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Abstract

Slepian models are derived describing the distributional form of a stochastic process observed at level crossings of a moving average driven by a Laplace noise. The approach is through a Gibbs sampler of a Slepian model for the Laplace noise and it allows for simultaneously studying a number of stochastic characteristics observed at the level crossing instants. A method of sampling from the corresponding biased sampling distribution of the underlying gamma process is also obtained from the same Gibbs sampler. This is used for efficient simulations of the behavior of a random processes sampled at crossings of a non-Gaussian moving average process. In particular, it facilitates comparisons of the behavior when a Gaussian process vs. a non-Gaussian processes are crossing a level. It is observed that the behavior of the process at high level crossings is fundamentally different from that in the Gaussian case, which is in line with some recent theoretical results on the subject.

Keywords: Rice formula, level crossings, generalized Laplace distribution, moving average process, extreme episodes, biased sampling distribution, tilted Rayleigh distribution, generalized inverse Gaussian distribution.

1. Introduction

In the physical world, a random function is often described as a sequence of local maxima or minima, constituting a series of random waves. In fact, not only the visual impression of the process but also many technologically important implications in such fields as metal fatigue caused by random vibrations, failure caused by excess load on a construction, etc., depend on the character of the process represented in such a random wave form. The basic objects in this theory are level crossings and local extremes, see [26] for computation of various crossing distributions, [7] for spatial wave characteristics, and [11] for an overview of other engineering applications.

In analysis of extreme behavior of a time series or a stochastic process, the Rice formula is often used to obtain the distribution of the process at the instants of high level crossings. For the purpose of simulation or analysis it is convenient to have a Slepian model corresponding to the behavior of the original process sampled at the level crossings. Here a Slepian model is understood as any explicitly defined stochastic process that is distributed according to the crossing level biased sampling distribution and thus is random function representation of the conditional behavior of a stochastic process after events defined by level crossings. The original Slepian model developed for a stationary Gaussian process is very helpful in analyzing behavior of the process at extreme levels. In general, a Slepian model contains one regression term with random coefficients which describe the dependence on initial conditions such as the slope at the crossing, the value of the process at a predetermined point, etc, and one residual term, which describes the deviations from the path set out by the initial conditions.

In its classical form, such a model was first introduced in [27] to describe the behavior of a stationary Gaussian process after a zero crossing. The model found applications in more theoretical studies of various asymptotic sample path properties of a Gaussian random process, or a function of a vector valued Gaussian process, see [19], [5], or [23]. Considerable work has been done on studying sample properties of Gaussian or related fields around high local maximum or level set, see [29] and, for more recent work, [6].

Typically, a Slepian model is defined for ergodic processes when the distribution of the model coincides with the long-run empirical distribution of stochastic outcomes observed at crossing instants. However,

a Slepian model was also defined for a non-stationary Gaussian process to study properties of the process under conditioning that a local maximum occurs at time zero, see [12] and also [15] for an engineering application. It is worth to mention here that in the approach presented in this paper, we analyze non-Gaussian models obtained by a random distortion of the Brownian motion through conditioning on the distortion which leads to (conditionally) non-stationary Gaussian process and through this our work connects with [12].

In many practical situations, the assumption of Gaussianity is not supported by empirical data and therefore derivation of a Slepian model for non-Gaussian processes is desirable. This need has driven growing interest in studying level crossing distributions for non-Gaussian models, for example, see [3] for results on the high level crossings and [28] for an example of practical context in which a Slepian model for data exhibiting non-Gaussian features is of interest. In this paper, we present an approach to obtain an effective Slepian models for a class of non-Gaussian models driven by a Laplace motion, i.e. a non-Gaussian Lévy motion obtained by subordination of Brownian motion by a gamma process. This class has proven to be sufficiently flexible to account for most non-Gaussian features observed in practical applications and some work has been done on the level crossing distributions derived from a generalized Rice formula applied to these processes, see [1], [2], [13].

The focus of this work is two-fold, firstly, we propose derivation general Slepian models by obtaining Slepian models of the noise that is driving the considered models, secondly, we show how conditioning on a variable or process can help in derivation of a convienient Slepian model for a non-Gaussian model. The novelity of the approach is its focus on the Slepian models of the noise. An alternative approach to building a Slepian model would be through a hierarichical approach to which one could employ a non-stationary Slepian models as discussed in [12]. We prefer to consider a Slepian model of noise for which we found a convenient simulation method through a Gibbs sampler. One advantage of the presented approach is a possibility of simultaneous studies of various random functionals of such a noise without necessity of constructing a separate Slepian model for each of the functionals. These benefits are illustrated by examples and numerical studies. Our interest in simulations of Slepian models is paralleling applied engineering papers on this subject [21]. They are useful, in particular, to study non-linear dynamical systems where such Slepian models can be considered as input to the system to study their responses at particular crossing events. For such an application our approach is more direct than the one presented in [3], and here this is illustrated for stochastic road profile models.

2. Preliminaries

We consider a stationary random process X having almost surely absolutely continuous samples and their derivatives. Moreover, it is assumed that the joint probability density function (pdf) $f_{X,\dot{X}}$ of $X(0), \dot{X}(0)$ exists. For $u \in \mathbb{R}$, the u-level upcrossing set within interval $[0,1]$ is defined as

$$\mathcal{C}(u) = \{s \in [0,1] : X(s) = u, \dot{X}(s) > u\}.$$

Let $N(u)$ be the number of elements in $\mathcal{C}(u)$. The generalized Rice's formula yields

$$E[N(u)] = \int_0^{+\infty} z f_{\dot{X},X}(z,u)\, dz,$$

where, for shortness, X denotes $X(0)$ and $\dot{X}$ denotes $\dot{X}(0)$. Equivalently

$$E[N(u)] = \int\int_0^{+\infty} z f_{\dot{X},X|K}(z,u|k) f_K(k)\, dz\, dk, \tag{1}$$

where conditioning on the random variable or vector K is used to simplify evaluation of the integral. Here and in what follows whenever the limits of integration are not shown in the notation they are understood to be over the entire set of possible values of the corresponding variable. The focus of this paper is on the Laplace moving average (LMA) processes for which K is a certain, possibly vector valued, functional of the gamma process that serves as the subordinator in the representation of the Laplace motion as a subordinated Brownian motion, [20].

For a properly defined statement A on trajectories of another stationary stochastic process Y, define $N(A|u)$ to be the number of $s \in \mathcal{C}(u)$ such that $Y(s+\cdot) \in A$. Similarly as for $N(u)$ one can consider

$$E[N(A|u)] = \int_0^{+\infty} P(Y \in A|\dot{X} = z, X = u) \cdot z f_{\dot{X},X}(z,u)\, dz = \\ = \int\int_0^{+\infty} P(Y \in A|\dot{X} = z, X = u, K = k) \cdot z f_{\dot{X},X|K}(z,u|k) f_K(k)\, dz\, dk$$

and use this to evaluate the u-level upcrossing distribution P^u of Y, see [30] for a derivation of this formula for a general class of stochastic processes. The upcrossing distribution P_u is defined as the ratio of the average number of the u-upcrossings at which a trajectory event occurs to the average number of all u-uppcrosings

$$P^u(A) = \frac{E[N(A|u)]}{E[N(u)]}. \tag{2}$$

Consequently, one has the following representation of u-level upcrossing distributions involving the conditioning on K:

$$P^u(A) = \frac{\int\int_0^{+\infty} P(Y \in A|\dot{X} = z, X = u, K = k) \cdot z f_{\dot{X},X|K}(z,u|k) f_K(k)\, dz\, dk}{\int\int_0^{+\infty} z f_{\dot{X},X|K}(z,u|k) f_K(k)\, dz\, dk} = \\ = \frac{\int\int_0^{+\infty} P(Y \in A|\dot{X} = z, X = u, K = k) \cdot z f_{\dot{X}|K,X}(z|k,u) f_{K|X}(k|u)\, dz\, dk}{\int\int_0^{+\infty} z f_{\dot{X}|K,X}(z|k,u) f_{K|X}(k|u)\, dz\, dk}. \tag{3}$$

A stochastic process Y_u such that its finite dimensional distributions correspond to these given by the upcrossing distribution is referred to as a Slepian model of Y at the u-up-crossings, i.e. for each measurable A in the space of trajectories, Y_u satisfies

$$P(Y_u \in A) = P^u(A).$$

If a Slepian model can be expressed in an explicit form, it can be used for deriving approximations for probabilities of interest as well as simulating trajectories interpreted as samples of the original process observed at instants of the u-level up-crossings. It can also help to analyze asymptotic behavior of the process crossing a high level and thus providing an insight into behavior of the process at extremal episodes.

Example 1. If one takes $Y = X$, then a Slepian process X_u describes behavior of X at its own up-crossings of u.

Example 2. Another case is to take $Y = K$ for which a Slepian model K_u has distribution given by the density

$$f_{K_u}(k) \propto \int_0^{+\infty} z f_{\dot{X}|K,X}(z|k,u) f_{K|X}(k|u)\, dz.$$

Example 3. A joint Slepian model for $\dot{X}$ and K is a random vector with the distribution given by

$$f_{K_u,\dot{X}_u}(k,z) \propto z f_{\dot{X}|K,X}(z|k,u) f_{K|X}(k|u).$$

We observe that the distribution of K_u given $\dot{X}_u = z$ is the same as the distribution of K given $\dot{X} = z$ and $X = u$.

Generalizing Example 3 one can conveniently write a scheme for obtaining a Slepian model for a process Y. The following structural result is quite obvious and sets the foundation for many results of this work.

Proposition 1. *Let a process $Y(\cdot|k,z,u)$ have distribution equal to that of Y conditionally on $K=k$, $\dot{X}=z$ and $X=u$ and $(K_u,\dot{X}_u)$ be a joint Slepian model for $(K,\dot{X}(0))$. Then a Slepian model for Y_u is given by*

$$Y_u(\cdot)=Y(\cdot|K_u,\dot{X}_u,u). \tag{4}$$

In this note we discuss a Slepian model of a certain stochastic process Y at the crossings of the Laplace moving average (LMA)

$$X(t)=\int g(s-t)\ dL(s),$$

where a Laplace noise dL is defined through a Laplace motion that can be defined through $L(t)=B(\Gamma(t))$, where B is the Brownian motion (BM) and Γ is a gamma process, see [20]. Our approach consists of two steps. In the first one, we obtain effective sample (l,γ,z) from the biased sampling joint distribution of the motion L_u, the gamma process Γ_u and $\dot{X}_u$, observed at the u level up-crossings of the moving average process X. Once this sample is available, in the second step, we consider a suitable model $Y(t|l,x,u)$, $l=l(\cdot)$, $z>0$, $u\in\mathbb{R}$, the distribution of which coincides with the conditional distribution of the process Y given that $L(\cdot)=l(\cdot)$, $\dot{X}(0)=z$ and $X(0)=u$. The Slepian model Y_u for Y is obtained by substituting the sample l,x from the first stage, for $l(\cdot)$ and z in $Y(t|l,z,u)$ of the second stage as formally stated in Proposition 1.

Clearly, the key to the approach is effective sampling of $(L_u,\Gamma_u,\dot{X}_u)$. Here, we propose a Gibbs sampler that uses the gaussianity of the conditional process $L(t|\gamma,z,u)$ given that $\Gamma=\gamma$, $X(0)=u$ and $\dot{X}(0)=z$. In the above $\gamma=\gamma(\cdot)$ stands for a realization of the process $\Gamma(t)$, $t\in\mathbb{R}$. This main contribution is presented in Section 4.

Our approach provides an alternative to computing crossing level distributions as compared to the results presented in [13], [1], [2], where approximations to the joint distribution of the process and derivative at zero were used for the purpose. The benefits of considering Slepian models is that they provide a unified frame for handling joint level crossing distributions for a variety of the variables and functionals of stochastic processes. The next section illustrates our approach by showing how the Slepian model in the Gaussian case can be presented through the Slepian noise model.

