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PÄR JOHANNESSON
KRZYSZTOF PODGÓRSKI
IGOR RYCHLIK

Department of Mathematical Sciences
Division of Mathematical Statistics
CHALMERS UNIVERSITY OF TECHNOLOGY
UNIVERSITY OF GOTHENBURG
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Pär Johannesson, Krzysztof Podgórski
and Igor Rychlik

Department of Mathematical Sciences
Division of Mathematical Statistics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Gothenburg, Sweden

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Modelling roughness of road profiles on parallel tracks using roughness indicators

PÄR JOHANNESSON* KRZYSZTOF PODGÓRSKI** AND IGOR RYCHLIK***

Addresses:

* SP Technical Research Institute of Sweden, Box 24036, SE-400 22 Göteborg, Sweden
Par.Johannesson@sp.se (Corresponding author)

** Department of Statistics, Lunds University, SE-220 07 Lund, Sweden
podgorski@lu.se

*** Mathematical Sciences, Chalmers University of Technology, SE-412 96 Göteborg, Sweden
rychlik@chalmers.se

Abstract: The vertical road input is the most important load for durability assessments of vehicles. We focus on stochastic modelling of the parallel road profiles with the aim to find a simple but still accurate model for such bivariate records. A model is proposed that is locally Gaussian with randomly gamma distributed variances leading to a generalized Laplace distribution of the road profile. This Laplace model is paired with the ISO spectrum and is specified by only three parameters. Two of them can be estimated directly from a sequence of roughness indicators, such as IRI or ISO roughness coefficient. The third parameter needed to define the cross spectrum between the left and right road profiles is estimated from the sample correlation. Explicit approximations for the expected fatigue damage for the proposed Laplace-ISO model are developed and its usefulness is validated using measured road profiles.

Keywords: Road surface profile; road roughness; road irregularity; generalized Laplace distribution; non-Gaussian process; power spectral density (PSD); ISO spectrum; roughness coefficient; international roughness index (IRI); vehicle durability; fatigue damage.

Biographical notes: Pär Johannesson received his PhD in Mathematical Statistics in 1999 at Lund Institute of Technology, with a thesis on statistical load analysis for fatigue. He has published about 10 papers in international journals, and is a co-editor of the handbook "*Guide to Load Analysis for Durability in Vehicle Engineering*". During 2000 and 2001 he had a position as PostDoc at Mathematical Statistics, Chalmers within a joint project with PSA Peugeot Citroën, where he stayed one year at the Division of Automotive Research and Innovations in Paris. From 2002 to 2010 he was an applied researcher at the Fraunhofer-Chalmers Research Centre for Industrial Mathematics in Göteborg, and in 2010 he was a guest researcher at Chalmers. He is currently working as a research engineer at SP Technical Research Institute of Sweden, mainly within industrial and research projects on statistical methods for load analysis, reliability and fatigue.

Krzysztof Podgórski is Professor in Statistics at Lund University. He earned his PhD in Statistics at Michigan State University, East Lansing, in 1993, with a thesis entitled *Resampling methods for linear models*. His research interests spread between stochastic models and statistical methods as well as applications of these in engineering, geosciences, and economics. In particular, he is interested in spatial and temporal models that involve non-Gaussian stochastic models. He has published about 50 papers in peer reviewed journals in probability theory, statistics and applications and is the co-author of the monograph "*The Laplace Distribution and Generalizations. A Revisit with New Applications*". He has had positions at University of Northern Carolina, Chapel Hill; Indiana University – Purdue University, Indianapolis; University of Nevada, Reno; University of Limerick, Ireland.

Igor Rychlik is Professor in Mathematical Statistics at Chalmers University of Technology. He earned his PhD in 1986, with a thesis entitled *Statistical wave analysis with application to fatigue*. His main research interest is in fatigue analysis, wave climate modelling and in general engineering applications of the theory of stochastic processes, especially in safety analysis of structures interacting with the environment, for example through wind pressure, ocean waves, or temperature variations. He has published more than 80 papers in international journals, is the co-author of the text book "*Probability and Risk Analysis. An Introduction for Engineers*", and has been visiting professor (long term visits) at Department of Statistics, Colorado State University; Center for Stochastic Processes, University of North Carolina at Chapel Hill; Center for Applied Mathematics, Cornell University, Ithaca; Department of Mathematics, University of Queensland, Brisbane, Australia.

1 Introduction – parallel road tracks roughness

Modelling of road profiles is an important area of transportation engineering as durability studies of vehicle components often require a customer or market specific load description. The most desired properties of the models are robustness and simplicity, so that only a small number of parameters is used to describe short homogeneous parts of the road.

Considering just a single path along the road is an oversimplification, as any four-wheeled vehicle is subjected to excitations due to road roughness in the left as well as the right wheel paths. Thus, accounting for both the paths should be an important aspect of vehicle fatigue assessment. Hence, it is natural to search for an adequate and effective bivariate stochastic model corresponding to parallel road tracks.

Homogeneous Gaussian loads have been extensively studied in literature and applied as models for road roughness. Early applications of Gaussian processes to model road surface roughness can be found in (Dodds & Robson, 1973). Direct Gaussian models are convenient since linear filter responses to them are Gaussian processes as well. However, the authors of that paper were aware that Gaussian processes cannot "exactly reproduce the profile of a real road".

In (Charles, 1993), a non-homogeneous model is proposed that is constructed as a sequence of independent Gaussian processes of varying variances but the same standardized spectrum. This approach was further developed in (Bruscella et al., 1999; Rouillard, 2004, 2009). The variability of variances was modelled by a discrete distribution taking few values. In (Rouillard, 2004) random lengths of constant variance section were also considered. In those papers the problem of connecting the segments with constant variances into one signal was not addressed and thus the response was modelled as a sequence of homogeneous Gaussian processes, i.e. by a process of the same type as the model of road surface. Another approach has been proposed in (Bogsjö, 2007*a*) where a bivariate road model was constructed based on a Gaussian process with added random irregularities.

Although models based on Gaussian distributions are standard in the field (see, e.g., (Sun et al., 2001) and also (Múčka, 2004) for some recent studies), most experts of vehicle engineering agree that road surfaces are not, in fact, accurately represented by a Gaussian distribution, see (Dodds & Robson, 1973). One reason for this is that the actual roads contain short sections with above-average irregularity. As shown in (Bogsjö, 2007*b*), such irregularities cause most of the vehicle fatigue damage.

In the current paper a simple model for road profiles along two parallel paths taken by vehicle left and right wheels is presented. The work is an extension to parallel track of the single track models presented in (Johannesson & Rychlik, 2013; Bogsjö et al., 2012). It can be interpreted as a Gaussian vector valued process where the local variance is randomly varying according to a gamma distribution. More precisely, if the left and right road profiles are denoted by $Z_L(x)$ and $Z_R(x)$, respectively, then their variances over short road segments, say 100 m, are assumed to have constant value and they are Gaussian otherwise. The value of the variance is random between the short road segments and varies according to a gamma distribution. The so constructed process has a generalized Laplace distribution for its marginal, see (Kotz et al., 2001), motivating the name *Laplace road model*. The novelty of this paper lies in modelling dependence between two profiles $Z_L(x)$ and $Z_R(x)$ and providing a way to evaluate the accuracy of the damage prediction for a mechanical system subjected to available real road profiles.

The paper is organized as follows. First in Section 2 homogeneous Gaussian model having the ISO spectrum and a normalized cross spectrum function is reviewed. Further, means to estimate parameters in the spectrum from IRI sequences are also given. In Sections 4 and 5 non-homogeneous Gaussian and Laplace models, respectively, will be given. Then in Section 6 the damage index is introduced. In Section 7 means to evaluate the expected damage indices from the model parameters are given. Section 8 is devoted to validation of the proposed model using 70 km measured parallel road profiles. The paper closes with conclusions, acknowledgments, references and three appendixes. In the first appendix some details of derivations of the proposed estimate of expected damage index are given, while in the second one, an algorithm and MATLAB code to simulate from the Laplace model for parallel road profiles are given. Finally, in the third appendix some useful properties of the gamma distributed AR(1)-process are presented.

