac{1}{\lambda_1^2\lambda_2} & -\frac{\lambda_4\kappa_2}{\lambda_1\lambda_2} & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda_4}{\lambda_1\lambda_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda_2\lambda_3^2\kappa_4\kappa_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda_4}{\lambda_2\lambda_3^2\kappa_4\kappa_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda_2\lambda_3^2\kappa_4\kappa_1} & 0 & -\frac{\lambda_1}{\lambda_3\kappa_4\kappa_1} & 0 \\ \frac{\lambda_4\kappa_1}{\lambda_1\lambda_2} & -\frac{\lambda_4}{\lambda_1\lambda_2} & 0 & 0 & -\frac{\lambda_4\kappa_1}{\kappa_4\kappa_1} & \frac{\lambda_4\lambda_1}{\kappa_4\kappa_1} \end{pmatrix}$$

and then rearrange the new equation system to get our result. Thus, we are now in a position to formulate the main result in this section. We have

Theorem 4.5 *Let S be a G^1-continuous 4-patch surface, composed of the $C^2_\#$-patches $\mathbf{r}^{(i)}$ for $i = 1, \ldots, 4$, which meet at the vertex point V. Necessary and sufficient conditions for the surface S to be G^2 are that there exist continuously differentiable functions λ_i, κ_i and continuous functions μ_i, ν_i for $i = 1, \ldots, 4$, fulfilling the relations (4.2), and the coefficients $\lambda_i = \lambda_i(0)$, $\kappa_i = \kappa_i(0)$, $\mu_i = \mu_i(0)$, $\nu_i = \nu_i(0)$, $\lambda_i' = \lambda_i'(0)$ and $\kappa_i' = \kappa_i'(0)$ for $i = 1, \ldots, 4$, with $\kappa_1\kappa_2 \neq 0$, satisfy the equations*

$$\begin{aligned}
&2\lambda_2\kappa_1\lambda_1' + 2\lambda_1^2\lambda_3\kappa_1\kappa_2' - \lambda_2\mu_1 - \lambda_1^3\lambda_2\mu_3 - \tfrac{\lambda_1\kappa_1^2}{\lambda_2}\nu_2 - \tfrac{\lambda_1^2\kappa_2}{\lambda_3\lambda_4}\nu_3 = 0 \\
&2\lambda_3\kappa_2\lambda_2' + 2\lambda_2^2\lambda_4\kappa_2\kappa_3' - \lambda_3\mu_2 - \lambda_2^3\lambda_3\mu_4 - \tfrac{\lambda_2\kappa_2^2}{\lambda_3}\nu_3 - \tfrac{\lambda_2^2\kappa_3}{\lambda_4\lambda_1}\nu_4 = 0 \\
&2\lambda_4\kappa_3\lambda_3' + 2\lambda_3^2\lambda_1\kappa_3\kappa_4' - \lambda_4\mu_3 - \lambda_3^3\lambda_4\mu_1 - \tfrac{\lambda_3\kappa_3^2}{\lambda_4}\nu_4 - \tfrac{\lambda_3^2\kappa_4}{\lambda_1\lambda_2}\nu_1 = 0 \\
&2\lambda_1\kappa_4\lambda_4' + 2\lambda_4^2\lambda_2\kappa_4\kappa_1' - \lambda_1\mu_4 - \lambda_4^3\lambda_1\mu_2 - \tfrac{\lambda_4\kappa_4^2}{\lambda_1}\nu_1 - \tfrac{\lambda_4^2\kappa_1}{\lambda_2\lambda_3}\nu_2 = 0 \\
&2\lambda_4^2\kappa_1^2\lambda_3' + 2\lambda_1\lambda_3\kappa_1\lambda_4' + 2\lambda_4\kappa_1\kappa_3' + \tfrac{\lambda_4\kappa_1}{\lambda_3}\mu_3 + \lambda_3\kappa_1^2\mu_4 + \tfrac{1}{\lambda_3}\nu_3 + \lambda_3\lambda_4\nu_1 = 0 \\
&2\lambda_1^2\kappa_2^2\lambda_4' + 2\lambda_2\lambda_4\kappa_2\lambda_1' + 2\lambda_1\kappa_2\kappa_4' + \tfrac{\lambda_1\kappa_2}{\lambda_4}\mu_4 + \lambda_4\kappa_2^2\mu_1 + \tfrac{1}{\lambda_4}\nu_4 + \lambda_4\lambda_1\nu_2 = 0.
\end{aligned}$$

An observation worth noticing is that there is no absolute ordering of the patches, i.e., we can number them in any order.

In order to complete our study in the 4-patch surface case we must consider two different subcases. First, the case $\kappa_1 = \kappa_1(0) = 0$, but $\kappa_2 = \kappa_2(0) \neq 0$. This implies that also $\kappa_3 = 0$ and $\kappa_4 \neq 0$. Second, the case $\kappa_1 = \kappa_2 = 0$, which implies that $\kappa_i = 0$ for $i = 1, 2, 3, 4$.

4.2.1 The first subcase

In this subsection we consider the case where $\kappa_1 = \kappa_1(0) = 0$ and $\kappa_2 = \kappa_2(0) \neq 0$. We use the equation systems (2.8) and (2.9) restricted to $n = 4$ to get

$$\begin{aligned}
\mathbf{r}_{vv}^{(1)} &= \lambda_4^2\mathbf{r}_{vv}^{(3)} + 2\lambda_4\kappa_4\mathbf{r}_{uv}^{(4)} + \kappa_4^2\mathbf{r}_{vv}^{(4)} + \mu_4\mathbf{r}_v^{(3)} + \nu_4\mathbf{r}_v^{(4)} \\
\mathbf{r}_{vv}^{(2)} &= \lambda_1^2\mathbf{r}_{vv}^{(4)} + \mu_1\mathbf{r}_v^{(4)} + \nu_1\mathbf{r}_v^{(1)} \\
\mathbf{r}_{vv}^{(3)} &= \lambda_2^2\mathbf{r}_{vv}^{(1)} + 2\lambda_2\kappa_2\mathbf{r}_{uv}^{(2)} + \kappa_2^2\mathbf{r}_{vv}^{(2)} + \mu_2\mathbf{r}_v^{(1)} + \nu_2\mathbf{r}_v^{(2)} \\
\mathbf{r}_{vv}^{(4)} &= \lambda_3^2\mathbf{r}_{vv}^{(2)} + \mu_3\mathbf{r}_v^{(2)} + \nu_3\mathbf{r}_v^{(3)}
\end{aligned} \tag{4.8}$$

and

$$
\begin{aligned}
\mathbf{r}_{uv}^{(2)} &= \lambda_1 \mathbf{r}_{uv}^{(1)} + \lambda_1' \mathbf{r}_v^{(4)} + \kappa_1' \mathbf{r}_v^{(1)} \\
\mathbf{r}_{uv}^{(3)} &= \lambda_2 \mathbf{r}_{uv}^{(2)} + \kappa_2 \mathbf{r}_{vv}^{(2)} + \lambda_2' \mathbf{r}_v^{(1)} + \kappa_2' \mathbf{r}_v^{(2)} \\
\mathbf{r}_{uv}^{(4)} &= \lambda_3 \mathbf{r}_{uv}^{(3)} + \lambda_3' \mathbf{r}_v^{(2)} + \kappa_3' \mathbf{r}_v^{(3)} \\
\mathbf{r}_{uv}^{(1)} &= \lambda_4 \mathbf{r}_{uv}^{(4)} + \kappa_4 \mathbf{r}_{vv}^{(4)} + \lambda_4' \mathbf{r}_v^{(3)} + \kappa_4' \mathbf{r}_v^{(4)}.
\end{aligned}
\tag{4.9}
$$

We can now formulate the above relations in the next matrix equation as

$$
\begin{pmatrix}
\lambda_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa_1' & 0 & 0 & \lambda_1' \\
0 & \lambda_2 & -1 & 0 & 0 & \kappa_2 & 0 & 0 & \lambda_2' & \kappa_2' & 0 & 0 \\
0 & 0 & \lambda_3 & -1 & 0 & 0 & 0 & 0 & 0 & \lambda_3' & \kappa_3' & 0 \\
-1 & 0 & 0 & \lambda_4 & 0 & 0 & 0 & \kappa_4 & 0 & 0 & \lambda_4' & \kappa_4' \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & \lambda_1^2 & \nu_1 & 0 & 0 & \mu_1 \\
0 & 2\lambda_2\kappa_2 & 0 & 0 & \lambda_2^2 & \kappa_2^2 & -1 & 0 & \mu_2 & \nu_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_3^2 & 0 & -1 & 0 & \mu_3 & \nu_3 & 0 \\
0 & 0 & 0 & 2\lambda_4\kappa_4 & -1 & 0 & \lambda_4^2 & \kappa_4^2 & 0 & 0 & \mu_4 & \nu_4
\end{pmatrix}
\begin{pmatrix} \mathbf{r}_{uv} \\ \mathbf{r}_{vv} \\ \mathbf{r}_v \end{pmatrix} = 0.
$$

