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An approximate globally convergent algorithm in a frequency domain for reconstruction of dielectrics in Maxwell equations

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We present a new model of an approximate globally convergent method in a frequency domain for reconstruction of dielectric permittivity in Maxwell equations. We consider data which are given only at the backscattered side of the medium which should be reconstructed. We formulate new approximately globally convergent algorithm for the reconstruction of dielectric permittivity function under the assumption that the magnetic permeability is a known constant.

1 Introduction

In this paper we present a new frequency domain model of an approximate globally convergent method for the solution of Multidimensional Coefficient Inverse Problem (MCIP) for the time-dependent Maxwell equations with backscattered data. We use the same approach as in [5]. Main new element of this work is that this new model uses the Fourier transform instead of the previously considered in [5] Laplace transform for the solution of the underlying time-dependent problem. As a result we obtain a new system of non-linear integral-differential equations on the frequency interval instead of the previously considered in [5] pseudo-frequency interval. The novelty of this paper is also in the new representation of the computation of the so-called tail function which includes in the integral-differential equation of the approximate globally convergent method.

A MCIP is a problem of the reconstruction of one or many unknown coefficients of a PDE inside the domain of interest from a boundary measurements. We consider the problems only with a single measurement data, or thus problems which use a single source location or a single direction of the propagation of incident plane wave to generate the data at the boundary.

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Approximate globally convergent method answers to the question: how to obtain unknown coefficient in the small neighborhood of the exact solution without a priori knowledge of any information about this solution ? Convexification algorithm of [8, 9, 14] is an approximate globally convergent method of the first generation. Approximate globally convergent method of the second generation is developed in [3, 5, 6, 10, 11, 12, 13] and is a different approach for solution of MCIP. This method uses layer stripping procedure with respect to the pseudo-frequency for solution of MCIPs. Based on our recent numerical experience [5] we can conclude that an approximate globally convergent method is numerically efficient and quantitative method, and thus, can be applied in real-life reconstruction of coefficients in MCIPs.

In the current work we propose new generation of approximate globally convergent methods which uses layer stripping in the frequency domain instead of the pseudo-frequency domain. This is the more realistic case of real-life applications with single measurement data.

2 Statements of Forward and Inverse Problems with backscattered data

In this section we present an approximately globally convergent method for an MCIP for time-dependent Maxwell equations. For complete theory of an approximate global convergence we refer to Chapter 1 of [5].

2.1 Statement of Forward problem

Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$ and $T = const. > 0$ be the observation time. We define $\Omega_T := \Omega \times (0,T)$, $\partial\Omega_T := \partial\Omega \times (0,T)$ and consider propagation of an electromagnetic field in a homogeneous anisotropic nonmagnetic dielectric medium Ω governed by the Maxwell equations:

$$
\begin{aligned}
\frac{\partial D}{\partial t} - \nabla \times H &= -J, \text{ in } \Omega_T, \\
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \text{ in } \Omega_T, \\
D &= \varepsilon E, \\
B &= \mu H, \\
E(x,0) &= \delta(x - x_0), \\
H(x,0) &= 0.
\end{aligned}
\tag{1}
$$

with perfectly conducting boundary conditions

$$
\begin{aligned}
n \times E &= 0, \quad \text{on } \partial\Omega_T \\
H \cdot n &= 0, \quad \text{on } \partial\Omega_T.
\end{aligned} \tag{2}
$$

Here, n denotes the outward normal on $\partial\Omega$, $E(x,t)$ and $H(x,t)$ are the electric and magnetic fields, $D(x,t)$ and $B(x,t)$ are the electric and magnetic inductions, respectively, $\forall x = (x_1, x_2, x_3)$ in the computational domain Ω. We assume that the magnetic permeability $\mu(x) := \mu = const. > 0$ and the dielectric permittivity $\varepsilon(x) > 0$ is the function in the anisotropic medium Ω. We assume that the current density $J(x,t) = 0 \in \mathbb{R}^3$. The electric and magnetic inductions satisfy to the equations

$$
\nabla \cdot D = \rho, \ \nabla \cdot B = 0 \quad \text{in } \Omega_T. \tag{3}
$$

Here $\rho(x,t)$ is a given charge density.

