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Functional Calculus and Bishop's Property (β) for Several Commuting Operators

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ABSTRACT. Let a be a commuting tuple of bounded operators on a Banach space. Let g be holomorphic in a neighbourhood of the Taylor spectrum of a and let f be holomorphic in a neighbourhood of the Taylor spectrum of $g(a)$. In the first paper the identity $f \circ g(a) = f(g(a))$ is proved using integral formulas.

In the second paper a generalisation of the so-called resolvent identity to several commuting operators is given. Using this identity we prove that Taylor's holomorphic functional can be extended to functions whose $\bar{\partial}$-derivative can be controlled by forms that define the resolvent.

Let D be a strictly pseudoconvex domain in $\mathbb{C}^m$ with smooth boundary and let g be a tuple of bounded holomorphic functions on D. Consider the tuple T_g of operators on the Hardy space $H^p(D)$ defined by $T_g f = gf$. In the third paper we prove that T_g has property $(\beta)_{\mathcal{E}}$ if $p < \infty$ and in the case $m = 1$ then also if $p = \infty$. As a corollary we have that T_g has Bishop's property (β).

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This thesis consists of an introduction and the following papers:

[AS1] Andersson, M. and Sandberg, S., *A constructive proof of the composition rule for Taylor's functional calculus*, Studia Math., **142**, no.1, 65-69 (2000).

[S1] Sandberg, S., *On non-holomorphic functional calculus for commuting operators*, submitted.

[S2] Sandberg, S., *Property* $(\beta)_{\mathcal{E}}$ *for Toeplitz operators with* H^∞*-symbol*, submitted.

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INTRODUCTION

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1. Spectrum of several commuting operators

Let X be a Banach space and denote by $L(X)$ the set of all continuous linear maps from X to X. An operator a in $L(X)$ is said to be non-singular if it is bijective. Now suppose that we have two commuting operators $a_1, a_2 \in L(X)$. Then the pair (a_1, a_2) is called non-singular if the following three conditions are satisfied.

$$\forall x \in X : \exists x_1, x_2 \in X : x = a_1 x_1 + a_2 x_2. \tag{1}$$

$$\forall x_1, x_2 \in X : a_1 x_1 = a_2 x_2 \implies \exists x \in X : x_1 = a_2 x, x_2 = a_1 x. \tag{2}$$

$$\forall x \in X : 0 = a_1 x \text{ and } 0 = a_2 x \implies x = 0. \tag{3}$$

If one of the operator a_1 and a_2 is non-singular then the pair (a_1, a_2) is non-singular. To see this let $x = a_1^{-1} x_2$ in (2); then $a_2 x = a_1^{-1} a_2 x_2 = x_1$.

The next example shows that the conditions (1), (2) and (3) are independent of each other.

Example 1.1. Let $X = l^2(\mathbb{N}^2)$. Define the shift operator $S_k \in L(X)$ by

$$S_k(x)(i,j) = x(i+k,j) \text{ if } 0 \leq i+k \text{ and } S_k(x)(i,j) = 0 \text{ otherwise.}$$

Similarly define $T_k \in L(X)$ by

$$T_k(x)(i,j) = x(i,j+k) \text{ if } 0 \leq j+k \text{ and } T_k(x)(i,j) = 0 \text{ otherwise.}$$

Thus S_k (or T_k) is injective but not surjective if $k < 0$, surjective but not injective if $0 < k$ and bijective if $k = 0$. Furthermore S_k and T_l commute. Let $e_{0,0}(i,j) = 1$ if $i = 0, j = 0$ and $e_{0,0}(i,j) = 0$ otherwise. If $k, l < 0$ then the conditions (2) and (3) hold but (1) fails (let $x = e_{0,0}$ in (1)). If $k < 0 < l$ then (1) and (3) hold but (2) fails (consider $x_1 = 0$ and $x_2 = e_{0,0}$ in (2)). If $0 < k, l$ then (1) and (2) hold but (3) fails (let $x = e_{0,0}$ in (3)).

We now extend the definition of non-singularity to tuples consisting of an arbitrary number of commuting operators. Denote by $\Lambda = \oplus_{p=0}^{n} \Lambda^p$ the exterior algebra of $\mathbb{C}^n$ over $\mathbb{C}$, and let $s_1, s_2 \ldots, s_n$ be the canonical basis of $\mathbb{C}^n$. Let $\Lambda^p X = X \otimes_{\mathbb{C}} \Lambda^p$. Thus $\Lambda^p X$ consists of sums of elements of the form $x s_{i_1} \wedge s_{i_2} \wedge \ldots \wedge s_{i_p}$, where $x \in X$. Consider

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an n-tuple $a = (a_1, a_2, \dots, a_n)$ of commuting operators in $L(X)$. The Koszul complex, $K_\bullet(a, X)$, is defined as the complex

$$0 \to \Lambda^n X \xrightarrow{\delta_a} \Lambda^{n-1} X \xrightarrow{\delta_a} \cdots \xrightarrow{\delta_a} \Lambda^0 X \to 0,$$

where

$$\delta_a(xs_{i_1} \wedge s_{i_2} \wedge \cdots \wedge s_{i_p}) = 2\pi i \sum_k (-1)^{k+1} a_{i_k} x s_{i_1} \wedge \cdots \widehat{s}_{i_k} \cdots \wedge s_{i_p}.$$

The hat over s_{i_k} in the definition of δ_a means that s_{i_k} is to be omitted. We have that

$$\delta_a \circ \delta_a = (2\pi i)^2 \sum_{k<l} (-1)^{k+l} \left(a_{i_k} a_{i_l} - a_{i_l} a_{i_k}\right) x s_{i_1} \wedge \cdots \widehat{s}_{i_k} \cdots \widehat{s}_{i_l} \cdots \wedge s_{i_p},$$

and thus $\delta_a \circ \delta_a = 0$ since a is commuting. The homology groups of $K_\bullet(a, X)$ are

$$H_p(a, X) = \frac{\text{Ker } \delta_a : \Lambda^p X \to \Lambda^{p-1} X}{\text{Im } \delta_a : \Lambda^{p+1} X \to \Lambda^p X}.$$

The tuple a is called non-singular if $K_\bullet(a, X)$ is exact, that is if

$$H_p(a, X) = 0$$

for all p. If a is not non-singular then it is called singular. Denote by 1 the identity operator on $L(X)$. The next proposition gives two conditions for non-singularity.

Proposition 1.1. *Suppose that there exists an element s in $\Lambda^1 L(X)$ such that $\delta_a s = 1$ and $sa = as$. Then a is non-singular. If (a, b) is a commuting tuple and a is non-singular then (a, b) is non-singular.*

Let the spectrum of a commuting n-tuple a be defined by

$$\sigma(a) = \{z \in \mathbb{C}^n : z - a \text{ is singular}\}.$$

Proposition 1.2. *The spectrum is non-empty if $X \neq \{0\}$ and it is compact. Moreover it has the projection property, that is*

$$\pi(\sigma(a, a_{n+1})) = \sigma(a)$$

if (a, a_{n+1}) is a commuting $n+1$-tuple and $\pi(z, z_{n+1}) = z$.

A calculation shows that the spectra of the operators in Example 1 are

$$\sigma(S_k) = \sigma(T_l) = \bar{D}, \quad \sigma(S_k, T_l) = \bar{D} \times \bar{D},$$

where $\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, if $k, l \neq 0$.

In general we have the inclusion

$$\sigma(a, b) \subset \sigma(a) \times \sigma(b),$$

as seen from the projection property.

Example 1.2. Suppose that $a_1, a_2 \in L(X)$ and define the operators $A_1, A_2 \in L(X^2)$ by $A_1(x, y) = (a_1 x, 0)$ and $A_2(x, y) = (0, a_2 x)$. Then

$$\sigma(A_1, A_2) = (\sigma(a_1), 0) \cup (0, \sigma(a_2)).$$

That any compact set in $\mathbb{C}^n$ is the spectrum of some tuple, follows from the next example.

Example 1.3. Suppose that $a_i \in L(C(K))$, K a compact set in $\mathbb{C}^n$, is defined by $a_i(f)(z) = z_i f(z)$. If $w \in K$ then

$$\delta_{w-a}\Lambda^1 C(K) \subset \{f \in C(K) : f(w) = 0\}$$

and thus $K \subset \sigma(a)$. If $w \notin K$ and $u \in \Lambda^p C(K)$ such that $\delta_{w-a}u = 0$ then

$$\delta_{w-a} \sum (\bar{w}_i - \bar{z}_i) s_i \wedge u(z) / |w - z|^2 = u(z).$$

Hence $\sigma(a) = K$.

The complex $K_\bullet(a, X)$ is called split if there are continuous linear operators $h : \Lambda^p X \to \Lambda^{p+1} X$ such that $\delta_a h + h\delta_a$ =identity. The split spectrum is defined as the set

$$sp(a) = \{z \in \mathbb{C}^n : K_\bullet(z - a, X) \text{ is not split}\}.$$

If X is a Hilbert space then $sp(a) = \sigma(a)$. By an example of Müller [10] there is a Banach space X and a commuting pair of operators a in $L(X)$ such that $\sigma(a)$ is strictly included in $sp(a)$.

The spectrum discussed in this section was defined by Taylor [16] in 1970.

2. Functional calculus

Suppose that U is an open subset of $\mathbb{C}^n$. Let $\mathcal{E}(U, X)$ be the Frechet space of all infinitely differentiable function on U with values in X, where differentiable means that for each $z \in U$ there is $\mathbb{R}$-linear map $f'(z)$ from $\mathbb{C}^n$ to X such that

$$\frac{1}{|h|} \|f(z + h) - f(z) - f'(z)h\|_X \to 0$$

as $h \to 0$. Denote by $\mathcal{O}(U, X)$ the Frechet space of all holomorphic function with values in X, that is $f \in \mathcal{O}(U, X)$ if $f \in \mathcal{E}(U, X)$ and $f'(z)$ is $\mathbb{C}$-linear. By $\mathcal{E}_{p,q}(U, X)$ we denote the space of smooth (p, q)-forms, that is $f \in \mathcal{E}_{p,q}(U, X)$ if f can be written as a sum of element of the form

$$g\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where $g \in \mathcal{E}(U, X)$. If $X = \mathbb{C}$ then we omit X.

Suppose that $f \in \mathcal{O}(\mathbb{C}^n)$ and that $a = (a_1, \ldots, a_n)$ is a tuple of commuting operators in $L(X)$. We can then form a new operator $f(a)$ by replacing z_i in the Taylor series of f by a_i. We get a continuous algebra homomorphism

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha \mapsto \sum_{\alpha \in \mathbb{N}^n} c_\alpha a^\alpha : \mathcal{O}(\mathbb{C}^n) \to L(X). \tag{4}$$

In Taylor [17] the following theorem is proved.

Theorem 2.1. *The mapping* (4) *extends to a continuous algebra homomorphism*

$$f \mapsto f(a) : \mathcal{O}(U) \to L(X)$$

for all open sets U *such that* $\sigma(a) \subset U$. *The operator* $f(a)$ *commutes with any operator that commutes with* a. *If* $f = (f_1, \ldots, f_m) \in \mathcal{O}(U, \mathbb{C}^m)$, $\sigma(a) \subset U$, *then*

$$f(\sigma(a)) = \sigma(f(a)),$$

where $f(a) = (f_1(a), \ldots, f_m(a))$.

The first proof of Theorem 2.1 was based on the Cauchy-Weil integral. In [18] Taylor generalised Theorem 2.1 to a more abstract setting with aid of homological methods. The composition rule,

$$f(g(a)) = f \circ g(a)$$

holds for all function $g \in \mathcal{O}(U, \mathbb{C}^k)$, $\sigma(a) \subset U$, and $f \in \mathcal{O}(V)$, $\sigma(g(a)) \subset U$. This was proved by Putinar in [12]. In the article [AS1] the composition rule is proved using the definition of $f(a)$ in (6) below. The mapping in Theorem 2.1 is unique, see [13].

The exactness of $K_\bullet(z-a, X)$, $z \in \mathbb{C}^n \setminus \sigma(a)$ implies that

$$0 \to \mathcal{E}_{n,q}(U, X) \xrightarrow{\delta_{z-a}} \cdots \xrightarrow{\delta_{z-a}} \mathcal{E}_{0,q}(U, X) \to 0 \tag{5}$$

is exact for all q and all open sets $U \subset \mathbb{C}^n \setminus \sigma(a)$, see Taylor [17]. Since $\delta_{z-a}\bar{\partial} = -\bar{\partial}\delta_{z-a}$ we get a double complex $(\mathcal{E}_{-p,q}(U, X), \delta_{z-a}, \bar{\partial})$. Let $\mathcal{L}^m(U, X) = \oplus_{k+l=m}\mathcal{E}_{-k,l}(U, X)$ be the total complex. The exactness of (5) implies that

$$0 \to \mathcal{L}^{-n}(U, X) \xrightarrow{\delta_{z-a}-\bar{\partial}} \cdots \xrightarrow{\delta_{z-a}-\bar{\partial}} \mathcal{L}^{n}(U, X) \to 0$$

is exact for all open sets $U \subset \mathbb{C}^n \setminus \sigma(a)$.

The resolvent of a single operator $a \in L(X)$ is defined as the operator valued function

$$z \mapsto (z-a)^{-1} : \mathbb{C} \setminus \sigma(a) \to L(X).$$

The following is a generalisation to several commuting operators. Let $a \in L(X)^n$ be a commuting tuple. Let $u_x \in \mathcal{L}^{-1}(\mathbb{C}^n \setminus \sigma(a), X)$ be a solution to the equation

$$(\delta_{z-a} - \bar{\partial})u_x = x.$$

The resolvent of a, $\omega_{z-a}x$, is defined as the $\bar{\partial}$-cohomology class of the component of degree $(n, n-1)$ of u_x. In Andersson [1] Theorem 2.1 is proved, based on the integral representation

$$f(a)x = \int_{\partial D} f(z)\omega_{z-a}x, \tag{6}$$

where f is holomorphic in a neighbourhood of D and $\sigma(a) \subset D$. Let u_x be a solution to the equation $(\delta_{z-a} - \bar{\partial})u_x = x$, let v_x be a solution to the equation $(\delta_{w-a} - \bar{\partial})v_x = x$, and let

$$c \in \mathcal{L}^{-2}(\mathbb{C}^n \setminus \sigma(a) \times \mathbb{C}^n \setminus \sigma(a), X)$$

be a solution to the equation $(\delta_{z-a,w-a} - \bar{\partial})c = v_x - u_x$. The $\bar{\partial}$-cohomology class $\omega_{z-a} \wedge \omega_{w-a}x$ is defined as the class of the component of degree $(2n, 2n-2)$ of c.

It is easy to see that $f \mapsto f(a)$, where $f(a)$ defined by (6), is linear. The multiplication property, $fg(a) = f(a)g(a)$ is harder. To prove the multiplication property for functions depending on one variable one can make use of the resolvent identity,

$$(z-a)^{-1}(w-a)^{-1} = (z-w)^{-1}(w-a)^{-1} - (z-w)^{-1}(z-a)^{-1}. \tag{7}$$

We can reformulate (7) as

$$\frac{dz}{z-a} \wedge \frac{dw}{w-a} + \frac{dw}{w-a} \wedge \frac{dz}{z-w} - \frac{dw}{z-w} \wedge \frac{dz}{z-a} = 0.$$

In the article [S1] the following generalisation of the resolvent identity to commuting tuples is given. Let $\widetilde{\omega}_{z-a}x$ be a representative of $\omega_{z-a}x$, let $\widetilde{\omega}_{w-a}x$ be a representative of $\omega_{w-a}x$, and let $\widetilde{\omega}_{z-a} \wedge \widetilde{\omega}_{w-a}x$ be a representative of $\omega_{z-a} \wedge \omega_{w-a}x$. Moreover, let

$$\widetilde{\omega}_{z-w} = \frac{1}{(2\pi i)^n} \frac{\partial |z-w|^2}{|z-w|^2} \wedge \left(\bar{\partial} \frac{\partial |z-w|^2}{|z-w|^2} \right)^{n-1}.$$

The current

$$\widetilde{\omega}_{z-a} \wedge \widetilde{\omega}_{w-a}x + \widetilde{\omega}_{w-a} \wedge \widetilde{\omega}_{z-w}x + \widetilde{\omega}_{z-w} \wedge \widetilde{\omega}_{z-a}x \tag{8}$$

defined on

$$(\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma(a))$$

is $\bar{\partial}$-closed if and only if $i^*\widetilde{\omega}_{z-a}x = i^*\widetilde{\omega}_{w-a}x$, where $i : \mathbb{C}^n \to \mathbb{C}^{2n}$ is defined by $i(\tau) = (\tau, \tau)$. If $i^*\widetilde{\omega}_{z-a}x = i^*\widetilde{\omega}_{w-a}x$ then (8) is $\bar{\partial}$-exact.

It is desirable to be able to extend the holomorphic functional calculus to a wider class of functions. Suppose that $a \in L(X)$ and that f is holomorphic on D, where D is open and $\sigma(a) \subset D$. We have the formula

$$f(a) = -\frac{1}{2\pi i} \int f(z) \bar{\partial}\varphi(z) \wedge (z-a)^{-1} dz,$$

where φ is compactly supported in D and equal to 1 on a neighbourhood of $\sigma(a)$. By using the formula

$$f(a) = -\frac{1}{2\pi i} \int \bar{\partial} f(z) \wedge (z-a)^{-1} dz$$

Dynkin [4] extended the holomorphic functional calculus to functions such that $\bar{\partial} f$ is controlled by the resolvent. In Nugyen [11] this has been generalised to tuples in a commuting Banach algebra. In Droste

[5] the holomorphic functional calculus is extended to ultradifferentiable functions in the case when the spectrum is contained in a totally real submanifold in $\mathbb{C}^n$. In [S1] we extend the holomorphic functional calculus to functions $f \in C_c^2(\mathbb{C}^n)$ such that

$$\sum_{i=1}^{n} \left\| \frac{\bar{\partial} f \wedge s \wedge (\bar{\partial} s)^{i-1}}{d(z,E)} \right\|_\infty + \sum_{k+l=n} \left\| \frac{\bar{\partial} f \wedge s \wedge (\bar{\partial} s)^{k-1} \wedge s \wedge (\bar{\partial} s)^{l-1}}{d(z,E)} \right\|_\infty < \infty,$$

where $d(z,E)$ is the distance between z and E, and E is a set such that there is solution $s \in \mathcal{E}_{1,0}(\mathbb{C}^n \setminus E, L(X))$ to the equation $\delta_{z-a} s = 1$ such that $as = sa$. The spectral mapping theorem

$$f(\sigma(a)) = \sigma(f(a))$$

holds for these functions too.

3. Bishop's property (β)

Let x be an element in X. The local spectrum $\sigma_a(x)$ of a at x is defined as the set of all $z \in \mathbb{C}^n$ such that there do not exist an open neighbourhood U of z and a function $f \in \Lambda^1 \mathcal{O}(U,X)$ such that $x = \delta_{w-a} f(w)$, $w \in U$. Thus $\sigma_a(x)$ is closed. From the pointwise exactness of $K_\bullet(z,a)$ it follows that

$$0 \to \Lambda^n \mathcal{O}(U,X) \xrightarrow{\delta_{z-a}} \cdots \xrightarrow{\delta_{z-a}} \Lambda^0 \mathcal{O}(U,X) \to 0$$

is exact for all open pseudoconvex sets U in $\mathbb{C}^n \setminus \sigma(a)$, see [7]. Hence the local spectrum of a at x is contained in $\sigma(a)$. For a subset $M \subset \mathbb{C}^n$, define the spectral subspace $X_a(M)$ by

$$X_a(M) = \{x \in X : \sigma_a(x) \subset M\}.$$

The spectral subspaces are invariant under any operator that commutes with a.

The tuple a is said to satisfy the condition SVEP (single-valued extension property) if

$$H_p(z-a, \mathcal{O}(U,X)) = 0$$

for all open pseudoconvex sets U in $\mathbb{C}^n$ and $p > 0$. If a satisfies SVEP then one can show that there is a solution $u_x \in \mathcal{L}^{-1}(\mathbb{C}^n \setminus \sigma_a(x))$ to the equation $(\delta_{z-a} - \bar{\partial}) u_x = x$. The local spectral mapping theorem, that is $f(\sigma_a(x)) = \sigma_{f(a)}(x)$, where f is holomorphic in a neighbourhood of $\sigma(a)$, holds if a has SVEP, see [6]. If a has SVEP then $X_a(\emptyset) = 0$; if a consists of only one operator then the converse also holds, see [8]. If $X_a(F)$ is closed for all closed sets F then a is said to satisfy Dunford's property (C).