Multiplying the above matrix equation with the the invertible matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\lambda_4\lambda_3\lambda_2 & \lambda_3\lambda_2 & \lambda_2 & 1 & 0 & 0 & \kappa_4 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \lambda_1^2 & 0 \\
0 & -2\kappa_2 & 2\lambda_2\kappa_4 & 0 & 0 & 1 & \lambda_2^2\kappa_4^2 & \lambda_2^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

we get simplified matrices as follows

$$
A \begin{pmatrix} \mathbf{r}_{uv} \\ \mathbf{r}_{vv} \end{pmatrix} + B\mathbf{r}_v = 0, \tag{4.10}
$$

where

$$A = \begin{pmatrix} \lambda_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & -1 & 0 & 0 & \kappa_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^2 & 0 & -1 \\ 0 & 0 & 0 & 2\lambda_4\kappa_4 & -1 & 0 & \lambda_4^2 & \kappa_4^2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \kappa_1' & 0 & 0 & \lambda_1' \\ \lambda_2' & \kappa_2' & 0 & 0 \\ 0 & \lambda_3' & \kappa_3' & 0 \\ \frac{\kappa_1'}{\lambda_1} + \lambda_3\lambda_4\lambda_2' & \lambda_3\lambda_4\kappa_2' + \lambda_4\lambda_3' + \kappa_4\mu_3 & \lambda_4' + \lambda_4\kappa_3' + \kappa_4\nu_3 & \kappa_4' + \frac{\lambda_1'}{\lambda_1} \\ \nu_1 & \lambda_1^2\mu_3 & \lambda_1^2\nu_3 & \mu_1 \\ \mu_2 - 2\kappa_2\lambda_2' & \nu_2 - 2\kappa_2\kappa_2' + 2\lambda_2\kappa_4\lambda_3' + \lambda_2^2\kappa_4^2\mu_3 & \lambda_2^2\mu_4 + 2\lambda_2\kappa_4\kappa_3' + \lambda_2^2\kappa_4^2\nu_3 & \lambda_2^2\nu_4 \\ 0 & \mu_3 & \nu_3 & 0 \\ 0 & 0 & \mu_4 & \nu_4 \end{pmatrix}.$$

Let $\bar{B}$ be defined by restricting B to the rows four to six, i.e.,

$$\bar{B} = \begin{pmatrix} \frac{\kappa_1'}{\lambda_1} + \lambda_3\lambda_4\lambda_2' & \lambda_3\lambda_4\kappa_2' + \lambda_4\lambda_3' + \kappa_4\mu_3 & \lambda_4' + \lambda_4\kappa_3' + \kappa_4\nu_3 & \kappa_4' + \frac{\lambda_1'}{\lambda_1} \\ \nu_1 & \lambda_1^2\mu_3 & \lambda_1^2\nu_3 & \mu_1 \\ \mu_2 - 2\kappa_2\lambda_2' & \nu_2 - 2\kappa_2\kappa_2' + 2\lambda_2\kappa_4\lambda_3' + \lambda_2^2\kappa_4^2\mu_3 & \lambda_2^2\mu_4 + 2\lambda_2\kappa_4\kappa_3' + \lambda_2^2\kappa_4^2\nu_3 & \lambda_2^2\nu_4 \end{pmatrix}.$$

By writing $\mathbf{r}_v$ in terms of the independent vectors $\mathbf{r}_u^{(1)}$ and $\mathbf{r}_v^{(1)}$ as in (4.5), it follows from (4.10) that

$$\bar{B} \begin{pmatrix} 0 & 1 \\ \lambda_1 & 0 \\ \lambda_1\kappa_2 & \lambda_2 \\ 1 & 0 \end{pmatrix} = 0.$$

Explicitly this means

$$
\begin{aligned}
&\lambda_1(\lambda_3\lambda_4\kappa_2' + \lambda_4\lambda_3' + \kappa_4\mu_3) + \lambda_1\kappa_2(\lambda_4' + \lambda_4\kappa_3' + \kappa_4\nu_3) + \kappa_4' + \tfrac{\lambda_1'}{\lambda_1} = 0\\
&\tfrac{\kappa_1'}{\lambda_1} + \lambda_3\lambda_4\lambda_2' + \lambda_2(\lambda_4' + \lambda_4\kappa_3' + \kappa_4\nu_3) = 0\\
&\lambda_1\lambda_1^2\mu_3 + \lambda_1\kappa_2\lambda_1^2\nu_3 + \mu_1 = 0\\
&\nu_1 + \lambda_2\lambda_1^2\nu_3 = 0\\
&\lambda_1(\nu_2 - 2\kappa_2\kappa_2' + 2\lambda_2\kappa_4\lambda_3' + \lambda_2^2\kappa_4^2\mu_3) + \lambda_1\kappa_2(\lambda_2^2\mu_4 + 2\lambda_2\kappa_4\kappa_3' + \lambda_2^2\kappa_4^2\nu_3) + \lambda_2^2\nu_4 = 0.\\
&\mu_2 - 2\kappa_2\lambda_2' + \lambda_2(\lambda_2^2\mu_4 + 2\lambda_2\kappa_4\kappa_3' + \lambda_2^2\kappa_4^2\nu_3) = 0.
\end{aligned}
$$

After multiplying the above equation system by the inverse matrix

$$
\begin{pmatrix}
0 & 0 & 0 & \kappa_2^2 & 0 & 1\\
0 & 2\lambda_4\kappa_4 & 0 & 0 & 0 & \lambda_4^3\\
2\lambda_1\kappa_4 & 0 & -\lambda_3\kappa_4^2 & 0 & -\lambda_4^2\lambda_1 & 0\\
0 & 0 & 0 & 0 & -\lambda_3^2 & \lambda_3\lambda_4\kappa_2\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

we finally arrive with the next result. We have

Corollary 4.6 *Let S be a G^1-continuous 4-patch surface, composed of the $C^2_{\#}$-patches $\mathbf{r}^{(i)}$ for $i = 1, \ldots, 4$, which meet at the vertex point V. Necessary and sufficient conditions for the surface S to be G^2 are that there exist continuously differentiable functions λ_i, κ_i and continuous functions μ_i, ν_i for $i = 1, \ldots, 4$, fulfilling the relations (4.2), and the coefficients $\lambda_i = \lambda_i(0)$, $\kappa_i = \kappa_i(0)$, $\mu_i = \mu_i(0)$, $\nu_i = \nu_i(0)$, $\lambda_i' = \lambda_i'(0)$ and $\kappa_i' = \kappa_i'(0)$ for $i = 1, \ldots, 4$, with $\kappa_1 = \kappa_1(0) = 0$, $\kappa_2 = \kappa_2(0) \neq 0$, satisfy the equations*

$$
\begin{aligned}
&2\kappa_2\lambda_2' + 2\lambda_1\lambda_2\kappa_2\kappa_3' - \mu_2 - \lambda_2^3\mu_4 - \lambda_2^3\kappa_4^2\nu_3 = 0\\
&2\kappa_4\lambda_4' + 2\lambda_3\lambda_4\kappa_4\kappa_1' - \mu_4 - \lambda_4^3\mu_2 - \lambda_4^3\kappa_2^2\nu_1 = 0\\
&2\kappa_4\lambda_1' + 2\lambda_1^2\kappa_2\kappa_4\lambda_4' + 2\lambda_1\kappa_4\kappa_4' - \lambda_3\kappa_4^2\mu_1 - \lambda_1^2\kappa_2\mu_4 - \lambda_4^2\lambda_1^2\nu_2 - \lambda_1\nu_4 = 0\\
&2\kappa_2\lambda_3' + 2\lambda_3^2\kappa_4\kappa_2\lambda_2' + 2\lambda_3\kappa_2\kappa_2' - \lambda_1\kappa_2^2\mu_3 - \lambda_3^2\kappa_4\mu_2 - \lambda_2^2\lambda_3^2\nu_4 - \lambda_3\nu_2 = 0\\
&\mu_1 + \lambda_1^3\mu_3 + \lambda_1^3\kappa_2\nu_3 = 0\\
&\nu_1 + \lambda_1^2\lambda_2\nu_3 = 0.
\end{aligned}
$$

4.2.2 The second subcase

Just as in the previous subsection we start with (4.8) and (4.9) further restricted by $\kappa_2 = 0$. We end up, as above, in the following equation system

$$\begin{aligned}
&\lambda_1(\lambda_3\lambda_4\kappa_2' + \lambda_4\lambda_3') + \kappa_4' + \tfrac{\lambda_1'}{\lambda_1} = 0\\
&\tfrac{\kappa_1'}{\lambda_1} + \lambda_3\lambda_4\lambda_2' + \lambda_2(\lambda_4' + \lambda_4\kappa_3') = 0\\
&\lambda_1\lambda_1^2\mu_3 + \mu_1 = 0\\
&\nu_1 + \lambda_2\lambda_1^2\nu_3 = 0\\
&\lambda_1\nu_2 + \lambda_2^2\nu_4 = 0\\
&\mu_2 + \lambda_2\lambda_2^2\mu_4 = 0.
\end{aligned}$$

Thus, we have

Corollary 4.7 *Let S be a G^1-continuous 4-patch surface, composed of the $C^2_{\#}$-patches $\mathbf{r}^{(i)}$ for $i = 1,\ldots,4$, which meet at the vertex point V. Necessary and sufficient conditions for the surface S to be G^2 are that there exist continuously differentiable functions λ_i, κ_i and continuous functions μ_i, ν_i for $i = 1,\ldots,4$, fulfilling the relations (4.2), and the coefficients $\lambda_i = \lambda_i(0)$, $\kappa_i = \kappa_i(0)$, $\mu_i = \mu_i(0)$, $\nu_i = \nu_i(0)$, $\lambda_i' = \lambda_i'(0)$ and $\kappa_i' = \kappa_i'(0)$ for $i = 1,\ldots,4$, with $\kappa_1 = \kappa_2 = 0$, satisfy the equations*

$$\begin{aligned}
&\lambda_3\lambda_4\lambda_2' + \lambda_2\lambda_4' + \lambda_3\kappa_1' + \kappa_3' = 0\\
&\lambda_4\lambda_1\lambda_3' + \lambda_3\lambda_1' + \lambda_4\kappa_2' + \kappa_4' = 0\\
&\mu_1 + \lambda_1^3\mu_3 = 0\\
&\mu_2 + \lambda_2^3\mu_4 = 0\\
&\nu_1 + \lambda_1^2\lambda_2\nu_3 = 0\\
&\nu_2 + \lambda_2^2\lambda_3\nu_4 = 0.
\end{aligned}$$

5 A 5-patch Surface

In this section we consider a 5-patch surface. As before, we want to study which compatibility conditions have to be satisfied in order for the surface to be tangent plane continuous and curvature continuous respectively. We start to consider the case of geometric continuity of order one.