2.2 Statement of Inverse Problem

Let the coefficient $\varepsilon(x)$ of equation (1) belongs to the set of admissible parameters M_ε which is defined as

$$
M_\varepsilon = \{\varepsilon(x) : |\varepsilon(x)| \in [1,d], \ \varepsilon(x) = 1 \ \forall x \in \mathbb{R}^3 \setminus \Omega, \varepsilon(x) \in C^2\left(\mathbb{R}^3\right)\}. \tag{4}
$$

Here, $|d| > 1$ is a given number. We are interested in the case of the *backscattered* data. In this case the data $g_1(x,t)$ are given only at a part of the boundary Γ of the computational domain Ω, where

$$
\begin{aligned}
&\Omega \subset \{x = (x_1, x_2, x_3) : x_3 > 0\}, \\
&\Gamma = \partial\Omega \cap \{x_3 = 0\} \neq \varnothing.
\end{aligned}
$$

More precisely, we assume that the function $g_1(x,t)$ is known

$$
g_1(x,t) = \begin{cases} g_0(x,t), (x,t) \in \Gamma \times (0,\infty), \\ r_0(x,t) \in (\partial\Omega \setminus \Gamma) \times (0,\infty), \end{cases} \tag{5}
$$

where the function r_0 can be obtained computationally by solving the problem (1) with $\varepsilon = \mu = 1$, see [5]. Our inverse problem is following:

Inverse Problem with backscattered data (IPB). Suppose that the coefficient $\varepsilon(x)$ *satisfies (4) and it is unknown in the domain* Ω. *Determine the function* $\varepsilon(x)$ *for* $x \in \Omega$, *assuming that the function* $g_1(x,t)$ *in (5) is known for a single source position* $x_0 \in \{x_3 < 0\}$.

Remarks 2.2.1:

1. The formulation for **IPB** is similar for the case when we initialize a plane wave instead of the delta-function in (1).

2. Uniqueness of the problem **IPB** is an open problem which can be solved via the method of Carleman estimates [7] in the case of replacing of delta-function in (1) with it approximation. Nevertheless, we assume that uniqueness holds for **IPB**.

2.3 An approximately globally convergent method

To obtain an approximately globally convergent method in frequency domain we consider the Fourier transforms of the functions $E(x,t), H(x,t)$,

$$\begin{aligned} E(x,\omega) &= \int_0^\infty E(x,t)e^{-i\omega t}dt, \text{ for } \omega > \underline{\omega} = const. > 0, \\ H(x,\omega) &= \int_0^\infty H(x,t)e^{-i\omega t}dt, \text{ for } \omega > \underline{\omega} = const. > 0, \end{aligned} \tag{6}$$

where ω is a circular frequency and $\underline{\omega}$ is a large number. Applying Fourier transform in time to the system (1) and noting that $x_0 \notin \overline{\Omega}$ we get the Maxwell equations in frequency domain

$$\begin{aligned} \nabla \times H(x,\omega) &= i\omega\varepsilon(x)E(x,\omega) && \text{in } \Omega, \\ \nabla \times E &= -i\omega\mu H(x,\omega) && \text{in } \Omega, \\ H \cdot n &= 0 && \text{on } \partial\Omega, \\ n \times E &= f(x,\omega) && \text{on } \partial\Omega, \end{aligned} \tag{7}$$

where the function $f(x,\omega)$ is obtained after applying Fourier transform to the function $g_1(x,t)$.