List of abbreviations

IRI	- International Roughness Index
ISO	- International Organization for Standardization
PSD	- Power Spectral Density

List of symbols and notation

α	- damage factor
b	- scale parameter in exponential normalized cross spectrum function
C	- roughness coefficient [m^3/rad]
$D_\theta(k)$	- damage index
$\mathsf{E}[X]$	- expectation of random variable X
$\mathcal{F}$	- Fourier transform
$g(x)$	- kernel for moving averages [$\mathrm{m}^{1/2}$]
$\Gamma(\cdot)$	- gamma function
$h_{ij}(x;v)$	- kernels defining responses
h_{rfc}	- rainflow cycle range
$H_v(\Omega)$	- transfer function of force response filter at speed v
IRI	- International Roughness Index [mm/m]
k	- damage exponent
$K(\Omega)$	- normalized cross spectrum
L	- length of road segments [m]
L_p	- length of a road profile [m]
λ_i	- spectral moments
ρ_{LR}	- correlation between parallel road profiles
r_j	- factors describing variability of variances
$r_{\mathrm{LR}}(\tau)$	- covariance function between left and right profiles
$S(\Omega)$	- road profile model spectrum [m^3/rad]
$\tilde{S}(\Omega)$	- normalized road profile model spectrum [m/rad]
$S_\theta(\Omega)$	- spectrum of vehicle force response [Nm/rad]
$S_{\mathrm{LR}}(\Omega)$	- cross spectrum
v	- vehicle speed [m/s]
$\mathsf{V}[X]$	- variance of random variable X
x	- position of a vehicle [m]
$Y(x)$	- force-response of a vehicle [N]
$Y_\theta(x)$	- linear combination of responses
θ	- angle defining the linear combination $Y_\theta(x)$
w	- waviness parameter in ISO spectrum
$Z(x)$	- road profile [m]
$Z_{\mathrm{L}}(x)$, $Z_{\mathrm{R}}(x)$	- left, right road profile [m]
κ	- kurtosis of road profile
ν	- scale parameter in gamma distribution
σ^2	- variance of road profile [m^2]
ω	- angular frequency [rad/s]
Ω	- spatial angular frequency [rad/m]
Ω_L , Ω_R	- cutoff frequencies defining the ISO spectrum [rad/m]

2 Road models with ISO spectrum

In this section, the Gaussian model of road profiles roughness along two parallel lines representing paths taken by vehicle left and right wheels are reviewed. Some material on the variability of the local roughness is also presented. More detailed presentations can be found in (Bogsjö, 2007*a*) and (Johannesson & Rychlik, 2013).

2.1 ISO road spectrum

The vertical road variability consists of the slowly changing landscape (topography), the road surface unevenness (road roughness), and the high variability components (road texture). For fatigue applications, the road roughness is the relevant part of the spectrum. Often one assumes that the energy for frequencies < 0.01 m^{-1} (wavelengths above 100 metres) represents landscape variability, which does not affect the vehicle dynamics and hence can be removed from the spectrum. Similarly high frequencies > 10 m^{-1} (wavelengths below 10 cm) are filtered out by the tire and thus are not included in the spectrum.

Following the ISO 8608 standard (ISO 8608, 1995), let us introduce the limits on the spectrum band of interest, viz.

$$\Omega_L = 2\pi/90 \text{ rad/m}, \qquad \Omega_R = 2\pi/0.35 \text{ rad/m}. \tag{1}$$

Further, the ISO standard uses a two parameter spectrum to describe the road profile $Z(x)$

$$S(\Omega) = C\left(\frac{\Omega}{\Omega_0}\right)^{-w}, \quad \Omega_L \le \Omega \le \Omega_R, \quad \text{and zero otherwise,} \tag{2}$$

where Ω is the spatial angular frequency, and $\Omega_0 = 1$ rad/m. The spectrum is parameterized by the degree of unevenness C, here called the roughness coefficient, and the waviness w. The ISO spectrum is often used for quite short road section (in the order of 100 metres). For road classification the ISO standard uses a fixed waviness $w = 2$. This simplified ISO spectrum has only one parameter, the roughness coefficient C. The ISO standard and classification of roads have been discussed by many authors, e.g. recently in (González et al., 2008; Ngwangwa et al., 2010).

The simplicity of the ISO spectrum makes it attractive to use in vehicle development. However, often the spectrum parameterized as in ISO 8608 does not provide an accurate description of real road spectra, and therefore many different parameterizations have been proposed, see e.g. (Andrén, 2006) where several spectral densities $S(\Omega)$ for modelling road profiles were compared. Further, in a study of roads in the USA, see (Kropáč & Múčka, 2008), estimated waviness values between 1 and 4 were found, with an average of $w = 2.5$. A typical waviness value of about 2.5 has also been reported by others, see e.g. (Andrén, 2006) for a study of Swedish roads and (Braun & Hellenbroich, 1991) for German roads. Therefore, two values of waviness will be used in this work, namely $w = 2$ and $w = 2.5$.

We will also use an alternative parametrization of the ISO spectrum by introducing the normalized ISO spectrum

$$\tilde{S}(\Omega) = C_0(\Omega/\Omega_0)^{-w}, \qquad C_0^{-1} = \left(\Omega_L^{1-w} - \Omega_R^{1-w}\right)/(w-1) \tag{3}$$

such that $\int \tilde{S}(\Omega)\, d\Omega = 1$. Here $C_0 = 0.0694$ m^3 if waviness $w = 2$, and $C_0 = 0.0273$ m^3 if $w = 2.5$. If we denote by σ^2 the fraction C/C_0, then the ISO spectrum can be written as

$$S(\Omega) = \sigma^2\, \tilde{S}(\Omega). \tag{4}$$

For simplicity, in the following we will assume that the left and the right road profiles have the same spectrum.

2.2 IRI - International Roughness Index

When monitoring road quality, segments of measured longitudinal road profiles are often condensed into a sequence of IRI values, see (Gillespie et al., 1986). The IRI is calculated using a quarter-car vehicle model whose suspension motion at speed 80 km/h is accumulated to yield a roughness index with units of slope (e.g. mm/m). Since its introduction in 1986, IRI has become the road roughness index most commonly used worldwide for evaluating and managing road systems. Thus, IRI parameters are often available from road databases maintained by road agencies and are typically reported for each 20 or 100 metres. For completeness, we give formulas to estimate the parameter C in the ISO spectrum (2) if the IRI value is known, viz.

$$\hat{C}_j = 10^{-6} \cdot \left(\frac{\hat{I}_j}{2.21}\right)^2, \quad \hat{C}_j = 10^{-6} \cdot \left(\frac{\hat{I}_j}{1.91}\right)^2 \tag{5}$$

for $w = 2, 2.5$, respectively, where $\hat{I}_j$ is an estimate of IRI in unit mm/m and $\hat{C}_j$ has unit m^3. A detailed presentation can be found in e.g. (Sun et al., 2001; Kropáč & Múčka, 2004, 2007; Johannesson & Rychlik, 2013).

2.3 Local road roughness

Suppose that for a 10 km long road section IRI values are available for each 100 metres, giving a sequence of estimates of the local roughness $\hat{C}_j$, $j = 1, \ldots, 100$, using Eq. (5). The observed variability of estimates $\hat{C}_j$ could be caused by statistical estimation errors and hence could be neglected. In such a case the road is homogeneous having PSD (2) with mean roughness C estimated as the average of M roughness values

$$\hat{C} = \frac{1}{M} \sum_{j=1}^{M} \hat{C}_j. \tag{6}$$

Consequently the variance σ^2 in (4) is estimated by

$$\hat{\sigma}^2 = \hat{C}/C_0, \tag{7}$$

while factors

$$r_j = \hat{C}_j/\hat{C}. \tag{8}$$

The variability of r_j is measured by means of variance, denoted by ν and estimated by

$$\hat{\nu} = \frac{1}{M-1} \sum_{j=1}^{M} (r_j - 1)^2. \tag{9}$$

Based on an extensive simulation study of homogeneous Gaussian road profiles it was found that $\hat{\nu}$ is only negligibly biased, with mean bias of 0.017. Hence, if $\hat{\nu}$ is less than 0.02 then we may assume that the variance σ^2 is constant for the road profile and homogeneous Gaussian model can be used, see Section 3.

However, quite often the variability of r_j is too large to be explained by solely statistical estimation errors and should be treated in a suitable way. For example, one could use different ISO spectra for each of the 100 metre road segments giving 100 parameters to describe the spectral properties of a 10 km long road, giving a non-homogeneous Gaussian model as presented in Section 4. An alternative is to propose a stochastic model for the variability of the factors r_j. Here the approach will be to model r_j as gamma distributed random variables, having mean one and variance ν, which results in the Laplace model presented in Section 5.

2.4 Correlation between road profiles

Recall that $Z_{\mathrm{R}}(x)$ and $Z_{\mathrm{L}}(x)$ denote the right and the left track elevations, respectively, at the location x. Consider a homogeneous road section. As before we assume that the right and the left tracks have the same distribution and ISO spectrum $S(\Omega)$ defined in Eqs. (2,4). In order to fully define the correlation structure of the bivariate process, besides the PSD one needs also to describe the cross-spectrum which is defined through the cross-covariance $r_{\mathrm{LR}}(\tau) = \mathbb{E}[Z_{\mathrm{L}}(x+\tau) \cdot Z_{\mathrm{R}}(x)]$, as follows

$$S_{\mathrm{LR}}(\Omega) = \int_{-\infty}^{+\infty} r_{\mathrm{LR}}(\tau) e^{-i\Omega\tau}\, d\tau.$$

Following (Bogsjö, 2007*a*; Bogsjö, 2008) we assume that for the parallel tracks the cross spectrum is real valued and we define the normalized cross spectrum $K(\Omega)$, say, by

$$K(\Omega) = \frac{S_{\mathrm{LR}}(\Omega)}{S(\Omega)}. \tag{10}$$

The correlation between $Z_{\mathrm{R}}(x)$ and $Z_{\mathrm{L}}(x)$ is denoted by $\rho_{\mathrm{LR}} = r_{\mathrm{LR}}(0)/\sigma^2$ and is now given by

$$\rho_{\mathrm{LR}} = \int \tilde{S}(\Omega)\, K(\Omega)\, d\Omega. \tag{11}$$

Note that $|K(\Omega)|^2$ is equal to the so-called (squared) coherence function.