3. Slepian noise model for the Gaussian case

A moving average process is a convolution of a kernel function g, say, with an infinitesimal "white noise" process having variance equal to the discretization step, say ds. Throughout the paper normalization of the process in its value and its argument is assumed so that the variances of the process and of its derivative are equal to one or, equivalently, $\int g^2=\int\dot{g}^2=1$. Here and in what follows, $\int f$ stands for the Lebesgue integral of a function f, i.e. $\int f=\int_{\mathbb{R}} f(t)\,dt$

The Gaussian moving average (GMA) model is given by

$$X(t)=\int_{-\infty}^{\infty} g(s-t)\,dB(s) \tag{5}$$

and its derivative $\dot{X}$ is given as the moving average with $-\dot{g}$ as the kernel. Consider a fixed level $u\in\mathbb{R}$ and the probability distribution P^u on events A in the space of real continuous functions on $\mathbb{R}$ as defined in (2). For a stationary Gaussian process X with variance one and variance of its derivative also equal to one the Slepian model process X_u around u-upcrossing of X is given by

$$X_u(t)=u\,r(t)-R\dot{r}(t)+\Delta(t)=u\,r(t)-\dot{X}_u(0)\,\dot{r}(t)+\Delta(t), \tag{6}$$

where $r(t)=g*\tilde{g}$ is the covariance function of X with $\tilde{g}(s)=g(-s)$, R is a standard Rayleigh variable independent from a non-stationary Gaussian process Δ having covariance

$$r(t,s)=r(t-s)-r(t)r(s)-r'(t)r'(s).$$

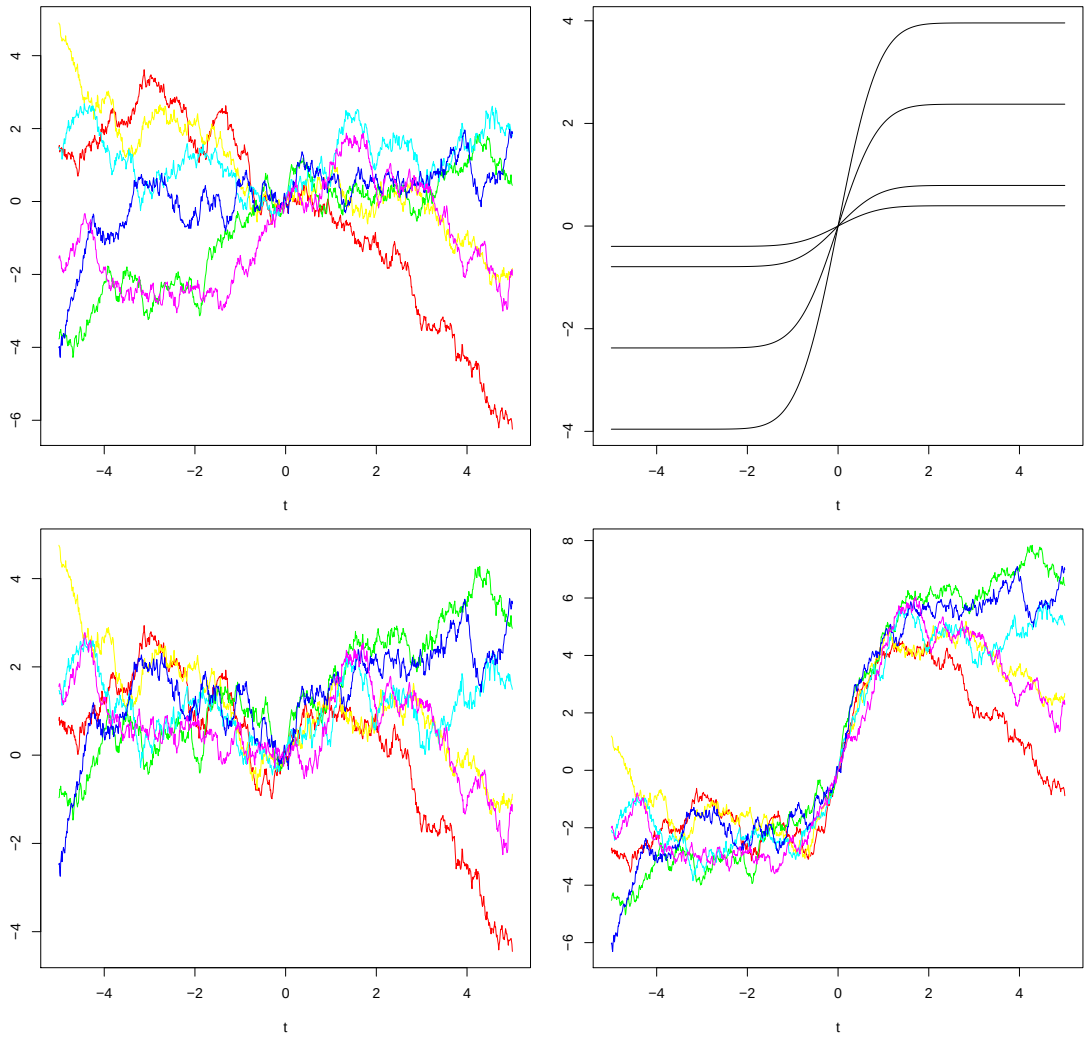


FIGURE 1. *Top-Left:* Six BM samples used in computing samples from the Slepian model $B_u(t)$. *Top-Right:* Deterministic part $F_{u,g}$ for levels: $u = 0.5, 1, 3, 5$. *Bottom:* Six samples of B_u corresponding to the BM samples; crossing levels: $u = 0.5$ *(left)* and $u = 5$ *(right)*. A single value for the Rayleigh variable is used for all these samples.

This form follows easily from the Rice formula that yields the Rayleigh distribution for $\dot{X}_u$ by application of Proposition 1 because $X(t|z,u) = u\,r(t) - z\,\dot{r}(t) + \Delta(t)$. A more detailed presentation of this Slepian model for Gaussian case can be found in [22] and [24].

For the purpose of this presentation, we derive another while equivalent Slepian model that explicitly use the moving average form of the underlying process. We first ask for a Slepian model $dB_u(x)$ for the noise $dB(x)$ at the crossing levels u of X. As argued in the Appendix, the biased sampling distribution of $dB(x)$ is represented by the distribution of the following stochastic process $B_u(t)$, $t \in \mathbb{R}$:

$$B_u(t) = u \int_0^t g - R \int_0^t \dot{g} - \int_0^t g \cdot \int g\, dB - \int_0^t \dot{g} \cdot \int \dot{g}\, dB + B(t),\ t \in \mathbb{R}, \tag{7}$$

where random variable R has the Rayleigh distribution and is independent of $dB(t)$, while $B_u(t)$ is understood as a random measure of $[0,t]$, with the convention that for $t < 0$, the measure is understood as minus the measure of $[t,0]$.

From this representation we can distinguish three components of the Slepian model for the noise. Firstly, the level and kernel dependent non-random component

$$F_{u,g}(t) = u \int_0^t g.$$

Secondly, the kernel only dependent random component

$$G_g(t) = \left(\int \dot{g}\ dB + R \right) \cdot (g(0) - g(t)) - \int g\ dB \cdot \int_0^t g,$$

and, finally, purely random noise represented by Brownian motion $B(t)$. We note that G_g and B are stochastically dependent and G_g conditionally on B is a linear combination of non-random functions with one random coefficient distributed according to a Rayleigh distribution.

Example 4. Let us consider (normalized) kernel $g(t) = (2/\pi)^{1/4}\ e^{-t^2}$, $t \in \mathbb{R}$. Direct calculations lead to the following form of the Slepian model

$$B_u(t) = F_{u,g}(t) + G_g(t) + B(t),$$

with

$$F_{u,g}(t) = \sqrt[4]{2\pi}\ u\ \Phi_0\left(\sqrt{2}t\right),$$
$$G_g(t) = \left(\sqrt[4]{\frac{2}{\pi}}\ R - \sqrt{\frac{8}{\pi}} \int s e^{-s^2}\ dB(u) \right) \cdot \left(1 - e^{-t^2}\right) - \sqrt{2} \int e^{-s^2}\ dB(s) \cdot \Phi_0\left(\sqrt{2}t\right),$$

where $\Phi_0(s) = (2\pi)^{-1/2} \int_0^s e^{-u^2/2}\ du$.

In Figure 1, we show simulations of this Slepian noise for different levels u and compare them with corresponding samples from a regular BM. We can observe how the behavior of $B_u(t)$ depends on the value of a level u. In particular, for a high level u the main contribution to B_u comes from the deterministic part. Since term $G_g(t)$ does not depend on u it is not shown in the figure.

There are several benefits of looking at the level crossing distributions through the Slepian model of the noise. The discussed biased sampling representation of the noise allows for the Slepian model for any process that is a functional of the BM by simply replacing B with B_u in the conditional representation of such a process given B. Decomposition into three components: level depending, kernel depending, and noise, allows separate studies of different aspect of process behavior at the crossing levels. This is particularly beneficiary if the process under consideration is a linear functional of the noise. More precisely, consider a vector of stochastic processes $\mathbf{Y}(t) = (Y_1(t), \ldots, Y_n(t))$, $t \in \mathbb{R}$ such that they arise as a result of some functionals acting on B:

$$Y_i(t) = H_i(t, B),\ i = 1, \ldots, n,$$

where for a given trajectory $B = b$, $H_i(t, b)$ can be random but independent of B (and thus independent of $\dot{X}(0)$ and $X(0)$). Then the joint Slepian model $\mathbf{Y}_u(t)$ for $\mathbf{Y}(t)$ at the instants when the moving average process $X(t)$ up-crosses level u is obtained by considering

$$Y_{u,i}(t) = H_i(t, B_u),\ i = 1, \ldots, n.$$

In particular, if functionals $H_i(t, B)$ are linear in B, we obtain a joint decomposition

$$Y_{i,u}(t) = u \cdot H_i(t, \textstyle\int_0^t g) - \left(\int \dot{g} dB + R \right) \cdot H_i(t, \dot{g}) - \int g dB \cdot H_i(t, \textstyle\int_0^t g) + Y_i(t),\ i = 1, \ldots, n. \tag{8}$$

Example 5. To illustrate the convenience of the approach, we consider a pair of linear functionals of dB, $\mathbf{Y} = (Y_1, Y_2)$, defined as follows.