Let us consider now the transverse electric (TE) waves

$$\begin{aligned} E(x,\omega) &= (0, E_{x_2}, 0)^T, \\ H(x,\omega) &= (H_{x_1}, 0, H_{x_3})^T. \end{aligned} \tag{8}$$

where $E_{x_2} = E_y(x_1,x_2,x_3)$, $H_{x_1} = H_{x_1}(x_1,x_2,x_3)$, $H_{x_3} = H_x(x_1,x_2,x_3)$. Substituting (8) into equations (7) we obtain the following system

$$
\begin{aligned}
&\frac{\partial H_{x_3}}{\partial x_2} = 0,\\
&\frac{\partial H_{x_1}}{\partial x_3} - \frac{\partial H_{x_3}}{\partial x_1} = i\omega\varepsilon E_{x_2},\\
&\frac{\partial H_{x_1}}{\partial x_2} = 0,\\
&\frac{\partial E_{x_2}}{\partial x_3} = i\omega\mu H_{x_1},\\
&\frac{\partial E_{x_2}}{\partial x_1} = -i\omega\mu H_{x_3}.
\end{aligned} \tag{9}
$$

From the system above follows that H_{x_1}, H_{x_3} and thus E_{x_2} does not depend on x_2. Noting that $x_0 \notin \overline{\Omega}$ and thus, $\delta(x-x_0) = 0$, we can easily obtain from the equations (9) the single equation for the electric field

$$
\begin{aligned}
\triangle w(x,\omega) + \omega^2 \mu \varepsilon(x) w(x,\omega) = 0,\\
\lim_{|x|\to\infty} w(x,\omega) = 0,
\end{aligned} \tag{10}
$$

where we denoted by $w(x,\omega) := E_{x_2}(x,\omega)$. Similarly with [4, 5] we can prove that $\lim_{|x|\to\infty} w(x,\omega) = 0$ for $\underline{\omega}, \omega > \underline{\omega}$.

Next, as in [5] we eliminate the unknown coefficient $\varepsilon(x)$ from equation (10). Similarly with the Theorem 4.1 of [6] can be proven that the function $w(x,\omega) > 0$. Because of that we can introduce the new function $v(x,\omega)$,

$$
v(x,\omega) = \frac{\ln w}{\omega^2}. \tag{11}
$$

Then equation (10) transforms to the following equation

$$
\Delta v + \omega^2 |\nabla v|^2 = -\varepsilon(x)\mu, \;\; x \in \Omega, \tag{12}
$$

$$
v|_{\partial\Omega} = \varphi(x,\omega), \quad \forall\omega \in [\underline{\omega}, \overline{\omega}], \tag{13}
$$

where the function $\varphi(x,\omega)$ is generated by the function $g_1(x,t)$ in (5). Now, by knowing the function $v(x,\omega)$, the function $\varepsilon(x)$ can be computed explicitly from the equations (12)-(13) as

$$
\varepsilon(x) = \frac{1}{\mu}(-\Delta v - \omega^2 |\nabla v|^2). \tag{14}
$$

Next, we differentiate both sides of (12) with respect to ω. Since $\varepsilon(x)$ and μ does not depend on ω we eliminate $\varepsilon(x)$ from (12). All further steps in derivation of the globally convergent algorithm are similar to ones in pseudo-frequency domain, see [5]. We briefly outline these steps below. Let us denote

$$
q(x,\omega) = \partial_\omega v(x,\omega).
$$

In the following consideration we use the asymptotic behavior of the function $w(x,\omega)$ at $\omega \to \infty$ which follows from Lemma in [9]:

Lemma . *Assume that conditions (4) are satisfied. Let the function* $w(x,\omega) \in C^3\left(\mathbb{R}^3 \diagdown \{|x-x_0| < \varepsilon\}\right), \forall \varepsilon > 0$ *be the solution of the problem (10). Assume that geodesic lines, generated by the eikonal equation which corresponds to the function* $\varepsilon(x)$ *are regular, i.e. any two points in* $\mathbb{R}^3$ *can be connected by a single geodesic line. Let* $l(x,x_0)$ *be the length of the geodesic line connecting points* x *and* x_0. *Then the following asymptotic behavior of the function* w *and its derivatives takes place for* $|\beta| \leq 3, k = 0,1, x \neq x_0$

$$D_x^\beta D_\omega^k w(x,\omega) = D_x^\beta D_\omega^k \left\{ \frac{\exp\left[-(x,x_0)\right]}{f(x,x_0)} \left[1 + O\left(\frac{1}{\omega}\right)\right]\right\}, \omega \to \infty, \tag{15}$$

where $f(x,x_0)$ *is a certain function and* $f(x,x_0) \neq 0$ *for* $x \neq x_0$.