As demonstrated in (Bogsjö, 2008), a function of the form

$$K(\Omega) = \exp(-b|\Omega|) \tag{12}$$

describes the correlation between the tracks in many measured signals rather well and will be used to model the measured road profiles in Section 8. Further, in that section the relation (11) will be used to estimate the parameter b, viz.

$$\hat{\rho}_{\mathrm{LR}} = \int \tilde{S}(\Omega)\, e^{-\hat{b}\,|\Omega|}\, d\Omega, \tag{13}$$

where $\hat{\rho}_{\mathrm{LR}}$ is the estimated correlation between $Z_{\mathrm{R}}(x)$ and $Z_{\mathrm{L}}(x)$. Unfortunately, there is no direct way to estimate the correlation ρ_{LR} from the measured IRI sequence.

3 Homogeneous Gaussian road profiles

A zero mean homogeneous bivariate Gaussian processes is completely defined by its two power spectra and the coherence function. There are several ways to generate Gaussian sample paths. The algorithm proposed in (Shinozuka, 1971) is often used in engineering. It is based on the spectral representation of a homogeneous process. Here we use an alternative way that expresses a Gaussian process as a moving average of white noise.

Roughly speaking a moving average process is a convolution of a kernel function $g(x)$, say, with a infinitesimal "white noise" process having variance equal to the spatial discretization step, say $\mathrm{d}x$. Consider a kernel function $g(x)$, then a Gaussian process can be approximated by

$$Z(x) \approx \sum g(x - x_i)\, Z_i \sqrt{\mathrm{d}x}, \tag{14}$$

where the Z_i's are independent standard Gaussian random variables, while $\mathrm{d}x$ is the discretization step, here reciprocal of the sampling frequency (often $\mathrm{d}x = 5$ cm for road profiles). An appropriate choice of the length of the increment $\mathrm{d}x$ is related to smoothness of the kernel. The kernel $g(x)$ is conveniently defined by its Fourier transform, viz.

$$(\mathcal{F}g)(\Omega) = \sqrt{2\pi\, S(\Omega)}, \tag{15}$$

where $\mathcal{F}$ stands for Fourier transform.

As can be seen in Figure 3 the parallel road profiles $Z_{\mathrm{R}}(x)$ and $Z_{\mathrm{L}}(x)$ are strongly correlated. Now we give a method to generate bivariate Gaussian processes having a real valued normalized cross spectrum. There are several means to do it and the algorithm proposed in (Shinozuka, 1971) is most commonly used. Here we present a method proposed in (Kozubowski et al., 2013) valid only for processes with real valued normalized cross spectrum $K(\Omega)$, however also applicable to non-Gaussian moving averages.

Let us introduce two kernels $\tilde{g}_1(x)$ and $\tilde{g}_2(x)$ by means of the Fourier transforms

$$\begin{aligned}(\mathcal{F}\tilde{g}_1)(\Omega) &= \left(\sqrt{1+K(\Omega)} + \sqrt{1-K(\Omega)}\right)/2,\\ (\mathcal{F}\tilde{g}_2)(\Omega) &= \left(\sqrt{1+K(\Omega)} - \sqrt{1-K(\Omega)}\right)/2,\end{aligned} \tag{16}$$

that will be used to introduce the correlation between tracks. As above, the kernel $g(x)$, defined in Eq. (15), will be used to get the desired PSD. Now define the two kernels $g_1(x)$ and $g_2(x)$ by

$$(\mathcal{F}g_1)(\Omega) = (\mathcal{F}g)(\Omega)\cdot(\mathcal{F}\tilde{g}_1)(\Omega), \qquad (\mathcal{F}g_2)(\Omega) = (\mathcal{F}g)(\Omega)\cdot(\mathcal{F}\tilde{g}_2)(\Omega). \tag{17}$$

Gaussian processes having spectrum $S(\Omega)$ and coherence $|K(\Omega)|^2$ are given by the Gaussian moving averages

$$\begin{aligned}Z_{\mathrm{L}}(x) &\approx \left(\sum g_1(x-x_i)\, Z_{1i} + \sum g_2(x-x_i)\, Z_{2i}\right)\sqrt{\mathrm{d}x},\\ Z_{\mathrm{R}}(x) &\approx \left(\sum g_2(x-x_i)\, Z_{1i} + \sum g_1(x-x_i)\, Z_{2i}\right)\sqrt{\mathrm{d}x}\end{aligned} \tag{18}$$

where Z_{1i}, Z_{2i}'s are independent standard Gaussian random variables and with equality in limit as $\mathrm{d}x$ tends to zero.

4 Non-homogeneous Gaussian road profiles

Homogeneous Gaussian loads have been extensively studied in literature and applied as models for road roughness, see e.g. (Dodds & Robson, 1973) for an early application. However, the authors of that paper were aware that Gaussian processes cannot "exactly reproduce the profile of a real road". In (Charles, 1993) a non-homogeneous model was proposed, constructed as a sequence of independent Gaussian processes of varying variances but the same standardized spectrum. Similar approaches were used in (Bruscella et al., 1999; Rouillard, 2004, 2009). Here an alternative approach to derive non-homogeneous bivariate Gaussian profiles will be presented.

The homogeneous Gaussian model for $Z_{\mathrm{L}}(x), Z_{\mathrm{R}}(x)$, presented in Section 3, are essentially filters of sequences of independent standard Gaussian variables Z_{1i}, Z_{2i}, which serve as two Gaussian noise sequences. Here we will use a similar constructions but allow Z_{1i}, Z_{2i} to have variable variances. Then the profiles $Z_{\mathrm{L}}(x), Z_{\mathrm{R}}(x)$, $x \in [0, L]$, will be defined using the same algorithms as described in Eq. (18). Consequently, in the special case when Z_{1i}, Z_{2i} have constant variance, the homogeneous Gaussian model will be retrieved. We begin with the definition of the non-homogeneous Gaussian noise.

Similarly as in (Bogsjö et al., 2012; Johannesson & Rychlik, 2013), let us assume that the road consists of M equally long segments of length $L = L_p/M$, where the j:th segment has the same normalized spectrum $\tilde{S}(\Omega)$ and variance $r_j\sigma^2$, where σ^2 is the average variance of the road profile, while r_j's are positive factors. Consequently, the spectrum of the j:th segment is $S_j(\Omega) = r_j\sigma^2\,\tilde{S}(\Omega)$.

Let Z_{1i}, Z_{2i} be sequences of independent standard Gaussian variables. Further, let $\mathrm{d}x$ be the sampling step of the process and $[s_{j-1}, s_j]$, $s_j - s_{j-1} = L$, the interval where the road profile model have PSD $S_j(\Omega) = r_j\sigma^2\,\tilde{S}(\Omega)$, viz. $s_0 = 0 < s_1 < \ldots < s_M = L_p$. Now define the non-homogeneous Gaussian noise sequences $\tilde{Z}_{1i}$, $\tilde{Z}_{2i}$ as follows

$$
\begin{aligned}
\tilde{Z}_{1i} &= \sqrt{r_j}\,Z_{1i}, \quad \text{if} \quad s_{j-1} < x_i \le s_j, \\
\tilde{Z}_{2i} &= \sqrt{r_j}\,Z_{2i}, \quad \text{if} \quad s_{j-1} < x_i \le s_j.
\end{aligned} \tag{19}
$$

The non-homogeneous Gaussian processes, $Z_{\mathrm{L}}(x), Z_{\mathrm{R}}(x)$, $x \in [0, L_p]$, having locally PSDs $S_j(\Omega) = r_j\sigma^2\tilde{S}(\Omega)$ and coherence $|K(\Omega)|^2$ are given by the moving averages

$$
\begin{aligned}
Z_{\mathrm{L}}(x) &\approx \Big(\sum g_1(x - x_i)\,\tilde{Z}_{1i} + \sum g_2(x - x_i)\,\tilde{Z}_{2i}\Big)\sqrt{\mathrm{d}x}, \\
Z_{\mathrm{R}}(x) &\approx \Big(\sum g_2(x - x_i)\,\tilde{Z}_{1i} + \sum g_1(x - x_i)\,\tilde{Z}_{2i}\Big)\sqrt{\mathrm{d}x}
\end{aligned} \tag{20}
$$

where the kernels $g_1(x), g_2(x)$ are defined in Eq. (17).

5 Non-homogeneous Laplace road profiles

The non-homogeneous Gaussian model requires M parameters in order to model the variability in the variance, i.e. the varying local roughness. In order to reduce the number of parameters it is desirable to use a stochastic model for the sequence of r_j-values. A non-homogeneous Laplace process is obtained by modelling the r_j's as gamma distributed random variables, thus, only one parameter is needed to model the variability in the variance, namely, the Laplace shape parameter ν.

By definition of the factors r_j in Eq. (8), the gamma factors have mean one and variance ν and hence the gamma distribution has the following probability density function

$$
f(r) = \frac{1}{\Gamma(1/\nu)\nu^{1/\nu}}\, r^{1/\nu - 1}\exp(-r/\nu), \tag{21}
$$

where $\Gamma(\cdot)$ is the gamma function. In order to completely define the Laplace model one also needs to specify the dependence structure of the sequence of factors r_j. The simplest

Laplace model is obtained by assuming that the factors are independent, i.e. the roughness of the segments vary in an independent manner. Then, the non-homogeneous Laplace model can be described by one extra parameter ν, compared to the homogeneous Gaussian model.