5.1 Tangent plane continuity – $\mathbf{G}^1$

In the case of a 5-patch surface the relations (2.3) are

$$\begin{array}{rcl} \mathbf{r}_v^{(2)}(t,0) & = & \lambda_1(t)\mathbf{r}_u^{(1)}(0,t) + \kappa_1(t)\mathbf{r}_v^{(1)}(0,t) \\ \mathbf{r}_v^{(3)}(t,0) & = & \lambda_2(t)\mathbf{r}_u^{(2)}(0,t) + \kappa_2(t)\mathbf{r}_v^{(2)}(0,t) \\ \mathbf{r}_v^{(4)}(t,0) & = & \lambda_3(t)\mathbf{r}_u^{(3)}(0,t) + \kappa_3(t)\mathbf{r}_v^{(3)}(0,t) \\ \mathbf{r}_v^{(5)}(t,0) & = & \lambda_4(t)\mathbf{r}_u^{(4)}(0,t) + \kappa_4(t)\mathbf{r}_v^{(4)}(0,t) \\ \mathbf{r}_v^{(1)}(t,0) & = & \lambda_5(t)\mathbf{r}_u^{(5)}(0,t) + \kappa_5(t)\mathbf{r}_v^{(5)}(0,t) \end{array} \tag{5.1}$$

for $t \in [0,1]$. The restriction of this equation system to the vertex $t = 0$ gives us the analogy to equation (2.5). That is

$$\begin{pmatrix} \lambda_2 & \kappa_2 & -1 & 0 & 0 \\ 0 & \lambda_3 & \kappa_3 & -1 & 0 \\ 0 & 0 & \lambda_4 & \kappa_4 & -1 \\ -1 & 0 & 0 & \lambda_5 & \kappa_5 \\ \kappa_1 & -1 & 0 & 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ \lambda_1\kappa_2 & \kappa_1\kappa_2 + \lambda_2 \\ \lambda_1(\lambda_3 + \kappa_2\kappa_3) & \kappa_1(\lambda_3 + \kappa_2\kappa_3) + \kappa_3\lambda_2 \\ 1 & 0 \end{pmatrix} = 0.$$

Equivalently, we have

$$\begin{aligned} &\lambda_4\lambda_1\kappa_2 + \kappa_4\lambda_1(\lambda_3 + \kappa_2\kappa_3) - 1 = 0 \\ &\lambda_4(\kappa_1\kappa_2 + \lambda_2) + \kappa_4(\kappa_1(\lambda_3 + \kappa_2\kappa_3) + \kappa_3\lambda_2) = 0 \\ &\lambda_5\lambda_1(\lambda_3 + \kappa_2\kappa_3) + \kappa_5 = 0 \\ &-1 + \lambda_5(\kappa_1(\lambda_3 + \kappa_2\kappa_3) + \kappa_3\lambda_2) = 0. \end{aligned}$$

After a minor rewriting of the above equation system, or more precisely multiply the equation system with the invertible matrix

$$\begin{pmatrix} \lambda_5 & 0 & \kappa_4 & 0 \\ 0 & 0 & -1 & \lambda_1 \\ 0 & \lambda_5 & 0 & \kappa_4 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

we can now formulate our first result in this subsection.

Theorem 5.1 *Let S be a continuous 5-patch surface, composed of the $C^1_\#$-patches $\mathbf{r}^{(i)}$, for $i = 1, \ldots, 5$, which meet at the vertex point V. Necessary and sufficient conditions for the surface S to be G^1 are that there exist continuous functions λ_i and κ_i, $i = 1, \ldots, 5$, fulfilling* *the relations (5.1) and the coefficients* $\lambda_i = \lambda_i(0)$ *and* $\kappa_i = \kappa_i(0)$, $i = 1, \ldots, 5$, *satisfy the equations*

$$\lambda_4\lambda_5\lambda_1\kappa_2 - \kappa_4\kappa_5 = \lambda_5$$
$$\lambda_5\lambda_1\lambda_2\kappa_3 - \kappa_5\kappa_1 = \lambda_1$$

and

$$\lambda_4\lambda_5(\kappa_1\kappa_2 + \lambda_2) + \kappa_4 = 0$$
$$\lambda_5\lambda_1(\kappa_2\kappa_3 + \lambda_3) + \kappa_5 = 0.$$

From the above Theorem the next Corollary easily follows. This result meets the assumption in Lemma 2.2. We have

Corollary 5.2 *Suppose that the relations in Theorem 5.1 hold, then*

$$\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = -1.$$

Proof

$$\begin{aligned} 0 &= \lambda_4\lambda_5\lambda_1\kappa_2 - \kappa_4\kappa_5 - \lambda_5 = \lambda_4\lambda_5\lambda_1\kappa_2 + \kappa_5\lambda_4\lambda_5(\lambda_2 + \kappa_1\kappa_2) - \lambda_5 \\ &= \lambda_4\lambda_5\kappa_2(\lambda_1 + \kappa_5\kappa_1) + \kappa_5\lambda_4\lambda_5\lambda_2 - \lambda_5 = \lambda_4\lambda_5\kappa_2\lambda_5\lambda_1\lambda_2\kappa_3 + \kappa_5\lambda_4\lambda_5\lambda_2 - \lambda_5 \\ &= \lambda_5(\lambda_4\lambda_2(\lambda_5\lambda_1\kappa_2\kappa_3 + \kappa_5) - 1) = \lambda_5(\lambda_4\lambda_2(-\lambda_5\lambda_1\lambda_3) - 1) \\ &= -\lambda_5(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 + 1). \end{aligned}$$

□

Because of symmetry the next two Corollaries are obvious, but anyhow we prove the results here. Thus

Corollary 5.3 *If the compatibility conditions in Theorem 5.1 hold, then*

$$\lambda_1\lambda_2\lambda_3\kappa_4 - \kappa_1\kappa_2 = \lambda_2$$
$$\lambda_2\lambda_3\lambda_4\kappa_5 - \kappa_2\kappa_3 = \lambda_3$$

and

$$\lambda_2\lambda_3(\lambda_5 + \kappa_4\kappa_5) + \kappa_2 = 0$$
$$\lambda_3\lambda_4(\lambda_1 + \kappa_5\kappa_1) + \kappa_3 = 0.$$

Proof The above identities follows immediately from the compatibility conditions. We have

$$\begin{aligned} \lambda_4\lambda_5(\lambda_1\lambda_2\lambda_3\kappa_4 - \kappa_1\kappa_2 - \lambda_2) &= -\kappa_4 - \lambda_4\lambda_5(\kappa_1\kappa_2 + \lambda_2) = 0 \\ \lambda_5\lambda_1(\lambda_2\lambda_3\lambda_4\kappa_5 - \kappa_2\kappa_3 - \lambda_3) &= -\kappa_5 - \lambda_5\lambda_1(\kappa_2\kappa_3 + \lambda_3) = 0 \\ \lambda_4\lambda_5\lambda_1(\lambda_2\lambda_3(\lambda_5 + \kappa_4\kappa_5) + \kappa_2) &= -\lambda_5 - \kappa_4\kappa_5 + \lambda_4\lambda_5\lambda_1\kappa_2 = 0 \\ \lambda_3\lambda_4\lambda_5(\lambda_1\lambda_2(\lambda_4 + \kappa_3\kappa_4) + \kappa_1) &= -\lambda_4 - \kappa_3\kappa_4 + \lambda_3\lambda_4\lambda_5\kappa_1 = 0. \end{aligned}$$

□

Corollary 5.4 *If the above compatibility conditions hold also the last symmetry conditions hold, i.e.,*

$$\lambda_3\lambda_4\lambda_5\kappa_1 - \kappa_3\kappa_4 = \lambda_4$$
$$\lambda_1\lambda_2(\lambda_4 + \kappa_3\kappa_4) + \kappa_1 = 0.$$

