CHALMERS | UNIVERSITY OF GOTHENBURG

PREPRINT 2013:21

On Streamline Diffusion Schemes for the One and One-half Dimensional Relativistic Vlasov–Maxwell System

CHRISTOFFER STANDAR

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
UNIVERSITY OF GOTHENBURG
Gothenburg Sweden 2013

Preprint 2013:21

On Streamline Diffusion Schemes for the One and One-half Dimensional Relativistic Vlasov−Maxwell System

Christoffer Standar

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Gothenburg, Sweden
Gothenburg, December 2013

Preprint 2013:21
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2013

ON STREAMLINE DIFFUSION SCHEMES FOR THE ONE AND ONE-HALF DIMENSIONAL RELATIVISTIC VLASOV-MAXWELL SYSTEM

CHRISTOFFER STANDAR

ABSTRACT. We study streamline diffusion schemes applied for numerical solution of the one and one-half dimensional relativistic Vlasov-Maxwell system arising in modeling plasma of particles governed by electro-magnetic fields. We derive stability estimates and prove optimal convergence rates, due to the maximal available regularity of the exact solution.

1. INTRODUCTION

Our main concern will be approximation of the Vlasov-Maxwell system by a semi-classical finite element approach of Petrov-Galerkin type, namely *the streamline diffusion method*. For the exact solution in the Sobolev space H^{k+1} (see Adams [1] for the definitions), the classical finite element method for the Vlasov-Maxwell system (hyperbolic equations) will have an optimal convergence rate only of order $\mathcal{O}(h^k)$. Whereas, with the same regularity (H^{k+1}) the corresponding optimal convergence rate for the elliptic and parabolic problems is $\mathcal{O}(h^{k+1})$. Streamline diffusion (SD) method is, roughly, a weak formulation where a multiple of the convection term is added to the test function. This resembles to the add of artificial diffusion to the continuous problem which enhance the regularity. Using the SD strategy would improve the convergence rate of the hyperbolic problems by an order of 1/2, i.e. $\mathcal{O}(h^{k+1/2})$. Then, by interpolation space techniques one can show that for the hyperbolic problems, a convergence beyond this rate (for the exact solution $f \in H^{k+1}$; a convergence rate of $\mathcal{O}(h^{k+1/2})$) is not achievable. The same result is obtained for approximation with the streamline diffusion based discontinuous Galerkin finite element method. Both approaches are investigated below.

The Vlasov-Maxwell (VM) system which describes the time evolution of collisionless plasma is formulated as

$$\begin{aligned} \partial_t f + \hat{v}\cdot\nabla_x f + q(E + c^{-1}v\times B)\cdot\nabla_v f = 0, \\ \partial_t E = c\nabla_x\times B - j, \qquad \nabla_x\cdot E = \rho, \\ \partial_t B = -c\nabla_x\times E, \qquad \nabla_x\cdot B = 0. \end{aligned} \tag{1.1}$$

Here f is the density, in phase space, of particles with charge q, mass m and velocity

$$\hat{v} = (m^2 + c^{-2}|v|^2)^{-1/2}v \qquad (v \text{ is momentum}).$$

Further, c is the speed of light and the charge and current densities ρ and j are given by

$$\rho(t,x) = 4\pi\int qf\,dv \quad \text{and} \quad j(t,x) = 4\pi\int qf\hat{v}\,dv. \tag{1.2}$$

A proof for the existence and uniqueness of the solution to VM system can be obtained using Schauder fixed point theorem: Insert an assumed and given g for f in (1.2). Compute the corresponding ρ_g and j_g and insert the results in the Maxwell

Key words and phrases. Streamline Diffusion, Discontinuous Galerkin, Vlasov-Maxwell system
.

equations to get E_g and B_g. Then insert, such obtained, E_g and B_g in the Vlasov equation to get f_g via an operator Λ, i.e. $f_g = \Lambda g$. A fixed point of Λ is the solution of the Vlasov equation in the Vlasov-Maxwell system. For the discretized version one should, instead, use the Brouwer fixed point theorem. Both these proofs are rather technical and non-trivial.

The Vlasov-Maxwell system, as well as Vlasov-Poisson (VP), having similar physical structures (physically VM is an extension of VP), are studied by several authors in various settings. The analytic approaches relevant to this study can be found in, e.g., Glassey and Schaeffer [8] (and the references therein) for the one and one-half dimensional relativistic model. Other geometrics are considered by Glass and Han-Kwan in [7], who studied the controllability of the relativistic Vlasov-Maxwell system on a two dimensional torus. The discontinuous Galerkin method is extensively studied by Cheng et al. in [4], deriving the same convergence rates as in the present work, however, with much more involved and excessive calculus and without involving the streamline diffusion strategy. Other numerical approaches commonly used for kinetic type equations are the particle methodes that are studied e.g., by Wollman for the Vlasov-Maxwell-Fokker-Planck system in [11] (see also the references therein). In this paper, we study streamline diffusion based finite element methods for the Vlasov-Maxwell system, prove existence and uniqueness for both the continuous problem and its discretized version. We also derive convergence rates and prove stability estimates. Assuming asymptotically vanishing solutions and data, we shall consider initial boundary value problem with compactly supported phase-space functions.

The VM system we are going to discretize is the *relativistic Vlasov-Maxwell model (RVM)* in *one and one-half dimensional geometry* ($x \in \mathbb{R}, v \in \mathbb{R}^2$):

$$\begin{cases} \partial_t f + \hat{v}_1 \partial_x f + (E_1 + \hat{v}_2 B)\partial_{v_1} f + (E_2 - \hat{v}_1 B)\partial_{v_2} f = 0 \\ \partial_x E_1 = \int f dv - \mathbf{n}(x) = \rho(t,x) \\ \partial_t E_1 = -\int \hat{v}_1 f dv = -j_1(t,x) \\ \partial_t E_2 + \partial_x B = -\int \hat{v}_2 f dv = -j_2(t,x) \\ \partial_t B + \partial_x E_2 = 0. \end{cases} \tag{1.3}$$

The system (1.3) is assigned with the Cauchy data

$$f(0,x,v) = f^0(x,v) \geq 0, \qquad E_2(0,x) = E_2^0(x), \qquad B(0,x) = B^0(x)$$

and

$$E_1(0,x) = \int_{-\infty}^{x} \left(\int f^0(y,v)dv - \mathbf{n}(y) \right) dy = E_1^0(x).$$

This is the only initial data that leads to a finite-energy solution (see [8]). In (1.3) we have for simplicity set all constants equal to one. We assume that the background density $\mathbf{n}(x)$ is smooth, has compact support and is neutralizing. This yields

$$\int_{-\infty}^{\infty} \rho(0,x)\, dx = 0. \tag{1.4}$$

To carry out discrete analysis, we need global existence of a classical solution. We have the following Theorem from [8].