To summarize, the bivariate Laplace-ISO model with spectrum $S(\Omega)$ and normalized cross spectrum $K(\Omega)$ is described by only three parameters, namely, the mean roughness C, the Laplace shape parameter ν, and the correlation between tracks ρ_{LR}. The parameters are estimated according to Eqs. (6,9,13). Laplace profiles $Z_{\text{L}}(x), Z_{\text{R}}(x)$, $x \in [0, L]$, can be constructed in the same way as the non-homogeneous Gaussian profiles, by using Eqs. (19,20) but replacing the r_j-factors in Eq. (19) by random numbers generated from the gamma distribution in Eq. (21). A MATLAB script to simulate parallel road profiles is given in Appendix II.

5.1 Laplace model with correlated variances

It seems reasonable to believe that the quality of the road surface varies slowly and hence the factors r_j defined in Eq. (8) are likely dependent between themselves. The degree of dependence is a function of the chosen length of the constant variance segments (here 100 metres). In this section we present an autoregressive model for random variances r_j.

The classical Gaussian AR(1)-process x_j, having mean zero and variance one, is defined by a recursion

$$x_j = \rho\, x_{j-1} + \sqrt{1-\rho^2}\epsilon_j, \tag{22}$$

where ϵ_j are independent zero mean variance one Gaussian variables (Gaussian white noise). Further, the parameter ρ is the correlation between x_{j-1} and x_j. Defining a gamma distributed AR(1)-process for is a more difficult problem which is discussed in Appendix III. A MATLAB script to simulate the gamma AR(1)-process is given in Appendix II. Here we just outline the construction.

The gamma AR(1)-process r_j can be defined by a recursion similar to Eq. (22) viz.

$$r_j = \rho\, k_j r_{j-1} + (1-\rho)\epsilon_j, \tag{23}$$

where ϵ_j are independent gamma distributed variables with mean one and variance equal to the parameter ν. Here k_j are random factors dependent on r_{j-1} and which in average are equal one. The factors can be derived using the following relation

$$k_j = \frac{1}{m}\sum_{i=0}^{N} W_i,$$

where W_i are independent random variables, $W_0 = 0$ while, for $i > 0$, W_i are exponentially distributed. Further, N is a Poisson distributed integer having mean m equal to

$$m = \frac{\rho/\nu}{1-\rho} r_{j-1}. \tag{24}$$

We observe two simple consequences of the above

$$\begin{aligned} \mathsf{E}\,[k_j|r_{j-1}] &= 1, \\ \mathsf{V}\,[k_j|r_{j-1}] &= \frac{2\nu(1-\rho)}{r_{j-1}\rho}. \end{aligned}$$

Note that, for $\rho = 0$ Eq. (24) equals $m = 0$, giving $k_j = 0$, and consequently $r_j = \epsilon_j$ becomes a sequence of independent gamma factors (gamma white noise).

It follows from the properties listed in Appendix III that the r_j's also have the same gamma distribution and their autocorrelation function has the form

$$\rho(j) = \rho^j, \quad j = 0, 1, 2, \ldots.$$

The r_j's can be utilized by taking their one step autocorrelation coefficient $\hat{\rho}$ as an estimate of the autoregressive coefficient ρ

5.2 A simulation example

If the parameter ν is moderate then the contribution to the damage of transients caused by changes of the normalized variance r_j should be negligible. Thus, the expected damage index will be the same for models with independent factors r_j as with the correlated ones. However, the variance of the damage index will not be the same.

In the Figure 1 two simulated road profiles are illustrated, both having ISO spectrum with waviness parameter $w = 2.5$ and standard deviation $\sigma = 0.02$ m. The factors r_j are gamma distributed with parameter $\nu = 0.18$, see Eq. (21). In the top plot the Laplace model has independent gamma distributed variances, while in the bottom plot the variances form an AR(1)-process with correlation $\rho = 0.9$. One can see that regions with larger/smaller variability of road profiles are longer for correlated variances. Such a region may last for several kilometres.

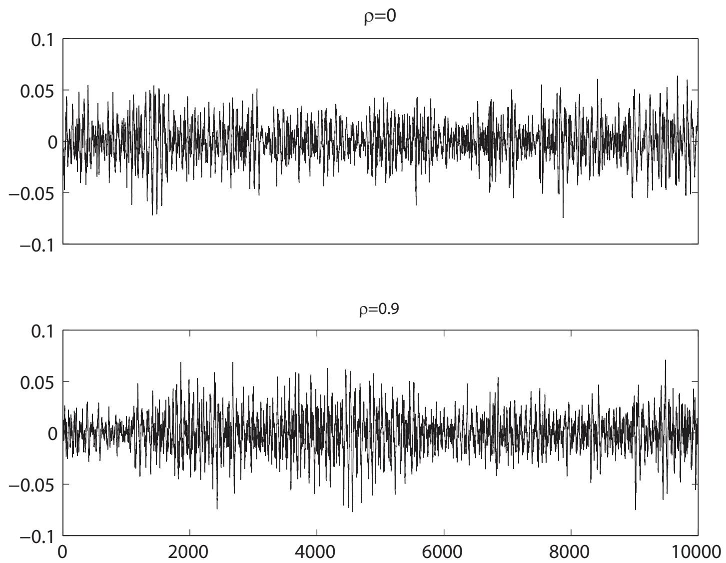


Figure 1: Comparison of 10 km simulated Laplace models having ISO spectrum, with waviness parameter $w = 2.5$ and parameters $\sigma = 0.02$ m and $\nu = 0.18$, values are taken from Table 3 sixth column. The variance of the factors r_j is $\nu = 0.18$. Top graph has independent random factors ($\rho = 0$) and the bottom graph has correlated AR(1) factors with $\rho = 0.9$.

6 Damage index

Models of roads along parallel tracks are often used to investigate more complex responses of a vehicle, for example vertical and rolling motions which are then used to evaluate stresses in some components, see (Johannesson & Speckert, 2013) for a more detailed presentation. For a stiff structure, stresses used to predict fatigue damage are linear combinations of forces and/or moments. Here, the left and right responses are simply denoted by $Y_1(x)$, $Y_2(x)$, respectively. Subsequently, we use the rainflow damage estimates computed for

$$Y_\theta(x) = \cos(\theta)Y_1(x) + \sin(\theta)Y_2(x). \tag{25}$$

to evaluate accuracy of the proposed model for the bivariate road profiles $Z_{\mathrm{L}}(x)$ and $Z_{\mathrm{R}}(x)$.

The fatigue damage accumulated in a material is expressed using a fatigue (damage) index defined by means of the rainflow method. It is computed in the following two steps. First rainflow ranges h_n^{rfc}, $n = 1, \dots, N$ in $Y_\theta(x)$ are found, then the rainflow damage is computed according to Palmgren-Miner rule (Palmgren, 1924; Miner, 1945), viz.

$$D_k(Y_\theta) = \alpha \sum_{i=1}^{N} (h_i^{rfc})^k, \tag{26}$$

see (Rychlik, 1987) for details. For compactness of the notation, we abbreviate $D_k(Y_\theta)$ to $D_k(\theta)$. Various choices of the damage exponent k can be considered, e.g. $k = 3$ which is the standard value for the crack growth process, is often used. For comparison we consider also $k = 5$, that is often used when a fatigue process is dominated by the crack initiation phase. The index $D_k(\theta)$ is often called the multi-axial damage and was first introduced in (Beste et al., 1992), see also (Rychlik, 1993*a*). A good model for $Z_{\mathrm{L}}(x), Z_{\mathrm{R}}(x)$ requires good accuracy of damage predictors for any choice of the angle θ.

For stationary loads, the damage intensity, measuring the average speed the damage grow, is used as a damage index. For stationary Gaussian responses the damage index depends only on the PSD $S_\theta(\Omega)$ of the response $Y_\theta(x)$ and it will be denoted by

$$\mathcal{D}_k(S_\theta) = \mathsf{E}\left[D_k(Y_\theta)\right]/L_p, \tag{27}$$

where L_p is the length of a road profile.

There are many approximation of the $\mathcal{D}_k(S_\theta)$ proposed in the literature for stationary Gaussian responses, see (Bengtsson & Rychlik, 2009) for comparison of the methods. Here we will use the narrow band bound, proposed first by (Bendat, 1964) and (Rychlik, 1993*b*) where it was demonstrated that the approximation is actually a bound, viz. for a Gaussian response with PSD $S(\Omega)$

$$\mathcal{D}_k(S) \le \alpha\, \lambda_0^{(k-1)/2} \lambda_2^{1/2}\, 2^{3k/2}\, \Gamma(1 + k/2)/2\pi. \tag{28}$$

Here λ_0, λ_2 are spectral moments of $S(\Omega)$ defined by

$$\lambda_i = \int_0^\infty \Omega^i\, S(\Omega)\, d\Omega. \tag{29}$$

Means to evaluate damage intensity for non-Gaussian is not as well developed. However some works do exists, see e.g. (Bogsjö et al., 2012), (Rychlik, 2013) or (Kvanström et al., 2013) for the multivalued case.

