Thesis for the Degree of Doctor of Philosophy

Generalized Patterns in Words and Permutations

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Abstract

The thesis consists of the following nine papers:

I *Multi-avoidance of generalized patterns.* (*Discrete Mathematics*, to appear) Recently, Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. We investigate simultaneous avoidance of two or more 3-patterns without internal dashes, that is, where the pattern corresponds to a contiguous subword in a permutation.

II *Generalized pattern avoidance with additional restrictions.* (Séminaire Lotharingien de Combinatoire, to appear) We consider n-permutations that avoid the generalized pattern 1-32 and whose k rightmost letters form an increasing subword. The number of such permutations is a linear combination of Bell numbers. We find a bijection between these permutations and all partitions of an $(n-1)$-element set with one subset marked that satisfy certain additional conditions. Also we find the e.g.f. for the number of permutations that avoid a generalized 3-pattern with no dashes and whose k leftmost or k rightmost letters form either an increasing or decreasing subword. Moreover, we find a bijection between n-permutations that avoid the pattern 132 and begin with the pattern 12 and increasing rooted trimmed trees with $n+1$ nodes.

III *Simultaneous avoidance of generalized patterns* (joint work with Toufik Mansour). In [Kit1] Kitaev considered simultaneous avoidance (multi-avoidance) of two or more 3-patterns with no internal dashes, that is, where the patterns correspond to contiguous subwords in a permutation. There either an explicit or a recursive formula was given for all but one case of simultaneous avoidance of more than two patterns. In this paper we find the exponential generating function for the remaining case. Also we consider permutations that avoid a pattern of the form x-yz or xy-z and begin with one of the patterns $12\ldots k$, $k(k-1)\ldots 1$, $23\ldots k1$, $(k-1)(k-2)\ldots 1k$ or end with one of the patterns $12\ldots k$, $k(k-1)\ldots 1$, $1k(k-1)\ldots 2$, $k12\ldots(k-1)$. For each of these cases we find either the ordinary or exponential generating functions or a precise formula for the number of such permutations. Besides we generalize some of the obtained results as well as some of the results given in [Kit3]: we consider permutations avoiding certain generalized 3-patterns and beginning (ending) with an arbitrary pattern having either the greatest or the least letter as its rightmost (leftmost) letter.

IV *On multi-avoidance of generalized patterns* (joint work with Toufik Mansour). In [Kit1] Kitaev discussed simultaneous avoidance of two 3-patterns with no internal dashes, that is, where the patterns correspond to contiguous subwords in a permutation. In three essentially different cases, the numbers of such n-permutations are 2^{n-1}, the number of involutions

in $\mathcal{S}_n$, and $2E_n$, where E_n is the n-th Euler number. In this paper we give recurrence relations for the remaining three essentially different cases.

To complete the descriptions in [Kit3] and [KitMans1], we consider avoidance of a pattern of the form x-y-z (a classical 3-pattern) and beginning or ending with an increasing or decreasing pattern. Moreover, we generalize this problem: we demand that a permutation must avoid a 3-pattern, begin with a certain pattern and end with a certain pattern simultaneously. We find the number of such permutations in case of avoiding an arbitrary generalized 3-pattern and beginning and ending with increasing or decreasing patterns.

V *Partially Ordered Generalized Patterns.* (*Discrete Mathematics*, to appear) We introduce partially ordered generalized patterns (POGPs), which further generalize the generalized permutation patterns (GPs) introduced by Babson and Steingrímsson. A POGP p is a GP some of whose letters are incomparable. Thus, in an occurrence of p in a permutation π, two letters that are incomparable in p pose no restrictions on the corresponding letters in π. We describe many relations between POGPs and GPs and give general theorems about the number of permutations avoiding certain classes of POGPs. These theorems have several known results as corollaries but also give many new results. We also give the generating function for the entire distribution of the maximum number of non-overlapping occurrences of a pattern p with no dashes, provided we know the e.g.f. for the number of permutations that avoid p.

VI *Partially ordered generalized patterns and k-ary words* (joint work with Toufik Mansour). We study the generating functions (g.f.) for the number of k-ary words avoiding some POGPs. We give analogues, extend and generalize several known results, as well as get some new results. In particular, we give the g.f. for the entire distribution of the maximum number of non-overlapping occurrences of a pattern p with no dashes (that allowed to have repetition of letters), provided we know the g.f. for the number of k-ary words that avoid p.

VII *Counting the occurrences of generalized patterns in words generated by a morphism* (joint work with Toufik Mansour). We count the number of occurrences of certain patterns in given words. We choose these words to be the set of all finite approximations of a sequence generated by a morphism with certain restrictions. The patterns in our considerations are either classical patterns 1-2, 2-1, 1-1-$\cdots$-1, or arbitrary generalized patterns without internal dashes, in which repetitions of letters are allowed. In particular, we find the number of occurrences of the patterns 1-2, 2-1, 12, 21, 123 and 1-1-$\cdots$-1 in the words obtained by iterations of the morphism $1 \to 123$, $2 \to 13$, $3 \to 2$, which is a classical example of a morphism generating a nonrepetitive sequence.

VIII *The Peano curve and counting occurrences of some patterns* (joint work with Toufik Mansour). We introduce *Peano words*, which are words corresponding to finite approximations of the Peano space filling curve. We then find the number of occurrences of certain patterns in these words.

IX *The sigma-sequence and counting occurrences of some patterns, subsequences and subwords.* We consider *sigma-words*, which are words used by Evdokimov in the construction of the sigma-sequence [Evdok1]. We then find the number of occurrences of certain patterns, subsequences and subwords in these words.

Key words and phrases. Generalized pattern avoidance, partially ordered generalized patterns, occurrence of a pattern in a word or permutation, iterated morphism, Peano curve, sigma-sequence, Dragon curve

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Sergey Kitaev
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Contents

In mathematics, if a pattern occurs, we can go on to ask, Why does it occur? What does it signify? And we can find answers to these questions. In fact, for every pattern that appears, a mathematician feels he ought to know why it appears.

– W. W. Sawyer

Introduction

In the last decade a wealth of papers has been written on the subject of pattern avoidance in permutations, also known as the study of "restricted permutations" and "permutations with forbidden subsequences." This topic is the main focus of the present thesis (the first five papers are about this). In the sixth paper, which extends and generalizes the fifth paper, we study certain patterns in k-ary words. The last three papers are dedicated to counting occurrences of certain patterns in certain words related to sequences generated by *morphisms*, the *Peano curve* and the *sigma-sequence*, respectively.

0.1 Permutation patterns

We write permutations as words $\pi = a_1a_2\cdots a_n$, whose letters are distinct and usually consist of the integers $1, 2, \ldots, n$.

An occurrence of a pattern τ in a permutation π is "classically" defined as a subsequence in π (of the same length as τ) whose letters are in the same relative order as those in τ. Formally speaking, for $r \leq n$, we say that a permutation σ in the symmetric group $\mathcal{S}_n$ has an occurrence of the pattern $\tau \in \mathcal{S}_r$ if there exist $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that $\tau = \sigma(i_1)\sigma(i_2)\ldots\sigma(i_r)$ in reduced form. The *reduced form* of a permutation σ on a set $\{j_1, j_2, \ldots, j_r\}$, where $j_1 < j_2 < \cdots < j_r$, is the permutation σ_1 obtained by renaming the letters of the permutation σ so that j_i is renamed i for all $i \in \{1, \ldots, r\}$. For example, the reduced form of the permutation 3651 is 2431.

We denote by $\mathcal{S}_n(\tau)$ the set of all permutations in $\mathcal{S}_n$ which *avoid* τ, that is have no occurrences of τ. If $R = \{\tau_1, \tau_2, \ldots, \tau_k\}$, we let

$$\mathcal{S}_n(R) = \bigcap_{1 \leq i \leq k} \mathcal{S}_n(\tau_i).$$

The *reverse* $\mathcal{R}(\pi)$ of a permutation $\pi = a_1a_2\ldots a_n$ is the permutation $a_na_{n-1}\ldots a_1$. The *complement* $C(\pi)$ is the permutation $b_1b_2\ldots b_n$ where $b_i = n + 1 - a_i$. Also, $\mathcal{R} \circ C$ is the composition of $\mathcal{R}$ and C. For example, $\mathcal{R}(13254) = 45231$, $C(13254) = 53412$ and $\mathcal{R} \circ C(13254) = 21435$. We call these bijections of $\mathcal{S}_n$ to itself *trivial*, and it is easy to see that for any pattern τ the number $|\mathcal{S}_n(\tau)|$ of permutations avoiding the pattern τ is the same as for the patterns $\mathcal{R}(\tau)$, $C(\tau)$ and $\mathcal{R} \circ C(\tau)$. For example, the number of permutations that avoid the pattern 132 is the same as the number of permutations that avoid the pattern 231. This property holds for sets of patterns as well. If we apply one of the trivial bijections to all patterns of a set R, then we get a set R' for which $|\mathcal{S}_n(R')|$ is equal to $|\mathcal{S}_n(R)|$. For example, the number of permutations avoiding $\{123, 132\}$ equals the number of those avoiding $\{321, 312\}$ because the second set is obtained from the first one by complementing each pattern.

Fundamental questions are to determine $|\mathcal{S}_n(R)|$ viewed as a function of n, and if $|\mathcal{S}_n(R)| = |\mathcal{S}_n(R')|$ to find an explicit bijection between $\mathcal{S}_n(R)$ and $\mathcal{S}_n(R')$. It is also interesting to find relations between $\mathcal{S}_n(R)$ and other combinatorial structures. By determining $|\mathcal{S}_n(R)|$ we mean finding an explicit formula, or ordinary or exponential generating functions (*g.f.* and *e.g.f.* respectively).

In cases when one does not succeed in finding $|\mathcal{S}_n(R)|$, there appear other questions. For example, does there exist a constant c such that $|\mathcal{S}_n(R)| < c^n$? (see [Bona3]). One more example is the following question: is $|\mathcal{S}_n(R)|$ *P-recursive*? We recall that a function $f : \mathbb{N} \to \mathbb{C}$ is called *P-recursive* if there exist polynomials $P_0, P_1, \ldots, P_k \in \mathbb{C}[n]$, so that

$$P_k(n)f(n+k) + P_{k-1}(n)f(n+k-1) + \cdots + P_0(n)f(n) = 0$$

for all $n \in \mathbb{N}$ (see [Bona4, NooZeil]). However, in the present thesis we only deal with the fundamental questions.

The most studied case has been to forbid a single pattern of length 3. Because of obvious symmetry arguments, namely the trivial bijections, there are only two essentially distinct cases to enumerate, $|\mathcal{S}_n(123)|$ and $|\mathcal{S}_n(132)|$. As it happens, these two functions are equal to the nth Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$, which was shown by Knuth [Knuth]. The first bijection between the two cases was presented by Simion and Schmidt [SimSch], a second one was given by Richards [Rich]; West described in [West1] a construction using trees; and recently, Krattenthaler [Krat] connected the 123-avoiding and 132-avoiding permutations via Dyck paths.

While there are 24 permutation patterns of length 4, for many of them the sequences $|\mathcal{S}_n(\tau)|$ are identical. In fact, there are only three different classes of patterns from this point of view [West, Stank]. The patterns 1342, 1234 and 1324 are distinct representatives of these classes. Table 1 shows the present state of research on permutations avoiding given patterns of length 4, where

$$(\star) = 2\sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}^2 \frac{3k^2+2k+1-n-2kn}{(k+1)^2(k+2)(n-k+1)} \quad \text{and}$$

$$(\star\star) = \frac{7n^2-3n-2}{2}\cdot(-1)^{n-1} + 3\sum_{i=2}^{n} 2^{i+1}\cdot\frac{(2i-4)!}{i!(i-2)!}\binom{n-i+2}{2}(-1)^{n-i}.$$

The second column there corresponds to the question of existence of a constant c such that $|\mathcal{S}_n(\tau)| < c^n$. Stanley and Wilf conjectured that such a constant exists for any pattern τ.

For the patterns of length greater than 4, the following result by Regev [Regev] is worth mention.

Theorem 1. *For all n, the number $N_n(12\ldots k)$ of permutations in $\mathcal{S}_n$ that avoid the pattern $12\ldots k$ is asymptotically equal to*

$$\lambda_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}.$$

Here

$$\lambda_k = {\gamma_k}^2 \int_{x_1\geq x_2}\int_{x_2\geq x_3}\cdots\int_{x_{k-1}\geq x_k} [D(x_1,x_2,\ldots,x_k)\cdot e^{-(k/2)x^2}]^2\, dx_1 dx_2 \ldots dx_k,$$

pattern p	$\lvert\mathcal{S}_n(p)\rvert < c^n$	**formula for** $N_n(p)$	P**-recursive**
1234	yes Regev [Regev]	$(\star)$ Gessel [Gessel]	yes Zeilberger [Zeil]
1342	yes Bóna [Bona]	$(\star\star)$ Bóna [Bona1]	yes Bóna [Bona1]
1324	yes Bóna [Bona]	open	open

Table 1: Present state of research on avoidance of patterns of length 4

where $D(x_1, x_2, \ldots, x_k) = \prod_{i<j}(x_i - x_j)$, *and* $\gamma_k = (1/\sqrt{2\pi})^{k-1} \cdot k^{k^2/2}$.

Another general result, involving generating functions, is due to Gessel [Gessel].

Theorem 2. *Let* $\ell_k(n) = |\mathcal{S}_n(12\ldots k)|$; *then*

$$L_k(x) = \sum_{n\geq 0} \ell_k(n)\frac{x^{2n}}{n!} = det(I_{|i-j|}(x))_{1\leq i,j\leq k},$$

where $I_i(x)$ *is a Bessel function:*

$$I_i(x) = \sum_{n\geq 0} \frac{x^{2n+i}}{n!(n+i)!} = \sum_{n\geq 0} \binom{2n+i}{n} \frac{x^{2n+i}}{(2n+i)!}.$$

This result was later explained in terms of lattice walks by Gessel, Weinstein and Wilf [GWW].

A natural question is the consideration of those permutations that avoid two or more patterns simultaneously. This problem was solved completely for the patterns from $\mathcal{S}_3$ (see [SimSch]). We summarize some of the results from that paper in Table 2. The trivial bijections break the set of all possibilities into 12 classes of equivalence; we pick one representative from each class.

For the case of simultaneous avoidance of two patterns τ_1 and τ_2, where $\tau_1 \in \mathcal{S}_3$ and $\tau_2 \in \mathcal{S}_4$ see [West2]. We summarize the known results in Table 3.

The results in Table 4 were given by West.

For the case of simultaneous avoidance of two patterns in $\mathcal{S}_4$, see [Bona2, Kremer] and references therein. Several recent papers [ChowWest, MV1, Krat,

patterns	**enumeration**
$\{123, 132\}$	2^{n-1}
$\{123, 231\}$	$\binom{n}{2} + 1$
$\{123, 321\}$	zero for $n > 4$
$\{132, 213\}$	2^{n-1}
$\{132, 231\}$	2^{n-1}
$\{132, 312\}$	2^{n-1}
$\{123, 132, 213\}$	Fibonacci numbers
$\{123, 132, 231\}$	n
$\{123, 132, 312\}$	n
$\{123, 132, 321\}$	zero for $n > 4$
$\{123, 231, 312\}$	n
$\{132, 213, 231\}$	n

Table 2: Simultaneous avoidance of patterns of length 3 ([SimSch])

restrictions	**formula**	**author**
$S_n(123, 4321)$	0	West
$S_n(123, 3421)$	$\binom{n}{4} + 2\binom{n}{3} + n$	West
$S_n(132, 4321)$	$2\binom{n}{4} + \binom{n}{3} + \binom{n}{2} + 1$	West
$S_n(123, 4231)$	$\binom{n}{5} + 2\binom{n}{4} + \binom{n}{3} + \binom{n}{2} + 1$	West
$S_n(123, 3241)$	$3 \cdot 2^{n-1} - \binom{n+1}{2} - 1$	West
$S_n(123, 3412)$	$2^{n+1} - \binom{n+1}{3} - 2n - 1$	Stanley
$S_n(132, 4231)$	$1 + (n-1)2^{n-2}$	Guibert
$S_n(132, 3421)$	$1 + (n-1)2^{n-2}$	West
$S_n(132, 3214)$	g.f.: $\frac{(1-x)^3}{1-4x+5x^2-3x^3}$	West

Table 3: Simultaneous avoidance of a 3-pattern and a 4-pattern

restrictions	restrictions	formula
$S_n(123, 2143)$	$S_n(312, 1342)$	
$S_n(123, 2413)$	$S_n(312, 3241)$	
$S_n(132, 2314)$	$S_n(312, 3214)$	
$S_n(132, 2341)$	$S_n(123, 3214)$	F_{2n}
$S_n(312, 2314)$	$S_n(312, 4321)$	(Fibonacci number)
$S_n(132, 3412)$	$S_n(312, 3421)$	
$S_n(312, 1432)$	$S_n(132, 3241)$	
$S_n(3142, 2413)$	$S_n(4132, 4231)$	the $(n-1)$-st Schröder number g.f.: $\frac{1-x-\sqrt{1-6x+x^2}}{2x}$

Table 4: Some results given by West

MV3, MV2] deal with the case $\tau_1 \in S_3$, $\tau_2 \in S_k$ for various pairs τ_1, τ_2. Erdős and Szekeres [ErdSze] gave the following general result.

Theorem 3. *For all* $n \geq (\ell - 1)(m - 1) + 1$,

$$|\mathcal{S}_n(12 \ldots \ell, m \ldots 21)| = 0.$$

0.2 Generalized permutation patterns

In [BabStein] Babson and Steingrímsson introduced *generalized permutation patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a "classical" pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if this pattern occurs in a permutation π, then the letters in π that correspond to 3 and 1 are adjacent. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563. If we use "[" in a pattern, for example if we write $p =$ [1-2), we indicate that in an occurrence of p, the letter corresponding to the 1 must be the first letter of the permutation, whereas if we write, say, $p =$ (1-2], then the letter corresponding to 2 must be the last (rightmost) letter of the permutation. Thus, a parenthesis at either end of a pattern corresponds to a dash, and a square bracket corresponds to the absence of a dash. However, when a pattern begins *and* ends with a parenthesis, we omit these parentheses, writing simply 123 instead of (123).

The motivation for introducing these patterns in [BabStein] was the study of Mahonian statistics. A number of interesting results on generalized patterns were obtained in [Claes]. Relations to several well studied combinatorial structures, such as set partitions, Dyck paths, Motzkin paths and involutions, were

patterns P	$\|\mathcal{S}_n(P)\|$	description
1-23	B_n	partitions of $[n]$
1-32	B_n	partitions of $[n]$
2-13	C_n	Dyck paths of length $2n$
1-23, 12-3	$B_n^\star$	non-overlapping partitions of $[n]$
1-23, 1-32	I_n	involutions in $\mathcal{S}_n$
1-23, 13-2	M_n	Motzkin paths of length n

Table 5: Generalized pattern avoidance ([Claes])

shown there. The main results from that paper are given in Table 5, where B_n is the n-th Bell number, C_n is the n-th Catalan number, and $B_n^\star$ is the n-th Bessel number.

For some other results on generalized permutation patterns see [ClaesMans1, ClaesMans2, Kit1, Kit2, Kit3, KitMans1, KitMans2]

Paper I. In Paper I ([Kit1]) we consider 3-patterns without internal dashes, that is, generalized patterns of the form xyz. Thus, such patterns correspond to contiguous subwords anywhere in a permutation. For example the permutation $\pi = 12345$ has 3 occurrences of the pattern 123 but 10 occurrences of the classical pattern 1-2-3. Patterns without internal dashes were considered by Elizalde and Noy in [ElizNoy]. In that paper, there is a number of results on the distribution of several classes of patterns without internal dashes. In particular, formulas are given for the bivariate exponential generating functions that count permutations by the number of occurrences of any given 3-pattern.

As in the paper by Simion and Schmidt [SimSch], dealing with the classical patterns, Claesson [Claes] considered a number of cases when permutations have to avoid two or more generalized patterns simultaneously (see Table 5). However, except for the simultaneous avoidance of the patterns 123 and 132, and three more pairs each of which is essentially equivalent to one of these, there were no other results for multi-avoidance of the patterns without internal dashes. In Paper I we give either an explicit formula or a recursive formula for almost all cases of simultaneous avoidance of more than two patterns. We also mention what is known about double restrictions. There are 18 classes of equivalence. As we did before, we choose a representative from each class and record all the known results in Table 6, where we define the *double factorial* $n!!$ by $0!! = 1$, and, for $n > 0$,

$$n!! = \begin{cases} n \cdot (n-2) \cdots 3 \cdot 1, & \text{if } n \text{ is odd,} \\ n \cdot (n-2) \cdots 4 \cdot 2, & \text{if } n \text{ is even.} \end{cases}$$

Besides, in order to complete the description of simultaneous avoidance of two generalized patterns without internal dashes, we put in the same table some results from papers III ([KitMans1]) and IV ([KitMans2]).

Paper II. In Paper II ([Kit3]) we consider avoidance of some generalized 3-patterns with additional restrictions. The restrictions consist of demanding that a permutation begin or end with the pattern $12\dots k$ or the pattern $k(k-1)\dots 1$. We observe that avoidance of some pattern with the additional restrictions described above in fact is equivalent to simultaneous avoidance of several patterns. For example, beginning with the pattern 12 is equivalent to the avoidance of the pattern [21) in the Babson-Steingrímsson notation. Thus avoidance of the pattern 132 and beginning with the pattern 12 is equivalent to simultaneous avoidance of the patterns 132 and [21). Also, ending with the pattern 123 is equivalent to simultaneously avoiding the patterns (132], (213], (231], (312] and (321]. So, demanding that a permutation must begin or end with some pattern is equivalent to simultaneous avoidance of a set of generalized patterns. A motivation for considering additional restrictions such as beginning or ending with some patterns is their connection to some classes of trees mentioned below.