Theorem 1.1 (Glassey, Schaeffer). *Assume that* $\mathbf{n}$ *is neutralizing,*

$$(i)\ 0 \leq f^0(x,v) \in C_0^k(\mathbb{R}^3), \qquad (ii)\ E_2^0 \in C_0^{k+1}(\mathbb{R}),\ \ B^0 \in C_0^{k+1}(\mathbb{R}).$$

for $k \geq 1$. *Then, there exists a global* C^k *solution of RVM.*

The theorem is an existence result. For $k \geq 2$ we can differentiate to get

$$\begin{cases} \partial_t E_2 + \partial_x B = -j_2 \quad \text{w.r.t } x \Rightarrow \quad \partial_x \partial_t E_2 + \partial_{xx}^2 B = -\partial_x j_2 \\ \quad \partial_t B + \partial_x E_2 = 0 \quad \text{w.r.t } t \Rightarrow \quad \partial_{tt}^2 B + \partial_t \partial_x E_2 = 0. \end{cases}$$

Subtracting the resulting equations, one can show that B has solution of *d'Alembert* type and satisfies the one dimensional wave equation. Similarly the same is valid for E_2. The closed form solution for E_1 is yet simpler. Hence (by uniqueness of the solution for the wave equation) we have now both existence and uniqueness in the one and one-half dimensional case, when $k \geq 2$.

An outline of this paper is as follows. In Section 2 we introduce some notations. In Sections 3 and 4 we study the streamline diffusion method for the Vlasov-Maxwell equations. This is done by first looking at the Maxwell part and then using the results for the Vlasov part. The fifth and sixth Sections are devoted to a streamline diffusion based discontinuous Galerkin scheme for the Vlasov-Maxwell equations.

Throughout this note C will denote a generic constant, not necessarily the same at each occurrence, and independent of the parameters in the equations, unless otherwise explicitly specified.

2. Assumptions and notations

Let $\Omega_x \subset \mathbb{R}$ and $\Omega_v \subset \mathbb{R}^2$ denote the space and velocity domains, respectively. We assume that $f(t,x,v)$, $E_2(t,x)$, $B(t,x)$ and $\mathbf{n}(x)$ have compact supports in Ω_x and that $f(t,x,v)$ has compact support in Ω_v. Since we have assumed that $\int \rho(0,x)dx = 0$, it follows that also E_1 has compact support in Ω_x.

Now we will introduce a finite element structure on $\Omega_x \times \Omega_v$. Let $T_h^x = \{\tau_x\}$ and $T_h^v = \{\tau_v\}$ be finite elements subdivision of Ω_x with elements τ_x and Ω_v with elements τ_v, respectively. Then $T_h = T_h^x \times T_h^v = \{\tau_x \times \tau_v\} = \{\tau\}$ is a subdivision of $\Omega_x \times \Omega_v$. Let $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ be a partition of $[0,T]$ into subintervals $I_m = (t_m, t_{m+1}]$, $m = 0, \ldots, M-1$. Further let $\mathcal{C}_h$ be the corresponding subdivision of $Q_T = [0,T] \times \Omega_x \times \Omega_v$ into elements $K = I_m \times \tau$, with $h = \operatorname{diam} K$ as the mesh parameter. Introduce $\tilde{\mathcal{C}}_h$ as the finite element subdivision of $[0,T] \times \Omega_x$. Before we define our finite dimensional spaces we need to define

$$\mathcal{H}_0 = \prod_{m=0}^{M-1} H_0^1(I_m \times \Omega_x \times \Omega_v) \quad \text{and} \quad \tilde{\mathcal{H}}_0 = \prod_{m=0}^{M-1} H_0^1(I_m \times \Omega_x),$$

where

$$H_0^1 = \{w \in H^1; w = 0 \text{ on } \partial\Omega\}.$$

Here Ω stands for either Ω_x or $\Omega_x \times \Omega_v$. For $k = 0, 1, 2, \ldots$, we define the finite element spaces

$$V_h = \{w \in \mathcal{H}_0; w|_K \in P_k(\tau) \times P_k(I_m), \forall K = \tau \times I_m \in \mathcal{C}_h\}$$

and

$$\tilde{V}_h = \{g \in [\tilde{\mathcal{H}}_0]^3; g_i|_{\tilde{K}} \in P_k(\tau_x) \times P_k(I_m), \forall \tilde{K} = \tau_x \times I_m \in \tilde{\mathcal{C}}_h,\, i = 1,2,3\},$$

where $P_k(\cdot)$ is the set of polynomial with degree at most k on the given set. We shall also use some notation, viz

$$(f,g)_m = (f,g)_{S_m}, \qquad \|g\|_m = (g,g)_m^{1/2}$$

and

$$\langle f,g\rangle_m = (f(t_m,\ldots), g(t_m,\ldots))_\Omega, \qquad |g|_m = \langle g,g\rangle_m^{1/2},$$

where $S_m = I_m \times \Omega$, is the slab at m-th level, $m = 0, \ldots, M-1$.

3. The Maxwell equations

We start looking at the Maxwell part, therefore in this section $\Omega = \Omega_x$. Set

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and let $W = (E_1, E_2, B)^T$, $W^0 = (E_1^0, E_2^0, B^0)^T$ and $b = (\rho, -j_1, -j_2, 0)^T$. Then, the Maxwell equations can be written as

$$\begin{cases} M_1 W_t + M_2 W_x = b \\ W(0,x) = W^0(x). \end{cases} \tag{3.1}$$

The streamline diffusion method for the Maxwell part can now be formulated as: Find $W^h \in \tilde{V}_h$ such that for $m = 0, 1, \dots, M-1$,

$$(M_1 W_t^h + M_2 W_x^h, \hat{g} + \delta(M_1 g_t + M_2 g_x))_m + \langle W_+^h, g_+ \rangle_m = \\ = (b, \hat{g} + \delta(M_1 g_t + M_2 g_x))_m + \langle W_-^h, g_+ \rangle_m, \quad \forall\, g \in \tilde{V}_h,$$

where $\hat{g} = (g_1, g_1, g_2, g_3)^T$, $g_\pm(t,x) = \lim_{s \to 0^\pm} g(t+s, x)$ and δ is a multiple of h. Now we define the bilinear form

$$\tilde{\mathcal{B}}(W,g) = \sum_{m=0}^{M-1} (M_1 W_t + M_2 W_x, \hat{g} + \delta(M_1 g_t + M_2 g_x))_m + \sum_{m=1}^{M-1} \langle [W], g_+ \rangle_m + \langle W_+, g_+ \rangle_0$$

and the linear form

$$\tilde{\mathcal{L}}(g) = \sum_{m=0}^{M-1} (b, \hat{g} + \delta(M_1 g_t + M_2 g_x))_m + \langle W^0, g_+ \rangle_0,$$

where $[W] = W_+ - W_-$. Then, the streamline diffusion can be formulated as: Find $W^h \in \tilde{V}_h$ such that

$$\tilde{\mathcal{B}}(W^h, g) = \tilde{\mathcal{L}}(g) \quad \forall\, g \in \tilde{V}_h. \tag{3.2}$$

We also have that the solution of (3.1) satisfies

$$\tilde{\mathcal{B}}(W, g) = \tilde{\mathcal{L}}(g) \quad \forall\, g \in \tilde{V}_h.$$

