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ABSTRACT. We construct and analyze a finite volume scheme for numerically solving a three-dimensional Poisson equation. This is an extension of a two-dimensional approach by Süli [26]. We derive optimal convergence rates in the discrete H^1 norm and sub-optimal in the maximum norm, using the maximal available regularity of the exact solution and with minimal smoothness requirement on the source term. We also find a gap in the proof of a key estimate in a reference in [26] and present a modified and completed proof. Finally, the theoretical results derived in the paper are justified through implementing some canonical examples in 3D.

Keywords: Finite volume method, Poisson's equation, stability estimates, convergence rates.

1. INTRODUCTION

Our motivation for the numerical study of the classical Poisson equation stems from its appearance in coupled system of PDEs involving the Vlasov type equations of plasma physics with a wide range of application areas, especially in modelling plasma of Coulomb particles. The common approach has been to consider a continuous Poisson solver and focus the approximation strategy on the study of the associated hyperbolic equations of the system of, e.g. Vlasov-Poisson-Fokker-Planck (VPFP) or Vlasov-Maxwell-Fokker-Planck (VMFP) equations. However, in a discrete scheme including the continuous Poisson solution, which is in an infinite dimensional space, requires an unrealistically fine degree of resolution for the mesh size and hence an excessive amount of computational cost. Even with availability of a computational environment, a miss-match will appear due to the lack of compatibility between the resolution degree for the continuous Poisson solver and a flexible numerical scheme for the discretized equations in the system. The present study concerns numerical approximations of the Poisson equation that can both complete the previous semi-analytic/semi-discrete schemes, for the Vlasov-type systems, and meanwhile are accurate enough to be comparable with the fully discrete numerical schemes for the hyperbolic system of PDEs. To this approach, we construct and analyze a finite volume scheme, prove stability estimates, and also derive optimal convergence rates in the discrete H^1 (corresponding to an order of $\mathcal{O}(h^2)$ for the exact solution in the Sobolev space $H_0^2(\Omega)$) as well as suboptimal convergence rates in the maximum norm (the maximum norm estimates are optimal in 2D) for the Dirichlet problem for the following three dimensional Poisson equation

$$\begin{cases} -\nabla^2 u &=& f & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = (0,1) \times (0,1) \times (0,1)$.

Problem (1.1) is a simplified form of the general Poisson equation formulated as

$$\begin{cases} -\nabla(A\nabla u) &=& f & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{cases} \qquad (1.2)$$

where A is a conductivity matrix and Ω is a bounded convex domain in $\mathbb{R}^3$. To simplify the calculus we have assumed that $A = I$ (the identity matrix) and considered the cubic domain $\Omega = (0,1)^3$. Considering Problem (1.2) with a variable coefficient matrix A will be much more involved and shift the focus away from the study of the Poisson operator. On the other hand, e.g. for a unifying finite element approach for VPFP, transferring the Poisson equation to a hyperbolic system yields the simple but less advantageous problem, (see. e.g. [3]),

$$\begin{cases} v &=& -\nabla u \\ \text{div } v &=& f. \end{cases} \qquad (1.3)$$

Therefore, considering the finite volume method (FVM) for the Dirichlet problem (1.1) we can also circumvent these inconvenient issues.

The convergence results for Problem (1.1) here, being for a cell-centered finite volume scheme in a quasi-uniform mesh, may be compared with that of a finite element scheme with no quadrature procedure. A finite element scheme combined with a quadrature would cause a reduce in the convergence rate by an order of $\sim \mathscr{O}(h^{1/2})$. In this aspect and compared to standard finite elements, the usual finite volume method (as the finite difference) is quasi-optimal.

The main advantage of the finite volume method is its *local conservativity property* of the numerical flux. This property makes the finite volume method an attractive tool in modelling problems emphasizing the flux, like most hyperbolic PDEs, e.g. fluid problems and conservation laws, see [14] for further details. In higher dimensions, a draw-back is that the calculus, being *seemingly involved*, cannot be represented by simple, short-hand notation. The finite volume method has been studied for both the Poisson equation, fluid problems and other PDEs by several authors in various settings: e.g. the discontinuous finite volume method for second-order elliptic problems in two-dimensions is considered in [7], where the closeness of the FVM to the interior penalty method is demonstrated and optimal error estimates are derived in L_2- and L_∞-norms. A three dimensional discrete duality finite volume scheme for nonlinear elliptic equations is considered in [12], where well-posedness and a priori L_p-error bounds are discussed. These are convergence analysis, with no particular consideration of their optimality. A more computation oriented, second-order finite volume scheme in three dimensions is considered in [28]. This paper deals with computing eigenvalues of a Schrödinger type operator in 3D. As another computational exposition we mention [24] where the authors construct a shape interface FVM for elliptic equations on Cartesian grids in three dimensions with second order accuracy in L_2- and L_∞-norms. This paper concerns variable coefficients as well with a particular piecewise trilinear ansatz. As for the fluid problems, e.g. in [2] a 3D finite volume scheme is presented for the ideal magnetohydrodynamics. Theoretical analysis for the upwind FVM on the counter-example of Peterson, for a two-dimensional, time dependent advection problem, can be found in [9]. For a detailed study of the finite volume method for compressible flow see [22].

However, the most relevant to our study are the works by Süli et. al., e.g. [26], for a two-dimensional version of our work, and [27] and [23], considering the accuracy of cell-vertex FVM for time-dependent advection- and convection-diffusion problems, respectively. Finally, a thorough theoretical study for the numerical solutions of general, linear, nonlinear and quasilinear elliptic problems are given by Böhmer in [8], where most numerical methods are rigorously featured.

Below, for the sake of completeness, we recall some classical results concerning regularities connecting the solution and the data for Problem (1.1) in different geometries. First

we state these results for $\mathbb{R}^n$ and then for an open set $\Omega \subset \mathbb{R}^n$ with smooth boundary. For details we refer the reader to, e.g. Folland [15].

Proposition 1.1. *Suppose $f \in L_1(\mathbb{R}^n)$, and also that $\int_{|x|>1} |f(x)| \log|x| \, dx < \infty$ in case $n = 2$. Let N be the fundamental solution of the $-\nabla^2$ operator: $-\nabla^2 N = \delta$. Then $u = f * N$ is locally integrable and is a distribution solution of $-\nabla^2 u = f$.*

Proposition 1.2. *If f satisfies the conditions of proposition 1.1 and in addition f is $\mathscr{C}^\alpha(\Omega)$ for some $\alpha \in (0,1)$ on some open set Ω, then $u = f * N$ is $\mathscr{C}^{2+\alpha}$ on Ω.*

Corollary 1.1. *If $f \in \mathscr{C}^{k+\alpha}(\Omega)$ for some integer k and $\alpha \in (0,1)$ then $u \in \mathscr{C}^{k+2+\alpha}(\Omega)$.*

To express in, L_2-based, Sobolev spaces (see Adams[1] for details) we have

Proposition 1.3. *If $f \in H^k(\Omega)$ then $u \in H_0^1(\Omega) \cap H^{2+k}(\Omega)$.*

In propositions 1.1-1.3, Ω is assumed to have a smooth boundary. For a general bounded convex domain Ω, by Dirichlet principle, given $f \in H^{-1}(\Omega)$, there exists a unique solution, $u \in H_0^1(\Omega)$, satisfying (1.1), and the mapping $f \longmapsto u$ is a Hilbert space isomorphism from $H^{-1}(\Omega)$ onto $H_0^1(\Omega)$. This is crucial in our study where, in order to derive optimal convergence with minimum smoothness requirement on the exact solution, we shall assume the data f belong to H^{-1}, the dual of $H_0^1(\Omega)$. Then for $f \in H^\sigma(\Omega)$, we have $u \in H^{\sigma+2}(\Omega)$ where $-1 \le \sigma < 1$. For justification of preserving regularity we refer the reader to approaches based on Green's function, see, e.g. [16] and [21].

The purpose of this study is to generalize the two dimensional results in [26] for the rectangular domain $\Omega = (0,1) \times (0,1)$ to the cube $\Omega = (0,1) \times (0,1) \times (0,1)$. The study of the finite volume scheme in three dimensions is somewhat different from a straightforward generalization of the two dimensional case and there are adjustments that need to be made for the dimension. We also provide a corrected (cf. [13]) proof of Theorem 4.2 (in [26]) utilized for the convergence of the finite volume method.

For Problem (1.1), existence uniqueness and regularity studies are extensions of two-dimensional results in [17]: $f \in H^{-1}(\Omega)$ implies that: there exists a unique solution $u \in H_0^1(\Omega)$, and for $f \in H^s(\Omega)$, with $-1 \le s < 1$, $s \ne \pm 1/2$, $u \in H^{s+2}(\Omega)$. The finite volume scheme can be described as: exploiting divergence from the differential equation (1.1) integrating over disjoint "volumes" and using Gauss' divergence theorem to convert volume-integrals to surface-integrals, and then discretizing to obtain the approximate solution u_h, with h denoting the mesh size. Here, the finite volume method is defined on the Cartesian product of non-uniform meshes as a Petrov-Galerkin method using piecewise trilinear trial functions on a *finite element* mesh and piecewise constant test functions on the dual box mesh. The main result of this paper: Theorem 1.1, together with the optimal finite element estimate in Theorem 1.2, justifies the sharpness of our estimate in L_2. The L_∞ estimate in three dimensions is suboptimal.

Theorem 1.1. *The finite volume error estimates for general non-uniform and quasi-uniform meshes in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, are given by*

$$\|u - u_h\|_{1,h} \le Ch^s |u|_{H^{s+1}}, \qquad \|u - u_h\|_\infty \le Ch^{s+1-d/2} |\log h| |u|_{H^{s+1}}, \quad 1/2 < s \le 2. \quad (1.4)$$

whereas the corresponding finite element estimates can be read as:

Theorem 1.2. *(cf [19])*
a) For the finite element solution of the Poisson problem (1.1)*, in two dimensions, with a quasiuniform triangulation we have the error estimate:*

$$\|u - u_h\|_{1,\infty} \le Ch^r |\log h| \times \|u\|_{r+1,\infty}, \qquad r \le 2$$

b) $\forall \varepsilon \in (0,1)$ small, $\exists C_\varepsilon$ such that $\|u - u_h\|_{1,\infty} \ge C_\varepsilon h^{r-\varepsilon} |\log h|$.