It turns out that the number of permutations that avoid the pattern 1-32 and end with the pattern $12\dots k$ is a linear combination of the Bell numbers. We find a bijection between these permutations and all partitions of an $(n-1)$-element set with one subset marked that satisfy certain additional conditions. In particular, we get that the total number of partitions of an $(n-1)$-element set with one part marked, is equal to the number of (1-32)-avoiding n-permutations that end with a 12-pattern. Also, we get an identity involving the Bell numbers and the Stirling numbers of the second kind, which seems to be new. Besides, we prove that the number of 132-avoiding n-permutations that begin with the pattern 12 is equal to the number of *increasing rooted trimmed trees* with $n+1$ nodes. In an increasing rooted tree, the nodes are numbered and the numbers increase as we move away from the root. A trimmed tree is a tree where no node has a single leaf as a child (every leaf has a sibling).

In Sections 4–7 of Paper II, we give a complete description, in terms of exponential generating functions, for the number of permutations that avoid a pattern of the form xyz and begin or end with the pattern $12\dots k$ or the pattern $k(k-1)\dots 1$. We record all the results concerning these e.g.f. in Table 7. The case $k=1$ is equivalent to the absence of the additional restriction. This case was considered in [ElizNoy] and Paper I.

Paper III. As mentioned above, Paper II dealt with the avoidance of a generalized 3-pattern p with no dashes and, at the same time, beginning or ending with an increasing or decreasing pattern. Theorem 2 in Paper III ([KitMans1]) generalizes some of these results to the case of beginning (resp. ending) with an arbitrary pattern p that has the greatest or least letter as the rightmost (resp. leftmost) letter. To write down this theorem, we need the following definitions. Let $E_q^p(x)$ denote the exponential generating function for the number of permutations that avoid the pattern q and begin with the pattern p. Also, Γ_k^{min} (resp. Γ_k^{max}) denotes the set of all k-patterns with no dashes such that the least

class	restrictions	formula
1	123, 321, 231, 213	2
2	321, 213, 231, 312	2
3	132, 231, 213, 312	2
4	123, 321, 132, 231	2 if $n = 3$; 0 if $n > 3$
5	231, 312, 213, 123	$n - 1$
6	123, 321, 132, 213	$2C_k$, if $n = 2k + 1$ $C_k + C_{k-1}$, if $n = 2k$
7	231, 312, 321	$\binom{n}{\lfloor n/2 \rfloor}$
8	123, 213, 312	n
9	132, 213, 312	$1 + 2^{n-2}$
10	123, 213, 231	recursive formula: $A(0) = 1$; $A(1) = 1$; $A(n) = \sum_i \binom{n-i-1}{i} A(n - 2i - 1) + ((n + 1) \bmod 2)$ the first few numbers: 1, 1, 2, 3, 6, 13, 29, 72, 185...
11	123, 321, 231	$(n - 1)!! + (n - 2)!!$
12	123, 231, 312	e.g.f.: $1 + x(\sec x + \tan x)$, Paper III
13	321, 132	recurrence relation, Paper IV
14	213, 231	recurrence relation, Paper IV
15	132, 213	recurrence relation, Paper IV
16	123, 321	$2E_n$, where E_n is the n-th Euler number
17	321, 231	the number of involutions in $\mathcal{S}_n$, Claesson [Claes]
18	132, 231	2^{n-1}

Table 6: Simultaneous avoidance of generalized 3-patterns (mostly Paper I)

	avoid	begin	end	e.g.f.
	123	$12\dots k$	–	
1	123	–	$12\dots k$	$\frac{\sqrt{3}}{2}\frac{e^{x/2}}{\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}$, if $k=1$
	321	$k\dots 21$	–	$\frac{\sqrt{3}}{2}e^{x/2}\sec(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})-\frac{1}{2}-\frac{\sqrt{3}}{2}\tan(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})$, if $k=2$
	321	–	$k\dots 21$	0, if $k\geq 3$
	123	$k\dots 21$	–	
2	123	–	$k\dots 21$	$\frac{\sqrt{3}}{2}\frac{e^{x/2}}{\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}$, if $k=1$
	321	$12\dots k$	–	
	321	–	$12\dots k$	$\frac{e^{x/2}\int_0^x e^{-t/2}t^{k-1}\sin(\frac{\sqrt{3}}{2}t+\frac{\pi}{6}))\ dt}{(k-1)!\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}$, if $k\geq 2$
	132	$12\dots k$	–	
3	213	–	$12\dots k$	$(1-\int_0^x e^{-t^2/2}\ dt)^{-1}$, if $k=1$
	312	$k\dots 21$	–	$e^{-x^2/2}(1-\int_0^x e^{-t^2/2}\ dt)^{-1}-x-1$, if $k=2$
	231	–	$k\dots 21$	$(1-\int_0^x e^{-t^2/2}\ dt)^{-1}\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_2}(e^{-t_1^2/2}-$
				$(t_1+1)(1-\int_0^{t_1} e^{-t^2/2}\ dt))dt_1dt_2\cdots dt_{k-2}$, if $k\geq 3$
	132	$k\dots 21$	–	
4	213	–	$k\dots 21$	$(1-\int_0^x e^{-t^2/2}\ dt)^{-1}$, if $k=1$
	312	$12\dots k$	–	$\frac{1}{(k-1)!(1-\int_0^x e^{-t^2/2}\ dt)}\int_0^x t^{k-1}e^{-t^2/2}\ dt$, if $k\geq 2$
	231	–	$12\dots k$	
	213	$12\dots k$	–	$(1-\int_0^x e^{-t^2/2}\ dt)^{-1}$, if $k=1$
5	132	–	$12\dots k$	
	231	$k\dots 21$	–	$\int_0^x\int_0^t\frac{s^{k-2}e^{T(t)-T(s)}}{(k-2)!(1-\int_0^t e^{-m^2/2}dm)}\ dsdt$, if $k\geq 2$, where
	312	–	$k\dots 21$	$T(x)=-x^2/2+\int_0^x\frac{e^{-t^2/2}}{1-\int_0^t e^{-s^2/2}ds}\ dt$
	213	$k\dots 21$	–	$(1-\int_0^x e^{-t^2/2}\ dt)^{-1}$, if $k=1$
6	132	–	$k\dots 21$	$-\frac{x^{k-1}}{(k-1)!}+\sum_{n=0}^{k-2}\int_0^x\int_0^{t_n}\cdots\int_0^{t_1}\frac{C_{k-n}(t)+\delta_{n,k-2}}{1-\int_0^t e^{-m^2/2}dm}dtdt_1\cdots dt_n,$
	231	$12\dots k$	–	if $k\geq 2$, where $C_k(x)=e^{T(x)}\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_1}e^{-T(t)}$.
	312	–	$12\dots k$	$\left(\frac{e^{-t^2/2}}{1-\int_0^t e^{-m^2/2}dm}-t-1\right)dtdt_1\cdots dt_{k-2}$ and $T(x)$ as above

Table 7: Avoiding a pattern xyz with additional restrictions (Paper II)

(resp. greatest) letter of a pattern is the rightmost letter. Now, we formulate Theorem 2 from Paper III:

Theorem 4. *Suppose* $p_1, p_2 \in \Gamma_k^{min}$ *and* $p_1 \in S_k(132)$, $p_2 \in S_k(123)$. *Thus, the complements* $C(p_1), C(p_2) \in \Gamma_k^{max}$ *and* $C(p_1) \in S_k(312)$, $C(p_2) \in S_k(321)$. *Then, for* $k \geq 2$,

$$E_{132}^{p_1}(x) = E_{312}^{C(p_1)}(x) = \frac{\int_0^x t^{k-1}e^{-t^2/2}\ dt}{(k-1)!(1-\int_0^x e^{-t^2/2}\ dt)}$$

and

$$E_{123}^{p_2}(x) = E_{321}^{C(p_2)}(x) = \frac{e^{x/2}\int_0^x e^{-t/2}t^{k-1}\sin(\frac{\sqrt{3}}{2}t + \frac{\pi}{6}))\ dt}{(k-1)!\cos(\frac{\sqrt{3}}{2}x + \frac{\pi}{6})}.$$

Propositions 4–15 (resp. 16–27) in Paper III give a complete description for the number of permutations avoiding a pattern of the form $x\text{-}yz$ or $xy\text{-}z$ and beginning with one of the patterns $12\dots k$ or $k(k-1)\dots 1$ (resp. $23\dots k1$ or $(k-1)(k-2)\dots 1k$). For each of these cases we find either the ordinary or exponential generating function or a precise formula for the number of such permutations. Theorem 28 in Paper III generalizes some of these results:

Theorem 5. *Suppose* $p_1, p_2 \in \Gamma_k^{min}$ *and* $p_1 \in S_k(1\text{-}23)$, $p_2 \in S_k(1\text{-}32)$. *Thus, the complements* $C(p_1), C(p_2) \in \Gamma_k^{max}$ *and* $C(p_1) \in S_k(1\text{-}23)$, $C(p_2) \in S_k(3\text{-}12)$. *Then, we have*

$$E_{1\text{-}23}^{p_1}(x) = E_{3\text{-}21}^{C(p_1)}(x) = E_{1\text{-}32}^{p_2}(x) = E_{3\text{-}12}^{C(p_2)}(x) =$$

$$\begin{cases} (e^{e^x}/(k-1)!)\int_0^x t^{k-1}e^{-e^t+t}\ dt, & \text{if } k \geq 2, \\ e^{e^x-1}, & \text{if } k = 1. \end{cases}$$

Moreover, the results from Propositions 4–27 in Paper III give a complete description for the number of permutations that avoid a pattern of the form $x\text{-}yz$ or $xy\text{-}z$ and end with one of the patterns $12\dots k$, $k(k-1)\dots 1$, $1k(k-1)\dots 2$ and $k12\dots(k-1)$. To get the last one of these we only need to apply the reverse operation defined above.

Paper IV. In Paper IV ([KitMans2]) we continue consideration of generalized pattern avoidance with additional restriction. In Section 4 of Paper IV, we consider avoidance of a pattern $x\text{-}y\text{-}z$, and beginning or ending with an increasing or decreasing pattern. This completes the results given in Paper III, which concerns the number of permutations that avoid a generalized 3-pattern and begin or end with an increasing or decreasing pattern.

In Sections 5–8 of Paper IV, we consider stronger restrictions, which generalize many results from Papers II, III, IV. Namely, we give enumeration for the number of permutations that avoid a generalized 3-pattern, and begin *and* end with increasing or decreasing patterns. We record our results in terms of either generating functions, or exponential generating functions, or formulas for the numbers in question.

In Section 9 of Paper IV, we discuss possible directions for generalization of the results from Sections 5–8. The first direction is to consider avoidance of more than one pattern, beginning with some pattern and ending with another pattern. The second direction concerns permutations in $\mathcal{S}_n$ containing a pattern τ exactly r times, beginning with some pattern and ending with another pattern.

0.3 Partially ordered generalized patterns

Suppose we are interested in finding the number of permutations that avoid all patterns from the set $\{12\text{-}4\text{-}3, 13\text{-}4\text{-}2, 23\text{-}4\text{-}1\}$ simultaneously. There is a way to code these three patterns into one pattern, and instead of considering three patterns to consider one. This is done by allowing some letters of a pattern to be incomparable. Thus the set of patterns above can be replaced by the pattern $p = 1'2'\text{-}3\text{-}1''$, where in an occurrence of p in a permutation π the letter corresponding to the $1''$ in p can be either larger or smaller than the letters corresponding to $1'2'$, but all of them must be less than the letter corresponding to the 3 in p. Such patterns are discussed in Papers V ([Kit2]) and VI ([KitMans3]). These patterns allow us to determine the distribution of *non-overlapping* occurrences of patterns without internal dashes.

Paper V. In Paper V ([Kit2]) we introduce a further generalization of generalized patterns (GPs)—namely *partially ordered generalized patterns (POGP)*. A POGP is a GP some of whose letters are incomparable. For instance, if we write $p = 1\text{-}1'2'$ then we mean that in an occurrence of p in a permutation π the letter corresponding to the 1 in p can be either larger or smaller than the letters corresponding to $1'2'$. Thus, the permutation 31254 has three occurrences of p, namely 3-12, 3-25, and 1-25.

We consider two particular classes of POGPs—*shuffle patterns* and *multi-patterns*. A multi-pattern is of the form $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ and a shuffle pattern is of the form $p = \sigma_0\text{-}a_1\text{-}\sigma_1\text{-}a_2\text{-}\cdots\text{-}a_k\text{-}\sigma_k$, where for any i and j, the letter a_i is greater than any letter of σ_j and for any $i \neq j$ each letter of σ_i is incomparable to any letter of σ_j These patterns are investigated in Sections 4 and 5. A corollary to Theorem 13 is the result of Claesson [Claes, Proposition 2] that the number of n-permutations that avoid the pattern 12-3 is the n-th Bell number.

Let p and q be two patterns. An occurrence of p *overlaps* an occurrence of q in a permutation π if these two occurrences share a letter in π. For example, if $p = 123$, $q = 231$ and $\pi = 623514$ then 235 and 351, being occurrences of the patterns p and q respectively, overlap.

Let $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ be an arbitrary multi-pattern and let $A_i(x)$ be the exponential generating function (e.g.f.) for the number of permutations that avoid σ_i for each i. In Theorem 28 we find the e.g.f., in terms of the $A_i(x)$, for the number of permutations that *avoid* p.

Theorem 6. *Let $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ be a multi-pattern and let $A_i(x)$ be the e.g.f. for the number of permutations that avoid σ_i. Then the e.g.f. $B(x)$ for* *the number of permutations that avoid p is*

$$B(x) = \sum_{i=1}^{k} A_i(x) \prod_{j=1}^{i-1} ((x-1)A_j(x) + 1).$$

In fact, this allows us to find the e.g.f. for the *entire distribution* of the maximum number of non-overlapping occurrences of a pattern p with no dashes, if we only know the e.g.f. for the number of permutations that avoid p:

Theorem 7. *Let p be a GP with no dashes. Let $A(x)$ be the e.g.f. for the number of permutations that avoid p. Let $D(x,y) = \sum_{\pi} y^{N(\pi)} \frac{x^{|\pi|}}{|\pi|!}$ where $N(\pi)$ is the maximum number of non-overlapping occurrences of p in π. Then*

$$D(x,y) = \frac{A(x)}{1 - y((x-1)A(x) + 1)}.$$

In many cases, this theorem gives nice generating functions. The following two examples are corollaries to Theorem 7. We recall that a descent in a permutation $\pi = a_1a_2\dots a_n$ is an i such that $a_i > a_{i+1}$. Two descents i and j *overlap* if $j = i+1$.

Example 1. If we consider descents then $A(x) = e^x$, hence the distribution of the maximum number of non-overlapping descents is given by the formula

$$D(x,y) = \frac{e^x}{1 - y(1 + (x-1)e^x)}.$$

The reader might want to compare this result with some known results related to descents. To this end we recall the following. The number of descents in a permutation π is denoted $\operatorname{des}\pi$ (and is equivalent to the generalized pattern 21). Any statistic with the same distribution as des is said to be *Eulerian.* The *Eulerian numbers* $A(n,k)$ count permutations in the symmetric group $\mathcal{S}_n$ with k descents and they are the coefficients of the *Eulerian polynomials* $A_n(t)$ defined by $A_n(t) = \sum_{\pi\in\mathcal{S}_n} t^{1+\operatorname{des}\pi}$. The e.g.f. for Eulerian polynomials is given by

$$\sum_{n\geq 0} A_n(t)\frac{x^n}{n!} = \frac{t(1-t)e^x}{e^{xt} - te^x}.$$

Example 2. If we consider the maximum number of non-overlapping occurrences of the pattern 132 then the distribution of these numbers is given by the formula

$$D(x,y) = \frac{1}{1 - yx + (y-1)\int_0^x e^{-t^2/2}\,dt}.$$

We will talk about *bivariate generating functions*, or *b.g.f.*, exclusively as generating functions of the form

$$A(u,z) = \sum_{\pi} u^{p(\pi)} \frac{z^{|\pi|}}{|\pi|!} = \sum_{n,k \geq 0} A_{n,k} u^k \frac{x^n}{n!},$$

were $A_{n,k}$ is the number of n-permutations with k occurrences of the pattern p.

In order to apply the last two theorems, as well as some other results from Paper V, we need to know how many patterns avoid a given ordinary GP with no dashes. We are also interested in different approaches to studying these patterns. There is a number of results on the distribution of several classes of patterns with no dashes. These results can be used as building blocks for some of the results in Paper V. The most important of these is the following result by Elizalde and Noy:

Theorem 8. ([ElizNoy, Theorem 3.4]) *Let m and a be positive integers with $a \leq m$, let $\sigma = 12\dots a\tau(a+1) \in \mathcal{S}_{m+2}$, where τ is any permutation of the letters $\{a+2, a+3, \dots, m+2\}$, and let $A(u,z)$ be the b.g.f. for permutations where u marks the number of occurrences of σ and z marks the length of the permutation. Then $A(u,z) = 1/w(u,z)$, where w is the solution of*

$$w^{a+1} + (1-u)\frac{z^{m-a+1}}{(m-a+1)!}w' = 0$$

with $w(0) = 1$, $w'(0) = -1$ and $w^{(k)}(0) = 0$ for $2 \leq k \leq a$. In particular, the distribution does not depend on τ.

In Paper V we give alternative proofs, using inclusion-exclusion, of some of the results of Elizalde and Noy [ElizNoy]. Our proofs result in explicit formulas for the coefficients of the e.g.f. whereas Elizalde and Noy obtained differential equations for the same e.g.f..

Paper VI. From now on we are not discussing permutations and generalized permutation patterns. Instead we consider k-ary words and occurrences of patterns in them. First of all we need some definitions, most of which are intuitively clear from the preceding discussion.

Let $[k]^n$ denote the set of all the words of length n over the (totally ordered) alphabet $[k] = \{1, 2, \dots, k\}$. We refer to these words as *n-long k-ary words.* A *generalized pattern* τ is a word in $[\ell]^m$ (possibly with dashes between some letters) that contains each letter from $[\ell]$ (possibly with repetitions). We say that the word $\sigma \in [k]^n$ *contains* a generalized pattern τ if σ contains a subsequence order-isomorphic to τ in which the entries corresponding to consecutive entries of τ that are not separated by a dash must be adjacent. Otherwise, we say that σ *avoids* τ and write $\sigma \in [k]^n(\tau)$. Thus, $[k]^n(\tau)$ denotes the set of all the words in $[k]^n$ that avoid τ. Moreover, if P is a set of generalized patterns then $|[k]^n(P)|$ denotes the set all the words in $[k]^n$ that avoid all patterns from P simultaneously. For example, a word $\pi = a_1a_2\dots a_n$ avoids the pattern 13-2 if

π has no subsequence $a_i a_{i+1} a_j$ with $j > i+1$ and $a_i < a_j < a_{i+1}$. Also, π avoids the pattern 121 if it has no subword $a_i a_{i+1} a_{i+2}$ such that $a_i = a_{i+2} < a_{i+1}$.

Burstein [Burstein] considered patterns without repeated letters on words instead of permutations. In particular, he found the number $|[k]^n(P)|$ of words of length n in a k-letter alphabet that avoid all patterns from a set $P \subseteq \mathcal{S}_3$ simultaneously. Burstein and Mansour [BurMans1] (resp. [BurMans2, BurMans3]) considered forbidden patterns (resp. generalized patterns) with repeated letters.

In Paper VI ([KitMans3]) we introduce a further generalization of the generalized patterns, namely *partially ordered generalized patterns in words* (*POGPs*), which are analogues of POGPs in permutations [Kit2]. A POGP is a generalized pattern some of whose letters are incomparable. For example, if we write $\tau = 1\text{-}1'2'$, then we mean that in an occurrence of τ in a word $\sigma \in [k]^n$ the letter corresponding to the 1 in τ can be either larger than, smaller than, or equal to the letters corresponding to $1'2'$. Thus, the word $113425 \in [5]^6$ contains seven occurrences of τ, namely 113, 134 twice, 125 twice, 325, and 425.

Following Paper V, we consider two particular classes of POGPs — *shuffle patterns* and *multi-patterns*, which allows us to give an analogue for all the main results of [Kit2] for k-ary words.

Let $\tau = \tau_0\text{-}\tau_1\text{-}\cdots\text{-}\tau_s$ be an arbitrary multi-pattern and let $A_{\tau_i}(x;k)$ be the ordinary generating function (g.f.) for the number of words in a k-letter alphabet that avoid τ_i for each i. In Theorem 4.7 of Paper VI we find the g.f., in terms of the $A_{\tau_i}(x;k)$, for the number of k-ary words that avoid τ:

Theorem 9. *Let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ be a multi-pattern. Then*

$$A_\tau(x;k) = \sum_{j=1}^{s} A_{\tau_j}(x;k) \prod_{i=1}^{j-1}((kx-1)A_{\tau_i}(x;k)+1).$$

In particular, this allows us to find the g.f. for the entire distribution of the maximum number of non-overlapping occurrences of a pattern τ with no dashes, if we only know the g.f. for the number of k-ary words that avoid τ:

Theorem 10. *Let τ be a generalized pattern with no dashes. Then, for all $k \geq 1$,*

$$\sum_{n\geq 0} \sum_{\sigma\in[k]^n} y^{N_\tau(\sigma)} x^n = \frac{A_\tau(x;k)}{1 - y((kx-1)A_\tau(x;k)+1)},$$

where $N_\tau(\sigma)$ is the maximum number of non-overlapping occurrences of τ in σ.