Subtracting (3.2) from this equation, we end up with the Galerkin orthogonality relation

$$\tilde{\mathcal{B}}(W - W^h, g) = 0 \quad \forall\, g \in \tilde{V}_h, \tag{3.3}$$

which is of vital importance in the error analysis. Now we will define the norm

$$|||g|||_M^2 = \frac{1}{2}\left(|g_+|_0^2 + |g_-|_M^2 + \sum_{m=1}^{M-1} |[g]|_m^2 + 2\delta \sum_{m=0}^{M-1} \|M_1 g_t + M_2 g_x\|_m^2 \right).$$

Lemma 3.1. *We have*

$$\tilde{\mathcal{B}}(g,g) = |||g|||_M^2 \quad \forall g \in \tilde{\mathcal{H}}_0.$$

Proof. By definition of $\tilde{\mathcal{B}}$ we have that

$$\tilde{\mathcal{B}}(g,g) = \sum_{m=0}^{M-1} ((M_1 g_t + M_2 g_x, \hat{g})_m + \delta \|M_1 g_t + M_2 g_x\|_m^2) + \sum_{m=1}^{M-1} \langle [g], g_+ \rangle_m + |g_+|_0^2.$$

Integrating by parts we get that

$$\sum_{m=0}^{M-1} (M_1 g_t, \hat{g}) + \sum_{m=1}^{M-1} \langle [g], g_+ \rangle_m + |g_+|_0^2 = \frac{1}{2}\left(\sum_{m=1}^{M-1} |[g]|_m^2 + |g_-|_M^2 + |g_+|_0^2 \right)$$

and

$$\sum_{m=0}^{M-1}(M_2 g_x, \hat{g})_m = 0,$$

since $g(t,x) = 0$ on $I \times \partial\Omega_x$. Then, the proof follows immediately through adding all above terms. □

Lemma 3.2. *For any positive constant C we have that for $g \in \tilde{\mathcal{H}}_0$,*

$$\|g\|_m^2 \leq \left(|g_-|_{m+1}^2 + \frac{1}{C}\|M_1 g_t + M_2 g_x\|_m^2\right) he^{2Ch}.$$

Proof. For $t_m < t < t_{m+1}$, we may write

$$\begin{aligned}
\|g(t)\|_{\Omega_x}^2 =& |g_-|_{m+1}^2 - \int_t^{t_{m+1}} \frac{d}{dt}\|g(s)\|_{\Omega_x}^2 ds = \\
=& |g_-|_{m+1}^2 - 2\int_t^{t_{m+1}} (M_1 g_t + M_2 g_x, \hat{g})_{\Omega_x} ds \leq \\
\leq& |g_-|_{m+1}^2 + \frac{1}{C}\|M_1 g_t + M_2 g_x\|_m^2 + C\int_t^{t_{m+1}} \|\hat{g}\|_{\Omega_x}^2 \leq \\
\leq& |g_-|_{m+1}^2 + \frac{1}{C}\|M_1 g_t + M_2 g_x\|_m^2 + 2C\int_t^{t_{m+1}} \|g\|_{\Omega_x}^2.
\end{aligned}$$

Now Grönwall's inequality gives us that

$$\|g(t)\|_{\Omega_x}^2 \leq \left(|g_-|_{m+1}^2 + \frac{1}{C}\|M_1 g_t + M_2 g_x\|_m^2\right) e^{2Ch}.$$

By integrating over $[t_m, t_{m+1}]$ we obtain the Lemma. □

Lemma 3.3. *For any $h > 0$ the problem* (3.2) *has a solution and if h is small enough the solution is unique.*

The proof of this Lemma is similar to the corresponding proof in the Vlasov part and is therefore omitted in here.

To proceed we define $F = E_2 + B$ and $G = E_2 - B$. By adding and subtracting the equations for E_2 and B, we get the following equations for F and G:

$$\begin{cases} \partial_t F + \partial_x F = -j_2(t,x), & F(0,x) = E_2^0(x) + B^0(x), \\ \partial_t G - \partial_x G = -j_2(t,x), & G(0,x) = E_2^0(x) - B^0(x) \end{cases}.$$

Solving these equations and using the facts that $E_2 = \frac{1}{2}(F+G)$ and $B = \frac{1}{2}(F-G)$, we get that

$$\begin{aligned}
E_2(t,x) = &\frac{1}{2}\left(E_2^0(x-t) + E_2^0(x+t) + B^0(x-t) - B^0(x+t)\right) - \\
&- \frac{1}{2}\int_0^t j_2(\tau, x+\tau-t) + j_2(\tau, x+t-\tau) d\tau
\end{aligned}$$

and

$$\begin{aligned}
B(t,x) = &\frac{1}{2}\left(E_2^0(x-t) - E_2^0(x+t) + B^0(x-t) + B^0(x+t)\right) - \\
&- \frac{1}{2}\int_0^t j_2(\tau, x+\tau-t) - j_2(\tau, x+t-\tau) d\tau.
\end{aligned}$$

Moreover, using the equations for E_1 we will get

$$E_1(t,x) = \int_{x_0}^x \left(\int f(t,y,v)dv - \mathbf{n}(y)\right) dy.$$

Then by some simple inequalities we end up with

$$\|E_1\|_{Q_T}^2 \leq C\left(\|f\|_{Q_T}^2 + T\int_{\Omega_x} |\mathbf{n}(x)|^2 dx\right) \tag{3.4}$$

and

$$\|E_2\|_{Q_T}^2 \leq CT\left(\int_{\Omega_x} |E_2^0(x)|^2 dx + \int_{\Omega_x} |B^0(x)|^2 dx + \|\hat{v}_2 f\|_{Q_T}^2\right). \tag{3.5}$$

In a similar way we get for $i = 1, 2$ that

$$\|\hat{v}_i B\|_{Q_T}^2 \leq CT\left(\int_{\Omega_x} |B^0(x)|^2 dx + \int_{\Omega_x} |E_2^0(x)|^2 dx + \|\hat{v}_2 f\|_{Q_T}^2\right). \tag{3.6}$$

Next we start the error analysis. First let $\tilde{W}$ be an interpolant of W in the finite dimensional discrete function space $\tilde{V}_h$. Then we represent the error as the following split

$$e = W - W^h = (W - \tilde{W}) - (W^h - \tilde{W}) = \eta - \xi.$$

Now we state and prove the convergence theorem.