Note that, in the two dimensional case, $s = 2$ in Theorem 1.1 corresponds to $r = 1$ in Theorem 1.2, whereas the optimal L_∞ estimate in 2D is not generalized to the 3D case.

2. The finite volume method in 3D

A version of the three–dimensional scheme construction has also been discussed in [6]. On our spatial domain Ω we construct an arbitrary (not necessarily uniform) mesh $\bar{\Omega}^h = \bar{\Omega}_x^h \times \bar{\Omega}_y^h \times \bar{\Omega}_z^h$ as a Cartesian product of three one–dimensional meshes,

$$
\begin{aligned}
\bar{\Omega}_x^h &= \{x_i, \, i=0,\dots,M_x : x_0 = 0, \, x_i - x_{i-1} = h_i^x, \, x_{M_x} = 1\} \\
\bar{\Omega}_y^h &= \{y_i, \, i=0,\dots,M_y : x_0 = 0, \, y_i - y_{i-1} = h_i^y, \, y_{M_y} = 1\} \\
\bar{\Omega}_z^h &= \{z_i, \, i=0,\dots,M_z : x_0 = 0, \, z_i - z_{i-1} = h_i^z, \, z_{M_z} = 1\}.
\end{aligned}
$$

We further define $\Omega_x^h := \bar{\Omega}_x^h \cap (0,1]$, $\Omega_y^h := \bar{\Omega}_y^h \cap (0,1]$, $\Omega_z^h := \bar{\Omega}_z^h \cap (0,1]$, $\partial\Omega_x^h := \{0,1\} \times \Omega_y^h \times \Omega_z^h$, $\partial\Omega_y^h := \Omega_x^h \times \{0,1\} \times \Omega_z^h$, $\partial\Omega_z^h := \Omega_x^h \times \Omega_y^h \times \{0,1\}$, $\Omega^h := \Omega \cap \bar{\Omega}^h$ and $\partial\Omega^h := \partial\Omega \cap \bar{\Omega}^h$. With each mesh point $(x_i, y_j, z_k) \in \Omega^h$ we associate the finite volume element

$$
\omega_{ijk} := (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}) \times (z_{k-1/2}, z_{k+1/2}),
$$

where

$$
\begin{aligned}
x_{i-1/2} &:= x_i - \frac{h_i^x}{2}, & x_{i+1/2} &:= x_i + \frac{h_{i+1}^x}{2}, \\
y_{j-1/2} &:= y_j - \frac{h_j^y}{2}, & y_{j+1/2} &:= y_j + \frac{h_{j+1}^y}{2}, \\
z_{k-1/2} &:= z_k - \frac{h_k^z}{2}, & z_{k+1/2} &:= z_k + \frac{h_{k+1}^z}{2},
\end{aligned}
$$

and denote the dimensions of the volume element ω_{ijk} by,

$$
\hbar_i^x := \frac{h_i^x + h_{i+1}^x}{2}, \quad \hbar_j^y := \frac{h_j^y + h_{j+1}^y}{2}, \quad \hbar_k^z := \frac{h_k^z + h_{k+1}^z}{2},
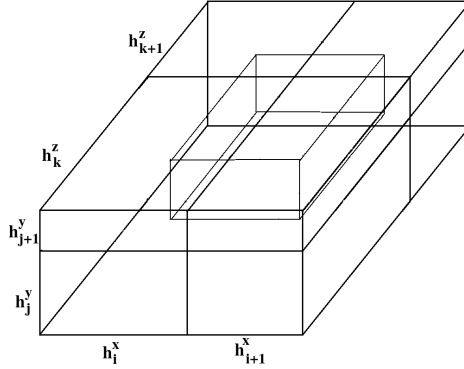$$

see Fig. 1.

FIGURE 1. Part of mesh showing the grid and finite volume ω_{ijk} (inside box) in three dimensions.

The characteristic function of the box ω_{ijk}, i.e. χ_{ijk} belongs to $H^\tau(\mathbb{R}^3)$ for all $\tau < 1/2$. This can be easily checked by the fact that the Fourier transform of the characteristic function of the unit interval $\chi_{(0,1)}$ is the sinc function $\sin\xi/\xi$. Thus using the Fourier transform we may determine the Sobolev class of χ_{ijk}. To this end, for each $s \in \mathbb{R}^+$ we recall the operator Λ^s defined as $(\Lambda^s \xi)\hat{\ }(\chi) = (1+|\xi|^2)^{s/2}\hat{\chi}(\xi)$ and the Soblolev norm of order s,

$$
\begin{aligned}
\|\chi_{ijk}\|_s^2 &= \|\Lambda^s \chi_{ijk}\|_{L_2(\mathbb{R}^3)}^2 \\
&= \int_{\mathbb{R}^3} (1+|\xi_1|^2+|\xi_2|^2+|\xi_3|^2)^s \left(\frac{\sin\xi_1}{\xi_1}\right)^2 \cdot \left(\frac{\sin\xi_2}{\xi_2}\right)^2 \cdot \left(\frac{\sin\xi_3}{\xi_3}\right)^2 d\xi.
\end{aligned} \tag{2.1}
$$

We split the above integral as

$$
\int_{\mathbb{R}^3} \bullet \, d\xi = \int_{|\xi|\le 1} \bullet \, d\xi + \int_{|\xi|>1} \bullet \, d\xi,
$$

and check for which s-values the integrals on the right hand side converge. For the first integral, since $\lim_{\xi_i \to 0} \sin \xi_i / \xi_1 = 1$, $i = 1, 2, 3$, we get an immediate bound. As for the second integral we have that,

$$
\int_{|\xi|>1} (1+|\xi_1|^2+|\xi_2|^2+|\xi_3|^2)^s \left(\frac{\sin \xi_1}{\xi_1}\right)^2 \cdot \left(\frac{\sin \xi_2}{\xi_2}\right)^2 \cdot \left(\frac{\sin \xi_3}{\xi_3}\right)^2 d\xi
$$

$$
\leq \int_{|\xi|>1} (1+|\xi_1|^2)^s (1+|\xi_2|^2)^s (1+|\xi_3|^2)^s \frac{1}{|\xi_1^2|} \cdot \frac{1}{|\xi_2^2|} \cdot \frac{1}{|\xi_3^2|} \, d\xi \qquad (2.2)
$$

$$
\leq \prod_{j=1}^{3} \int_{|\xi_j|>r_j} (1+|\xi_j|^2)^s \frac{1}{|\xi_j^2|} \, d\xi = \prod_{j=1}^{3} \int_{r_j}^{\infty} (1+r^2)^s \frac{1}{r^2} \, dr,
$$

which converges for $2s - 2 < -1$, i.e. $s < 1/2$. Since $\chi \in H^\tau(\Omega)$, $\tau < 1/2$, we may assume that $f \in H^\sigma(\Omega)$ for $\sigma > -1/2$. Then the convolution $\chi_{ijk} * f$ will be continuous on $\mathbb{R}^3$ and if we have $f \in L^1_{loc}(\Omega)$, then

$$
\frac{-2}{|\omega_{ijk}|} \int_{\partial \omega_{ijk}} \frac{\partial u}{\partial \mathbb{n}} \mathrm{d}s = \frac{1}{|\omega_{ijk}|} \left(\chi_{ijk} * f\right)(x_i, y_j, z_k), \qquad (2.3)
$$

where $|\omega_{ijk}| = \hbar_i^x \hbar_j^y \hbar_k^z$. Let now S_0^h be the set of piecewise continuous trilinear functions defined on the cubic rectangular partition of $\bar{\Omega}$ induced by $\bar{\Omega}^h$ and vanishing on $\partial \Omega$. We can now construct the finite volume approximation $u^h \in S_0^h$ of u as satisfying,

$$
\frac{-2}{\hbar_i^x \hbar_j^y \hbar_k^z} \int_{\partial \omega_{ijk}} \frac{\partial u^h}{\partial \mathbb{n}} \mathrm{d}s = \frac{1}{\hbar_i^x \hbar_j^y \hbar_k^z} \left(\chi_{ijk} * f\right)(x_i, y_j, z_k) \qquad \text{for} \quad (x_i, y_j, z_k) \in \Omega^h. \quad (2.4)
$$

Here the factor 2 appears due to the jump of χ_{ijk} across the inter-element boundaries on $\partial \omega_{ijk}$, and will not matter for any of the stability results and convergence rates as considered by [6] and [26] but only in numerical implementations of the scheme.

3. PROPERTIES OF THE SCHEME AND STABILITY ESTIMATES

To investigate the behavior of this scheme we will rewrite it as a finite difference scheme. To this end, we define the averaging operators (all are presented, due to miss-matches in indexing discrepancies they are not presentable in a single generic form)

$$
\begin{aligned}
\mu_{xy} u_{ijk} &:= \frac{1}{16\hbar_i^x \hbar_j^y} \Big(h_i^x h_j^y u_{i-1,j-1,k} + h_{i+1}^x h_j^y u_{i+1,j-1,k} + 12\hbar_i^x \hbar_j^y u_{ijk} \\
&\quad + h_i^x h_{j+1}^y u_{i-1,j+1,k} + h_{i+1}^x h_{j+1}^y u_{i+1,j+1,k} \Big), \\
\mu_{xz} u_{ijk} &:= \frac{1}{16\hbar_i^x \hbar_k^z} \Big(h_i^x h_k^z u_{i-1,j,k-1} + h_{i+1}^x h_k^z u_{i+1,j,k-1} + 12\hbar_i^x \hbar_k^z u_{ijk} \\
&\quad + h_i^x h_{k+1}^z u_{i-1,j,k+1} + h_{i+1}^x h_{k+1}^z u_{i+1,j,k+1} \Big), \\
\mu_{yz} u_{ijk} &:= \frac{1}{16\hbar_j^y \hbar_k^z} \Big(h_j^y h_k^z u_{i,j-1,k-1} + h_{j+1}^y h_k^z u_{i,j+1,k-1} + 12\hbar_j^y \hbar_k^z u_{ijk} \\
&\quad + h_j^y h_{k+1}^z u_{i,j-1,k+1} + h_{j+1}^y h_{k+1}^z u_{i,j+1,k+1} \Big),
\end{aligned} \qquad (3.1)
$$

and the divided differences,

$$
\begin{array}{llll}
\Delta_x^- u_{i,j,k} &= \frac{u_{i,j,k}-u_{i-1,j,k}}{h_i^x}, & \Delta_x^+ u_{i,j,k} &= \frac{u_{i+1,j,k}-u_{i,j,k}}{h_i^x}, \\[6pt]
\Delta_y^- u_{i,j,k} &= \frac{u_{i,j,k}-u_{i,j-1,k}}{h_j^y}, & \Delta_y^+ u_{i,j,k} &= \frac{u_{i,j+1,k}-u_{i,j,k}}{h_j^y}, \\[6pt]
\Delta_z^- u_{i,j,k} &= \frac{u_{i,j,k}-u_{i,j,k-1}}{h_k^z}, & \Delta_z^+ u_{i,j,k} &= \frac{u_{i,j,k+1}-u_{i,j,k}}{h_k^z}.
\end{array}
$$