The proof of this Lemma is not presented here but can be done similarly with the proof of the similar Lemma in [9]. Thus, using (15) we can obtain the asymptotic behavior for functions $v(x,\omega)$ and $q(x,\omega)$:

$$\|v\|_{C^{2+\alpha}(\overline{\Omega})} = O\left(\frac{1}{\omega}\right), \|q\|_{C^{2+\alpha}(\overline{\Omega})} = O\left(\frac{1}{\omega^2}\right), \omega \to \infty. \tag{16}$$

We should verify the asymptotic behavior (16) numerically as it is done in the subsection 7.2 of [5] and section 3.1.2 in [4].

Now using (16) we obtain

$$v(x,\omega) = -\int_\omega^\infty q(x,\tau)\,d\tau, \tag{17}$$

or

$$v(x,\omega) = -\int_\omega^{\overline{\omega}} q(x,\tau)\,d\tau + V(x,\overline{\omega}), \tag{18}$$

where the truncation number $\overline{\omega} > \underline{\omega}$ is a large frequency which should be chosen in numerical experiments. The function $V(x,\overline{\omega})$ we call “the tail function” and it is defined as

$$V(x,\overline{\omega}) = -\int_{\overline{\omega}}^\infty q(x,\tau)\,d\tau.$$

Using definition of the function $v(x,\omega)$ (11), we obtain an equivalent formula for the tail,

$$V(x,\overline{\omega}) = \frac{\ln w(x,\overline{\omega})}{\overline{\omega}^2}. \tag{19}$$

Next, using (11), (12) and (17) we get the following nonlinear integral differential equation

$$\Delta q-2\omega^{2}\nabla q\cdot\int\limits_{\omega}^{\overline{\omega}}\nabla q\left(x,\tau\right)d\tau+2\omega\left[\int\limits_{\omega}^{\overline{\omega}}\nabla q\left(x,\tau\right)d\tau\right]^{2}+2\omega^{2}\nabla q\nabla V$$
$$-4\omega\nabla V\cdot\int\limits_{\omega}^{\overline{\omega}}\nabla q\left(x,\tau\right)d\tau+2\omega\left(\nabla V\right)^{2}=0,\quad x\in\Omega,\omega\in\left[\underline{\omega},\overline{\omega}\right],$$
$$q\mid_{\partial\Omega}=\psi\left(x,\omega\right):=\partial_{\omega}\varphi\left(x,\omega\right).\tag{20}$$

The $\omega-$integrals as well as the tail function of the above equation leads to the nonlinearity. Now we apply asymptotic (16) to get

$$\left\Vert V\left(x,\overline{\omega}\right)\right\Vert_{C^{2+\alpha}\left(\overline{\Omega}\right)}=O\left(\frac{1}{\overline{\omega}}\right),\overline{\omega}\rightarrow\infty,\tag{21}$$

From (21) follows that the tail function $V(x,\bar{\omega})$ is small for large values of the truncation of frequency $\overline{\omega}$.

Our previous numerical studies on this method have demonstrated that quality of the reconstruction will be better if we will use the new model for the tail function which we present in the next section.

2.4 New model of the tail function

In this subsection we present new approximate mathematical model which is based on the another representation model of the tail function $V(x,\bar{\omega})$.