Thus, in order to apply our results from the last two theorems we need to know how many k-ary words avoid a given ordinary generalized pattern with no dashes. This question was examined, for instance, in [BurMans1, Sections 2 and 3], [BurMans2, Section 3] and [BurMans3, Section 3.3].

All of the following examples are corollaries to Theorem 10.

Example 3. If we consider rises (the pattern 12) then $A_{12}(x;k) = \frac{1}{(1-x)^k}$ (see [BurMans2]), hence the distribution of the maximum number of non-overlapping descents is given by the formula:

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{12}(\sigma)}x^n = \frac{1}{(1-x)^k + y(1-kx-(1-x)^k)}.$$

Example 4. The distribution of the maximum number of non-overlapping occurrences of the pattern 122 is given by the formula:

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{122}(\sigma)}x^n = \frac{x}{(1-x^2)^k + x - 1 + y(1-kx^2-(1-x^2)^k)},$$

since, according to [BurMans3, Theorem 3.10], $A_{122}(x;k) = \frac{x}{(1-x^2)^k-(1-x)}$.

Example 5. If we consider the pattern 212 then $A_{212}(x;k) = \left(1 - x\sum_{j=0}^{k-1}\frac{1}{1+jx^2}\right)^{-1}$ (see [BurMans3, Theorem 3.12]), hence the distribution of the maximum number of non-overlapping occurrences of the pattern 212 is given by the formula:

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{212}(\sigma)}x^n = \frac{1}{1 - x\sum_{j=0}^{k-1}\frac{1}{1+jx^2} + xy\left(\sum_{j=0}^{k-1}\frac{1}{1+jx^2} - k\right)}.$$

0.4 Counting occurrences of certain patterns in certain words

The most attention, in the papers on classical or generalized patterns, in particular in Papers I–VI, is paid to obtaining exact formulas and/or generating functions for the number of words or permutations avoiding, or having k occurrences of, certain patterns. In Papers VII–IX we suggest another problem, namely counting the occurrences of certain patterns in certain words. These words were chosen to be the set of all finite approximations of certain sequences.

In Paper VII ([KitMans4]) this is a sequence generated by a morphism (a system of substitutions, to be defined below) with certain restrictions. In Paper VIII ([KitMans5]) the sequence is obtained from the *Peano curve*. The Peano curve was studied by the Italian mathematician Giuseppe Peano in 1890 as an example of a continuous space filling curve. Finally, in Paper IX ([Kit4]) this sequence is the *sigma-sequence*, which was used by Evdokimov [Evdok1] to construct chains of maximal length in the n-dimensional unit cube.

Independent interest in the sigma-sequence appears in connection with the well-known *Dragon curve*, discovered by the physicist John E. Heighway and defined as follows: Fold a sheet of paper in half, then fold in half again (so that the folds are parallel), and again, etc. and then unfold in such a way that each crease created by the folding process is opened out into a 90-degree angle. The "curve" refers to the shape of the partially unfolded paper as seen edge on (see

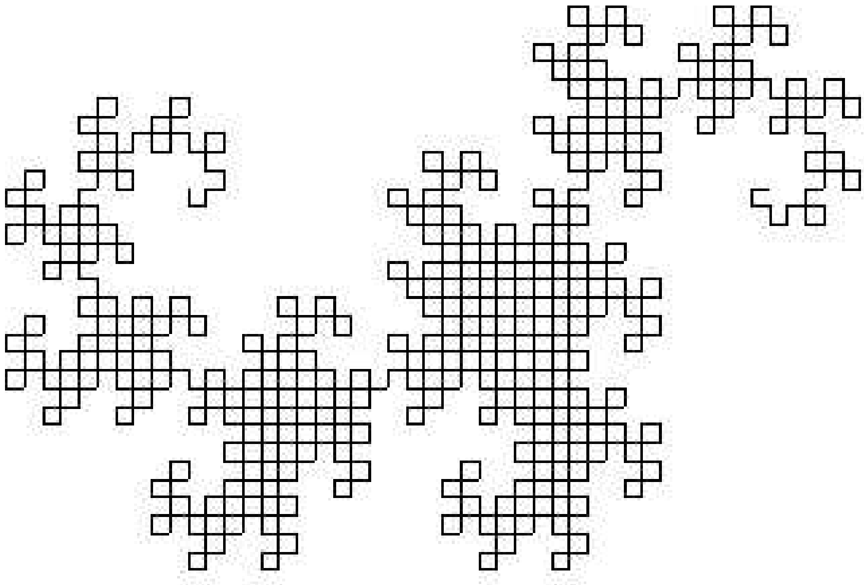

Figure 1: Dragon curve

Figure 1). If one travels along the curve, some of the creases will represent turns to the left and others turns to the right. Now if 1 indicates a turn to the right, and 2 to the left, and we start travelling along the curve indicating the turns, we get the sigma-sequence [Evdok].

Paper VII. In Paper VII ([KitMans4]) we count the occurrences of certain patterns in certain words. We choose these words to be a set of all finite approximations (to be defined below) of a sequence generated by a morphism with certain restrictions. The motivation is to study classes of sequences and words that are defined by iterative schemes [Lothaire, Salomaa]. The pattern τ in our considerations is either a classical pattern (with repeated letters allowed) from the set $\{1\text{-}2, 2\text{-}1, 1\text{-}1\text{-}\cdots\text{-}1\}$, or an arbitrary generalized pattern without internal dashes, in which repetitions of letters are allowed. In particular, we find that there are $(3 \cdot 4^{n-1} + 2^n)$ occurrences of the pattern 1-2 in the n-th finite approximation of the sequence w defined below, which is a classical example of a nonrepetitive sequence.

Let Σ be an alphabet and $\Sigma^\star$ the set of all words over Σ. A map $\varphi : \Sigma^\star \to \Sigma^\star$ is called a *morphism* if we have $\varphi(uv) = \varphi(u)\varphi(v)$ for any $u, v \in \Sigma^\star$. It is easy to see that a morphism φ can be defined by defining $\varphi(i)$ for each $i \in \Sigma$. The set of all rules $i \mapsto \varphi(i)$ is called a *substitution system*. We create words by starting with a letter from the alphabet Σ and iterating the substitution system. Such a substitution system is called a *D0L* (*Deterministic, with no* *context Lindenmayer) system* [LindRoz]. D0L systems are classical objects of formal language theory. They are interesting from a mathematical point of view [Frid], but also have applications in theoretical biology [Lind]. Let $|X|$ denote the length of a word X, that is the number of letters in X.

Suppose a word $\varphi(a)$ begins with a for some $a \in \Sigma$, and that the length of $\varphi^k(a)$ increases without bound. The symbolic sequence $\lim_{k\to\infty} \varphi^k(a)$ is said to be *generated* by the morphism φ. In particular, $\lim_{k\to\infty} \varphi^k(a)$ is a *fixed point* of φ. However, in this paper we are only interested in the *finite approximations* of $\lim_{k\to\infty} \varphi^k(a)$, that is in the words $\varphi^k(a)$ for $k = 1, 2, \ldots$.

An example of a sequence generated by a morphism is the following sequence w. We create words by starting with the letter 1 and iterating the substitution system ϕ_w: $1 \mapsto 123$, $2 \mapsto 13$, $3 \mapsto 2$. Thus, the initial letters of w are 123132123213.... This sequence was constructed in connection with the problem of constructing a nonrepetitive sequence on a 3-letter alphabet, that is, a sequence that does not contain any subwords of the type XX, where X is any non-empty word over a 3-letter alphabet. The sequence w has that property. The question of the existence of such a sequence, as well as the questions of the existence of sequences avoiding other kinds of repetitions, were studied in algebra [Adian, Justin, Kol], discrete analysis [Carpi, Dekk, Evdok2, Ker, Pleas] and in dynamical systems [MorseHedl]. In Examples 2.2, 2.6 and 3.3 of Paper VII we give the number of occurrences of the patterns 1-2, 2-1, 1-1-$\cdots$-1, 12, 123 and 21 in the finite approximations of w.

Suppose $N_\phi^\tau(n)$ denotes the number of occurrences of a pattern τ in a word generated by some morphism ϕ after n iterations. Suppose $W = AXBYC$, where A, X, B, Y, and C are some subwords. We say that an occurrence of a pattern τ in W is *external* for the pair of words (X, Y), if this occurrence starts somewhere in X and ends somewhere in Y. Also, an occurrence of τ in W is *internal* for the word X if this occurrence is a subsequence of X. For example, if $W = 12324265$, $A = 1$, $X = 23$, $B = 2$ and $Y = 426$ then an occurrence of the generalized pattern 213, namely 324 is external for (X, Y). On the other hand, the word $X = 231$ has two internal occurrences of the pattern 2-1, namely 21 and 31.

The following theorem was proved in Paper VII.

Theorem 11. *Let $\mathcal{A} = \{1, 2, \ldots, k\}$ be an alphabet, where $k \geq 2$ and a pattern $\tau \in \{1\text{-}2, 2\text{-}1\}$. Let X_1 begins with the letter 1 and consists of ℓ copies of each letter $i \in \mathcal{A}$ ($\ell \geq 1$). Let a morphism ϕ be such that*

$$1 \to X_1,\ 2 \to X_2,\ 3 \to X_3, \ldots, k \to X_k,$$

where we allow X_i to be the empty word ϵ for $i = 2, 3, \ldots, k$ (that is, ϕ may be an erasing morphism), $\sum_{i=2}^{k} |X_i| = k \cdot d$, and each letter from $\mathcal{A}$ appears in the word $X_2X_3\ldots X_k$ exactly d times. Besides, let $e_{i,j}$ (resp. e_i) be the number of external occurrences of τ for (X_i, X_j) (resp. (X_i, X_i)), where $i \neq j$. We assume *that* $e_{i,j} = e_{j,i}$ *for all* i *and* j. *Let* s_i *be the number of internal occurrences of* τ *in* X_i. *In particular,* $s_i = e_i = e_{i,j} = e_{j,i} = 0$, *whenever* $X_i = \epsilon$; *also,* $e_i = |X_i| \cdot (|X_i| - 1)/2$, *whenever there are no repetitive letters in* X_i. *Then* $N^{\tau}_{\phi}(1) = s_1$ *and for* $n \geq 2$, $N^{\tau}_{\phi}(n)$ *is given by*

$$\ell \cdot (d+\ell)^{n-2} \sum_{i=1}^{k} s_i + \binom{\ell \cdot (d+\ell)^{n-2}}{2} \sum_{i=1}^{k} e_i + \ell^2 \cdot (d+\ell)^{2n-4} \sum_{1 \leq i < j \leq k} e_{i,j}.$$

Paper VIII. Let us define the Peano curve and the Peano words. We follow [GelbOlm] and present a description of a curve that fills the unit square $S = [0,1] \times [0,1]$, given in 1891 by D. Hilbert.

As indicated in Figure 2, the idea is to subdivide S and the unit interval $I = [0,1]$ into 4^n closed subsquares and subintervals, respectively, and to set up a correspondence between subsquares and subintervals so that inclusion relationships are preserved (at each stage of subdivision, if a square corresponds to an interval, then its subsquares correspond to subintervals of that interval).

We now define the continuous mapping f of I onto S: If $x \in I$, then at each stage of subdivision x belongs to *at least* one closed subinterval. Select either one (if there are two) and associate it to the corresponding square. In this way a decreasing sequence of closed squares is obtained corresponding to a decreasing sequence of closed intervals. This sequence of closed squares has the property that there is exactly one point belonging to all of them. This point is defined to be $f(x)$. It can be shown that the point $f(x)$ is well-defined, that is, independent of any choice of intervals containing x; the range of f is S; and f is continuous.

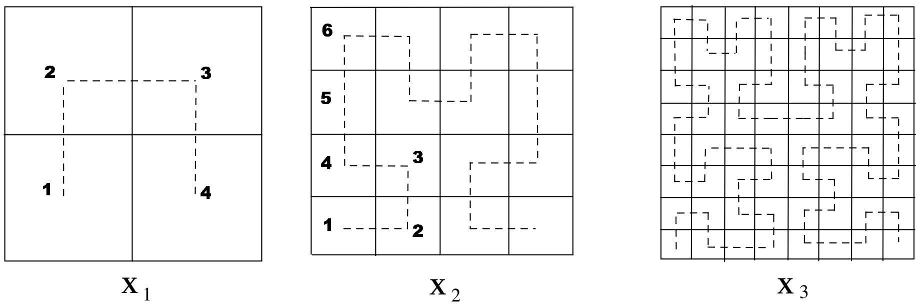

Figure 2: The Peano words

The following discrete analogue of the Peano curve was given by Evdokimov [Evdok]. For subdivision stage (iteration) n we construct a word X_n as follows: Go through the curve inside S starting at the point 1 (see Figure 2), and coding any movement "up" by 1, "right" by 2, "down" by 3, "left" by 4. Thus, we start

x	y	$N_{\tau_1(x,y)}(X_{2k+1})$	$N_{\tau_2(x,y)}(X_{2k+1})$	$N_{\tau_1(x,y)}(X_{2k+2})$	$N_{\tau_2(x,y)}(X_{2k+2})$
1	1	$\binom{4^{2k}-1}{\ell}$	$\binom{4^{2k}-1}{\ell}$	$\binom{4^{2k+1}+2^{2k+1}-1}{\ell}$	$\binom{4^{2k+1}+2^{2k+1}-1}{\ell}$
1	2	S_1	$\binom{4^{2k}}{\ell}+\binom{4^{2k}+2^{2k}-1}{\ell}$	S_2	$\binom{4^{2k+1}}{\ell}$
2	1	0	$\binom{4^{2k}-2^{2k}}{\ell}$	$\binom{4^{2k+1}}{\ell}$	S_2

Table 8: Generalized patterns having 2 letters (Paper VIII)

with the first iteration $X_1 = 123$, the second iteration is $X_2 = 214112321233432$. More generally, it is easy to see that the n-th iteration is given by

$$X_n = \varphi_1(X_{n-1})1X_{n-1}2X_{n-1}3\varphi_2(X_{n-1}),$$

where the function $\varphi_1(A)$ reverses the letters in the word A and makes the substitution corresponding to the permutation 4123, that is, 1 becomes 4 etc. The function φ_2 does the same, except with 4123 replaced by 2341. In this paper, we are interested in the words X_n, for $n = 1, 2, \ldots$, which appear as the subdivision stages of the Peano curve. We call these words the Peano words.

In Paper VIII ([KitMans5]) we consider the Peano words and find the number of occurrences of the patterns

$$12,\ 21,\ 1^\ell,\ \tau_1(x,y) = [x\text{-}y^\ell),\ \tau_2(x,y) = (x^\ell\text{-}y] \text{ and } \tau_3(x,y,z) = [x\text{-}y^\ell\text{-}z],$$

where $x, y, z \in \{1, 2, 3\}$, $y^\ell = y\text{-}y\text{-}\cdots\text{-}y$ (ℓ times), and we recall that "[" in $p = [x - w)$ indicates that in an occurrence of p, the letter corresponding to the x must be the first letter of the word. For example, the number of occurrences of the pattern 12 in X_n, according to Theorem 4 in Paper VIII, is equal to either $\frac{2}{5}(4 \cdot 16^k + 1)$ or $\frac{2}{5}(16^{k+1} - 1)$ depending on whether n is odd or even.

Let $N_\tau(W)$ denote the number of occurrences of the pattern τ in the word W. Let S_1 and S_2 denote the following:

$$S_1 = \binom{4^{2k} - 2^{2k}}{\ell} + \binom{4^{2k}}{\ell} + \binom{4^{2k} + 2^{2k} - 1}{\ell}, \quad S_2 = \binom{4^{2k+1}}{\ell} + \binom{4^{2k+1} - 2^{2k+1}}{\ell}.$$

Tables 8 and 9 give all the results concerning the patterns $\tau_1(x,y)$, $\tau_2(x,y)$ and $\tau_3(x,y,z)$ except those triples (x,y,z), for which $N_{\tau_3(x,y,z)}(X_n) = 0$ for all n.

Paper IX. Let us define the sigma-sequence and the sigma-words. In [Evdok1, Yab], Evdokimov constructed chains of maximal length in the n-dimensional unit cube using the *sigma-sequence.* The sigma-sequence w_σ was defined there by the following recursive scheme:

x	y	z	$N_{\tau_3(x,y,z)}(X_{2k+1})$	$N_{\tau_3(x,y,z)}(X_{2k+2})$
1	1	1	0	$\binom{4^{2k+1}-2}{\ell}$
1	1	2	$\binom{4^{2k}-1}{\ell}$	0
1	2	1	0	S_2
1	2	2	$\binom{4^{2k}-1}{\ell}$	0
2	1	2	0	$\binom{4^{2k+1}}{\ell}$
1	2	3	$\binom{4^{2k}+2^{2k}-1}{\ell}$	0
1	3	2	$\binom{4^{2k}-2^{2k}}{\ell}$	0

Table 9: Generalized patterns having 3 letters (Paper VIII)

$$C_1 = 1, \qquad D_1 = 2$$
$$C_{k+1} = C_k 1 D_k, \qquad D_{k+1} = C_k 2 D_k$$
$$k = 1, 2, \ldots$$

and $w_\sigma = \lim_{k\to\infty} C_k$. Thus, the initial letters of w_σ are $11211221112212\ldots$. We call the words C_k the *sigma words.* The first four values of the sequence $\{C_k\}_{k\geq 1}$ are 1, 112, 1121122, 112112211122122.

In [Kit] an equivalent definition of w_σ was given: any natural number $n \neq 0$ can be presented unambiguously as $n = 2^t(4s + \sigma)$, where $\sigma < 4$, and t is the greatest natural number such that 2^t divides n. If n runs through the natural numbers then σ runs through some sequence consisting of 1s and 3s. If we substitute 2 for 3 in this sequence, we get w_σ.

In Paper IX ([Kit4]) we give either an explicit formula or recurrence relation for the number of occurrences for some classes of patterns, subwords and subsequences in the sigma-words. In particular, Theorem 4 allows us to find the number of occurrences of an arbitrary generalized pattern without internal dashes of length ℓ, provided we know certain four numbers that can be easily calculated for the words C_k, D_k, C_{k+1} and D_{k+1}, where $k = \lceil \log_2 \ell \rceil$. Theorem 9 gives a recurrence relation for counting occurrences of patterns of the form τ_1-τ_2. In Section 6 we discuss occurrences of patterns of the form τ_1-τ_2-$\cdots$-τ_k, where the pattern τ_i does not overlap with the patterns τ_{i-1} and τ_{i+1} for $i = 1, 2, \ldots, k-1$. Finally, Section 7 deals with patterns of the form $[\tau_1$-τ_2-$\cdots$-$\tau_k]$, $[\tau_1$-τ_2-$\cdots$-$\tau_k)$ and $(\tau_1$-τ_2-$\cdots$-$\tau_k]$ in the Babson-Steingrímsson notation.

To formulate some of the results from Paper IX we need the following definitions.

Suppose a word $W = AaB$, where A and B are some words of *the same length*, and a is a single letter. We define the *kernel of order k* for the word

W to be the subword consisting of the $k-1$ rightmost letters of A, the letter a, and the $k-1$ leftmost letters of B. We denote it by $\mathcal{K}_k(W)$. For example, $\mathcal{K}_3(111211221) = 12112$. If $|A| < k-1$ then we set $\mathcal{K}_k(W) = \epsilon$, that is the kernel in this case is the empty word. Also, $m_k(\tau, W)$ denotes the number of occurrences of the pattern τ in $\mathcal{K}_k(W)$.

The following theorems are proved in Paper IX.

Theorem 12. *Let $\tau = \tau_1\tau_2\dots\tau_\ell$ be an arbitrary generalized pattern without internal dashes that consists of 1s and 2s. Suppose $k = \lceil \log_2 \ell \rceil$, $a = m_\ell(\tau, D_k 1 C_k)$, and $b = m_\ell(\tau, D_k 2 C_k)$. Then for $n > k+1$, we have*

$$c_n^\tau = (a + b + c_{k+1}^\tau + d_{k+1}^\tau) \cdot 2^{n-k-2} - b,$$

$$d_n^\tau = (a + b + c_{k+1}^\tau + d_{k+1}^\tau) \cdot 2^{n-k-2} - a.$$

Theorem 13. *Let $p = \tau_1\text{-}\tau_2$ be a generalized pattern such that $|\tau_1| = k_1$ and $|\tau_2| = k_2$. Suppose $k = \lceil \log_2(k_1 + k_2 - 1) \rceil$. Let the following denote the number of occurrences of the subwords τ_1 and τ_2 in the kernels (recall that by definitions $|C_n| = |D_n|$):*

$$\begin{array}{ll} a_{\tau_1} = m_{k_1}(\tau_1, D_k 1 C_k) & a_{\tau_2} = m_{k_2}(\tau_2, D_k 1 C_k) \\ b_{\tau_1} = m_{k_1}(\tau_1, D_k 2 C_k) & b_{\tau_2} = m_{k_2}(\tau_2, D_k 2 C_k) \end{array}$$

Also, let r_1^a (resp. r_2^a, r_1^b, r_2^b) denote the number of occurrences of overlapping subwords τ_1 and τ_2 in the word $D_k 1 C_k$ (resp. $D_k 1 C_k$, $D_k 2 C_k$, $D_k 2 C_k$), where $\tau_1 \in \mathcal{K}_{k_1}(D_k 1 C_k)$ and $\tau_2 \in C_k$ (resp. $\tau_1 \in D_k$ and $\tau_2 \in \mathcal{K}_{k_2}(D_k 1 C_k)$, $\tau_1 \in \mathcal{K}_{k_1}(D_k 2 C_k)$ and $\tau_2 \in C_k$, $\tau_1 \in D_k$ and $\tau_2 \in \mathcal{K}_{k_2}(D_k 2 C_k)$).