6.1 Definition of responses

In general the responses $Y_1(x), Y_2(x)$, are computed by linearly filtering signals $Z_{\text{L}}(x)$, $Z_{\text{R}}(x)$. The responses depend on the velocity of the moving vehicle and could be computed as follows

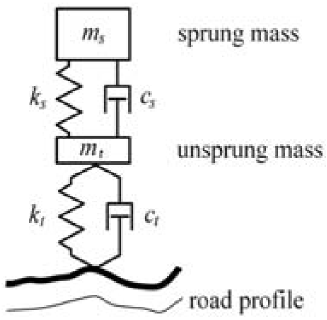
$$Y_i(x) = \int_{-\infty}^{x} h_{i1}(x-u;v) Z_{\text{L}}(u)\,\mathrm{d}u + \int_{-\infty}^{x} h_{i2}(x-u;v) Z_{\text{R}}(u)\,\mathrm{d}u, \tag{30}$$

where $h_{ij}(\cdot;v)$, $i,j=1,2$ are impulse responses of the filter at a given speed v.

In this work the damage index is used to quantify the severity of road conditions and not to actually predict the fatigue life of components. Consequently, we propose to use a very simple uncoupled symmetrical filter to describe responses $Y_i(x)$ satisfying

$$h_{11}(u;v) = h_{22}(u;v) = h_v(u), \qquad h_{12}(u;v) = h_{21}(u;v) = 0 \tag{31}$$

where $h_v(u)$ and $H_v(u)$ denote impulse response and transfer function, respectively, of the quarter truck driving with constant speed v. Thus $Y_1(x)$, $Y_2(x)$ are responses of two quarter-vehicles travelling with the same speed on left and right road profile. Such a simplification of a physical vehicle cannot be expected to predict loads exactly, but it will highlight the most important road characteristics as far as fatigue damage accumulation is concerned. The parameters in the model are set to mimic heavy vehicle dynamics, following (Bogsjö, 2007*b*), see Figure 2.



Quarter Truck

Symbol	Value	Unit
m_s	3 400	kg
k_s	270 000	N/m
c_s	6 000	Ns/m
m_t	350	kg
k_t	950 000	N/m
c_t	300	Ns/m

Figure 2: Quarter vehicle model.

Neglecting possible "jumps", which occur when a vehicle looses contact with the road surface, the response of the quarter-vehicle, i.e. the force $Y(x) = m_s \ddot{X}_s(x)$, as a function of vehicle location x, can be computed through linear filtering of the road profile. The filter at speed v has the following transfer function

$$H_v(\Omega) = \frac{m_s\omega^2(k_t + i\omega c_t)}{k_t - \dfrac{(k_s + i\omega c_s)\omega^2 m_s}{-m_s\omega^2 + k_s + i\omega c_s} - m_t\omega^2 + i\omega c_t}\left(1 + \frac{m_s\omega^2}{k_s - m_s\omega^2 + i\omega c_s}\right), \tag{32}$$

where $\omega = \Omega \cdot v$ is the angular frequency having units rad/s.

7 Expected damage index for the Laplace model

Neglecting the contribution to damage from transients caused by variability of local variances of road profiles leads to the following approximation

$$\mathsf{E}\left[D_k(Y_\theta)\right] \approx L_p\,\mathcal{D}_k(S_\theta)\,\mathsf{E}\left[R^k\right].$$

Since R is gamma distributed according to Eq. (21)

$$\mathsf{E}\left[D_k(Y_\theta)\right] \approx L_p\, \mathcal{D}_k(S_\theta)\, \nu^{k/2} \frac{\Gamma(k/2+1/\nu)}{\Gamma(1/\nu)}, \tag{33}$$

where the PSD $S_\theta(\Omega)$ is given by

$$S_\theta(\Omega) = \frac{C}{C_0}\, |H_v(\Omega)|^2 \tilde{S}(\Omega)\, (1+\sin(2\theta)K(\Omega))\,. \tag{34}$$

The PSD $S_\theta(\Omega)$ is a function of a number of loading parameters; C the mean roughness coefficient of the ISO model of the road profile; w the waviness parameter in the ISO spectrum; ρ_{LR} the correlation between left and right road profiles, and v the speed of the quarter vehicle. In addition the expected damage index depends on parameters α, k in S-N curve. For uncertainty analysis it is desirable to have an explicit formula factorizing the dependence of $\mathsf{E}\left[D_k(Y_\theta)\right]$ on the parameters. This is achieved by combining the narrow-band bound (28) for $\mathcal{D}_k(S_\theta)$ with Eq. (33) leading to the following factorization of the expected damage index for the Laplace model of parallel road profiles

$$\mathsf{E}\left[D_k(Y_\theta)\right] \approx L_p{\cdot}\alpha{\cdot}\left(\frac{C}{C_0}\right)^{k/2}{\cdot} c_k(\theta,\rho_{\mathrm{LR}},w){\cdot}\nu^{k/2}\frac{2^{3k/2}\,\Gamma(1+k/2)\Gamma(k/2+1/\nu)}{2\pi\,\Gamma(1/\nu)}{\cdot}\left(\frac{v}{v_0}\right)^{k(w-1)/2-1} \tag{35}$$

where v_0 is the reference vehicle speed, here $v_0 = 10$ m/s. An explicit formula for the factor $c_k(\theta,\rho_{\mathrm{LR}},w)$ is derived in Appendix I.

8 Validation of models

8.1 Measured road profile

The data are measurements of surface irregularity along two parallel tracks. The tracks are separated by 2.4 metres and are about 70 km long. The signals are filtered and the harmonics with wavelength longer than 100 metiers are removed. The removed signal is assumed to represent landscape variability which does not influence the fatigue accumulation in heavy vehicles. The left and right tracks are presented in Figure 3. Plots presented in Figure 3 indicate that the data are likely non-homogeneous. Therefore, the data are divided into seven 10 km long records. In this section some statistics for the measure profiles are presented, checking whether the left and right measured profilers seems to have the same statistical properties. Then the Laplace-ISO model will be fitted to the data. All calculations have been performed using MATLAB together with the toolbox WAFO (Wave Analysis for Fatigue and Oceanography), see (Brodtkorb et al., 2000; WAFO Group, 2011*a*), which can be downloaded free of charge, (WAFO Group, 2011*b*).

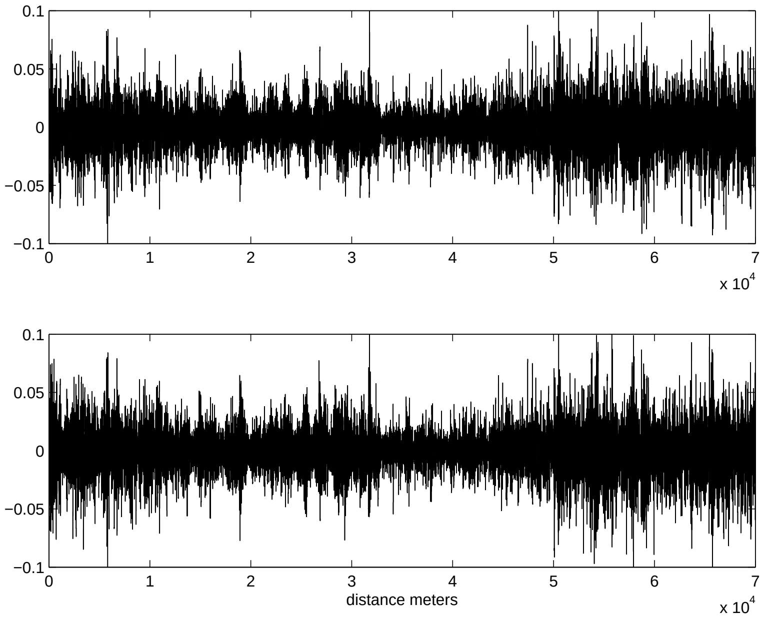


Figure 3: Road roughness for left and right tracks for 70 km of a Swedish road.

In Tables 1 and 2 some basic statistics for the seven segments are given, and the observed damage indices D_k^{obs} are also presented. The parameter α is chosen in such a way that $D_k^{obs}(0) + D_k^{obs}(\pi/2) = 2$ for the 70 km long road profile. One can see that most of the damage is accumulated in road profiles 1, 6 and 7. It is important that the proposed model works well for these roads. Based on the presented statistics one can conclude that the same statistical model can be used for left and right road profiles.

Table 1: Statistics for left profiles for the seven road segments. In rows 5 and 6 relative damages for damage exponents $k = 3, 5$, respectively, are given.

Left profile							
road segment	1	2	3	4	5	6	7
std [m]	0.023	0.017	0.017	0.014	0.017	0.026	0.023
correlation ρ_{LR}	0.89	0.92	0.91	0.86	0.92	0.83	0.80
skewness	-0.11	0.04	0.08	0.27	0.29	0.13	-0.09
kurtosis	3.84	3.45	3.17	7.05	3.85	3.65	4.12
$D_3^{obs}(0)$	0.15	0.04	0.02	0.08	0.05	0.38	0.29
$D_5^{obs}(0)$	0.28	0.02	0.00	0.11	0.02	0.44	0.24

Table 2: Statistics for right profiles for the seven road segments. In rows 4 and 5 relative damages for damage exponents $k = 3, 5$, respectively, are given.