If $\delta_{z-a}\Lambda^1\mathcal{O}(U,X)$ is closed in $\mathcal{O}(U,X)$ for all pseudoconvex sets U in $\mathbb{C}^n$ and a has SVEP then a is said to possess Bishop's property (β). If a has Bishop's property (β) it has Dunford's property (C). There is an example of an operator a with Dunford's property property (C) but without Bishop's property (β), see [9]. Suppose that X, Y are Banach spaces and that $a \in L(X)^n$ and $b \in L(Y)^n$ are commuting tuples with Bishop's property (β). Suppose that a and b are quasi-similar, that is there are injective, continuous and linear operators $S : X \to Y$ and $T : Y \to X$ with dense range such that $Sa = bS$ and $Tb = aT$. Then we have that $\sigma(a) = \sigma(b)$, see [14].

A tuple a is called decomposable if it has Bishop's property (β) and

$$X_a(U_1 \cup U_2) = X_a(U_1) + X_a(U_2)$$

for all open sets $U_1, U_2 \subset \mathbb{C}^n$. Let A be an algebra of functions defined on some subset in $\mathbb{C}^n$. Suppose that $1, z_i \in A$, that if $f \in A$ and $\lambda \in \mathbb{C}^n \setminus \operatorname{supp} f$ then $f = \delta_{\lambda - z} g$ for some $g \in \Lambda^1 A$ and that A admits partition of unity. If there is an algebra homomorphism $\Psi : A \to L(X)$ such that $\Psi(1) = 1$ and $\Psi(z_i) = a_i$ then a is decomposable, see [7] and [8]. The operator a is decomposable if and only if a' (here $a' = (a'_1, \ldots, a'_n)$ denotes the tuple of dual operators) is decomposable, which happens if and only if a and a' have Bishop's property (β), see [7]. A decomposable resolution of a is an exact complex of Banach spaces,

$$0 \to X \xrightarrow{d} X_0 \xrightarrow{d} \cdots \xrightarrow{d} X_r \to 0,$$

such that $da = a_0 d$ and $da_i = a_{i+1} d$, where $a_i \in L(X)^n$ is decomposable. The tuple a has Bishop's property (β) if and only if it has a decomposable resolution, see [7].

If

$$H_p(z - a, \mathcal{E}(\mathbb{C}^n, X)) = 0$$

for all $p > 0$ and $\delta_{z-a}\Lambda^1\mathcal{E}(\mathbb{C}^n, X)$ is closed in $\mathcal{E}(\mathbb{C}^n, X)$ then a is said to possess property $(\beta)_{\mathcal{E}}$. If a has property $(\beta)_{\mathcal{E}}$ then a has Bishop's property (β). A commuting tuple a is called generalised scalar if there exists a continuous algebra homomorphism $\mathcal{E}(\mathbb{C}^n) \to L(X)$ which extends the holomorphic functional calculus. Suppose that there is an exact complex of Banach spaces,

$$0 \to X \xrightarrow{d} X_0 \xrightarrow{d} \cdots \xrightarrow{d} X_r \to 0, \tag{9}$$

such that $da = a_0 d$ and $da_i = a_{i+1} d$, where $a_i \in L(X)^n$ are generalised scalars. By simple homological algebra it then follows that a has property $(\beta)_{\mathcal{E}}$.

Let D be a strictly pseudoconvex domain in $\mathbb{C}^m$ with smooth boundary and ρ be a strictly plurisubharmonic defining function such that $D = \{\zeta : \rho(\zeta) < 0\}$. The Banach space of all bounded functions in $\mathcal{O}(D)$ is denoted by $H^\infty(D)$. The Hardy space $H^p(D)$, $p < \infty$, is defined as the Banach space of all functions f in $\mathcal{O}(D)$ such that

$$\|f\|_{H^p(D)} = \sup_{\varepsilon>0} \int_{\rho(\zeta)=-\varepsilon} |f(\zeta)|^p \, d\sigma(\zeta) < \infty,$$

where σ is the surface measure.

For $g \in H^\infty(D)^n$, define the tuple

$$T_g : H^p(D) \to H^p(D)^n$$

by $T_g(f) = gf$. In [3] it is proved that $\sigma(T_g) = \overline{g(D)}$ if $p < \infty$. The case $p = \infty$ is equivalent to the corona theorem. We now claim that T_g has Dunford's property (C). Let $f \in H^p(D)$. We calculate $\sigma_{T_g}(f)$. Define D' as the set of all $\zeta \in D$ such that there is no neighbourhood U of ζ where $f = 0$ on U. The set D' is open by the identity theorem. Since D' also is closed, D' consists of the components of D where f is not identical equal to zero. Suppose that $z \notin \sigma_{T_g}(f)$. Let U be an open neighbourhood of z such that there is a function $h \in \Lambda^1\mathcal{O}(U, H^p(D))$ that satisfies

$$f = \delta_{w-T_g} h(w), \quad w \in U.$$

Hence $f = 0$ on $g^{-1}(U)$ and thus $g^{-1}(z) \subset D \setminus D'$. Therefore $z \notin g(D')$ and hence $g(D') \subset \sigma_{T_g}(f)$. Conversely, we have that

$$\sigma_{T_g}(f) = \sigma_{T_{g|_{D'}}}(f|_{D'}) \subset \sigma(T_{g|_{D'}}) = \overline{g(D')}.$$

Hence $\sigma_{T_g}(f) = \overline{g(D')}$. Suppose that F is a closed set in $\mathbb{C}^n$. We have that $g(D') \subset F$ if and only if $g^{-1}(\mathbb{C}^n \setminus F) \subset D \setminus D'$. To see this suppose that $\zeta \in g^{-1}(\mathbb{C}^n \setminus F)$. Then $g(\zeta) \in \mathbb{C}^n \setminus F \subset \mathbb{C}^n \setminus g(D')$ and thus $\zeta \in D \setminus D'$. Conversely suppose that $z \in g(D')$. Then there is a $\zeta \in D'$ such that $z = g(\zeta)$. Thus $\zeta \notin g^{-1}(\mathbb{C}^n \setminus F)$ and therefore $z \notin \mathbb{C}^n \setminus F$ and hence $z \in F$. Thus

$$H^p(D)_{T_g}(F) = \{f \in H^p(D) : f = 0 \text{ on } g^{-1}(\mathbb{C}^n \setminus F)\}$$

and hence T_g has Dunford's property (C).

In the article [S2] it is proved that T_g has property $(\beta)_\mathcal{E}$ for $1 \leq p < \infty$ and if $m = 1$ also for the case $p = \infty$. Under additional conditions on g (boundedness of the derivative of g) this theorem has been proved before, see [15] and [19]. The proof of the statement that T_g has property $(\beta)_\mathcal{E}$ amounts to the construction of a smooth resolution of $H^p(D)$, (9). However a resolution such as (9) was not found. As a substitute we have the following.

Let $X = H^p(D)$ and $a = T_g$. In [S2] we define a complex

$$(10) \qquad 0 \to X \xrightarrow{i} X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} X_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{m-1}} X_m \to 0$$

of Banach spaces. Let $a_i \in L(X_i)^n$ be generalised scalar tuples such that $ia = a_0 i$ and $d_i a_i = a_{i+1} d_i$. The complex (10) is exact at X_i, $i > 2$, but not necessarily at X_1 and X_2. Instead we find Banach spaces X_0' and X_1' and continuous mappings

$$r : X_0 \to X_0' \text{ and } d_1' : X_1' \to X_2.$$

If $d_2 x_2 = 0$ then there is a solution $x_1' \in X_1'$ such that $d_1' x_1' = x_2$. We have commuting tuples $a_0' \in L(X_0')^n$ and $a_1' \in L(X_1')^n$ such that $r a_0 = a_0' r$ and $d_1' a_1' = a_2 d_1'$. Moreover there is a relation $d_0' x_0' = x_1 + x_1'$, where $x_0' \in X_0'$, $x_1 \in X_1$ and $x_1' \in X_1'$ which satisfies the following conditions. If $x_0 \in X_0$ then $d_0' r x_0 = d_0 x_0$, d_0' is linear and if $d_0' x_0' = x_1 + x_1'$ then $d_0' a_0 x_0' = a_1 x_1 + a_1' x_1'$. Moreover if $f_1 \in \mathcal{E}(\mathbb{C}^n, X_1)$ and $f_1' \in \mathcal{E}(\mathbb{C}^n, X_1')$ then there is a function $f_0' \in \mathcal{E}(\mathbb{C}^n, X_0')$ such that $d_0' f_0' = f_1 + f_1'$. We also have that ri is an isomorphism from X to $\{x_0' \in X_0' : d_0' x_0' = 0\}$. By a similar proof to the one that the existence of a resolution (9) implies that a has property $(\beta)_{\mathcal{E}}$ it now follows that a has property $(\beta)_{\mathcal{E}}$. The complex (10) has its origin in the Wolff-type proof of the H^p-corona problem.

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A CONSTRUCTIVE PROOF OF THE COMPOSITION RULE FOR TAYLOR'S FUNCTIONAL CALCULUS

MATS ANDERSSON & SEBASTIAN SANDBERG

Abstract. We give a new constructive proof of the composition rule for Taylor's functional calculus for commuting operators on a Banach space.

1. Introduction

Let X be a Banach space, let $\mathcal{L}(X)$ denote the space of bounded operators on X, and suppose that $a_1, \dots, a_n \in \mathcal{L}(X)$ are commuting. If $p(z) = p(z_1, \dots, z_n)$ is a polynomial then $p(a) = p(a_1, \dots, a_n)$ has a welldefined meaning. Since the polynomials are dense in $\mathcal{O}(C^n)$ there is a continuous algebra homomorphism

$$\mathcal{O}(\mathbb{C}^n) \to (a) \subset \mathcal{L}(X), \tag{1.1}$$

where (a) denotes the closed subalgebra of $\mathcal{L}(X)$ that is generated by $a_1, \dots, a_n$. The proper notion of joint spectrum $\sigma(a)$ of the operators $a_1, \dots, a_n$ was found by Taylor, [9]; it is a compact subset of $\mathbb{C}^n$. Let $(a)''$ denote the subalgebra of $\mathcal{L}(X)$ consisting of all operators that commute with all operators that commute with each a_k. It is easy to see that $(a)''$ is commutative. Taylor proved the following fundamental result in [10].

Theorem 1.1 (Taylor). *Let $a_1, \dots, a_n$ be commuting operators on a Banach space with joint spectrum $\sigma(a)$. There is a continuous algebra homomorphism $g \mapsto g(a)$ from $\mathcal{O}(\sigma(a))$ into $(a)''$ that extends* (1.1). *Moreover, if $g = (g_1, \dots, g_m)$ is a holomorphic mapping, $g_j \in \mathcal{O}(\sigma(a))$, and $g(a) = (g_1(a), \dots, g_m(a))$, then*

$$\sigma(g(a)) = g(\sigma(a)). \tag{1.2}$$

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Taylor's original proof of this theorem was based on representation of holomorphic functions by means of Cauchy–Weyl formulas. Later on, in [11] and [12], he made the construction with homological methods. Suppose that $g \in \mathcal{O}(\sigma(a))$ is a holomorphic mapping and that $f \in \mathcal{O}(g(\sigma(a)))$. In view of (1.2), both $f \circ g(a)$ and $f(g(a))$ has meaning and it is natural to ask if they coincide. Putinar proved in [7] by homological methods that this question has an affirmative answer.

Theorem 1.2 (Putinar). *Suppose that $g = (g_1, \dots, g_m)$ is a mapping, $g_j \in \mathcal{O}(\sigma(a))$, $g(a) = (g_1(a), \dots, g_m(a))$ and that $f \in \mathcal{O}(g(\sigma(a)))$. Then*

$$f(g(a)) = f \circ g(a). \tag{1.3}$$

A simplified proof appeared in [6].

If a is one single operator and $f \in \mathcal{O}(\sigma(a))$, then $f(a)$ is given by the formula

$$f(a) = \int_{\partial D} f(z)\omega_{z-a}, \tag{1.4}$$

where ω_{z-a} is the resolvent

$$\omega_{z-a} = \frac{1}{2\pi i}(z-a)^{-1}dz.$$

In the case with several commuting operators, the resolvent ω_{z-a} is an $(a)''$-valued cohomology class in $\mathbb{C}^n \setminus \sigma(a)$. In [1] we gave a new constructive proof of Taylor's theorem. From the very definition of the spectrum $\sigma(a)$ we defined, for each $x \in X$, a closed X-valued $(n, n-1)$-form in $\mathbb{C}^n \setminus \sigma(a)$ that represents the class $\omega_{z-a}x$. Then $f(a)$ can be defined by the formula (1.4). The form $\omega_{z-a}x$ is sort of an abstract Cauchy–Fantappie–Leray kernel. In special situations, for instance outside any Stein compact set that contains $\sigma(a)$, the form $\omega_{z-a}x$ can be realized as a classical Cauchy–Fantappie–Leray kernel. (Contrary to the convention in [1] we include the factor $(2\pi i)^{-n}$ in the definition of resolvent class here.) This constructive approach is natural if one wants to extend the functional calculus to larger classes of functions. In one variable this was done by Dynkin in [5]; in several variabels partial results have been obtained by e.g. Droste; see also the forthcoming papers [3] and [8].

The purpose of this note is to give a proof of Theorem 1.2 along the lines of [1] and [2]. It can be viewed as a continuation of these papers and we keep the same notation.

2. Some auxiliary results

It is sometimes convenient to replace the boundary integral in (1.4) by a smoothed out integral. If $f \in \mathcal{O}(V)$, V a neighborhood of $\sigma(a)$, and if ϕ is a cutoff function that is identically 1 in a neighborhood of $\sigma(a)$ and has support in V, then

$$f(a) = -\int f(z)\bar{\partial}\phi(z) \wedge \omega_{z-a}. \tag{2.1}$$

This immediately follows from (1.4) and Stokes' theorem.

Lemma 2.1. *Suppose that* $T\colon \mathbb{C}^n \to \mathbb{C}^m$ *is linear and* $\phi \in \mathcal{O}(\sigma(Ta))$. *Then* $\phi(Ta) = T^*\phi(a)$.

We already know from Theorem 1.1 that $T\sigma(a) = \sigma(Ta)$ so both sides make sense. The lemma is the special case of Theorem 1.2 when g is linear; for a proof see, e.g., [1] Theorem 3.1.

Let us now consider commuting operators $a_1, \dots, a_n, b_1, \dots, b_m$. It follows from Lemma 2.1 that $\sigma(a,b) \subset \sigma(a) \times \sigma(b)$.

We will now recall from [1] and [2] how the resolvent class $\omega_{z-a,w-b}$ in $\mathbb{C}^n \times \mathbb{C}^m \setminus \sigma(a,b)$ for (a,b) can be represented in terms of ω_{z-a} and ω_{w-b}. Let $\widetilde{\omega}_{z-a}x$ is an explicit form in $\mathbb{C}^n \setminus \sigma(a)$ that represents the class $\omega_{z-a}x$. There is a smooth $\bar{\partial}$-closed form $\widetilde{\omega}_{w-b} \wedge \widetilde{\omega}_{z-a}x$ in $\mathbb{C}^n \setminus \sigma(a) \times \mathbb{C}^m \setminus \sigma(b)$ which, for each fixed $z \in \mathbb{C}^n \setminus \sigma(a)$, represents the class $\omega_{w-b} \wedge \widetilde{\omega}_{z-a}x$. Let $\chi(z,w)$ be a function in $\mathbb{C}^n \times \mathbb{C}^m \setminus \sigma(a) \times \sigma(b)$ such that $\{\chi, 1-\chi\}$ is a partition of unity subordinate to the open cover

$$\{\mathbb{C}^n \setminus \sigma(a) \times \mathbb{C}^m, \mathbb{C}^n \times \mathbb{C}^m \setminus \sigma(b)\},$$

of $\mathbb{C}^n \times \mathbb{C}^m \setminus \sigma(a) \times \sigma(b)$. In the set $\mathbb{C}^n \times \mathbb{C}^m \setminus \sigma(a) \times \sigma(b)$, the class $\omega_{z-a,w-b}x$ is then represented by the form

$$\bar{\partial}\chi \wedge \widetilde{\omega}_{w-b} \wedge \widetilde{\omega}_{z-a}x,$$

cf., formula (3.6) in [1].

Let $\phi(z)$ and $\psi(w)$ be cutoff functions that are identically 1 in neighborhoods of $\sigma(a)$ and $\sigma(b)$ respectively. Moreover let $G(z,w)$ be holomorphic in a neighborhood of $\sigma(a) \times \sigma(b)$. Then (2.1) applied to the pair (a,b) gives

$$G(a,b)x = -\int_z\int_w G(z,w)\bar{\partial}(\phi \otimes \psi) \wedge \bar{\partial}\chi \wedge \widetilde{\omega}_{w-b} \wedge \widetilde{\omega}_{z-a}x.$$

Integration by parts in this formula yields, cf., formula (3.7) in [1],

$$G(a,b)x = \int_z\int_w G(z,w)\bar{\partial}\psi(w) \wedge \bar{\partial}\phi(z) \wedge \widetilde{\omega}_{w-b} \wedge \widetilde{\omega}_{z-a}x. \tag{2.2}$$

In particular, if $g_1 \in \mathcal{O}(\sigma(a))$, $g_2 \in \mathcal{O}(\sigma(b))$, and $G = g_1 \otimes g_2$, it follows by Fubini's theorem that

$$g_1 \otimes g_2(a,b) = g_1(a)g_2(b). \tag{2.3}$$

3. Proof of Theorem 1.2

We are now ready to prove Theorem 1.2. Let $a_1, \dots, a_n$ be a commuting n-tuple of operators, let $g = (g_1, \dots, g_m)$ be a holomorphic mapping, $g_j \in \mathcal{O}(\sigma(a))$, and let $b = g(a)$.

Lemma 3.1. *If ϕ is holomorphic at the origin of $\mathbb{C}^m$ and $\Phi(z,w) = \phi(w - g(z))$, then $\Phi \in \mathcal{O}(\sigma(a,b))$ and $\Phi(a,b) = \phi(0)$.*

Proof. It follows from the spectral mapping statement in Theorem 1.1 that $\sigma(a,b) = \{(z,w);\ z \in \sigma(a),\ w = g(z)\}$. Therefore $\Phi(z,w)$ is holomorphic in a neighborhood of $\sigma(a,b)$.

There are holomorphic functions $\phi_1, \dots, \phi_m$ at the origin so that $\phi(\xi) = \phi(0) + \sum \xi_j \phi_j(\xi)$. Therefore,

$$\Phi(z,w) = \phi(0) + \sum_{j=1}^{m} H_j(z,w)\Phi_k(z,w),$$

where $\Phi_j(z,w) = \phi_j(w - g(z))$ and $H_j(z,w) = w_j - g_j(z)$. Now $H_j(a,b) = b_j - g_j(a) = 0$, where the first equality follows from linearity and (2.3), and the second equality follows from our assumption. Since the functional calculus is multiplicative it follows that $\Phi(a,b) = \phi(0)$. □

We can now conclude the proof of Theorem 1.2. Assume that $f(w)$ is holomorphic in a neighborhood of $\sigma(b)$. Then $h(z,w,\xi) = f\big(\xi - (w - g(z))\big)$ is holomorphic in a neighborhood of $\sigma(a,b) \times \sigma(b) \subset \mathbb{C}^{2m} \times \mathbb{C}^m$, and in view of (2.2) above we can therefore write

$$h(a,b,b)x = \int_\xi \int_{z,w} f\big(\xi - (w - g(z))\big)\bar\partial\psi(z,w) \wedge \bar\partial\phi(\xi) \wedge \widetilde{\omega}_{z-a,w-b} \wedge \widetilde{\omega}_{\xi-b} x,$$

if $\psi(z,w)$ is 1 in a small neighborhood of $\sigma(a,b)$ and $\phi(\xi)$ is 1 in a small neighborhood of $\sigma(b)$. For each fixed ξ we can evaluate the inner integral by Lemma 3.1 and get that

$$h(a,b,b)x = -\int_\xi f(\xi)\bar\partial\phi(\xi) \wedge \omega_{\xi-b} x = f(b)x.$$

Thus $h(a,b,b) = f(b) = f(g(a))$. On the other hand, by the linear mapping $T\colon (z,\eta) \mapsto (z,w,\xi) = (z,\eta,\eta)$ and Lemma 2.1, we have that

$$h(a,b,b) = h(T(a,b)) = T^*h(a,b).$$

Now, $T^*h(z,\eta) = f \circ g(z) \otimes 1$, and hence $T^*h(a,b) = f \circ g(a)$ according to (2.3). Summing up, we get the desired equality $f(g(a)) = f \circ g(a)$.

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ON NON-HOLOMORPHIC FUNCTIONAL CALCULUS FOR COMMUTING OPERATORS

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ABSTRACT. We provide a general scheme to extend Taylor's holomorphic functional calculus for several commuting operators to classes of non-holomorphic functions. These classes of functions will depend on the growth of the operator valued forms that define the resolvent cohomology class. The proofs are based on a generalisation of the so-called resolvent identity to several commuting operators.