Theorem 3.4. *If W^h is a solution to* (3.2) *and the exact solution W for* (3.1) *satisfies*

$$\|W\|_{k+1} \leq C,$$

then there exists a constant C such that

$$|||W - W^h|||_M \leq Ch^{k+\frac{1}{2}}.$$

Proof. We have by Lemma 3.1 and (3.3) that

$$|||\xi|||_M^2 = \tilde{\mathcal{B}}(\xi, \xi) = \tilde{\mathcal{B}}(\eta, \xi) =$$
$$= \sum_{m=0}^{M-1} (M_1\eta_t + M_2\eta_x, \hat{\xi} + \delta(M_1\xi_t + M_2\xi_x))_m + \sum_{m=1}^{M-1} \langle [\eta], \xi_+\rangle_m + \langle \eta_+, \xi_+\rangle_0.$$

Partial integration gives the identities

$$(M_1\eta_t, \hat{\xi})_m = (\eta_t, \xi)_m = \int_{\Omega_x} [\eta\xi]_{t=t_m}^{t_{m+1}} dx - (\eta, \xi_t)_m =$$
$$= \langle \eta_-, \xi_-\rangle_{m+1} - \langle \eta_+, \xi_+\rangle_m - (\eta, \xi_t)_m$$

and

$$(M_2\eta_x, \hat{\xi})_m = (\eta_x, \xi)_m = -(\eta, \xi_x)_m,$$

since η and ξ have compact supports in Ω_x. Inserting these equations into the expression for $\tilde{\mathcal{B}}(\eta, \xi)$ we end up with an estimate of the form

$$\tilde{\mathcal{B}}(\eta, \xi) \leq |\langle \eta_-, \xi_-\rangle_M - \sum_{m=1}^{M-1} \langle \eta_-, [\xi]\rangle_m +$$
$$+ \sum_{m=0}^{M-1} (\hat{\eta}, M_1\xi_t + M_2\xi_x)_m + \delta(M_1\eta_t + M_2\eta_x, M_1\xi_t + M_2\xi_x)_m|.$$

Further, using some standard inequalities it follows that

$$|||\xi|||_M^2 \leq \frac{1}{8}|||\xi|||_M^2 + \sum_{m=0}^{M-1} (4|\eta_-|_{m+1}^2 + \frac{8}{\delta}\|\eta\|_m^2 + 4\delta\|M_1\eta_t + M_2\eta_x\|_m^2).$$

Hiding the ξ-term on the right hand side in the ξ-term on the left hand side, implies the following inequality

$$|||\xi|||_M^2 \leq \frac{8}{7}\sum_{m=0}^{M-1} (4|\eta_-|_{m+1}^2 + \frac{8}{\delta}\|\eta\|_m^2 + 4\delta\|M_1\eta_t + M_2\eta_x\|_m^2).$$

Thus, we have estimated $|||\xi|||_M^2$ by terms only depending on η. This implies that

$$|||e|||_M^2 \leq |||\eta|||_M^2 + |||\xi|||_M^2 \leq$$
$$\leq C\left(|\eta_+|_0^2 + \sum_{m=0}^{M-1}(|\eta_-|_{m+1}^2 + \frac{1}{h}\|\eta\|_m^2 + h\|\eta_t\|_m^2 + h\|\eta_x\|_m^2) + \sum_{m=1}^{M-1}|[\eta]|_m^2\right),$$

where we have used that δ is a multiple of h. By standard interpolation theory we have that the interpolation error satisfies (see e.g. [5])

$$\left[h|\eta_+|_0^2 + \sum_{m=0}^{M-1}(h|\eta_-|_{m+1}^2 + \|\eta\|_m^2 + h^2\|\eta_t\|_m^2 + h^2\|\eta_x\|_m^2) + \sum_{m=1}^{M-1}h|[\eta]|_m^2\right]^{1/2} \leq$$
$$\leq Ch^{k+1}\|W\|_{k+1}.$$

Then, it now follows that

$$|||e|||_M \leq Ch^{k+\frac{1}{2}}\|W\|_{k+1}$$

and the proof is complete. □

We have the following Corollary of the Theorem, which will be useful for the error analysis for the Vlasov equation.

Corollary 3.5. *Assume that $E_2^0, B^0 \in C_0^2$ and that $f^0 \in C_0^1$. Assume also that the hypothesis in the previous Theorem hold true, then*

$$\int_{\Omega_x} |E_2^0(x) - E_{2,+}^h(0,x)|^2 + |B^0(x) - B_+^h(0,x)|^2 dx \leq Ch^{2h+1}.$$

Proof. We will prove the Corollary for the B-term. A similar proof works for the E_2-term. We start by noticing that $|e_+|_0^2$ is included in $|||e|||_M^2$. Theorem 1.1 states that if $E_2^0, B^0 \in C_0^2$ and $f^0 \in C_0^1$, then there exists a global solution to (1.3) such that f, E_1, E_2 and B are differentiable. Especially B is continuous, meaning $B_+(0,x) = B_-(0,x) = B^0(x)$. This will together with the previous Theorem imply that

$$\int_{\Omega_x} |B^0(x) - B_+^h(0,x)|^2 dx = |B_+ - B_+^h|_0^2 \leq |||e|||_M^2 \leq Ch^{2k+1}.$$

□

4. The Vlasov-Maxwell equations

Let us now return to the Vlasov equation given by

$$\begin{cases} \partial_t f + \hat{v}_1\partial_x f + (E_1 + \hat{v}_2 B)\partial_{v_1} f + (E_2 - \hat{v}_1 B)\partial_{v_2} f = 0 \\ f(0,x,v) = f^0(x,v) \geq 0 \end{cases}. \tag{4.1}$$

Introduce the notation

$$G(f) = (\hat{v}_1, E_1 + \hat{v}_2 B, E_2 - \hat{v}_1 B)$$

and

$$\nabla f = (\partial_x f, \partial_{v_1} f, \partial_{v_2} f).$$

We can write the Vlasov equation as

$$\partial_t f + G(f)\cdot\nabla f = 0.$$

In this section $\Omega = \Omega_x \times \Omega_v$. The streamline diffusion method for the Vlasov part can now be formulated as: Find $f^h \in V_h$ such that for $m = 0, 1, \ldots, M-1$,

$$(f_t^h + G(f^h)\cdot\nabla f^h, g + \delta(g_t + G(f^h)\cdot\nabla g))_m + \langle f_+^h, g_+\rangle_m = \langle f_-^h, g_+\rangle_m \quad \forall g \in V_h,$$

where

$$G(f^h) = (\hat{v}_1, E_1^h + \hat{v}_2 B^h, E_2^h - \hat{v}_1 B^h).$$

Define the bilinear form

$$\mathcal{B}(G;f,g)=\sum_{m=0}^{M-1}(f_t+G\cdot\nabla f,g+\delta(g_t+G(f^h)\cdot\nabla g))_m+\sum_{m=1}^{M-1}\langle[f],g_+\rangle_m+\langle f_+,g_+\rangle_0$$

and the linear form

$$\mathcal{L}(g)=\langle f^0,g_+\rangle_0.$$

Streamline diffusion can now be formulated as: Find $f^h\in V_h$ such that

$$\mathcal{B}(G(f^h);f^h,g)=\mathcal{L}(g)\quad\forall g\in V_h. \tag{4.2}$$

Here we need to define a new norm, viz

$$|||g|||_V^2=\frac{1}{2}(|g_+|_0^2+|g_-|_M^2+\sum_{m=1}^{M-1}|[g]|_m^2+2\delta\sum_{m=0}^{M-1}\|g_t+G(f^h)\cdot\nabla g\|_m^2).$$

Lemma 4.1. *We have that*

$$\mathcal{B}(G(f^h);g,g)=|||g|||_V^2\quad\forall g\in\mathcal{H}_0.$$

The proof of this is similar to the proof of Lemma 3.1 and will therefore be omitted.