Besides, we assume that we know $c_n^{\tau_i}$ and $d_n^{\tau_i}$ for $n > n_i$, $i = 1, 2$. Then for $n > \max(k+1, n_1+1, n_2+1)$, c_n^τ and d_n^τ are given by the following recurrence:

$$\begin{pmatrix} c_n^\tau \\ d_n^\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_{n-1}^\tau \\ d_{n-1}^\tau \end{pmatrix} + \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix},$$

where

$$\alpha_n = (c_{n-1}^{\tau_1} + a_{\tau_1} - r_1^a) d_{n-1}^{\tau_2} + (a_{\tau_2} - r_2^a) c_{n-1}^{\tau_1}$$

and

$$\beta_n = (c_{n-1}^{\tau_1} + b_{\tau_1} - r_1^b) d_{n-1}^{\tau_2} + (b_{\tau_2} - r_2^b) c_{n-1}^{\tau_1}.$$

Theorem 14. *Let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k$ be a generalized pattern such that $|\tau_i| = k_i$ for $i = 1, 2, \dots, k$. We assume that for $i = 1, 2, \dots, k-1$, the subword τ_i does not overlap with the subwords τ_{i-1} and τ_{i+1} in the following sense: no suffix of τ_{i-1} is a prefix of τ_i and no suffix of τ_i is a prefix of τ_{i+1}.*

Suppose $\ell_i = \lceil \log_2 k_i \rceil$*,* $\ell = \max_i \ell_i$*, and for the subwords* τ_i *we have* $a_i = m_{k_i}(\tau_i, D_{\ell_i} 1 C_{\ell_i})$ *and* $b_i = m_{k_i}(\tau_i, D_{\ell_i} 2 C_{\ell_i})$*, for* $i = 1, 2, \ldots, k$*.*

We assume that we know $c_{n-1}^{\tau_1\text{-}\cdots\text{-}\tau_i}$ *and* $d_{n-1}^{\tau_{i+1}\text{-}\cdots\text{-}\tau_k}$ *for each* $1 \le i \le k-1$ *and for all* $n > n^\star$*. Then for all* $n > \max(\ell+1, n^\star+1)$*,* c_n^τ *and* d_n^τ *are given by the following recurrence:*

$$\begin{pmatrix} c_n^\tau \\ d_n^\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_{n-1}^\tau \\ d_{n-1}^\tau \end{pmatrix} +$$

$$\sum_{i=1}^{k-1} \begin{pmatrix} c_{n-1}^{e(i)} \cdot d_{n-1}^{f(i)} \\ c_{n-1}^{e(i)} \cdot d_{n-1}^{f(i)} \end{pmatrix} + \sum_{i=1}^{k} \begin{pmatrix} a_i \cdot c_{n-1}^{e(i-1)} \cdot d_{n-1}^{f(i)} \\ b_i \cdot c_{n-1}^{e(i-1)} \cdot d_{n-1}^{f(i)} \end{pmatrix},$$

where $e(i) = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_i$ *and* $f(i) = \tau_{i+1}\text{-}\tau_{i+2}\text{-}\cdots\text{-}\tau_k$*.*

Theorem 15. *Suppose* τ_1 *and* τ_2 *are two patterns without internal dashes such that* $|\tau_1| = k_1$ *and* $|\tau_2| = k_2$*. Also, suppose* $\ell_1 = \log_2(k_1+1)$*,* $\ell_2 = \log_2(k_2+1)$ *and* $\ell = \log_2(k_1+k_2+1)$*.*

Let $a(\tau_1, \tau_2)$ *be the number of overlapping subwords* τ_1 *and* τ_2 *in* C_ℓ *such that* τ_1 *consists of the* k_1 *leftmost letters of* C_ℓ*;* $b(\tau_1, \tau_2)$ *is the number of overlapping subwords* τ_1 *and* τ_2 *in* C_ℓ *such that* τ_2 *consists of the* k_2 *rightmost letters of* C_ℓ*.*

We assume that we know $c_n^{\tau_i}$ *and* $d_n^{\tau_i}$ *for* $i = 1, 2$ *and for all* $n > n^\star$*.*

i. For $n \ge \max(\ell_1, n^\star)$*,*

$$c_n^{[\tau_1\text{-}\tau_2)} = \begin{cases} c_n^{\tau_2} - a(\tau_1, \tau_2), & \text{if } C_{\ell_1} \text{ begins with } \tau_1, \\ 0, & \text{otherwise.} \end{cases}$$

ii. For $n \ge \max(\ell_2, n^\star)$*,*

$$c_n^{(\tau_1\text{-}\tau_2]} = \begin{cases} c_n^{\tau_1} - b(\tau_1, \tau_2), & \text{if } C_{\ell_2} \text{ ends with } \tau_2, \\ 0, & \text{otherwise.} \end{cases}$$

iii. For $n \ge \ell$*,*

$$c_n^{[\tau_1\text{-}\tau_2]} = \begin{cases} 1, & \text{if } C_\ell \text{ begins with } \tau_1 \text{ and ends with } \tau_2, \\ 0, & \text{otherwise.} \end{cases}$$

iv. For $n \ge \max(\ell_1, n^\star)$*,*

$$d_n^{[\tau_1\text{-}\tau_2)} = \begin{cases} d_n^{\tau_2} - a(\tau_1, \tau_2), & \text{if } D_{\ell_1} \text{ begins with } \tau_1, \\ 0, & \text{otherwise.} \end{cases}$$

v. For $n \geq \max(\ell_2, n^\star)$,

$$d_n^{(\tau_1\text{-}\tau_2]} = \begin{cases} d_n^{\tau_1} - b(\tau_1, \tau_2), & \textit{if } D_{\ell_2} \textit{ ends with } \tau_2, \\ 0, & \textit{otherwise.} \end{cases}$$

vi. For $n \geq \ell$,

$$d_n^{[\tau_1\text{-}\tau_2]} = \begin{cases} 1, & \textit{if } D_\ell \textit{ begins with } \tau_1 \textit{ and ends with } \tau_2, \\ 0, & \textit{otherwise.} \end{cases}$$

So, in Paper IX we count occurrences of certain patterns, subsequences and subwords in the sigma-words, which are particular initial subwords of w_σ. However, the challenging question is to find the number of occurrences of patterns, subsequences and subwords in an arbitrary initial subword of w_σ, or more generally, in a subword of w_σ starting in position i and ending in position j.

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Paper I

Multi-avoidance of generalised patterns

Multi-Avoidance of Generalised Patterns

Sergey Kitaev [1]

Abstract

Recently, Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. We investigate simultaneous avoidance of two or more 3-patterns without internal dashes, that is, where the pattern corresponds to a contiguous subword in a permutation.

1.1 Introduction and Background

We write permutations as words $\pi = a_1 a_2 \cdots a_n$, whose letters are distinct and usually consist of the integers $1, 2, \ldots, n$.

An occurrence of a pattern p in a permutation π is "classically" defined as a subsequence in π (of the same length as the length of p) whose letters are in the same relative order as those in p. Formally speaking, for $r \leq n$, we say that a permutation σ in the symmetric group $\mathcal{S}_n$ has an occurrence of the pattern $p \in \mathcal{S}_r$ if there exist $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that $p = \sigma(i_1)\sigma(i_2)\ldots\sigma(i_r)$ in reduced form. The *reduced form* of a permutation σ on a set $\{j_1, j_2, \ldots, j_r\}$, where $j_1 < j_2 < \cdots < j_r$, is a permutation σ_1 obtained by renaming the letters of the permutation σ so that j_i is renamed i for all $i \in \{1, \ldots, r\}$. For example, the reduced form of the permutation 3651 is 2431.

In [1] Babson and Steingrímsson introduced *generalised permutation patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a "classical" pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if this pattern occurs in the permutation, then the letters in the permutation that correspond to 3 and 1 are adjacent. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563.

The motivation for introducing these patterns in [1] was the study of Mahonian statistics. A number of interesting results on generalised patterns were obtained in [5]. Relations to several well studied combinatorial structures, such as set partitions, Dyck paths, Motzkin paths and involutions, were shown there.

In this paper we consider 3-patterns without internal dashes, that is, generalised patterns of the form xyz. Thus, such patterns correspond to contiguous subwords anywhere in a permutation. For example the permutation $\pi = 12345$ has 3 occurrences of the pattern 123 but 10 occurrences of the classical pattern 1-2-3. Patterns without internal dashes were considered by Elizalde and Noy

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in [6]. In that paper, there is a number of results on the distribution of several classes of patterns without internal dashes. In particular, formulas are given for the bivariate exponential generating functions that count permutations by the number of occurrences of any given 3-pattern. Those formulas give rise to the exponential generating functions for the number of permutations that avoid any 3-pattern.

As in the paper by Simion and Schmidt [11], dealing with the classical patterns, one can consider the case when permutations have to avoid two or more generalised patterns simultaneously. A number of such cases were considered in [5]. However, except for the simultaneous avoidance of the patterns 123 and 132, and three more pairs that are essentially equivalent to this, there are no other results for multi-avoidance of the patterns without internal dashes. In this paper we give either an explicit formula or a recursive formula for almost all cases of simultaneous avoidance of more than two patterns. We also mention what is known about double restrictions.

1.2 Preliminaries

Since we only treat patterns of length 3, and permutations of length 1 or 2 avoid all such patterns, we always assume that our permutations have length $n \geq 3$.

Obviously, no permutation avoids all six patterns of length three. Only the increasing permutation $12 \dots n$ avoids all 3-patterns but 123, and only the decreasing permutation avoids all but 321.

Consider now permutations that avoid all but one 3-pattern, different from 123 and 321, Obviously, there is exactly one such 3-permutation. However, for $n \geq 4$ there is no such permutation. Indeed, if the permutation $\pi = a_1a_2 \dots a_n$ avoids the patterns 123 and 321, then the letters of π alternate in size. That means that $a_1a_2a_3$ and $a_2a_3a_4$ form different patterns and thus π has an occurrence of a forbidden pattern.

There are, of course, $\binom{6}{k}$ sets consisting of k different 3-patterns, so we have 15 sets of two 3-patterns, 20 with three 3-patterns and 15 with four. So we have 50 different sets having more than one restriction. But we can simplify our work by partitioning the sets into equivalence classes in the way shown below and it will be enough to consider only 18 sets of restrictions.

The *reverse* $R(\pi)$ of a permutation $\pi = a_1a_2 \dots a_n$ is the permutation $a_na_{n-1} \dots a_1$. The *complement* $C(\pi)$ is the permutation $b_1b_2 \dots b_n$ where $b_i = n+1-a_i$. Also, $R \circ C$ is the composition of R and C. For example, $R(13254) = 45231$, $C(13254) = 53412$ and $R \circ C(13254) = 21435$. We call these bijections of S_n to itself *trivial*, and it is easy to see that for any pattern p the number $A_p(n)$ of permutations avoiding the pattern p is the same as for the patterns $R(p)$, $C(p)$ and $R \circ C(p)$. For example, the number of permutations that avoid the pattern 132 is the same as the number of permutations that avoid the pattern 231. This property holds for sets of patterns as well. If we apply one of the trivial bijections to all patterns of a set G, then we get a set G' for which $A_{G'}(n)$ is equal to $A_G(n)$. For example, the number of permutations avoiding

$\{123, 132\}$ equals the number of those avoiding $\{321, 312\}$ because the second set is obtained from the first one by complementing each pattern.

So up to equivalence modulo the trivial bijections we need to investigate 18 sets of restrictions that are represented in the table below.

We define the *double factorial* $n!!$ by $0!! = 1$, and, for $n > 0$,

$$n!! = \begin{cases} n \cdot (n-2) \cdots 3 \cdot 1, & \text{if } n \text{ is odd,} \\ n \cdot (n-2) \cdots 4 \cdot 2, & \text{if } n \text{ is even.} \end{cases}$$

Recall that the *n-th Catalan number* is defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Instead of writing $A_G(n)$ for a set G of patterns, we will write $A(n)$ since it will be unambiguous what set of patterns is under consideration.

Class	**Restrictions**	**Formula**
1	123, 321, 132, 312 123, 321, 231, 213	2
2	123, 312, 132, 213 321, 213, 231, 312 123, 231, 231, 132 321, 132, 312, 231	2
3	132, 231, 213, 312	2
4	123, 321, 132, 231 123, 321, 312, 213	2, if $n = 3$ 0, if $n > 3$
5	132, 213, 312, 321 231, 312, 213, 123 213, 132, 231, 321 312, 231, 132, 123	$n - 1$
6	123, 321, 132, 213 123, 321, 231, 312	$2C_k$, if $n = 2k + 1$ $C_k + C_{k-1}$, if $n = 2k$
7	123, 132, 213 231, 312, 321	$\binom{n}{\lfloor n/2 \rfloor}$
8	123, 132, 231 123, 213, 312 132, 231, 321 213, 312, 321	n

Class	Restrictions	Formula
9	132, 213, 231 132, 213, 312 132, 231, 312 213, 231, 312	$1+2^{n-2}$
10	123, 132, 312 123, 213, 231 132, 312, 321 213, 231, 321	Recursive Formula: $A(0)=1;\ A(1)=1;$ $A(n)=\sum_i \binom{n-i-1}{i} A(n-2i-1)+((n+1) \bmod 2)$ The first few numbers: 1, 1, 2, 3, 6, 13, 29, 72, 185...
11	123, 321, 132 123, 321, 231 123, 321, 312 123, 321, 213	$(n-1)!!+(n-2)!!$
12	123, 231, 312 132, 213, 321	?
13	123, 231 321, 132 321, 213 123, 312	?
14	213, 231 312, 132	?
15	132, 213 231, 312	?
16	123, 321	$2E_n$, where E_n is the n-th Euler number
17	123, 132 321, 231 321, 312 123, 213	the number of involutions in S_n (Claesson, [5])
18	132, 231 312, 213	2^{n-1}

We now give proofs and comments for the results represented in the table.

1.3 Proofs, remarks, comments

From now on, when talking about class **i**, we mean the first set of patterns in the equivalence class **i** according to the table above. Thus, for instance, **8** will be taken to refer to the set of patterns $\{123, 132, 231\}$.

Let us consider class **1**. There are only two patterns, namely 231 and 213, that are *allowed* to occur. Suppose a permutation $\pi = a_1a_2\dots a_n$ avoids the patterns from **1**. If $a_1a_2a_3$ forms a 231-pattern then $a_2a_3a_4$ has to form a 213-pattern since $a_2 > a_3$. It is easy to see that $a_3a_4a_5$ has to form the pattern

231 and so on. Moreover, if we consider the letters in even positions from left to right then we get an increasing sequence any element of which is greater then any element in an odd position; letters in odd positions form a decreasing sequence when read from left to right. From this we see that there is a unique such permutation in which the letters $\{1, 2, \ldots, \lfloor(n+1)/2\rfloor\}$ are in the odd positions in decreasing order, and all other letters are in the even positions in increasing order.

By the same argument there is only one permutation that avoids **1** and begins with a 213-pattern. Thus, in this case $A(n) = 2$.

For class **2** there are only two permutations that avoid it, namely $\pi_1 = n(n-1)(n-2)\ldots 1$ and $\pi_2 = (n-1)n(n-2)(n-3)\ldots 1$. This is because n has to be either in the leftmost position or in the second position from the left, for otherwise we have either an occurrence of the pattern 123 or of the pattern 213 that involves n. To the right of n we have to have decreasing order because otherwise we have an occurrence of a 312- or a 213-pattern. Moreover, if n is in the second position from the left then in the leftmost position we must have the letter $(n-1)$ because otherwise $(n-1)$ must be in the third place and the first three letters form a 132-pattern.

There are obviously only two permutations that avoid class **3**. They are $\pi_1 = 12\ldots n$ and $\pi_2 = n(n-1)\ldots 1$.

For class **4**, only the patterns 213 and 312 are allowed. Obviously, for $n = 3$ we have $A(n) = 2$. Suppose $n > 3$. If a permutation $\pi = a_1a_2\ldots a_n$ avoids **4**, then it has to be that $a_2 < a_3$, because $a_1a_2a_3$ forms either a 213- or a 312-pattern. But this means that $a_2a_3a_4$ cannot form a 213- or a 312-pattern, whence $A(n) = 0$.

For class **5**, n has to be either in the rightmost position or in the second position from the right, for otherwise we have an occurrence of a 312- or a 321-pattern. Moreover we must have increasing order to the left of n because otherwise we have an occurrence of a 213- or a 312-pattern. Thus there is only one permutation with n in the rightmost position.
If n is in the second position from the right then $(n-1)$ cannot be in the rightmost position, because in this case we have an occurrence of a 132-pattern that involves n and $(n-1)$. So in this case $(n-1)$ has to be in the third position from the right, and we can put any letter i other than $n-1$ and n in the rightmost position. This means that $A(n) = 1 + (n-2) = n-1$.

Class **6** will be considered in Theorem 2 below.

Theorem 1. *For class* **7** *we have* $A(n) = \binom{n}{\lfloor n/2\rfloor}$.

Proof. Let us construct a permutation that avoids class **7** by inserting the numbers $1, 2, \ldots, n$ into n slots and observing the following:

The number 1 can be placed either in the rightmost slot or in the second slot from the right, since otherwise, independently of what we have to the right of 1 in the permutation, we get either a 123- or a 132-pattern, which is prohibited. If 1 has already been placed then 2 must be placed in such way that:

1. The two slots immediately to the right of 2 are not both empty, for otherwise we will get an occurrence of either a 123- or a 132-pattern involving 2;

2. If 1 is not in the rightmost slot then 2 cannot be immediately to the left of 1, because in this case we will get an occurrence of a 213-pattern involving the letters 1 and 2.

In general it is easy to see that if i letters have been placed then for some j such that $0 \leq j \leq i$ the rightmost j slots are non-empty and the $2 \cdot (i-j)$ slots immediately to the left of these j slots are alternatingly empty and non-empty. By an argument analogous to the above we can only place the letter $(i+1)$ into either

0) the rightmost empty slot or

1) the second empty slot to the left of the leftmost non-empty slot.

If we place 1 next to the rightmost slot we assume that we use option 1).

Let us call the leftmost two slots *critical* slots. When we fill one of the critical slots, there is only one way to place the remaining letters, using option 0), since in this case, option 1) can not be applied any more.

So any permutation with the right properties can be written as a sequence of 0s and 1s according to which option we use in placing the ith letter $(i = 1, 2, \ldots)$ and we stop writing a (0,1)-sequence whenever we reach one of the critical slots.

Let us call the (0,1)-sequences thus constructed *legal sequences.*

Example 1. Let $n = 6$. The (0,1)-sequence 01101 is a legal sequence that corresponds to the permutation 5736241. But 1111 is not a legal sequence, because after 3 steps, namely 111, we are already in a critical slot and must stop writing the (0,1)-sequence.

Since obviously there is a bijection between legal sequences and permutations in class **7**, our problem is to count all possible legal sequences. We prove by induction on n that the number of such sequences is equal to $\binom{n}{\lfloor n/2 \rfloor}$.

It is easy to check this for $n = 3$.

Assuming that for all $i < n$ we have $A(i) = \binom{i}{\lfloor i/2 \rfloor}$, we prove the statement for $A(n)$. We consider separately the cases when n is even and odd.

Suppose n is even. The number of legal sequences that begin with 0 is obviously equal to

$$A(n-1) = \binom{n-1}{\lfloor (n-1)/2 \rfloor} = \binom{n-1}{(n-2)/2}.$$

Now we prove that the number of legal sequences beginning with 1 is equal to the number of legal sequences beginning with 0. We shall show that a bijection between these legal sequences is given by the correspondence $0X \leftrightarrow 1X$, where $0X$ is any legal (0,1)-sequence of length ℓ, $\frac{n}{2} \leq \ell \leq n-1$, that starts with 0. From this it follows that

$$A(n) = 2A(n-1) = \binom{n-1}{(n-2)/2} + \binom{n-1}{(n-2)/2} =$$

$$= \binom{n-1}{(n-2)/2} + \binom{n-1}{n/2} = \binom{n}{n/2} = \binom{n}{\lfloor n/2 \rfloor}.$$

So the problem is to prove that $0X \leftrightarrow 1X$ is a bijection.

We use induction on even n. If $n = 2$ then we only have the critical slots and thus there are only two legal sequences possible, namely 0 and 1. In this case $X = \emptyset$ and we have that $0X \leftrightarrow 1X$ is a bijection.

Suppose for all even m less than n the correspondence $0X \leftrightarrow 1X$ is a bijection. We consider the case $m = n$. Recall that n is even.

By *n-permutation* we mean a permutation of elements $1, 2, \ldots, n$.