Right profile							
road segment	1	2	3	4	5	6	7
std [m]	0.024	0.018	0.018	0.015	0.016	0.027	0.023
skewness	-0.15	-0.02	0.06	0.28	0.24	0.05	-0.02
kurtosis	3.95	3.68	3.01	6.43	4.00	3.60	3.80
$D_3^{obs}(\pi/2)$	0.18	0.05	0.03	0.06	0.05	0.32	0.31
$D_5^{obs}(\pi/2)$	0.12	0.03	0.01	0.05	0.02	0.35	0.29

The Laplace models with ISO spectrum have been fitted to the seven road segments. The parameters are presented in Table 3. Note that the standard deviations $\sigma = \sqrt{C/C_0}$ of the Laplace model for road profiles given in rows 3 and 6 of Table 3 are smaller than the estimated standard deviations in measured road profiles given in Tables 1 and 2. This is not a bias or an error but a consequence of the fact that the ISO spectra have been fitted using IRI. This estimation method fits the ISO spectrum to the observed road spectrum in the frequency region which is amplified by the golden car response filter. Finally, in the last row of Table 3 the correlations ρ between the successive factors r_j are presented. The correlations are not very strong.

Table 3: Parameters in the Laplace-ISO models for road profiles along parallel tracks. The parameter b in the normalized cross spectrum $K(\Omega)$, see Eq. (12), was estimated using Eq. (13) with correlation coefficient between left and right road profiles ρ_{LR} given in Table 1, row 2.

road segment	1	2	3	4	5	6	7
$C/10^{-6}$ [m^3/rad] when $w = 2$	2.90	1.21	1.10	1.31	2.34	7.06	6.31
σ [m] when $w = 2$	0.007	0.004	0.004	0.004	0.006	0.01	0.01
b when $w = 2$	0.41	0.27	0.31	0.56	0.29	0.72	0.91
$C/10^{-6}$ [m^3/rad] when $w = 2.5$	3.88	1.62	1.47	1.76	3.14	9.45	8.45
σ [m] when $w = 2.5$	0.012	0.008	0.007	0.008	0.011	0.019	0.018
b when $w = 2.5$	0.69	0.47	0.54	0.91	0.51	1.15	1.42
ν	0.48	0.76	0.44	1.18	0.34	0.33	0.24
ρ	0.15	0.58	0.46	0.33	0.55	0.41	0.21

8.2 Expected Damage index

In this section the observed damage indices $D_k^{obs}(\theta)$ will be compared to the expected damage indices estimated for the Laplace model using formula (35). The parameters of the model are presented in Table 3. The validation of the Laplace model will be done in two steps. Firstly, the univariate case will be considered, i.e. we will check if the expected damage evaluated for Laplace model agrees with the damage index estimated for left and right profiles separately. Secondly, the expected damage $\mathsf{E}\left[D_k(Y_\theta)\right]$ evaluated using Eq. (35) will be compared to $D_k^{obs}(\theta)$.

Checking accuracy of the univariate Laplace model: The left and right tracks are modelled by the same Laplace model $\mathsf{E}\left[D_k(Y_0)\right] = \mathsf{E}\left[D_k(Y_{\pi/2})\right]$ hence only $\mathsf{E}\left[D_k(Y_0)\right]$ will compared with the average observed damage index $(D_k^{obs}(0) + D_k^{obs}(\pi/2))/2$ by means of the ratio

$$d_k = \frac{\mathsf{E}\left[D_k(S_0)\right]}{(D_k^{obs}(0) + D_k^{obs}(\pi/2))/2}. \tag{36}$$

The values of d_k close to one means very good agreement between observed damages and the one computed using the Laplace model. Values between 0.5 and 2 would indicate a good agreement.

In Table 4 the damage ratios (36) are given for $k = 3, 5$ and two ISO models having waviness parameters $w = 2, 2.5$, respectively. One can see that for the damage exponent $k = 3$ both ISO models give a very good agreement between the indices computed using the model and the observed ones. For the higher damage exponent $k = 5$ the agreement is not as good. The damage indices for $k = 5$ have been underestimated. Further, one can see that the ISO spectrum with $w = 2.5$ is working better giving underestimation of the total damage to be 30% which is an acceptable accuracy for this kind of model.

Table 4: Relative damage indices d_k, see Eq. (36), for 10 km long road segments for waviness parameter $w = 2$ and $w = 2.5$ and two damage exponents $k = 3, 5$.

Relative damage index d_k, see Eq. (36), for 10 km long roads							
road segment	1	2	3	4	5	6	7
$k = 3$, $w = 2$	0.70	0.77	1.05	0.60	1.56	1.18	1.14
$k = 5$, $w = 2$	0.13	0.15	0.28	0.10	0.59	0.51	0.50
$k = 3$, $w = 2.5$	0.77	0.85	1.16	0.66	1.73	1.31	1.26
$k = 5$, $w = 2.5$	0.2	0.23	0.42	0.15	0.88	0.76	0.74

In (Bogsjö et al., 2012) and (Johannesson & Rychlik, 2013) the Laplace model was validated for a single road profile. Here we are again validating the univariate Laplace model on the data presented in Section 8.1. The difference to the previously presented studies is that now the average Laplace model fitted using both left and right profiles was employed.

Checking accuracy of bivariate Laplace model: For the univariate Laplace model the expected damage indices were independent of the correlation ρ_{LR} between the two tracks. Basically, in the univariate case the factor c_k in Eq. (35) is constant and

$$c_k(0, \rho_{\text{LR}}, w) = c_k(0, 0, w)$$

for all values ρ_{LR}. In contrast, all other factors building up Eq. (35) are independent of the values of ρ_{LR} or θ. Hence, the accuracy of the multivariate part of the Laplace model will be validated by checking the accuracy of the factor $c_k(\theta, \rho_{\text{LR}}, w)$ for observed values of ρ_{LR} and all θ between 0 and 2π. This will be achieved by comparing the normalized factor

$$\tilde{c}_k(\theta, \rho_{\text{LR}}, w) = \frac{c_k(\theta, \rho_{\text{LR}}, w)}{c_k(0, 0, w)} \tag{37}$$

to the corresponding observed factor

$$\tilde{c}_k^{obs}(\theta, \rho_{\mathrm{LR}}, w) = \frac{D_k^{obs}(\theta)}{(D_k^{obs}(0) + D_k^{obs}(\pi/2))/2} \tag{38}$$

for $k = 3, 5$ and $w = 2, 2.5$.

The results are shown in Figures 4 and 5 where the normalized factors $\tilde{c}_k(\theta, \rho_{\mathrm{LR}}, w)$ and $\tilde{c}_k^{obs}(\theta, \rho_{\mathrm{LR}}, w)$ are compared. As can be seen, the observed factors $\tilde{c}_k^{obs}$ (solid lines) are close to $\tilde{c}_k$ (dashed lines) for both damage exponents $k = 3, 5$ and ISO spectra with waviness parameter $w = 2, 2.5$.

9 Conclusions

A statistical model for road profiles along parallel tracks has been proposed and validated. The proposed Laplace model involve a modification of a Gaussian model by introducing a random variance process. The variance process is assumed to have a marginal gamma distribution and its autocovariance follows an AR(1)-model. All parameters in the Laplace-ISO model can be directly estimated from a sequence of IRI measurements and the correlation coefficient between the profiles. A method to stochastically reconstruct road profiles has been given. An estimate of the expected damage due to a Laplace road with ISO spectrum was given in Eq. (35).

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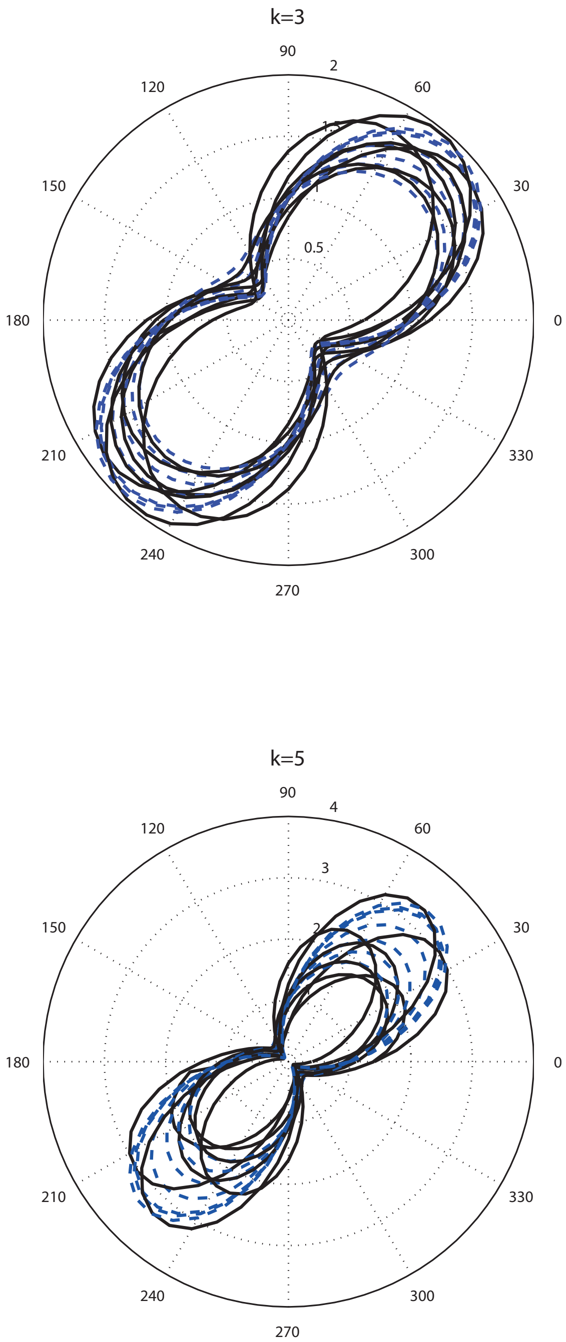
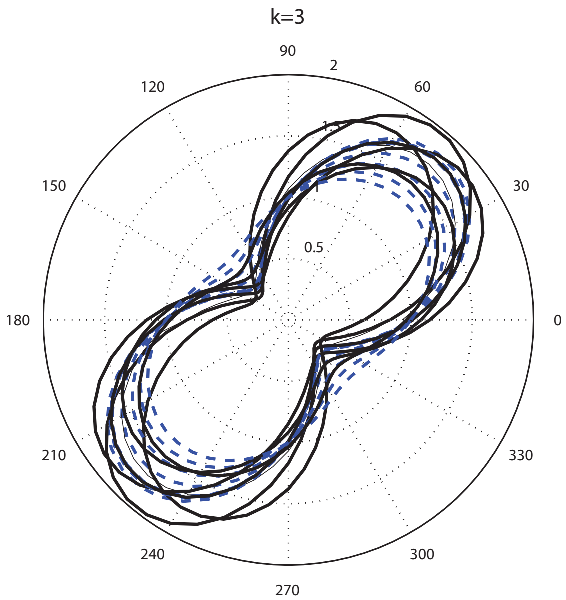