The first component $Y_1(t)$, $t \in \mathbb{R}$, is a filtered original process $X(t)$ by means of a filter $h(t)$, i.e. the output from a linear system that is described by $h(t)$ when the input is $X(t)\ dt$. Thus we have

$$Y_1(t) = \int h(s - t) X(s)\ ds = \int h * g(s - t)\ dB(s).$$

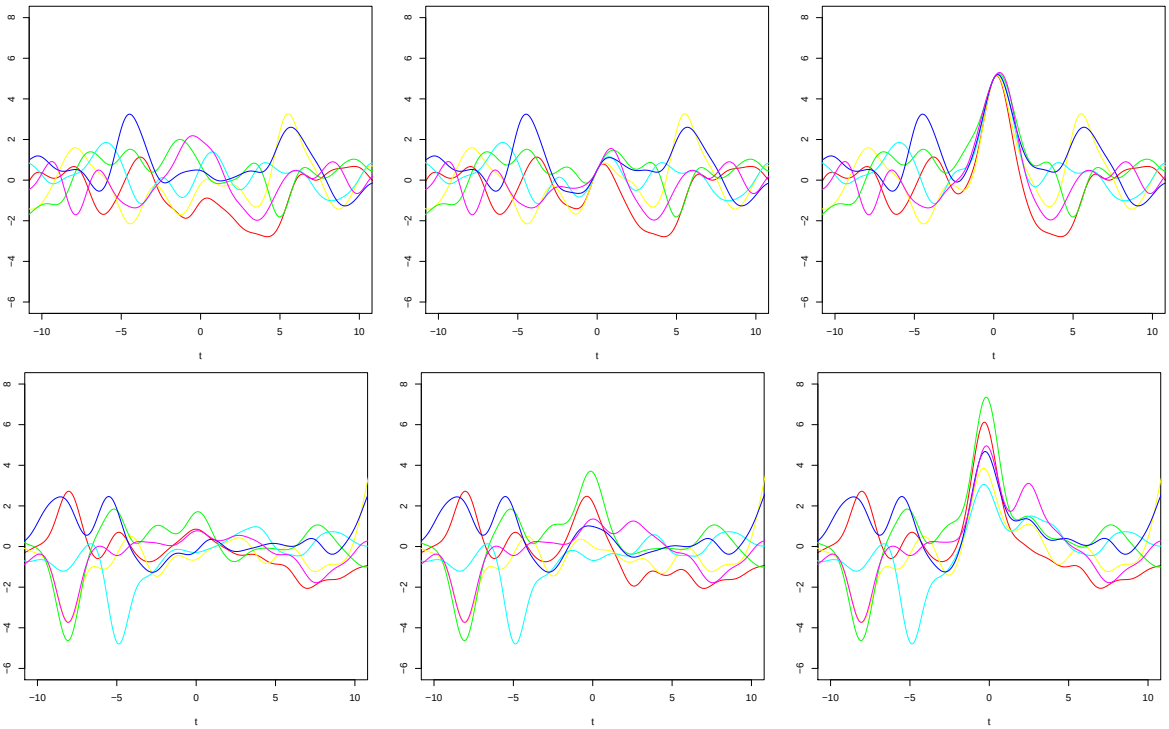


FIGURE 2. *Left:* Six samples from the Gaussian moving average X – *(top)* and corresponding samples from the Laplace moving average Y – *(bottom)*. Samples are based on six samples of the Brownian motion and a single sample of the gamma process that is used for process Y. *Middle:* Samples from the joint Slepian model at the crossings of X at level: $u = 0.5$. *Right:* Analogous samples at the crossings of level $u = 5$.

The second component, $Y_2(t)$, $t \in \mathbb{R}$ is a far more complex functional of B as it arises from a linear scheme that alter Gaussian distribution of the moving average process. Namely, we consider the moving average driven by a Lévy motion build upon the Laplace distribution – the Laplace motion. The Laplace motion is obtained through subordination of the original BM to a gamma motion. For a kernel f and the Lévy process Γ such that $\Gamma(1)$ has the gamma distribution with shape τ and scale $1/\tau$ (for negative t, the process $-\Gamma(t)$ is an independent copy of $\Gamma(t), t \geq 0$), we define the Laplace moving average (LMA)

$$Y_2(t) = \int f(s-t)\, dB \circ \Gamma(s).$$

It is clear that although both the processes are linear in B, their mutual dependence is a fairly complex due to randomness of the gamma subordinator. The direct approach to the joint distribution of (Y_1, Y_2) at up-crossings of X would require analysis of the joint distribution of (Y_1, Y_2) together with the distribution of $X(0)$ and $\dot{X}(0)$. This is not straightforward due to non-Gaussianity of Y_2 and a complex nature of its dependence on Y_1 and X. However, our approach through the Slepian model of

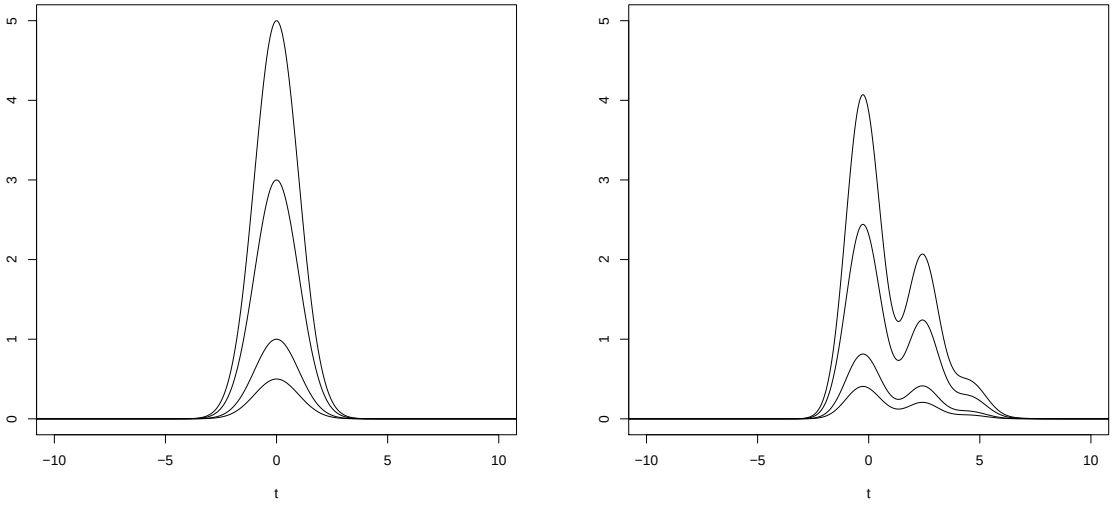


FIGURE 3. Level dependent components in the Slepian model, $u = 0.5, 1, 3, 5$ for Gaussian (*left*) and Laplace (*right*) moving averages. Irregularity in the right picture is due to the gamma process Γ that is present in this component – only one sample of this process is used for all the presented levels.

dB as given in (8) yields

$$\begin{aligned}
Y_{1u}(t) =& u \cdot h * r(t) - \left(\int \dot{g}\ dB + R \right) \cdot h * \dot{g} * \tilde{g}(t) - \int g\ dB \cdot h * r(t) + Y_1(t), \\
Y_{2u}(t) =& u \cdot \int f(s-t)\ dG \circ \Gamma(s) \\
& - \left(\int \dot{g}\ dB + R \right) \cdot \int f(s-t)\ dg \circ \Gamma(s) - \int g\ dB \cdot \int f(s-t)\ dG \circ \Gamma(s) + Y_2(t),
\end{aligned}$$

where $G(t) = \int_0^t g$, $\tilde{g}(t) = g(-t)$ and $r = g * \tilde{g}$ is the covariance of X. The obtained decomposition reveals complex dependence structure between processes.

For a graphical illustration, we take X as in Example 4 and consider $Y_1 = X$, while $Y_2 = Y$, where

$$Y(t) = \int g(s-t)\ dB \circ \Gamma(s), \tag{9}$$

which could be viewed as a modified X obtained by random distortion of time represented by gamma process Γ. We have the following formulas

$$\begin{aligned}
X_u(t) =& u \cdot e^{-t^2/2} + \left(R - 2 \left(\frac{2}{\pi} \right)^{1/4} \int s e^{-s^2} dB(s) \right) \cdot t e^{-t^2/2} - \left(\frac{2}{\pi} \right)^{1/4} \int e^{-s^2} dB(s) \cdot e^{-t^2/2} + X(t), \\
Y_u(t) =& (2\pi)^{1/4}\ u \cdot \int e^{-(s-t)^2}\ d\Phi_0\left(\sqrt{2}\Gamma(s) \right) - \sqrt{\frac{2}{\pi}} \left(R - 2 \left(\frac{2}{\pi} \right)^{1/4} \int s e^{-s^2} dB(s) \right) \cdot \int e^{-(s-t)^2}\ de^{-\Gamma^2(s)} \\
& - \frac{2^{3/4}}{\pi^{1/4}} \int e^{-s^2} dB(s) \cdot \int e^{-(s-t)^2}\ d\Phi_0\left(\sqrt{2}\Gamma(s) \right) + Y(t),
\end{aligned}$$

Using the above relation, we illustrate particular components of the Slepian model for the joint upcrossing distribution of (Y_1, Y_2). We have chosen $\tau = 0.5$ for the shape parameter of the gamma process. The samples of underlying Brownian motion are the same as those in Figure 1.

In Figure 2 *(top)*, we observe samples simulated from bivariate process $(X(t), Y(t))$ (to facilitate better visual comparison we have used the same sample of the underlying gamma process for all six samples of the Laplace moving average). They reveal complex leptikurtic behavior of Y, which shows much larger extreme values than X. In the middle and right columns we see a sample from the Slepian model at level $u = 0.5$ and $u = 5$, respectively. The level crossing occurs at $t = 0$ as seen at the top middle/right plots. We observe in the bottom graphs that the random time change introduced by the gamma motion is adding to variability of Y at the crossing instants of X. For large level u the variability relatively to the level is reduced however the process Y still significantly overshoots the crossing value $u = 5$.

Our approach allows for investigating the role of particular components in the model. For example, the level dependent components are presented in Figure 3, where we see that for the non-Gaussian case this component is randomly affected by the presence of gamma process Γ.

4. Slepian noise model at crossings of a non-Gaussian moving averages

We have discussed a Slepian model for the Gaussian noise at crossings of a stationary Gaussian moving averages. Our interest will turn now to the case of crossings by a moving average driven by a non-Gaussian noise

$$X(t) = \int g(s-t)\ dL(s) = \int g(s-t)\ dB \circ \Gamma(s), \tag{10}$$

where, as before, $\Gamma(t)$ is a gamma process with shape τ and scale $1/\tau$, i.e. $\Gamma(1)$ has the gamma distribution with these two parameters. The choice of a gamma process as a subordinator is dictated by convenience of a simple parameterization and an available convenient Gibbs sampler (see the Appendix), but in general one can consider other classes of non-negative second order Lévy processes.

Let us consider an arbitrary process Y and a process $Y(\cdot|\gamma, z, u)$ with the distribution equal to conditional distribution of Y given $\Gamma = \gamma$, ($\gamma = \gamma(\cdot)$ is a trajectory of Γ), $\dot{X}(0) = z$, and $X(0) = u$. Then, as stated in Proposition 1, if one have a joint Slepian model $(\Gamma_u, \dot{X}_u)$ for $(\Gamma, \dot{X})$, then a Slepian model for Y can be obtained through

$$Y_u(t) = Y(t|\Gamma_u, \dot{X}_u, u),$$

where for shortness $\dot{X}_u = \dot{X}_u(0)$. This approach splits finding a Slepian model for Y into two separate tasks: firstly, finding $Y(\cdot|\gamma, z, u)$, then, secondly finding a Slepian model $(\Gamma_u, \dot{X}_u)$. While finding $Y(\cdot|\gamma, z, u)$ is specific to a given process Y and need to be addressed in each case of Y individually, the obtaining a Slepian model $(\Gamma_u, \dot{X}_u)$ is universal in the sense that it has the same structure dependent only on the elements defining the moving average X but independent of the choice of Y. A Slepian model $(\Gamma_u, \dot{X}_u)$ is considered next.

It is easier to consider an extended model $(L_u, \Gamma_u, \dot{X}_u)$ and express a Slepian model for sampling from the crossing level distribution of this vector by a convenient Gibbs sampler. Namely, the model will based on alternate samples from Γ_u conditionally on $L_u, \dot{X}_u$ and $L_u, \dot{X}_u$ conditionally on Γ_u. As shown in the Appendix in (19), these two conditional distributions, given by through the Bayes relation

$$\begin{aligned} f_{\Gamma_u|L_u,\dot{X}_u}(\gamma|l,z) &\sim f_{L_u|\Gamma_u,\dot{X}_u}(l|\gamma,z) f_{\dot{X}_u|\Gamma_u}(z|\gamma) \\ f_{L_u,\dot{X}_u|\Gamma_u}(l,z|\gamma) &\sim f_{L_u|\Gamma_u,\dot{X}_u}(l|\gamma,z) f_{\dot{X}_u|\Gamma_u}(z|\gamma), \end{aligned}$$

can be simulated in a straightforward fashion.