Then, we can write

$$
\hbar_i^x \hbar_j^y \hbar_k^z \left(\Delta_x^+ \Delta_x^- \mu_{yz} + \Delta_y^+ \Delta_y^- \mu_{xz} + \Delta_z^+ \Delta_z^- \mu_{xy}\right) u_{i,j,k} = \int_{\partial \omega_{ijk}} \frac{\partial u}{\partial \mathbb{n}} \mathrm{d}s.
$$

This allows us to restate the finite volume scheme, (2.4) as the following finite difference scheme,

$$-2\left(\Delta_x^+\Delta_x^-\mu_{yz}+\Delta_y^+\Delta_y^-\mu_{xz}+\Delta_z^+\Delta_z^-\mu_{xy}\right)u^h = T_{111}f \qquad \text{in } \Omega^h,$$
$$u^h = 0 \qquad \text{on } \partial\Omega^h, \qquad (3.2)$$

where

$$(T_{111}f)_{ijk} = \frac{1}{h_i^x h_j^y h_k^z}\left(\chi_{ijk}*f\right)(x_i,y_j,z_k).$$

To extend (3.2) to higher than three dimensions, the same scheme will apply, however the definition of μ will change. If we look at carefully how this averaging operator works then it appears that the main difference will be what will correspond to the factor 12 appearing as the coefficient of the central term in (3.1). In fact if we denote by d the dimension then,

$$\mu_{x_1 x_2 \ldots x_{d-1}}u_{i_1\ldots i_d} = \frac{1}{2^{d+1}}\frac{1}{\hbar_{i_1}^{x_1}\ldots\hbar_{i_{d-1}}^{x_{d-1}}}\left(3\cdot 2^{d-1}\cdot\hbar_{i_1}^{x_1}\ldots\hbar_{i_{d-1}}^{x_{d-1}}u_{i_1\ldots i_d}\right.$$
$$\left.+h_{i_1}^{x_1}\ldots h_{i_{d-1}}^{x_{d-1}}u_{i_1-1,\ldots,i_{d-1}-1,i_d}+\ldots+h_{i_1+1}^{x_1}\ldots h_{i_{d-1}+1}^{x_{d-1}}u_{i_1+1,\ldots,i_{d-1}+1,i_d}\right). \qquad (3.3)$$

We will study the behavior of the scheme defined by (3.2) in the discrete H^1 norm $\|\cdot\|_{1,h}$,

$$\|v\|_{1,h}=\sqrt{\|v\|^2+|v|_{1,h}^2},$$

where $\|\cdot\|$ is the discrete L_2-norm over Ω^h (we suppressed h in the discrete L_2), i.e.,

$$\|v\| = \sqrt{(v,v)}, \qquad (v,w)=\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1}\sum_{z=1}^{M^z-1}\hbar_i^x\hbar_j^y\hbar_k^z v_{i,j,k}w_{i,j,k},$$

and $|\cdot|_{1,h}$ is the discrete H^1-seminorm given by

$$|v|_{1,h}=\sqrt{\|\Delta_x^- v\|_x^2+\|\Delta_y^- v\|_y^2+\|\Delta_z^- v\|_z^2},$$

with

$$\|v\|_x^2 = (v,v)_x, \qquad (v,w)_x = \sum_{i=1}^{M^x}\sum_{j=1}^{M^y-1}\sum_{k=1}^{M^z-1}h_i^x\hbar_j^y\hbar_k^z v_{i,j,k}w_{i,j,k},$$

$$\|v\|_y^2 = (v,v)_y, \qquad (v,w)_y = \sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y}\sum_{k=1}^{M^z-1}\hbar_i^x h_j^y\hbar_k^z v_{i,j,k}w_{i,j,k},$$

$$\|v\|_z^2 = (v,v)_z, \qquad (v,w)_z = \sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1}\sum_{k=1}^{M^z}\hbar_i^x\hbar_j^y h_k^z v_{i,j,k}w_{i,j,k}.$$

In addition we define the discrete H^{-1} norm as,

$$\|v\|_{-1,h}=\sup_{w\in H_0^{1,h}(\bar\Omega^h)}\frac{|(v,w)|}{\|w\|_{1,h}},$$

where the supremum is over all non–zero mesh functions on $\bar\Omega^h$ vanishing on $\partial\bar\Omega^h$.

We will now state and prove two *coercivity-type* estimates describing relationships between the above and our operators. These are essentially the same as Lemmas 3.1 and 3.2 in [26] with the coefficients adjusted for the three dimensional case.

Lemma 3.1. *Let v be a mesh function on $\bar\Omega^h$. If $v=0$ on $\partial\Omega_{\alpha\beta}^h$, then $(\mu_{\alpha\beta}v,v)_\gamma\geq\frac{5}{8}\|v\|_\gamma^2$, in the following three cases:*

(i) $\alpha\beta:=xy$, $\gamma:=z$, (ii) $\alpha\beta:=xz$, $\gamma:=y$, and (iii) $\alpha\beta:=yz$, $\gamma:=x$.

Proof. We only prove *i*) here as all of the rest will be done in the same way. Note, in particular, that $v=0$ on $\partial\Omega_{xy}^h$, and we shall also use $a^2/2+2ab+b^2/2\geq-a^2/2-b^2/2$. To proceed let

$$\mathscr{A}_1:=\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1}\left(\hbar_i^x\hbar_j^y v_{ijk}^2+h_i^x h_j^y v_{i-1,j-1,k}v_{ijk}+h_{i+1}^x h_j^y v_{i+1,j-1,k}v_{ijk}\right.$$
$$\left.+h_i^x h_{j+1}^y v_{i-1,j+1,k}v_{ijk}+h_{i+1}^x h_{j+j}^y v_{i+1,j+1,k}v_{ijk}\right).$$

Then we use the shift law, vanishing boundary conditions, and split the terms in $\mathscr{A}_1$ at the end-point indices to obtain

$$
\begin{aligned}
\mathscr{A}_1 = & \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y \left(v_{i-1,j-1,k} + v_{ijk}\right) v_{ijk} + \sum_{j=1}^{M^y-1} (h_1^x h_j^y v_{1jk}^2) + \sum_{i=1}^{M^x-1} (h_i^x h_1^y v_{i1k}^2) \\
& + \sum_{i=1}^{M^x-2} \sum_{j=2}^{M^y-1} h_{i+1}^x h_j^y \left(v_{i+1,j-1,k} + v_{ijk}\right) v_{ijk} + \sum_{j=1}^{M^y-1} (h_{M^x}^x h_j^y v_{M^x jk}^2) \\
& + \sum_{i=1}^{M^x-2} (h_{i+1}^x h_1^y v_{i1k}^2) + \sum_{i=2}^{M^x-1} \sum_{j=1}^{M^y-2} h_i^x h_{j+1}^y \left(v_{i-1,j+1,k} + v_{ijk}\right) v_{ijk} \\
& + \sum_{j=1}^{M^y-1} (h_1^x h_j^y v_{1jk}^2) + \sum_{i=2}^{M^x-1} (h_i^x h_{M^y}^y v_{iM^y k}^2) \\
& + \sum_{i=1}^{M^x-2} \sum_{j=2}^{M^y-2} h_{i+1}^x h_{j+1}^y \left(v_{i+1,j+1,k} + v_{ijk}\right) v_{ijk} + \sum_{j=1}^{M^y-1} (h_{M^x}^x h_{j+1}^y v_{M^x jk}^2) \\
& + \sum_{i=1}^{M^x-2} (h_{i+1}^x h_{M^y}^y v_{iM^y k}^2).
\end{aligned}
$$

Since the single sums in above identity are all nonnegative, removing them it follows that

$$
\begin{aligned}
\mathscr{A}_1 \geq & \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{i-1,j-1,k} v_{ijk} + \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{ijk}^2 + \sum_{i=1}^{M^x-2} \sum_{j=2}^{M^y-1} h_{i+1}^x h_j^y v_{i+1,j-1,k} v_{ijk} \\
& + \sum_{i=1}^{M^x-2} \sum_{j=2}^{M^y-1} h_{i+1}^x h_j^y v_{ijk}^2 + \sum_{i=2}^{M^x-1} \sum_{j=1}^{M^y-2} h_i^x h_{j+1}^y v_{i-1,j+1,k} v_{ijk} + \sum_{i=2}^{M^x-1} \sum_{j=1}^{M^y-2} h_i^x h_{j+1}^y v_{ijk}^2 \\
& + \sum_{i=1}^{M^x-2} \sum_{j=1}^{M^y-2} h_{i+1}^x h_{j+1}^y v_{i+1,j+1,k} v_{ijk} + \sum_{i=1}^{M^x-2} \sum_{j=1}^{M^y-2} h_{i+1}^x h_{j+1}^y v_{ijk}^2 =: \mathscr{B}_1.
\end{aligned}
$$