1. INTRODUCTION

Let X, Y be two Banach spaces. We denote by $L(X, Y)$ the Banach space of all continuous linear operators from X to Y and we let $L(X) = L(X, X)$. We denote by e the identity operator of $L(X)$. For a subset $A \subset L(X)$ we let A'' denote the bicommutant, that is the Banach algebra of all operators in $L(X)$ which commute with every operator $b \in L(X)$ such that $ab = ba$ for all $a \in A$.

Suppose that $a \in L(X)$. The spectrum of a is then defined as

$$\sigma(a) = \{z \in \mathbb{C} : z - a \text{ is not invertible}\},$$

where $z - a$ is the operator $ze - a$. If f is a holomorphic function in a neighbourhood of $\sigma(a)$ then one can define the operator $f(a)$ by the integral

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} f(z)(z - a)^{-1} dz, \tag{1.1}$$

where D is an appropriate neighbourhood of $\sigma(a)$. This expression defines a continuous algebra homomorphism

$$f \mapsto f(a) : \mathcal{O}(\sigma(a)) \to (a)'',$$

such that $1(a) = e$ and $z(a) = a$, called the Riesz functional calculus. We want to extend this algebra homomorphism to functions not necessarily holomorphic in a neighbourhood of the spectrum. Following Dynkin [6] we define $f(a)$ by

$$f(a) = -\frac{1}{2\pi i} \int \bar{\partial} f(z) \wedge (z - a)^{-1} dz \tag{1.2}$$

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for all $f \in S_a$, where S_a is defined by

$$S_a = \left\{ f \in C_c^1(\mathbb{C}) : \|f\|_a := \left\| \bar{\partial} f(z) \wedge (z-a)^{-1} dz \right\|_\infty < \infty \right\}.$$

It is evident that $f(a)$ is a bounded linear operator on X which commutes with each operator that commutes with a, that is $f(a) \in (a)''$. By Stokes theorem the definition of $f(a)$ only depends on the behaviour of f near $\sigma(a)$. Suppose that D is an open set such that $\sigma(a) \subset D$ and that $f \in \mathcal{O}(D)$. Then if $\phi \in C_c^1(D)$ is equal to 1 in a neighbourhood of $\sigma(a)$, we have that $\phi f \in S_a$ and $\phi f(a)$ defined by (1.2) equals $f(a)$ defined by (1.1).

We now prove that $f \mapsto f(a)$ is an algebra homomorphism and that the spectral mapping theorem holds for function in S_a. This is done in Dynkin [6], we provide proof as we will generalize these theorems to the several operator case.

Theorem 1.1. *The mapping*

$$f \mapsto f(a) : S_a \to (a)'',$$

where $a \in L(X)$, is a continuous algebra homomorphism that continuously extends the holomorphic functional calculus. Moreover, if $f \in S_a$ then $\sigma(f(a)) = f(\sigma(a))$.

Proof. The map $f \mapsto f(a)$ is obviously linear and continuous. We have the so-called resolvent identity,

$$(w-z)(z-a)^{-1}(w-a)^{-1} = (z-a)^{-1} - (w-a)^{-1} \tag{1.3}$$

where $z, w \in \mathbb{C}$. The multiplicative property then follows,

$$f(a)g(a) = \frac{1}{(2\pi i)^2} \int_z \int_w \bar{\partial} f(z) \wedge (z-a)^{-1} dz \wedge \bar{\partial} g(w) \wedge (w-a)^{-1} dw$$

$$= \frac{1}{(2\pi i)^2} \int_z \int_w \bar{\partial} f(z) \wedge (z-a)^{-1} dz \wedge \bar{\partial} g(w) \wedge (w-z)^{-1} dw$$

$$+ \frac{1}{(2\pi i)^2} \int_z \int_w \bar{\partial} f(z) \wedge (z-w)^{-1} dz \wedge \bar{\partial} g(w) \wedge (w-a)^{-1} dw$$

$$= -\frac{1}{2\pi i} \int_z g(z) \bar{\partial} f(z) \wedge (z-a)^{-1} dz$$

$$-\frac{1}{2\pi i} \int_w f(w) \bar{\partial} g(w) \wedge (w-a)^{-1} dw = fg(a),$$

by Fubini-Tonelli's theorem.

Suppose that D is an open neighbourhood of $\sigma(a)$ and that $f_n \in \mathcal{O}(D)$ is a sequence such that $f_n \to 0$ uniformly on compacts. Then if $\phi \in C_c^1(D)$ is a function equal to 1 in a neighbourhood of $\sigma(a)$ we have that $\|f_n \phi\|_a \to 0$. Thus the mapping $f \mapsto f(a)$ continuously extends the holomorphic functional calculus.

If $w \notin f(\sigma(a))$ and $\phi \in C_c^1(\mathbb{C})$ is equal to 1 in an appropriate neighbourhood of $g(\sigma(a))$, then

$$\frac{\phi}{w-f} \in S_a,$$

and hence $w - f(a)$ is invertible and thus $w \notin \sigma(f(a))$. Therefore we have the inclusion $\sigma(f(a)) \subset f(\sigma(a))$. Suppose that $w \in f(\sigma(a))$ and assume that $w = 0$. Then $0 = f(\zeta)$ for some $\zeta \in \sigma(a)$. Let

$$g(z) = \frac{f(z)}{z-\zeta}.$$

Then

$$\begin{aligned} f(a) &= -\frac{1}{2\pi i}\int_z (z-\zeta)\bar{\partial} g(z) \wedge (z-a)^{-1} dz \\ &= (\zeta - a)\frac{1}{2\pi i}\int_z \bar{\partial} g(z) \wedge (z-a)^{-1} dz \\ &- \frac{1}{2\pi i}\int_z (z-a)\bar{\partial} g(z) \wedge (z-a)^{-1} dz. \end{aligned}$$

The last integral equals $f(\zeta)$, which is 0, and hence $0 \in \sigma(f(a))$ since otherwise $\zeta - a$ would be invertible. Therefore $f(\sigma(a)) \subset \sigma(f(a))$, and hence the theorem is proved. □

Furthermore, we have a rule of composition for this functional calculus.

Theorem 1.2. ***(Rule of composition)*** *If $g \in S_a$ and f is a holomorphic function in a neighbourhood of $\sigma(a)$, then $\phi(f \circ g) \in S_a$ and $f(g(a)) = \phi(f \circ g)(a)$, if $\phi \in C_c^1(\mathbb{C})$ is equal to 1 in a neighbourhood of $\sigma(a)$.*

Proof. Suppose that $\psi \in C_c^1(\mathbb{C})$ is equal to 1 in a neighbourhood of $\sigma(g(a))$. There is a function $\phi \in C_c^1(\mathbb{C})$ such that ϕ is equal to 1 in a neighbourhood of $\sigma(a)$ and

$$h = \frac{\phi}{w-g} \in S_a$$

for each fixed $w \in \operatorname{supp} \left|\bar{\partial}\psi\right|$. The function $\phi(f \circ g)$ is in S_a since

$$\frac{\partial(\phi(f \circ g))}{\partial \bar{z}} = f \circ g \frac{\partial \phi}{\partial \bar{z}} + \phi \frac{\partial f}{\partial w}\frac{\partial g}{\partial \bar{z}}.$$

We have that

$$\begin{aligned} f(g(a)) &= -\frac{1}{2\pi i}\int_w f(w)\bar{\partial}_w \psi(w) \wedge (w - g(a))^{-1} dw \\ = \frac{1}{(2\pi i)^2}&\int_w\int_z f(w)\bar{\partial}_w\psi(w) \wedge dw \wedge \bar{\partial}_z h(z) \wedge (z-a)^{-1} dz \\ = \frac{1}{(2\pi i)^2}&\int_z \bar{\partial}_z \int_w f(w)\bar{\partial}_w \psi(w) \wedge \frac{\phi(z) dw}{w - g(z)} \wedge (z-a)^{-1} dz \end{aligned}$$

$$= -\frac{1}{2\pi i}\int_z \bar{\partial}_z\left(\phi f \circ g\right) \wedge (z-a)^{-1} dz = \phi(f \circ g)(a),$$

and hence the theorem is proved. □

For further results regarding this functional calculus, see Dynkin [6].

Now to the notion of spectrum of a commuting tuple of operators. Suppose that $a = (a_1, \ldots, a_n) \in L(X)^n$ is a commuting tuple of operators, that is $a_i a_j = a_j a_i$ for all i and j. Denote by

$$\Lambda = \oplus_{p=0}^{n} \Lambda^p$$

the exterior algebra of $\mathbb{C}^n$ over $\mathbb{C}$. If $s_1, \cdots, s_n$ is a basis of $\mathbb{C}^n$ then Λ has the basis

$$s_\emptyset = 1, \quad s_I = s_{i_1} \wedge \cdots \wedge s_{i_p}, \quad I = \{i_1, \cdots, i_p\},$$

where $i_1 < \cdots < i_p$ and $1 \leq p \leq n$, and we denote $\Lambda = \Lambda(s)$ in this case. We let $K_\bullet(a, X)$ be the Koszul complex induced by a,

$$\cdots \to K_{p+1}(a, X) \xrightarrow{\delta_{p+1}} K_p(a, X) \xrightarrow{\delta_p} K_{p-1}(a, X) \to \cdots,$$

where

$$K_p(a, X) = \Lambda^p(s, X) = X \otimes_{\mathbb{C}} \Lambda^p(s)$$

and

$$\delta_p(x s_I) = 2\pi i \sum_{k=1}^{p} (-1)^{k-1} a_{i_k} x s_{i_1} \wedge \cdots \wedge \widehat{s}_{i_k} \wedge \cdots \wedge s_{i_p}.$$

If $K_\bullet(a, X)$ is exact then a is called non-singular, otherwise singular. The spectrum is defined as

$$\sigma(a) = \{z \in \mathbb{C}^n : z - a \text{ is singular}\}.$$

One also defines the split spectrum as

$$sp(a) = \{z \in \mathbb{C}^n : K_\bullet(z - a, X) \text{ is not split}\},$$

where split means that for every integer p there are operators h and k such that $e = \delta_{p+1} h + k \delta_p$. If X is a Hilbert space or $n = 1$ then $sp(a) = \sigma(a)$. In general we have that $\sigma(a) \subset sp(a)$, but not the reverse inclusion, see Müller [11].

We will consider operators parametrized by a variable z, such as $z \mapsto z - a$. In that case the boundary map δ_p depends on z and we will henceforth suppress the index p and write δ_p as δ_{z-a} for every p. We also let $s_i = dz_i$.

Now suppose that $T \in L(X, Y)$ has closed range and let $k(T)$ be the norm of the inverse of T considered as a map from $X/\operatorname{Ker} T$ to $\operatorname{Im} T$. The next lemma is Lemma 2.1.3 of [7], and it implies that if a_0 is a non-singular tuple then a is non-singular if $\|a_0 - a\|$ is small enough.

Lemma 1.3. *Suppose that X, Y, Z are Banach spaces, $\alpha_0 \in L(X,Y)$, $\beta_0 \in L(Y,Z)$, $\operatorname{Im}\beta_0$ closed and $\operatorname{Ker}\beta_0 = \operatorname{Im}\alpha_0$, that is*

$$X \xrightarrow{\alpha_0} Y \xrightarrow{\beta_0} Z$$

is exact. Let r be a number such that $r > \max\{k(\alpha_0), k(\beta_0)\}$. If $\alpha \in L(X,Y)$, $\beta \in L(Y,Z)$, $\operatorname{Im}\alpha \subset \operatorname{Ker}\beta$ and $\|\alpha - \alpha_0\|, \|\beta - \beta_0\| < 1/6r$ then $\operatorname{Im}\alpha = \operatorname{Ker}\beta$ and $k(\alpha) \le 4r$.

Hence $\sigma(a)$ is closed. Furthermore, the spectrum has the projection property, see Theorem 2.5.4 of [7].

Theorem 1.4. *If $a \in L(X)^n$ and $a' = (a, a_{n+1}) \in L(X)^{n+1}$ are commuting and $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is defined by $\pi(z, z_{n+1}) = z$ then $\pi(\sigma(a')) = \sigma(a)$.*

It follows that

$$\sigma(a) \subset \sigma(a_1) \times \cdots \times \sigma(a_n)$$

and hence $\sigma(a)$ is bounded. Thus $\sigma(a)$ is a compact subset of $\mathbb{C}^n$. Conversely, any compact set K in $\mathbb{C}^n$ can arise as the spectrum of a commuting tuple of operators. This one sees by letting the operators a_k to be multiplication by z_k on the Banach space $C(K)$ of continuous functions on $K \subset \mathbb{C}^n$.

The next theorem says that pointwise exactness is equivalent to continuous exactness, see Corollary 2.1.4 of [7].

Theorem 1.5. *Suppose that X, Y, Z are Banach spaces and that Ω is a paracompact topological space. Furthermore suppose that $\alpha \in C(\Omega, L(X,Y))$ and $\beta \in C(\Omega, L(Y,Z))$ such that $\operatorname{Im}\beta(\lambda)$ is closed and $\operatorname{Ker}\beta(\lambda) = \operatorname{Im}\alpha(\lambda)$ for all $\lambda \in \Omega$. Then*

$$\operatorname{Ker}\left(C(\Omega, Y) \xrightarrow{\beta} C(\Omega, Z)\right) = \operatorname{Im}\left(C(\Omega, X) \xrightarrow{\alpha} C(\Omega, Y)\right).$$

Moreover for each point $\lambda \in \Omega$ and vector $x \in \operatorname{Ker}\alpha(\lambda)$ there is a function $f \in C(\Omega, X)$ with $\alpha f = 0$ and $f(\lambda) = x$.

Thus the complex

$$K_\bullet(a, C(\mathbb{C}^n \setminus \sigma(a), X))$$

is exact. The next theorem is more complicated to prove, see Taylor [16], Theorem 2.16 and Eschmeier and Putinar [7], Section 6.4.

Theorem 1.6. *Suppose that U is an open subset of $\mathbb{C}^n$, Y_p are Banach spaces, $\alpha_p \in \mathcal{O}(U, L(Y_p, Y_{p-1}))$ and that*

$$\cdots \to Y_{p+1} \xrightarrow{\alpha_{p+1}(z)} Y_p \xrightarrow{\alpha_p(z)} Y_{p-1} \to \cdots$$

is exact for all $z \in U$. Then the complex

$$\cdots \to C^\infty(U, Y_{p+1}) \xrightarrow{\alpha_{p+1}} C^\infty(U, Y_p) \xrightarrow{\alpha_p} C^\infty(U, Y_{p-1}) \to \cdots$$

is exact.

Hence the complex

$$K_\bullet(a, C^\infty(\mathbb{C}^n \setminus \sigma(a), X))$$

is exact.

This notion of joint spectrum for a commuting tuple of operator was introduced by Taylor, [15], in 1970. Furthermore, he proved the holomorphic functional calculus and the spectral mapping theorem for this spectrum in [16]. His first proof of the holomorphic functional calculus was based on the Cauchy-Weil integral. Using homological algebra he generalized the construction to not necessarily commuting tuples of operators in [18]. See Kisil and Ramirez de Arellano [9] for more recent developments of non-commuting functional calculus. In [1, 2] Andersson proved the holomorphic functional calculus for commuting operators using Cauchy-Fantappie-Leray formulas.

The purpose of this paper is to study generalisations of Theorem 1.1 to the case of several commuting operators. Suppose that E is a set such there is a smooth function s such that $\delta_{z-a}s = e$ outside E. In that case we can use the integral representation from [1] to extend the holomorphic functional calculus. The main difficulty is to show the multiplication property; for this we will generalize the resolvent identity (1.3) to several commuting operators. In case E is a convex set we can use approximation by holomorphic functions to show that the map $f \mapsto f(a)$ extends. Similiar results to the one in this paper has been proved in Nguyen [13]. In the setting where one has a tuple a of elements in a commutative Banach algebra (or more general a b-algebra) he extends the holomorphic functional calculus. The method of the proof of the multiplication property in [13] is to show that $f(a)g(a) = f \otimes g(Ta) = f \otimes g \circ T(a) = fg(a)$, where $T(z) = (z, z)$. In Droste [5] the holomorphic fuctional calculus is extended to ultradifferentiable functions in the case when the spectrum is contained in a totally real submanifold in $\mathbb{C}^n$. His method of proof is to use the denseness of the holomorphic functions in the algebra ultradifferentiable functions.

I would like to thank my supervisor Mats Andersson for valuable discussions about the results of this paper.

2. Holomorphic functional calculus

Remember that X is a Banach space, $a \in L(X)^n$ is a tuple of commuting operators on X, and $z \in \mathbb{C}^n$ is a variable. Remember also the fact that if the complex $K_\bullet(z - a, X)$ is exact for every z in an open set U then there is a smooth solution u in U to the equation $\delta_{z-a}u = f$ if f is a closed and smooth X-valued form in U.

We now construct the resolvent on $\mathbb{C}^n \setminus \sigma(a)$. We have that

$$\delta_{z-a}\bar{\partial}\sum_k f_k dz_k = -2\pi i\sum_{k,l}(z_k - a_k)\frac{\partial f_k}{\partial \bar{z}_l}d\bar{z}_l = -\bar{\partial}\delta_{z-a}\sum_k f_k dz_k,$$

and therefore $\delta_{z-a}\bar{\partial} = -\bar{\partial}\delta_{z-a}$ for 1-forms and hence for all forms since δ_{z-a} and $\bar{\partial}$ are anti-derivations. Suppose that $K_\bullet(z-a, X)$ is exact and $x \in X$. Then we can define a sequence u_i in $\mathbb{C}^n \setminus \sigma(a)$ by

$$\delta_{z-a}u_1 = x, \quad \delta_{z-a}u_{i+1} = \bar{\partial}u_i, \tag{2.1}$$

since $\bar{\partial}$ and δ_{z-a} anti-commute. If this sequence starts with $x = 0$ then there is a form w_n such that $u_n = \bar{\partial}w_n$, this follows from the fact that we successively can find w_i such that

$$w_1 = 0, \quad \delta_{z-a}w_{i+1} = \bar{\partial}w_i - u_i. \tag{2.2}$$

Thus if one has two sequences u_i and u'_i as in (2.1) then the difference $u_n - u'_n$ is exact. Hence u_n defines a Dolbeault cohomology class $\omega_{z-a}x$ of bidegree $(n, n-1)$, which is called the resolvent cohomology class.

Suppose we have two cohomology classes, $\omega_{z-a}x$ and $\omega_{w-b}x$, where $z, w \in \mathbb{C}^n$, $a, b \in L(X)^n$, corresponding to sequences u_i and v_i, respectively. Then one defines the X-valued cohomology class $\omega_{z-a} \wedge \omega_{w-b}x$ as the class of c_{2n}, where c_i solve

$$c_1 = 0, \quad \delta_{z-a,w-b}c_{i+1} = \bar{\partial}c_i + v_i - u_i. \tag{2.3}$$

To see that this really is a well defined cohomology class, let u'_i, v'_i and c'_i be other choices of sequences. Let w^u_i and w^v_i be the sequences given by (2.2) for the sequences $u_i - u'_i$ and $v_i - v'_i$ respectively. Then we have that

$$c_1 - c'_1 + w^v_1 - w^u_1 = 0$$

and

$$\delta_{z-a,w-b}\left(c_{i+1} - c'_{i+1} + w^v_{i+1} - w^u_{i+1}\right) = \bar{\partial}\left(c_i - c'_i + w^v_i - w^u_i\right).$$

Hence, by (2.2) again, there exists a sequence w^c_i such that $c_{2n} - c'_{2n} = \bar{\partial}w^c_{2n}$.

Now suppose that we instead have operator valued forms, u_i, such that

$$\delta_{z-a}u_1 = e, \quad \delta_{z-a}u_{i+1} = \bar{\partial}u_i, \tag{2.4}$$

so that u_n represents the operator valued cohomology class ω_{z-a}. Then we have that $\omega_{z-a}\wedge\omega_{w-b}x$ is the class of $u_n \wedge v_n$, where v_i is an X-valued sequence defining $\omega_{w-b}x$. This follows from the fact

$$\delta_{z-a}\left(u_1 \wedge v_n\right) = v_n, \quad \delta_{z-a}\left(u_{i+1} \wedge v_n\right) = \bar{\partial}\left(u_i \wedge v_n\right)$$

and the following proposition.

Proposition 2.1. *If v_i is a sequence defining $\omega_{w-b}x$ and*

$$\delta_{z-a} f_1 = v_n, \quad \delta_{z-a} f_{i+1} = \bar{\partial} f_i,$$

then f_n represents $\omega_{z-a} \wedge \omega_{w-b} x$.