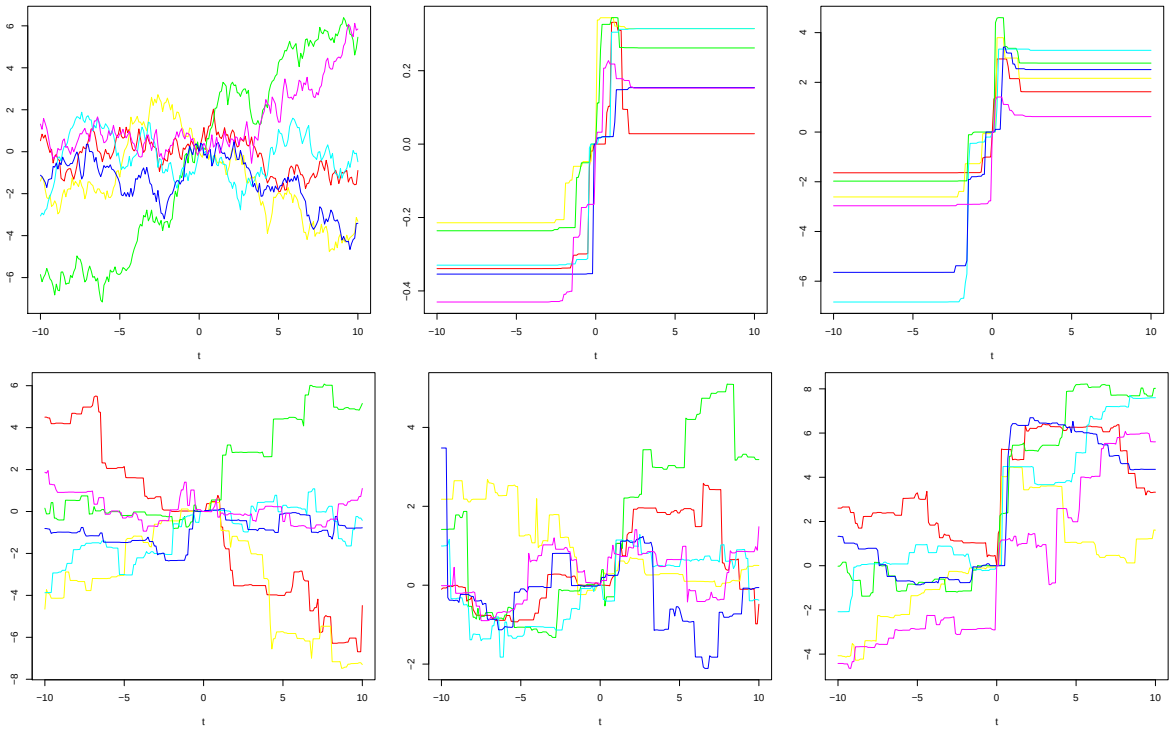


FIGURE 4. *Left:* Six samples from a regular Brownian motion (top) and corresponding samples for Laplace motion (bottom), $\tau = 0.5$ (the same gamma trajectory was used for subordination in all six cases). *Middle:* Level dependent part $F_{u,g,\gamma}$ of the Slepian model obtained for $u = 0.5$ (top) and the corresponding Slepian noise (bottom). *Right:* The same but for level $u = 5$.

It follows directly by the above conditional arguments and properties (18) and (20), stated in the Appendix, that a Slepian model of L at the u-crossings of Y can be written in the form

$$L_u(t) = \frac{G_\gamma(t) + r_\gamma \dot{G}_\gamma(t)}{1 - r_\gamma^2} \frac{u}{\sqrt{\int g^2 d\gamma}} - \frac{\dot{G}_\gamma(t) + r_\gamma G_\gamma(t)}{1 - r_\gamma^2} \frac{\dot{X}_u + \int \dot{g} dB \circ \gamma}{\sqrt{\int \dot{g}^2 d\gamma}} - \frac{r_\gamma \dot{G}_\gamma(t) + G_\gamma(t)}{1 - r_\gamma^2} \frac{\int g \; dB \circ \gamma}{\sqrt{\int g^2 d\gamma}} + B \circ \gamma(t), \tag{11}$$

where $\gamma = \Gamma_u$ and $r_\gamma = -\int g\dot{g} d\gamma / \sqrt{\int \dot{g}^2 d\gamma \int g^2 d\gamma}$, $G_\gamma(t) = \int_0^t g d\gamma / \sqrt{\int g^2 d\gamma}$, $\dot{G}_\gamma(t) = \int_0^t \dot{g} d\gamma / \sqrt{\int \dot{g}^2 d\gamma}$.

Although the structure of the Slepian model is more complex as compared with the Gaussian case, one can still identify similar components as before. We have the level and $\Gamma_u = \gamma$ depending component

$$F_{u,g,\gamma}(t) = \frac{G_\gamma(t) + r_\gamma \dot{G}_\gamma(t)}{1 - r_\gamma^2} \frac{u}{\sqrt{\int g^2 d\gamma}},$$

a linear combination of functions depending only on Γ_u, where the coefficients are random variables

$$G_{\dot{X}_u, g, \gamma, B}(t) = -\frac{\dot{G}_\gamma(t) + r_\gamma G_\gamma(t)}{1 - r_\gamma^2} \frac{\dot{X}_u + \int \dot{g} \; dB \circ \gamma}{\sqrt{\int \dot{g}^2 d\gamma}} - \frac{r_\gamma \dot{G}_\gamma(t) + G_\gamma(t)}{1 - r_\gamma^2} \frac{\int g \; dB \circ \gamma}{\sqrt{\int g^2 d\gamma}},$$

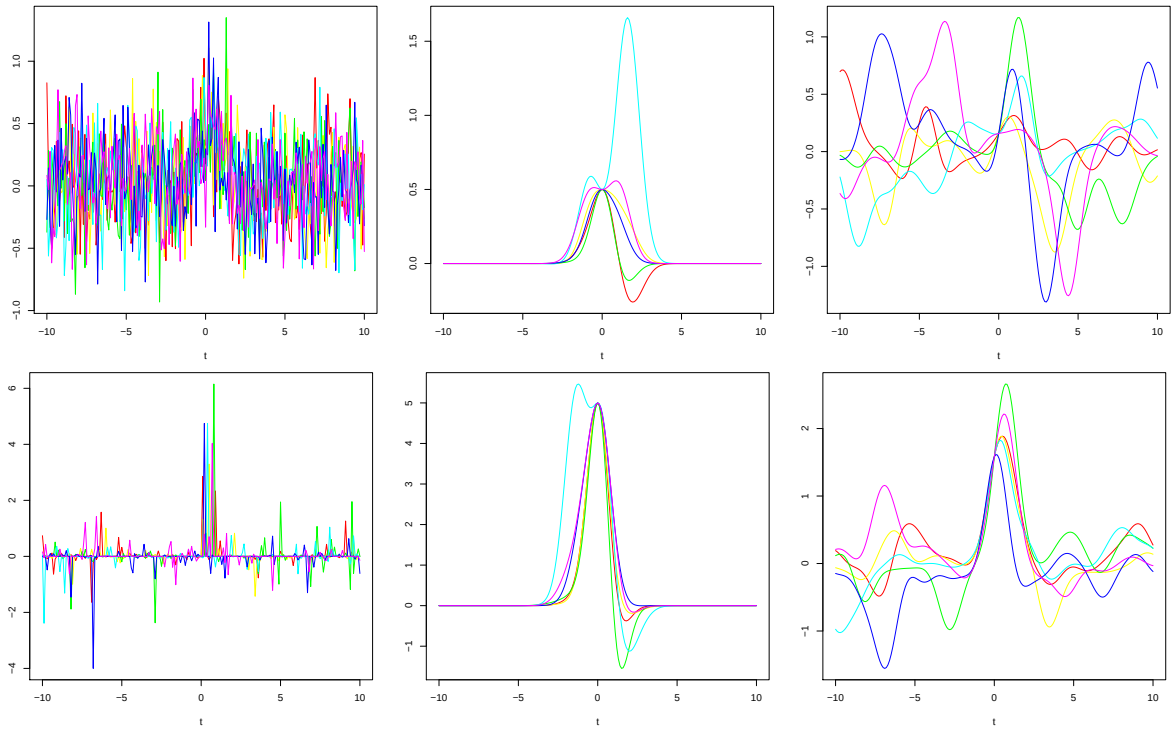


FIGURE 5. *Left:* The samples from the Slepian noise for the Gaussian (top) and the Laplace (bottom), both for the crossing level $u = 5$. *Middle:* The explicit level dependent component of the Slepian model for the Laplace moving average for $u = 0.5$ (*top*) and $u = 5$ (*bottom*). *Right:* The Slepian model itself for the LMA, $u = 0.5$ (*top*) and $u = 5$ (*bottom*).

and, finally, the time distorted random noise

$$B \circ \gamma(t).$$

Using this notation, we can write

$$L_u = F_{u,g,\Gamma_u} + G_{\dot{X}_u,g,\Gamma_u,B} + B \circ \Gamma_u. \tag{12}$$

We observe that in this decomposition, in contrast to the stationary Gaussian case, all terms are dependent on the level u. The Gibbs sampler given in (19) of the Appendix simulates from $(\dot{X}_u, \Gamma_u, L_u)$ and thus allows for evaluation each of the three components in (12).

Example 6 (Crossings by LMA). To illustrate our approach, we consider the u-level crossings of the Laplace moving average defined by (9) in the Gaussian kernel example. Six samples from such a process are presented in Figure 2 *(top-right)*. We want compare how the noise and its Slepian model differ from the case when the crossings were taken by the GMA given in (5) and discussed in Example 4. As before, the shape parameter for the Laplace noise is $\tau = 0.5$.

Using the Gibbs sampler, six samples $(\dot{X}_u, \Gamma_u, L_u)$ are obtained for two different levels: $u = 0.5$ and $u = 5$, and the components F_{u,g,Γ_u}, $G_{\dot{X}_u,g,\Gamma_u,B}$, $B \circ \Gamma_u$ are presented in Figures 4 and 5.

In Figure 4 *(left)*, we see six samples from the Brownian motion *(top)* and the corresponding gamma subordinated samples for the Laplace motion *(bottom)* (a single trajectory of the Gamma process has been used in all samples). For comparison, in the next two columns the components in the decomposition of the Slepian model (12) are shown, first for the level $u = 0.5$ *(middle)* and then for $u = 5$ *(right)*. We note the different vertical scales used in the plots for different crossing levels.

Further, we illustrate how by simulating from the Slepian model for noise, we can obtain the Slepian model Y_u. Using (12) we obtain the following decomposition of Y_u:

$$Y_u(t) = \int g(s-t)\ dL_u(s) = \int g(s-t)\ dF_{u,g,\Gamma_u} + \int g(s-t)\ dG_{\dot{X}_u,g,\Gamma_u,B} + \int g(s-t)\ dB \circ \Gamma_u$$

In Figure 5, we illustrate the components of Y_u and their dependence on level u. This figure should be compared with the analogous figures obtained for crossings of the Gaussian moving average process in the top row of Figure 2. At the first sight the behavior of the Slepian at the high level crossing $u = 5$ shown at Figure 2 *(right-top)* for the GMA and at Figure 5 *(right-bottom)* for the LMA appears to be very similar in these two cases. However, this initial impression is misleading as explained next.