For simplicity we denoted the right hand side above by $\mathscr{B}_1$. Below, once again using the shift law, we make $\mathscr{B}_1$ uniformly indexed, i.e. with all sums having the same index range. Then we can easily verify that

$$
\begin{aligned}
\mathscr{B}_1 = & \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{i-1,j-1,k} v_{ijk} + \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{i,j-1,k} v_{i-1,j,k} \\
& + \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{i-1,j,k} v_{i,j-1,k} + \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{i,j,k} v_{i-1,j-1,k} \\
& + \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{ijk}^2 + \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{i-1,j,k}^2 \\
& + \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{i,j-1,k}^2 + \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{i-1,j-1,k}^2 \\
= & \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y \left(v_{i-1,j-1,k}^2 + 2v_{i-1,j-1,k} v_{ijk} + v_{ijk}^2 + v_{i,j-1,k}^2 + \right. \\
& \left. + 2v_{i,j-1,k} v_{i-1,j,k} + v_{i-1,j,k}^2\right) \\
\geq & -\frac{1}{4} \left(\sum_{i=1}^{M^x-2} \sum_{j=1}^{M^y-2} h_{i+1}^x h_{j+1}^y v_{ijk}^2 + \sum_{i=2}^{M^x-1} \sum_{j=2}^{M^y-1} h_i^x h_j^y v_{ijk}^2 \right. \\
& \left. + \sum_{i=1}^{M^x-2} \sum_{j=2}^{M^y-1} h_{i+1}^x h_j^y v_{ijk}^2 + \sum_{i=2}^{M^x-1} \sum_{j=1}^{M^y-2} h_i^x h_{j+1}^y v_{ijk}^2 \right).
\end{aligned}
$$

Now, recalling the definition of $\mathscr{A}_1$ and using the bound for $\mathscr{B}_1$ iteratively, we can derive the following chain of estimates

$$
\begin{aligned}
\frac{1}{16} \sum_{i=1}^{M^x-1} \sum_{j=1}^{M^y-1} &\Big(12\hbar_i^x\hbar_j^y v_{ijk}^2 + h_i^x h_j^y v_{i-1,j-1,k}v_{ijk} + h_{i+1}^x h_j^y v_{i+1,j-1,k}v_{ijk} \\
&\quad + h_i^x h_{j+1}^y v_{i-1,j+1,k}v_{ijk} h_{i+1}^x h_{j+j}^y v_{i+1,j+1,k}v_{ijk} \Big) \\
\geq \frac{1}{16} \sum_{i=1}^{M^x-1} \sum_{j=1}^{M^y-1} &\Bigg(11\hbar_i^x\hbar_j^y v_{ijk}^2 - \frac{1}{4}\Bigg(\sum_{i=1}^{M^x-2}\sum_{j=1}^{M^y-2} h_{i+1}^x h_{j+1}^y v_{ijk}^2 \\
&\quad + \sum_{i=2}^{M^x-1}\sum_{j=2}^{M^y-1} h_i^x h_j^y v_{ijk}^2 \sum_{i=1}^{M^x-2}\sum_{j=2}^{M^y-1} h_{i+1}^x h_j^y v_{ijk}^2 + \sum_{i=2}^{M^x-1}\sum_{j=1}^{M^y-2} h_i^x h_{j+1}^y v_{ijk}^2 \Bigg)\Bigg) \\
\geq \frac{10}{16} \sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1} \hbar_i^x\hbar_j^y v_{ijk}^2 &+ \frac{1}{16}\Bigg(\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1} \frac{h_i^x+h_{i+1}^x}{4}\frac{h_j^y+h_{j+1}^y}{4}v_{ijk}^2 - \sum_{i=1}^{M^x-2}\sum_{j=1}^{M^y-2}\frac{h_{i+1}^x h_{j+1}^y}{4}v_{ijk}^2 \\
&\quad - \sum_{i=2}^{M^x-1}\sum_{j=2}^{M^y-1}\frac{h_i^x h_j^y}{4}v_{ijk}^2 - \sum_{i=1}^{M^x-2}\sum_{j=2}^{M^y-1}\frac{h_{i+1}^x h_j^y}{4}v_{ijk}^2 - \sum_{i=2}^{M^x-1}\sum_{j=1}^{M^y-2}\frac{h_i^x h_{j+1}^y}{4}v_{ijk}^2 \Bigg) \\
= \frac{10}{16} \sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1} \hbar_i^x\hbar_j^y v_{ijk}^2 &+ \frac{1}{64}v_{ijk}^2 \Bigg(h_i^x h_j^y \Bigg(\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1} 1 - \sum_{i=2}^{M^x-1}\sum_{j=2}^{M^y-1} 1 \Bigg) \\
&\quad + h_i^x h_{j+1}^y \Bigg(\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1} 1 - \sum_{i=2}^{M^x-1}\sum_{j=1}^{M^y-2} 1 \Bigg) + h_{i+1}^x h_j^y \Bigg(\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1} 1 - \sum_{i=1}^{M^x-2}\sum_{j=2}^{M^y-1} 1 \Bigg) \\
&\quad + h_{i+1}^x h_{j+1}^y \Bigg(\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1} 1 - \sum_{i=1}^{M^x-2}\sum_{j=1}^{M^y-2} 1 \Bigg) \Bigg) \geq \frac{10}{16}\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1} \hbar_i^x\hbar_j^y v_{ijk}^2,
\end{aligned}
$$

where in the last step we used that all the differences of the sums are positive. Note in particular the role of the coefficient 12 in the central differencing term and the splits in this term. Finally, recalling the definition of $(\mu_{xy}v,v]_z$, we multiply the above estimate by $\hbar_k^z$ and sum over k to obtain.

$$
(\mu_{xy}v,v]_z \geq \frac{10}{16}\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1}\sum_{k=1}^{M^z} \hbar_i^x\hbar_j^y\hbar_k^z v_{ijk}^2 = \frac{10}{16}\|v\|_z^2 \geq \frac{1}{2}\|v\|_z^2.
$$

This completes the proof of the first assertion $i)$ of the lemma. The other two estimates are derived by similar calculus, alternating the relevant sub- and super-indices and therefore are omitted. $\qquad\square$

In the general case of d dimensions we can see that the coefficient will become $\frac{3 \cdot 2^{d-2}-1}{2^d}$. The general ratio above is linked to the coefficient of the central term in the finite difference case (3.3).

Lemma 3.2. *Let v be a mesh function on $\bar{\Omega}^h$ that vanishes on $\partial\Omega^h$, then*

$$
\|v\|^2 \leq \frac{1}{3}|v|_{1,h}^2.
$$

Proof. Using the definitions of the divided differences and following the notation, the desired result is obtained through the successive estimates below

$$
\|v\|^2 = \sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1}\sum_{k=1}^{M^z-1} \hbar_i^x \hbar_j^y \hbar_k^z v_{ijk}^2 = \frac{1}{3}\left(\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1}\sum_{k=1}^{M^z-1} \hbar_i^x \hbar_j^y \hbar_k^z |\sum_{m=1}^{i} h_m^x \Delta_x^- v_{mjk}|^2 \right.
$$

$$
+ \sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1}\sum_{k=1}^{M^z-1} \hbar_i^x \hbar_j^y \hbar_k^z |\sum_{m=1}^{j} h_m^y \Delta_y^- v_{imk}|^2 + \left. \sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1}\sum_{k=1}^{M^z-1} \hbar_i^x \hbar_j^y \hbar_k^z |\sum_{m=1}^{k} h_m^z \Delta_z^- v_{ijm}|^2 \right)
$$

$$
\leq \frac{1}{3}\left(\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1}\sum_{k=1}^{M^z-1} \hbar_i^x \hbar_j^y \hbar_k^z \left(\left(\sum_{m=1}^{i} h_m^x\right)\left(\sum_{m=1}^{i} h_m^x |\Delta_x^- v_{mjk}|^2\right) \right.\right.
$$

$$
\left.\left. + \left(\sum_{m=1}^{i} h_m^x\right)\left(\sum_{m=1}^{j} h_m^y |\Delta_y^- v_{imk}|^2\right) + \left(\sum_{m=1}^{i} h_m^x\right)\left(\sum_{m=1}^{k} h_m^z |\Delta_z^- v_{ijm}|^2\right) \right)\right)
$$

$$
= \frac{1}{3}\left(\sum_{m=1}^{M^x}\sum_{j=1}^{M^y-1}\sum_{k=1}^{M^z-1} h_m^x |\Delta_x^- v_{mjk}|^2 \hbar_j^y \hbar_k^z \right)\left(\sum_{i=1}^{M^x-1} \hbar_i^x \sum_{m=1}^{i} h_m^x \right)
$$

$$
+ \frac{1}{3}\left(\sum_{i=1}^{M^x-1}\sum_{m=1}^{M^y}\sum_{k=1}^{M^z-1} h_m^y |\Delta_y^- v_{imk}|^2 \hbar_i^x \hbar_k^z \right)\left(\sum_{j=1}^{M^y-1} \hbar_j^y \sum_{m=1}^{j} h_m^y \right)
$$

$$
+ \frac{1}{3}\left(\sum_{i=1}^{M^x-1}\sum_{j=1}^{M^y-1}\sum_{m=1}^{M^z} h_m^z |\Delta_z^- v_{ijm}|^2 \hbar_i^x \hbar_j^y \right)\left(\sum_{k=1}^{M^z-1} \hbar_k^z \sum_{m=1}^{k} h_m^z \right)
$$

$$
\leq \frac{1}{3}\left(\|\Delta_x^- v\|_x^2 + \|\Delta_y^- v\|_y^2 + \|\Delta_z^- v\|_z^2 \right) = \frac{1}{3}|v|_{1,h}^2 \leq \frac{1}{2}|v|_{1,h}^2.
$$

$\square$

In the general case of d dimensions the coefficient $1/3$ above, becomes $1/d$.

Based on these estimates we can prove the counterparts of Theorems 3.1 and 3.2 in [26] in three (as well as higher) dimensions.