Let the function $\varepsilon^{*}(x)$ be the exact solution of **IPB** for the exact data g^{*} in (5). Let $V^{*}\left(x,\bar{s}\right)$ be the exact tail function defined as

$$V^{*}\left(x,\overline{\omega}\right)=\frac{\ln w^{*}\left(x,\overline{\omega}\right)}{\overline{\omega}^{2}}.\tag{22}$$

Let $q^{*}\left(x,\omega\right)$ and $\psi^{*}\left(x,\omega\right)$ be the exact functions for q and ψ in (20), respectively. These functions are defined from the following nonlinear integral differential equation

$$\Delta q^{*}-2\omega^{2}\nabla q^{*}\cdot\int\limits_{\omega}^{\overline{\omega}}\nabla q^{*}\left(x,\tau\right)d\tau+2\omega\left[\int\limits_{\omega}^{\overline{\omega}}\nabla q^{*}\left(x,\tau\right)d\tau\right]^{2}+2\omega^{2}\nabla q^{*}\nabla V^{*}$$
$$-4\omega\nabla V^{*}\cdot\int\limits_{\omega}^{\overline{\omega}}\nabla q^{*}\left(x,\tau\right)d\tau+2\omega\left(\nabla V^{*}\right)^{2}=0,\quad x\in\Omega,\omega\in\left[\underline{\omega},\overline{\omega}\right],\tag{23}$$
$$q^{*}\mid_{\partial\Omega}=\psi^{*}\left(x,\omega\right):=\partial_{\omega}\varphi^{*}\left(x,\omega\right)\quad\forall\left(x,\omega\right)\in\partial\Omega\times\left[\underline{\omega},\overline{\omega}\right],$$

$$q^*(x,\omega) \in C^{2+\alpha}\left(\overline{\Omega}\right) \times C^1\left[\underline{\omega},\overline{\omega}\right]. \tag{24}$$

Let us assume that the functions V^* and q^* have the following asymptotic behavior

$$\begin{aligned} V^*(x,\overline{\omega}) &= \frac{p^*(x)}{\overline{\omega}} + O\left(\frac{1}{\overline{\omega}^2}\right) \approx \frac{p^*(x)}{\overline{\omega}},\ \overline{\omega} \to \infty, \\ q^*(x,\overline{\omega}) &= \partial_{\overline{\omega}} V^*(x,\overline{\omega}) = -\frac{p^*(x)}{\overline{\omega}^2} + O\left(\frac{1}{\overline{\omega}^3}\right) \approx -\frac{p^*(x)}{\overline{\omega}^2},\ \overline{\omega} \to \infty. \end{aligned} \tag{25}$$

Taking $\omega = \overline{\omega}$ in (23) we obtain the following equation

$$\begin{aligned} \Delta q^* + 2\overline{\omega}^2 \nabla q^* \nabla V^* + 2\overline{\omega} |\nabla V^*|^2 = 0, \quad x \in \Omega, \\ q^* \mid_{\partial\Omega} = \psi^*(x,\bar{\omega}) \quad \forall x \in \partial\Omega. \end{aligned} \tag{26}$$

Substituting the first terms in the asymptotic behavior (25) for the exact tail $V^*(x,\overline{\omega}) = \frac{p^*(x)}{\overline{\omega}}$ and for the exact function $q^*(x,\overline{\omega}) = -\frac{p^*(x)}{\overline{\omega}^2}$ into (26) we obtain

$$\begin{aligned} -\frac{\Delta p^*}{\overline{\omega}^2} - 2\bar{\omega}^2 \frac{\nabla p^*}{\overline{\omega}^2}\frac{\nabla p^*}{\overline{\omega}} + 2\bar{\omega}\frac{(\nabla p^*)^2}{\overline{\omega}^2} = 0, \quad x \in \Omega, \\ q^* \mid_{\partial\Omega} = \psi^*(x,\bar{\omega}) \quad \forall x \in \partial\Omega. \end{aligned} \tag{27}$$

From the equation above we get the following *approximate* Dirichlet boundary value problem for the function $p^*(x)$

$$\Delta p^* = 0 \text{ in } \Omega,\ p^* \in C^{2+\alpha}\left(\overline{\Omega}\right), \tag{28}$$

$$p^*|_{\partial\Omega} = -\overline{\omega}^2 \psi^*(x,\overline{\omega}). \tag{29}$$