For a large value of the crossing level $u = 5$, the Slepian model for the Laplace motion is having large jumps at the crossing level and thus this jumps convoluted with the kernel are responsible for the shape of the process at the crossing. The jumps in the Slepian model for the noise at level $u = 5$ are shown in Figure 5 in the right-bottom graph are accumulating near the crossing instant. This is in contrast to the Gaussian case where the convolution results in the non-random covariance function that is shown in Figure 3 *(left)*. Irregularities of the Slepian noise around the crossing instant for the Gaussian case is spread over many values as presented at the right-top graph of that figure and result with convolution of the kernel with itself. The corresponding function for the Laplace motion Slepian model is random as seen in Figure 5 *(middle-bottom)*, and it is narrower at its base due to the fact that the kernel is proportional to e^{-t^2} while covariance is proportional to $e^{-t^2/2}$ (convolution of the kernel with itself). The difference would be more profound if the convolution of the kernel with its symmetrization around origin would be more distinct from the kernel itself, as it is the case in the next application, see Figure 8. It would be also interesting to investigate theoretically the observed behavior and obtain an asymptotic result for large level crossings in the Laplace case, in the spirit of some previous work, see [5], [3], [29]. Finally it should be noted that not only behavior at the high level crossing is different but also the frequency of reaching that level is for the Laplace case higher. For example using (1), one can evaluate the intensity of crossing of the level $u = 5$ for the LMA which is approximately 4 times per 10^4 time units much higher than that for the GMA which is less than 3 times per 10^5 time units.

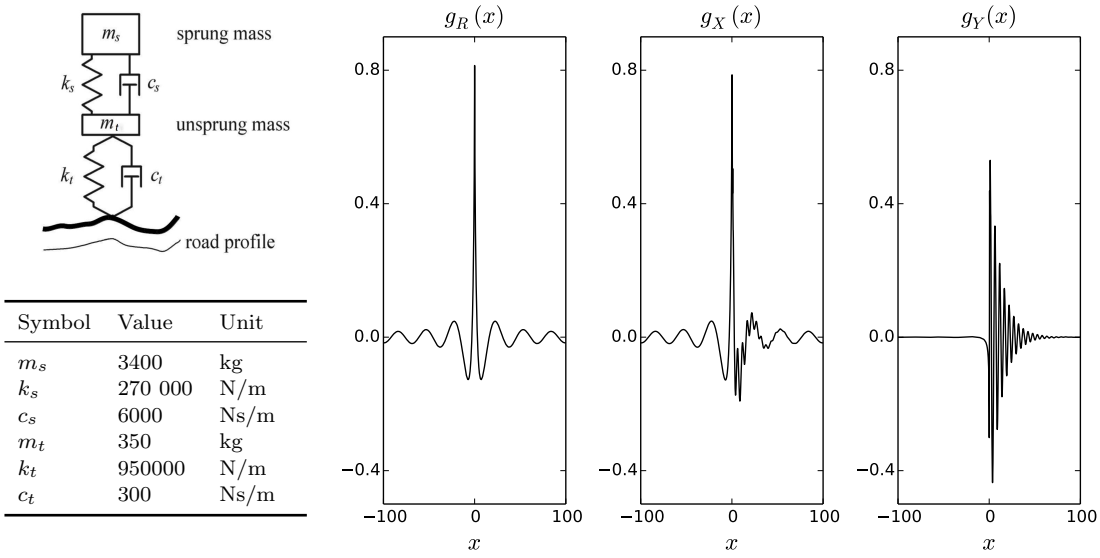
5. Application

Durability applications of vehicle components often requires a customer or market specific load description. It is desirable to have a model of load environment that is vehicle independent and which may consist of many components, like driving habits, encountered road roughness, hilliness, curve radius, cargo loading, and others. Of all the mentioned factors the road surface roughness is one of the most important.

The road profile roughness is often quantified by means of the response of a quarter-vehicle model traveling at a constant velocity on road profiles, see Figure 6 *(left)*. Such a simplification of a physical vehicle cannot be expected to predict loads exactly, but it will highlight the most important road characteristics as far as durability is concerned. Often one choses the force acting on the sprung mass m_s as the response $Y(x)$ which then is used to compute suitable indexes to classify severity of road roughness. The parameters in the model are set to mimic heavy vehicle dynamics developed in SCANIA. Here the parameters have the following physical interpretation. Properties of the tire are described by k_t, c_t, which relate to vertical stiffness and damping of the tire, while properties of the suspension are given by vertical stiffness and damping k_s, c_s, respectively.

By $R(x)$ we denote the road surface elevation at location x (in $[m]$) and by $X(x)$, $U(x)$ the positions of masses m_t, m_s, respectively, relatively to the height for the vehicle at rest. The response $Y(x)$ becomes a variable force acting on the mass m_s, viz. $Y = m_s\ddot{U}$. Quantities $X(x)$, $U(x)$, and $Y(x)$ are all linear functionals of road profile $R(x)$.

Modeling of true loads acting on components are difficult since tires filter nonlinearly the road profile and the filters parameters depends on very uncertain factors, e.g. tire's pressure, wear etc. Consequently the response $X(x)$ is difficult to model and one sometimes considers $X(x)$ as an external



Symbol	Value	Unit
m_s	3400	kg
k_s	270 000	N/m
c_s	6000	Ns/m
m_t	350	kg
k_t	950000	N/m
c_t	300	Ns/m

FIGURE 6. *Left:* Quarter vehicle model. *Three right hand side plots:* Kernels $g_R(x)$ given in (15) and the pair $g_X(x), g_Y(x)$ given in (16).

input to calculate loads acting on the vehicle components, see [17] for a more detailed presentation. Hence it is of interest to study and model both the response and the road profile at locations when X reaches some extreme level. In what follows, we shall give Slepian models for X_u and R_u and Y_u, when X upcrosses u. We consider two moving average models of the road profile variability, namely, the Gaussian and Laplace ones.

In the literature, many models for the power spectral density (psd) of road profiles have been proposed, see [4] for a review. Here, for simplicity, we choose a Matérn type psd fited to a measured road profile for a road in Sweden given in [9] viz.

$$S_R(\omega) = \frac{C\, c^{w_1}}{(1 + (c\omega/\omega_0)^2)^{w_1/2}}, \quad 0.02\pi \le \omega \le 5.65\pi, \tag{13}$$

and zero otherwise. The parameter values are $\omega_0 = 1$, $w_1 = 3$, $C = 10^{-4}$ and $c = 10$. Further one often assumes that the energy for frequencies < 0.01 $[m^{-1}]$ (wavelengths above 100 $[m]$) represents landscape variability, which does not affect the vehicle dynamics and hence can be removed from the spectrum. Similarly high frequencies > 10 $[m^{-1}]$ (wavelengths below 10 $[cm]$) are filtered out by the tire and thus are not included in the spectrum.

In order to simplify the discussion, we have chosen to normalize the spectrum as follows

$$S(\omega) = \sigma^2\, a\, S_R(a\omega), \qquad \sigma^2 = 46.5, a = 0.226, \tag{14}$$

so that $\int S(\omega)\, d\omega = \int \omega^2\, S(\omega)\, d\omega = 1$. The road profile $R(x)$ will be modeled as moving average having a symmetrical kernel $g_R(x)$ which is introduced here through its Fourier transform

$$G_R(\omega) = \mathcal{F}g(\omega) = \sqrt{2\pi S(\omega)}. \tag{15}$$

The responses $X(x)$, $Y(x)$, defined in Figure 6 by means of a mechanical system, are obtained by linearly filtering road profile $R(x)$. The filters transfer functions will be given next.

First, by writing the equations of motion for the two masses, we obtain the transfer function for X:

$$H_X(\omega) = \frac{i\,\omega_v\, c_t + k_t}{H_t(\omega_v) - m_s\omega_v^2(i\,\omega_v\, c_t + k_t)/H_t(\omega_v)}.$$

where $\omega_v = \omega \cdot v$ and ω is a wave number and v is vehicle speed in $[m/s]$.

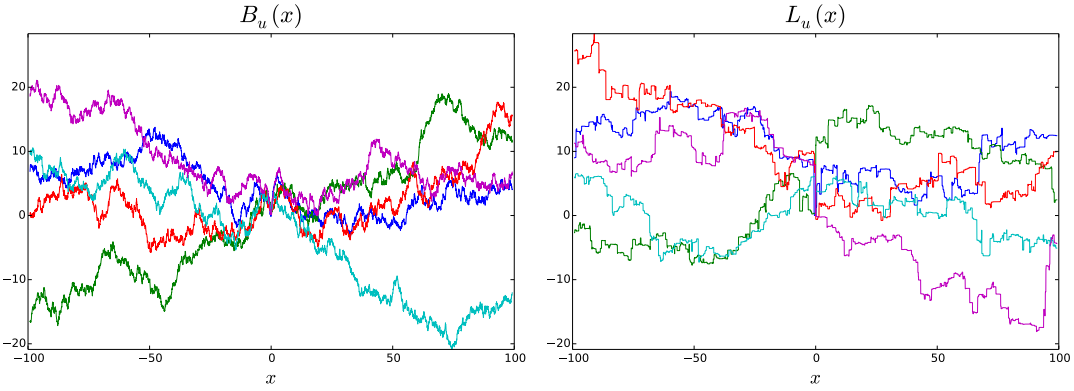


FIGURE 7. Slepian model for noise at a $u = 4.5$ upcrossing of $X(x)$; Brownian motion $B_u(x)$ for the Gaussian model *(left)* and Laplace motion $L_u(x)$ for the LMA *(right)*.

Now, the response $Y(t)$ is filtered $X(t)$ by a filter having the following transfer function

$$H_Y(\omega) = \frac{-m_s\omega_v^2(i\,\omega_v\,c_s + k_t s)}{H_s(\omega_v)},$$

where

$$H_t(\omega) = -m_t\,\omega^2 + i\,\omega\,c_t + k_t, \qquad H_s(\omega) = -m_s\,\omega^2 + i\,\omega\,c_s + k_s.$$

Consequently, the processes $X(x)$ and $Y(x)$ are moving averages with kernels g_X and g_Y defined through theirs Fourier transforms

$$G_X(\omega) = H_X(\omega)G_R(\omega), \qquad G_Y(\omega) = 4\cdot 10^{-6}H_Y(\omega)G_X(\omega), \tag{16}$$

respectively. Here, to ease comparisons, we scaled $Y(x)$ by a factor $4\cdot 10^{-6}$. The kernels g_R, g_X and g_Y are shown in Figure 6 *(three right hand side plots)*.

The Gaussian moving average is commonly used to model the road profile variability. Although it is well known that Gaussian processes do not describe the road profiles well, see [8] and references therein, they are used because many tools are available for fast computations of probabilities of interest for durability evaluations. In [9], the LMA road profile model was proposed and it was demonstrated that it gives much more accurate than the GMA predictions of fatigue damage accumulations in vehicle components.

Next, we shall illustrate some properties LMA that can be useful for a design of components. Our application is kept simple for transparency of the example but it can be easily developed further to address more realistic situations by changing kernels g_X, g_Y and to include additional responses, linear or even nonlinear functionals of X. The purpose of the example is to illustrate quantitative differences between Gaussian and Laplace modeling.

Notably the GMA and LMA models for X, R and Y have the same mean and power spectral densities, and hence variances but they differ in statistical distributions. The Laplace model of road surface has kurtosis equal to 5, which is a typical value for this type of roads in Sweden. For kernel g_X the kurtosis 5 gives the Laplace motion parameter $\tau = 0.37$. We are interested in properties of extremal episodes, here defined as evolution of the system responses, and properties of the road profile when the wheel position reaches an extreme level (for simplicity we assume that the wheel can not lose contact with the road).