A (0,1)-sequence $p_0 = 00X'$ is a legal sequence that corresponds to some n-permutation avoiding **7** if and only if $p_0' = X'$ is a legal sequence that corresponds to some $(n-2)$-permutation. To see this we observe that after the first two steps, p_0 fills in the two rightmost slots. We can strike them and forget about the first two steps of p_0; by this, we are left with the (0,1)-sequence X' that can be investigated (if it is a legal sequence) with respect to $(n-2)$-permutations.

By the same reasoning, a (0,1)-sequence $p_1 = 10X'$ is a legal sequence that corresponds to some n-permutation avoiding **7** if and only if $p_1' = X'$ is a legal sequence that corresponds to some $(n-2)$-permutation.

From these arguments we conclude, that if $X = 0X'$ then the correspondence $0X \leftrightarrow 1X$ is a bijection.

For any natural number k, we write (k) instead of writing k consecutive letters 1. In particular $(0) = \emptyset$.

Suppose $X = (k)0X'$ and $k \geq 1$. Reasoning as before, $p_0 = 0(k)0X'$ is a legal sequence with respect to n-permutations if and only if $p_0' = 0(k-1)X'$ is a legal sequence with respect to $(n-2)$-permutations. Also, $p_1 = 1(k)0X'$ is a legal sequence with respect to n-permutations if and only if $p_1' = 1(k-1)X'$ is a legal sequence with respect to $(n-2)$-permutations. By induction, for $(n-2)$-permutations, the correspondence $0Y \leftrightarrow 1Y$ between legal sequences $0Y$ and $1Y$ is a bijection, thus the correspondence $0X \leftrightarrow 1X$, when $X = (k)0X'$, is a bijection for n-permutations as well.

The last thing we need to observe is that since n is even, $p_0 = 0(k)$ is a legal sequence if and only if $p_1 = 1(k)$ is a legal sequence.
This proves that the correspondence $0X \leftrightarrow 1X$ is a bijection.

Suppose n is odd. If a legal sequence begins with 0, then we obviously have that there are $A(n-1) = \binom{n-1}{(n-1)/2}$ such legal sequences. So to prove the statement we need to prove that the number of legal sequences that begin with 1 is equal to $\binom{n-1}{(n+1)/2}$ because if it is so then we have

$$A(n) = \binom{n-1}{(n-1)/2} + \binom{n-1}{(n+1)/2} = \binom{n}{(n-1)/2} = \binom{n}{\lfloor n/2 \rfloor}.$$

If a legal sequence begins with 1 then either

i) the number of 1s always exceeds the number of 0s, or

ii) at some point the number of 1s is equal to the number of 0s.

Let us consider case **i)**. Here we deal with Catalan numbers, which, among many other things, count the *Dyck paths.* A Dyck path of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ that never goes below the x-axis. Let us explain why in case **i)** we have $\frac{1}{(n-1)/2}\binom{n-3}{(n-3)/2}$ legal sequences with the right properties.

We can see that the number of ones is fixed in this case and equal to $(n-1)/2$. We can complete our (0,1)-sequence with 0s if necessary (in order to complete a Dyck path that corresponds to the (0,1)-sequence under consideration). Moreover, we can forget about the leftmost letter 1 because we know that it is followed by another letter 1, so we have $(n-3)/2$ ones. We thus substitute $k = (n-3)/2$ in the formula for the Catalan numbers, $C_k = \frac{1}{k+1}\binom{2k}{k}$, which completes the consideration of **i)**.

In case **ii)** we apply induction. Let us consider the first time, say step i, when the number of 0s is equal to the number of 1s. Obviously it can occur at any even step (and not at any odd one). Moreover, because it is the first such time, if we consider initial subsequences of length less then i, we always have that in such subsequences the number of 1s exceeds the number of 0s. So in case **ii)**, if we apply the induction hypothesis to the $A(n-i)$, the number of legal sequences is equal to

$$\sum_{\substack{i=2 \\ i \text{ is even}}}^{n-3} \frac{1}{i/2}\binom{i-2}{(i-2)/2} A(n-i) = \sum_{\substack{i=2 \\ i \text{ is even}}}^{n-3} \frac{1}{i/2}\binom{i-2}{(i-2)/2}\binom{n-i}{(n-i-1)/2}.$$

So to complete the case when n is odd we need only check the following equality:

$$\binom{n-1}{(n+1)/2} = \sum_{\substack{i=2 \\ i \text{ is even}}}^{n-3} \frac{1}{i/2}\binom{i-2}{(i-2)/2}\binom{n-i}{(n-i-1)/2} + \frac{1}{(n-1)/2}\binom{n-3}{(n-3)/2}.$$

The last term can be moved inside the sum. Since n is odd, we have $n = 2m+1$ and the equation above can be rewritten as

$$\binom{2m}{m+1} = \sum_{i=1}^{m} \frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(m-i)+1}{m-i}.$$

We give a combinatorial proof of this identity. We observe that the left hand side of it counts the number of all lattice paths from $(0,0)$ to $(2m,-2)$ with steps $(1,1)$ and $(1,-1)$.

The i-th term in the right hand side counts the number of such paths whose first step below the x-axis is just after step $2(i-1)$. Now the first $2(i-1)$ steps of any such path determine a Dyck path of length $2(i-1)$. So there are $\binom{2(i-1)}{i-1}/i$ possibilities for a such path to pass the point $(2(i-1),0)$ and come to the point $(2i-1,-1)$ with the $(1,-1)$ step. We multiply this number with $\binom{2(m-i)+1}{m-i}$ which counts the number of all lattice paths from $(2i-1,-1)$ to $(2m,-2)$ with steps $(1,1)$ and $(1,-1)$. Thus, the right hand side counts the same paths as the left hand side.

This completes the case when n is odd and thereby the proof. □

Example 2. For $n = 4$ there are indeed $\binom{4}{2} = 6$ permutations avoiding class **7**. In the table below we show these permutations and legal sequences that correspond to them.

Permutation	**Corresponding legal sequence**
4321	0000
3421	001
4231	01
4312	100
3412	101
2413	11

Theorem 2. *For class* **6** *we have*

$$A(n) = \begin{cases} 2C_k, & \text{if } n = 2k+1, \\ C_k + C_{k-1}, & \text{if } n = 2k, \end{cases}$$

where C_k is the k-th Catalan number.

Proof. We consider n empty slots. If we fill the slots successively with the letters $1, 2, \ldots, n$ then we always have one or two possibilities, namely, either

0) we place the current number in the rightmost empty slot, or

1) we place it in the second empty slot left of the leftmost non-empty slot.

Observe that we can use option 0), except in the first step, only if there is a non-empty slot to the left of the rightmost empty slot. This is a crucial difference between classes **6** and **7**.

As in the proof of Theorem 1 we can consider the critical slots as well as (0,1)-sequences that appear in the obvious way (we have always one or two possibilities until we reach a critical slot and uniquely place all remaining numbers). After that we can associate the (0,1)-sequences with Dyck paths and apply the formula for the number of Dyck paths.

The number of legal sequences that correspond to the permutations avoiding class **6**, whose rightmost letter is 1, is equal to

$$\frac{1}{\lfloor (n-1)/2 \rfloor + 1} \binom{2 \cdot \lfloor (n-1)/2 \rfloor}{\lfloor (n-1)/2 \rfloor}.$$

The number of legal sequences that correspond to the permutations avoiding class **6**, with the second letter from the right equals 1, is equal to

$$\frac{1}{\lfloor n/2 \rfloor + 1} \binom{2 \cdot \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor}.$$

From these facts we have that

$$A(n) = \frac{1}{\lfloor n/2 \rfloor + 1} \binom{2 \cdot \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} + \frac{1}{\lfloor (n-1)/2 \rfloor + 1} \binom{2 \cdot \lfloor (n-1)/2 \rfloor}{\lfloor (n-1)/2 \rfloor}.$$

Substituting n by $2k+1$ and $2k$, respectively, completes the proof. □

For class **8**, 1 must be either in the rightmost position or in the second position from the right. It is easy to see that the letters to the left of 1 must be in decreasing order. So there are n ways to choose the rightmost element of a permutation and all other elements can be placed uniquely, so there are n permutations avoiding **8**.

For class **9**, if 1 is in the rightmost position then we must place all other letters in decreasing order, so in this case we have the permutation $\pi = n(n-1)\ldots 21$ that avoids class **9**.

Assume that 1 is not in the rightmost position. The letters to the left of 1 must be in decreasing order. On the other hand it is easy to see that the letters to the right of 1 must be in increasing order (the set of such elements is non-empty). But 2 can not be to the left of 1 since in this case we obviously have an occurrence of a 213-pattern in the permutation that involves the letters 1 and

2. So 2 is immediately right of 1. Thus, to determine a permutation in class **9** is equivalent to partitioning the letters $\{3, 4, \ldots, n\}$ into two blocks. There are 2^{n-2} ways of doing it. One of the blocks is all elements of a permutation to the right of 12, and the other one is all elements to the left of 12. So there are $1 + 2^{n-2}$ permutations avoiding class **9**.

Let us consider class **10**. We explain how to get a recurrence relation for $A(n)$ in this case.

It is easy to see that 1 is either in the rightmost position or in the second position from the right. In the first case there are $A(n-1)$ permutations that avoid **10**. In the second case we can place the letter 2 either in the position immediately left of 1 or in the second position left of 1.

In the first of these cases we choose from the remaining $(n-2)$ letters a candidate for the rightmost position. One can do this in $(n-2)$ ways. Then we multiply this by $A(n-3)$ since three of rightmost positions do not affect to placement of all other letters in a permutation.

So we need to consider the case when 2 is in the second position left of 1. In general, we need to consider the case when the letters $1, 2, \ldots, i$ have been already placed in such way that $2i$ rightmost positions are alternatingly empty and non-empty, the rightmost position is empty, and these i letters are in decreasing order from the left to the right. If we place $(i+1)$ immediately left of the leftmost non-empty position then we choose i elements from the remaining $(n-i-1)$ elements in order to fill in i of rightmost empty positions. We observe that we must fill in the chosen elements in increasing order from the left to the right, otherwise we get an occurrence of a 312-pattern that is prohibited. Then we multiply this by $A(n-2i-1)$ because in this case the $(2i+1)$ rightmost letters do not affect the placement of the other letters in the permutation. So we need to consider the case when $(i+1)$ is in the second position left of i and so on.

So we have

$$A(n) = \sum_i \binom{n-i-1}{i} A(n-2i-1) + ((n+1) \bmod 2).$$

The last term appears because if n is odd we have to consider the permutation

$$\pi = \frac{n+1}{2}\frac{n-1}{2}\frac{n+3}{2}\frac{n-3}{2}\ldots 2(n-1)1n,$$

which avoids **10** and which is not counted in the sum.

As initial conditions one can take $A(0) = 1$, $A(1) = 1$.

Theorem 3. *For class* **11** *we have* $A(n) = (n-1)!! + (n-2)!!$.

Proof. Since the patterns 123 and 321 can not occur in the permutations avoiding class **11**, such permutations are *alternating* or *reverse alternating*, that is, of the form $a_1 > a_2 < a_3 > \cdots$ or $a_1 < a_2 > a_3 < \cdots$, with one more restriction. One can easily see that 1 is either in the rightmost position or next to this position, for otherwise we have an occurrence of a 123- or 132-pattern. If we go from the right to the left starting from 1 and jumping over one element then we get an increasing sequence of letters because otherwise we have an occurrence of the pattern 132.

Let $P_1(n)$ be the number of permutations having 1 in the rightmost position and let $P_2(n)$ be the number of permutations having 1 in the next to the rightmost position. Then obviously

$$A(n) = P_1(n) + P_2(n).$$

It is easy to see that

$$P_1(n) = P_2(n-1),$$
$$P_2(n) = (n-1)P_2(n-2)$$

whence $P_1(n) = (n-2)!!$ and $P_2(n) = (n-1)!!$. □

Class **16** is a classically studied object. Permutations that avoid **16** are the alternating and the reverse alternating permutations. It is well known that the exponential generating function for the number of such permutations is $2(\tan x + \sec x)^2$. The initial values for $A(n)$ are $1, 2, 4, 10, 32, 122, 544, 2770, \ldots$

For the result on class **17** we refer the reader to Porism 10 in [5].

Finally, for class **18** we can observe that to the left of 1 in such a permutation we must have a decreasing subword and to the right of 1 we must have an increasing subword, since otherwise we have either a 132- or a 231-pattern. Thus we can choose the elements to the right of 1 from the set $\{2, 3, \ldots, n\}$ in 2^{n-1} ways and then arrange uniquely the right hand side and the left hand side (elements of a permutation to the left of 1). So there are 2^{n-1} permutations that avoid class **18**.

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Paper II

Generalized pattern avoidance with additional restrictions

Generalized Pattern Avoidance with Additional Restrictions

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Abstract

Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. We consider n-permutations that avoid the generalized pattern $1-32$ and whose k rightmost letters form an increasing subword. The number of such permutations is a linear combination of Bell numbers. We find a bijection between these permutations and all partitions of an $(n-1)$-element set with one subset marked that satisfy certain additional conditions. Also we find the e.g.f. for the number of permutations that avoid a generalized 3-pattern with no dashes and whose k leftmost or k rightmost letters form either an increasing or decreasing subword. Moreover, we find a bijection between n-permutations that avoid the pattern 132 and begin with the pattern 12 and increasing rooted trimmed trees with $n+1$ nodes.

2.1 Introduction and Background

All permutations in this paper are written as words $\pi = a_1a_2\cdots a_n$, where the a_i consist of all the integers $1, 2, \ldots, n$.

A *pattern* is a word on some alphabet of letters, where some of the letters may be separated by dashes. In our notation, the classical permutation patterns, first studied systematically by Simion and Schmidt [SchSim], are of the form $p = 1 - 3 - 2$, the dashes indicating that the letters in a permutation corresponding to an occurrence of p do not have to be adjacent. In the classical case, an occurrence of a pattern p in a permutation π is a subsequence in π (of the same length as the length of p) whose letters are in the same relative order as those in p. For example, the permutation 264153 has only one occurrence of the pattern $1-2-3$, namely the subsequence 245. Note that a classical pattern should, in our notation, have dashes at the beginning and end. Since most of the patterns considered in this paper satisfy this, we suppress these dashes from the notation.

In [BabStein] Babson and Steingrímsson introduced *generalized permutation patterns* (*GPs*) where two adjacent letters in a pattern may be required to be adjacent in the permutation. Such an adjacency requirement is indicated by the absence of a dash between the corresponding letters in the pattern. Thus, a pattern with no dashes corresponds to a contiguous subword anywhere in a permutation. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subword 564, but the pattern 2-3-1 occurs also in the subwords 562 and 563. The motivation for introducing these patterns in [BabStein] was the study of Mahonian statistics.

A number of interesting results on GPs were obtained by Claesson [Claes]. Relations to several well studied combinatorial structures, such as set partitions, Dyck paths, Motzkin paths and involutions, were shown there. In [Kit1] the present author investigated simultaneous avoidance of two or more 3-letter GPs with no dashes. Also there is a number of works concerning GPs by Mansour (see for example [Mans1, Mans2]).

In this paper we consider avoidance some generalized 3-patterns with additional restrictions. The restrictions consist of demanding that a permutation begin or end with the pattern $12\dots k$ or the pattern $k(k-1)\dots 1$.

It turns out that the number of permutations that avoid the pattern $1-32$ and end with the pattern $12\dots k$ is a linear combination of the Bell numbers. The n-th Bell number is the number of ways a set of n elements can be partitioned into nonempty subsets. We find a bijection between these permutations and all partitions of an $(n-1)$-element set with one subset marked that satisfy certain special conditions. In particular, in Theorem 1, we investigate the case $k = 2$. We get that the total number of partitions of an $(n-1)$-element set with one part marked, is equal to the number of $(1-32)$-avoiding n-permutations that end with a 12-pattern. Lemma 1 gives us an identity involving the Bell numbers and the Stirling numbers of the second kind, which seems to be new. In Theorem 3 we prove that the number of 132-avoiding n-permutations that begin with the pattern 12 is equal to the number of increasing rooted trimmed trees with $n+1$ nodes.

In Sections 4 – 7, we give a complete description (in terms of *exponential generating functions (e.g.f.)*) for the number of permutations that avoid a pattern of the form xyz and begin or end with the pattern $12\dots k$ or the pattern $k(k-1)\dots 1$. We record all the results concerning these e.g.f. in the table in Section 7. The case $k = 1$ is equivalent to the absence of the additional restriction. This case was considered in [ElizNoy] and [Kit2].

We observe that avoidance of some pattern with the additional restrictions described above, in fact is equivalent to simultaneous avoidance of several patterns. For example, beginning with the pattern 12 is equivalent to the avoidance of the pattern [21) in the Babson-Steingrímsson notation. Thus avoidance of the pattern 132 and beginning with the pattern 12 is equivalent to simultaneous avoidance of the patterns 132 and [21). Also, ending with the pattern 123 is equivalent to simultaneously avoiding the patterns (132], (213], (231], (312] and (321].

2.2 Set partitions and pattern avoidance

We recall some basic definitions.

A *partition* of a set S is a family, $\pi = \{A_1, A_2, \dots, A_k\}$, of pairwise disjoint non-empty subsets of S such that $S = \cup_i A_i$. The total number of partitions of an n-element set is called a *Bell number* and is denoted B_n.

The *Stirling number of the second kind* $S(n,k)$ is the number of ways a set with n elements can be partitioned into k disjoint, non-empty subsets.

Proposition 1. *Let $P(n,k)$ be the number of n-permutations that avoid the pattern $1-32$ and end with the pattern $12\dots k$. Then*

$$P(n,k) = \sum_{i=0}^{n-k} \binom{n-1}{i} B_i.$$

Proof. Suppose a permutation $\pi = \sigma 1 \tau$ avoids the pattern $1-32$ and ends with the pattern $12\dots k$. The letters of τ must be in increasing order, since otherwise we have an occurrence of the pattern $1-32$ involving 1. Also, σ must avoid $1-32$. If $|\sigma| = i$ then obviously $0 \le i \le n-k$ and we can choose the letters of σ in $\binom{n-1}{i}$ ways. By [Claes, Proposition 5], the number of i-permutations that avoid the pattern $1-32$ is equal to B_i, hence there are B_i ways to form σ. □

Lemma 1. *We have* $\displaystyle\sum_{i=0}^{n-1} \binom{n}{i} B_i = \sum_{i=0}^{n} i \cdot S(n,i).$

Proof. The identity can be proved from the recurrences for $S(n,k)$ and B_n, but we give a combinatorial proof.

The left-hand side of the identity is the number of ways to choose i elements from an n-element set, and then to make all possible partitions of the chosen elements.

The right-hand side is the number of ways to partition a set with n elements into i disjoint non-empty subsets ($i = 1, 2, \dots, n$) and mark one of the subsets. For example if $n = 4$ then $\overline{1} - 24 - 3$ and $1 - \overline{24} - 3$ are two different partitions, where the marked subset is overlined.

A bijective correspondence between these combinatorial interpretations is given by the following: For the left-hand side, after partitioning the i chosen elements, let the remaining $n-i$ elements form the marked subset in the partition. □

The formula for $P(n,k)$ in Proposition 1, applied to $k = 2$, and Lemma 1 now give the following theorem:

Theorem 1. *The total number of partitions of an $(n-1)$-element set with one part marked, is equal to the number of $(1-32)$-avoiding n-permutations that end with the pattern 12.*

We give now a direct combinatorial proof of this theorem.

Proof. Suppose $P = S_1 - S_2 - \cdots - S_k$ is a partition of an $(n-1)$-element set into k subsets with one marked subset and T_i is the word that consists of all elements of S_i in increasing order. We may, without loss of generality, assume that $\min(S_i) < \min(S_j)$ if $i > j$. In particular, $1 \in S_k$. There are two cases possible:

1) $S_k = \{1\}$ (S_k is not marked set);

2) Either $S_k = \overline{1}$ or $1 \in S_k$ and $|S_k| \geq 2$.

In the first case, to a partition $P = S_1 - S_2 - \cdots - \overline{S_i} - \cdots - S_{k-1} - 1$ we associate the permutation $\pi(P) = nT_1T_2 \ldots T_{i-1}T_{i+1} \ldots T_{k-1}1T_i$, which is $(1-32)$-avoiding and ends with the pattern 12 since $S_i \neq \emptyset$. For example $4 - \overline{23} - 1 \mapsto 54123$.

In the second case we adjoin n to a marked subset, and then consider the permutation $\pi(P) = T_1T_2 \ldots T_k$. This permutation is obviously $(1-32)$-avoiding since $\min(S_i) < \min(S_j)$ if $i > j$ and the letters in T_i are in increasing order. Also it ends with the pattern 12. For example $5 - \overline{34} - 12 \mapsto 534612$, and $5 - 234 - \overline{1} \mapsto 523416$.

Obviously in both cases we have an injection.

Now it is easy to see that the correspondence above is a surjection as well. Indeed, for any $(1-32)$-avoiding permutation π that ends with the pattern 12, we can check if π begins with n or not and according to this we have either case 1) or 2). In the first case, we remove n, then read π from left to right and consider all maximal increasing intervals. The elements of each such interval correspond to some subset, and we let all the letters to the right of 1 constitute the marked subset. In the second case, we divide π into maximal increasing intervals, and let the letters of each interval correspond to a subset. Then we let the interval containing n be the marked subset. Thus we have a surjection. So the correspondence is a bijection and the theorem is proved. □

The following theorem generalizes Theorem 1.