Proof Let us consider the first equation. Using an identity in Corollary 5.3 we first get

$$\begin{aligned}\lambda_3\lambda_4\lambda_5\kappa_1 - \kappa_3\kappa_4 - \lambda_4 &= \lambda_3\lambda_4\lambda_5\kappa_1 + \kappa_3\lambda_4\lambda_5(\kappa_1\kappa_2 + \lambda_2) - \lambda_4 \\ &= \lambda_3\lambda_4\lambda_5\kappa_1 + \kappa_3\lambda_4\lambda_5\lambda_2 + \lambda_4\lambda_5\kappa_1\kappa_2\kappa_3 - \lambda_4 \\ &= \lambda_3\lambda_4\lambda_5\kappa_1 + \kappa_3\lambda_4\lambda_5\lambda_2 + \lambda_4\lambda_5\kappa_1(\lambda_2\lambda_3\lambda_4\kappa_5 - \lambda_3) - \lambda_4 \\ &= \kappa_3\lambda_4\lambda_5\lambda_2 + \lambda_4\lambda_5\kappa_1\lambda_2\lambda_3\lambda_4\kappa_5 - \lambda_4 \\ &= -\frac{1}{\lambda_1\lambda_3}(\kappa_3 + \lambda_3\lambda_4(\kappa_5\kappa_1 + \lambda_1)) = 0.\end{aligned}$$

From this first equality the other also follows immediately. We have

$$0 = \lambda_1\lambda_2(\lambda_3\lambda_4\lambda_5\kappa_1 - \kappa_3\kappa_4 - \lambda_4) = -\kappa_1 - \lambda_1\lambda_2(\kappa_3\kappa_4 + \lambda_4),$$

which completes the proof. □

5.2 Curvature continuity – G^2

In this subsection we will study what conditions that are needed to be satisfied in order for a 5-patch surface to be regular of order two. Besides fulfilling

$$\begin{aligned}\mathbf{r}^{(1)}_{vv}(t,0) &= \lambda_5^2\mathbf{r}^{(5)}_{uu}(0,\cdot) + 2\lambda_5\kappa_5\mathbf{r}^{(5)}_{uv}(0,\cdot) + \kappa_5^2\mathbf{r}^{(5)}_{vv}(0,\cdot) + \mu_5\mathbf{r}^{(5)}_{u}(0,\cdot) + \nu_5\mathbf{r}^{(5)}_{v}(0,\cdot)|_t \\ \mathbf{r}^{(2)}_{vv}(t,0) &= \lambda_1^2\mathbf{r}^{(1)}_{uu}(0,\cdot) + 2\lambda_1\kappa_1\mathbf{r}^{(1)}_{uv}(0,\cdot) + \kappa_1^2\mathbf{r}^{(1)}_{vv}(0,\cdot) + \mu_1\mathbf{r}^{(1)}_{u}(0,\cdot) + \nu_1\mathbf{r}^{(1)}_{v}(0,\cdot)|_t \\ \mathbf{r}^{(3)}_{vv}(t,0) &= \lambda_2^2\mathbf{r}^{(2)}_{uu}(0,\cdot) + 2\lambda_2\kappa_2\mathbf{r}^{(2)}_{uv}(0,\cdot) + \kappa_2^2\mathbf{r}^{(2)}_{vv}(0,\cdot) + \mu_2\mathbf{r}^{(2)}_{u}(0,\cdot) + \nu_2\mathbf{r}^{(2)}_{v}(0,\cdot)|_t \\ \mathbf{r}^{(4)}_{vv}(t,0) &= \lambda_3^2\mathbf{r}^{(3)}_{uu}(0,\cdot) + 2\lambda_3\kappa_3\mathbf{r}^{(3)}_{uv}(0,\cdot) + \kappa_3^2\mathbf{r}^{(3)}_{vv}(0,\cdot) + \mu_3\mathbf{r}^{(3)}_{u}(0,\cdot) + \nu_3\mathbf{r}^{(3)}_{v}(0,\cdot)|_t \\ \mathbf{r}^{(5)}_{vv}(t,0) &= \lambda_4^2\mathbf{r}^{(4)}_{uu}(0,\cdot) + 2\lambda_4\kappa_4\mathbf{r}^{(4)}_{uv}(0,\cdot) + \kappa_4^2\mathbf{r}^{(4)}_{vv}(0,\cdot) + \mu_4\mathbf{r}^{(4)}_{u}(0,\cdot) + \nu_4\mathbf{r}^{(4)}_{v}(0,\cdot)|_t\end{aligned} \tag{5.2}$$

for $t \in [0,1]$, the equation (2.12) must be satisfied for $n = 5$. In order to solve this equation we start by considering $\Gamma - H\,\Lambda^{-1}\,\kappa$ on page 8. We have

$$\Gamma - H\,\Lambda^{-1}\,\kappa$$

$$= \begin{pmatrix} 1-\lambda_2\ldots\lambda_5\kappa_5\kappa_1 & -\lambda_3\lambda_4\lambda_5\kappa_5\kappa_2 & -\lambda_4\lambda_5\kappa_5\kappa_3 & -\lambda_5^2-\lambda_5\kappa_5\kappa_4 & 0 \\ 0 & 1-\lambda_3\ldots\lambda_1\kappa_1\kappa_2 & -\lambda_4\lambda_5\lambda_1\kappa_1\kappa_3 & -\lambda_5\lambda_1\kappa_1\kappa_4 & -\lambda_1^2-\lambda_1\kappa_1\kappa_5 \\ -\lambda_2^2-\lambda_2\kappa_2\kappa_1 & 0 & 1-\lambda_4\ldots\lambda_2\kappa_2\kappa_3 & -\lambda_5\lambda_1\lambda_2\kappa_2\kappa_4 & -\lambda_1\lambda_2\kappa_2\kappa_5 \\ -\lambda_2\lambda_3\kappa_3\kappa_1 & -\lambda_3^2-\lambda_3\kappa_3\kappa_2 & 0 & 1-\lambda_5\lambda_1\lambda_2\lambda_3\kappa_3\kappa_4 & -\lambda_1\lambda_2\lambda_3\kappa_3\kappa_5 \\ -\lambda_2\lambda_3\lambda_4\kappa_4\kappa_1 & -\lambda_3\lambda_4\kappa_4\kappa_2 & -\lambda_4^2-\lambda_4\kappa_4\kappa_3 & 0 & 1-\lambda_1\ldots\lambda_4\kappa_4\kappa_5 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_5\lambda_2\kappa_3 & -\lambda_3\lambda_4\lambda_5\kappa_5\kappa_2 & -\lambda_4\lambda_5\kappa_5\kappa_3 & \frac{\lambda_5\kappa_2}{\lambda_2\lambda_3} & 0 \\ 0 & \lambda_1\lambda_3\kappa_4 & -\lambda_4\lambda_5\lambda_1\kappa_1\kappa_3 & -\lambda_5\lambda_1\kappa_1\kappa_4 & \frac{\lambda_1\kappa_3}{\lambda_3\lambda_4} \\ \frac{\lambda_2\kappa_4}{\lambda_4\lambda_5} & 0 & \lambda_2\lambda_4\kappa_5 & -\lambda_5\lambda_1\lambda_2\kappa_2\kappa_4 & -\lambda_1\lambda_2\kappa_2\kappa_5 \\ -\lambda_2\lambda_3\kappa_3\kappa_1 & \frac{\lambda_3\kappa_5}{\lambda_5\lambda_1} & 0 & \lambda_3\lambda_5\kappa_1 & -\lambda_1\lambda_2\lambda_3\kappa_3\kappa_5 \\ -\lambda_2\lambda_3\lambda_4\kappa_4\kappa_1 & -\lambda_3\lambda_4\kappa_4\kappa_2 & \frac{\lambda_4\kappa_1}{\lambda_1\lambda_2} & 0 & \lambda_4\lambda_1\kappa_2 \end{pmatrix}$$

$$= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix},$$

where f_i, $i = 1, 2, \ldots, 5$, are defined by the above equality. Supposing that $\kappa_3\kappa_4\kappa_5 \neq 0$, it is easily seen that the above matrix equality can be reduced as follows

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 - \frac{\kappa_4}{\lambda_4\lambda_5^2\kappa_3}\,f_1 + \frac{\lambda_2\lambda_3\lambda_4\kappa_5\kappa_2}{\kappa_3}\,f_2 \\ f_4 - \frac{\kappa_5}{\lambda_5\lambda_1^2\kappa_4}\,f_2 + \frac{\lambda_3\lambda_4\lambda_5\kappa_1\kappa_3}{\kappa_4}\,f_3 \\ f_5 - \frac{\kappa_1}{\lambda_1\lambda_2^2\kappa_5}\,f_3 + \frac{\lambda_4\lambda_5\lambda_1\kappa_2\kappa_4}{\kappa_5}\,f_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.3}$$