To choose a level u which upcrossed by X will define the center of an extreme episode, we consider the level that is crossed about once per 600 km for the stationary road considered here. In Figure 8 *(left column)*, some sample paths of the Gaussian and Laplace moving average models for road profile R and responses X, Y are shown. One can see that the Laplace model reaches more extreme values than the Gaussian model does. For example it can be evaluated using the Rice formula that the frequency of

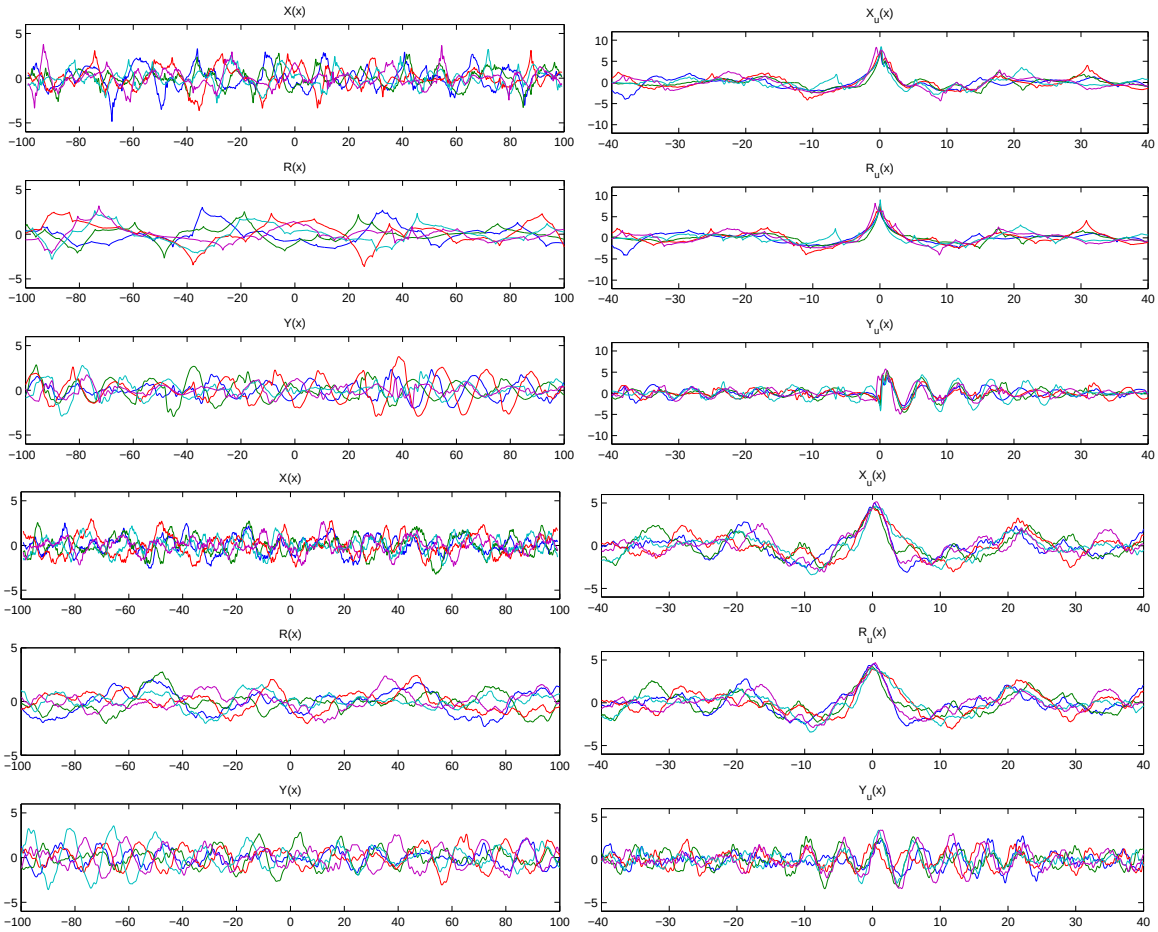


FIGURE 8. Simulation study. The Laplace case shown in the top three graphs of each column and the Gaussian case shown in the bottom three graphs of each column. *Left:* Road profile $R(x)$ (middle) and responses $X(x)$ (top), $Y(x)$ (bottom). *Right:* Slepian models $X_u(x)$, $R_u(x)$ and $Y_u(x)$ around the $u = 7$ upcrossing of $X(x)$ in the Laplace case *(top three graphs)* and around the $u = 4.5$ upcrossing of $X(x)$ for the Gaussian case *(bottom three graphs)*.

upcrossings of level $u = 4.5$ (measured in standard deviations of the process) by the Gaussian process is about the same at the frequency of upcrossings of $u = 7$ (also in the standard deviation unit) by the Laplace process. This happens rarely but still frequently enough to be of importance in durability analysis. Note that many components are designed to hold 200 thousands km with high probability. In the following, the level $u = 4.5, 7$ has been chosen for the GMA, LMA models of X, respectively.

The Slepian models for "noises" $B_u(x)$ and $L_u(x)$ are shown in Figure 7. The processes seems to vary in the same range but in L_u one can easily see jumps. The difference is actually very influential on the properties of the Slepian models for processes X, R, Y around u level upcrossings by X, i.e. X_u, R_u and Y_u, which can be observed in Figure 8 (the plots in the right column).

First by studying kernels g_X and g_R, given in Figure 6 *(three right hand side plots)*, we expect that the paths of X_u and R_u are very similar. There will be some extra vibrations in the tire after passing a large bump but those are relatively small. In contrast the kernel g_Y is very different from kernels g_R, g_X. It is oscillatory and asymmetric. These oscillations are characteristic for the shape of the Y_u for the LMA model of a road profile which, in contrast to the Gaussian case, is not a time reversible process.

Basically, the shape of extreme episodes Y_u resembles the (asymmetric) kernel while for Gaussian model the shape is given by the correlation function of Y which is symmetric in time.

Appendix

Slepian model for homogeneous Gaussian noise. Here we derive a Slepian model for the homogenous Gaussian noise that is driving a moving average process. Such a noise can be viewed as a stochastic measure defined through a Brownian motion $B(t)$, $t \in \mathbb{R}$, obtained from a regular Brownian motion by reflecting it independently at $t = 0$, so that $B(t)$ represents the measure of $[0, t]$ for $t > 0$ (for negative t it equals to minus the measure of $[t, 0]$). This identification of the measures and processes is kept throughout the paper. The biased sampling distribution for the finite dimensional distributions of the Gaussian process B at the u-level up-crossings of $X(t) = \int g(s-t)\, dB(s)$ are obtained by considering the conditional distribution of $B(t), B(s)$, for some fixed t and s, $|t| \geq |s|$, given $X(0) = u$ and $\dot{X}(0) = z$.

The covariance matrix of normally distributed vector $(B(t), B(s), X(0), \dot{X}(0))$ is given by

$$\mathbf{\Sigma} = \begin{bmatrix} |t| & a & G(t) & g(0)-g(t) \\ a & |s| & G(s) & g(0)-g(s) \\ G(t) & G(s) & 1 & 0 \\ g(0)-g(t) & g(0)-g(s) & 0 & 1 \end{bmatrix},$$

where $a = |s|$ if s and t have the same sign and zero otherwise, while $G(t) = \int_0^t g$. The 'ones' on the diagonal are the consequence of the assumption: $\int |g|^2 = \int |\dot{g}|^2 = 1$. Direct verification of the covariances leads to the following representation of Gaussian noise B given $X(0) = u$, $\dot{X}(0) = z$:

$$B(t|u, z) = u \cdot G(t) + z \cdot (g(0) - g(t)) - G(t) \int g dB + (g(0) - g(t)) \int \dot{g} dB + B(t).$$

From the Rice formula it follows that this is a Slepian process for B at the up-crossing level distribution given that the derivative $\dot{X}_u$ at the up-crossings is equal to z. Taking into account that the Rayleigh distribution is representing the biased sampling distribution of the derivative and using (4), a Slepian model for the noise is given by

$$B_u(t) = u \cdot G(t) + R \cdot (g(0) - g(t)) - G(t) \int g dB + (g(0) - g(t)) \int \dot{g} dB + B(t),$$

where random variable R has the Rayleigh distribution and is independent of B.

Conditional Slepian model of noise given subordinator, and derivative. An extension of the representation of the noise from the previous section to a non-homogeneous Gaussian noise is important for derivation of the Slepian model when a moving average with respect to the Laplace noise is crossing a level u. Formally, we are interested in a conditional Slepian model L_u of non-Gaussian noise L given that $\Gamma_u = \gamma$ and $\dot{X}_u = z$, where Γ_u and $\dot{X}_u$ are some Slepian models for subordinator Γ and the derivative $\dot{X}(0)$. Here, the crossing levels are marked by the non-Gaussian moving average given in (10).

As presented in (4), this conditional distribution is equivalent to that of L given $\Gamma = \gamma$, $\dot{X} = z$, and $X = u$. This in turn can be presented through conditioning a non-stationary Gaussian moving average process

$$X_\gamma(t) = \int g(s-t)\ dB(\gamma(s)),$$

where $\gamma(s)$ is a non-decreasing function, i.e. we consider a moving-average integral with respect to stochastic measure B_γ defined on intervals as

$$B_\gamma(s, s+ds] = B(\gamma(s), \gamma(s+ds)] = \sqrt{d\gamma_s} Z_s,$$

where $d\gamma_= \gamma(s+ds) - \gamma(s)$ and $Z_s = B_\gamma(s, s+ds]/\sqrt{d\gamma_s}$ is a standard Gaussian variable that are independent for different s as long as the intervals $(\gamma(s), \gamma(s+ds)]$ do not overlap. We use the notation

$B(\gamma(s),\gamma(s+ds)] = B(\gamma(s+ds)) - B(\gamma(s)$ to denote the corresponding independent increment process B_γ. In what follows, we use $\int f d\gamma$ for $\int f(t) d\gamma(t)$.

The joint distribution of $(B_\gamma(t), B_\gamma(s), \dot{X}(0), X(0))$ has the covariance matrix

$$\mathbf{\Sigma}_\gamma = \begin{bmatrix} |\gamma(t)| & a & \int_0^t g d\gamma & \int_0^t \dot{g} d\gamma \\ a & |\gamma(s)| & \int_0^s g d\gamma & \int_0^s \dot{g} d\gamma \\ \int_0^t g d\gamma & \int_0^s g d\gamma & \int g^2 d\gamma & \int g\dot{g} d\gamma \\ \int_0^t \dot{g} d\gamma & \int_0^s \dot{g} d\gamma & \int g\dot{g} d\gamma & \int \dot{g}^2 d\gamma \end{bmatrix},$$

where $a = |\gamma(s)|$ if the signs of $\gamma(t)$ and $\gamma(s)$ are the same and zero otherwise. For compactness of the presentation, let set

$$\begin{aligned} r_\gamma &= -\frac{\int g\dot{g} d\gamma}{\sqrt{\int \dot{g}^2 d\gamma \int g^2 d\gamma}}, \\ g_\gamma &= \frac{g}{\sqrt{\int g^2 d\gamma}},\ \dot{g}_\gamma = \frac{\dot{g}}{\sqrt{\int \dot{g}^2 d\gamma}}, \\ G_\gamma(t) &= \int_0^t g_\gamma d\gamma,\ \dot{G}_\gamma(t) = \int_0^t \dot{g}_\gamma d\gamma. \end{aligned}$$

Direct verification of the covariances proves the following conditional Slepian model for the non-homogenous noise L_u conditionally on $\dot{X}_u = z, \Gamma_u = \gamma$:

$$\begin{aligned} (17)\quad B_\gamma(t|u,z) = {} & \frac{G_\gamma(t) + r_\gamma \dot{G}_\gamma(t)}{1-r_\gamma^2} \frac{u}{\sqrt{\int g^2 d\gamma}} - \frac{\dot{G}_\gamma(t) + r_\gamma G_\gamma(t)}{1-r_\gamma^2} \frac{z}{\sqrt{\int \dot{g}^2 d\gamma}} \\ & - \frac{G_\gamma(t)}{1-r_\gamma^2} \int (r_\gamma \dot{g}_\gamma + g_\gamma)\, dB_\gamma - \frac{\dot{G}_\gamma(t)}{1-r_\gamma^2} \int (r_\gamma g_\gamma + \dot{g}_\gamma)\, dB_\gamma + B_\gamma(t). \end{aligned}$$

From this model samples can be taken from a Gibbs sampler scheme as discussed next.