Approximate Mathematical Model

There exists a function $p^*(x) \in C^{2+\alpha}\left(\overline{\Omega}\right)$ such that the exact tail function $V^*(x)$ has the form

$$V^*(x,\omega) := \frac{p^*(x)}{\omega},\ \forall \omega \ge \overline{\omega}. \tag{30}$$

Using (19) we assume that

$$V^*(x,\omega) = \frac{p^*(x)}{\overline{\omega}} = \frac{\ln w^*(x,\overline{\omega})}{\overline{\omega}^2}. \tag{31}$$

Since $q^*(x,\omega) = \partial_\omega V^*(x,\omega)$ for $\omega \ge \overline{\omega}$, we can get from (30)

$$q^*(x,\overline{\omega}) = -\frac{p^*(x)}{\overline{\omega}^2}. \tag{32}$$

Then we have the following formulas for the reconstruction of the coefficient $\varepsilon^*(x)$

$$\varepsilon^*(x) = \frac{1}{\mu}(-\Delta v^* - \omega^2 |\nabla v^*|^2),$$

$$v^* = -\int_{\omega}^{\overline{\omega}} q^*(x,\tau)\, d\tau + \frac{p^*(x)}{\overline{\omega}}.$$

Using the new approximate mathematical model we take the function

$$V_{1,1}(x) := \frac{p(x)}{\overline{\omega}}. \tag{33}$$

as the first guess for the tail function. Here, $p(x)$ is the solution of the problem (28)-(29) for $p^* = p$.

2.5 The layer stripping procedure with respect to the frequency

We consider a layer stripping procedure with respect to the frequency ω by dividing the interval $[\underline{\omega}, \overline{\omega}]$ into N small subintervals such that every interval has the step size $h = \omega_{n-1} - \omega_n$ in the frequency such that

$$\underline{\omega} = \omega_n < \omega_{n-1} < ... < \omega_0 = \overline{\omega}. \tag{34}$$

Now we approximate the function $q(x,\omega)$ as a piecewise constant function with respect to ω, $q(x,\omega) = q_n(x)$ for $\omega \in [\omega_n, \omega_{n-1})$. Then we set $q_0 = 0$ to get the following approximation for the integrals in (20):

$$\int_{\omega}^{\overline{\omega}} \nabla q(x,\tau)\, d\tau \approx (\omega_{n-1} - \omega)\nabla q_n(x) + h\sum_{j=0}^{n-1} \nabla q_j(x). \tag{35}$$

Next, we introduce the $\omega-$dependent Carleman Weight Function (CWF)

$$\mathscr{C}_{n,\lambda}(\omega) = \exp\left[\lambda\left(\omega - \omega_{n-1}\right)\right], \tag{36}$$

where $\lambda > 1$ is a large parameter, which is chosen computationally. Next we multiply both sides of equation (20) by $\mathscr{C}_{n,\lambda}(\omega)$ and integrating over (ω_n, ω_{n-1}), we obtain following system of equations with respect to the frequency for $x \in \Omega$

$$
\begin{aligned}
L_n(q_n) \; : \; &= \Delta q_n - A_{1,n}\left(h\sum_{j=0}^{n-1}\nabla q_j - \nabla V_n\right)\nabla q_n \\
&= 2B_n(\nabla q_n)^2 - A_{2,n}h^2\left(\sum_{j=0}^{n-1}\nabla q_j\right)^2 \\
&\quad + 2A_{2,n}\nabla V_n\left(h\sum_{j=0}^{n-1}\nabla q_j\right) - A_{2,n}(\nabla V_n)^2, \\
q_n \mid_{\partial\Omega} &= \psi_n(x) := \frac{1}{h}\int\limits_{\omega_n}^{\omega_{n-1}}\psi(x,\omega)\,d\omega,\; n=1,...,N.
\end{aligned}
\tag{37}
$$