Theorem 2. *Let $P = S_1 - S_2 - \cdots - S_\ell$ be a partition of $\{1, 2, \ldots, n-1\}$ into ℓ subsets with subset S_i marked. We assume also that $1 \in S_\ell$. Then $P(n,k)$ counts all possible marked partitions of $\{1, 2, \ldots, n-1\}$ that satisfy the following conditions:*

1) if $i = \ell$ (the last subset is marked) then $|S_\ell| \geq k-1$;

2) if $i \neq \ell$ and $|S_\ell| \neq 1$ then $|S_\ell| \geq k$;

3) if $i \neq \ell$ and $|S_\ell| = 1$ then $|S_i| \geq k-1$.

Proof. A proof of this theorem is similar to the proof of Theorem 1. We assume that $\min(S_i) < \min(S_j)$ for $i > j$ and consider three cases.

If a partition satisfies 1), that is $P = S_1 - S_2 - \cdots - \overline{S_\ell}$ and $|S_\ell| \geq k-1$, then adjoining n to S_ℓ guarantees that the permutation $\pi(P) = T_1T_2 \ldots T_\ell$, which is $(1-32)$-avoiding, ends with k letters in increasing order.

In case 2), we adjoin n to the marked subset and consider $\pi(P) = T_1T_2 \ldots T_\ell$. This permutation avoids the pattern $1-32$ and ends with the pattern $12\ldots k$ since $|S_\ell| \geq k$.

In case 3), to a partition $P = S_1 - S_2 - \cdots - \overline{S_i} - \cdots - S_{k-1} - 1$ we associate the permutation $\pi(P) = nT_1T_2 \ldots T_{i-1}T_{i+1} \ldots T_{k-1}1T_i$, which is $(1-32)$-avoiding and ends with at least k letters in increasing order since $|S_i| \geq k-1$.

That this correspondence is a bijection can be shown in a way similar to the proof of Theorem 1. □

2.3 Increasing rooted trimmed trees and pattern avoidance

In an *increasing rooted tree*, nodes are numbered and the numbers increase as we move away from the root. A *trimmed tree* is a tree where no node has a single leaf as a child (every leaf has a sibling).

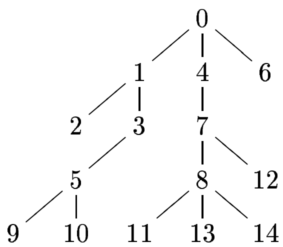
Theorem 3. *Let A_n denote the set of all n-permutations that avoid the pattern* 132 *and begin with the pattern* 12*. The number of permutations in A_n is equal to the number of increasing rooted trimmed trees (IRTTs) with $n+1$ nodes.*

Proof. A *right-to-left minimum* of a permutation π is an element a_i such that $a_i < a_j$ for every $j > i$.

We describe a bijective correspondence F between the permutations in A_n and IRTTs with $n+1$ nodes.

Suppose $\pi \in A_n$ and $\pi = P_0 a_0 P_1 a_1 \dots P_k a_k$, where a_i are the right-to-left minima of π and P_j are (possibly empty) subwords of π. We construct a IRTT $T = F(\pi)$ with $n+1$ nodes as follows. The root of T is labelled by 0 and $a_0, a_1, \dots, a_k$ are the labels of the root's children if we read them from left to right. Then we let the right-to-left minima of P_i be the labels of the children of a_i and so on. It is easy to see that, since π avoids 132 and begins with 12, T avoids limbs of length 2. Also, T is an increasing rooted tree and hence T is a IRTT. For instance,

$F(2\ 9\ 10\ 5\ 3\ 1\ 11\ 13\ 14\ 8\ 12\ 7\ 4\ 6) =$



Obviously, the correspondence F is an injection.

To see, that F is a surjection, we show how to construct the permutation $\pi \in A_n$ that corresponds to a given IRTT T. The main rule is the following: If a_i and a_j are siblings, and $a_i < a_j$, then the labels of the nodes of the subtree below a_j, are all the letters in π between a_i and a_j, that is, $a_{i+1}, a_{i+2}, \dots, a_{j-1}$. If a_i is a single child, then the labels of the nodes of the subtree below a_i appear immediately left of a_i in π. That is, if there are k nodes in the subtree below a_i then the k corresponding labels form the subword $a_{i-k} a_{i-k+1} \dots a_{i-1}$. We now start from the first level of T, which consists of the root's children, and apply this rule. After that we consider the second level and so on. The fact that T is a IRTT ensures that π avoids the pattern 132 and begins with the pattern 12. Thus, F is a bijection. $\square$

2.4 Avoiding 132 and beginning with $12\dots k$ or $k(k-1)\dots 1$

Let $E_q^p(x)$ denote the e.g.f. for the number of permutations that avoid the pattern q and begin with the pattern p.

If $k=1$, then there is no additional restriction, that is, we are dealing with avoidance of the pattern 132 (no dashes) and thus

$$E_{132}^{1}(x) = \frac{1}{1-\int_0^x e^{-t^2/2}\,dt}, \tag{2.1}$$

since this result is a special case of [ElizNoy, Theorem 4.1] and [Kit2, Theorem 12].

Theorem 4. *We have*

$$E_{132}^{12}(x) = \frac{e^{-x^2/2}}{1-\int_0^x e^{-t^2/2}\,dt} - x - 1,$$

and for $k \geq 3$

$$E_{132}^{12\dots k}(x) = E_{132}^{1}(x)\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_2}\left(e^{-t_1^2/2} - \frac{t_1+1}{E_{132}^{1}(t_1)}\right)dt_1dt_2\cdots dt_{k-2}.$$

Proof. Let $E_{n,k}$ denote the number of n-permutations that avoid the pattern 132 and begin with an increasing subword of length $k>0$. Let π be such a permutation of length $n+1$. Also, suppose $k\neq 2$. If $\pi=\sigma 1\tau$ then either $\sigma=\epsilon$ or $\sigma\neq\epsilon$ where ϵ denotes the empty word. If $\sigma=\epsilon$ then τ must avoid 132 and begin with an increasing subword of length $k-1$. Otherwise σ must avoid 132 and begin with an increasing subword of length k, whereas τ must begin with the pattern 12, or be a single letter (there are n ways to choose this letter), or be ϵ. This leads to the following:

$$E_{n+1,k} = E_{n,k-1} + \sum_{i\geq 0}\binom{n}{i}E_{i,k}E_{n-i,2} + nE_{n-1,k} + E_{n,k}. \tag{2.2}$$

Multiplying both sides of the equality with $x^n/n!$ and summing over all n we get the following differential equation

$$\frac{d}{dx}E_{132}^{12\dots k}(x) = (E_{132}^{12}(x)+x+1)E_{132}^{12\dots k}(x) + E_{132}^{12\dots(k-1)}(x), \tag{2.3}$$

with the initial conditions $E_{132}^{12\dots k}(0)=0$ for $k\geq 3$.

Observe that equality (2.3) is not valid for $k=2$. Indeed, if $k=2$, then it is incorrect to add the term $E_{n,k-1}=E_{n,1}$ in (2.2), since this term counts the number of permutations $\pi = 1\tau$ with the only restriction for τ that it must avoid 132. The absence of an additional restriction for τ means that the 3 leftmost letters of π could form the pattern 132. However, we can use (2.3) to find $E_{132}^{12}(x)$ by letting k equal 1. In this case we have

$$\frac{d}{dx}E_{132}^{1}(x) = (E_{132}^{12}(x) + x + 1)E_{132}^{1}(x),$$

which gives

$$E_{132}^{12}(x) = \frac{e^{-x^2/2}}{1 - \int_0^x e^{-t^2/2}\,dt} - x - 1. \tag{2.4}$$

For the case $k \geq 3$, it is convenient to write $E_{132}^{12}(x)$ in the form

$$E_{132}^{12}(x) = B'(x) - x - 1,$$

where $B(x) = -\ln(1 - \int_0^x e^{-t^2/2}\,dt)$ and thus $B'(x) = \exp(B(x) - \frac{x^2}{2})$. So (2.3) is equivalent to the differential equation

$$\frac{d}{dx}E_{132}^{12\dots k}(x) = B'(x)E_{132}^{12\dots k}(x) + E_{132}^{12\dots(k-1)}(x)$$

which has the solution

$$E_{132}^{12\dots k}(x) = e^{B(x)}\int_0^x e^{-B(t)}E_{132}^{12\dots(k-1)}(t)\,dt =$$

$$E_{132}^{1}(x)\int_0^x \frac{E_{132}^{12\dots(k-1)}(t)}{E_{132}^{1}(t)}dt =$$

$$E_{132}^{1}(x)\int_0^x\int_0^{t_2} \frac{E_{132}^{12\dots(k-2)}(t_1)}{E_{132}^{1}(t_1)}dt_1dt_2 =$$

$$E_{132}^{1}(x)\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_2} \frac{E_{132}^{12}(t_1)}{E_{132}^{1}(t_1)}dt_1dt_2\cdots dt_{k-2}.$$

Using (2.1) and (2.4) we now get the desired result. □

Using the formula for $E_{132}^{12\dots k}(x)$ in Theorem 4 one can derive, in particular, that

$$E_{132}^{123}(x) = -\frac{1}{2} - x - \frac{x^2}{2} + \frac{\left(1+\frac{x}{2}\right)e^{-x^2/2} - \frac{1}{2}}{1 - \int_0^x e^{-t^2/2}\,dt}.$$

Theorem 5. *For $k \geq 2$*

$$E_{132}^{k(k-1)\dots 1}(x) = \frac{E_{132}^{1}(x)}{(k-1)!}\int_0^x t^{k-1}e^{-t^2/2}\,dt.$$

Proof. We proceed as in the proof of Theorem 4.

Let $R_{n,k}$ denote the number of n-permutations that avoid the pattern 132 and begin with a decreasing subword of length $k > 1$ and let π be such a permutation of length $n+1$. Suppose also that $\pi = \sigma 1 \tau$. If $\tau = \epsilon$ then, obviously, there are $R_{n,k}$ ways to choose σ. If $|\tau| = 1$, that is, 1 is in the second position from the right in π, then there are n ways to choose the rightmost letter in π and we multiply this by $R_{k,n-1}$, which is the number of ways to choose σ. If $|\tau| > 1$ then τ must begin with the pattern 12, otherwise the letter 1 and the two leftmost letters of τ form the pattern 132, which is forbidden. So, in this case there are $\sum_{i\geq 0}\binom{n}{i}R_{i,k}E_{n-i,2}$ such permutations with the right properties, where i indicates the length of σ and $E_{n-i,2}$ is defined in the proof of Theorem 4. In the last formula, of course, $R_{i,k} = 0$ if $i < k$. Finally we have to consider the situation when 1 is in the k-th position. In this case we can choose the letters of σ in $\binom{n}{k-1}$ ways, write them in decreasing order and then choose τ in $E_{n-k+1,2}$ ways. Thus

$$R_{n+1,k} = R_{n,k} + nR_{n-1,k} + \sum_{i\geq 0}\binom{n}{i}R_{i,k}E_{n-i,2} + \binom{n}{k-1}E_{n-k+1,2}. \quad (2.5)$$

We observe that (2.5) is not valid for $n = k-1$ and $n = k$. Indeed, if 1 is in the k-th position in these cases, the term $\binom{n}{k-1}E_{n-k+1,2}$, which counts the number of such permutations, is zero, whereas there is one "good" $(n+1)$-permutation in the case $n = k-1$ and n "good" $(n+1)$-permutations in case $n = k$. Multiplying both sides of the equality with $x^n/n!$, summing over n and using the observation above (which gives the term $x^{k-1}/(k-1)! + kx^k/k!$ in the right-hand side of Equalion (2.6)), we get

$$\frac{d}{dx}E_{132}^{k(k-1)\dots 1}(x) = (E_{132}^{12}(x) + x + 1)\left(E_{132}^{k(k-1)\dots 1}(x) + \frac{x^{k-1}}{(k-1)!}\right), \quad (2.6)$$

with the initial condition $E^{132}_{k(k-1)\dots 1}(0) = 0$. We solve the equation in the way proposed in Theorem 4 and get

$$E_{132}^{k(k-1)\dots 1}(x) = \frac{E_{132}^{1}(x)}{(k-1)!}\int_0^x \frac{(E_{132}^{12}(t)+t+1)t^{k-1}}{E_{132}^{1}(t)}\,dt = \frac{E_{132}^{1}(x)}{(k-1)!}\int_0^x t^{k-1}e^{-t^2/2}\,dt.$$

□

For instance,

$$E_{132}^{21}(x) = \frac{1-e^{-x^2/2}}{1-\int_0^x e^{-t^2/2}\,dt} \quad \text{and} \quad E_{132}^{321}(x) = \frac{1}{2}\left(-1 + \frac{1-xe^{-x^2/2}}{1-\int_0^x e^{-t^2/2}\,dt}\right).$$

Moreover, the integral $\int_0^x t^{k-1}e^{-t^2/2}\,dt$ from the formula for $E_{132}^{k(k-1)\dots 1}(x)$ can be solved to show that $E_{132}^{k(k-1)\dots 1}(x)$ equals

$$\frac{(k/2-1)!2^{k/2-1}}{(k-1)!(1-\sqrt{\frac{\pi}{2}}\,\mathrm{erf}(x))}\left(1-e^{-x^2/2}\sum_{i=0}^{k/2-1}\frac{x^{2i}}{2^i i!}\right),$$

if k is even, and

$$\frac{1}{(k-1)!!}\left(-1+\frac{1}{1-\sqrt{\frac{\pi}{2}}\,\mathrm{erf}(x)}\left(1-e^{-x^2/2}\sum_{i=0}^{(k-3)/2}\frac{x^{2i+1}}{(2i+1)!!}\right)\right)$$

if k is odd.

In the formula above, $\mathrm{erf}(x)$ is the *error function*:

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

2.5 Avoiding 123 and beginning with $k(k-1)\dots 1$ or $12\dots k$

If $k=1$, we have no additional restrictions and, according to [ElizNoy, Theorem 4.1],

$$E_{123}^{1}(x)=\frac{\sqrt{3}}{2}\frac{e^{x/2}}{\cos\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)}.$$

Theorem 6. *For $k\geq 2$*

$$E_{123}^{k(k-1)\dots 1}(x)=\frac{e^{x/2}}{(k-1)!\cos\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)}\int_0^x e^{-t/2}t^{k-1}\sin\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right)\,dt.$$

In particular,

$$E_{123}^{21}(x)=\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)-x-\frac{1}{2}.$$

Proof. Let $P_{n,k}$ denote the number of n-permutations that avoid the pattern 123 and begin with a decreasing subword of length k. We observe that we can use arguments similar to the proof of Theorem 5 to get the recurrence formula for $P_{n,k}$. Indeed, we only need to write the letter P instead of R and E in (2.5):

$$P_{n+1,k}=P_{n,k}+nP_{n-1,k}+\sum_{i\geq 0}\binom{n}{i}P_{i,k}P_{n-i,2}+\binom{n}{k-1}P_{n-k+1,2}. \quad (2.7)$$

This formula is valid for $k > 1$. Multiplying both sides of the equality with $x^n/n!$, summing over n and reasoning as in the proof of Theorem 5, we get:

$$\frac{d}{dx}E_{123}^{k(k-1)\dots 1}(x) = (E_{123}^{21}(x) + x + 1)\left(E_{123}^{k(k-1)\dots 1}(x) + \frac{x^{k-1}}{(k-1)!}\right), \qquad (2.8)$$

with the initial condition $E_{123}^{k(k-1)\dots 1}(0) = 0$. To solve (2.8), we need to know $E_{123}^{21}(x)$. To find it, we consider the case $k = 1$. In this case we have almost the same recurrence as we have in (2.7), but we must remove the last term in the right-hand side:

$$P_{n+1,1} = P_{n,1} + nP_{n-1,1} + \sum_{i \geq 0}\binom{n}{i}P_{i,k}P_{n-i,2}.$$

After multiplying both sides of the last equality with $x^n/n!$ and summing over n, we have

$$\frac{d}{dx}E_{123}^{1}(x) = (E_{123}^{21}(x) + x + 1)E_{123}^{1}(x),$$

and thus

$$E_{123}^{21}(x) = \frac{\frac{d}{dx}E_{123}^{1}(x)}{P_1(x)} - x - 1 = \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}x + \frac{\pi}{6}\right) - x - \frac{1}{2}.$$

Now we solve (2.8) in the way we solved Equalion (2.6) and get

$$E_{123}^{k(k-1)\dots 1}(x) = \frac{e^{x/2}}{(k-1)!\cos\left(\frac{\sqrt{3}}{2}x + \frac{\pi}{6}\right)}\int_0^x e^{-t/2}t^{k-1}\sin\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{3}\right)\,dt.$$

□

The following theorem is straightforward to prove.

Theorem 7. *We have $E_{123}^{12\dots k}(x) = 0$ for $k \geq 3$ and*

$$E_{123}^{12}(x) = E_{123}^{1}(x) - E_{123}^{21}(x) = \frac{\sqrt{3}}{2}\frac{e^{x/2}}{\cos\left(\frac{\sqrt{3}}{2}x + \frac{\pi}{6}\right)} - \frac{1}{2} - \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}x + \frac{\pi}{6}\right).$$

2.6 Avoiding 213 and beginning with $k(k-1)\dots 1$ or $12\dots k$

If $k = 1$, then by [ElizNoy, Theorem 4.1] or [Kit2, Theorem 12]

$$E_{213}^{1}(x) = \frac{1}{1 - \int_0^x e^{-t^2/2}\,dt}.$$

Theorem 8. *For $k \geq 2$*

$$E_{213}^{12\dots k}(x) = \int_0^x \int_0^t \frac{s^{k-2}e^{T(t)-T(s)}}{(k-2)!(1-\int_0^t e^{-m^2/2}dm)}\, dsdt,$$

where $T(x) = -x^2/2 + \int_0^x \frac{e^{-t^2/2}}{1-\int_0^t e^{-s^2/2}ds}\, dt$.

Proof. Let A_n denote the number of n-permutations that avoid the pattern 213 and let B_n denote the number of n-permutations that avoid 213 and begin with the pattern $12\dots k$. Let C_n denote the number of n-permutation that avoid 213, begin with the pattern $12\dots k$ and end with the pattern 12 and let D_n denote the number of n-permutations that avoid 213 and end with the pattern 12. Also, let $A(x)$, $B(x)$, $C(x)$ and $D(x)$ denote the e.g.f. for the numbers A_n, B_n, C_n and D_n respectively.

We observe, that

$$D(x) = E_{132}^{12}(x) = e^{-x^2/2}/(1 - \int_0^x e^{-t^2/2}dt) - x - 1,$$

since, by using the reverse and complement discussed in the next section, there are as many permutations that avoid the pattern 213 and end with the pattern 12 as those that avoid the pattern 132 and begin with the pattern 12. Also, $A(x) = E_{312}^{1}(x)$ and $B(x) = E_{213}^{12\dots k}(x)$.

Suppose now that $\pi = \sigma(n+1)\tau$ is an $(n+1)$-permutation that avoids the pattern 213 and begins with the pattern $12\dots k$. So σ must avoid 213, begin with $12\dots k$, but also end with the pattern 12 since otherwise the two rightmost letters of σ together with the letter $(n+1)$ form the pattern 213, which is forbidden. For τ, there is only one restriction — avoidance of 213. So if $|\sigma| = i$ then we can choose the letters of σ in $\binom{n}{i}$ ways, which gives $\sum_{i\geq 0}\binom{n}{i}C_iA_i$ permutations that avoid the pattern 213 and begin with the pattern $12\dots k$. Moreover, it is possible for $(n+1)$ to be in the kth position, in which case we choose the letters of σ in $\binom{n}{k-1}$ ways and arrange them in increasing order. Thus

$$B_{n+1} = \sum_{i\geq 0}\binom{n}{i}C_iA_{n-i} + \binom{n}{k-1}A_{n-(k-1)}.$$

Multiplying both sides of this equality with $x^n/n!$ and summing over n, we get

$$B'(x) = \left(C(x) + \frac{x^{k-1}}{(k-1)!}\right)A(x), \tag{2.9}$$

with the initial condition $B(0) = 0$.

To solve (2.9) we need to find $C(x)$. Let $\pi = \sigma(n+1)\tau$ be an $(n+1)$-permutation that avoids the pattern 213, begins with the pattern $12\dots k$ and ends with the pattern 12. Reasoning as above, σ must avoid the pattern 213, begin with the pattern $12\dots k$ and end with the pattern 12, whereas τ must avoid 213 and end with the pattern 12. This gives $\sum_{i\geq 0}\binom{n}{i}C_iD_{n-i}$ permutations counted by C_{n+1}. Also, the letter $(n+1)$ can be in the kth position, which gives $\binom{n}{k-1}D_{n-(k-1)}$ permutations, and this letter can be in the $(n+1)$st position, which gives C_n permutations that avoid the pattern 213, begin with the pattern $12\dots k$ and end with the pattern 12. Also, if $n+1=k$ and all the letters are arranged in increasing order, then $(n+1)$ is in the $(n+1)$st position, but this permutation is not counted by C_n above. So

$$C_{n+1}=\sum_{i\geq 0}\binom{n}{i}C_iD_{n-i}+\binom{n}{k-1}D_{n-(k-1)}+C_n+\delta_{n,k-1},$$

where $\delta_{n,k}$ is the Kronecker delta, that is,

$$\delta_{n,k}=\begin{cases}1, & \text{if } n=k,\\ 0, & \text{else.}\end{cases}$$

Multiplying both sides of the equality with $x^n/n!$ and summing over n, we get

$$C'(x)=(D(x)+1)C(x)+(D(x)+1)\frac{x^{k-1}}{(k-1)!}. \tag{2.10}$$

To solve (2.10), it is convenient to introduce the function $T(x)$ such that $T'(x)=D(x)+1$. Thus

$$T(x)=x+\int_0^x D(t)dt=-x^2/2+\int_0^x\frac{e^{-t^2/2}}{1-\int_0^t e^{-s^2/2}ds}\,dt,$$

and we need to solve the equation

$$C'(x)=T'(x)C(x)+T'(x)\frac{x^{k-1}}{(k-1)!},$$

with $C(0)=0$.