Slepian model of noise, subordinator, and derivative based on Gibbs sampler. Here, we discuss a Slepian model $(L_u, \Gamma_u, \dot{X}_u)$ that is based on a Gibbs sampler. Let us consider gamma process Γ, i.e. the Lévy process such that $\Gamma(1)$ has the gamma distribution with shape τ and scale $1/\tau$ (for negative t, the process $-\Gamma(t)$ is an independent copy of $\Gamma(t), t \geq 0$). For the computational and practical reasons it is more convenient to consider a discretized version of the problem. We consider a uniformly spaced grid dt (for compactness of the notation, we use dt both for the grid and for its diameter) and assume that stochastic measures are approximated by the Lebesgue measure multiplied by independently sampled random increments of the considered stochastic measure. In particular, dL is a vector of values of the motion increments over this grid, $d\Gamma$ are random gamma variances distributed with shape τdt and scale $1/\tau$ while Z is a vector of independent standard normal random variables so that $dL = \sqrt{d\Gamma} Z$, where the multiplication is coordinate-wise. We also use $d\Gamma_u$ and dL_u as notation for Slepian models of $d\Gamma$ and dL, respectively.

Further if g is a function, then $\int g$ is a vector of values of the Lebesgue integrals of g over the cells of the grid dt. With the assumed discretization, we write $\int g\, dL = \int g \cdot dL$, where $\cdot$ stands for the inner product of the two vectors. Consequently, we write $X = \int f dL = \int f \sqrt{d\Gamma} Z$ and $\dot{X} = \int \dot{f} dL = \int \dot{f} \sqrt{d\Gamma} Z$.

We first notice that $(\Gamma_u | L_u = l, \dot{X}_u = z) \stackrel{d}{=} (\Gamma | L = l, \dot{X} = z, X = u)$, where each of the sides denotes a conditional distribution. Since, both $\dot{X}$ and X are deterministic functions of dL thus we can assume from now on that $\int \dot{f} dl = z$ and $\int f dl = u$. The Bayes formula yields

$$(18)\qquad f_{\Gamma_u | L_u, \dot{X}_u}(\gamma | l, z) \propto f_{L|\Gamma}(l|\gamma) f_\Gamma(\gamma) \propto \left(\prod_{i=1}^N \gamma_i^{\tau dt - 3/2} e^{-\left(2\gamma_i/\tau - dl_i^2/\gamma_i\right)/2} \right),$$

which corresponds to the distribution of a vector of independent variables distributed as $GIG(\tau dt - 1/2, 2/\tau, dl_i^2)$.

The Gibbs sampler from the Slepian model $(L_u, \Gamma_u, \dot{X}_u)$ will be based on alternate samples from the conditional distributions

$$\begin{aligned} \gamma^{(j)} &\sim (\Gamma_u | L_u = l^{(j-1)}, \dot{X}_u = z^{(j-1)}) \\ \left(l^{(j)}, z^{(j)}\right) &\sim (L_u, \dot{X}_u | \Gamma_u = \gamma^{(j)}). \end{aligned} \tag{19}$$

As we have seen above, the first sampling is reduced to sampling from independent GIG distributions for which there exists a uniformly bounded rejection algorithm, see [16]. Let us next discuss how to sample from $L_u, \dot{X}_u$ given that $\Gamma_u = \gamma$.

We note that it is equivalent to sampling z from $\dot{X}_u$ given that $\Gamma_u = \gamma$ and, then l from L_u given that $\dot{X}_u = z, \Gamma_u = \gamma$ which is the same as sampling from L given that $\dot{X} = z, \Gamma = \gamma, X = u$, which was discussed in the previous section, see (17).

To express the distribution of $\dot{X}_u$ given Γ_u, we further extend our notation for the discretized model. For any integrable function g we write $\int g \, d\gamma$ as the inner product between the vector of the Lebesgue integrals of g over the grid cells and the vector of values of the gamma vector $d\Gamma = d\gamma$. Using this convention we define

$$r = -\frac{\int g\dot{g} \, d\gamma}{\sqrt{\int \dot{g} \, d\gamma \int g \, d\gamma}}.$$

The density of the distribution of $(\dot{X}_u | \Gamma_u = \gamma)$ is proportional to the joint density of $(\dot{X}_u, \Gamma_u)$ as a function of the first argument, say z, and the value for the second argument fixed to γ. By the Rice formula as written in (3), with $Y = (\dot{X}_u, \Gamma_u)$, we observe that this joint density as a function of z is proportional to

$$\begin{aligned} f_{\dot{X}_u|\Gamma_u}(z|\gamma) &\sim z f_{\dot{X},X|\Gamma}(z, u|\gamma) f_\Gamma(\gamma) \\ &\sim z f_{\dot{X},X|\Gamma}(z, u|\gamma) \\ &\sim z \cdot \exp\left(-\frac{\left(z - ur\sqrt{\int \dot{g}^2 \, d\gamma / \int g^2 \, d\gamma}\right)^2}{2(1-r^2)\int \dot{g}^2 \, d\gamma} \right) \end{aligned}$$

or, in a more compact way,

$$f_{\dot{X}_u|\Gamma_u}(z|\gamma) \sim z \cdot \exp\left(-\frac{(z-a)^2}{b} \right) \tag{20}$$

where

$$\begin{aligned} a &= u \cdot r \cdot \sqrt{\int \dot{g}^2 \, d\gamma} / \sqrt{\int g^2 \, d\gamma}, \\ b &= 2(1-r^2) \int \dot{g}^2 \, d\gamma. \end{aligned}$$

This distribution, that we term the tilted Rayleigh distribution, can be sampling by a rejection algorithm that is presented next.

Conditional Slepian derivative model – the tilted Rayleigh distribution. We have seen that the distribution of $\dot{X}_u$ conditionally on $L_u = l$ and $\Gamma_u = \gamma$ is given by a scaled random variable distributed according to the density

$$f(x; a) = C_a \cdot x e^{-(x-a)^2}, \ x > 0, \tag{21}$$

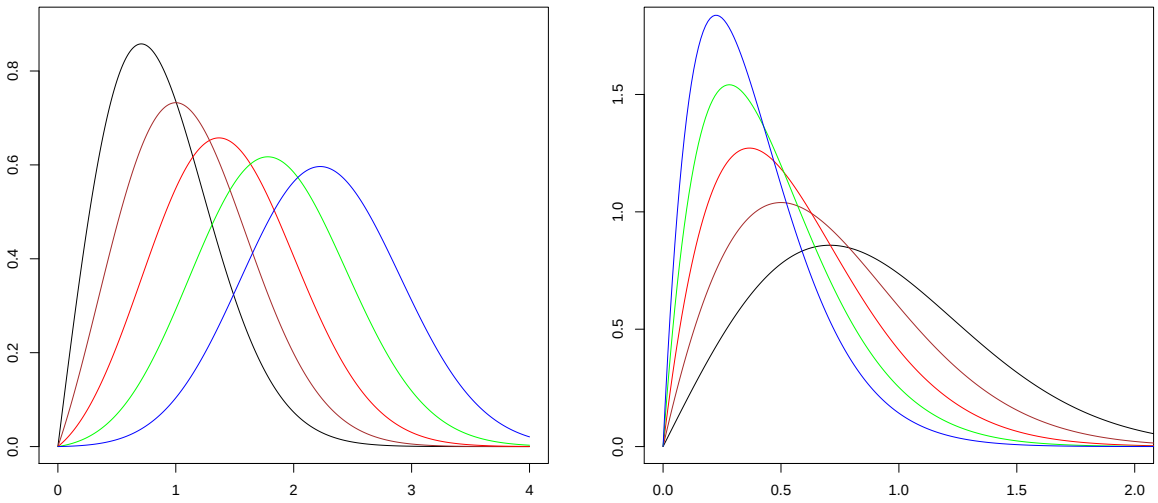


FIGURE 9. Tilted Rayleigh densities: $a = 0, 0.5, 1, 1.5, 2$ (left) and $a = 0, -0.5, -1, -1.5, -2$ (right), the densities are more shifted toward left for smaller values of a.

where $a \in \mathbb{R}$ is a parameter (in our problem this parameter as well as the scaling parameter, which is not considered here explicitly, will be determined by the model). The normalizing constant equals

$$C_a = \frac{2}{e^{-a^2} + 2\sqrt{\pi} a \Phi(\sqrt{2}a)}.$$

Observe that, for $a = 0$ the density corresponds to the classical Rayleigh distribution and thus we will refer to the distribution given by (21) as the tilted Rayleigh distribution with the tilting parameter a. In Figure 9, we show few densities from this class of distributions for positive values *(left)* and negative values *(right)* of a.

Direct calculations lead to the cumulative distribution function

$$F(x) = 1 - \frac{e^{-(x-a)^2} + 2\sqrt{\pi} a \Phi(\sqrt{2}(a-x))}{e^{-a^2} + 2\sqrt{\pi} a \Phi(\sqrt{2}a)}, \ x > 0. \tag{22}$$

In Algorithm 1 we state an accept-reject algorithm for which the rejection rate is a uniformly bounded function of a. The main result is stated in the next theorem which is a direct consequence of the two lemmas that are presented later along with the details of the algorithms. The acceptance rate obtained in the theorem is illustrated graphically in Figure 10. In the formulation of the result we use the following constant

$$K_a = \frac{2}{e^{-a^2} - 1 + \sqrt{\pi} a \operatorname{erf}(a)},$$

where $\operatorname{erf}(x) = 2\Phi_0(\sqrt{2}x)$.

ALGORITHM 1. Rejection algorithm for sampling tilted Rayleigh distribution

Require: parameter $a > 0$
generate $U \sim \mathcal{U}[0,1]$
$\lambda_1 \leftarrow \frac{1}{e^{-a^2}+2\sqrt{\pi}a\Phi(\sqrt{2}a)}$
$\lambda_2 \leftarrow \sqrt{\pi}a\lambda_1$
if $U \le \lambda_1$ **then**
  generate $U_1 \sim \mathcal{U}[0,1]$
  $X \leftarrow a + \sqrt{-\log(U_1)}$
else if $U \le \lambda_1 + \lambda_2$ **then**
  $Z \sim N(0,1)$
  $X \leftarrow a + \frac{|Z|}{\sqrt{2}}$
else
  if $a \le \sqrt{\pi}$ **then**
    repeat
      generate $U_1 \sim \mathcal{U}[0,1]$ and $U_2 \sim \mathcal{U}[0,1]$
      $X \leftarrow a\sqrt{U_2}$
    until $U_1 \le \exp\left(-(X-a)^2\right)$
  else
    repeat
      generate $U_1 \sim \mathcal{U}[0,1]$ and $Z \sim N(0,1)$
      $X \leftarrow a - \frac{|Z|}{\sqrt{2}}$
    until $U_1 \le \frac{X}{a}$
  end if
end if
return X

Require: parameter $a \le 0$
if $a > -\frac{1}{\sqrt{2}}$ **then**
  repeat
    generate $U_1 \sim \mathcal{U}[0,1]$ and $U_2 \sim \mathcal{U}[0,1]$
    $X \leftarrow 2^{-1/2}\sqrt{-2\log(U_2)}$
  until $U_1 \le e^{2aX}$
else
  repeat
    generate $X \sim \Gamma(2, (-2a)^{-1})$ and $U_1 \sim \mathcal{U}[0,1]$
  until $U_1 \le e^{-X^2}$
end if
return X

Theorem 1. *Algorithm 1 generates a sample from the tilted Rayleigh distribution, with the acceptance probability equal to*

$$\alpha(a) = \begin{cases} \dfrac{4a^2e^{a^2}}{C_a} & \text{if } a < -\frac{\sqrt{2}}{2}, \\ \dfrac{2e^{a^2}}{C_a} & \text{if } -\frac{\sqrt{2}}{2} \le a \le 0, \\ C_a\left(\dfrac{1}{2} + \dfrac{\sqrt{\pi}a}{2} + \dfrac{2/C_a - 1 - \sqrt{\pi}a}{a^2K_a}\right) & \text{if } 0 < a < \sqrt{\pi}, \\ C_a\left(\dfrac{1}{2} + \dfrac{\sqrt{\pi}a}{2} + \dfrac{2/C_a - 1 - \sqrt{\pi}a}{\sqrt{\pi}aK_a}\right) & \text{if } \sqrt{\pi} \le a, \end{cases}$$

which is bounded below by 0.345.