Lemma 4.2. *For any constant C we have for $g\in\mathcal{H}_0$,*

$$\|g\|_m^2\le\left(|g_-|_{m+1}^2+\frac{1}{C}\|g_t+G(f^h)\cdot\nabla g\|_m^2\right)he^{Ch}.$$

This proof will also be omitted since it is similar to the proof of Lemma 3.2.

Lemma 4.3. *For any $h>0$ there exists a solution for the problem (4.2). If h is small enough and we assume that*

$$\|G(f)\|_\infty+\|\nabla f\|_\infty\le C$$

for some constant C, then the solution is unique.

Proof. We will start by proving the existence part of the Lemma by following the proof of Lemma 2.4 in [10]. Let $V^m=\{v|_{I_m};v\in V_h\}$ and define $P^m:V^m\to V^m$ by

$$[P^mv,\theta]=(v_t+G(f^h)\cdot\nabla v,\theta+\delta(\theta_t+G(f^h)\cdot\nabla\theta))_m+\langle v_+,\theta_+\rangle_m-\langle v_-,\theta_+\rangle_m\quad\forall\theta\in V^m.$$

Notice that $P^mv=0$ if and only if $v=f^h$ satisfies (4.2). We also need to define the new scalar product

$$[v,\theta]=(v,\theta)_m+\langle v_-,\theta_-\rangle_{m+1}+\langle v_+,\theta_+\rangle_m$$

and the corresponding norm $\lfloor v\rfloor^2=[v,v]$. With respect to this norm P^m is continuous. We have that

$$[P^mv,v]=\frac{1}{2}|v_-|_{m+1}^2+\frac{1}{2}|v_+|_m^2+\delta\|v_t+G(f^h)\cdot\nabla v\|_m^2-\langle v_-,v_+\rangle_m.$$

Using Lemma 4.2 and the Cauchy-Schwarz inequality it follows that

$$[P^mv,v]\ge\frac{1}{4}|v_-|_{m+1}^2+\frac{1}{2}|v_+|_m^2+C(h)\|v\|_m^2-|v_-|_m|v_+|_m\ge C(h)\lfloor v\rfloor^2-|v_-|_m\lfloor v\rfloor.$$

For a large enough r, we have that $[P^mv,v]\ge0$ when $\lfloor v\rfloor=r$. Using a Brouwer's fixed point theorem argument implies that there exists $v=f^m\in V^m$ with $\lfloor v\rfloor\le r$ such that $P^mv=0$ (see [6] for more details). An induction argument completes the proof of the existence.

To prove uniqueness, suppose that f_1 and f_2 are two solutions of (4.2). For any m, write $f = f_1 - f_2$ and assume that $f(t_m, \cdot)_- = 0$. Subtracting the equations for f_1 and f_2 we get

$$(\partial_t f, f)_m + |f_+|_m^2 + \delta(\partial_t f_1, \partial_t f + G(f_1)\cdot\nabla f)_m + (G(f_1)\cdot\nabla f_1, f + \delta(\partial_t f + G(f_1)\cdot\nabla f))_m - \\ - \delta(\partial_t f_2, \partial_t f + G(f_2)\cdot\nabla f)_m - (G(f_2)\cdot\nabla f_2, f + \delta(\partial_t f + G(f_2)\cdot\nabla f))_m = 0.$$

After some simplifications by adding and subtracting some auxiliary terms we end up with the following equation

$$\frac{1}{2}|f_-|_{m+1}^2 + \frac{1}{2}|f_+|_m^2 + \delta\|\partial_t f + G(f_1)\cdot\nabla f\|_m^2 + (G(f_1)\cdot\nabla f, f)_m + \\ + ((G(f_1) - G(f_2))\cdot\nabla f_2, f + \delta(\partial_t f + G(f_2)\cdot\nabla f))_m + \\ + \delta(\partial_t f_2 + G(f_1)\cdot\nabla f_2, \partial_t f + (G(f_1) - G(f_2))\cdot\nabla f)_m = 0.$$

Moving the last three terms of the equation to the right hand side and taking absolute values yields

$$\frac{1}{2}|f_-|_{m+1}^2 + \frac{1}{2}|f_+|_m^2 + \delta\|\partial_t f + G(f_1)\cdot\nabla f\|_m^2 \le \\ \le \|G(f_1)\|_\infty\cdot\|\nabla f\|_\infty\cdot\|f\|_m + \|G(f_1) - G(f_2)\|_\infty\cdot\|\nabla f\|_\infty(\|f\|_m + \delta\|\partial_t f + G(f_2)\cdot\nabla f\|_m) + \\ + \delta(\|\partial_t f_2\|_\infty + \|G(f_1)\|_\infty\cdot\|\nabla f_2\|_\infty)(\|\partial_t f\|_m + \|G(f_1) - G(f_2)\|_\infty\cdot\|\nabla f\|_m).$$

By the assumption

$$\|G(f_i)\|_\infty + \|\nabla f_i\|_\infty \le C, \quad i = 1, 2,$$

and inverse inequalities, we get the inequality

$$\frac{1}{2}|f_-|_{m+1}^2 + \frac{1}{2}|f_+|_m^2 + \delta\|\partial_t f + G(f_1)\cdot\nabla f\|_m^2 \le C\|f\|_m^2$$

for some constant C. This inequality together with Lemma 4.2 implies that

$$h^{-1}\|f\|_m \le C\|f\|_m.$$

If h is small enough, this inequality implies that $\|f\|_m = 0$. Hence $f_1 = f_2$ and the uniqueness follows. □

Let us now start with the error analysis. First we let $\tilde{f}$ be an interpolant of f. Then we set

$$e = f - f^h = (f - \tilde{f}) - (f^h - \tilde{f}) = \eta - \xi.$$

We have the following convergence theorem.