Proof. Let c_i be any sequence that defines $\omega_{z-a} \wedge \omega_{w-b} x$, so that c_i satisfies (2.3). Denote by $c_i^{k,l}$ the component of c_i which is of degree k in dz and degree l in dw. We have that $\delta_{z-a} c_i^{0,i} = 0$, so there is a form f such that $c_i^{0,1} = \delta_{z-a} f$. This gives

$$\delta_{z-a,w-b} c_i = \delta_{z-a,w-b} \left(c_i - c_i^{0,1} - \delta_{w-b} f \right),$$

and hence we can assume that the component $c_i^{0,i}$ vanishes. We have that

$$\delta_{z-a} c_{n+1}^{1,n} = v_n, \quad \delta_{z-a} c_{n+i+1}^{i+1,n} = \bar{\partial} c_{n+i}^{i,n},$$

and therefore there is a form w_n such that

$$f_n - c_{2n}^{n,n} + \bar{\partial} w_n = 0.$$

Since $c_{2n} = c_{2n}^{n,n}$ the proposition is proved. □

In one variable there is only one possible representative for $\omega_{z-a}x$, $a \in L(X)$,

$$\omega_{z-a} x = \frac{1}{2\pi i}(z-a)^{-1} dz\, x,$$

and we have that ω_{z-a} is operator valued. The key part of the proof of the holomorphic functional calculus in one variable is the resolvent identity (1.3), which we can reformulate as

$$\omega_{z-a} \wedge \omega_{w-a} + \omega_{w-a} \wedge \omega_{z-w} + \omega_{w-z} \wedge \omega_{z-a} = 0.$$

We will now generalize this equality to several commuting operators. Let $\Delta = \{(z,w) \in \mathbb{C}^{2n} : z = w\}$ be the diagonal in what follows.

Lemma 2.2. *For every $x \in X$, we have the equality*

$$\omega_{z-a} \wedge \omega_{w-a} x + \omega_{w-a} \wedge \omega_{z-w} x + \omega_{w-z} \wedge \omega_{z-a} x = 0, \tag{2.5}$$

on $((\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma(a))) \setminus \Delta$.

Proof. Define the sequence m_k by

$$m_k = \frac{1}{(2\pi i)^k} \frac{\partial |z-w|^2}{|z-w|^2} \wedge \left(\bar{\partial} \frac{\partial |z-w|^2}{|z-w|^2} \right)^{k-1}. \tag{2.6}$$

The equalities,

$$\delta_{z-a,w-a} m_1 = \frac{1}{2\pi i\, |z-w|^2} \delta_{z-a,w-a} \partial |z-w|^2 = 1, \tag{2.7}$$

$$\delta_{z-a,w-a} m_{k+1} = \frac{1}{(2\pi i)^k} \left(\bar{\partial} \frac{\partial |z-w|^2}{|z-w|^2} \right)^k = \bar{\partial} m_k, \tag{2.8}$$

for all $k \leq n$, and $m_k = 0$ for all $k > n$, hold on $\mathbb{C}^{2n} \setminus \Delta$. Let u_i be a sequence as in (2.1) that defines $\omega_{z-a}x$. Define u_i^1 and u_i^2 by $u_i^1 = \pi_1^* u_i$ and $u_i^2 = \pi_2^* u_i$, where $\pi_1(z,w) = z$ and $\pi_2(z,w) = w$ are the projections. Let c_i be a sequence that satisfies the equalities

$$c_1 = 0, \quad \delta_{z-a,w-a} c_{l+1} = \bar{\partial} c_l + u_l^2 - u_l^1. \tag{2.9}$$

Using the equalities (2.7), (2.8) and (2.9) (for $l \geq n$), we get that

$$\begin{aligned} -\bar{\partial} \sum_{k+l=2n} m_k \wedge c_l &= \delta_{z-a,w-a} \sum_{k+l=2n+1} m_k \wedge c_l - \bar{\partial} \sum_{k+l=2n} m_k \wedge c_l \\ &= \sum_{l=n+1}^{2n} \delta_{z-a,w-a} m_{2n+1-l} \wedge c_l - \sum_{l=n}^{2n-1} \bar{\partial} m_{2n-l} \wedge c_l \\ &\quad + \sum_{k=1}^{n} m_k \wedge \left(\bar{\partial} c_{2n-k} - \delta_{z-a,w-a} c_{2n+1-k} \right) \\ &= -\bar{\partial} m_n \wedge c_n + c_{2n} + m_n \wedge \left(u_n^1 - u_n^2 \right). \end{aligned}$$

Thus

$$-\bar{\partial} \sum_{k+l=2n} m_k \wedge c_l = c_{2n} + u_n^2 \wedge m_n + m_n \wedge u_n^1 \tag{2.10}$$

outside the diagonal. We have that the component of m_n which does not contain dw and $d\bar{w}$ represents ω_{z-w} and that the component of m_n which does not contain dz or $d\bar{z}$ represents ω_{w-z}. Since c_{2n} represents $\omega_{z-a} \wedge \omega_{w-a}x$, the lemma follows from (2.10). $\square$

Choose representatives $\widetilde{\omega}_{z-a}x$, $\widetilde{\omega}_{w-a}x$ and $\widetilde{\omega}_{z-a} \wedge \widetilde{\omega}_{w-a}x$ for $\omega_{z-a}x$, $\omega_{w-a}x$ and $\omega_{z-a} \wedge \omega_{w-a}x$ respectively on $(\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma(a))$. Let $\widetilde{\omega}_{z-w} = m_n$. Then (2.5) says that the form

$$\widetilde{\omega}_{z-a} \wedge \widetilde{\omega}_{w-a}x + \widetilde{\omega}_{w-a} \wedge \widetilde{\omega}_{z-w}x + \widetilde{\omega}_{z-w} \wedge \widetilde{\omega}_{z-a}x, \tag{2.11}$$

defined on $((\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma(a))) \setminus \Delta$, is exact. We want this expression to be an exact current over Δ as well. Suppose that (2.11) is exact on $(\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma(a))$. We have that $[\Delta] = \bar{\partial} \widetilde{\omega}_{z-w}$, where $[\Delta]$ denotes the current of integration over Δ. If we apply $\bar{\partial}$ to (2.11), interpreted as a current, we get

$$0 = -\widetilde{\omega}_{w-a}x \wedge [\Delta] + [\Delta] \wedge \widetilde{\omega}_{z-a}x = [\Delta] \wedge (\widetilde{\omega}_{z-a}x - \widetilde{\omega}_{w-a}x)$$

since (2.11) is supposed to be exact and therefore is closed. Hence $i^* \left(\widetilde{\omega}_{z-a}x - \widetilde{\omega}_{w-a}x \right) = 0$, where i is a function defined by $i(\tau) = (\tau, \tau)$. The next theorem gives the desired result in the case where we have $i^* \widetilde{\omega}_{z-a}x = i^* \widetilde{\omega}_{w-a}x$.

Theorem 2.3. ***(Resolvent identity)*** *Suppose that $\widetilde{\omega}_{z-a}x$, $\widetilde{\omega}_{w-a}x$ and $\widetilde{\omega}_{z-a} \wedge \widetilde{\omega}_{w-a}x$ are representatives for $\omega_{z-a}x$, $\omega_{w-a}x$ and $\omega_{z-a} \wedge \omega_{w-a}x$, respectively. Let $\widetilde{\omega}_{z-w} = m_n$, where m_n is defined in (2.6). Then the current*

$$\widetilde{\omega}_{z-a} \wedge \widetilde{\omega}_{w-a}x + \widetilde{\omega}_{w-a} \wedge \widetilde{\omega}_{z-w}x + \widetilde{\omega}_{z-w} \wedge \widetilde{\omega}_{z-a}x$$

defined on $(\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma(a))$ *is exact if and only if* $i^*\widetilde{\omega}_{z-a}x = i^*\widetilde{\omega}_{w-a}x$*, where* $i : \mathbb{C}^n \to \mathbb{C}^{2n}$ *is defined by* $i(\tau) = (\tau, \tau)$.

Proof. The necessity of having $i^*\left(\widetilde{\omega}_{z-a}x - \widetilde{\omega}_{w-a}x\right) = 0$ has already been proved. Now suppose that $i^*\left(\widetilde{\omega}_{z-a}x - \widetilde{\omega}_{w-a}x\right) = 0$. Let u_i^1, u_i^2, m_i and c_i be the sequences in the proof of Lemma 2.2. Let $\delta = \delta_{z-a,w-a}$. Then we have that $i^*\delta = \delta_{\tau-a}i^*$ by induction, since

$$\begin{aligned} i^*\delta\left(f dz_k + g dw_l\right) &= (\tau_k - a_k)\, f(\tau,\tau) + (\tau_l - a_l)\, g(\tau,\tau) \\ &= \delta_{\tau-a} i^*\left(f dz_k + g dw_l\right) \end{aligned}$$

and

$$i^*\delta(u \wedge v) = i^*\delta u \wedge i^*v - i^*u \wedge i^*\delta v = \delta_{\tau-a}i^*\left(u \wedge v\right),$$

if u is a 1-form. Thus

$$i^*c_1 = 0, \quad \delta_{\tau-a}i^*c_{i+1} = \bar{\partial} i^* c_i$$

and hence, by (2.2), there is a form w_n of τ such that $i^*c_n = \bar{\partial}w_n$. For all test forms f we have the identity

$$\bar{\partial}m_n \wedge c_n.f = \int_\Delta i^*\left(c_n \wedge f\right) = \int_\Delta \bar{\partial}w_n \wedge i^*f = \int_\Delta w_n \wedge i^*\bar{\partial}f.$$

Therefore the calculation in the proof of Lemma 2.2 gives the equality

$$-\bar{\partial}\left([\Delta] \wedge w_n + \sum_{k=1}^{n} m_k \wedge c_{2n-k}\right) = c_{2n} + u_n^2 \wedge m_n + m_n \wedge u_n^1. \tag{2.12}$$

Since $\widetilde{\omega}_{\tau-a}x$ and u_n represent the same cohomology class, there is a form q such that $\widetilde{\omega}_{\tau-a}x - u_n = \bar{\partial}q$. Let $q^1 = \pi_1^*q$ and $q^2 = \pi_2^*q$. Then

$$\begin{aligned} &\widetilde{\omega}_{z-w} \wedge \left(\widetilde{\omega}_{z-a}x - \widetilde{\omega}_{w-a}x - (u_n^1 - u_n^2)\right) \\ &= \widetilde{\omega}_{z-w} \wedge \left(\bar{\partial}q^1 - \bar{\partial}q^2\right) = [\Delta] \wedge \left(q^1 - q^2\right) - \bar{\partial}\left(\widetilde{\omega}_{z-w} \wedge \left(q^1 - q^2\right)\right) \\ &= -\bar{\partial}\left(\widetilde{\omega}_{z-w} \wedge \left(q^1 - q^2\right)\right). \end{aligned}$$

Thus, since $\widetilde{\omega}_{z-a} \wedge \widetilde{\omega}_{w-a}x - c_{2n}$ is an exact current, the theorem is proved. □

Now we give the definition of $f(a)$. If f is a holomorphic function in a neighbourhood of $\sigma(a)$ then we define $f(a)$ by the formula

$$f(a)x = -\int f\bar{\partial}\phi \wedge \omega_{z-a}x \quad \text{for all } x \in X, \tag{2.13}$$

where $\phi \in C_c^\infty$ is equal to 1 in a neighbourhood of $\sigma(a)$. This definition is independent of the choice of ϕ. To see this, suppose that $\varphi \in C_c^\infty$ is equal to 0 in a neighbourhood of the spectrum. Then we have that

$$\int \bar{\partial}\varphi \wedge \omega_{z-a}x = \int \bar{\partial}\varphi \wedge u_n = \int \bar{\partial}\left(\varphi \wedge u_n\right) = 0,$$

if u_n is a smooth form in $\mathbb{C}^n \setminus \sigma(a)$ representing $\omega_{z-a}x$. Note also that, by Stokes theorem, we have the equality

$$-\int f\bar{\partial}\phi \wedge \omega_{z-a}x = \int_{\partial D} f\omega_{z-a}x,$$

where D is a small enough neighbourhood of $\sigma(a)$. We now prove that $f(a) \in (a)''$.

Lemma 2.4. *If $f(a)$ is defined by the formula (2.13), then $f(a) \in (a)''$.*

Proof. Suppose that $x, y \in X$ and $c, d \in \mathbb{C}$. Denote by u_i^x the sequence (2.1). Then

$$\delta_{z-a}\left(u_1^{cx+dy} - cu_1^x - du_1^y\right) = 0$$

and

$$\delta_{z-a}\left(u_{i+1}^{cx+dy} - cu_{i+1}^x - du_{i+1}^y\right) = \bar{\partial}\left(u_i^{cx+dy} - cu_i^x - du_i^y\right),$$

so u_n^{cx+dy} and $cu_n^x + du_n^y$ define the same cohomology class. Therefore the resolvent is linear, i.e.,

$$\omega_{z-a}(cx + dy) = c\omega_{z-a}x + d\omega_{z-a}y,$$

and hence $f(a)$ is a linear operator.

The map δ_{z-a} is linear, continuous and surjective between the Frechet space of all $C^\infty_{p+1,q}(U, X)$ forms to the Frechet space of all δ_{z-a}-closed $C^\infty_{p,q}(U, X)$ forms, where $U = \mathbb{C}^n \setminus \sigma(a)$. Let $K_1 \subset \mathbb{C}^n \setminus \sigma(a)$ be a given compact set and let $t_1 = 0$. Then the open mapping theorem gives the existence of a sequence of compact sets $K_i \subset \mathbb{C}^n \setminus \sigma(a)$ and natural numbers t_i such that the equation $\delta_{z-a}u = v$ has a solution u, which satisfies

$$\|u\|_{K_i, t_i+1} \leq C\|v\|_{K_{i+1}, t_{i+1}}$$

for all closed v. Thus we can choose the sequence (2.1) so that

$$\|u_1\|_{K_n, t_n+1} \leq C\|x\|_{K_{n+1}, t_{n+1}} = C\|x\|$$

and

$$\|u_{i+1}\|_{K_{n-i}, t_{n-i}+1} \leq C\left\|\bar{\partial}u_i\right\|_{K_{n-i+1}, t_{n-i+1}} \leq C\|u_i\|_{K_{n-i+1}, t_{n-i+1}+1}.$$

Hence

$$\|f(a)x\| \leq \int \left\|f\bar{\partial}\phi \wedge u_n\right\| \leq C\,|f|_{\text{supp }\phi}\,\|x\| \tag{2.14}$$

and thus the operator $f(a)$ is bounded.

Suppose that $b \in L(X)$ is an operator which commutes with the tuple a. Then

$$\delta_{z-a}bu_1^x = bx, \quad \delta_{z-a}bu_{i+1}^x = \bar{\partial}bu_i^x,$$

so bu_n^x and u_n^{bx} defines the same cohomology class. Therefore

$$b\omega_{z-a}x = \omega_{z-a}bx$$

and thus $f(a) \in (a)''$. □

We can now prove Taylor's theorem.

Theorem 2.5. ***(Taylor)*** *The mapping*

$$f \mapsto f(a) : \mathcal{O}(\sigma(a)) \to (a)'' \tag{2.15}$$

is a continuous algebra homomorphism such that $1(a) = e$ *and* $z_k(a) = a_k$.

Proof. The map $f \mapsto f(a)$ is continuous by (2.14). We now prove that $f(a)g(a) = fg(a)$. Let u_i, u_i^1, u_i^2 and c_i be as in Lemma 2.2. By the proof of Proposition 2.1 we can assume that the component $c_i^{0,i}$ vanishes. Since

$$\delta_{z-a} c_{n+1}^{1,n} = u_n(w), \quad \delta_{z-a} c_{n+i+1}^{i+1,n} = \bar{\partial} c_{n+i}^{i,n},$$

we have that c_{2n} represents $\omega_{z-a} u_n(w)$ and thus we have that

$$f(a)u_n(w) = -\int_z f(z)\bar{\partial}\phi_1(z) \wedge \omega_{z-a} u_n(w) = -\int_z f(z)\bar{\partial}\phi_1(z) \wedge c_{2n}.$$

Multiplying this equality by $g(w)\bar{\partial}\phi_2(w)$ and integrating with respect to w we get

$$f(a)g(a)x = \int_w \int_z f(z)g(w)\bar{\partial}\phi_2(w) \wedge \bar{\partial}\phi_1(z) \wedge c_{2n}.$$

The resolvent identity (2.12) then gives that the right hand side is equal to

$$\iint fg\bar{\partial}\phi_1 \wedge \bar{\partial}\phi_2 \wedge m_n \wedge u_n^1 + \iint fg\bar{\partial}\phi_1 \wedge \bar{\partial}\phi_2 \wedge u_n^2 \wedge m_n,$$

and hence we get, by the Bochner-Martinelli integral formula,

$$-\int \left(fg\phi_2\bar{\partial}\phi_1 + f\phi_1 g\bar{\partial}\phi_2\right) \wedge u_n = -\int fg\bar{\partial}\left(\phi_1\phi_2\right) \wedge u_n = fg(a)x,$$

since $u_n^1 = \pi_1^* u_n$ and $u_n^2 = \pi_2^* u_n$. Since the map (2.15) obviously is linear, it is an algebra homomorphism.

It remains to prove that $1(a) = e$ and $z_k(a) = a_k$. The first equality follows by representing ω_{z-a} by

$$\frac{1}{(2\pi i)^n}\left(|z|^2 e - \bar{z}a\right)^{-n} \partial|z|^2 \wedge \left(\bar{\partial}\partial|z|^2\right)^{n-1},$$

cf. [1], and integrating against $\bar{\partial}\phi$, where ϕ is a radial cutoff function which is equal to 1 in a neighbourhood of $\sigma(a)$. The second equality follows from the first equality and

$$\left(z_k - a_k\right) u_n = \frac{1}{2\pi i}\left(\delta_{z-a} u_n\right) \wedge dz_k = \frac{1}{2\pi i}\bar{\partial}\left(u_{n-1} \wedge dz_k\right),$$

where u_i is a sequence that satisfies (2.1). □

Taylor also proved the spectral mapping theorem; if $f \in \mathcal{O}(\sigma(a))$ then $f(\sigma(a)) = \sigma(f(a))$. Suppose that a is a commuting tuple and that D is an open set such that $\sigma(a) \subset D$. Then there exist a $\delta > 0$ such that $\sigma(b) \subset D$ if $\|a-b\| < \delta$. This follows from Lemma 1.3. In Newburgh [12] it is proved that the spectrum of one operator is continuous under commutative pertubations; the next proposition says that the same is true for the Taylor spectrum.

Proposition 2.6. *If a and b are tuples of operators such that a, b is commuting then*

$$\sup_{z\in\sigma(a)} \inf_{w\in\sigma(b)} |z-w| + \sup_{w\in\sigma(b)} \inf_{z\in\sigma(a)} |z-w| \leq 2 \sup_{z\in\sigma(a-b)} |z| \leq 2\,\|a-b\|\,.$$

Proof. Suppose that $u \in \sigma(a)$. Since $P\sigma(a,b) = \sigma(a)$, where $P(z,w) = z$, there is a v in $\sigma(b)$ such that $(u,v) \in \sigma(a,b)$. Since $T\sigma(a,b) = \sigma(a-b)$, where $T(z,w) = z-w$, we have that $u - v \in \sigma(a-b)$. Thus

$$\sup_{z\in\sigma(a)} \inf_{w\in\sigma(b)} |z-w| \leq \sup_{z\in\sigma(a-b)} |z|$$

and by symmetry the proposition is proved. □

The next theorem says what happens when one has a norm convergent sequence in $L(X)^n$. Notice that if $\sigma(a) = sp(a)$ then the conclusion would be that $f(a_k) \to f(a_0)$ in operator norm.

Theorem 2.7. *Suppose that $a_k \in L(X)^n$ are commuting tuples (not necessarily commuting with each other) for $k \geq 0$ and that $\|a_k - a_0\| \to 0$ as $k \to \infty$. If f is holomorphic in a neighbourhood of $\cup_{k\geq 0}\sigma(a_k)$, then $f(a_k)x \to f(a_0)x$ for every $x \in X$.*

Proof. Consider the Banach space

$$c(X) = \left\{(x_k)_{k=0}^{\infty} : \lim_{k\to\infty} \|x_k - x_0\| = 0\right\}$$

with norm $\|(x_k)_{k=0}^{\infty}\|_{\infty} = \sup_{k\geq 0} \|x_k\|$ and the tuple of n operators $a' \in L(c(X))^n$ defined by $a'(x_k)_{k=0}^{\infty} = (a_k x_k)_{k=0}^{\infty}$. Suppose that a_k is a non-singular tuple for every $k \geq 0$. Let f be a closed $c(X)$-form, that is $\delta_{a'} f = 0$. Then $\delta_{a_k} f_k = 0$ for every $k \geq 0$. Hence there is a solution u_0 of the equation $\delta_{a_0} u_0 = f_0$ since a_0 is non-singular. Lemma 1.3 gives a uniform constant C and v_k such that $\delta_{a_k} v_k = \delta_{a_k} u_0 - f_k$ and

$$\|v_k\| \leq C\|\delta_{a_k} u_0 - f_k\| \leq C\|\delta_{a_k} - \delta_{a_0}\|\|u_0\| + C\|f_0 - f_k\|.$$

Thus $u_k = u_0 - v_k$ solve the equations $\delta_{a_k} u_k = f_k$ and $u_k \to u_0$ if $k \to \infty$. Hence $u = (u_k)_{k=0}^{\infty}$ is a solution of $\delta_{a'} u = f$ and the complex $K_\bullet(a', c(X))$ is exact, and thus a' is non-singular. That is, we have proved the inclusion

$$\sigma(a') \subset \bigcup_{k\geq 0} \sigma(a_k).$$

Let u_i be smooth $c(X)$-forms defined on $\mathbb{C}^n \setminus \sigma(a')$ by the equations

$$\delta_{z-a'}u_1 = x, \quad \delta_{z-a'}u_{i+1} = \bar{\partial}u_i.$$

Thus $(u_n)_k$ represent $\omega_{z-a_k}x$ for all $k > 0$ and $(u_n)_0 = \lim_{k\to\infty}(u_n)_k$ represents $\omega_{z-a_0}x$. Suppose that $\phi \in C_c^\infty$ is equal to 1 in a neighbourhood the union of $\sigma(a_k)$. Then

$$\lim_{k\to\infty} f(a_k)x = -\lim_{k\to\infty}\int f\bar{\partial}\phi \wedge (u_n)_k = -\int f\bar{\partial}\phi\wedge(u_n)_0 = f(a_0)x$$

for all $x \in X$, and hence the theorem is proved. □

3. Non-holomorphic functional calculus

In this section we will extend the holomorphic functional calculus of Section 2 to functions such that $\left|\bar{\partial}f(z)\right|$ tends to zero when z approches the spectrum. If f is a C^1-function with compact support, we define whenever possible

$$f(a)x = -\int \bar{\partial}f \wedge u_n^x,$$

where u_n^x is a form that represents $\omega_{z-a}x$.