Here numbers $A_{1,n}, A_{2,n}, B_n := \frac{I_{1,n}}{I_0}$ can be computed explicitly via formulas

$$
\begin{aligned}
I_0 : &= I_0(\lambda,h) = \int_{\omega_n}^{\omega_{n-1}} \mathscr{C}_{n,\lambda}(\omega)\,d\omega, \\
I_{1,n} : &= I_{1,n}(\lambda,h) = \int_{\omega_n}^{\omega_{n-1}} \omega(\omega_{n-1}-\omega)[\omega-(\omega_{n-1}-\omega)]\mathscr{C}_{n,\lambda}(\omega)\,d\omega, \\
A_{1,n} : &= A_{1,n}(\lambda,h) = \frac{2}{I_0}\int_{\omega_n}^{\omega_{n-1}} \omega[\omega-2(\omega_{n-1}-\omega)]\mathscr{C}_{n,\lambda}(\omega)\,d\omega, \\
A_{2,n} : &= A_{2,n}(\lambda,h) = \frac{2}{I_0}\int_{\omega_n}^{\omega_{n-1}} \omega\mathscr{C}_{n,\lambda}(\omega)\,d\omega.
\end{aligned}
\tag{38}
$$

In (37) functions V_n are determined from the iterative procedure described in the next section.

2.6 An Approximate Globally Convergent Algorithm

In this section we describe the algorithm for the numerical solution of (37). By index k we denote the number of iterations inside every frequency interval $[\omega_n, \omega_{n-1})$.

On the Step 0 we describe iterations with respect to the nonlinear term $(\nabla q_n)^2$ in (37). These iterations can be omitted since the nonlinear term is very small, see for details [5]. However, we include this step in the algorithm for the sake of completeness.

Step 0 Iteration $(n,1), n \geq 1$.
Suppose that the initial tail function $V_{n,0}(x,\overline{\omega}) \in C^{2+\alpha}(\overline{\Omega})$ is determined from (33). Suppose also that functions $q_{1,1}^0, ..., q_{n,1}^0 \in C^{2+\alpha}(\overline{\Omega})$ are already constructed. Then, we solve iteratively with respect to the nonlinear term the following problems, for $k = 1, 2, ...$

$$\begin{aligned}
&\Delta q_{n,1}^{k}-A_{1n}\left(h\sum_{j=1}^{n-1}\nabla q_{j}\right)\cdot\nabla q_{n,1}^{k}+A_{1n}\nabla q_{n,1}^{k}\cdot\nabla V_{n,0}\\
&\quad=2B_{1,n}\left(\nabla q_{n,1}^{k-1}\right)^{2}-A_{2n}h^{2}\left(\sum_{j=1}^{n-1}\nabla q_{j}\left(x\right)\right)^{2}\\
&\quad\quad+2A_{2n}\nabla V_{n,0}\cdot\left(h\sum_{j=1}^{n-1}\nabla q_{j}\left(x\right)\right)-A_{2n}\left(\nabla V_{n,0}\right)^{2},\\
&q_{n,1}^{k}=\overline{\Psi}_{n}\left(x\right),\qquad x\in\partial\Omega.
\end{aligned}$$

We obtain the function $q_{n,1}:=\lim_{k\to\infty}q_{n,1}^{k}$ such that $q_{n,1}\in C^{2+\alpha}(\overline{\Omega})$.

Step 1 Compute $\varepsilon_{n,1}$ via backwards calculations using the finite element formulation of the equation (10), see details in Chapter 3 of [5], or via the finite difference discretization of (14) as

$$\varepsilon_{n,1}\left(x\right)=\frac{1}{\mu}(-\Delta v_{n,1}-\omega_{n}^{2}\left|v_{n,1}\right|^{2}),x\in\Omega,$$

where functions $v_{n,1}$ are defined as

$$v_{n,1}\left(x\right)=-hq_{n,1}-h\sum_{j=0}^{n-1}q_{j}+V_{n,1}\left(x\right).$$

Step 2 Solve the forward problem (1) with $\varepsilon_{n}(x):=\varepsilon_{n,1}\left(x\right)$, calculate the Fourier transform and the function $w_{n,1}\left(x,\overline{\omega}\right)$.