Theorem 4.4. *Let f^h be a solution to* (4.2) *and assume that the exact solution f of* (4.1) *is in the Sobolev class $H^{k+1}(I \times \Omega)$ and satisfies the bound*

$$\|\nabla f\|_\infty + \|G(f)\|_\infty + \|\nabla\eta\|_\infty \le C.$$

Also assume that h is sufficiently small, then there exists a constant C such that

$$|||f - f^h|||_V \le Ch^{k+\frac{1}{2}}.$$

Proof. By (4.2) and Lemma 4.1 we get that

$$|||\xi|||_V^2 = \mathcal{B}(G(f^h); \xi, \xi) = \mathcal{L}(\xi) - \mathcal{B}(G(f^h); \tilde{f}, \xi) = T_1 + T_2,$$

where

$$T_1 = \mathcal{B}(G(f^h); \eta, \xi)$$

and

$$T_2 = \mathcal{B}(G(f); f, \xi) - \mathcal{B}(G(f^h); f, \xi).$$

We will estimate these two terms separately. Starting with T_1, estimations are done similar to the proof of Theorem 3.4. After partial integration and using some standard inequalities it follows that

$$|T_1| \leq \frac{4}{\delta}\|\eta\|^2_{Q_T} + 4\delta\|\eta_t + G(f^h)\cdot\nabla\eta\|^2_{Q_T} + 4\sum_{m=1}^{M}|\eta_-|^2_m + \frac{1}{8}|||\xi|||^2_V.$$

To estimate T_2 we use the definition of $\mathcal{B}$ and write

$$T_2 = \sum_{m=1}^{M-1}((G(f)-G(f^h))\cdot\nabla f, \xi + \delta(\xi_t + G(f^h)\cdot\nabla\xi))_m.$$

From some standard inequalities it follows that

$$\begin{aligned}|T_2| \leq \frac{1}{2C_1}\|G(f)-G(f^h)\|^2_{Q_T}\cdot\|\nabla f\|^2_\infty + \frac{C_1}{2}\|\xi\|^2_{Q_T} + \\ + 2\delta\|G(f)-G(f^h)\|^2_{Q_T}\|\nabla f\|^2_\infty + \frac{\delta}{8}\|\xi_t + G(f^h)\cdot\nabla\xi\|^2_{Q_T}. \quad (4.3)\end{aligned}$$

To proceed we need to estimate $\|G(f)-G(f^h)\|^2_{Q_T}$. By the definition of $G(f)$ and $G(f^h)$ we have that

$$G(f)-G(f^h) = (0, E_1 - E_1^h + \hat{v}_2(B-B^h), E_2 - E_2^h - \hat{v}_1(B-B^h)).$$

Therefore

$$\begin{aligned}\|G(f)-G(f^h)\|^2_{Q_T} \leq 2(\|E_1 - E_1^h\|^2_{Q_T} + \|\hat{v}_2(B-B^h)\|^2_{Q_T} + \\ + \|E_2 - E_2^h\|^2_{Q_T} + \|\hat{v}_1(B-B^h)\|^2_{Q_T}).\end{aligned}$$

Using (3.4) - (3.6) we thus obtain

$$\begin{aligned}\|G(f)-G(f^h)\|^2_{Q_T} \leq C\|e\|^2_{Q_T} + CT\|\hat{v}_2 e\|^2_{Q_T} + \\ + CT\int_{\Omega_x}|\mathbf{n}(x) - \tilde{\mathbf{n}}(x)|^2 + |E_2^0(x) - E^h_{2,+}(0,x)|^2 + |B^0(x) - B^h_+(0,x)|^2 dx,\end{aligned}$$

where $\tilde{\mathbf{n}}(x)$ is an interpolant of $\mathbf{n}(x)$ on the space

$$\{w \in H^1_0(\Omega_x); w|_{\tau_x} \in P_k(\tau_x), \forall \tau_x \in T^x_h\}.$$

We can now use Corollary 3.5 and get

$$\|G(f)-G(f^h)\|^2_{Q_T} \leq C(T)(\|e\|^2_{Q_T} + \int_{\Omega_x}|\mathbf{n}(x)-\tilde{\mathbf{n}}(x)|^2 dx + h^{2k+1}). \quad (4.4)$$

This inequality together with (4.3) and the assumption $\|\nabla f\|_\infty \leq C$, leads to the following inequality

$$|T_2| \leq C(T)(\|e\|^2_{Q_T} + \int_{\Omega_x}|\mathbf{n}(x)-\tilde{\mathbf{n}}(x)|^2 dx + h^{2k+1}) + \frac{C_1}{2}\|\xi\|^2_{Q_T} + \frac{\delta}{8}|||\xi|||^2_V.$$

Let us now estimate

$$\|\eta_t + G(f^h)\cdot\nabla\eta\|_{Q_T} \leq \|\eta_t\|_{Q_T} + \|G(f)\|_\infty\|\nabla\eta\|_{Q_T} + \|\nabla\eta\|_\infty\|G(f^h)-G(f)\|_{Q_T}.$$

Using the assumption $\|G(f)\|_\infty + \|\nabla\eta\|_\infty \leq C$ and (4.4), we get that

$$\begin{aligned}\|\eta_t + G(f^h)\cdot\nabla\eta\|_{Q_T} \leq \|\eta_t\|_{Q_T} + C\|\nabla\eta\|_{Q_T} + \\ + C(T)(\|e\|^2_{Q_T} + \int_{\Omega_x}|\mathbf{n}(x)-\tilde{\mathbf{n}}(x)|^2 dx + h^{2k+1}). \quad (4.5)\end{aligned}$$

Combining the inequalities for T_1 and T_2 together with (4.4) and (4.5), we have

$$|||\xi|||_V^2 \leq \frac{4}{\delta}\|\eta\|_{Q_T}^2 + 4\delta\|\eta_t\|_{Q_T} + C\delta\|\nabla\eta\|_{Q_T} + 4\sum_{m=1}^{M}|\eta_-|_m^2 + \frac{1}{4}|||\xi|||_V^2 + \\ + C(T)(\|e\|_{Q_T}^2 + \int_{\Omega_x}|\mathbf{n}(x) - \tilde{\mathbf{n}}(x)|^2 dx + h^{2k+1}) + C_1\|\eta\|_{Q_T}^2.$$

Hiding all ξ-terms in the norm on the left hand side, will give us the following inequality

$$|||\xi|||_V^2 \leq \frac{C}{\delta}\|\eta\|_{Q_T}^2 + C\|\eta\|_{Q_T}^2 + C\delta\|\eta_t\|_{Q_T} + C\delta\|\nabla\eta\|_{Q_T} + C\sum_{m=1}^{M}|\eta_-|_m^2 + \\ + C(T)(\|e\|_{Q_T}^2 + \int_{\Omega_x}|\mathbf{n}(x) - \tilde{\mathbf{n}}(x)|^2 dx + h^{2k+1}).$$

We can now estimate $|||e|||_V$ in the following way

$$|||e|||_V^2 \leq 2|||\xi|||_V^2 + 2|||\eta|||_V^2 \leq \frac{C}{\delta}\|\eta\|_{Q_T}^2 + C\|\eta\|_{Q_T}^2 + C\delta\|\eta_t\|_{Q_T} + C\delta\|\nabla\eta\|_{Q_T} + \\ + C\sum_{m=1}^{M}|\eta_-|_m^2 + C(T)(\|e\|_{Q_T}^2 + \int_{\Omega_x}|\mathbf{n}(x) - \tilde{\mathbf{n}}(x)|^2 dx + h^{2k+1}) + |\eta_+|_0^2 + \sum_{m=1}^{M-1}|[\eta]|_m^2.$$

By some standard interpolation inequalities and using Lemma 4.2 with an adequate choice of C, we get that

$$|||e|||_V^2 \leq \frac{1}{2}|||e|||_V^2 + hC(T)\sum_{m=1}^{M}|e_-|_m^2 + C(T)(h^{2k+1} + h^{2k+2}).$$