Theorem 3.1. *Let* $L^h v = -(\Delta_x^+ \Delta_x^- \mu_{yz} + \Delta_y^+ \Delta_y^- \mu_{xz} + \Delta_z^+ \Delta_z^- \mu_{xy})v$, *then*

$$
\|v\|_{1,h} \leq \frac{32}{15}\|L^h v\|_{-1,h}.
$$

Proof. Evidently, we have the identities

$$
(-\Delta_x^+ w, v) = (w, \Delta_x^- v]_x, \quad (-\Delta_y^+ w, v) = (w, \Delta_y^- v]_y, \quad (-\Delta_z^+ w, v) = (w, \Delta_z^- v]_z.
$$

Therefore, using Lemmas 3.1 and 3.2 yields

$$
\begin{aligned}
(L^h v, v) &= (-(\Delta_x^+ \Delta_x^- \mu_{yz} + \Delta_y^+ \Delta_y^- \mu_{xz} + \Delta_z^+ \Delta_z^- \mu_{xy})v, v) \\
&= (\Delta_x^- \mu_{yz} v, \Delta_x^- v]_x + (\Delta_y^- \mu_{xz} v, \Delta_y^- v]_y + (\Delta_z^- \mu_{xy} v, \Delta_z^- v]_z \\
&\geq \frac{5}{8}\left(\|\Delta_x^- v]_x + \|\Delta_y^- v]_y + \|\Delta_z^- v]_z \right) = \frac{5}{8}|v|_{1,h}^2 \geq \frac{15}{32}\|v\|_{1,h}^2.
\end{aligned}
$$

Thus, by the definition of $\|\cdot\|_{-1,h}$ we obtain,

$$
\|v\|_{1,h} \leq \frac{32}{15}\|L^h v\|_{-1,h}.
$$

$\square$

In d dimensions following the same procedure we obtain

$$
\|v\|_{1,h} \leq \frac{2^d (1+d)}{d(3 \cdot 2^{d-2} - 1)}\|L^h v\|_{-1,h}.
$$

Theorem 3.2. *If* $f \in H^\sigma(\Omega)$, $\sigma > -1/2$, *then the convolution* T_{111} *is continuous and the equation* (3.2) *has a unique solution* u^h. *Further,*

$$
\|u^h\|_{1,h} \leq \frac{32}{30}\|T_{111}f\|_{-1,h}.
$$

Proof. Follows directly from Eq. (3.2) and Theorem 3.1. $\square$

In d dimensions we will obtain

$$\|u^h\|_{1,h} \le \frac{2^d\,(1+d)}{2d\,(3\cdot 2^{d-2}-1)}\|T_{1...1}f\|_{-1,h}.$$

4. CONVERGENCE ANALYSIS

In this section we derive convergence rate of the proposed finite volume scheme. Most of the results in here hold true for the corresponding finite difference- and finite element-schemes as well. In the convergence rate proofs, we shall use the following classical result:

Theorem 4.1. *Let W be a Banach space and W_1 a normed linear space. Let $A : W \to W_1$ be a compact linear operator and let $S_1 : W \to \mathbb{R}$ and $S_2 : W \to \mathbb{R}$ denote two bounded sublinear functionals (i.e. $S_i(\alpha u + \beta v) \le |\alpha| S_i(u) + |\beta| S_i(v)$ for , α, $\beta \in \mathbb{R}$ and , $u, v \in W$). Further, assume that there exists a constant C_0 such that,*

$$\|v\|_W \le C_0\,(\|Av\|_{W_1} + S_2(v))\ \forall v \in W,$$

and that $\mathrm{Ker}(S_2) \subset \mathrm{Ker}(S_1)$. Then

  i) *$P := \mathrm{Ker}(S_2)$ is a finite dimensional vector space,*
  ii) *there exists a constant C_1 such that $\inf_{p\in P}\|v - p\|_W \le C_1 S_2(v)$ $\forall v \in W$,*
  iii) *there exists a constant C_2 such that $S_1(v) \le C_2 S_2(v)$.*

Proof. Follows directly from Theorem 5.1 (see Appendix) by taking $E = W$, E_0, E_1, $F = W_1$, $S_1 = L$ and $S_2 = A_1$. □

Theorem 4.2. *If $u \in H^{1+\sigma}(\Omega)$, $1/2 < \sigma \le 2$, then*

$$\|u - u^h\|_{1,h} \le Ch^\sigma |u|_{H^{1+\sigma}(\Omega)},$$

where $h = \max_{i,j,k}(h_i^x, h_j^y, h_k^z)$ and the constant $C > 0$ does not depend on u and the discretization parameters.

This is an optimal result corresponding to a finite element approach without a quadrature (gives an L_2-estimate of order $\mathcal{O}(h^{\sigma+1})$). With the same regularity, i.e. $u \in H^{1+\sigma}(\Omega)$, the corresponding L_2-estimate for the finite element method with quadrature rule, and the finite difference method, would have a lower convergence rate of order $\mathcal{O}(h^{\sigma+1/2})$.

Proof. For a cuboid $\omega = \Pi_{i=1}^d \omega_i := \Pi_{i=1}^d [a_i, b_i] \subset \mathbb{R}^d$ and a d-dimensional multi-index $\alpha := (\alpha_1, \ldots, \alpha_d)$, for $i = 1, \ldots, d$, we use the notation $\alpha^i := (0, \ldots, 0, \alpha_i, \ldots, 0)$ (only the i-th coordinate is non–zero) and set $\omega_{-i} := \omega \setminus \omega_i$. Further we denote by x_{-i} the $(d-1)$ dimensional vector $x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_d)$. Then we define $H^\alpha(\omega)$, the anisotropic Sobolev space, that consists of all functions $u \in L_2(\omega)$ such that

$$\|u\|_{H^{\alpha^i}(\omega)} = \Big(\int_{\omega_{-i}} |u(x_{-i})|^2_{H^{\alpha_i}(\omega_i)}\, dx\Big)^{1/2} < \infty.$$

$H^\alpha(\omega)$ is a Banach space with the norm,

$$\|u\|_{H^\alpha(\omega)} = \Big(\|u\|^2_{L^2(\omega)} + |u|^2_{H^\alpha(\omega)}\Big)^{1/2} = \Big(\|u\|^2_{L^2(\omega)} + \sum_{i=1}^d |u|^2_{H^{\alpha^i}(\omega)}\Big)^{1/2},$$

see, e.g. [20]. Further we denote the global error function $z = u - u^h$, then as $T_{111}f = L^h u^h$ and $f = -\Delta u$ we have,

$$L^h z = \Big(T_{111}\frac{\partial^2 u}{\partial x^2} - \Delta_x^+\Delta_x^- \mu_{yz}u\Big) + \Big(T_{111}\frac{\partial^2 u}{\partial y^2} - \Delta_y^+\Delta_y^- \mu_{xz}u\Big) + \Big(T_{111}\frac{\partial^2 u}{\partial z^2} - \Delta_z^+\Delta_z^- \mu_{xy}u\Big).$$

We can easily verify that

$$
\begin{aligned}
(T_{111}\tfrac{\partial^2 u}{\partial x^2})_{ijk} &= \tfrac{1}{2}\tfrac{1}{\hbar_i^x \hbar_j^y \hbar_k^z}\left(\chi_{ijk} * \tfrac{\partial^2 u}{\partial x^2}\right)(x_{ijk})\\
&= \tfrac{1}{2}\tfrac{1}{\hbar_i^x \hbar_j^y \hbar_k^z}\int\limits_{z_{k-1/2}}^{z_{k+1/2}}\int\limits_{y_{j-1/2}}^{y_{j+1/2}} \tfrac{\partial u}{\partial x}(x_{i+1/2},y,z) - \tfrac{\partial u}{\partial x}(x_{i-1/2},y,z)\mathrm{d}y\mathrm{d}x\\
&= \tfrac{1}{2}\Delta_x^+(T_{011}^-\tfrac{\partial u}{\partial x})_{ijk},
\end{aligned}
$$

where,

$$
(T_{011}^- w)_{ijk} = \frac{1}{\hbar_j^y \hbar_k^z}\int\limits_{z_{k-1/2}}^{z_{k+1/2}}\int\limits_{y_{j-1/2}}^{y_{j+1/2}} w(x_{i-1/2},y,z)\mathrm{d}y\mathrm{d}z.
$$

$(T_{111}\tfrac{\partial^2 u}{\partial y^2})_{ijk}$, and $(T_{111}\tfrac{\partial^2 u}{\partial y^2})_{ijk}$ are treated in analogous fashion, e.g.

$$
\begin{aligned}
(T_{101}^- w)_{ijk} &= \tfrac{1}{\hbar_i^x \hbar_k^z}\int\limits_{z_{k-1/2}}^{z_{k+1/2}}\int\limits_{x_{i-1/2}}^{x_{i+1/2}} w(x,y_{j-1/2},z)\mathrm{d}x\mathrm{d}z (T_{101}^- w)_{ijk}\\
&= \tfrac{1}{\hbar_i^x \hbar_j^y}\int\limits_{y_{j-1/2}}^{y_{j+1/2}}\int\limits_{x_{i-1/2}}^{x_{i+1/2}} w(x,y,z_{k-1/2})\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

This gives us

$$
\begin{cases}
L^h z &= \Delta_x^+ \eta_1 + \Delta_y^+ \eta_2 + \Delta_z^+ \eta_3 & \text{in} \quad \Omega^h,\\
z &= 0 & \text{on} \quad \partial\Omega^h,
\end{cases}
\tag{4.1}
$$

with

$$
\begin{aligned}
\eta_1 &= \tfrac{1}{2}T_{011}^-\tfrac{\partial u}{\partial x} - \Delta_x^-\mu_{yz}u,\\
\eta_2 &= \tfrac{1}{2}T_{101}^-\tfrac{\partial u}{\partial y} - \Delta_y^-\mu_{xz}u,\\
\eta_3 &= \tfrac{1}{2}T_{110}^-\tfrac{\partial u}{\partial z} - \Delta_z^-\mu_{xy}u.
\end{aligned}
$$

Now from Eq. (4.1) and Theorem 3.1 we can derive

$$
\|z\|_{1,h} \le \frac{32}{15}\|\Delta_x^+ \eta_1 + \Delta_y^+ \eta_2 + \Delta_z^+ \eta_3\|_{-1,h}.
$$

We can also show that for certain mesh functions (e.g. shape regular) defined on $\bar{\Omega}^h$ and vanishing on $\partial\Omega_h$ we have $(-\Delta_{(\cdot)}^+ w, g) = (w, \Delta_{(\cdot)}^- g]$. Hence

$$
\begin{aligned}
|(\Delta_x^+ \eta_1 + \Delta_y^+ \eta_2 + \Delta_z^+ \eta_3, w)| &= |(\eta_1, \Delta_x^- w]_x + (\eta_2, \Delta_y^- w]_y + (\eta_3, \Delta_z^- w]_z|\\
&\le |\|\eta_1|]_x\|\Delta_x^- w\|_x + \|\eta_2|]_y\|\Delta_y^- w\|_y + \|\eta_3|]_z\|\Delta_z^- w\|_z|\\
&\le (\|\eta_1|]_x + \|\eta_2|]_y + \|\eta_3|]_z)\,\|w\|_{1,h}.
\end{aligned}
$$