Several problems occur. There is a problem with the possible dependence of the choice of representative u_n^x of the class $\omega_{z-a}x$. Other problems are to investigate whether

$$f(a) \in (a)'', \quad f(a)g(a) = fg(a), \quad \sigma(f(a)) = f(\sigma(a)),$$

$$g(f(a)) = g \circ f(a)$$

and whether $f(a) = 0$ if $f = 0$ on $\sigma(a)$. We will prove that $f(a)g(a) = fg(a)$, $f(a) \in (a)''$ and $\sigma(f(a)) = f(\sigma(a))$ for a certain algebra S_a (3.7) of functions. In order to do this, we will need a slightly stronger condition on $\bar{\partial}f$ than in the case $n = 1$. To begin with, we will see what is needed for the muliplicative property to hold.

Suppose that $E \supset \sigma(a)$ is a compact set such that there exists a sequence u_i on $\mathbb{C}^n \setminus E$ satisfying (2.4). Then we have that u_n is operator valued and represents ω_{z-a} in $\mathbb{C}^n \setminus E$. The definition of $f(a)$ in this case is

$$f(a) = -\int \bar{\partial}f \wedge u_n.$$

Define a sequence c_l by

$$c_1 = 0, \quad \delta_{z-a,w-a}c_{l+1} = \bar{\partial}c_l + u_l^2 - u_l^1, \tag{3.1}$$

where $u_l^1 = \pi_1^* u_l$ and $u_l^2 = \pi_2^* u_l$. Then we have that c_{2n} represents $\omega_{z-a} \wedge \omega_{w-a}$. We now prove the multiplicative property.

Proposition 3.1. *Let u_i be a sequence defined on $\mathbb{C}^n \setminus E$, where $E \supset \sigma(a)$ is a compact set, as in (2.4), and suppose that c_l, $n \le l \le 2n$ are forms that satisfies the condition,*

$$i^* c_n = 0, \quad \delta_{z-a,w-a} c_{l+1} = \bar\partial c_l + u_l^2 - u_l^1, \quad c_{2n} = u_n^1 \wedge u_n^2, \tag{3.2}$$

where $i(\tau) = (\tau, \tau)$. Moreover suppose that $f, g \in C_c^2$ such that

$$\int \left\| \bar\partial f \wedge u_n \right\| < \infty, \quad \int \left\| \bar\partial g \wedge u_n \right\| < \infty$$

and

$$\int_z \int_w \frac{\left\| \bar\partial f(z) \wedge \bar\partial g(w) \wedge c_l \right\|}{d\,(z, E)\, d\,(w, E)\, |z - w|^{2(2n-l)-1}} < \infty, \tag{3.3}$$

for all l such that $n \le l < 2n$. Then $f(a)g(a) = fg(a)$.

Proof. First note that

$$f(a)g(a) = -\int_z \int_w \bar\partial f(z) \wedge \bar\partial g(w) \wedge u_n^1 \wedge u_n^2$$

and that, by the Bochner-Martinelli integral formula,

$$fg(a) = -\int \left(g\bar\partial f + f\bar\partial g \right) \wedge u_n$$

$$= \int_z \int_w \bar\partial f(z) \wedge \bar\partial g(w) \wedge m_n \wedge u_n^1 - \int_z \int_w \bar\partial f(z) \wedge \bar\partial g(w) \wedge m_n \wedge u_n^2.$$

Let χ_ε be the convolution of the characteristic function of the set

$$\{(z, w) : d((z, w), E \times \mathbb{C}^n \cup \mathbb{C}^n \times E) \ge 2\varepsilon\}$$

and the function $\varepsilon^{-4n}\rho(\cdot/\varepsilon)$, where ρ is a non-negative smooth function with compact support in the unit ball of $\mathbb{C}^{2n}$ such that its integral is equal to 1. Since

$$\left\| \bar\partial f(z) \wedge \bar\partial g(w) \wedge \left(u_n^1 \wedge u_n^2 + m_n \wedge u_n^1 - m_n \wedge u_n^2 \right) \right\|$$

is integrable, we must prove that

$$\lim_{\varepsilon \to 0} \int_z \int_w \chi_\varepsilon \bar\partial f(z) \wedge \bar\partial g(w) \wedge \left(u_n^1 \wedge u_n^2 + m_n \wedge u_n^1 - m_n \wedge u_n^2 \right) = 0.$$

The resolvent identity (2.10) gives that

$$-\bar\partial \sum_{k+l=2n} m_k \wedge c_l + [\Delta] \wedge c_n = u_n^1 \wedge u_n^2 + m_n \wedge u_n^1 - m_n \wedge u_n^2$$

in the sense of currents (note that the proof of this formula only made use of the forms c_l for $l \ge n$). Hence, since $i^* c_n = 0$, we must prove that

$$\lim_{\varepsilon \to 0} \int_z \int_w \chi_\varepsilon \bar\partial f(z) \wedge \bar\partial g(w) \wedge \bar\partial \sum_{k+l=2n} m_k \wedge c_l = 0. \tag{3.4}$$

Integration by parts gives that (3.4) is equivalent to

$$\lim_{\varepsilon\to 0}\int_z\int_w \bar\partial\chi_\varepsilon\wedge\bar\partial f(z)\wedge\bar\partial g(w)\wedge\sum_{k+l=n} m_k\wedge c_l=0. \tag{3.5}$$

Note that $\left|\bar\partial\chi_\varepsilon\right|\le C\varepsilon^{-1}$ and that $\left|\bar\partial\chi_\varepsilon\right|$ has support in

$$\varepsilon\le d((z,w),E\times\mathbb{C}^n\cup\mathbb{C}^n\times E)\le 3\varepsilon.$$

We also have that

$$\begin{aligned} d\left((z,w),E\times\mathbb{C}^n\cup\mathbb{C}^n\times E\right) &\ge \min\left\{d(z,E),d(w,E)\right\}\\ &\ge Cd\left(z,E\right)d\left(w,E\right)\end{aligned}$$

on a bounded set, where $C>0$ is a constant (depending on the bound). Thus (3.5) follows since

$$\int_z\int_w \frac{\left\|\bar\partial f(z)\wedge\bar\partial g(w)\wedge\sum_{k+l=2n} m_k\wedge c_l\right\|}{d((z,w),E\times\mathbb{C}^n\cup\mathbb{C}^n\times E)}<\infty$$

by (3.3). Hence the proposition is proved. □

To be able to separate the condition (3.3) we will assume that u_i commute with a. We can then choose the sequence c_i in the following way.

Proposition 3.2. *Suppose that u_i is a sequence as in (2.4) and that $au_i=u_ia$. Then*

$$c_i=\sum_{k+l=i} u_k^1\wedge u_l^2$$

satisfies (3.1).

Proof. We have that $c_1=0$, and since a and u_i commute,

$$\begin{aligned}\delta c_{i+1}-\bar\partial c_i &= \sum_{k+l=i+1}\left(\delta u_k^1\wedge u_l^2-u_k^1\wedge\delta u_l^2\right)\\ &-\sum_{k+l=i}\left(\delta u_{k+1}^1\wedge u_l^2-u_k^1\wedge\delta u_{l+1}^2\right)=u_i^2-u_i^1,\end{aligned}$$

where $\delta=\delta_{z-a,w-a}$. Thus c_i satisfies (3.1). □

Unfortunately, the sequence c_i in Proposition 3.2 does not necessarily satisfy $i^*c_n=0$. However, by the proof of Theorem 2.3 we have that i^*c_n is exact.

We have an explicit choice of sequence that satisfies (2.4). Suppose that s satisfies the equalities $\delta_{z-a}s=e$ and $as=sa$. Then

$$\delta_{z-a}s=e,\quad \delta_{z-a}\left(s\wedge\left(\bar\partial s\right)^i\right)=\left(\bar\partial s\right)^i=\bar\partial\left(s\wedge\left(\bar\partial s\right)^{i-1}\right)$$

and hence $u_i=s\wedge\left(\bar\partial s\right)^{i-1}$ satisfies (2.4). The sequence c_i of Proposition 3.2 is then

$$c_i=\sum_{k+l=i} s^1\wedge\left(\bar\partial s^1\right)^{k-1}\wedge s^2\wedge\left(\bar\partial s^2\right)^{l-1}, \tag{3.6}$$

where $s^1 = \pi_1^* s$ and $s^2 = \pi_2^* s$. Note that if $s \wedge s = 0$ then $s \wedge (\bar{\partial} s) = (\bar{\partial} s) \wedge s$ and hence $i^* c_n = 0$.

Let $E \supset \sigma(a)$ be a compact set and let s be a given form such that s is defined on $\mathbb{C}^n \setminus E$, $\delta_{z-a} s = e$ and $as = sa$. Define the class S_a by

$$S_a = \left\{ f \in C_c^2(\mathbb{C}^n) : \|f\|_a < \infty \right\}, \tag{3.7}$$

where

$$\|f\|_a = \sum_{i=1}^{n} \left\| \frac{\bar{\partial} f \wedge s \wedge (\bar{\partial} s)^{i-1}}{d(z,E)} \right\|_\infty$$
$$+ \sum_{k+l=n} \left\| \frac{\bar{\partial} f \wedge s \wedge (\bar{\partial} s)^{k-1} \wedge s \wedge (\bar{\partial} s)^{l-1}}{d(z,E)} \right\|_\infty .$$

Note that the second sum vanishes if $s \wedge s = 0$. This is always the case if $n = 2$ since then $\delta_{z-a}(s \wedge s) = s - s = 0$ and δ_{z-a} injective. If $n = 1$ then S_a defined by (3.7) is a slightly smaller class than S_a defined in the introduction. This is because the left hand side in the resolvent identity (2.10) is 0 if $n = 1$. If $f \in S_a$ then $f(a)$ is defined by

$$f(a) = -\int \bar{\partial} f \wedge s \wedge (\bar{\partial} s)^{n-1} .$$

Of course we have that $f(a) \in L(X)$ if $f \in S_a$. Note that S_a is an algebra. In the next lemma we will use Proposition 3.1 to prove that $f(a)g(a) = fg(a)$ if $f, g \in S_a$.

Lemma 3.3. *If $f, g \in S_a$ then $f(a)g(a) = fg(a)$.*

Proof. Let c_i be the sequence defined by (3.6) and let

$$d_i = \sum_{k+l=i} s^2 \wedge (\bar{\partial} s^2)^{k-1} \wedge s^2 \wedge (\bar{\partial} s^2)^{l-1} .$$

By a computation similar to the proof of Proposition 3.2, we see that the sequence d_i satisfies the relation

$$\delta_{z-a,w-a} d_{i+1} = \bar{\partial} d_i,$$

and hence that $\bar{\partial} d_n = 0$. For every $l > n$ define c_l' by $c_l' = c_l$ and define c_n' by $c_n' = c_n - d_n$. Then c_l' satisfies the condition (3.2) since $\bar{\partial} d_n = 0$ and $i^* c_n = i^* d_n$. We have that $|z-w|^{-2n+1}$ is a locally integrable function on $\mathbb{C}^{2n}$ and hence

$$\int_z \int_w \frac{\left\| \bar{\partial} f(z) \wedge \bar{\partial} g(w) \wedge c_i \right\|}{d(z,E)\, d(w,E)\, |z-w|^{2n-1}}$$

$$\leq \sum_{k+l=i} \int_z \int_w \frac{\left\| \bar{\partial} f(z) \wedge s^1 \wedge (\bar{\partial} s^1)^{k-1} \right\| \left\| \bar{\partial} g(w) \wedge s^2 \wedge (\bar{\partial} s^2)^{l-1} \right\|}{d(z,E)\, d(w,E)\, |z-w|^{2n-1}} < \infty.$$

Similarly, we have that

$$\int_z \int_w \frac{\left\|\bar{\partial} f(z) \wedge \bar{\partial} g(w) \wedge d_n\right\|}{d\,(z,E)\,d\,(w,E)\,|z-w|^{2n-1}} < \infty,$$

since $\|g\|_a < \infty$. Thus the statement follows from Proposition 3.1. $\square$

In order to prove that $f(a) \in (a)''$ we construct the resolvent $\omega_{z-a,w-b}$ and use the multiplicative property of the functional calculus of the tuple (a, b), where $b \in L(X)$ commutes with a.

Lemma 3.4. *If $f \in S_a$ then $f(a) \in (a)''$.*

Proof. Suppose that $b \in L(X)$ is an operator such that $ab = ba$. Define the form

$$v(w) = \frac{1}{2\pi i}\,(w-b)^{-1}\,dw$$

Define the sequence c_k by

$$c_1 = 0, \quad c_k = v \wedge s \wedge \left(\bar{\partial} s\right)^{k-2}.$$

Then we have the equations

$$c_1 = 0, \quad \delta_{z-a,w-b} c_2 = s - v$$

and

$$\delta_{z-a,w-b} c_{k+1} = s \wedge \left(\bar{\partial} s\right)^{k-1} - v \wedge \left(\bar{\partial} s\right)^{k-1} = \bar{\partial} c_k + s \wedge \left(\bar{\partial} s\right)^{k-1}.$$

Let χ be a smooth cutoff function such that $\{\chi, 1-\chi\}$ is a partition of unity subordinate the cover

$$\{\{(z,w) : z \notin E, |w| < 3\,\|b\|\}\,, \{(z,w) : |w| > 2\,\|b\|\}\}$$

of $\mathbb{C}^n \times \mathbb{C} \setminus E \times \{w : |w| \leq 2\,\|b\|\}$. This is a special choice of function χ used in Lemma 3.2 of [1] which enables us to avoid an integration by parts. Define the sequence a_k outside $E \times \{w : |w| \leq 2\,\|b\|\}$ by

$$a_1 = \chi s + (1-\chi)\,v, \quad a_k = \chi s \wedge \left(\bar{\partial} s\right)^{k-1} - \bar{\partial}\chi \wedge c_k.$$

We then have that

$$\delta_{z-a,w-b} a_1 = e, \quad \delta_{z-a,w-b} a_2 = \chi \bar{\partial} s + \bar{\partial}\chi \wedge (s - v) = \bar{\partial} a_1$$

and that

$$\delta_{z-a,w-b} a_{k+1} = \chi\left(\bar{\partial} s\right)^k + \bar{\partial}\chi \wedge \left(\bar{\partial} c_k + s \wedge \left(\bar{\partial} s\right)^{k-1}\right) = \bar{\partial} a_k,$$

and thus

$$a_{n+1} = -\bar{\partial}\chi \wedge v \wedge s \wedge \left(\bar{\partial} s\right)^{n-1}$$

represents $\omega_{z-a,w-b}$. Choose $\phi \in C_c^\infty(\mathbb{C})$ which is 1 in a neighbourhood of $\{w \in \mathbb{C} : |w| < 3\,\|b\|\}$. Then we have that

$$\begin{aligned}(\phi f)(a,b) &= -\int_w \int_z \bar{\partial}\left(\phi(w) f(z)\right) \wedge a_{n+1}(z,w) \\ &= \iint f \bar{\partial}_w \phi \wedge \bar{\partial}_z \chi \wedge v \wedge s \wedge \left(\bar{\partial} s\right)^{n-1}\end{aligned}$$

$$+\iint \phi\bar{\partial}_z f \wedge \bar{\partial}_w \chi \wedge v \wedge s \wedge \left(\bar{\partial} s\right)^{n-1} = -\int \bar{\partial} f \wedge s \wedge \left(\bar{\partial} s\right)^{n-1} = f(a).$$

Let $a_k^1 = \pi_1^* a_k$ and $a_k^2 = \pi_2^* a_k$, where

$$\pi_1(z_1, w_1, z_2, w_2) = (z_1, w_1) \text{ and } \pi_2(z_1, w_1, z_2, w_2) = (z_2, w_2).$$

Define the sequence c_i' by

$$c_1' = 0, \quad c_i' = \sum_{k+l=i} a_k^1 \wedge a_l^2$$

so that by Proposition 3.2,

$$c_1 = 0, \quad \delta_{z_1-a, w_1-b, z_2-a, w_2-b} c_{i+1}' = \bar{\partial} c_i' + a_i^2 - a_i^1.$$

Let $F = E \times \{w : |w| \leq 2\,\|b\|\}$. Define the function g by $g(z,w) = w\psi(z,w)$ where $\psi \in C_c^\infty$ is equal to 1 in a neighbourhood of F. We have that

$$\left\| \frac{\bar{\partial}(\phi f) \wedge a_k}{d\left((z,w), F\right)} \right\|_\infty$$
$$\leq \left\| \frac{\chi \bar{\partial}(\phi f) \wedge s \wedge \left(\bar{\partial} s\right)^{k-1}}{d(z,E)} \right\|_\infty$$
$$+ \left\| \frac{\bar{\partial}(\phi f) \wedge \bar{\partial}\chi \wedge v \wedge s \wedge \left(\bar{\partial} s\right)^{k-2}}{d(z,E)} \right\|_\infty < \infty$$

since $f \in S_a$. Hence we have that

$$\iint \frac{\left\|\bar{\partial}(\phi(w_1) f(z_1)) \wedge \bar{\partial} g(z_2, w_2) \wedge c_l'\right\|}{d\left((z_1, w_1), F\right) d\left((z_2, w_2), F\right) \left|(z_1, w_1) - (z_2, w_2)\right|^{2n+1}} < \infty$$

for all l. Define the forms c_l'' by the equations $c_l'' = c_l'$ if $l > n+1$ and

$$c_{n+1}'' = c_{n+1}' - \sum_{k+l=n+1} a_k^2 \wedge a_l^2.$$

Then we have that c_{n+1}'' satisfies $i^* c_{n+1}'' = 0$ and hence by Proposition 3.1 we have that $(\phi f)(a,b) g(a,b) = g(a,b)(\phi f)(a,b)$ since

$$\iint \frac{\left\|\bar{\partial}(\phi(w_1) f(z_1)) \wedge \bar{\partial} g(z_2, w_2) \wedge \sum_{k+l=n+1} a_k^2 \wedge a_l^2\right\|}{d\left((z_1, w_1), F\right) d\left((z_2, w_2), F\right) \left|(z_1, w_1) - (z_2, w_2)\right|^{2n+1}} < \infty.$$

Thus $f(a)b = bf(a)$ since $g(a,b) = b$ by the holomorphic functional calculus. □

We can now prove a generalisation of the holomorphic functional calculus.

Theorem 3.5. ***(Non-holomorphic functional calculus)*** *Suppose that a is an n-tuple of commuting operators and that $E \supset \sigma(a)$ is compact such that it exists a smooth form s defined on $\mathbb{C}^n \setminus E$ with*

$\delta_{z-a}s = e$ and $as = sa$. Let S_a be the class defined by (3.7) and let $f(a)$, $f \in S_a$, be the operator defined by

$$f(a) = -\int \bar{\partial} f \wedge s \wedge \left(\bar{\partial} s\right)^{n-1}.$$

Then we have that the map $f \mapsto f(a) : S_a \to (a)''$ is a continuous algebra homomorphism that continuously extends the map $f \mapsto f(a) : \mathcal{O}(E) \to (a)''$.

Proof. By Lemma 3.4 the map $f \mapsto f(a) : S_a \to (a)''$ is well defined. The map is continuous and linear. Lemma 3.3 gives that the map is multiplicative, and thus the map is an algebra homomorphism. To see that it continuously extends the map $f \mapsto f(a) : \mathcal{O}(E) \to (a)''$, suppose that we have a sequence $f_n \in \mathcal{O}(U)$, where U is an open neighbourhood of E, and that $f_n \to 0$ uniformly on compacts. Then

$$\|f_n \phi\|_a \to 0,$$

where $\phi \in C_c^\infty(U)$ is a function equal to 1 in a neighbourhood of E. $\square$

We now go on and prove the spectral mapping theorem for this functional calculus. To do this, we need the following lemma which shows that $f(w)$ acts as $f(a)$ on $H_p(w-a, c, X)$.