Next we have, see page 9,

$$F + H\,\Lambda^{-1}\,G$$

$$= \left(\begin{matrix} \lambda_3\lambda_4\lambda_5\kappa_5(\lambda_2\kappa_1' + \lambda_2') & \lambda_4\lambda_5\kappa_5(\lambda_3\kappa_2' + \lambda_3') & \lambda_5\kappa_5(\lambda_4\kappa_3' + \lambda_4') \\ \lambda_3\ldots\lambda_1\kappa_1\lambda_2' - \kappa_1\kappa_1' + \nu_1 & \lambda_4\lambda_5\lambda_1\kappa_1(\lambda_3\kappa_2' + \lambda_3') & \lambda_5\lambda_1\kappa_1(\lambda_4\kappa_3' + \lambda_4') \\ \kappa_2(\lambda_2\kappa_1' - \lambda_2') + \mu_2 & \lambda_4\ldots\lambda_2\kappa_2\lambda_3' - \kappa_2\kappa_2' + \nu_2 & \lambda_5\lambda_1\lambda_2\kappa_2(\lambda_4\kappa_3' + \lambda_4') \\ \lambda_3\kappa_3(\lambda_2\kappa_1' + \lambda_2') & \kappa_3(\lambda_3\kappa_2' - \lambda_3') + \mu_3 & \lambda_5\ldots\lambda_3\kappa_3\lambda_4' - \kappa_3\kappa_3' + \nu_3 \\ \lambda_3\lambda_4\kappa_4(\lambda_2\kappa_1' + \lambda_2') & \lambda_4\kappa_4(\lambda_3\kappa_2' + \lambda_3') & \kappa_4(\lambda_4\kappa_3' - \lambda_4') + \mu_4 \end{matrix}\right.$$

$$\left.\begin{matrix} \kappa_5(\lambda_5\kappa_4' - \lambda_5') + \mu_5 & \lambda_2\ldots\lambda_5\kappa_5\lambda_1' - \kappa_5\kappa_5' + \nu_5 \\ \lambda_1\kappa_1(\lambda_5\kappa_4' + \lambda_5') & \kappa_1(\lambda_1\kappa_5' - \lambda_1') + \mu_1 \\ \lambda_1\lambda_2\kappa_2(\lambda_5\kappa_4' + \lambda_5') & \lambda_2\kappa_2(\lambda_1\kappa_5' + \lambda_1') \\ \lambda_1\lambda_2\lambda_3\kappa_3(\lambda_5\kappa_4' + \lambda_5') & \lambda_2\lambda_3\kappa_3(\lambda_1\kappa_5' + \lambda_1') \\ \lambda_1\ldots\lambda_4\kappa_4\lambda_5' - \kappa_4\kappa_4' + \nu_4 & \lambda_2\lambda_3\lambda_4\kappa_4(\lambda_1\kappa_5' + \lambda_1') \end{matrix}\right) = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{pmatrix}.$$

Here we have defined h_i, $i = 1, 2, \ldots, 5$, by the above equality. Because of the structure of the matrix in (5.3), we restrict ourselves to consider only the three lower rows in that matrix equation. Thus, we have reduced our interest to the matrix

$$\begin{pmatrix} h_3 - \frac{\kappa_4}{\lambda_4\lambda_5^2\kappa_3}\,h_1 + \frac{\lambda_2\lambda_3\lambda_4\kappa_5\kappa_2}{\kappa_3}\,h_2 \\ h_4 - \frac{\kappa_5}{\lambda_5\lambda_1^2\kappa_4}\,h_2 + \frac{\lambda_3\lambda_4\lambda_5\kappa_1\kappa_3}{\kappa_4}\,h_3 \\ h_5 - \frac{\kappa_1}{\lambda_1\lambda_2^2\kappa_5}\,h_3 + \frac{\lambda_4\lambda_5\lambda_1\kappa_2\kappa_4}{\kappa_5}\,h_4 \end{pmatrix} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix}, \qquad (5.4)$$

where we at the same time have defined the column vectors q_i, for $i = 1, 2, \ldots, 5$.

Since our goal is to solve equation (2.12), we see from above that this equation is now reduced to the next equation

$$\begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ \kappa_2\lambda_1 & -\frac{\kappa_4}{\lambda_4\lambda_5} \\ -\frac{\kappa_5}{\lambda_5} & \frac{1}{\lambda_5} \\ 1 & 0 \end{pmatrix} = 0. \qquad (5.5)$$

The column vectors q_i, $i = 1, 2, \ldots, 5$, are defined in equation (5.4). A direct calculation gives

$$q_1 = \begin{pmatrix} \kappa_2(\lambda_2\kappa_1' - \lambda_2') + \mu_2 - \frac{\kappa_4}{\lambda_4\lambda_5^2\kappa_3}\lambda_3\lambda_4\lambda_5\kappa_5(\lambda_2\kappa_1' + \lambda_2') + \frac{\lambda_2\lambda_3\lambda_4\kappa_5\kappa_2}{\kappa_3}(-\frac{\kappa_1}{\lambda_2}(\lambda_2' + \lambda_2\kappa_1') + \nu_1) \\ \lambda_3\kappa_3(\lambda_2\kappa_1' + \lambda_2') - \frac{\kappa_5}{\lambda_5\lambda_1^2\kappa_4}(-\frac{\kappa_1}{\lambda_2}(\lambda_2' + \lambda_2\kappa_1') + \nu_1) + \frac{\lambda_3\lambda_4\lambda_5\kappa_1\kappa_3}{\kappa_4}(\kappa_2(\lambda_2\kappa_1' - \lambda_2') + \mu_2) \\ \lambda_3\lambda_4\kappa_4(\lambda_2\kappa_1' + \lambda_2') - \frac{\kappa_1}{\lambda_1\lambda_2^2\kappa_5}(\kappa_2(\lambda_2\kappa_1' - \lambda_2') + \mu_2) + \frac{\lambda_4\lambda_5\lambda_1\kappa_2\kappa_4}{\kappa_5}\lambda_3\kappa_3(\lambda_2\kappa_1' + \lambda_2') \end{pmatrix}$$

$$q_2 = \begin{pmatrix} -\frac{\kappa_2}{\lambda_3}(\lambda_3' + \lambda_3\kappa_2') + \nu_2 - \frac{\kappa_4}{\lambda_4\lambda_5^2\kappa_3}\lambda_4\lambda_5\kappa_5(\lambda_3\kappa_2' + \lambda_3') + \frac{\lambda_2\lambda_3\lambda_4\kappa_2\kappa_5}{\kappa_3}\lambda_4\lambda_5\lambda_1\kappa_1(\lambda_3\kappa_2' + \lambda_3') \\ \kappa_3(\lambda_3\kappa_2' - \lambda_3') + \mu_3 - \frac{\kappa_5}{\lambda_5\lambda_1^2\kappa_4}\lambda_4\lambda_5\lambda_1\kappa_1(\lambda_3\kappa_2' + \lambda_3') + \frac{\lambda_3\lambda_4\lambda_5\kappa_1\kappa_3}{\kappa_4}(-\frac{\kappa_2}{\lambda_3}(\lambda_3' + \lambda_3\kappa_2') + \nu_2) \\ \lambda_4\kappa_4(\lambda_3\kappa_2' + \lambda_3') - \frac{\kappa_1}{\lambda_1\lambda_2^2\kappa_5}(-\frac{\kappa_2}{\lambda_3}(\lambda_3' + \lambda_3\kappa_2') + \nu_2) + \frac{\lambda_4\lambda_5\lambda_1\kappa_2\kappa_4}{\kappa_5}(\kappa_3(\lambda_3\kappa_2' - \lambda_3') + \mu_3) \end{pmatrix}$$

$$q_3 = \begin{pmatrix} \lambda_5\lambda_1\lambda_2\kappa_2(\lambda_4\kappa_3' + \lambda_4') - \frac{\kappa_4}{\lambda_4\lambda_5^2\kappa_3}\lambda_5\kappa_5(\lambda_4\kappa_3' + \lambda_4') + \frac{\lambda_2\lambda_3\lambda_4\kappa_5\kappa_2}{\kappa_3}\lambda_5\lambda_1\kappa_1(\lambda_4\kappa_3' + \lambda_4') \\ -\frac{\kappa_3}{\lambda_4}(\lambda_4' + \lambda_4\kappa_3') + \nu_3 - \frac{\kappa_5}{\lambda_5\lambda_1^2\kappa_4}\lambda_5\lambda_1\kappa_1(\lambda_4\kappa_3' + \lambda_4') + \frac{\lambda_3\lambda_4\lambda_5\kappa_1\kappa_3}{\kappa_4}\lambda_5\lambda_1\lambda_2\kappa_2(\lambda_4\kappa_3' + \lambda_4') \\ \kappa_4(\lambda_4\kappa_3' - \lambda_4') + \mu_4 - \frac{\kappa_1}{\lambda_1\lambda_2^2\kappa_5}\lambda_5\lambda_1\lambda_2\kappa_2(\lambda_4\kappa_3' + \lambda_4') + \frac{\lambda_4\lambda_5\lambda_1\kappa_2\kappa_4}{\kappa_5}(-\frac{\kappa_3}{\lambda_4}(\lambda_4' + \lambda_4\kappa_3') + \nu_3) \end{pmatrix}$$