Here we need also to assume that h is sufficiently small. Otherwise, we do not get $\frac{1}{2}|||e|||_V^2$ on the right hand side. To sum up it follows directly that

$$|||e|||_V^2 \leq hC(T)\sum_{m=1}^{M}|e_-|_m^2 + C(T)(h^{2k+1} + h^{2k+2}). \tag{4.6}$$

We shall now use the following discrete Grönwall estimate. If

$$y_n \leq C + \sum_{k=1}^{n} g_k y_k,$$

then

$$y_n \leq Ce^{\sum_{k=1}^{n-1} g_k}.$$

From the definition of $|||e|||_V$ it follows that

$$|e_-|_M^2 \leq C_1 h^{2k+1} + C_2 h\sum_{m=1}^{M}|e_-|_m^2,$$

with $C_1 = C(T)(1+h)$ and $C_2 = C(T)$. The discrete Grönwall estimate implies that

$$|e_-|_M^2 \leq C_1 h^{2k+1} e^{C_2 h(M-1)}.$$

Using this inequality on the right hand side of (4.6), give us that

$$|e_-|_{M-1}^2 \leq C_1 h^{2k+1}(1 + he^{C_2 h(M-1)}) + C_2 h\sum_{m=1}^{M-1}|e_-|_m^2.$$

Once again applying the discrete Grönwall estimate leads to

$$|e_-|_{M-1}^2 \leq C_1 h^{2k+1}(1 + he^{C_2 h(M-1)})e^{C_2 h(M-2)}.$$

Continuing in this way yields the following estimate

$$|||e|||_V^2 \le Ch^{2k+1} + \mathcal{O}(h^{2k+2}),$$

which is the desired result. □

5. SD-based DG for the Maxwell equations

We are now going to prove the same error estimates for the streamline diffusion based discontinuous Galerkin method. We start with the Maxwell equations. Define

$$\partial\tilde{K} = \{(t,x) \in \partial\tilde{K}; n_t(t,x) + n_x(t,x) < 0\}$$

for $\tilde{K} \in \tilde{\mathcal{C}}_h$ and where (n_t, n_x) denotes the outward unit normal to $\tilde{K}$. Introduce

$$\tilde{\mathcal{W}}_h = \{g \in [L_2(Q_T)]^3; g_i|_{\tilde{K}} \in P_k(\tilde{K}), \forall \tilde{K} \in \tilde{\mathcal{C}}_h, i = 1,2,3\}.$$

The discontinuous Galerkin finite element method for (3.1) can now be formulated as follows: find $W^h \in \tilde{\mathcal{W}}_h$ such that

$$(M_1 W_t^h + M_2 W_x^h, \hat{g} + \delta(M_1 g_t + M_2 g_x))_{Q_T} + \sum_{\tilde{K}\in\tilde{\mathcal{C}}_h} \int_{\partial\tilde{K}_-} [W^h] g_+ d\sigma =$$

$$= (b, \hat{g} + \delta(M_1 g_t + M_2 g_x))_{Q_T} \quad \forall g \in \tilde{\mathcal{W}}_h. \quad (5.1)$$

We introduce the bilinear form

$$\tilde{\mathcal{B}}(W,g) = (M_1 W_t + M_2 W_x, \hat{g} + \delta(M_1 g_t + M_2 g_x))_{Q_T} + \sum_{\tilde{K}} \int_{\partial\tilde{K}'_-} [W] g_+ d\sigma + \langle W_+, g_+\rangle_0$$

and the linear form

$$\tilde{\mathcal{L}}(g) = (b, \hat{g} + \delta(M_1 g_t + M_2 g_x))_{Q_T} + \langle W_0, g_+\rangle_0,$$

where $\partial\tilde{K}'_- = \partial\tilde{K}_- \backslash \{0\} \times \Omega_x$. We can now reformulate (5.1) in a more compact form: find $W^h \in \tilde{\mathcal{W}}_h$ such that

$$\tilde{\mathcal{B}}(W^h, g) = \tilde{\mathcal{L}}(g) \quad \forall g \in \tilde{\mathcal{W}}_h.$$

Define the norm

$$|||g|||_M^2 = \frac{1}{2}(|g_-|_M^2 + |g_+|_0^2 + \sum_{\tilde{K}} \int_{\partial\tilde{K}'_-} [g]^2 d\sigma + 2\delta \|M_1 g_t + M_2 g_x\|_{Q_T}^2).$$

Lemma 5.1. *We have*

$$\tilde{\mathcal{B}}(g,g) = |||g|||_M^2 \quad \forall g \in \tilde{\mathcal{W}}_h$$

Proof. The proof of this Lemma is very similar to the proof of Lemma 3.1. Here we use the equality

$$(M_1 g_t + M_2 g_x, \hat{g})_{\tilde{K}} + |g_+|_0^2 + \sum_{\tilde{K}} \int_{\partial\tilde{K}'_-} [g] g_+ d\sigma =$$

$$= \frac{1}{2}(|g_-|_M^2 + |g_+|_0^2 + \sum_{\tilde{K}} \int_{\partial\tilde{K}'_-} [g]^2 d\sigma),$$

which follows from partial integration. □

Lemma 5.2. *For any constant $C > 0$ we have for $g \in \tilde{\mathcal{W}}_h$*

$$\|g\|_{Q_T} \le \left(\frac{1}{C}\|M_1 g_t + M_2 g_x\|_{Q_T}^2 + \sum_{m=1}^{M} |g_-|_m^2 + \sum_{\tilde{K}} \int_{\partial\tilde{K}''_-} [g]^2 d\sigma\right) h e^{Ch},$$

where $\partial\tilde{K}''_- = \{(t,x) \in \partial\tilde{K}'_-; n_t(t,x) = 0\}$.

Proof. This proof is similar to the proof of Lemma 3.2. In this proof we have to prove an estimate for every $\tilde{K} \in \tilde{\mathcal{C}}_h$ (local estimates) and then adding them together to get the desired inequality. □

We have the following theorem for the error estimate.

Theorem 5.3. *If W^h is a solution to* (5.1) *and the exact solution W of* (3.1) *satisfies*

$$\|W\|_{k+1} \leq C,$$

then there exists a constant C such that

$$|||W - W^h|||_M \leq Ch^{k+\frac{1}{2}}.$$

Proof. We have as in the proof of Theorem 3.4,

$$|||\xi|||_M^2 = \tilde{\mathcal{B}}(\eta, \xi),$$

where ξ and η are the same as in Section 3. Integration by parts leads to appearance of a term of the form

$$T_0 = \sum_{\tilde{K}} \int_{\partial \tilde{K}''_-} [\xi]\eta_+ d\sigma.$$

We can estimate this term with

$$|T_0| \leq 4 \sum_{\tilde{K}} \int_{\partial \tilde{K}''_-} |\eta_+|^2 d\sigma + \frac{1}{8} \sum_{\tilde{K}} \int_{\partial \tilde{K}''_-} [\xi]^2 d\sigma.$$

By standard interpolation theory the first sum can be estimated by $Ch^{2k+1}\|W\|_{k+1}^2$, while the second sum can be hidden in $|||\xi|||_M^2$. This together with the proof of Theorem 3.4 completes the proof. □