Thus, by the definition of the dual norm, we get

$$
\begin{aligned}
|(\Delta_x^+ \eta_1 + \Delta_y^+ \eta_2 + \Delta_z^+ \eta_3, w)|\|w\|_{1,h}^{-1} &\le \|\eta_1|]_x + \|\eta_2|]_y + \|\eta_3|]_z\\
\|(\Delta_x^+ \eta_1 + \Delta_y^+ \eta_2 + \Delta_z^+ \eta_3, w)\|_{-1,h} &\le \|\eta_1|]_x + \|\eta_2|]_y + \|\eta_3|]_z.
\end{aligned}
$$

Therefore,

$$
\|u - u^h\|_{1,h} \le \frac{32}{15}\left(\|\eta_1|]_x + \|\eta_2|]_y + \|\eta_3|]_z\right).
\tag{4.2}
$$

We now have to bound the right–hand side of (4.2). Here we only consider the η_1-term as the other two can be treated in the same way. To this end, for a fixed x let $I_{yz}w(x,\cdot,\cdot)$ denote the piecewise interpolant of $w(x,\cdot,\cdot)$ on the mesh $\bar{\Omega}_{yz}^h$ and

$$
\begin{aligned}
(\mu_{yz}u)(x,y_j,z_k) = \tfrac{1}{16}\tfrac{1}{\hbar_j^y \hbar_k^z}\Big(& h_j^y h_k^z u(x,y_{j-1},z_{k-1}) + h_{j+1}^y h_k^z u(x,y_{j+1},z_{k-1})\\
&+ 12\hbar_j^y \hbar_k^z u(x,y_j,z_k) + h_j^y h_{k+1}^z u(x,y_{j-1},z_{k+1}) + h_{j+1}^y h_{k+1}^z u(x,y_{j+1},z_{k+1})\Big),
\end{aligned}
$$

then

$$
(\mu_{yz}u)(x,y_j,z_k) = \frac{1}{\hbar_j^y \hbar_k^z}\int\limits_{y_{j-1/2}}^{y_{j+1/2}}\int\limits_{z_{k-1/2}}^{z_{k+1/2}} (I_{yz}u)(x,y,z)\mathrm{d}z\mathrm{d}y.
$$

Further, using

$$
\begin{aligned}
(\mu_{yz}u)_{ijk} - (\mu_{yz}u)_{i-1,j,k} &= \int_{x_{i-1}}^{x_i} \frac{\partial}{\partial x}(\mu_{yz}u)(x,y_j,x_k)\mathrm{d}x \\
&= \int_{x_{i-1}}^{x_i} \frac{\partial}{\partial x}\frac{1}{\hbar_j^y \hbar_k^z} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{z_{k-1/2}}^{z_{k+1/2}} (I_{yz}u)(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\hbar_j^y \hbar_k^z} \int_{x_{i-1}}^{x_i} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{z_{k-1/2}}^{z_{k+1/2}} \frac{\partial}{\partial x}(I_{yz}u)(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\hbar_j^y \hbar_k^z} \int_{x_{i-1}}^{x_i} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{z_{k-1/2}}^{z_{k+1/2}} I_{yz}\left(\frac{\partial u}{\partial x}\right)(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x,
\end{aligned}
$$

we can write $(\eta_1)_{ijk}$ as

$$
(\eta_1)_{ijk} = \frac{1}{\hbar_i^x \hbar_j^y \hbar_k^z} \int_{x_{i-1}}^{x_i} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{z_{k-1/2}}^{z_{k+1/2}} \left(\frac{1}{2}\frac{\partial u}{\partial x}(x_{i-1/2},y,z) - T_{011}\left(\frac{\partial u}{\partial x}\right)(x,y,z)\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x.
$$

Now we split $(\eta_1)_{ijk}$ into a sum of four terms:

$$
\begin{aligned}
(\eta_{11})_{ijk} &= \frac{1}{\hbar_i^x \hbar_j^y \hbar_k^z} \int_{x_{i-1}}^{x_i} \int_{y_j}^{y_{j+1/2}} \int_{z_k}^{z_{k+1/2}} \left(\tfrac{1}{2}\frac{\partial u}{\partial x}(x_{i-1/2},y,z) - T_{011}\left(\frac{\partial u}{\partial x}\right)(x,y,z)\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x, \\
(\eta_{12})_{ijk} &= \frac{1}{\hbar_i^x \hbar_j^y \hbar_k^z} \int_{x_{i-1}}^{x_i} \int_{y_j}^{y_{j+1/2}} \int_{z_{k-1/2}}^{z_k} \left(\tfrac{1}{2}\frac{\partial u}{\partial x}(x_{i-1/2},y,z) - T_{011}\left(\frac{\partial u}{\partial x}\right)(x,y,z)\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x, \\
(\eta_{13})_{ijk} &= \frac{1}{\hbar_i^x \hbar_j^y \hbar_k^z} \int_{x_{i-1}}^{x_i} \int_{y_{j-1/2}}^{y_j} \int_{z_k}^{z_{k+1/2}} \left(\tfrac{1}{2}\frac{\partial u}{\partial x}(x_{i-1/2},y,z) - T_{011}\left(\frac{\partial u}{\partial x}\right)(x,y,z)\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x, \\
(\eta_{14})_{ijk} &= \frac{1}{\hbar_i^x \hbar_j^y \hbar_k^z} \int_{x_{i-1}}^{x_i} \int_{y_{j-1/2}}^{y_j} \int_{z_{k-1/2}}^{z_k} \left(\tfrac{1}{2}\frac{\partial u}{\partial x}(x_{i-1/2},y,z) - T_{011}\left(\frac{\partial u}{\partial x}\right)(x,y,z)\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

Thus, to estimate η_1 it suffices to estimate η_{11}, η_{13}, η_{13} and η_{14}. Here, we only show how to estimate η_{11} as the other three terms will follow in the same fashion. We introduce the change of variables

$$
x = x_{i-1/2} + s h_i^x, \ -\frac{1}{2} \le s \le \frac{1}{2}; \quad y = y_j + t h_{j+1}^y, \ 0 \le t \le 1; \quad z = z_k + r h_{k+1}^z, \ 0 \le r \le 1,
$$

and define

$$
\tilde{v}(s,t,r) := h_i^x \frac{\partial u}{\partial x}(x(s),y(t),z(r)).
$$

This gives us

$$
(\eta_{11})_{ijk} = \frac{h_{j+1}^y h_{k+1}^z}{h_i^x \hbar_j^y \hbar_k^z} \tilde{\eta}_{11},
$$

with

$$
\tilde{\eta}_{11} = \int_{-1/2}^{1/2} \int_0^{1/2} \int_0^{1/2} \frac{1}{2}\tilde{v}(0,t,r) - \left(\tilde{v}(s,0,0)(1-t-r) + \tilde{v}(s,1,0)t + \tilde{v}(s,0,1)r\right)\mathrm{d}r\mathrm{d}t\mathrm{d}s.
$$

Note that $\tilde{v}(0,t,r) = \frac{\partial u}{\partial x}(x_{i-1/2},y,z)$ and

$$
\tilde{v}(s,0,0)(1-t-r) + \tilde{v}(s,1,0)t + \tilde{v}(s,0,1)r = (I_{yz}\frac{\partial u}{\partial x})(x,y,z).
$$

Hence we treat $\tilde{\eta}_{11}$ as a linear functional with the argument $\tilde{v}$ defined on $H^\sigma(\tilde{\omega})$ $\sigma > 1/2$, where $\tilde{\omega} = (-\frac{1}{2},\frac{1}{2}) \times (0,1) \times (0,1)$. Note that $\sigma > 1/2$ is due to the fact that all η components, defined by T_{111}, are convolutions with the characteristic function $\xi_{i,j,k}$. Since $\xi \in H^\tau(\mathbb{R}^3)$, $\tau < 1/2$, continuity requires $\sigma > 1/2$. Notice further that, for a given $\tilde{v}$,

$\tilde{\eta}_{11}$ is constant and its value on the boundary is the same as anywhere inside the domain. Therefore, by the trace theorem we have

$$|\tilde{\eta}_{11}| \le C \|\tilde{v}\|_{H^\sigma(\tilde{\omega})}, \qquad \sigma > 1/2,$$

and using Theorem 4.1 with $W = H^\sigma(\tilde{\omega})$, $W_1 = L_2(\tilde{\omega})$, $S_1 = |\tilde{\eta}_{11}|$, $S_2 = \left(|\cdot|_{H^{\sigma,0,0}(\tilde{\omega})} + |\cdot|_{H^{0,\sigma,0}(\tilde{\omega})} + |\cdot|_{H^{0,0,\sigma}(\tilde{\omega})}\right)^{0.5}$ and with $A : H^\sigma(\tilde{\omega}) \to L_2(\tilde{\omega})$ being the compact embedding operator we obtain

$$|\tilde{\eta}_{11}(\tilde{v})| \le C \left(|\tilde{v}|^2_{H^{\sigma,0,0}(\tilde{\omega})} + |\tilde{v}|^2_{H^{0,\sigma,0}(\tilde{\omega})} + |\tilde{v}|^2_{H^{0,0,\sigma}(\tilde{\omega})}\right)^{0.5}$$

for $\sigma > 1/2$. We let now $\omega_{ijk}^{++} = (x_{i-1}, x_i) \times (y_j, y_{j+1}) \times (z_k, z_{k+1})$, then returning to the original variables we obtain

$$|\tilde{\eta}_{11}|^2 \le C \left(\frac{h_i^{x^2}}{h_i^{x^{2\sigma}}} h_i^x h_{j+1}^y h_k^z |\frac{\partial u}{\partial x}|^2_{H^{\sigma,0,0}(\omega_{ijk}^{++})} + \frac{h_i^{x^2} h_{j+1}^{y^{2\sigma}}}{h_i^x h_{j+1}^y h_k^z} |\frac{\partial u}{\partial x}|^2_{H^{0,\sigma,0}(\omega_{ijk}^{++})} + \frac{h_i^{x^2} h_{k+1}^{z^{2\sigma}}}{h_i^x h_{j+1}^y h_k^z} |\frac{\partial u}{\partial x}|^2_{H^{0,0,\sigma}(\omega_{ijk}^{++})} \right).$$