Lemma 3.6. *Suppose that there is an operator valued form s outside E such that $\delta_{z-a}s = e$ and $sa = as$. Furthermore, suppose that $c \in ((a)'')^m$, $w \in \sigma(a)$ and $k \in K_p(w-a, c, X)$ (with respect to a basis $dw_1, \ldots, dw_n, e_{n+1}, \ldots, e_{n+m}$ of $\mathbb{C}^{n+m}$) such that $\delta_{w-a,c}k = 0$. If $f \in S_a$, then*

$$(f(a) - f(w))\, k = \delta_{w-a,c} \int_z \bar{\partial} f(z) \wedge \sum_{l=1}^{n} m''_{n+1-l} \wedge s \wedge \left(\bar{\partial} s\right)^{l-1} \wedge k,$$

where m''_i is defined in the proof.

Proof. We have that

$$\delta_{z-a,w-a} m_1 = e, \quad \delta_{z-a,w-a} m_{i+1} = \bar{\partial} m_i,$$

by (2.7) and (2.8), where m_i is defined by (2.6). We also have that

$$\delta_{z-a,w-a} s = e, \quad \delta_{z-a,w-a}\left(s \wedge \left(\bar{\partial} s\right)^i\right) = \bar{\partial}_z \left(s \wedge \left(\bar{\partial} s\right)^{i-1}\right),$$

where s only depends on z. Therefore the same calculation as in the proof of Proposition 3.2 shows that

$$\delta_{z-a,w-a} \sum_{k+l=i+1} m_k \wedge s \wedge \left(\bar{\partial} s\right)^{l-1} - \bar{\partial} \sum_{k+l=i} m_k \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}$$

$$= s \wedge \left(\bar{\partial} s\right)^{i-1} - m_i.$$

Let $i = n$ and identify the component without any dw and $d\bar{w}$ in this expression to get,

$$\delta_{w-a} \sum_{k+l=n+1} m''_k \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}$$

$$= s \wedge (\bar{\partial} s)^{n-1} - m'_n + \bar{\partial}_z \sum_{k+l=n} m'_k \wedge s \wedge \left(\bar{\partial} s\right)^{l-1},$$

where

$$m'_k = \frac{1}{(2\pi i)^k} \frac{\partial_z |z-w|^2}{|z-w|^2} \wedge \left(\bar{\partial}_z \frac{\partial_z |z-w|^2}{|z-w|^2}\right)^{k-1}$$

and m''_k is the component of m_k with one dw and no $d\bar{w}$. Let χ_ε be the convolution of the characteristic function of the set

$$\{z : d(z, E) \geq 2\varepsilon\}$$

and the function $\varepsilon^{-2n}\rho(\cdot/\varepsilon)$, where ρ is a non-negative smooth function with compact support in the unit ball of $\mathbb{C}^n$ such that its integral is equal to 1. We have that

$$\int_z \bar{\partial}_z f(z) \wedge \bar{\partial}_z \sum_{k+l=n} m'_k \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}$$

$$= \lim_{\varepsilon \to 0} \int_z \chi_\varepsilon \bar{\partial}_z f(z) \wedge \bar{\partial}_z \sum_{k+l=n} m'_k \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}$$

$$= \lim_{\varepsilon \to 0} \int_z \bar{\partial}_z \chi_\varepsilon \wedge \bar{\partial}_z f(z) \wedge \sum_{k+l=n} m'_k \wedge s \wedge \left(\bar{\partial} s\right)^{l-1} = 0$$

since $\left|\bar{\partial}\chi_\varepsilon\right| \leq C\varepsilon^{-1}$ and $\left|\bar{\partial}\chi_\varepsilon\right|$ has support in $\varepsilon \leq d(z, E) \leq 3\varepsilon$. Hence we have that

$$f(a) - f(w) = \int_z \bar{\partial} f(z) \wedge \left(s \wedge \left(\bar{\partial} s\right)^{n-1} - m'_n\right)$$

$$= \delta_{w-a} \int_z \bar{\partial} f(z) \wedge \sum_{k+l=n+1} m''_k \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}.$$

Therefore,

$$\left(f(a) - f(w)\right) k = \delta_{w-a,c} \int_z \bar{\partial} f(z) \wedge \sum_{l=1}^{n} m''_{n+1-l} \wedge s \wedge \left(\bar{\partial} s\right)^{l-1} \wedge k,$$

since $(w - a, c)$ and s commute. □

We can now prove the spectral mapping theorem.

Theorem 3.7. ***(Spectral mapping theorem)*** *If f is tuple of functions in S_a, where S_a is defined by (3.7), then $\sigma(f(a)) = f(\sigma(a))$.*

Proof. Suppose that we can prove the statement; if $z \in \sigma(a)$ then $(z-a, f(a))$ is non-singular if and only if $f(z) \neq 0$. In that case $(z-a, w-f(a))$ is non-singular if and only if $w - f(z) \neq 0$ and hence

$$\sigma\left(f(a)\right) = \pi_2 \sigma\left(a, f(a)\right) = \pi_2 \left\{(z,w) : w = f(z), z \in \sigma(a)\right\} = f\left(\sigma(a)\right)$$

by Theorem 1.4.

Suppose that $z \in \sigma(a)$. We have the induction hypothesis that if m is a natural number then the tuple $(z-a, f(a))$ is non-singular if and only if $f(z) \neq 0$ for all m-tuples f of functions in S_a. The case $m = 0$ follows from Lemma 3.6. Assume that the hypothesis has been proved for m. Given $f' = (f_1, \ldots, f_{m+1})$ let $f = (f_1, \ldots, f_m)$. Then there is a long exact sequence

$$\ldots \to H_p(z-a, f(a), X) \to H_p(z-a, f'(a), X)$$

$$\to H_{p-1}(z-a, f(a), X) \xrightarrow{f_{m+1}(a)} H_{p-1}(z-a, f(a), X) \to \ldots,$$

for this see Taylor [15], Lemma 1.3. Lemma 3.6 gives that the last homomorphism is equal to $f_{m+1}(z)$. Hence

$$H_p(z-a, f'(a), X) = 0$$

if $f_{m+1}(z) \neq 0$ and

$$\operatorname{Im} H_p(z-a, f'(a), X) = H_{p-1}(z-a, f(a), X)$$

if $f_{m+1}(z) = 0$. Therefore the induction hypothesis hold for $m+1$ and hence the theorem follows. □

We now consider a case where we can answer all the question we set up in the beginning of this section. Denote by $ch(E)$ the convex hull of the set E.

Theorem 3.8. *Let h be a positive decreasing function on $[0, \infty)$. If there is differential form u^x on $\mathbb{C}^n \setminus ch\left(\sigma(a)\right)$ such that $\|u^x(z)\| \leq \|x\|\, h(d(z,E))$ then we define the class $S_h(a)$ by*

$$S_h(a) = \left\{f \in C_c^1(\mathbb{C}^n) : \left\| \left|\bar{\partial} f(z)\right| h\left(d\left(z, ch\left(\sigma(a)\right)\right)\right)\right\|_{L^\infty} < \infty\right\}.$$

Let the norm of functions in $S_h(a)$ be given by

$$\|f\|_{S_h(a)} = \left\| \left|\bar{\partial} f(z)\right| h\left(d\left(z, ch\left(\sigma(a)\right)\right)\right)\right\|_{L^1}.$$

Then the map

$$f \mapsto f(a) : S_h(a) \to (a), \text{ where } f(a)x = -\int \bar{\partial} f \wedge u^x,$$

is a continuous algebra homomorphism. If $f \in S_h(a)$ then $\sigma(f(a)) = f(\sigma(a))$ and $f(a) = 0$ if $f = 0$ on $ch(\sigma(a))$. Furthermore, if $f \in S_h(a)$, $g \in S_{h_1}(f(a))$ (or $g \in \mathcal{O}(\sigma(f(a)))$), where h_1 is a decreasing function such that $h(y/\sup|df|) \leq Ch_1(y)$, $y \in [0, \infty)$, and $g(0) = 0$ then $g(f(a)) = g \circ f(a)$.

Proof. Suppose that $0 \in ch(\sigma(a))$ and let $f_t(z) = f(tz)$ for $t < 1$. Since $h(d(z, ch(\sigma(a)))) \leq h(d(tz, ch(\sigma(a))))$ we see that $f_t \to f$ in $S_h(a)$ by dominated convergence. We know that all the conclusions in the theorem holds for functions that are holomorphic in a neighbourhood of the spectrum. Since $\|f(a)\|_{L(X)} \leq \|f\|_{S_h(a)}$ we will be able to prove the theorem using the approximation f_t. Consider especially the spectral mapping property. By Proposition 2.6 $\sigma(f_t(a))$ deforms continuously to $\sigma(f(a))$. Since $f_t(\sigma(a))$ also deforms continuously to $f(\sigma(a))$, we get that $\sigma(f(a)) = f(\sigma(a))$. For the composition rule we have that

$$\|g(f(a))x - g \circ f(a)x\| \leq \|g(f(a))x - g_s(f(a))x\|$$
$$+ \|g_s(f(a))x - g_s(f_t(a))x\| + \|g_s \circ f_t(a)x - g_s \circ f(a)x\|$$
$$+ \|g_s \circ f(a)x - g \circ f(a)x\| \leq \|g - g_s\|_{S_{h_1}(f(a))} \|x\|$$
$$+ \|g_s(f(a))x - g_s(f_t(a))x\| + \|g_s \circ f_t - g_s \circ f\|_{S_h(a)} \|x\|$$
$$+ \|g_s \circ f - g \circ f\|_{S_h(a)} \|x\| \to 0,$$

by Theorem 2.7 and

$$\left|\bar{\partial}\left(g_s \circ f - g \circ f\right)\right| h(d(z, ch(\sigma(a))))$$
$$\leq \left|\partial\left(g_s - g\right)\right| \left|\bar{\partial} f\right| h(d(z, ch(\sigma(a))))$$
$$+ \left|\bar{\partial}\left(g_s - g\right)\right| \left|\partial f\right| h_1(d(f(z), ch(f(\sigma(a))))).$$

□

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PROPERTY $(\beta)_{\mathcal{E}}$ FOR TOEPLITZ OPERATORS WITH H^∞-SYMBOL

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ABSTRACT. Suppose that g is a tuple of bounded holomorphic functions on a strictly pseudoconvex domain D in $\mathbb{C}^m$ with smooth boundary. Viewed as a tuple of operators on the Hardy space $H^p(D)$, $1 \le p < \infty$, g is shown to have property $(\beta)_{\mathcal{E}}$ and therefore g possess Bishop's property (β). In the case $m = 1$ it is proved that the same result also holds when $p = \infty$.

1. INTRODUCTION

Suppose that X is a Banach space and that $a = (a_1, \dots, a_n)$ is a commuting tuple of bounded linear operators on X. Let E be one of spaces X, $\mathcal{E}(\mathbb{C}^n, X)$ or $\mathcal{O}(U, X)$, where $U \subset \mathbb{C}^n$. Denote by $K_\bullet(z-a, E)$ the Koszul complex

$$0 \to \Lambda^n E \xrightarrow{\delta_{z-a}} \Lambda^{n-1} E \xrightarrow{\delta_{z-a}} \cdots \xrightarrow{\delta_{z-a}} \Lambda^0 E \to 0,$$

with boundary map

$$\delta_{z-a}(f s_I) = 2\pi i \sum_{k=1}^{p} (-1)^{k-1} (z_{i_k} - a_{i_k}) f s_{i_1} \wedge \cdots \wedge \widehat{s}_{i_k} \wedge \cdots \wedge s_{i_p},$$

where $I = (i_1, \dots, i_p)$ and p is an integer. Let $H_\bullet(z - a, E)$ be the corresponding homology groups.

The Taylor spectrum of a, $\sigma(a)$, is defined as the set of all $z \in \mathbb{C}^n$ such that $K_\bullet(z - a, X)$ is not exact. If for all Stein open sets U in $\mathbb{C}^n$ the natural quotient topology of $H_0(z - a, \mathcal{O}(U, X))$ is Hausdorff and $H_p(z - a, \mathcal{O}(U, X)) = 0$ for all $p > 0$, then a is said to have Bishop's property (β). It has property $(\beta)_{\mathcal{E}}$ if the natural quotient topology of $H_0(z - a, \mathcal{E}(\mathbb{C}^n, X))$ is Hausdorff and if $H_p(z - a, \mathcal{E}(\mathbb{C}^n, X)) = 0$ for all $p > 0$.

By Theorem 6.2.4 in [9], the tuple a has Bishop's property (β) if and only if there exists a decomposable resolution, that is, if and only if there are Banach spaces X_i and decomposable tuples (see [9] for the definition) of operators a_i on X_i such that

$$0 \to X \xrightarrow{d} X_0 \xrightarrow{d} \cdots \xrightarrow{d} X_r \to 0$$

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is exact, $da = a_0 d$ and $da_i = a_{i+1} d$. Property $(\beta)_{\mathcal{E}}$ is equivalent to the existence of a resolution of Fréchet spaces with Mittag-Leffler inverse limit of generalized scalar tuples (that is tuples which admit a continuous $C^\infty(\mathbb{C}^n)$-functional calculus), see Theorem 6.4.15 in [9]. Property $(\beta)_{\mathcal{E}}$ implies Bishop's property (β), see [9].

Suppose that D is a strictly pseudoconvex domain in $\mathbb{C}^m$ with smooth boundary. We consider the tuple $T_g = (T_{g_1}, \dots, T_{g_n})$, $g_k \in H^\infty(D)$, of operators on $H^p(D)$ defined by $T_{g_k} f = g_k f$, $f \in H^p(D)$. The main theorem of this paper is the following.

Theorem 1.1. *Suppose that D is a bounded strictly pseudoconvex domain in $\mathbb{C}^m$ with C^∞-boundary and that $g \in H^\infty(D)^n$. Then the tuple T_g of Toeplitz operators on $H^p(D)$, $1 \le p < \infty$, satisfies property $(\beta)_{\mathcal{E}}$, and thus Bishop's property (β).*

In case g has bounded derivative this theorem has previously been proved in [14, 16, 17]. In case D is the unit disc in $\mathbb{C}$, Theorem 1.1 also holds when $p = \infty$; this is proved in Section 4. As a corollary to Theorem 1.1 we have that T_g on the Bergman space $\mathcal{O}L^p(D)$ has property $(\beta)_{\mathcal{E}}$, see Corollary 3.4.

Let us recall how one can prove that T_g on the Bergman space $\mathcal{O}L^2(D)$ has property $(\beta)_{\mathcal{E}}$ under the extra assumption that g has bounded derivative. Define the Banach spaces B_k as the spaces of locally integrable $(0,k)$-forms u such that

$$\|u\|_{B_k} := \|u\|_{L^2(D)} + \left\|\bar\partial u\right\|_{L^2(D)} < \infty.$$

Since g has bounded derivate we have the inequality

$$\|(\varphi \circ g)u\|_{B_k} \lesssim \sup_{z \in g(D)} \left(|\varphi(z)| + \left|\bar\partial \varphi(z)\right|\right) \|u\|_{B_k}$$

for all $\varphi \in C^\infty(\mathbb{C}^n)$. Hence $\varphi \mapsto T_{\varphi \circ g}$ is a continuous $C^\infty(\mathbb{C}^n)$-functional calculus, where $T_{\varphi\circ g}$ denotes multiplication by $\varphi \circ g$ on B_k. Since we have the resolution

$$0 \to \mathcal{O}L^2(D) \to B_0 \xrightarrow{\bar\partial} B_1 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} B_m \to 0$$

by Hörmander's L^2-estimate of the $\bar\partial$ equation, the tuple T_g on $\mathcal{O}L^2(D)$ has property $(\beta)_{\mathcal{E}}$ by the above mentioned Theorem 6.4.15 in [9].

To prove Theorem 1.1 we will construct a complex

$$0 \to H^p(D) \xrightarrow{i} B_0 \xrightarrow{\bar\partial} B_1 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} B_m \to 0, \tag{1}$$

where B_k are Banach spaces of $(0,k)$-forms on D. The spaces B_k are defined in terms of tent norms. We prove that $\varphi \mapsto T_{\varphi\circ g}$ is a continuous $C^\infty(\mathbb{C}^n)$-functional calculus, where $T_{\varphi\circ g}$ denotes multiplication by $\varphi \circ g$ on B_k. If the complex (1) were exact the proof of Theorem 1.1 would be finished. As we can solve the $\bar\partial$-equation with appropiate estimates we will be able to prove that T_g on H^p has property $(\beta)_{\mathcal{E}}$ anyway. More precisely (1) is exact at B_k, $k \ge 3$. If $f \in B_2$ and $\bar\partial f = 0$ then there is a function u in another Banach space B_1' such that $\bar\partial u = f$.

Mutiplication by g is a bounded operator on B_1'. If $f \in B_1$ and $f' \in B_1'$ such that $\bar{\partial} f + \bar{\partial} f' = 0$ then there is a solution $u \in L^p(\partial D)$ to the equation $\bar{\partial}_b u = f + f'$.

The construction of the complex (1) in the case $p < \infty$ is inspired by the construction in [5] and in the case $p = \infty$ and $m = 1$ it is inspired by Tom Wolff's proof of the corona theorem. Let us recall the proof of the H^p-corona theorem in the unit disc of $\mathbb{C}$. Suppose that $g = (g_1, \ldots, g_n) \in H^\infty(D)^n$, where D is the unit disc in $\mathbb{C}$, and that $0 \notin \overline{g(D)}$. Consider the complex (1); the definitions of the B_k-spaces can be found in the beginning of Section 3 and Section 4. Suppose that $f \in H^p(D)$. Then the equation $\delta_g u_1 = f$ has a solution in $K_1(g, B_0)$, namely $u_1 = \sum_k \bar{g}_k f s_k / |g|^2$. Hence $\delta_g \bar{\partial} u_1 = 0$ as δ_g and $\bar{\partial}$ anticommute, and we can solve the equation $\delta_g u_2 = \bar{\partial} u_1$ by defining $u_2 \in K_2(g, B_1)$ as $u_1 \wedge \bar{\partial} u_1$. Since u_2 satisfies the condition

$$\|(1-|z|)\, u_2\|_{T_2^p} + \left\|(1-|z|)^2\, \partial u_2\right\|_{T_1^p} < \infty,$$

by a Wolff type estimate there is a solution v in $K_2(g, L^p(\partial D))$ to the equation $\bar{\partial}_b v = u_2$ (here T_2^p and T_1^p denote certain tent spaces). Let $u_1' = u_1^* - \delta_g v \in K_1(g, L^p(\partial D))$, where u_1^* is the boundary values of u_1. Since $\bar{\partial}_b u_1' = 0$ there is a holomorphic extension U_1' of u_1' to D which satisfies the equation $\delta_g U_1' = f$.

The above proof also yields that $\sigma(T_g) = \overline{g(D)}$; the exactness of higher order in the Koszul complex follows by similar resoning. That $\sigma(T_g) = \overline{g(D)}$ is proved in [5] for the case D strictly pseudoconvex and $p < \infty$. One main difference of the proof of that T_g has property $(\beta)_{\mathcal{E}}$ and the proof of that $\sigma(T_g) = \overline{g(D)}$ is the following. As a substitution of the explicit choices of u_1 and u_2 one uses the fact that T_g considered as an operator on B_k has property $(\beta)_{\mathcal{E}}$, which in turn follows from the fact that T_g on B_k has a $C^\infty(\mathbb{C}^n)$-functional calculus.

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2. Preliminaries

Suppose that D is a bounded strictly pseudoconvex domain in $\mathbb{C}^m$ with C^∞-boundary given by a strictly plurisubharmonic defining function ρ. Let $r = -\rho$. All norms below are with respect to the metric

$$\Omega = r i \partial \bar{\partial} \log(1/r),$$

and we have

$$|f|^2 \sim r^2 |f|_\beta^2 + r\, |f \wedge \partial r|_\beta^2 + r \left| f \wedge \bar{\partial} r \right|_\beta^2 + \left| f \wedge \partial r \wedge \bar{\partial} r \right|_\beta^2,$$

where $\beta = i\partial\bar{\partial} r$, which is equivalent to the Euclidean metric.

The Hardy space H^p is the Banach space of all holomorphic functions, f, on D such that

$$\|f\|_{H^p} = \sup_{\varepsilon>0} \int_{r(z)=\varepsilon} |f(z)|^p \, d\sigma(z) < \infty,$$

where σ is the surface measure. It is wellknown that a function u in $L^p(\partial D)$ is the boundary value of a function U in H^p if and only

$$\int_{\partial D} uh = 0$$

for all $h \in C^\infty_{n,n-1}(\bar{D})$ such that $\bar{\partial} h = 0$.