Figure 4: Comparisons between the factors; $\tilde{c}_k(\theta, \rho_{\mathrm{LR}}, w)$, Eq. (37) (dashed line) and $\tilde{c}_k^{obs}(\theta, \rho_{\mathrm{LR}}, w)$, Eq. (38) (solid lines) for the seven 10 km long roads having ISO spectrum with waviness parameter $w = 2$. Top $k = 3$, bottom $k = 5$.



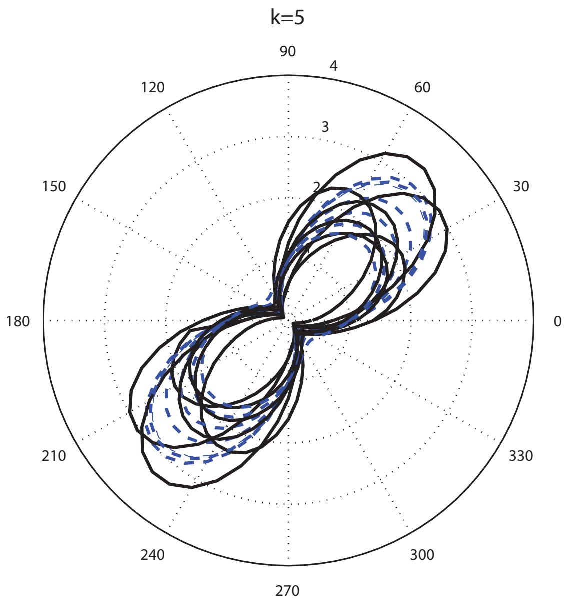


Figure 5: Comparisons between the factors; $\tilde{c}_k(\theta, \rho_{\text{LR}}, w)$, Eq. (37) (dashed line) and $\tilde{c}_k^{obs}(\theta, \rho_{\text{LR}}, w)$, Eq. (38) (solid lines) for the seven 10 km long roads having ISO spectrum with waviness parameter $w = 2.5$. Top $k = 3$, bottom $k = 5$.

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Appendix I - Derivation of approximation (35)

The damage intensity $\mathcal{D}_k(S_\theta)$ for a Gaussian response having PSD S_θ that appears in Eq. (35) can be bounded using Eq. (28), viz.

$$\mathcal{D}_k(S_\theta) \le \sigma^k \, c_k(\theta, \rho_{\text{LR}}, w) \, \frac{2^{3k/2} \, \Gamma(1+k/2)}{2\pi} \, \left(\frac{v}{v_0}\right)^{k(w-1)/2-1},$$

where factor

$$c_k(\theta, \rho_{\text{LR}}, w) = \lambda_0^{(k-1)/2}(\theta)\lambda_2(\theta)^{1/2} \tag{39}$$

and

$$\lambda_i(\theta) = \int_0^\infty \Omega^i \, |H_{v_0}(\Omega)|^2 \tilde{S}(\Omega) \left(1 + \sin(2\theta) e^{-b|\Omega|}\right) \, d\Omega. \tag{40}$$

Finally, some simple calculus leads to

$$\lambda_i(\theta) = \lambda_i(\pi/4)\sin(2\theta) + \lambda_i(0)(1 - \sin(2\theta)),$$

where

$$\lambda_i(0) = \int_0^\infty \Omega^i \, |H_{v_0}(\Omega)|^2 \tilde{S}(\Omega) \, d\Omega, \quad \lambda_i(\pi/4) = \int_0^\infty \Omega^i \, |H_{v_0}(\Omega)|^2 \tilde{S}(\Omega) \left(1 + e^{-b|\Omega|}\right) \, d\Omega.$$

Appendix II - simulation of bivariate Laplace model

A zero mean bivariate homogeneous Gaussian processe is completely defined by its two power spectra and the normalized cross spectrum function. There are several means to simulate sample paths, and the algorithm proposed in (Shinozuka, 1971) is most commonly used. Here we present a method proposed in (Kozubowski et al., 2013) valid only for processes with real valued normalized cross spectrum $K(\Omega)$ and the same power spectral density $S(\Omega)$.

Let us introduce two kernels $g_1(x)$ and $g_2(x)$ by means of the Fourier transforms

$$
\begin{aligned}
(\mathcal{F}g_1)(\Omega) &= \left(\sqrt{2\pi S(\Omega)(1+K(\Omega))} + \sqrt{2\pi S(\Omega)(1-K(\Omega))}\right)/2,\\
(\mathcal{F}g_2)(\Omega) &= \left(\sqrt{2\pi S(\Omega)(1+K(\Omega))} - \sqrt{2\pi S(\Omega)(1-K(\Omega))}\right)/2.
\end{aligned}
\tag{41}
$$

Then two correlated Gaussian moving averages $Z_{\mathrm{L}}(x), Z_{\mathrm{R}}(x)$ are given by

$$
\begin{aligned}
Z_{\mathrm{L}}(x) &\approx \left(\sum g_1(x-x_i)\,Z_{1i} + \sum g_2(x-x_i)\,Z_{2i}\right)\sqrt{\mathrm{d}x},\\
Z_{\mathrm{R}}(x) &\approx \left(\sum g_2(x-x_i)\,Z_{1i} + \sum g_1(x-x_i)\,Z_{2i}\right)\sqrt{\mathrm{d}x}
\end{aligned}
\tag{42}
$$

where Z_{1i}, Z_{2i}'s are independent standard Gaussian random variables, with equality in limit as $\mathrm{d}x$ tends to zero.

Essentially, Gaussian processes $Z_{\mathrm{L}}(x), Z_{\mathrm{R}}(x)$ are filters of sequences of independent standard Gaussian variables Z_{1i}, Z_{2i}, which serve as two Gaussian noise sequences. The Laplace processes are constructed in a similar way, the difference is that now we allow Z_{1i}, Z_{2i} to have variable gamma distributed variances, see Section 5 and Eqs. (19,20). For completeness we give MATLAB code to simulate the Laplace model. Parameters estimated for the sixth 10 km long road section given in the sixth column of Table 3 rows 5-9 are used. In the code some functions from the WAFO (Brodtkorb et al., 2000; WAFO Group, 2011*a*) toolbox are used, which can be downloaded free of charge, (WAFO Group, 2011*b*). The statistical functions `rndnorm`, `rndpois` and `rndgam` are also available in the MATLAB statistics toolbox through `normrnd`, `poissrnd` and `gamrnd`. Note that WAFO also contains functions to find rainflow ranges used to estimate fatigue damage.

```
>> b=1.15; C=9.45e-6; nu=0.33;
>> rho=0.41;
>> Lp=10000; L=100; dx=0.05;
>> N=Lp/L; M=L/dx;
>> riid=nu*rndgam(1/nu,1,N,1);
>> rAR1=riid;
>> for i=2:N
>>     m=rho/(1-rho)/nu*rAR1(i-1);
>>     NPois=rndpois(m);
>>     if NPois>0, W=rndexp(1,NPois,1); else W=0; end
>>     kj=sum(W)/m;
>>     rAR1(i)=rho*kj*rAR1(i-1)+(1-rho)*riid(i);
>> end
>> Noise=[]; NoiseAR1=[];
>> for i=1:N
>>     Nois=rndnorm(0,1,2,M)';
>>     Noise=[Noise; sqrt(riid(i))*Nois];
>>     NoiseAR1=[NoiseAR1; sqrt(rAR1(i))*Nois];
>> end
>> wL=0.011*2*pi; wR=2*pi*2.83;
>> ww=2.5; C0=1/((1/wL.^(ww-1) - 1/wR.^(ww-1))/(ww-1));
>> sigma=sqrt(C/C0);
>> NM=N*M;
```

```
>> w=pi/dx*linspace(-1,1,NM)'; dw=w(2)-w(1);
>> indw=find(abs(w)>wL & abs(w)<wR);
>> S=zeros(NM,1); S(indw)=C/2*1./abs(w(indw)).^ww;
>> K=zeros(NM,1); K=exp(-b*abs(w));
>> G1=sqrt(2*pi*S)/dx.*(sqrt(1+K)+sqrt(1-K))/2;
>> G2=sqrt(2*pi*S)/dx.*(sqrt(1+K)-sqrt(1-K))/2;
>> G1=ifftshift(G1); G2=ifftshift(G2);
>> zLiid = real(ifft(fft(Noise(:,1)).*G1)+ifft(fft(Noise(:,2)).*G2))*sqrt(dx);
>> zRiid = real(ifft(fft(Noise(:,1)).*G2)+ifft(fft(Noise(:,2)).*G1))*sqrt(dx);
>> zLAR1 = real(ifft(fft(NoiseAR1(:,1)).*G1)+ifft(fft(NoiseAR1(:,2)).*G2))*sqrt(dx);
>> zRAR1 = real(ifft(fft(NoiseAR1(:,1)).*G2)+ifft(fft(NoiseAR1(:,2)).*G1))*sqrt(dx);
>> figure(1), plot((1:NM)*dx,[zLiid zRiid])
>> figure(2), plot((1:NM)*dx,[zLAR1 zRAR1])
```