Thus

$$|(\eta_{11})_{ijk}|^2 \le C \left(\frac{h_{j+1}^y h_{k+1}^z h_i^{x^{2\sigma-1}}}{h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{\sigma,0,0}(\omega_{ijk}^{++})} + \frac{h_{j+1}^{y^{2\sigma+1}} h_{k+1}^z}{h_i^x h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{0,\sigma,0}(\omega_{ijk}^{++})} + \frac{h_{j+1}^y h_{k+1}^{z^{2\sigma+1}}}{h_i^x h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{0,0,\sigma}(\omega_{ijk}^{++})} \right).$$

Similar estimates are derived for

$$\begin{aligned}
\omega_{ijk}^{+-} &= (x_{i-1}, x_i) \times (y_j, y_{j+1}) \times (z_{k-1}, z_k), \\
\omega_{ijk}^{-+} &= (x_{i-1}, x_i) \times (y_{j-1}, y_j) \times (z_k, z_{k+1}), \\
\omega_{ijk}^{--} &= (x_{i-1}, x_i) \times (y_{j-1}, y_j) \times (z_{k-1}, z_k),
\end{aligned}$$

leading to

$$|(\eta_{12})_{ijk}|^2 \le C \left(\frac{h_{j+1}^y h_k^z h_i^{x^{2\sigma-1}}}{h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{\sigma,0,0}(\omega_{ijk}^{+-})} + \frac{h_{j+1}^{y^{2\sigma+1}} h_k^z}{h_i^x h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{0,\sigma,0}(\omega_{ijk}^{+-})} + \frac{h_{j+1}^y h_k^{z^{2\sigma+1}}}{h_i^x h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{0,0,\sigma}(\omega_{ijk}^{+-})} \right)$$

$$|(\eta_{13})_{ijk}|^2 \le C \left(\frac{h_j^y h_{k+1}^z h_i^{x^{2\sigma-1}}}{h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{\sigma,0,0}(\omega_{ijk}^{-+})} + \frac{h_j^{y^{2\sigma+1}} h_{k+1}^z}{h_i^x h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{0,\sigma,0}(\omega_{ijk}^{-+})} + \frac{h_j^y h_{k+1}^{z^{2\sigma+1}}}{h_i^x h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{0,0,\sigma}(\omega_{ijk}^{-+})} \right)$$

$$|(\eta_{14})_{ijk}|^2 \le C \left(\frac{h_j^y h_k^z h_i^{x^{2\sigma-1}}}{h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{\sigma,0,0}(\omega_{ijk}^{--})} + \frac{h_j^{y^{2\sigma+1}} h_k^z}{h_i^x h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{0,\sigma,0}(\omega_{ijk}^{--})} + \frac{h_j^y h_k^{z^{2\sigma+1}}}{h_i^x h_j^{y^2} h_k^{z^2}} |\frac{\partial u}{\partial x}|^2_{H^{0,0,\sigma}(\omega_{ijk}^{--})} \right).$$

Writing $h = \max_{i,j,k}(h_i^x, h_j^y, h_k^z)$, by the super–additivity of the Sobolev norm on a family of disjoint Lebesgue measurable subsets of Ω,

$$\begin{aligned}
\|\eta_1\|_x^2 &\le Ch^{2\sigma} \left(|\frac{\partial u}{\partial x}|^2_{H^{\sigma,0,0}(\Omega)} + |\frac{\partial u}{\partial x}|^2_{H^{0,\sigma,0}(\Omega)} + |\frac{\partial u}{\partial x}|^2_{H^{0,0,\sigma}(\Omega)}\right), \\
\|\eta_2\|_y^2 &\le Ch^{2\sigma} \left(|\frac{\partial u}{\partial y}|^2_{H^{\sigma,0,0}(\Omega)} + |\frac{\partial u}{\partial y}|^2_{H^{0,\sigma,0}(\Omega)} + |\frac{\partial u}{\partial y}|^2_{H^{0,0,\sigma}(\Omega)}\right), \qquad (4.3) \\
\|\eta_3\|_z^2 &\le Ch^{2\sigma} \left(|\frac{\partial u}{\partial z}|^2_{H^{\sigma,0,0}(\Omega)} + |\frac{\partial u}{\partial z}|^2_{H^{0,\sigma,0}(\Omega)} + |\frac{\partial u}{\partial z}|^2_{H^{0,0,\sigma}(\Omega)}\right).
\end{aligned}$$

All together we arrive at,

$$\|u - u^h\|_{1,h} \le Ch^\sigma |u|_{H^{1+\sigma}(\Omega)},$$

for $1/2 < \sigma \le 2$. □

From the above calculus we can see that the proof will also carry over to the d–dimensional case, but then C will depend on d.

In [26] it is shown that on a two–dimensional quasi–uniform mesh (i.e. there is a constant C_* such that $h := \max_{i,j}(h_i^x, h_j^y) \le C_* \min_{i,j}(h_i^x, h_j^y)$) the finite volume method of Eq. (2.4) is almost optimally accurate in the discrete (over the mesh points) maximum norm $\|\cdot\|_\infty$, i.e. for $u \in H^{1+\sigma}(\Omega), \frac{1}{2} < \sigma \le 2$ we have

$$\|u - u^h\|_\infty \le Ch^\sigma \sqrt{|\log h|} |u|_{H^{1+\sigma}(\Omega)},$$

where C depends on C_*. This does not hold in the three–dimensional case as,

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W_p^k(\Omega)}, \qquad k > n/p,$$

here k is the number of derivatives and p is the parameter of the L_p-space (Ω should be Lipschitz, as it is in our case). But $n = 3$ and $p = 2$, requires $k > 3/2$, and if we use the inverse estimate to go down half of a derivative to $H^1(\Omega)$, then we need to pay with half a power of h. Thus in three–dimensional case the result is $h^{\sigma-1/2}$, rather than $h^\sigma \sqrt{|\log h|}$.

5. NUMERICAL EXAMPLE

We implemented the finite volume scheme described by equation (2.4) according to the finite difference scheme for the equation (3.2) in C++. The code is available from URL: `http://www.math.chalmers.se/~mohammad`. The implementation is general and allows for any dimension of the problem, a user defined mesh (through an external text file) and a user defined data function, f. The data functions are implemented in an external dynamically linked library and can be parametrized. The user can provide the values of the parameters via a text file at execution. Therefore the user is completely free to specify a data function. Furthermore the program allows for comparison of the solution to a user defined function (similarly in an external dynamically linked library and it can be parametrized through a text file).

We use the uBLAS Boost and umfpack libraries for matrix operations. This has the one consequence that the sparse solver collapses in the three dimensional case if we increase the mesh size above 54 points in all directions. In the two dimensional case we did not observe any problems with the sparse solver. For multidimensional numerical integration we use the Cuba library [18]. We tested our code for a number of different functions based on the normal distribution density and on mollifier functions. We define the shrunk to the unit cube Gaussian function in k dimensions as,

$$u(\mathbf{x}) = \exp(-\sum_{i=1}^k \frac{1}{\tan(\pi x_i)^2})\mathbf{1}_{\text{unit cube}}(\mathbf{x})$$

a mollifier function shrunk to the unit cube in k dimensions as,

$$u(\mathbf{x}) = \exp((1-4\|\mathbf{x}-(0.5,\ldots,0.5)\|_2^2)^{-1})\mathbf{1}_{\text{unit cube}}(\mathbf{x})$$

and a multidimensional Hicks–Henne sine bump function as,

$$u(\mathbf{x}) = \left(\sin\left(2\pi\left(0.25-\|\mathbf{x}\|^2\right)\right)\right)^3 \mathbf{1}_{\{\mathbf{x}:\sum(x_i-0.5)^2\leq 0.25\}}(\mathbf{x}).$$

We considered the following as the difference of two functions G_1 and G_2 for $\mathbf{x}$ in the unit cube,

$$u(\mathbf{x}) = G_1(\mathbf{x}) - 3\cdot G_2(2\mathbf{x}-0.5).$$

G_1 and G_2 were both either a Gaussian, mollifier or Hicks–Henne sine bump.

The mesh points were randomly distributed in all dimensions. We present graphs of L_2, H_1 and relative errors of our implementation in Figs. 2, 3 and 4 .

APPENDIX

Theorem 5.1. *Let E be a Banach space and let E_0, E_1 and F be three normed linear spaces, A_0, A_1 and L be linear continuous operators from E into E_0, E_1 and F respectively. If*

  i)

$$\|g\|_E = C_0\left(\|A_0 g\|_{E_0} + \|A_1 g\|_{E_1}\right), \tag{5.1}$$

  ii) $Lg = 0$ if $A_1 g = 0$, i.e. $\text{Ker}(L) \subset \text{Ker}(A_1)$,
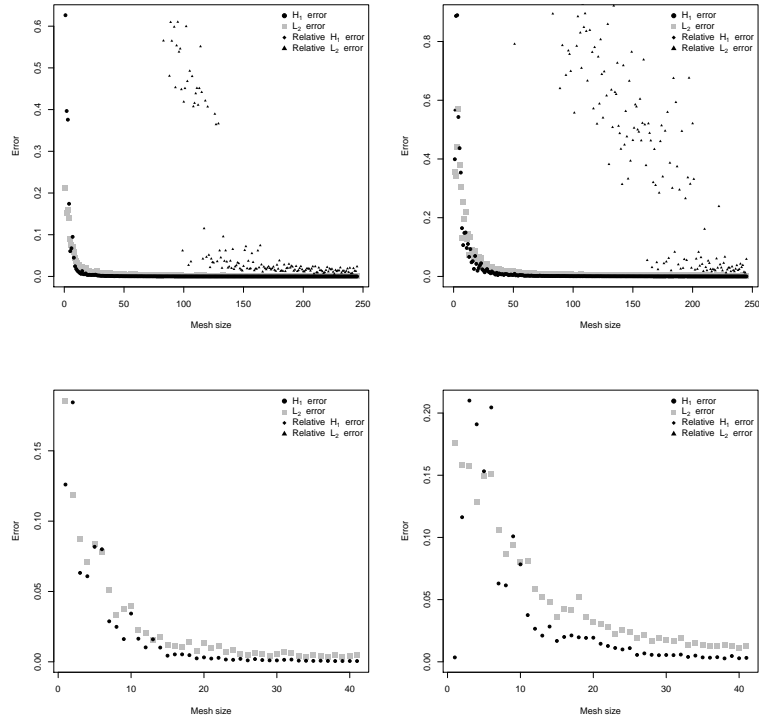 *iii) A_0 is compact,*