$$q_4 = \begin{pmatrix} \lambda_1\lambda_2\kappa_2(\lambda_5\kappa_4' + \lambda_5') - \frac{\kappa_4}{\lambda_4\lambda_5^2\kappa_3}(\kappa_5(\lambda_5\kappa_4' - \lambda_5') + \mu_5) + \frac{\lambda_2\lambda_3\lambda_4\kappa_5\kappa_2}{\kappa_3}\lambda_1\kappa_1(\lambda_5\kappa_4' + \lambda_5') \\ \lambda_1\lambda_2\lambda_3\kappa_3(\lambda_5\kappa_4' + \lambda_5') - \frac{\kappa_5}{\lambda_5\lambda_1^2\kappa_4}\lambda_1\kappa_1(\lambda_5\kappa_4' + \lambda_5') + \frac{\lambda_3\lambda_4\lambda_5\kappa_1\kappa_3}{\kappa_4}\lambda_1\lambda_2\kappa_2(\lambda_5\kappa_4' + \lambda_5') \\ -\frac{\kappa_4}{\lambda_5}(\lambda_5' + \lambda_5\kappa_4') + \nu_4 - \frac{\kappa_1}{\lambda_1\lambda_2^2\kappa_5}\lambda_1\lambda_2\kappa_2(\lambda_5\kappa_4' + \lambda_5') + \frac{\lambda_4\lambda_5\lambda_1\kappa_2\kappa_4}{\kappa_5}\lambda_1\lambda_2\lambda_3\kappa_3(\lambda_5\kappa_4' + \lambda_5') \end{pmatrix}$$

$$q_5 = \begin{pmatrix} \lambda_2\kappa_2(\lambda_1\kappa_5' + \lambda_1') - \frac{\kappa_4}{\lambda_4\lambda_5^2\kappa_3}(-\frac{\kappa_5}{\lambda_1}(\lambda_1' + \lambda_1\kappa_5') + \nu_5) + \frac{\lambda_2\lambda_3\lambda_4\kappa_5\kappa_2}{\kappa_3}(\kappa_1(\lambda_1\kappa_5' - \lambda_1') + \mu_1) \\ \lambda_2\lambda_3\kappa_3(\lambda_1\kappa_5' + \lambda_1') - \frac{\kappa_5}{\lambda_5\lambda_1^2\kappa_4}(\kappa_1(\lambda_1\kappa_5' - \lambda_1') + \mu_1) + \frac{\lambda_3\lambda_4\lambda_5\kappa_1\kappa_3}{\kappa_4}\lambda_2\kappa_2(\lambda_1\kappa_5' + \lambda_1') \\ \lambda_2\lambda_3\lambda_4\kappa_4(\lambda_1\kappa_5' + \lambda_1') - \frac{\kappa_1}{\lambda_1\lambda_2^2\kappa_5}\lambda_2\kappa_2(\lambda_1\kappa_5' + \lambda_1') + \frac{\lambda_4\lambda_5\lambda_1\kappa_2\kappa_4}{\kappa_5}\lambda_2\lambda_3\kappa_3(\lambda_1\kappa_5' + \lambda_1') \end{pmatrix}.$$

We simplify equation (5.5) by multiplying it by two matrices as seen below. Since we are assuming that $\kappa_3\kappa_4\kappa_5 \neq 0$, it follows that both those matrices are invertible and we have the equivalent equation

$$\begin{pmatrix} 1 & 0 & \lambda_1\lambda_2\lambda_3 \\ 0 & 1 & \lambda_1\lambda_2\lambda_3\lambda_4 \\ \lambda_4\lambda_5 & 0 & 1 \end{pmatrix} \begin{pmatrix} \kappa_3 & 0 & 0 \\ 0 & \kappa_4 & 0 \\ 0 & 0 & \kappa_5 \end{pmatrix} \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda_1 & \kappa_1 \\ \lambda_1\kappa_2 & -\frac{\kappa_4}{\lambda_4\lambda_5} \\ -\frac{\kappa_5}{\lambda_5} & \frac{1}{\lambda_5} \\ 1 & 0 \end{pmatrix} = 0.$$

Considering the previous equation as a system of equations, we have

$$
\begin{aligned}
&-2\tfrac{\lambda_1\kappa_4\kappa_5}{\lambda_5}\lambda_3' - 2\tfrac{\kappa_4\kappa_5^2}{\lambda_4\lambda_5^3}\lambda_5' - 2\tfrac{\lambda_2\kappa_5}{\lambda_5\lambda_1}\lambda_1' - 2\tfrac{\lambda_1\kappa_4}{\lambda_2\lambda_4\lambda_5}\lambda_3\kappa_2' - 2\tfrac{\lambda_1\kappa_2\kappa_4\kappa_5}{\lambda_4\lambda_5}\lambda_4\kappa_3' + 2\tfrac{\kappa_4\kappa_5}{\lambda_4\lambda_5^2\lambda_1}\lambda_1\kappa_5' \\
&\quad +\lambda_2\lambda_3\lambda_4\kappa_5\kappa_2\mu_1 - \lambda_1^2\kappa_2\kappa_4\mu_3 + \lambda_1^2\lambda_2\lambda_3\kappa_5\kappa_2\mu_4 + \tfrac{\kappa_4\kappa_5}{\lambda_4\lambda_5^3}\mu_5 + \lambda_1(\kappa_3 - \tfrac{\lambda_3\kappa_1}{\lambda_2})\nu_2 \\
&\quad -\lambda_1^2\kappa_2^2\kappa_4\nu_3 - \tfrac{\lambda_1\lambda_2\lambda_3\kappa_5^2}{\lambda_5}\nu_4 - \tfrac{\kappa_4}{\lambda_4\lambda_5^2}\nu_5 = 0 \\
&2\lambda_1\lambda_3^2\kappa_4\lambda_2' - 2\tfrac{\kappa_4\kappa_5\kappa_1}{\lambda_5}\lambda_3' + 2\tfrac{\kappa_4\kappa_5}{\lambda_4\lambda_5^3}\lambda_5' + 2\lambda_2\lambda_3\lambda_4\kappa_5\lambda_2\kappa_1' - 2\tfrac{\kappa_1\kappa_4}{\lambda_2\lambda_4\lambda_5}\lambda_3\kappa_2' + 2\tfrac{\kappa_4^2\kappa_5}{\lambda_4^2\lambda_5^2}\lambda_4\kappa_3' \\
&\quad +(\kappa_3 - \tfrac{\lambda_3\kappa_1}{\lambda_2})\mu_2 - \lambda_1\kappa_4\kappa_1\kappa_2\mu_3 - \tfrac{\lambda_1\lambda_2\lambda_3\kappa_4\kappa_5}{\lambda_4\lambda_5}\mu_4 - \tfrac{\kappa_4}{\lambda_4\lambda_5^3}\mu_5 + \lambda_2\lambda_3\lambda_4\kappa_5\kappa_2\nu_1 \\
&\quad +\kappa_1(\kappa_3 - \tfrac{\lambda_3\kappa_1}{\lambda_2})\nu_2 + \tfrac{\lambda_1\kappa_2\kappa_4^2}{\lambda_4\lambda_5}\nu_3 + \tfrac{\lambda_1\lambda_2\lambda_3\kappa_5}{\lambda_5}\nu_4 = 0 \\
&-2\tfrac{\lambda_1\lambda_4\kappa_4\kappa_5}{\lambda_5}\lambda_3' + 2\tfrac{\kappa_5\kappa_1}{\lambda_5\lambda_1^2}\lambda_1' - 2\lambda_1\lambda_3\lambda_4\kappa_4\kappa_5\kappa_1\lambda_3\kappa_2' - 2\tfrac{\lambda_1\kappa_2\kappa_4\kappa_5}{\lambda_5}\lambda_4\kappa_3' - \tfrac{\kappa_5}{\lambda_5\lambda_1^2}\mu_1 \\
&\quad -\tfrac{\lambda_1\kappa_4^2\kappa_5}{\lambda_5}\mu_3 + \lambda_1^2\lambda_2\lambda_3\lambda_4\kappa_5\kappa_2\mu_4 + \tfrac{\lambda_3\lambda_4\kappa_5\kappa_1^2}{\lambda_2}\nu_2 - \tfrac{\lambda_1\kappa_4^2\kappa_5\kappa_2}{\lambda_5}\nu_3 - \tfrac{\lambda_1\lambda_2\lambda_3\lambda_4\kappa_5^2}{\lambda_5}\nu_4 = 0 \\
&2\tfrac{\lambda_3\kappa_4\kappa_5\kappa_1}{\lambda_5\lambda_1\lambda_2}\lambda_2' - 2\tfrac{\lambda_4\kappa_4\kappa_5\kappa_1}{\lambda_5}\lambda_3' + 2\tfrac{\kappa_5\kappa_1}{\lambda_5\lambda_1^2\lambda_2}\lambda_2\kappa_1' - 2\lambda_3\lambda_4\kappa_4\kappa_5\kappa_1^2\lambda_3\kappa_2' + 2\tfrac{\kappa_4^2\kappa_5}{\lambda_4\lambda_5^2}\lambda_4\kappa_3' \\
&\quad +\tfrac{\lambda_3\lambda_4\kappa_5\kappa_1^2}{\lambda_1\lambda_2}\mu_2 - \tfrac{\kappa_4^2\kappa_5\kappa_1}{\lambda_5}\mu_3 - \tfrac{\lambda_1\lambda_2\lambda_3\kappa_4\kappa_5}{\lambda_5}\mu_4 - \tfrac{\kappa_5}{\lambda_5\lambda_1^2}\nu_1 + \tfrac{\lambda_3\lambda_4\kappa_5\kappa_1^3}{\lambda_1\lambda_2}\nu_2 + \tfrac{\kappa_4^3\kappa_5}{\lambda_4\lambda_5^2}\nu_3 \\
&\quad +\tfrac{\lambda_1\lambda_2\lambda_3\lambda_4\kappa_5}{\lambda_5}\nu_4 = 0 \\
&2\lambda_4^2\lambda_5\lambda_1^2\kappa_2\lambda_3' + 2\lambda_5\lambda_1\kappa_2\lambda_4' - 2\tfrac{\lambda_4\lambda_1\kappa_5\kappa_2}{\lambda_5}\lambda_5' - 2\tfrac{\kappa_2\kappa_3}{\lambda_1\lambda_3}\lambda_1' + 2\tfrac{\kappa_1\kappa_2}{\lambda_2^2\lambda_3}\lambda_3\kappa_2' + 2\lambda_4\lambda_5\lambda_1^2\kappa_2^2\lambda_4\kappa_3' \\
&\quad -2\tfrac{\kappa_5}{\lambda_5}\lambda_5\kappa_4' + 2\lambda_4\kappa_2\lambda_1\kappa_5' + \lambda_2\lambda_3\lambda_4^2\lambda_5\kappa_5\kappa_2\mu_1 + \lambda_4\lambda_5\lambda_1^2\kappa_2\kappa_4\mu_3 + \lambda_1\kappa_2\kappa_5\mu_4 \\
&\quad +\tfrac{\kappa_4\kappa_5}{\lambda_5^2}\mu_5 + \lambda_1(\lambda_4\lambda_5\kappa_3 - \tfrac{\kappa_1}{\lambda_1\lambda_2^2})\nu_2 + \lambda_4\lambda_5\lambda_1^2\kappa_2^2\kappa_4\nu_3 - \tfrac{\kappa_5^2}{\lambda_5}\nu_4 - \tfrac{\kappa_4}{\lambda_5}\nu_5 = 0 \\
&2\tfrac{\kappa_1\kappa_2}{\lambda_1\lambda_2^2}\lambda_2' + 2\lambda_4^2\lambda_5\lambda_1\kappa_1\kappa_2\lambda_3' - 2\tfrac{\kappa_4}{\lambda_4}\lambda_4' + 2\tfrac{\lambda_4\lambda_1\kappa_2}{\lambda_5}\lambda_5' - 2\tfrac{\kappa_2\kappa_3}{\lambda_1\lambda_2\lambda_3}\lambda_2\kappa_1' + 2\tfrac{\kappa_1^2\kappa_2}{\lambda_1\lambda_2^2\lambda_3}\lambda_3\kappa_2' \\
&\quad -2\lambda_1\kappa_2\kappa_4\lambda_4\kappa_3' + 2\tfrac{1}{\lambda_5}\lambda_5\kappa_4' + (\lambda_4\lambda_5\kappa_3 - \tfrac{\kappa_1}{\lambda_1\lambda_2^2})\mu_2 + \lambda_4\lambda_5\lambda_1\kappa_1\kappa_2\kappa_4\mu_3 - \tfrac{\kappa_4\kappa_5}{\lambda_4\lambda_5}\mu_4 \\
&\quad -\tfrac{\kappa_4}{\lambda_5^2}\mu_5 + \lambda_2\lambda_3\lambda_4^2\lambda_5\kappa_5\kappa_2\nu_1 + \kappa_1(\lambda_4\lambda_5\kappa_3 - \tfrac{\kappa_1}{\lambda_1\lambda_2^2})\nu_2 - \lambda_1\kappa_2\kappa_4^2\nu_3 + \tfrac{\kappa_5}{\lambda_5}\nu_4 = 0.
\end{aligned}
$$