6. SD-based DG for the Vlasov-Maxwell equations

Before formulating the discontinuous Galerkin method for the Vlasov equation, we need to introduce some notations. For each $K \in \mathcal{C}_h$ define

$$\partial K_- = \{(t, x, v) \in \partial K; n_t(t, x, v) + n(t, x, v) \cdot G(f^h) < 0\},$$

where $(n_t, n) = (n_t, n_x, n_v)$ denotes the outward unit normal to ∂K. Let us also introduce for $k \geq 0$,

$$\mathcal{W}_h = \{g \in L_2(Q_T); g|_K \in P_k(K), \forall\, K \in \mathcal{C}_h\}.$$

We can now formulate the discontinuous Galerkin method as: find $f^h \in \mathcal{W}_h$ such that

$$(f_t^h + G(f^h) \cdot \nabla f^h, g + \delta(g_t + G(f^h) \cdot \nabla g))_{Q_T} + \\ + \sum_{K \in \mathcal{C}_h} \int_{\partial K_-} [f^h] g_+ |n_t + n \cdot G(f^h)| d\sigma = 0, \forall g \in \mathcal{W}_h. \quad (6.1)$$

Here $g_\pm = \lim_{s \to 0\pm} g(t + s, (x, v) + G(f^h)s)$. We introduce the bilinear form

$$\mathcal{B}(G; f, g) = (f_t + G \cdot \nabla f, g + \delta(g_t + G(f^h) \cdot \nabla g))_{Q_T} + \\ \sum_K \int_{\partial K'_-} [f] g_+ |n_t + n \cdot G(f^h) d\sigma + \langle f_+, g_+ \rangle_0$$

and the linear form

$$L(g) = \langle f_0, g_+ \rangle_0,$$

where $\partial K'_- = \partial K_- \backslash \{0\} \times \Omega$. Now we can write (6.1) as: find $f^h \in \mathcal{W}_h$ such that

$$\mathcal{B}(G(f^h); f^h, g) = \mathcal{L}(g) \quad \forall g \in \mathcal{W}_h.$$

Lemma 6.1. *We have*

$$\mathcal{B}(G(f^h); g, g) = |||g|||_V^2 \quad \forall g \in \mathcal{W}_h,$$

where

$$|||g|||_V^2 = \frac{1}{2}(|g_-|_M^2 + |g_+|_0^2 + \sum_K \int_{\partial K'_-} [g]^2 |n_t + n\cdot G(f^h)| d\sigma + 2h\|g_t + G(f^h)\cdot \nabla g\|_{Q_T}^2).$$

Lemma 6.2. *For any constant $C > 0$ we have*

$$\|g\|_{Q_T}^2 \leq \left(\frac{1}{C}\|g_t + G(f^h)\cdot \nabla g\|_{Q_T}^2 + \sum_{m=1}^{M} |g_-|_m^2 + \right.$$
$$\left. + \sum_K \int_{\partial K''_-} [g]^2 |n\cdot G(f^h)| d\sigma \right) he^{Ch} \quad \forall g \in \mathcal{W}_h,$$

where

$$\partial K''_- = \{(t,x,v) \in \partial K'_-; n_t(t,x,v) = 0\}.$$

The two lemmas are proven similar to Lemma 5.1 and Lemma 5.2, respectively. We end this section with the following theorem corresponding to Theorem 4.4.

Theorem 6.3. *Let f and f^h be as in Theorem 4.4 and $\|f\|_{k+1,\infty} \leq C$, then we have the following error estimate for the problem* (6.1),

$$|||f - f^h|||_V \leq Ch^{k+\frac{1}{2}},$$

where $\|\cdot\|_{k+1,\infty}$ denotes the $W_\infty^{k+1}(Q_T)$-norm.

Proof. As in the proof of Theorem 5.3, we only get one extra term of the form

$$\sum_K \int_{\partial K''_-} [\xi]\eta_+ |n\cdot G(f^h)| d\sigma$$

to estimate. Following the proof of Theorem 4.1 in [2] together with the proof of Theorem 4.4, the theorem follows. □

7. Conclusion

In summary, we have performed numerical studies of a relativistic Vlasov-Maxwell system of equations formulated for an one and one-half dimensional model (i.e. one space variable and two velocity variables). As for the existence and uniqueness of the analytic solution for the continuous problem, both the electric and magnetic fields have unique representations given by the initial data. Notice that this approach will not work in higher dimensions, therefore extensions to higher dimensions require a different type of investigation.

As for the numerical approach, we have applied the streamline diffusion finite element strategy and discretized the Vlasov-Maxwell system accordingly. We derived for the Maxwell equations and the Vlasov equation, optimal convergence rates for a priori error estimates using streamline diffusion as well as discontinuous Galerkin finite element methods. In a forthcoming paper we shall study a posteriori error estimates for the current problem. This will generalize the work by Johnson and Hansbo [9] and Asadzadeh et al. [3]. A natural extension thereafter is to study both a priori and a posteriori error estimates for higher dimensional cases.

Acknowledgment

This project was suggested and guided through by M. Asadzadeh.

References

[1] R. A. Adams: *Sobolev spaces*, Academic press, 1975.

[2] M. Asadzadeh: *Streamline diffusion methods for the Vlasov-Poisson equation*, Modelisation mathematique et analyse numerique, 24 (1990), p. 177-196.

[3] M. Asadzadeh, D. Rostamy, F. Zabihi: *A posteriori error estimates for a coupled wave system with a local damping*, Journal of Mathematical Sciences, 175 (2011), 228-248.

[4] Y. Cheng, I. M. Gamba, F. Li, P. J. Morrison: *Discontinuous Galerkin methods for the Vlasov-Maxwell equations*, arXiv:1302.2136.

[5] P. G. Ciarlet: *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[6] L. Evans: *Partial differential equations*, Providence, R.I. : American Mathematical Society, cop. 1998.

[7] O. Glass, D. Han-Kwan: *On the controllability of the relativistic Vlasov-Maxwell system*, arXiv:1211.7236.

[8] R. Glassey, J. Schaeffer: *On the One and One-half Dimensional Relativistic Vlasov-Maxwell System*, Mathematical Methods in the Applied Sciences, 13 (1990), p. 169-179.

[9] C. Johnson, P. Hansbo: *Adaptive finite element methods in computational mechanics*, Computer Methods in Applied Mechanics and Engineering, 101 (1992), p. 143-181.

[10] C. Johnson, J. Saranen: *Streamline Diffusion Methods for the Incompressible Euler and Navier-Stokes Equations*, Mathematics of Computation, 47 (1986), p. 1-18.

[11] S. Wollman: *Numerical approximation of the Vlasov-Maxwell-Fokker-Planck system in two dimensions*, Journal of Computational Physics, 231 (2012), no. 9, p. 3483-3517.

Department of Mathematics, Chalmers University of Technology and Göteborg University, SE–412 96, Göteborg, Sweden

E-mail address: standarc@chalmers.se