Let $d(\cdot,\cdot)$ be the Koranyi pseudometric on ∂D and let z' be the point on ∂D closest to $z \in D_\varepsilon$, where D_ε is a small enough neighbourhood of ∂D in D. For a point ζ on the boundary let

$$A_\zeta = \{z \in D_\varepsilon : d(z',\zeta) < r(z)\} \cup (D \setminus D_\varepsilon).$$

For a ball B defined by $B = \{z \in \partial D : d(z,\zeta) < t\}$ let, for small t,

$$\hat{B} = \{z \in D_\varepsilon : d(z',\zeta) < t - r(z)\},$$

and for large t

$$\hat{B} = \{z \in D_\varepsilon : d(z',\zeta) < t - r(z)\} \cup (D \setminus D_\varepsilon).$$

A function f is in the tent space T^p_q, where $p < \infty$ and $q < \infty$, if

$$\|f\|_{T^p_q} := \left(\int_{\partial D} \left(\int_{z\in A_\zeta} |f(z)|^q \, r(z)^{-m-1} \right)^{p/q} d\sigma(\zeta) \right)^{1/p} < \infty.$$

The function f is in T^p_∞ if f is continuous with limits along A_ζ at the boundary almost everywhere and such that

$$\|f\|_{T^p_\infty} := \left(\int_{\partial D} \sup_{z\in A_\zeta} |f(z)|^p \, d\sigma(\zeta) \right)^{1/p} < \infty.$$

A function f is in T^∞_q if

$$\|f\|_{T^\infty_q} := \left\| \sup_{\cdot \in B} \left(\frac{1}{|B|} \int_{z\in\hat{B}} |f(z)|^q \, r(z)^{-1} \right)^{1/q} \right\|_{L^\infty(\partial D)} < \infty.$$

Note that $f \in T^p_p$ if and only if $r^{-1/p} f \in L^p(D)$ by Fubini's theorem. From [8] we have the inequality

$$\int_D |fg| \, r^{-1} \lesssim \|f\|_{T^p_q} \, \|g\|_{T^{p'}_{q'}} \tag{2}$$

for $1 \le p, q \le \infty$, where p' and q' denote dual exponents. By [8] $T^{p'}_{q'}$, where $1 \le p < \infty$ and $1 < q < \infty$, is the dual of T^p_q with respect to the pairing

$$\langle f, g\rangle \to \int_D fgr^{-1}.$$

Suppose that $f \in T^p_{q_0}$, $g \in T^\infty_{q_1}$ and let $q = \left(q_0^{-1} + q_1^{-1}\right)^{-1}$. Then for all $h \in T^{p'}_{q'}$ we have

$$\int_D |fgh|\, r^{-1} \lesssim \|fh\|_{T^1_{q_1'}} \|g\|_{T^\infty_{q_1}} \le \|f\|_{T^p_{q_0}} \|g\|_{T^\infty_{q_1}} \|h\|_{T^{p'}_{q'}}$$

by (2) and Hölder's inequality. Thus by the duality for $T^{p'}_{q'}$ we get the inequality

$$\|fg\|_{T^p_q} \lesssim \|f\|_{T^p_{q_0}} \|g\|_{T^\infty_{q_1}} \tag{3}$$

for $1 < p$ and $1 < q < \infty$. Since the inequality (3) is equivalent to

$$\|fg\|_{T^{tp}_{tq}} \lesssim \|f\|_{T^{tp}_{tq_0}} \|g\|_{T^\infty_{tq_1}}$$

for $0 < t < \infty$, (3) holds if $0 < p, q_0, q_1$.

We will use the inequality (see [12])

$$\|f\|_{T^p_\infty} \lesssim \|f\|_{H^p}, \quad p > 0 \tag{4}$$

and (see e.g. [7] for $p < \infty$ and [3] for $p = \infty$)

$$\left\|r^{1/2}\partial f\right\|_{T^p_2} \lesssim \|f\|_{H^p}, \quad p > 0. \tag{5}$$

Moreover, we use that $|\partial f| \lesssim r^{-1/2}$ if $f \in H^\infty$.

There is an integral operator $K : C^\infty_{0,q+1}(\bar D) \to C_{0,q}(\bar D)$, $q \ge 0$, see [5], such that $\bar\partial K u + K\bar\partial u = u$, $u \in C^\infty_{0,s}(\bar D)$, $s \ge 1$,

$$\|r^\tau K u\|_{T^p_1} \lesssim \left\|r^{\tau+1/2} u\right\|_{T^p_1} \text{ and } \|Ku\|_{L^p(\partial D)} \lesssim \left\|r^{1/2} u\right\|_{T^p_1} \tag{6}$$

if $\tau > 0$ and $1 \le p < \infty$. Furthermore,

$$\|Ku\|_{L^p(\partial D)} \lesssim \left\|r^{1/2} u\right\|_{T^p_2} + \|r\partial u\|_{T^p_1}. \tag{7}$$

To see that the inequality (6) follows from [5], note that by the definition of $W^{1-1/p}$ in [1], $\|ru\|_{T^p_1} = \|u\|_{W^{1-1/p}}$. By [4] the adjoint P of K satisfies

$$\|P\psi\|_{L^\infty(D)} \lesssim \|\psi\|_{L^\infty(\partial D)} \text{ and } \left\|r^{1/2}\mathcal{L}P\psi\right\|_{L^2(D)} \lesssim \|\psi\|_{L^2(\partial D)}$$

(where $\mathcal{L}$ is an arbitrary smooth $(1,0)$-vectorfield). The L^2-result is proven by means of a $T1$-theorem of Christ and Journé. By [10] it now follows that

$$\|P\psi\|_{T^p_\infty} \lesssim \|\psi\|_{L^p(\partial D)}, \quad p > 1, \tag{8}$$

and

$$\|r\mathcal{L}P\psi\|_{T^p_2} \lesssim \|\psi\|_{L^p(\partial D)}, \quad p > 1. \tag{9}$$

The inequality (7) follows from (8) and (9).

In section 4 we use completed tensor products of locally convex Hausdorff spaces, see e.g. Appendix 1 in [9]. Suppose that E and F are locally convex Hausdorff spaces. We denote by $L(E,F)$ the space of all continuous and linear maps from E to F. The topology π on $E\otimes F$ is defined as the finest locally convex topology such that the canonical bilinear map $E\times F\to E\otimes F$ is continuous. We denote by $E\otimes_\pi F$, the space $E\otimes F$ with the topology π and we denote the completion of $E\otimes_\pi F$ with $E\hat{\otimes}_\pi F$. There is another topology on $E\otimes F$, the topology ϵ; in case E is nuclear this topology coincides with the topology π and we therefore omit the index π in this case. The Fréchet space $\mathcal{E}(\mathbb{C}^n)$ is nuclear and we have the isomorphism $\mathcal{E}\left(\mathbb{C}^n,E\right)\cong\mathcal{E}\left(\mathbb{C}^n\right)\hat{\otimes}E$.

3. Property $(\beta)_{\mathcal{E}}$ for Toeplitz operators with H^∞-symbol on H^p

First we need to define the sequence (1) and prove that there is a continuous $C^\infty(\mathbb{C}^n)$-functional calculus on each of the spaces B_k.

Define the norms $\|\cdot\|_{B_k}$, $k\geq 0$, by

$$\|u\|_{B_0}=\|u\|_{T^p_\infty}+\left\|r^{1/2}du\right\|_{T^p_2}+\left\|r\partial\bar{\partial}u\right\|_{T^p_1} \tag{10}$$

on $C^\infty(\bar{D})$,

$$\|u\|_{B_1}=\left\|r^{1/2}u\right\|_{T^p_2}+\|rdu\|_{T^p_1} \tag{11}$$

on $C^\infty_{0,1}(\bar{D})$ and

$$\|u\|_{B_k}=\left\|r^{k/2}u\right\|_{T^p_1}+\left\|r^{k/2+1/2}\bar{\partial}u\right\|_{T^p_1} \tag{12}$$

on $C^\infty_{0,k}(\bar{D})$ for $k\geq 2$. Let B_k be the completion of $C^\infty_{0,k}\left(\bar{D}\right)$ with respect to the norm $\|\cdot\|_{B_k}$. We also define B_1' as the completion of $C^\infty_{0,1}(\bar{D})$ with respect to the norm $\|\cdot\|_{B_1'}$, defined by

$$\|u\|_{B_1'}=\left\|r^{1/2}u\right\|_{T^p_1}+\left\|r\bar{\partial}u\right\|_{T^p_1}.$$

The injection $i:H^p\to B_0$ is well defined and continuous by (4) and (5). That $\bar{\partial}:B_k\to B_{k+1}$, $k\geq 0$ is continuous follows immediately from the definitions. Thus we have defined a complex

$$0\to H^p(D)\xrightarrow{i}B_0\xrightarrow{\bar{\partial}}B_1\xrightarrow{\bar{\partial}}\cdots\xrightarrow{\bar{\partial}}B_m\to 0. \tag{13}$$

Lemma 3.1. *Suppose that $g\in H^\infty(D)^n$. Then one can define $T_{g_i}:B_k\to B_k$ by $T_{g_i}u=g_iu$, $1\leq i\leq n$, for all $k\geq 0$. The tuple T_g on B_k, $k\geq 0$, has a continuous $C^\infty(\mathbb{C}^n)$-functional calculus and property $(\beta)_{\mathcal{E}}$.*

Proof. That T_{g_i} can be defined on B_k follows from the calculation below (let $\varphi(z) = z_i$ below). We begin with the case $k = 0$. Suppose that $\varphi \in C^\infty(\mathbb{C}^n)$ and $u \in C^\infty(\bar{D})$. From (3) we have

$$\left\| r^{1/2} u \partial g \right\|_{T_2^p} \lesssim \|u\|_{T_\infty^p} \left\| r^{1/2} \partial g \right\|_{T_2^\infty},$$

$$\|r\, |du|\, |\partial g|\|_{T_1^p} \lesssim \left\| r^{1/2} du \right\|_{T_2^p} \left\| r^{1/2} \partial g \right\|_{T_2^\infty}$$

and

$$\left\| r u \left|\partial g\right|^2 \right\|_{T_1^p} \lesssim \|u\|_{T_\infty^p} \left\| r \left|\partial g\right|^2 \right\|_{T_1^\infty}.$$

Since $\left\| r^{1/2} \partial g \right\|_{T_2^\infty} < \infty$ by the inequality (5) we thus get

$$\|(\varphi \circ g) u\|_{B_0} \leq \sup_{z \in g(D)} |\varphi(z)|\, \|u\|_{B_0} + \left\| r^{1/2} d(\varphi \circ g) u \right\|_{T_2^p} +$$
$$\left\| r \bar{\partial}(\varphi \circ g) \wedge \partial u \right\|_{T_1^p} + \left\| r \partial(\varphi \circ g) \wedge \bar{\partial} u \right\|_{T_1^p} + \left\| r \partial\bar{\partial}(\varphi \circ g) u \right\|_{T_1^p} \lesssim$$
$$\sup_{z \in g(D)} \left(|\varphi(z)| + |D\varphi(z)| + \left|D^2 \varphi(z)\right| \right) \|u\|_{B_0},$$

where $D\varphi$ and $D^2\varphi$ denotes all derivates of φ of order 1 and 2 respectively. Note that $(\varphi \circ g) u \notin C^\infty(\bar{D})$ in general. Let $g_l \in C^\infty(\bar{D})^n \cap \mathcal{O}(D)^n$ be such that $g_l \to g$ in $H^p(D)^n$ with g_l uniformly bounded as $l \to \infty$ and suppose that u is fixed. We have the equalities

$$d(\varphi \circ g_l - \varphi \circ g) = \sum_i \varphi_i \circ g_l \partial g_l^i - \varphi_i \circ g \partial g^i + \varphi_{\bar{i}} \circ g_l \overline{\partial g_l^i} - \varphi_{\bar{i}} \circ g \overline{\partial g^i}$$

and

$$\partial\bar{\partial}(\varphi \circ g_l - \varphi \circ g) = \sum_{i,j} \varphi_{\bar{i}j} \circ g_l \partial g_l^j \wedge \overline{\partial g_l^i} - \varphi_{\bar{i}j} \circ g \partial g^j \wedge \overline{\partial g^i},$$

where the index in φ_i denotes partial derivate and the upper index in g_l^i and g^i denotes i:th component. Hence we get

$$\left| d\left(\varphi \circ g_l - \varphi \circ g\right) \right| \leq |D\varphi \circ g_l|\, |\partial g_l - \partial g| + |D\varphi \circ g_l - D\varphi \circ g|\, |\partial g|,$$

and

$$\left| \partial\bar{\partial}\left(\varphi \circ g_l - \varphi \circ g\right) \right| \leq \left| D^2 \varphi \circ g_l \right| |\partial g_l - \partial g| \left(|\partial g_l| + |\partial g| \right) +$$
$$\left| D^2 \varphi \circ g_l - D^2 \varphi \circ g \right| |\partial g|^2.$$

By (4) we have

$$\left\| \left(\varphi \circ g_l - \varphi \circ g\right) u \right\|_{T_\infty^p} + \left\| r^{1/2} \left(\varphi \circ g_l - \varphi \circ g\right) du \right\|_{T_2^p}$$
$$\left\| r \left(\varphi \circ g_l - \varphi \circ g\right) \partial\bar{\partial} u \right\|_{T_1^p} \lesssim \|\varphi \circ g_l - \varphi \circ g\|_{T_\infty^p} \lesssim \|g_l - g\|_{T_\infty^p} \lesssim$$
$$\|g_l - g\|_{H^p}.$$

We also have that

$$\left\| r^{1/2} d\left(\varphi \circ g_l - \varphi \circ g\right) u \right\|_{T_2^p} + \left\| r \left| d\left(\varphi \circ g_l - \varphi \circ g\right) \right| |du| \right\|_{T_1^p} \lesssim$$
$$\left\| r^{1/2} d\left(\varphi \circ g_l - \varphi \circ g\right) \right\|_{T_2^p} \lesssim \left\| r^{1/2} \left| D\varphi \circ g_l \right| |\partial g_l - \partial g| \right\|_{T_2^p} +$$

$$\left\|r^{1/2}\left|D\varphi\circ g_l-D\varphi\circ g\right|\left|\partial g\right|\right\|_{T_2^p}\lesssim\left\|g_l-g\right\|_{H^p}$$

by (3),(4) and (5). Furthermore,

$$\left\|r\partial\bar\partial\left(\varphi\circ g_l-\varphi\circ g\right)u\right\|_{T_1^p}\lesssim\left\|r\left|D^2\varphi\circ g_l\right|\left|\partial g_l-\partial g\right|\left(\left|\partial g_l\right|+\left|\partial g\right|\right)\right\|_{T_1^p}+$$

$$\left\|r\left|D^2\varphi\circ g_l-D^2\varphi\circ g\right|\left|\partial g\right|^2\right\|_{T_1^p}\lesssim\left\|g_l-g\right\|_{H^p}$$

by (3),(4) and (5). Thus

$$\left\|\left(\varphi\circ g_l-\varphi\circ g\right)u\right\|_{B_0}\to 0$$

as $l\to\infty$ and therefore we have that $(\varphi\circ g)u$ is in the completion of $C^\infty(\bar D)$ with respect to the norm $\|\cdot\|_{B_0}$. We extend the map

$$u\mapsto(\varphi\circ g)u:C^\infty(\bar D)\to B_0$$

to a continuous map $\varphi(T_g):B_0\to B_0$, bounded by a constant times

$$\sup_{z\in g(D)}\left(|\varphi(z)|+|D\varphi(z)|+\left|D^2\varphi(z)\right|\right).$$

Hence T_g on B_0 has a continuous $C^\infty\left(\mathbb{C}^n\right)$-functional calculus.

Next we consider the case $k=1$. Suppose that $\varphi\in C^\infty(\mathbb{C}^n)$ and $u\in C^\infty_{0,1}(\bar D)$. From (3) and (5) we have the inequality

$$\left\|r\left|\partial g\right|\left|u\right|\right\|_{T_1^p}\lesssim\left\|r^{1/2}\partial g\right\|_{T_2^\infty}\left\|r^{1/2}u\right\|_{T_2^p}\lesssim\left\|r^{1/2}u\right\|_{T_2^p}.$$

Hence we get

$$\|(\varphi\circ g)u\|_{B_1}\leq\sup_{z\in g(D)}|\varphi(z)|\,\|u\|_{B_1}+\|rd(\varphi\circ g)\wedge u\|_{T_1^p}\lesssim$$

$$\sup_{z\in g(D)}\left(|\varphi(z)|+|D\varphi(z)|\right)\|u\|_{B_1}.$$

As in the case $k=0$ we prove that $(\varphi\circ g)u$ is in the completion of $C^\infty_{0,1}(\bar D)$. When we extend the map

$$u\mapsto(\varphi\circ g)u:C^\infty(\bar D)\to B_1$$

by continuity to a map $\varphi(T_g):B_1\to B_1$ bounded by

$$\sup_{z\in g(D)}\left(|\varphi(z)|+|D\varphi(z)|\right)$$

and hence we have proved that T_g on B_1 has a $C^\infty\left(\mathbb{C}^n\right)$-functional calculus.

In case $k\geq 2$ we suppose that $\varphi\in C^\infty(\mathbb{C}^n)$ and $u\in C^\infty_{0,k}(\bar D)$. Since $|\partial g|\lesssim r^{-1/2}$ we have

$$\|(\varphi\circ g)u\|_{B_k}\leq\sup_{z\in g(D)}|\varphi(z)|\,\|u\|_{B_k}+\left\|r^{k/2+1/2}\bar\partial(\varphi\circ g)\wedge u\right\|_{T_1^p}\lesssim$$

$$\sup_{z\in g(D)}\left(|\varphi(z)|+|D\varphi(z)|\right)\|u\|_{B_k}.$$

As in the case $k=0$ it follows that T_g on B_k, $k\geq 2$, has a $C^\infty\left(\mathbb{C}^n\right)$-functional calculus.

That each of the tuples T_g has property $(\beta)_{\mathcal{E}}$ now follows from Proposition 6.4.13 in [9]. □

We can extend the integral operator $K : C^\infty_{0,k+1}(\bar{D}) \to C_{0,k}(\bar{D})$, $k \geq 1$, to a continuous operator $K : B_{k+1} \to B_k$, $k \geq 2$, and a continuous operator $K : B_2 \to B_1'$. This because

$$\left\| r^{k/2} K u \right\|_{T_1^p} \lesssim \left\| r^{k/2+1/2} u \right\|_{T_1^p} \leq \|u\|_{B_{k+1}} \tag{14}$$

and

$$\left\| r^{k/2+1/2} \bar{\partial} K u \right\|_{T_1^p} = \left\| r^{k/2+1/2} (u - K \bar{\partial} u) \right\|_{T_1^p} \lesssim \|u\|_{B_{k+1}}$$

for all $u \in C^\infty_{0,k+1}(\bar{D})$ by (6), (12) and (14). Also observe that Ku is in the completion of $C^\infty_{0,k}(\bar{D})$ under the norm $\|\cdot\|_{B_k}$ (or $\|\cdot\|_{B_1'}$) by dominated convergence and the fact that one can find $f_l \in C^\infty_{0,k}(\bar{D})$ such that $f_l \to Ku, \bar{\partial} f_l \to \bar{\partial} K u$ pointwise and $|f_l|, \left|\bar{\partial} f_l\right| \lesssim 1$ (as $Ku, \bar{\partial} K u \in C(\bar{D})$). Approximation in B_{k+1} yields that $\bar{\partial} K u + K \bar{\partial} u = u$ for all $u \in B_{k+1}$, $k \geq 1$. Thus the complex (13) is exact in higher degrees.

Extend $K : C^\infty_{0,1}(\bar{D}) \to C(\partial D)$ to continuous maps $K : B_1 \to L^p(\partial D)$ and $K : B_1' \to L^p(\partial D)$, which is possible by (6) and (7). Define the $(1,0)$-vector field $\mathcal{L}$ by the equation

$$\mathcal{L} = \chi \sum |\partial r|^{-2} \frac{\partial r}{\partial \bar{z}_k} \frac{\partial}{\partial z_k},$$

where χ is equal to 1 in a neighbourhood of ∂D and 0 on the set where $\partial r = 0$. Suppose that $u \in C^\infty(\bar{D})$ and let $f = \bar{\partial} u$. By integration by parts we have

$$\int_{\partial D} uh = \int_D f \wedge h =: V(f,h)$$

and

$$\int_{\partial D} uh = \int_D f \wedge h = \int_D O(r) f \wedge h + \int_D r \mathcal{L} (f \wedge h) =: W(f,h)$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar{\partial} h = 0$. We extend V to elements f in B_1' and W to elements in B_1. We say that the equation $\bar{\partial}_b u = f + f'$, where $u \in L^p(\partial D), f \in B_1$ and $f' \in B_1'$, holds if and only if

$$\int_{\partial D} uh = W(f,h) + V(f',h)$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar{\partial} h = 0$.