In order to reduce the equations in such a way that we can see a pattern, we multiply the above equation system by still another invertible matrix. That matrix is

$$
\begin{pmatrix}
0 & 0 & 0 & \frac{\lambda_4\lambda_5^2\kappa_2}{\lambda_2\lambda_3\kappa_5} & 0 & -\lambda_4\lambda_5\kappa_4 \\
-\frac{\lambda_4\lambda_5^3\lambda_1}{\kappa_4} & 0 & \frac{\lambda_5^3\lambda_1}{\kappa_4} & 0 & 0 & 0 \\
0 & 0 & \frac{\lambda_5\lambda_1^2\lambda_2\kappa_1}{\kappa_5} & -\frac{\lambda_5\lambda_1^3\lambda_2}{\kappa_5} & 0 & 0 \\
-\frac{\lambda_2\kappa_2}{\lambda_1} & \frac{\kappa_2^2}{\lambda_1\lambda_3\kappa_4} & \frac{\lambda_2\kappa_2}{\lambda_1\lambda_4} & \frac{\lambda_2\lambda_5\kappa_2^2}{\kappa_4} & -\frac{\lambda_2^2\lambda_3\kappa_4\kappa_5}{\lambda_4\lambda_5\lambda_1} & \frac{\lambda_2\kappa_5\kappa_2}{\lambda_4\lambda_5\lambda_1} \\
0 & 0 & 0 & -\frac{1}{\lambda_3\lambda_4\kappa_5} & 0 & 0 \\
-\frac{\lambda_4\lambda_5\kappa_2}{\lambda_1\kappa_4} & 0 & \frac{\lambda_5\kappa_2}{\lambda_1\kappa_4} & 0 & \frac{\kappa_5}{\lambda_4\lambda_5\lambda_1^2} & 0
\end{pmatrix}.
$$

We get an equivalent equation system, which can be rewritten in the following way

$$
\begin{aligned}
&2\kappa_4^2\lambda_5\lambda_4' - 2\lambda_4^2\kappa_2\kappa_4\lambda_1\lambda_5' - 2\lambda_4\kappa_4\lambda_5\kappa_4' + 2\lambda_4^3\lambda_5^2\kappa_2\lambda_2\kappa_1' \\
&\qquad -\lambda_5\kappa_4\mu_4 + \tfrac{\lambda_4\kappa_4^2}{\lambda_5}\mu_5 + \lambda_4^3\lambda_5^2\mu_2 + \lambda_4\lambda_5\nu_4 + \lambda_4^3\lambda_5^2\kappa_2^2\nu_1 + \lambda_4^3\lambda_5^2\kappa_1\nu_2 = 0 \\
&2\kappa_5^2\lambda_1\lambda_5' - 2\lambda_5^2\kappa_3\kappa_5\lambda_2\lambda_1' - 2\lambda_5\kappa_5\lambda_1\kappa_5' + 2\lambda_5^3\lambda_1^2\kappa_3\lambda_3\kappa_2' \\
&\qquad -\lambda_1\kappa_5\mu_5 + \tfrac{\lambda_5\kappa_5^2}{\lambda_1}\mu_1 + \lambda_5^3\lambda_1^2\mu_3 + \lambda_5\lambda_1\nu_5 + \lambda_5^3\lambda_1^2\kappa_3^2\nu_2 + \lambda_5^3\lambda_1^2\kappa_2\nu_3 = 0 \\
&2\kappa_1^2\lambda_2\lambda_1' - 2\lambda_1^2\kappa_4\kappa_1\lambda_3\lambda_2' - 2\lambda_1\kappa_1\lambda_2\kappa_1' + 2\lambda_1^3\lambda_2^2\kappa_4\lambda_4\kappa_3' \\
&\qquad -\lambda_2\kappa_1\mu_1 + \tfrac{\lambda_1\kappa_1^2}{\lambda_2}\mu_2 + \lambda_1^3\lambda_2^2\mu_4 + \lambda_1\lambda_2\nu_1 + \lambda_1^3\lambda_2^2\kappa_4^2\nu_3 + \lambda_1^3\lambda_2^2\kappa_3\nu_4 = 0 \\
&2\kappa_2^2\lambda_3\lambda_2' - 2\lambda_2^2\kappa_5\kappa_2\lambda_4\lambda_3' - 2\lambda_2\kappa_2\lambda_3\kappa_2' + 2\lambda_2^3\lambda_3^2\kappa_5\lambda_5\kappa_4' \\
&\qquad -\lambda_3\kappa_2\mu_2 + \tfrac{\lambda_2\kappa_2^2}{\lambda_3}\mu_3 + \lambda_2^3\lambda_3^2\mu_5 + \lambda_2\lambda_3\nu_2 + \lambda_2^3\lambda_3^2\kappa_5^2\nu_4 + \lambda_2^3\lambda_3^2\kappa_4\nu_5 = 0 \\
&2\kappa_1\kappa_4\lambda_3\lambda_2' + 2\tfrac{\kappa_1\kappa_4}{\lambda_3\lambda_5}\lambda_3' + 2\kappa_1^2\kappa_4\lambda_3\kappa_2' + 2\tfrac{\lambda_1\lambda_2\kappa_4^2}{\lambda_4\lambda_5}\lambda_4\kappa_3' + 2\tfrac{\kappa_1}{\lambda_1}\lambda_2\kappa_1' \\
&\qquad -\tfrac{\kappa_1^2}{\lambda_1\lambda_2}\mu_2 + \tfrac{\kappa_4^2\kappa_1}{\lambda_3\lambda_4\lambda_5}\mu_3 + \tfrac{\lambda_1\lambda_2\kappa_4}{\lambda_4\lambda_5}\mu_4 - \tfrac{\kappa_1^3}{\lambda_1\lambda_2}\nu_2 - \tfrac{\kappa_4^3}{\lambda_3\lambda_4^2\lambda_5^2}\nu_3 - \tfrac{\lambda_1\lambda_2}{\lambda_5}\nu_4 - \tfrac{\lambda_2}{\lambda_1}\nu_1 = 0 \\
&2\kappa_2\kappa_5\lambda_4\lambda_3' + 2\tfrac{\kappa_2\kappa_5}{\lambda_4\lambda_1}\lambda_4' + 2\kappa_2^2\kappa_5\lambda_4\kappa_3' + 2\tfrac{\lambda_2\lambda_3\kappa_5^2}{\lambda_5\lambda_1}\lambda_5\kappa_4' + 2\tfrac{\kappa_2}{\lambda_2}\lambda_3\kappa_2' \\
&\qquad -\tfrac{\kappa_2^2}{\lambda_2\lambda_3}\mu_3 + \tfrac{\kappa_5^2\kappa_2}{\lambda_4\lambda_5\lambda_1}\mu_4 + \tfrac{\lambda_2\lambda_3\kappa_5}{\lambda_5\lambda_1}\mu_5 - \tfrac{\kappa_2^3}{\lambda_2\lambda_3}\nu_3 - \tfrac{\kappa_5^3}{\lambda_4\lambda_5^2\lambda_1^2}\nu_4 - \tfrac{\lambda_2\lambda_3}{\lambda_1}\nu_5 - \tfrac{\lambda_3}{\lambda_2}\nu_2 = 0.
\end{aligned}
$$