The solution to this equation is given by

$$C(x)=e^{T(x)}\int_0^x e^{-T(t)}T'(t)\frac{t^{k-1}}{(k-1)!}dt=-\frac{x^{k-1}}{(k-1)!}+e^{T(x)}\int_0^x e^{-T(t)}\frac{t^{k-2}}{(k-2)!}dt.$$

Now we substitute $C(x)$ into (2.9) to get the desired result. □

Theorem 9. *For $k\geq 2$*

$$E_{213}^{k(k-1)\dots 1}(x)=-\frac{x^{k-1}}{(k-1)!}+\sum_{n=0}^{k-2}\int_0^x\int_0^{t_n}\cdots\int_0^{t_1}\frac{C_{k-n}(t)+\delta_{n,k-2}}{1-\int_0^t e^{-m^2/2}dm}dtdt_1\cdots dt_n,$$

where

$$C_k(x)=e^{T(x)}\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_1}e^{-T(t)}\left(\frac{e^{-t^2/2}}{1-\int_0^t e^{-m^2/2}dm}-t-1\right)dtdt_1\cdots dt_{k-2},$$

with $T(x)=-x^2/2+\int_0^x\frac{e^{-t^2/2}}{1-\int_0^t e^{-s^2/2}ds}\,dt$.

Proof. Let A_n denote the number of n-permutations that avoid the pattern 213 and let $B_{n,k}$ denote the number of n-permutations that avoid 213 and begin with the pattern $k(k-1)\dots 1$ for $k \geq 2$. Let $C_{n,k}$ denote the number of n-permutation that avoid 213, begin with $k(k-1)\dots 1$ for $k \geq 2$ and end with the pattern 12 and let D_n denote the number of n-permutations that avoid 213 and end with the pattern 12. Also, let $A(x)$, $B_k(x)$, $C_k(x)$ and $D(x)$ denote the e.g.f. for the numbers A_n, $B_{n,k}$, $C_{n,k}$ and D_n respectively. In the proof of Theorem 8 it was shown that $D(x) = e^{-x^2/2}/(1 - \int_0^x e^{-t^2/2}dt) - x - 1$ and $A(x) = E^1_{312}(x)$. Moreover, $B_k(x) = E^{k(k-1)\dots 1}_{213}(x)$.

Suppose now that $\pi = \sigma(n+1)\tau$ is an $(n+1)$-permutation that avoids 213 and begins with the pattern $k(k-1)\dots 1$. So σ must avoid 213, begin with $k(k-1)\dots 1$, but also end with the pattern 12 if $|\sigma| \geq 2$, since otherwise the two rightmost letters of σ together with the letter $(n+1)$ form the pattern 213 which is forbidden. For τ, there is only one restriction - avoidance of 213. So if $|\sigma| = i$ then we can choose the letters of σ in $\binom{n}{i}$ ways, which gives $\sum_{i\geq 0}\binom{n}{i}C_{i,k}A_i$ permutations counted by $B_{n+1,k}$. Also, it is possible for $(n+1)$ to be the leftmost letter, in which case the remaining letters must form a n-permutation that avoids 213 and begins with the pattern $(k-1)(k-2)\dots 1$. Thus

$$B_{n+1,k} = \sum_{i\geq 0}\binom{n}{i}C_{i,k}A_{n-i} + B_{n,k-1}. \tag{2.11}$$

However, this formula is not valid when $k = 2$ and $n = 0$. Indeed, since $B_{0,1} = A_0 = 1$, it follows from the formula that $B_{1,2} = 1$, which is not true, since $B_{1,2}$ must be 0. So, in the right-hand side of (2.11), we need to subtract the term

$$\gamma_{n,k} = \begin{cases} 1, & \text{if } n = 0 \text{ and } k = 2, \\ 0, & \text{else.} \end{cases}$$

Multiplying both sides of the obtained equality by $x^n/n!$ and summing over n, we get, that for $k \geq 3$

$$\frac{d}{dx}B_k(x) = C_k(x)A(x) + B_{k-1}(x), \tag{2.12}$$

with the initial condition $B_k(0) = 0$, and

$$\frac{d}{dx}B_2(x) = C_2(x)A(x) + B_1(x) - 1, \tag{2.13}$$

with the initial condition $B_2(0) = 0$.

The solution to differential equations (2.12) and (2.13) is given by

$$B_k(x) = -\frac{x^{k-1}}{(k-1)!} + \sum_{n=0}^{k-2}\int_0^x\int_0^{t_n}\cdots\int_0^{t_1}\frac{C_{k-n}(t) + \delta_{n,k-2}}{1 - \int_0^t e^{-m^2/2}dm}dt\,dt_1\cdots dt_n.$$

So, to prove the theorem, we only need to find $C_k(x)$.

Suppose $\pi = \sigma(n+1)\tau$ be an $(n+1)$-permutation that avoids the pattern 213, begins with the pattern $k(k-1)\dots 1$ and ends with the pattern 12. It is clear that σ must avoid 213, begin with the pattern $k(k-1)\dots 1$ and end with the pattern 12, whereas τ must avoid 213 and end with the pattern 12. There are $\sum_{i\geq 0}\binom{n}{i}C_{i,k}D_{n-i}$ permutations with these properties. Also, the letter $(n+1)$ can be in the leftmost position, which gives $C_{n,k-1}$ permutations, and $(n+1)$ can be in the rightmost position, which gives $C_{n,k}$ permutations, since in this case, two letters immediately to the left of $(n+1)$ cannot form a descent. So,

$$C_{n+1,k} = \sum_{i\geq 0}\binom{n}{i}C_{i,k}D_{n-i} + C_{n,k-1} + C_{n,k}.$$

Multiplying both sides of the equality with $x^n/n!$ and summing over n, we get the following differential equation

$$C_k'(x) = (D(x)+1)C_k(x) + C_{k-1}(x). \tag{2.14}$$

As when solving Equation (2.10), it is convenient to introduce the function $T(x)$ such that $T'(x) = D(x)+1$. Moreover, Equation (2.14) is similar to Equation (2.3) and we can solve it in the same way. Also we observe that from the definitions, $C_1(t) = D(t)$, and thus

$$C_k(x) = e^{T(x)}\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_1} e^{-T(t)}C_1(t)dtdt_1\cdots dt_{k-2} =$$

$$e^{T(x)}\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_1} e^{-T(t)}\left(\frac{e^{-t^2/2}}{1-\int_0^t e^{-m^2/2}dm} - t - 1\right)dtdt_1\cdots dt_{k-2}.$$

□

2.7 Summarizing the results from sections 2.4, 2.5 and 2.6

We recall that the *reverse* $R(\pi)$ of a permutation $\pi = a_1a_2\dots a_n$ is the permutation $a_na_{n-1}\dots a_1$ and the *complement* $C(\pi)$ is the permutation $b_1b_2\dots b_n$ where $b_i = n+1-a_i$. Also, $R\circ C$ is the composition of R and C. We call these bijections of $\mathcal{S}_n$ to itself *trivial*. Let ϕ be an arbitrary trivial bijection. It is easy to see that, for example, there are as many permutations avoiding the pattern 132 as those avoiding the pattern $\phi(132)$. Moreover if, for instance, a permutation π begins with a decreasing pattern of length k, then depending on ϕ, $\phi(\pi)$ either begins with an increasing pattern, or ends with either a decreasing or increasing pattern of length k. This allows us to apply Theorems 6 – 11 to a number of other cases. We summarize all the obtained results concerning avoidance of a generalized 3-pattern with no dashes and beginning or ending with either increasing or decreasing subword, in the table below.

	avoid	begin	end	e.g.f.
	123	$12\ldots k$	–	
1	123	–	$12\ldots k$	$\frac{\sqrt{3}}{2}\frac{e^{x/2}}{\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}$, if $k=1$
	321	$k\ldots 21$	–	$\frac{\sqrt{3}}{2}\frac{e^{x/2}}{\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}-\frac{1}{2}-\frac{\sqrt{3}}{2}\tan(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})$, if $k=2$
	321	–	$k\ldots 21$	0, if $k\geq 3$
	123	$k\ldots 21$	–	
2	123	–	$k\ldots 21$	$\frac{\sqrt{3}}{2}\frac{e^{x/2}}{\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}$, if $k=1$
	321	$12\ldots k$	–	
	321	–	$12\ldots k$	$\frac{e^{x/2}\int_0^x e^{-t/2}t^{k-1}\sin(\frac{\sqrt{3}}{2}t+\frac{\pi}{6}))\ dt}{(k-1)!\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}$, if $k\geq 2$
	132	$12\ldots k$	–	
3	213	–	$12\ldots k$	$(1-\int_0^x e^{-t^2/2}\ dt)^{-1}$, if $k=1$
	312	$k\ldots 21$	–	$e^{-x^2/2}(1-\int_0^x e^{-t^2/2}\ dt)^{-1}-x-1$, if $k=2$
	231	–	$k\ldots 21$	$(1-\int_0^x e^{-t^2/2}\ dt)^{-1}\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_2}(e^{-t_1^2/2}-$
				$(t_1+1)(1-\int_0^{t_1} e^{-t^2/2}\ dt))dt_1dt_2\cdots dt_{k-2}$, if $k\geq 3$
	132	$k\ldots 21$	–	
4	213	–	$k\ldots 21$	$(1-\int_0^x e^{-t^2/2}\ dt)^{-1}$, if $k=1$
	312	$12\ldots k$	–	$\frac{1}{(k-1)!(1-\int_0^x e^{-t^2/2}\ dt)}\int_0^x t^{k-1}e^{-t^2/2}\ dt$, if $k\geq 2$
	231	–	$12\ldots k$	
	213	$12\ldots k$	–	$(1-\int_0^x e^{-t^2/2}\ dt)^{-1}$, if $k=1$
5	132	–	$12\ldots k$	
	231	$k\ldots 21$	–	$\int_0^x\int_0^t \frac{s^{k-2}e^{T(t)-T(s)}}{(k-2)!(1-\int_0^t e^{-m^2/2}dm)}\ dsdt$, if $k\geq 2$, where
	312	–	$k\ldots 21$	$T(x)=-x^2/2+\int_0^x \frac{e^{-t^2/2}}{1-\int_0^t e^{-s^2/2}ds}\ dt$
	213	$k\ldots 21$	–	$(1-\int_0^x e^{-t^2/2}\ dt)^{-1}$, if $k=1$
6	132	–	$k\ldots 21$	$-\frac{x^{k-1}}{(k-1)!}+\sum_{n=0}^{k-2}\int_0^x\int_0^{t_n}\cdots\int_0^{t_1}\frac{C_{k-n}(t)+\delta_{n,k-2}}{1-\int_0^t e^{-m^2/2}dm}dtdt_1\cdots dt_n$,
	231	$12\ldots k$	–	if $k\geq 2$, where $C_k(x)=e^{T(x)}\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_1}e^{-T(t)}$.
	312	–	$12\ldots k$	$\left(\frac{e^{-t^2/2}}{1-\int_0^t e^{-m^2/2}dm}-t-1\right)dtdt_1\cdots dt_{k-2}$ and $T(x)$ as above

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Paper III

Simultaneous avoidance of generalized patterns

Simultaneous avoidance of generalized patterns

Sergey Kitaev[1] and Toufik Mansour [2]

Abstract

In [BabStein] Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In [Kit1] Kitaev considered simultaneous avoidance (multi-avoidance) of two or more 3-patterns with no internal dashes, that is, where the patterns correspond to contiguous subwords in a permutation. There either an explicit or a recursive formula was given for all but one case of simultaneous avoidance of more than two patterns. In this paper we find the exponential generating function for the remaining case. Also we consider permutations that avoid a pattern of the form $x - yz$ or $xy - z$ and begin with one of the patterns $12\dots k$, $k(k-1)\dots 1$, $23\dots k1$, $(k-1)(k-2)\dots 1k$ or end with one of the patterns $12\dots k$, $k(k-1)\dots 1$, $1k(k-1)\dots 2$, $k12\dots(k-1)$. For each of these cases we find either the ordinary or exponential generating functions or a precise formula for the number of such permutations. Besides we generalize some of the obtained results as well as some of the results given in [Kit3]: we consider permutations avoiding certain generalized 3-patterns and beginning (ending) with an arbitrary pattern having either the greatest or the least letter as its rightmost (leftmost) letter.

3.1 Introduction and Background

Permutation patterns: All permutations in this paper are written as words $\pi = a_1a_2\dots a_n$, where the a_i consist of all the integers $1, 2, \dots, n$. Let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. We say that α *contains* τ if there exists a subsequence $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that $(\alpha_{i_1}, \dots, \alpha_{i_k})$ is order-isomorphic to τ, that is, for all j and m, $\tau_j < \tau_m$ if and only if $a_{i_j} < a_{i_m}$; in such a context τ is usually called a *pattern*. We say that α *avoids* τ, or is τ-*avoiding*, if α does not contain τ. The set of all τ-avoiding permutations in S_n is denoted by $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that α avoids T if α avoids each $\tau \in T$; the corresponding subset of S_n is denoted by $S_n(T)$.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns τ_1, τ_2. This problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [SchSim]), for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [W]), and for $\tau_1, \tau_2 \in S_4$ (see [B, K]

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and references therein). Several recent papers [CW, MV1, Kr, MV3, MV2] deal with the case $\tau_1 \in S_3$, $\tau_2 \in S_k$ for various pairs τ_1, τ_2.

Generalized permutation patterns: In [BabStein] Babson and Steingrímsson introduced *generalized permutation patterns* (*GPs*) where two adjacent letters in a pattern may be required to be adjacent in the permutation. Such an adjacency requirement is indicated by the absence of a dash between the corresponding letters in the pattern. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subword 564, but the pattern 2-3-1 occurs also in the subwords 562 and 563. Note that a classical pattern should, in our notation, have dashes at the beginning and end. Since most of the patterns considered in this paper satisfy this, we suppress these dashes from the notation. Thus, a pattern with no dashes corresponds to a contiguous subword anywhere in a permutation. The motivation for introducing these patterns was the study of Mahonian statistics. A number of results on GPs were obtained by Claesson, Kitaev and Mansour. See for example [Claes], [Kit1, Kit2, Kit3] and [Mans1, Mans2, Mans3].

As in [SchSim], dealing with the classical patterns, one can consider the case when permutations have to avoid two or more generalized patterns simultaneously. A complete solution for the number of permutations avoiding a pair of 3-patterns of type (1,2) or (2,1), that is, the patterns having one internal dash, is given in [ClaesMans]. In [Kit1] Kitaev gives either an explicit or a recursive formula for all but one case of simultaneous avoidance of more than two patterns. This is the case of avoiding the GPs 123, 231 and 312 simultaneously. In Theorem 1 we find the exponential generating function (e.g.f.) for the number of such permutations.

As it was discussed in [Kit3], if a permutation begins (resp. ends) with the pattern $p = p_1p_2\ldots p_k$, that is, the k leftmost (resp. rightmost) letters of the permutation form the pattern p, then this is the same as avoidance of $k! - 1$ patterns simultaneously. For example, beginning with the pattern 123 is equivalent to the simultaneous avoidance of the patterns (132], (213], (231], (312] and (321] in the Babson-Steingrímsson notation. Thus demanding that a permutation must begin or end with some pattern, in fact, we are talking about simultaneous avoidance of generalized patterns. The motivation for considering additional restrictions such as beginning or ending with some patterns is their connection to some classes of trees. An example of such a connection can be found in [Kit3, Theorem 5]. There it was shown that there is a bijection between n-permutations avoiding the pattern 132 and beginning with the pattern 12 and *increasing rooted trimmed trees* with $n+1$ nodes. We recall that a trimmed tree is a tree where no node has a single leaf as a child (every leaf has a sibling) and in an increasing rooted tree, nodes are numbered and the numbers increase as we move away from the root. The avoidance of a generalized 3-pattern p with no dashes and, at the same time, beginning or ending with an increasing or decreasing pattern was discussed in [Kit3]. Theorem 2 generalizes some of these results to the case of beginning (resp. ending) with an arbitrary pattern avoiding p and having the greatest or least letter as the rightmost (resp. leftmost) letter.

Propositions 4 – 15 (resp. 16 – 27) give a complete description for the number of permutations avoiding a pattern of the form $x - yz$ or $xy - z$ and beginning with one of the patterns $12\dots k$ or $k(k-1)\dots 1$ (resp. $23\dots k1$ or $(k-1)(k-2)\dots 1k$). For each of these cases we find either the ordinary or exponential generating functions or a precise formula for the number of such permutations. Theorem 3 generalizes some of these results. Besides, the results from Propositions 4–27 give a complete description for the number of permutations that avoid a pattern of the form $x - yz$ or $xy - z$ and end with one of the patterns $12\dots k$, $k(k-1)\dots 1$, $1k(k-1)\dots 2$ and $k12\dots(k-1)$. To get the last one of these we only need to apply the reverse operation discussed in the next section. The results of Theorems 2 and 3 can also be used to get the case of ending with a pattern from the sets Δ_k^{min} or Δ_k^{max} introduced in the next section.

Except for the empty permutation, every permutation ends and begins with the pattern $p = 1$. To simplify the discussion we assume that the empty permutation also begin with the pattern 1. This does not course any harm since, to count the generating functions in question for this, we need only subtract 1 from the generating functions obtained in this paper.

3.2 Preliminaries

The *reverse* $R(\pi)$ of a permutation $\pi = a_1a_2\dots a_n$ is the permutation $a_na_{n-1}\dots a_1$. The *complement* $C(\pi)$ is the permutation $b_1b_2\dots b_n$ where $b_i = n + 1 - a_i$. Also, $R \circ C$ is the composition of R and C. For example, $R(13254) = 45231$, $C(13254) = 53412$ and $R \circ C(13254) = 21435$. We call these bijections of S_n to itself *trivial*, and it is easy to see that for any pattern p the number $A_p(n)$ of permutations avoiding the pattern p is the same as for the patterns $R(p)$, $C(p)$ and $R \circ C(p)$. For example, the number of permutations that avoid the pattern 132 is the same as the number of permutations that avoid the pattern 231. This property holds for sets of patterns as well. If we apply one of the trivial bijections to all patterns of a set G, then we get a set G' for which $A_{G'}(n)$ is equal to $A_G(n)$. For example, the number of permutations avoiding $\{123, 132\}$ equals the number of those avoiding $\{321, 312\}$ because the second set is obtained from the first one by complementing each pattern.

In this paper we denote the nth Catalan number by C_n; the generating function for these numbers by $C(x)$; the nth Bell number by B_n.

Also, $N_q^p(n)$ denotes the number of permutations that avoid the pattern q and begin with the pattern p; $G_q^p(x)$ (resp. $E_q^p(x)$) denotes the ordinary (resp. exponential) generating function for the number of such permutations. Besides, Γ_k^{min} (resp. Γ_k^{max}) denotes the set of all k-patterns with no dashes such that the least (resp. greatest) letter of a pattern is the rightmost letter; Δ_k^{min} (resp. Δ_k^{max}) denotes the set of all k-patterns with no dashes such that the least (resp. greatest) letter of a pattern is the leftmost letter.

Recall the following properties of $C(x)$:

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x} = \frac{1}{1 - xC(x)}. \tag{3.1}$$

3.3 Simultaneous avoidance of 123, 231 and 312

The *Entringer numbers* $E(n,k)$ (see [SloPlo, Sequence A000111/M1492]) are the number of permutations on $1, 2, \ldots, n+1$, starting with $k+1$, which, after initially falling, alternately fall then rise. The Entringer numbers (see [Ent]) are given by

$$E(0,0) = 1, \qquad E(n,0) = 0,$$

together with the recurrence relation

$$E(n,k) = E(n,k+1) + E(n-1, n-k).$$

The numbers $E(n) = E(n,n)$, are the secant and tangent numbers given by the generating function

$$\sec x + \tan x.$$

The following theorem completes the consideration of multi-avoidance of more than two generalized 3-patterns with no dashes made in [Kit1].

Theorem 1. *Let $E(x)$ be the e.g.f. for the number of permutations that avoid* 123, 231 *and* 312 *simultaneously. Then*

$$E(x) = 1 + x(\sec(x) + \tan(x)).$$

Proof. Let $s(n; i_1, \ldots, i_m)$ denote the number of permutations $\pi \in S_n(123, 231, 312)$ such that $\pi_1\pi_2\ldots\pi_m = i_1 i_2 \ldots i_m$ and $f : S_n \to S_n$ be a map defined by

$$f(\pi_1\pi_2\ldots\pi_n) = (\pi_1 + 1)(\pi_2 + 1)\ldots(\pi_n + 1),$$

where the addition is modulo n. Using f one can see that for all $a = 1, 2, \ldots, n-1$,

$$s(n; a) = s(n; a+1). \tag{3.2}$$

Thus, $|S_n(123, 231, 312)| = ns(n; 1)$ and we only need to prove that $s(n; 1) = E_{n-1}$, where E_n is the nth Euler number (see [SloPlo, Sequence A000111/M1492]).