FIGURE 2. Errors for different functions. Top: Gaussian function (left) and difference of two Gaussians (right) in two dimensions and bottom: Gaussian function (left) and difference of two Gaussians (right) in three dimensions.
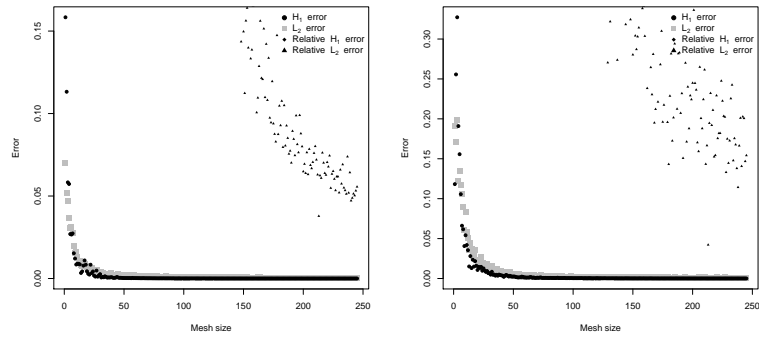

FIGURE 3. Errors for different functions. Left: mollifier function and right: difference of two mollifiers in two dimensions.

then, there exists a constant C such that,

$$\forall_{g \in E}, \ \|Lg\|_F \leq C \|A_1 g\|_{E_1}. \tag{5.2}$$

Proof. This theorem is an unpublished lemma of Tartar, mentioned as an exercise in [11] and cited in [26]. Both of the works indicate that its proof can be found in [10]. The proof starts by noticing that $P := \mathrm{Ker}(A_1)$ is finite dimensional however the argument for this
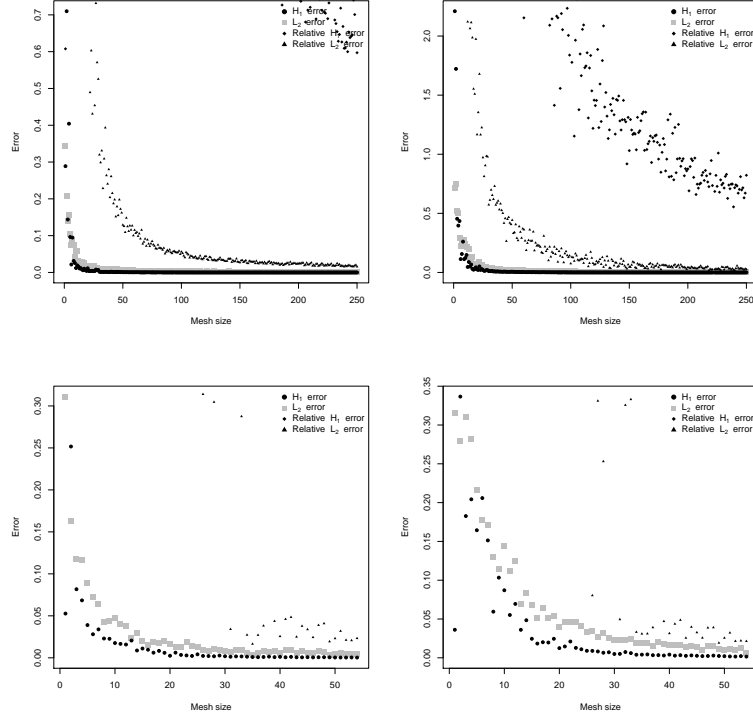
FIGURE 4. Errors for different functions. Top: Hicks–Henne sine bump
function (left) and difference of two Hicks–Henne sine bump functions
(right) in two dimensions, and bottom: Hicks–Henne sine bump function
(left) and difference of two Hicks–Henne sine bump functions (right) in
three dimensions

in [10] is that if weak sequential convergence implies norm convergence then it indicates
that P is finite dimensional. This argument is not clear however as due to Schur [25] we
have that in l^1, weak sequential convergence is equivalent to norm convergence. Below we
provide an alternative proof.

We will use the property that a unit ball is compact iff the subspace is finite dimensional.
Let us take $g \in P = \text{Ker}(A_1) \subset E$ and of course we have

$$\|g\|_P \equiv \|g\|_E \leq C_0 \left(\|A_0 g\|_{E_0} + \|A_1 g\|_{E_1} \right) = C_0 \|A_0 g\|_{E_0}$$

hence we can write

$$\forall_{g \in P}, \quad \|A_0 g\|_{E_0} \geq C \|g\|_E \equiv C \|g\|_P.$$

Let us assume that the kernel of A_1, P is infinite dimensional and then P is not bounded
in particular not totally bounded and hence will not have a finite ε–net, meaning,

$$\exists_{\varepsilon > 0}, \ \forall_n, \ \exists_{\substack{g_1,\ldots,g_n \in P, \\ \|g_i\|_P \leq 1}} \|g_i - g_j\|_P \geq \varepsilon.$$

We assumed that A_0 is compact so (denoting by K_E the unit ball in E and by K_0 unit ball
in E_0)

$$A_0(K_E) \subseteq \|A_0\| K_0,$$

due to

$$\forall_{x \in K_E}, \quad \|A_0 x\| \leq \|A_0\| \|x\|_E.$$

By P being infinite dimensional we can write,

$$\|A_0 g_i - A_0 g_j\|_{E_0} \geq C\|g_i - g_j\|_E \equiv C\|g_i - g_j\|_P \geq C\varepsilon.$$

This means that $A_0(K_E)$ does not have a finite ε–net so $A_0(K_E)$ would not be relatively compact contradicting that A_0 is compact. Hence P must be finite dimensional.

After establishing that $\dim P < \infty$ one can follow the proof found in [10] but, for the sake of completeness, we repeat it below.

The proof of Eq. (5.2) will be done in two steps. For all $g \in E$ we define $Q(g) := \inf_{p \in E} \|g - p\|_E$.

I: We show first that there exists a constant C_1 such that

$$\forall_{g \in E}, \ Q(g) \leq C_1 \|A_1 g\|_{E_1}. \tag{5.3}$$

II: Secondly we will show that there exists a constant C_2 such that

$$\forall_{g \in E}, \ \|Lg\|_F \leq C_2 Q(g), \tag{5.4}$$

giving Eq. (5.2) $\|Lg\|_F \leq C\|A_1 g\|_{E_1}$.

Proof of **I**. We show Eq. (5.3) by contradiction, assume that there is a sequence $\{g_n\} \subset E$ such that $\|A_1 g_n\|_{E_1} \to 0$ and $Q(g_n) = 1$, i.e.,

$$\forall_n, \ \exists_{\{g_n\}} : \quad Q(g_n) > n\|A_1 g_n\|_{E_1}$$

and for convenience we can rescale $1 > \frac{n}{Q(g_n)}\|A_1 g_n\|_{E_1}$, so we can take $Q(g_n) = 1$.

As P is finite dimensional and totally bounded (hence compact) there exists a sequence $\tilde{g}_n = g_n - p_n$ such that,

$$\|\tilde{g}_n\|_E = Q(g_n) = \inf_{p \in P} \|g_n - p\|_E = \|g_n - p_n\|_E.$$

Therefore we have $\|A_1 \tilde{g}_n\|_{E_1} = \|A_1 g_n\|_{E_1} \to 0$ as $A_1 p_n = 0$. Since the sequence $\{\tilde{g}_n\}$ is bounded in E ($\|\tilde{g}_n\|_E = 1$ as $Q(g_n) = 1$) it will contain a weakly convergent subsequence $\tilde{g}_{n_k} \rightharpoonup g^* \in E$ giving $A_0 \tilde{g}_{n_k} \xrightarrow{E_0} A_0 g^*$ and $A_1 g_{n_k} \xrightarrow{E_1} A_1 g^*$ implying $A_1 \tilde{g}_{n_k} \xrightarrow{E_1} A_1 g^* = 0$. Now by this and Eq. (5.1) $g_{n_k} \xrightarrow{E} g^*$ as

$$\|g_{n_k} - g^*\|_E \leq C_0 \left(\|A_0 g_{n_k} - A_0 g^*\|_{E_0} + \|A_1 g_{n_k} - A_1 g^*\|_{E_1} \right)$$

giving $\inf_{p \in P} \|\tilde{g}_{n_k} - p\|_E \leq \|\tilde{g}_{n_k} - g^*\|_E \to 0$ but this contradicts $Q(g_{n_k}) = 1$ and so there exists a constant C_1 such that $Q(g) \leq C_1\|A_1 g\|_{E_1}$.

Proof of **II**. We now turn to Eq. (5.4). As we assumed L is continuous and by the assumption of the theorem $Lp = 0$ we have,

$$\|Lg\|_F = \|Lg - Lp\|_F \leq C_2\|g - p\|_E.$$

Taking inf over $p \in P$ on both sides gives,

$$\|Lg\|_F \leq C_2 \inf_{p \in P} \|g - p\|_E = C_2 Q(g) \leq C_1 C_2 \|A_1 g\|_{E_1},$$

as desired.

$\square$

Conclusion. We construct and analyze a finite volume method for Poisson's equation, using a quasi-uniform mesh, in the three dimensional cube $\Omega = (0,1) \times (0,1) \times (0,1)$. We derive both stability and convergence estimates. The convergence rates are optimal in an L_2-setting, whereas the L_∞ error estimates are sub-optimal. This generalizes the two-dimensional result by Süli, [26] to three dimensions. We show that the underlying theory for the two-dimensional case, studied by Grisvard in [17], is extendable to three dimensions (with some draw-back for L_∞ error estimate). We also include a corrected proof of a classical result, cited in [26], and used in convergence analysis. Finally we have implemented the scheme in the C++ environment, for a general k-dimensional unit cube,

and for *Gaussians*, *mollifier* and multidimensional *Hicks-Henne sine bump* functions. The implementations are justifying the convergence rates both in L_2- and H^1- norms. The Figures 2-4 are showing the absolute and relative errors.

Acknowledgments

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