Before we formulate the main theorem in this subsection we consider what different cases that are possible to receive. First let us consider the case $\kappa_i = \kappa_i(0) \neq 0$ for $i = 1, 2, \ldots, 5$. Second, without loss of generality, we assume that $\kappa_1 = \kappa_1(0) = 0$ and all the other κ_i:s are non-zero. From Theorem (5.1) and its Corollaries it follows that

$$
\begin{aligned}
\kappa_3 &= \tfrac{1}{\lambda_5\lambda_2} \\
\kappa_4 &= \tfrac{1}{\lambda_1\lambda_3} \\
\lambda_2\kappa_5 + \lambda_1\kappa_2 &= \tfrac{1}{\lambda_4}.
\end{aligned} \tag{5.6}
$$

The third and last case is where $\kappa_2 = \kappa_2(0) = 0$ besides the fact that $\kappa_1 = \kappa_1(0) = 0$. From the relations (5.6) it now follows that

$$
\begin{aligned}
\kappa_3 &= \tfrac{1}{\lambda_5\lambda_2} \\
\kappa_4 &= \tfrac{1}{\lambda_1\lambda_3} \\
\kappa_5 &= \tfrac{1}{\lambda_2\lambda_4}.
\end{aligned}
$$

Now we are in the position to formulate the main Theorem. Since the conditions are rotion invariants, we start by changing indices in the previous equation system by $i \mapsto \tau(i+2) = (i+1) mod\ 5 + 1$. With this setting all the different cases are included in next formulation. We have

Theorem 5.5 *Let S be a G^1-continuous 5-patch surface, composed of the $C^2_\#$-patches $\mathbf{r}^{(i)}$ for $i = 1, \ldots, 5$, which meet at the vertex point V. Necessary and sufficient conditions for the surface S to be G^2 are that there exist continuously differentiable functions λ_i, κ_i and continuous functions μ_i, ν_i for $i = 1, \ldots, 5$, fulfilling the relations (5.2), and the coefficients*

$\lambda_i = \lambda_i(0)$, $\kappa_i = \kappa_i(0)$, $\mu_i = \mu_i(0)$, $\nu_i = \nu_i(0)$, $\lambda'_i = \lambda'_i(0)$ and $\kappa'_i = \kappa'_i(0)$ for $i = 1, \ldots, 5$, with $\kappa_i \neq 0$ for $i = 3, 4, 5$, satisfying the equations

$$
\begin{aligned}
&2\kappa_2^2\lambda_3\lambda'_2 - 2\lambda_2^2\kappa_5\kappa_2\lambda_4\lambda'_3 - 2\lambda_2\kappa_2\lambda_3\kappa'_2 + 2\lambda_2^3\lambda_3^2\kappa_5\lambda_5\kappa'_4 \\
&\qquad -\lambda_3\kappa_2\mu_2 + \tfrac{\lambda_2\kappa_2^2}{\lambda_3}\mu_3 + \lambda_2^3\lambda_3^2\mu_5 + \lambda_2\lambda_3\nu_2 + \lambda_2^3\lambda_3^2\kappa_5^2\nu_4 + \lambda_2^3\lambda_3^2\kappa_4\nu_5 = 0 \\
&2\kappa_3^2\lambda_4\lambda'_3 - 2\lambda_3^2\kappa_1\kappa_3\lambda_5\lambda'_4 - 2\lambda_3\kappa_3\lambda_4\kappa'_3 + 2\lambda_3^3\lambda_4^2\kappa_1\lambda_1\kappa'_5 \\
&\qquad -\lambda_4\kappa_3\mu_3 + \tfrac{\lambda_3\kappa_3^2}{\lambda_4}\mu_4 + \lambda_3^3\lambda_4^2\mu_1 + \lambda_3\lambda_4\nu_3 + \lambda_3^3\lambda_4^2\kappa_1^2\nu_5 + \lambda_3^3\lambda_4^2\kappa_5\nu_1 = 0 \\
&2\kappa_4^2\lambda_5\lambda'_4 - 2\lambda_4^2\kappa_2\kappa_4\lambda_1\lambda'_5 - 2\lambda_4\kappa_4\lambda_5\kappa'_4 + 2\lambda_4^3\lambda_5^2\kappa_2\lambda_2\kappa'_1 \\
&\qquad -\lambda_5\kappa_4\mu_4 + \tfrac{\lambda_4\kappa_4^2}{\lambda_5}\mu_5 + \lambda_4^3\lambda_5^2\mu_2 + \lambda_4\lambda_5\nu_4 + \lambda_4^3\lambda_5^2\kappa_2^2\nu_1 + \lambda_4^3\lambda_5^2\kappa_1\nu_2 = 0 \\
&2\kappa_5^2\lambda_1\lambda'_5 - 2\lambda_5^2\kappa_3\kappa_5\lambda_2\lambda'_1 - 2\lambda_5\kappa_5\lambda_1\kappa'_5 + 2\lambda_5^3\lambda_1^2\kappa_3\lambda_3\kappa'_2 \\
&\qquad -\lambda_1\kappa_5\mu_5 + \tfrac{\lambda_5\kappa_5^2}{\lambda_1}\mu_1 + \lambda_5^3\lambda_1^2\mu_3 + \lambda_5\lambda_1\nu_5 + \lambda_5^3\lambda_1^2\kappa_3^2\nu_2 + \lambda_5^3\lambda_1^2\kappa_2\nu_3 = 0 \\
&2\kappa_4\kappa_2\lambda_1\lambda'_5 + 2\tfrac{\kappa_4\kappa_2}{\lambda_1\lambda_3}\lambda'_1 + 2\kappa_4^2\kappa_2\lambda_1\kappa'_5 + 2\tfrac{\lambda_4\lambda_5\kappa_2^2}{\lambda_2\lambda_3}\lambda_2\kappa'_1 + 2\tfrac{\kappa_4}{\lambda_4}\lambda_5\kappa'_4 \\
&\qquad -\tfrac{\kappa_4^2}{\lambda_4\lambda_5}\mu_5 + \tfrac{\kappa_2^2\kappa_4}{\lambda_1\lambda_2\lambda_3}\mu_1 + \tfrac{\lambda_4\lambda_5\kappa_2}{\lambda_2\lambda_3}\mu_2 - \tfrac{\kappa_4^3}{\lambda_4\lambda_5}\nu_5 - \tfrac{\kappa_2^3}{\lambda_1\lambda_2^2\lambda_3^2}\nu_1 - \tfrac{\lambda_4\lambda_5}{\lambda_3}\nu_2 - \tfrac{\lambda_5}{\lambda_4}\nu_4 = 0 \\
&2\kappa_5\kappa_3\lambda_2\lambda'_1 + 2\tfrac{\kappa_5\kappa_3}{\lambda_2\lambda_4}\lambda'_2 + 2\kappa_5^2\kappa_3\lambda_2\kappa'_1 + 2\tfrac{\lambda_5\lambda_1\kappa_3^2}{\lambda_3\lambda_4}\lambda_3\kappa'_2 + 2\tfrac{\kappa_5}{\lambda_5}\lambda_1\kappa'_5 \\
&\qquad -\tfrac{\kappa_5^2}{\lambda_5\lambda_1}\mu_1 + \tfrac{\kappa_3^2\kappa_5}{\lambda_2\lambda_3\lambda_4}\mu_2 + \tfrac{\lambda_5\lambda_1\kappa_3}{\lambda_3\lambda_4}\mu_3 - \tfrac{\kappa_5^3}{\lambda_5\lambda_1}\nu_1 - \tfrac{\kappa_3^3}{\lambda_2\lambda_3^2\lambda_4^2}\nu_2 - \tfrac{\lambda_5\lambda_1}{\lambda_4}\nu_3 - \tfrac{\lambda_1}{\lambda_5}\nu_5 = 0.
\end{aligned}
$$

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