Step 3 Find a new approximation for the tail function

$$V_{n,1}\left(x\right)=\frac{\ln w_{n,1}\left(x,\overline{\omega}\right)}{\overline{\omega}^{2}}. \tag{39}$$

Step 4 Iterations (n,i), $i\geq 2,n\geq 1$. We now iterate with respect to the tails (39). Suppose that functions $q_{n,i-1},V_{n,i-1}\left(x,\overline{\omega}\right)\in C^{2+\alpha}\left(\overline{\Omega}\right)$ are already constructed.

Step 5 Solve the boundary value problem

$$\begin{aligned}
&\Delta q_{n,i}-A_{1n}\left(h\sum_{j=1}^{n-1}\nabla q_{j}\right)\cdot\nabla q_{n,i}+A_{1n}\nabla q_{n,i}\cdot\nabla V_{n,i-1}\\
&\quad=2B_{1,n}\left(\nabla q_{n,i-1}\right)^{2}-A_{2n}h^{2}\left(\sum_{j=1}^{n-1}\nabla q_{j}\left(x\right)\right)^{2}\\
&\quad\quad+2A_{2n}\nabla V_{n,i-1}\cdot\left(h\sum_{j=1}^{n-1}\nabla q_{j}\left(x\right)\right)-A_{2n}\left(\nabla V_{n,i-1}\right)^{2},\\
&q_{n,i}\left(x\right)=\overline{\Psi}_{n}\left(x\right),\qquad x\in\partial\Omega.
\end{aligned}$$

Step 6 Compute $\varepsilon_{n,i}$ by backwards calculations using the finite element formulation of the equation (10) or via the finite difference discretization of (14)

$$\varepsilon_{n,i}(x) = \frac{1}{\mu}(-\Delta v_{n,i} - \omega_n^2 |v_{n,i}|^2), x \in \Omega,$$

where functions $v_{n,i}$ are defined as

$$v_{n,i}(x) = -hq_{n,i} - h\sum_{j=0}^{n-1} q_j + V_{n,i}(x).$$

Step 7 Solve the forward problem (1) with $\varepsilon_{n,i}$, compute the Fourier transform and obtain the function $w_{n,i}(x,\overline{\omega})$.

Step 8 Find a new approximation for the tail function

$$V_{n,i}(x) = \frac{\ln w_{n,i}(x,\overline{\omega})}{\overline{\omega}^2}.$$

Step 9 Iterate with respect to i and stop iterations at $i = m_n$ such that $q_{n,m_n} := \lim_{i\to\infty} q_{n,i}^k$. Stopping criterion for computing functions $q_{n,i}^k$ is

$$\text{either } F_n^k \geq F_n^{k-1} \text{ or } F_n^k \leq \eta, \tag{40}$$

where η is a chosen tolerance and F_n^k are defined as

$$F_n^k = \frac{||q_{n,i}^k - q_{n,i}^{k-1}||_{L_2}}{||q_{n,i}^{k-1}||_{L_2}}$$

Step 10 Set

$$q_n := q_{n,m_n}, \qquad \varepsilon_n(x) := \varepsilon_{n,m_n}(x), \qquad V_{n+1,0}(x) := \frac{\ln w_{n,m_n}(x,\overline{\omega})}{\overline{\omega}^2}.$$

Step 11 We stop computing functions $\varepsilon_{n,i}^k$ when

$$\text{either } N_n \geq N_{n-1} \text{ or } N_n \leq \eta, \tag{41}$$

where

$$N_n = \frac{||\varepsilon_n^k - \varepsilon_n^{k-1}||_{L_2(\Omega)}}{||\varepsilon_n^{k-1}||_{L_2(\Omega)}}. \tag{42}$$

3 Summary

We have presented a new approximate globally convergent algorithm for the reconstruction of dielectrics in the Maxwell equations in a frequency domain. To do that we have modified approximately globally convergent method of [5] in a pseudo-frequency domain. We believe that future numerical experiments will confirm validity of our new approximate globally convergent algorithm.

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