Using the code, 10 km long left and right profiles having ISO spectrum with waviness parameter $w = 2.5$ sampled each 5 cm were simulated for independent and correlated factors r_j. Parts of the simulated profiles are shown in Figures 6 and 7. Since the factors are weekly correlated ($\rho = 0.41$) the signals are quite similar. Note that the same gamma noise is used in the both simulations. The resulting simulations can also be compared with two km long measured profiles from the sixth road given in Figure 8. The variability of the signals presented in Figures 6-8 looks very similar.

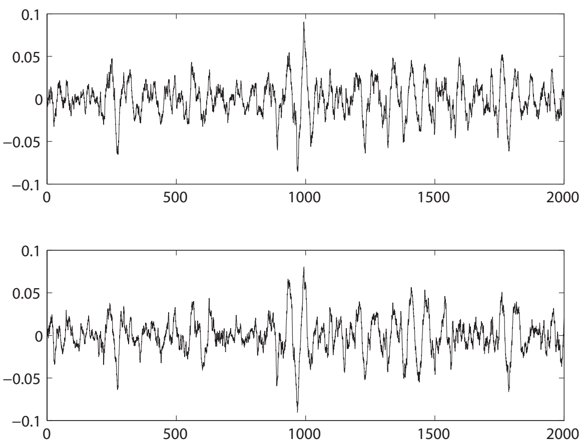

Figure 6: Two kilometres of simulated Laplace models having ISO spectrum, with waviness parameter $w = 2.5$, of left and right road profiles having parameters estimated for the sixth 10 km long road segment, top, bottom plots, respectively. The variances of 100 metre road segments are independent gamma distributed.

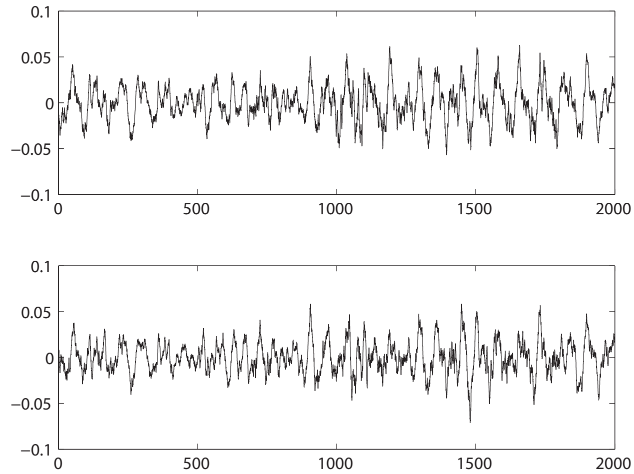

Figure 7: Two kilometres of simulated Laplace models having ISO spectrum, with waviness parameter $w = 2.5$, of left and right road profiles having parameters estimated for the sixth 10 km long road segment, top, bottom plots, respectively. The variances of 100 metre road segments are correlated gamma distributed variables.

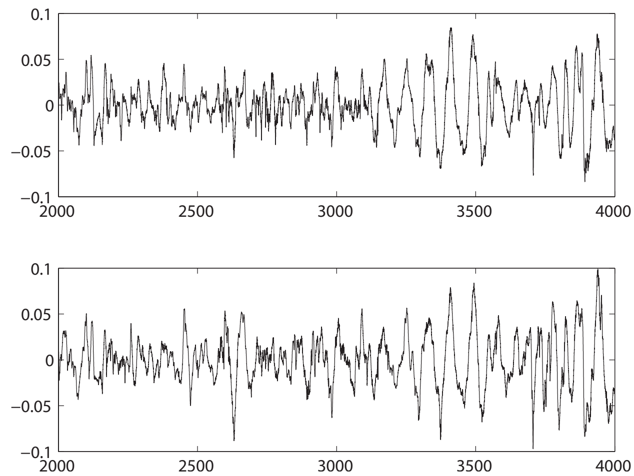

Figure 8: Two kilometres of left and right measured road profiles for the sixth 10 km long road segment, top, bottom plots, respectively.

Appendix III – gamma AR(1)-model for variances

There is a number of models in the literature of processes having a Gamma distribution for one dimensional marginal with autocorrelation function equal to the one for the classical AR(1)-model, (Kozubowski & Podgórski, 2008) and the references therein for a survey of the topic. Here we have chosen to follow the model introduced in (Sim, 1971) although in our approach we follow a certain invariance property shown in Proposition 2.6 of (Kozubowski & Podgórski, 2007) stating

$$\Gamma \stackrel{d}{=} \frac{1}{1+\beta}\left(\Gamma_0 \circ N_0 \circ \beta\Gamma + \Gamma_1\right), \tag{43}$$

where $\beta > 0$, $\stackrel{d}{=}$ stands for distributional equality of the processes, $\circ$ for the superposition of two functions so that $f \circ g(t) = f(g(t))$, Γ, Γ_1 and Γ_0 are independent standard gamma processes, i.e. ones that at time $t = 1$ are gamma distributed with the shape parameter and the scale both equal to one, N_0 is a standard Poisson process. We can rewrite it using a compound Poisson process with exponential compounding (EC) defined as

$$N(t) = \Gamma_0(N_0(t)) = \sum_{i=1}^{N_0(t)} W_i,$$

where $N_0(t)$ is a standard Poisson process and $W_i = \Gamma(i) - \Gamma(i-1)$ are iid with the standard exponential distribution independent of $N_0(t)$. We note the following fundamental relations for the moments

$$\begin{aligned} \mathsf{E}\left[N(t)\right] &= t,\\ \mathsf{V}\left[N(t)\right] &= 2t,\\ \mathsf{V}\left[\Gamma(t)\right] &= t,\\ \mathsf{V}\left[N(\beta\Gamma(t))\right] &= \beta(\beta+2)t,\\ \mathsf{Cov}\left(N(\beta\Gamma(t)), \Gamma(t)\right) &= \beta t. \end{aligned} \tag{44}$$

Moreover, $N(t)$ is a non-decreasing and non-negative process that starts at zero and has homogeneous and independent increments. We remark that $N(t)$ is a Lévy process that is related to the negative binomial process, see (Kozubowski & Podgórski, 2007, 2009) for details. By using $N(t)$ and taking $G(t) = \nu\Gamma(t)$, $\nu > 0$, the ν-scaled version of (43) can be written as

$$G(t) \stackrel{d}{=} \frac{\nu}{1+\beta} N\left(\beta G(t)/\nu\right) + \frac{1}{1+\beta} G_0(t). \tag{45}$$

If we consider gamma variables $G = G(1/\nu)$, $G_0 = G(1/\nu)$, so that they have mean one and variance ν, we can write (44) as

$$G \stackrel{d}{=} \frac{\beta}{1+\beta} K G + \frac{1}{1+\beta} G_0, \tag{46}$$

where a random scaling K is dependent on G and given by

$$K = \frac{N(\beta G/\nu)}{\beta G/\nu}.$$

The above can serve as a defining equation of the autoregressive time series of positive variables

$$\begin{aligned} Y_0 &= G^{(0)},\\ Y_n &= \rho K_n Y_{n-1} + (1-\rho) G_0^{(n)}, \end{aligned} \tag{47}$$

where independent innovations $G_0^{(n)}$ distributed as G_0 are added at each step, while

$$K_n = \frac{N^{(n)}(\beta Y_{n-1}/\nu)}{\beta Y_{n-1}/\nu},$$

with $N^{(n)}$'s being independent copies of N and $\rho = \beta/(1+\beta) \in [0,1]$.

The following property implies AR(1) form of the covariance

$$\mathsf{E}\left[Y_{n+k} | Y_m, m \leq n\right] = \rho^k Y_n + k(1-\rho).$$

Indeed, the autocovariance function is given by

$$\begin{aligned} \text{Cov}\left(Y_{n+k}, Y_n\right) &= \text{ Cov}\left(\mathsf{E}\left[Y_{n+k} | Y_n\right], Y_n\right) \\ &= \rho^k \mathsf{V}\left[Y_n\right]. \end{aligned}$$