Before proving the theorem we state different sampling schemes that are used in the algorithm. First we propose two sampling schemes for

$$h(x;a) = K_a\ xe^{-(x-a)^2},\ x \in [0,a],$$

The first scheme is

(1) Sample $X \sim 2a^{-2}x\ \mathbb{I}_{[0,a]}(x)$ using the cdf inverse method,
(2) Sample U uniformly over $[0,1]$ and if $U \le e^{-(X-a)^2}$ accept X, else goto step (1).

The second scheme is

(1) Sample $X \sim \frac{2}{\sqrt{\pi}}e^{-(x-a)^2}\mathbb{I}_{(-\infty,a]}(x)$ using a normal random variable,

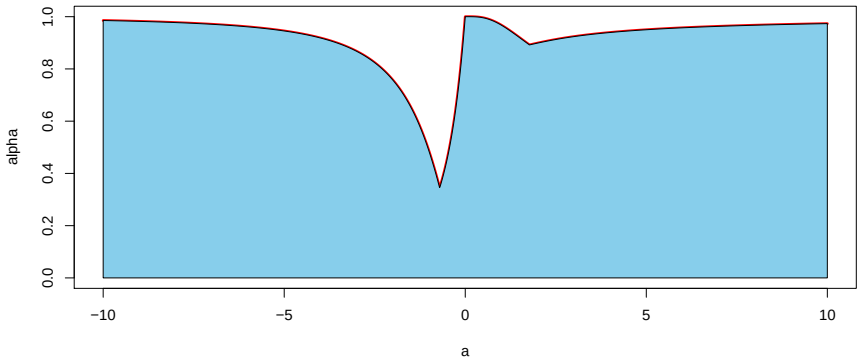


FIGURE 10. Acceptance rate as a function of tilting a.

(2) Sample U uniformly over $[0,1]$, and if $U \leq \frac{X}{a}$ accept X, or else go to step (1).

We get explicit acceptance probability for both the schemes formulated in the following Lemma.

Lemma 1. *Both scheme generates a random variables X with distribution $h(x;a)$. The acceptance probability for the first scheme is $2/(a^2K_a)$, which is a strictly decreasing function of a. For the second scheme the acceptance probability is $2/(\sqrt{\pi}aK_a)$, which is strictly increasing in a. Moreover, the two acceptance rates are equal to 0.63 at $a=\sqrt{\pi}$.*

Proof. Both schemes are standard accept reject algorithm, and the distribution of X is immediate. By a general property, see [10], the acceptance probability of the first scheme is given by

$$P\left(U \leq e^{-(X-a)^2}\right) = \frac{2}{a^2K_a}$$

and for the second algorithm by

$$P\left(U \leq \frac{X}{a}\right) = \frac{2}{\sqrt{\pi}aK_a}.$$

To see the monotonicity of the acceptance rates note that the derivative in a is given by

$$\frac{1}{K_a} = \frac{\sqrt{\pi}}{2}\int_0^a \operatorname{erf}(t)\ dt,$$

so that

$$\begin{aligned}\left(\frac{2}{a^2K_a}\right)' &= \sqrt{\pi}\left(\frac{\int_0^a \operatorname{erf}(t)\ dt}{a^2}\right)' \\ &= \sqrt{\pi}\ \frac{a\ \operatorname{erf}(a) - 2\int_0^a \operatorname{erf}(t)\ dt}{a^3} = -2\sqrt{\pi}\frac{\int_0^a (\operatorname{erf}(t)/t - \operatorname{erf}(a)/a)\ t\ dt}{a^3} \\ &< 0,\end{aligned}$$

where the inequality is because $\operatorname{erf}(x)/x$ is strictly increasing.

Similarly, for the second acceptance rate

$$\left(\frac{2}{\sqrt{\pi}aK_a}\right)' = \left(\frac{\int_0^a \operatorname{erf}(t)\ dt}{a}\right)' = \frac{a\ \operatorname{erf}(a) - \int_0^a \operatorname{erf}(t)\ dt}{a^2} > 0,$$

since $\mathrm{erf}(x) > 0$ if $x > 0$. □

For $a < 0$, note that

$$f(x;a) = C_{\tilde{a}} x e^{-x^2 - 2\tilde{a}x}, \tag{23}$$

where $\tilde{a} = -a > 0$, and

$$C_{\tilde{a}} = \frac{2e^{-\tilde{a}^2}}{e^{-\tilde{a}^2} - 2\sqrt{\pi}\tilde{a}(1 - \Phi(\sqrt{2}\tilde{a}))}.$$

Similarily, we also derive two sampling schemes for $f(x;a)$ when $a \leq 0$. The first scheme is

(1) Sample $X \sim Rayleigh(2^{-1/2})$,
(2) Sample $U \sim \mathcal{U}[0,1]$, if $U \leq e^{-2\tilde{a}X}$ accept X, else goto step (1).

The second scheme is

(1) Sample $X \sim \Gamma(2, (2\tilde{a})^{-2})$
(2) Sample $U \sim \mathcal{U}[0,1]$, if $U \leq e^{-X^2}$ accept X, else goto step (1).

For these two schemes we also get explicit acceptance probability:

Lemma 2. *Both scheme generates a random variables X with distribution $f(x;a)$ for $a \leq 0$. The acceptance probability of the first scheme is $2/C_{\tilde{a}}$ which is a strictly decreasing function of $\tilde{a}$. The acceptance probability for the second algorithm is $4\tilde{a}^2/C_{\tilde{a}}$, which is a strictly increasing function of $\tilde{a}$. The two algorithm have the same acceptance rate at $\tilde{a} = 2^{-1/2}$, which equals $1 - \sqrt{2\pi e}\bar{\Phi}(1) \approx 0.344$.*

Proof. Both schemes are based on the standard accept reject algorithm, and the distribution of X is immediate. We follow the same method of the proof in Lemma 5, and observe the respective acceptance rates

$$P(U \leq e^{-2\tilde{a}X}) = \frac{2}{C_{\tilde{a}}},\ P(U \leq e^{-X^2}) = \frac{4\tilde{a}^2}{C_{\tilde{a}}}.$$

To see that the function is strictly decreasing as a function of $\tilde{a}$, note that

$$2/C_{\tilde{a}} = 1 - 2\sqrt{\pi}\tilde{a}e^{\tilde{a}^2}(1 - \Phi(\sqrt{2}\tilde{a})) = 1 - \sqrt{\pi}\tilde{a}e^{\tilde{a}^2}(1 - \mathrm{erf}(\tilde{a})) = 1 - \sqrt{\pi}\tilde{a}e^{\tilde{a}^2}\,\mathrm{erfc}(\tilde{a}).$$

Thus,

$$\left(\frac{2}{C_{\tilde{a}}}\right)' = -\sqrt{\pi}(1 + 2\tilde{a}^2)e^{\tilde{a}^2}\,\mathrm{erfc}(\tilde{a}) + 2\tilde{a} = -2(1 + 2\tilde{a}^2)M(\tilde{a}) + 2\tilde{a} < 0,$$

where the last inequality comes from [14] as a relation for the Mills ratio

$$M(x) = e^{x^2}\int_x^\infty e^{-t^2}\,dt.$$

Similarily,

$$\left(\frac{\tilde{a}^2}{C_{\tilde{a}}}\right)' = \tilde{a}\left(2(\tilde{a}^2 + 1) - 2(2\tilde{a}^2 + 3)\tilde{a}M(\tilde{a})\right) > 0 \tag{24}$$

where the last inequality comes from [25] equation (7.8.5). □

We are now ready to state the proof of of Theorem 1.

Proof. We prove two complementary cases: $a \leq 0$ and $a > 0$.

For $a > 0$, we have representation

$$f(x;a) = \lambda_1 h_1(x - a) + \lambda_2 h_2(x - a) + (1 - \lambda_1 - \lambda_2)h(x;a),$$

where

$$\lambda_1 = \frac{1}{e^{-a^2} + 2\sqrt{\pi}a\Phi(\sqrt{2}a)},$$
$$\lambda_2 = \sqrt{\pi}a\lambda_1 = \frac{\sqrt{\pi}a}{e^{-a^2} + 2\sqrt{\pi}a\Phi(\sqrt{2}a)},$$

and

$$h_1(x) = 2xe^{-x^2},\ x > 0,$$
$$h_2(x) = \frac{2}{\sqrt{\pi}}e^{-x^2},\ x > 0,$$
$$h(x;a) = K_a\ xe^{-(x-a)^2},\ x \in [0,a].$$

Note that, h_1 is the density of the Rayleigh distribution, and h_2 is the density of the absolute value of a normal random variable with variance $\frac{1}{2}$. This together with Lemma 5 demonstrates that the algorithm generates tilted Rayleigh samples for $a > 0$. The acceptance rate function for $a > 0$ follows from multiplying $1 - \lambda_1 - \lambda_2$ by the rates from Lemma 5.

Both sampling and acceptance rate for $a \leq 0$ follow immediately from Lemma 2.

□

Generalized inverse Gaussian distribution. The generalized inverse Gaussian distribution with parameters $p \in \mathbb{R}$, $a \geq 0$, and $b \geq 0$, for shortness $GIG(p,a,b)$, is given by the pdf

$$f(x) = \frac{(a/b)^{p/2}}{2K_p\left(\sqrt{ab}\right)}x^{p-1}e^{-\frac{ax+b/x}{2}}.$$

The parameters satisfy

$$a > 0\,, b \geq 0\,,\ \text{if } p > 0,$$
$$a > 0\,, b > 0\,,\ \text{if } p = 0,$$
$$a \geq 0\,, b > 0\,,\ \text{if } p < 0.$$

The moment generating function of a GIG distribution is given by

$$M(t) = \left(\frac{a}{a-2t}\right)^{p/2}\frac{K_p\left(\sqrt{b(a-2t)}\right)}{K_p(\sqrt{ab})},\ t < a/2. \tag{25}$$

The following formulas for the expectations of a $GIG(p,a,b)$ random variable X hold

$$\mathbb{E}[X^\lambda] = (b/a)^{\lambda/2}\frac{K_{p+\lambda}\left(\sqrt{ab}\right)}{K_p\left(\sqrt{ab}\right)},\ \lambda \in \mathbb{R}$$
$$\mathbb{E}[\log(X)] = \log(\sqrt{a/b}) + \frac{\partial \log K_p}{\partial p}\left(\sqrt{ab}\right), \tag{26}$$

where $\partial \log K_p/\partial p(x)$ is the derivative of the Bessel function $K_p(x)$ with respect of its order p and evaluated at value (p,x), cf. [18]. Consequently, by setting $R_p(x) = K_{p+1}(x)/K_p(x)$ we obtain

$$\mathbb{E}[X] = \sqrt{\frac{b}{a}}\cdot R_p\left(\sqrt{ab}\right),$$
$$\mathbb{E}[X^{-1}] = \sqrt{\frac{a}{b}}\cdot\frac{1}{R_{p-1}\left(\sqrt{ab}\right)}. \tag{27}$$

This together with the following recurrence relation

$$R_p(x) = 2p/x + 1/R_{p-1}(x),$$

see [18], yields

$$\mathbb{E}[X^{-1}] = \frac{a}{b} \cdot (\mathbb{E}[X] - 2p/a)\,. \tag{28}$$

The special case of $p = \frac{1}{2}$ corresponds to the reciprocal inverse Gaussian and the following simple forms of expectations hold

$$\mathbb{E}[X] = \sqrt{\frac{b}{a}}\left(1 + \frac{1}{\sqrt{ab}}\right),$$
$$\mathbb{E}[X^{-1}] = \sqrt{\frac{a}{b}}.$$

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