Lemma 3.2. *If $f \in B_1$, $f' \in B_1'$ and $\bar{\partial} f + \bar{\partial} f' = 0$ then $u = Kf + Kf'$ solves the equation $\bar{\partial}_b u = f + f'$. Moreover, if $\varphi \in H^\infty(D)$ then $\bar{\partial}_b(\varphi u) = T_\varphi f + T_\varphi f'$.*

Proof. Suppose that $f, f' \in C^\infty_{0,1}(\bar D)$. Since $\bar\partial K(f+f') + K\bar\partial(f+f') = f+f'$ we have

$$(15)\quad \int_{\partial D}(Kf+Kf')h = W(f,h)+V(f',h) - \int_D K(\bar\partial f+\bar\partial f')\wedge h$$

for all $h \in C^\infty_{m,m-1}(\bar D)$ such that $\bar\partial h = 0$. For fixed h, we can estimate each term of the above equality by a constant times $\|f\|_{B_1} + \|f'\|_{B_1'}$. Thus approximation in B_1 and B_1' yields that if $f \in B_1$ and $f' \in B_1'$ then

$$\int_{\partial D} uh = W(f,h)+V(f',h) - \int_D K(\bar\partial f+\bar\partial f')\wedge h$$

for all $h \in C^\infty_{m,m-1}(\bar D)$ such that $\bar\partial h = 0$. Hence the equation $\bar\partial_b u = f+f'$ holds since we also have that $\bar\partial f + \bar\partial f' = 0$. Suppose that $\varphi_k \in C^\infty(\bar D)\cap \mathcal{O}(D)$ are chosen such that $\varphi_k \to \varphi$ in $H^1(D)$. Replace h in (15) by $\varphi_k h$ and approximate to get

$$\int_{\partial D}\varphi\,(Kf+Kf')\,h = W(f,h\varphi)+V(f',h\varphi) - \int_D \varphi K(\bar\partial f+\bar\partial f')\wedge h$$

for all $h \in C^\infty_{m,m-1}(\bar D)$ such that $\bar\partial h = 0$, if $f, f' \in C^\infty_{0,1}(\bar D)$. We estimate the terms to the right,

$$|W(f,h\varphi)| \lesssim \int_D r^{3/2}\,|f|\,|\varphi|\,r^{-1} + \int_D r\,|\partial f|\,|\varphi|\,r^{-1} + \int_D r\,|f|\,|\partial\varphi|\,r^{-1} \lesssim$$

$$\|f\|_{B_1}\,\|\varphi\|_{H^{p'}},$$

$$|V(f',h\varphi)| \lesssim \int_D r^{1/2}\,|f'|\,|\varphi|\,r^{-1} \lesssim \|f'\|_{B_1'}\,\|\varphi\|_{H^{p'}}$$

and

$$\left|\int_D \varphi K(\bar\partial f+\bar\partial f')\wedge h\right| \lesssim \left\|r^{1/2}K\left(\bar\partial f+\bar\partial f'\right)\right\|_{T_1^p}\|\varphi\|_{T_\infty^{p'}} \lesssim$$

$$\left\|\bar\partial f+\bar\partial f'\right\|_{B_2}\|\varphi\|_{H^{p'}} \lesssim \left(\|f\|_{B_1}+\|f'\|_{B_1'}\right)\|\varphi\|_{H^{p'}}$$

for fixed h by (2), (4) and (5). Hence approximation in B_1 and B_1' yields that

$$\int_{\partial D} u\varphi h = W(T_\varphi f,h)+V(T_\varphi f',h)$$

for all $f \in B_1$, $f' \in B_1'$ such that $\bar\partial f + \bar\partial f' = 0$ and $h \in C^\infty_{m,m-1}(\bar D)$ such that $\bar\partial h = 0$. □

Next we prove that functions in B_0 has boundary values in $L^p(\partial D)$.

Lemma 3.3. *There is a continuous and linear operator $u \mapsto u^*$ from B_0 to $L^p(\partial D)$ such that u^* is the restriction of u to ∂D if $u \in C^\infty(\bar D)$ and $(T_f u)^* = f^* u^*$ if $f \in H^\infty(D)$.*

Proof. Suppose that $u \in C^\infty(\bar{D})$. Then $\|u\|_{L^p(\partial D)} \le \|u\|_{B_0}$ and hence the restriction operator can be extended to a continuous operator from B_0 to $L^p(\partial D)$. Suppose that $u \in B_0$ and $f \in H^\infty(D)$. Let $u_l \in C^\infty(\bar{D})$ and $f_k \in C^\infty(\bar{D}) \cap \mathcal{O}(D)$ be such that $u_l \to u$ in B_0 and $f_k \to f$ in $H^p(D)$ with f_k uniformily bounded. Then

$$\|f^*u^* - (T_f u)^*\|_{L^p(\partial D)} \lesssim \|f^*u^* - f^*u_l^*\|_{L^p(\partial D)} + \|f^*u_l^* - f_k^*u_l^*\|_{L^p(\partial D)} + \|(f_k u_l)^* - (fu_l)^*\|_{L^p(\partial D)} + \|(fu_l)^* - (T_f u)^*\|_{L^p(\partial D)} \to 0$$

if one first let $k \to \infty$ and then $l \to \infty$. □

Note that if $u \in B_0$ then

$$\int_{\partial D} u^* h = W(\bar{\partial} u, h) \qquad (16)$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar{\partial} h = 0$ by approximation in B_0 and Lemma 3.3.

Proof of Theorem 1.1

We want to prove that the complex $K_\bullet(z - T_g, \mathcal{E}(\mathbb{C}^n, H^p))$ has vanishing homology groups of positive order and that

$$\sum_i (z_i - T_{g_i})\,\mathcal{E}(\mathbb{C}^n, H^p)$$

is closed in $\mathcal{E}(\mathbb{C}^n, H^p)$.

Suppose that $u^k \in K_1(z - T_g, \mathcal{E}(\mathbb{C}^n, H^p))$ and that $\delta_{z-g} u^k \to u_0$ in $\mathcal{E}(\mathbb{C}^n, H^p)$. By Lemma 3.1 there is a $u_1 \in K_1(z - T_g, \mathcal{E}(\mathbb{C}^n, B_0))$ such that $iu_0 = \delta_{z-T_g} u_1$. Again by Lemma 3.1 we can recursively find $u_i \in K_i(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{i-1}))$ such that $\delta_{z-T_g} u_{i+1} = \bar{\partial} u_i$ for $i \ge 1$. Then we have that $\bar{\partial} u_{m+1} = 0$. Define $v_{m+1} \in K_{m+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{m-2}))$ by $v_{m+1} = K u_{m+1}$. Recursively define v_i, $i \ge 2$, by $v_i = K u_i - K \delta_{z-T_g} v_{i+1}$. Thus $v_i \in K_i(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{i-2}))$ if $i \ge 4$, $v_3 \in \Lambda^3 \mathcal{E}(\mathbb{C}^n, B_1')$ and the equation $\bar{\partial} v_i = u_i - \delta_{z-T_g} v_{i+1}$ holds for $i \ge 3$. Furthermore $v_2 \in \Lambda^2 \mathcal{E}(\mathbb{C}^n, L^p(\partial D))$ satisfies the equation $\bar{\partial}_b v_2 = u_2 - \delta_{z-T_g} v_3$ by Lemma 3.2.

Let $u_1' = u_1^* - \delta_{z-g^*} v_2$. By Lemma 3.2 we have that $\bar{\partial}_b \delta_{z-g^*} v_2 = \delta_{z-T_g} u_2$ and thus

$$\int_{\partial D} \delta_{z-g^*} v_2 h = W(\delta_{z-T_g} u_2, h)$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar{\partial} h = 0$. Since by equation (16)

$$\int_{\partial D} u_1^* h = W(\bar{\partial} u_1, h)$$

we have proved that

$$\int_{\partial D} u_1' h = 0$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar\partial h = 0$. Thus $U_1' \in K(z-T_g, \mathcal{E}(\mathbb{C}^n, H^p))$, where U_1' is the unique holomorphic extension of u_1'. Since $u_0 = \delta_{z-T_g} U_1'$ by Lemma 3.3 we have proved that

$$\sum_i (z_i - T_{g_i})\, \mathcal{E}\left(\mathbb{C}^n, H^p\right)$$

is closed in $\mathcal{E}\left(\mathbb{C}^n, H^p\right)$.

Suppose that $u_k \in K_k(z-T_g, \mathcal{E}\left(\mathbb{C}^n, H^p\right))$ is δ_{z-T_g}-closed. Then there is a $u_{k+1} \in K_{k+1}(z-T_g, \mathcal{E}\left(\mathbb{C}^n, B_0\right))$ such that $u_k = \delta_{z-T_g} u_{k+1}$. Let $u_{i+1} \in K_{i+1}\left(z-T_g, \mathcal{E}(\mathbb{C}^n, B_{i-k})\right)$ solve the equation $\delta_{z-T_g} u_{i+1} = \bar\partial u_i$. Then we have that $\bar\partial u_{m+k+1} = 0$. Let $v_{m+k+1} = K u_{m+k+1}$ and $v_i = K u_i - K\delta_{z-T_g} v_{i+1}$. Thus $\bar\partial v_i = u_i - \delta_{z-T_g} v_{i+1}$ and $\bar\partial_b v_{k+2} = u_{k+2} - \delta_{z-T_g} v_{k+3}$ since $\bar\partial\left(u_i - \delta_{z-Tg} v_{i+1}\right) = 0$. Define u_{k+1}' by the equation $u_{k+1}' = u_{k+1}^* - \delta_{z-T_g} v_{k+2}$. As in the case above we see that U_{k+1}' is a solution of the equation $u_k = \delta_{z-T_g} U_{k+1}'$, and hence the theorem is proved.

□

We now prove the analogue of Theorem 1.1 with the Hardy space replaced by the Bergman space. In the case of when g has bounded derivate this is proved in Theorem 8.1.5 in [9].

Corollary 3.4. *Suppose that D is a bounded strictly pseudoconvex domain in $\mathbb{C}^m$ with C^∞-boundary and that $g \in H^\infty(D)^n$. Then the tuple T_g of Toeplitz operators on the Bergman space $\mathcal{O}L^p(D)$, $1 \le p < \infty$, satisfies property $(\beta)_{\mathcal{E}}$ and Bishop's property (β).*

Proof. Let ρ be a strictly plurisubharmonic defining function for D and let $\tilde{D} = \left\{(v,w) \in \mathbb{C}^{m+1} : \rho(v) + |w|^2 < 0\right\}$. Define the operators $P : H^p(\tilde{D}) \to \mathcal{O}L^p(D)$ and $I : \mathcal{O}L^p(D) \to H^p(\tilde{D})$ by $Pf(v) = f(v,0)$ and $If(v,w) = f(v)$ respectively. The operator P is continuous by the Carleson-Hörmander inequality since the measure with mass uniformly distributed on $\tilde{D} \cap \{w = 0\}$ is a Carleson measure. The operator I is continuous since

$$\int_{\partial\tilde{D}} |f(v)|^p \,\sigma(v,w) \sim \lim_{\varepsilon\to 0} \varepsilon \int_{\tilde{D}} \left(-\rho(v) - |w|^2\right)^{\varepsilon-1} |f(v)|^p \sim$$

$$\lim_{\varepsilon\to 0} \int_D \left(-\rho(v)\right)^\varepsilon |f(v)|^p = \int_D |f(v)|^p\,,$$

where σ is the surface measure. Let $\tilde{g}(v,w) = g(v)$. Then $T_{\tilde{g}}$ has property $(\beta)_{\mathcal{E}}$ and since $PI = id, T_{\tilde{g}} I = I T_g$ and $PT_{\tilde{g}} = T_g P$ it is easy to see that T_g has property $(\beta)_{\mathcal{E}}$. □

4. Property $(\beta)_{\mathcal{E}}$ for Toeplitz operators with H^∞-symbol on the unit disc

In this section we will use the Euclidean norm. Let $r(w) = 1 - |w|^2$ and let D be the unit disc in $\mathbb{C}$. Let B_0 be the Banach space of all functions $u \in L^\infty(D)$ such that

$$\|u\|_{B_0} = \|u\|_{L^\infty(D)} + \|rdu\|_{L^\infty(D)} + \|rdu\|_{T_2^\infty} + \left\|r^2\partial\bar{\partial}u\right\|_{T_1^\infty} < \infty.$$

Since $\|rdu\|_{L^\infty(D)} < \infty$, B_0 consists of continuous functions on D. We define B_1 as the Banach space of all locally integrable $(0,1)$-forms u such that

$$\|u\|_{B_1} = \|ru\|_{L^\infty(D)} + \|ru\|_{T_2^\infty} + \left\|r^2\partial u\right\|_{T_1^\infty} < \infty.$$

Suppose that $u \in C^\infty(\bar{D})$ and $h \in C^\infty(\partial D)$. Then the Wolff trick (see the proof of Theorem 1.1) yields

$$\int_{\partial D} uhdw = \int_D \bar{\partial}(uPhdw) =$$

$$\int_D O(r)\bar{\partial}(uPhdw) + \int_D r\mathcal{L}\bar{\partial}(uPhdw) := S(u,h),$$

where Ph is the Poisson integral of h.

As in Section 3 we need to know that functions in B_0 has well defined boundary values.

Lemma 4.1. *If $u \in B_0$ then there is a $u^* \in L^\infty(\partial D)$ such that*

$$\int_{\partial D} u^* hdw = S(u,h)$$

for all $h \in L^2(\partial D)$ and $(fu)^* = f^* u^*$ if $f \in H^\infty(D)$.

Proof. We have the estimate

$$|S(u,h)| \lesssim \|u\|_{B_0} \|h\|_{L^2(\partial D)}.$$

Hence there is a function $u^* \in L^2(\partial D)$ such that

$$\int_{\partial D} u^* hdw = S(u,h)$$

for all $h \in L^2(\partial D)$. Suppose that $h \in C^\infty(\partial D)$. Let u_t be the dilation $u_t(w) = u(tw)$. Since

$$|S(u_t - u, h)| \lesssim \int_D |u_t - u| + \int_D r\,|d(u_t-u)|^2 + \int_D r\,\left|\partial\bar{\partial}(u_t - u)\right|$$

for fixed h we have that

$$\int_{\partial D} u_t^* hdw \to \int_{\partial D} u^* hdw$$

as $t \nearrow 1$. Therefore $\|u^*\|_{L^\infty(\partial D)} \leq \|u\|_{B_0}$ since u_t^* is uniformly bounded by $\|u\|_{L^\infty(D)}$. Let $f_s(w) = f(sw)$ be the dilation of f. Then we have that

$$\int_{\partial D} f_s^* u_t^* h dw = \int_{\partial D} (f_s^* - f^*) u_t^* h dw + \int_{\partial D} f^* u_t^* h dw \to \int_{\partial D} f^* u^* h dw$$

as $s, t \nearrow 1$, by dominated convergence. Since we also have

$$\int_{\partial D} (fu)_t^* h dw \to \int_{\partial D} (fu)^* h dw$$

as $t \nearrow 1$ we see that $(fu)^* = f^* u^*$. □

Let

$$W(u,h) = \int_D O(r) u \wedge h dw + \int_D r \mathcal{L}(u \wedge h dw)$$

for $u \in B_1$ and $h \in H^1$, where $O(r)$ is the same $O(r)$ as in the definition of $S(u,h)$.

Lemma 4.2. *If $f \in \mathcal{E}(\mathbb{C}^n, B_1)$ then there is a $u \in \mathcal{E}(\mathbb{C}^n, L^\infty(\partial D))$ such that $\bar{\partial}_b u = f$, that is*

$$\int_{\partial D} u(z) h dw = W(f(z), h)$$

for all $h \in H^1(D)$ and $z \in \mathbb{C}^n$.

Proof. Consider the bilinear map $W : B_1 \times H^1 \to \mathbb{C}$. This map is continuous since we have the estimate

$$|W(f,h)| \lesssim \|f\|_{B_1} \|h\|_{H^1},$$

which is used in Wolff's proof of the corona theorem. By the universal property for π-tensor products (see 41.3.(1) in [13]) there is a corresponding linear and continuous map W_1 from $B_1 \hat{\otimes}_\pi H^1$ to $\mathbb{C}$. Since

$$\mathcal{E}(\mathbb{C}^n, B_1) \cong \mathcal{E}(\mathbb{C}^n) \hat{\otimes} B_1 \cong L(\mathcal{E}'(\mathbb{C}^n), B_1)$$

by Appendix 1 in [9], $f \otimes id$ is a continuous map $\mathcal{E}'(\mathbb{C}^n) \hat{\otimes} H^1 \to B_1 \hat{\otimes}_\pi H^1$. Compose with the map W_1 to get a continuous functional on $\mathcal{E}'(\mathbb{C}^n) \hat{\otimes} H^1$. The injection $\mathcal{E}'(\mathbb{C}^n) \hat{\otimes} H^1 \to \mathcal{E}'(\mathbb{C}^n) \hat{\otimes} L^1(\partial D)$ is a topological monomorphism, and hence we can extend with Hahn-Banach Theorem to a continuous functional on $\mathcal{E}'(\mathbb{C}^n) \hat{\otimes} L^1(\partial D)$. Since the dual of $\mathcal{E}'(\mathbb{C}^n) \hat{\otimes} L^1(\partial D)$ is isomorphic to $\mathcal{E}(\mathbb{C}^n, L^\infty(\partial D))$ by Theorem A1.12 in [9] we have a $u \in \mathcal{E}(\mathbb{C}^n, L^\infty(\partial D))$. If $h \in H^1$ then

$$\int u(z) h dw = W(f(z), h)$$

and thus u is a solution to the equation $\bar{\partial}_b u = f$ in the sense of this lemma. □

Theorem 4.3. *Let D be the unit disc in $\mathbb{C}$ and suppose that $g \in H^\infty(D)^n$. Then the tuple T_g of Toeplitz operators on $H^\infty(D)$ satisfies property $(\beta)_{\mathcal{E}}$, and thus Bishop's property (β).*

Proof. The tuple T_g considered as operators on B_0 or B_1 has a $C^\infty(\mathbb{C}^n)$-functional calculus (the proof of this is similar to Lemma 3.1). Hence they satisfies property $(\beta)_{\mathcal{E}}$ by Proposition 6.4.13 in [9]. Consider the well-defined complex

$$0 \to H^\infty \to B_0 \xrightarrow{\bar{\partial}} B_1 \to 0. \tag{17}$$

Suppose that $u^k \in \sum_i (z_i - T_{g_i})\mathcal{E}(\mathbb{C}^n, H^\infty)$ and $u^k \to u_0$ in $\mathcal{E}(\mathbb{C}^n, H^\infty)$. As T_g on B_0 has property $(\beta)_{\mathcal{E}}$ there is a $u_1 \in K_1(z - T_g, \mathcal{E}(\mathbb{C}^n, B_0))$ such that $u_0 = \delta_{z-T_g} u_1$. Since T_g on B_1 has property $(\beta)_{\mathcal{E}}$, there is a $u_2 \in K_2(z - T_g, \mathcal{E}(\mathbb{C}^n, B_1))$ such that $\delta_{z-T_g} u_2 = \bar{\partial} u_1$. By Lemma 4.2 there is a $v \in \Lambda^2 \mathcal{E}(\mathbb{C}^n, L^\infty(\partial D))$ such that

$$\int_{\partial D} vhdw = W(u_2, h)$$

for all $h \in H^1(D)$. Therefore we have that

$$\int_{\partial D} \delta_{z-g^*} vhdw = W(\delta_{z-T_g} u_2, h)$$

for all $h \in H^1(D)$. Define $u_1' \in K_1(z - g^*, \mathcal{E}(\mathbb{C}^n, L^\infty(\partial D)))$ by the equation $u_1' = u_1^* - \delta_{z-g^*} v$. Then

$$\int_{\partial D} u_1' hdw = 0$$

for all $h \in H^1$ since

$$\int_{\partial D} u_1^* hdw = S(u_1, h) = W(\bar{\partial} u_1, h)$$

by Lemma 4.1. Thus $U_1' \in K_1(z - T_g, \mathcal{E}(\mathbb{C}^n, H^\infty))$, where U_1' is the holomorphic extension. Since $u_0 = \delta_{z-T_g} U_1'$ by Lemma 4.1 we have proved that $\delta_{z-T_g} K_1(z - g, \mathcal{E}(\mathbb{C}^n, H^\infty))$ is closed.

Suppose that $u_k \in K_k(z - T_g, \mathcal{E}(\mathbb{C}^n, H^\infty))$ is δ_{z-T_g}-closed. Then there is a solution $u_{k+1} \in K_{k+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, B_0))$ to the equation $\delta_{z-T_g} u_{k+1} = u_k$ since T_g on B_0 has property $(\beta)_{\mathcal{E}}$. Continuing in exactly the same way as above we see that we can replace u_{k+1} with $U_{k+1}' \in K_{k+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, H^\infty))$ such that $\delta_{z-T_g} U_{k+1}' = u_k$. Thus the theorem is proved. □

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