Suppose $\pi \in S_n(123, 231, 312)$ is an n-permutation such that $\pi_1 = 1$ and $\pi_2 = t$. Since π avoids 123, we get $\pi_3 \leq t-1$ and it is easy to see that

$$s(n; 1, t) = \sum_{j=2}^{t-1} s(n; 1, t, j) = \sum_{j=1}^{t-2} s(n-1; t-1, j),$$

so

$$s(n; 1, t+1) = s(n; 1, t) + \sum_{j=1}^{t-1} s(n-1; t, j) - \sum_{j=1}^{t-2} s(n-1; t-1, j).$$

Using (3.2) we get

$$s(n;1,t+1) = s(n;1,t)+s(n-1;t,1)+\sum_{j=2}^{t-1} s(n-1;t-1,j-1)-\sum_{j=1}^{t-2} s(n-1;t-1,j),$$

and by (3.2) again, we have for all $t = 2,3,\dots,n-1$,

$$s(n;1,t+1) = s(n;1,t) + s(n-1;1,n-t+1).$$

Besides, by the definition, it is easy to see that $s(n;1,2) = 0$ for all $n \geq 3$, hence using the definition of Entringer numbers [Ent] we get $s(n;1) = \sum_{t=2}^{n} s_{n;1,t} = E_{n-1}$, as required. □

3.4 Avoiding a 3-pattern with no dashes and beginning with a pattern whose rightmost letter is the greatest or smallest

The following theorem generalizes Theorems 7 and 8 in [Kit3]. Recall the definition of $E_q^p(x)$ in Section 3.2.

Theorem 2. *Suppose $p_1, p_2 \in \Gamma_k^{min}$ and $p_1 \in S_k(132)$, $p_2 \in S_k(123)$. Thus, the complements $C(p_1), C(p_2) \in \Gamma_k^{max}$ and $C(p_1) \in S_k(312)$, $C(p_2) \in S_k(321)$. Then, for $k \geq 2$,*

$$E_{132}^{p_1}(x) = E_{312}^{C(p_1)}(x) = \frac{\int_0^x t^{k-1}e^{-t^2/2}\ dt}{(k-1)!(1-\int_0^x e^{-t^2/2}\ dt)}$$

and

$$E_{123}^{p_2}(x) = E_{321}^{C(p_2)}(x) = \frac{e^{x/2}\int_0^x e^{-t/2}t^{k-1}\sin(\frac{\sqrt{3}}{2}t+\frac{\pi}{6}))\ dt}{(k-1)!\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}.$$

Proof. To prove the theorem, it is enough to copy the proofs of Theorems 7 and 8 in [Kit3], since the fact that the first $k-1$ letters of p are possibly not in decreasing order is immaterial for the proofs of that theorems. Thus we can get the formula for $E_{132}^p(x)$ and $E_{123}^p(x)$, and automatically, using properties of the complement, the formula for $E_{312}^{C(p)}(x)$ and $E_{321}^{C(p)}(x)$, directly from these theorems. However we give here a proof of the formula for $E_{132}^p(x)$ and refer to [Kit3, Theorem 8] for a proof of the formula for $E_{123}^p(x)$.

If $k = 1$, we have no additional restrictions, that is, we are dealing only with the avoidance of 132 and, according to [ElizNoy, Theorem 4.1] or [Kit2, Theorem 12],

$$E_{132}^1(x) = \frac{1}{1-\int_0^x e^{-t^2/2}\ dt}.$$

Also, according to [Kit3, Theorem 6],

$$E^{12}_{132}(x) = \frac{e^{-x^2/2}}{1-\int_0^x e^{-t^2/2}\,dt} - x - 1.$$

Let $R_{n,k}$ (resp. $F_{n,k}$) denote the number of n–permutations that avoid the pattern 132 and begin with a decreasing (resp. increasing) subword of length $k > 1$ and let π be such a permutation of length $n+1$. Suppose $\pi = \sigma 1 \tau$. If $\tau = \epsilon$ then, obviously, there are $R_{n,k}$ ways to choose σ. If $|\tau| = 1$, that is, 1 is in the second position from the right, then there are n ways to choose the rightmost letter in π and we multiply this by $R_{k,n-1}$, which is the number of ways to choose σ. If $|\tau| > 1$ then τ must begin with the pattern 12, otherwise the letter 1 and the two leftmost letters of τ form the pattern 132, which is forbidden. So, in this case there are $\sum_{i\geq 0}\binom{n}{i}R_{i,k}F_{n-i,2}$ such permutations with the right properties, where i indicates the length of σ. In the last formula, of course, $R_{i,k} = 0$ if $i < k$. Finally we have to consider the situation when 1 is in the k-th position. In this case we can choose the letters of σ in $\binom{n}{k-1}$ ways, write them in decreasing order and then choose τ in $F_{n-k+1,2}$ ways. Thus

$$R_{n+1,k} = R_{n,k} + nR_{n-1,k} + \sum_{i\geq 0}\binom{n}{i}R_{i,k}F_{n-i,2} + \binom{n}{k-1}F_{n-k+1,2}. \qquad (3.3)$$

We observe that (3.3) is not valid for $n = k-1$ and $n = k$. Indeed, if 1 is in the k-th position in these cases, the term $\binom{n}{k-1}F_{n-k+1,2}$, which counts the number of such permutations, is zero, whereas there is one "good" $(n+1)$–permutation in case $n = k-1$ and n good $(n+1)$–permutations in case $n = k$. Multiplying both sides of the equality with $x^n/n!$, summing over n and using the observation above (which gives the term $x^{k-1}/(k-1)!+kx^k/k!$ in the right-hand side of equality (3.4)), we get

$$\frac{d}{dx}E^{p}_{132}(x) = (E^{12}_{132}(x)+x+1)E^{p}_{132}(x) + (E^{12}_{132}(x)+x+1)\frac{x^{k-1}}{(k-1)!}, \qquad (3.4)$$

with the initial condition $E^p_{132}(0) = 0$. We solve this equation and get

$$E^p_{132}(x) = \frac{E^1_{132}(x)}{(k-1)!}\int_0^x \frac{(E^{12}_{132}(t)+t+1)t^{k-1}}{E^1_{132}(t)}\,dt = \frac{E^1_{132}(x)}{(k-1)!}\int_0^x t^{k-1}e^{-t^2/2}\,dt.$$

□

Remark 1. *It is obvious that if in the previous theorem* $p_1 \notin S_k(132)$ *and* $p_2 \notin S_k(123)$*, then* $E^{p_1}_{132}(x) = E^{p_2}_{123}(x) = 0$.

3.5 Avoiding a pattern x-yz and beginning with an increasing or decreasing pattern

In this section we consider avoidance of one of the patterns $1-23$, $1-32$, $2-31$, $2-13$, $3-12$ and $1-32$ and beginning with a decreasing pattern. We get all the other cases, that is, avoidance of one of these patterns and beginning with an increasing pattern, by the complement operation. For instance, we have $E_{1-23}^{k(k-1)\ldots1}(x) = E_{3-21}^{12\ldots k}(x)$.

Proposition 1. *We have*

$$E_{1-23}^{k(k-1)\ldots1}(x) = E_{1-32}^{k(k-1)\ldots1}(x) = \begin{cases} (e^{e^x}/(k-1)!)\int_0^x t^{k-1}e^{-e^t+t}\,dt, & \text{if } k \geq 2, \\ e^{e^x-1}, & \text{if } k = 1. \end{cases}$$

Proof. We prove the statement for the pattern $1-23$. All the arguments we give for this pattern are valid for the pattern $1-32$. The only difference is that instead of decreasing order in τ (see below), we have increasing order.

Suppose $k \geq 2$. Let $B_{n,k}$ denote the number of n–permutations that avoid the pattern $1-23$ and begin with a decreasing subword of length k. Suppose $\pi = \sigma 1 \tau$ be one of such permutations of length $n+1$. Obviously, the letters of τ must be in decreasing order since otherwise we have an occurrence of $1-23$ in π starting from the letter 1. If $|\sigma| = i$ then we can choose the letters of σ in $\binom{n}{i}$ ways. Since the letters of τ are in decreasing order, they do not affect σ and thus there are $B_{i,k}$ possibilities to choose σ. Besides, if $|\sigma| = k-1$ and letters of σ are in decreasing order, we get $\binom{n}{k-1}$ additional possibilities to choose π. Thus

$$B_{n+1,k} = \sum_{i\geq0}\binom{n}{i}B_{i,k} + \binom{n}{k-1}.$$

Multiplying both sides of the equality with $x^n/n!$ and summing over n, we get the differential equation

$$\frac{d}{dx}E_{1-23}^{k(k-1)\ldots1}(x) = (E_{1-23}^{k(k-1)\ldots1}(x) + \frac{x^{k-1}}{(k-1)!})e^x$$

with the initial condition $E_{1-23}^{k(k-1)\ldots1}(0) = 0$. The solution to this equation is given by

$$E_{1-23}^{k(k-1)\ldots1}(x) = (e^{e^x}/(k-1)!)\int_0^x t^{k-1}e^{-e^t+t}\,dt. \tag{3.5}$$

If $k = 1$, then there is no additional restriction. According to [Claes, Prop. 2] (resp. [Claes, Prop. 5]), the number of n-permutations that avoid the pattern 1-23 (resp. 1-32) is the nth Bell number and the e.g.f. for the Bell numbers is e^{e^x-1}. However, all the arguments used for $k \geq 2$ remain the same for the case $k = 1$ except for the fact that we do not count the empty permutation, which, of course, avoids 1-23. So, if $k = 1$, we need to add 1 to the right-hand side of (3.5):

$$E^{1}_{1-23}(x) = e^{e^x} \int_0^x e^{-e^t+t}\, dt + 1 = e^{e^x-1}.$$

□

Proposition 2. *We have*

$$E^{k(k-1)\dots 1}_{3-12}(x) = \begin{cases} e^{e^x} \int_0^x e^{-e^t} \sum\limits_{n\ge k-1} \frac{t^n}{n!}\, dt, & \textit{if } k \ge 2, \\ e^{e^x-1}, & \textit{if } k = 1. \end{cases}$$

Proof. Suppose $k \ge 2$. Let $B_{n,k}$ denote the number of n–permutations that avoid the pattern $3-12$ and begin with a decreasing subword of length k. Suppose $\pi = \sigma(n+1)\tau$ be such a permutation of length $n+1$. Obviously, the letters of τ must be in decreasing order since otherwise we have an occurrence of the pattern $3-12$ in π starting from the letter $(n+1)$. If $|\sigma| = i$ then we can choose the letters of σ in $\binom{n}{i}$ ways. Since the letters of τ are in decreasing order, they do not affect σ and thus there are $B_{i,k}$ possibilities to choose σ. Besides, if $n \ge k-1$, then π can be decreasing, that is, $(n+1)$ can be in the leftmost position. Thus

$$B_{n+1,k} = \sum_{i\ge 0} \binom{n}{i} B_{i,k} + \delta_{n,k},$$

where

$$\delta_{n,k} = \begin{cases} 1, & \text{if } n \ge k-1, \\ 0, & \text{else.} \end{cases}$$

Multiplying both sides of the equality with $x^n/n!$ and summing over n, we get the differential equation

$$\frac{d}{dx} E^{k(k-1)\dots 1}_{3-12}(x) = e^x E^{k(k-1)\dots 1}_{3-12}(x) + \sum_{n\ge k-1} \frac{x^n}{n!}$$

with the initial condition $E^{k(k-1)\dots 1}_{3-12}(0) = 0$. The solution to this equation is given by

$$E^{k(k-1)\dots 1}_{3-12}(x) = e^{e^x} \int_0^x e^{-e^t} \sum_{n\ge k-1} \frac{t^n}{n!}\, dt. \tag{3.6}$$

If $k = 1$, then there is no additional restriction. In [Claes, Prop. 5] it is shown that $E^1_{1-32}(x) = e^{e^x-1}$. Using the complement, the number of n-permutations that avoid $1-32$ is equal to the number of n-permutations that avoid $3-12$. We get that $E^1_{3-12}(x) = e^{e^x-1}$. However, all the arguments used for the case $k \ge 2$ remain the same for the case $k = 1$ except the fact that we do not count the empty permutation, which avoids 3-12. So, if $k = 1$, we need to add 1 to the right-hand side of (3.6):

$$E^1_{3-12}(x) = e^{e^x} \int_0^x e^{-e^t} e^t\, dt + 1 = e^{e^x-1}.$$

□

Proposition 3. *We have*

$$E_{3-21}^{k(k-1)\dots 1}(x) = \begin{cases} 0, & \textit{if } k \geq 3, \\ e^{e^x} \int_0^x e^{-e^t}(e^t - 1)\ dt, & \textit{if } k = 2, \\ e^{e^x - 1}, & \textit{if } k = 1. \end{cases}$$

Proof. For $k \geq 3$, the statement is obviously true. If $k = 1$, then the statement follows from [Claes, Prop. 2] and the fact that there are as many n-permutations avoiding the pattern $1-23$, as n-permutations avoiding the pattern $3-21$. For the case $k = 2$, we can use exactly the same arguments as those in the proof of Proposition 2 to get the same recurrence relation and thus the same formula, which, however, is valid only for $k = 2$. □

Recall the definition of N_q^p in Section 3.2.

Proposition 4. *We have*

$$N_{2-13}^{k(k-1)\dots 1}(n) = \begin{cases} C_{n-k+1}, & \textit{if } n \geq k, \\ 0, & \textit{else.} \end{cases}$$

Proof. If $k = 1$, then the statement follows from [Claes, Prop. 22]. Suppose $k \geq 2$ and let $\pi = \sigma n \tau$ be an n-permutation avoiding $2-31$ and beginning with the pattern $k(k-1)\dots 1$. Suppose, without loss of generality that σ consists of the letters $1, 2, \dots, \ell$. Now ℓ must be the rightmost letter of σ, since otherwise ℓ, the rightmost letter of σ and n form the pattern $2-13$. Also, the letter $(\ell - 1)$ must be next to the rightmost letter of σ since otherwise the letter $(\ell - 1)$, next to the rightmost letter of σ and the letter ℓ form the pattern $2-13$. And so on. Thus σ must be increasing, which contradicts the fact that π must begin with a decreasing pattern of length greater than 1. So $|\sigma| = 0$ and τ must begin with the pattern $(k-1)(k-2)\dots 1$. Now, we can consider the letter $(n-1)$ and, by the same reasoning, get that it must be in the second position of π. Then we consider $(n-2)$, and so on up to the letter $(n-k+2)$. Finally, we get that $\pi = n(n-1)\dots(n-k+2)\pi'$, where π' must avoid the pattern $2-13$ and thus, there are C_{n-k+1} ways to choose π ([Claes, Prop. 22]). □

Recall that $C(x)$ is the generating function for the Catalan numbers. Also recall the definition of G_q^p in Section 3.2.

Proposition 5. *We have*

$$G_{2-31}^{k(k-1)\dots 1}(x) = \begin{cases} x^k C^{k+1}(x), & \textit{if } k \geq 2 \\ C(x), & \textit{if } k = 1. \end{cases}$$

Proof. If $k = 1$, then there is no additional restriction, and thus $G_{2-31}^1(x) = C(x)$ (applying the complement operation to [Claes, Prop. 22]).

Suppose $k \geq 2$. Using the reverse, we see that beginning with $k(k-1)\dots 1$ and avoiding $2-31$ is equivalent to ending with $12\dots k$ and avoiding $13-2$, which by [Claes] is equivalent to ending with $12\dots k$ and avoiding $1-3-2$.

Suppose $\pi = \pi' n \pi''$ ends with $12\dots k$ and avoids $1-3-2$. Each letter of π' must be greater than any letter of π'', since otherwise we have an occurrence of the pattern $1-3-2$ involving the letter n. Also, π' and π'' avoid the pattern $1-3-2$, and π'' ends with the pattern $12\dots k$. In terms of generating functions (the generating function for the number of permutations ending with $12\dots k$ and avoiding $1-3-2$ is, of course, $G_{2-31}^{k(k-1)\dots 1}(x)$) this means that

$$G_{2-31}^{k(k-1)\dots 1}(x) = xC(x)G_{2-31}^{k(k-1)\dots 1}(x) + xG_{2-31}^{(k-1)\dots 1}(x), \tag{3.7}$$

where the rightmost term corresponds to the case when π'' is empty. Now, (3.1) and (3.7) give

$$G_{2-31}^{k(k-1)\dots 1}(x) = x^k C(x)/(1-xC(x))^k = x^k C^{k+1}(x).$$

□

3.6 Avoiding a pattern xy-z and beginning with an increasing or decreasing pattern

First of all we state the following well-known binomial identity

$$\sum_{i=1}^{n-m-k+1} \binom{n-m-i}{k-1}\binom{m+i-1}{m} = \binom{n}{m+k}. \tag{3.8}$$

Let $s_q(n)$ denote the cardinality of the set $S_n(q)$ and $s_q(n; i_1, i_2, \dots i_m)$ denote the number of permutations $\pi \in S_n(q)$ with $\pi_1\pi_2\dots\pi_m = i_1 i_2 \dots i_m$.

In this section we consider avoidance of one of the patterns 12-3, 13-2 and 23-1 and beginning with an increasing or decreasing pattern. We get all the other cases, which are avoidance of one of the patterns 32-1, 31-2 and 21-3 and beginning with an increasing or decreasing pattern, by the complement operation. For instance, we have $N_{13-2}^{12\dots k}(n) = N_{31-2}^{k(k-1)\dots 1}(n)$.

3.6.1 The pattern $12-3$

We first consider beginning with the pattern $p = k\dots 21$. In [ClaesMans, Lemma 9] it was proved that

$$s_{12-3}(n; i) = \sum_{j=0}^{i-1} \binom{i-1}{j} s_{12-3}(n-2-j),$$

together with $s_{12-3}(n; n) = s_{12-3}(n; n-1) = s_{12-3}(n-1)$.

On the other hand, from the definitions, it is easy to see that

$$N_{12-3}^{k(k-1)\dots 1}(n) = \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} s_{12-3}(n-k+1; i).$$

Hence, using (3.8) and the fact shown in [Claes] that $s_{12-3}(n)$ equals B_n, we get the following proposition.

Proposition 6. *For all $n \geq k+1$, we have*

$$N_{12-3}^{k(k-1)\dots 1}(n) = (k+1)B_{n-k} + \\ + \sum_{j=0}^{n-k-2} \left(\binom{n}{k+j} - k\binom{n-k-1}{j} - \binom{n-k}{j} \right) B_{n-k-1-j},$$

together with $N_{12-3}^{k(k-1)\dots 1}(k) = 1$ and $N_{12-3}^{k(k-1)\dots 1}(n) = 0$ for all $n \leq k-1$.

Now, let us consider beginning with the pattern $p = 12\dots k$. From the definitions, it is easy to see that $N_{12-3}^{12\dots k}(n) = 0$ for all n, where $k \geq 3$, and $N_{12-3}^{1}(n) = s_{12-3}(n) = B_n$ (see [ClaesMans, Prop. 10]). Thus, we only need to consider the case $k = 2$.

Suppose $\pi \in S_{12-3}(n)$ is a permutation with $\pi_1 < \pi_2$. It is easy to see that $\pi_2 = n$. Hence $N_{12-3}^{12}(n) = (n-1)s_{12-3}(n-2)$, for all $n \geq 2$, and by [ClaesMans, Prop. 10], we get the truth of the following

Proposition 7. *We have*

$$E_{12-3}^{12\dots k}(x) = \begin{cases} 0, & \text{if } k \geq 3, \\ x^2 \sum_{j=0}^{k} (1-jx)^{-1} \sum_{d \geq 0} \frac{x^d}{(1-x)(1-2x)\dots(1-dx)}, & \text{if } k = 2, \\ \sum_{d \geq 0} \frac{x^d}{(1-x)(1-2x)\dots(1-dx)}, & \text{if } k = 1. \end{cases}$$

3.6.2 The pattern $13-2$

Let us introduce an object that plays an important role in the proof of the main result in this case. For $n \geq m+1 \geq 0$, we define

$$A(n; m) = \sum_{1 \leq i_m < \dots < i_2 < i_1 < n-1} s_{1-3-2}(n; i_1, i_2, \dots, i_m).$$

We extend this definition to $m = 0$ by $A(n; 0) = s_{1-3-2}(n)$.

Lemma 1. *For all $n \geq m \geq 0$,*

$$A(n; m) = \sum_{j \geq 0} (-1)^j \binom{m+1-j}{j} s_{1-3-2}(n-j).$$

Proof. For $m = 0$ the lemma holds by definitions. Let $m \geq 0$; so

$$\begin{aligned}
A(n;m) &= \sum_{1 \leq i_m < \cdots < i_2 < i_1 < n-1} \sum_{j=1}^{n} s_{1-3-2}(n; i_1, i_2, \ldots, i_m, j), \\
&= A(n;m+1) + \sum_{1 \leq i_m < \cdots < i_2 < i_1 < n-1} s_{1-3-2}(n; i_1, i_2, \ldots, i_m, n), \\
&= A(n;m+1) + \sum_{1 \leq i_m < \cdots < i_2 < i_1 < n-1} s_{1-3-2}(n-1; i_1, i_2, \ldots, i_m), \\
&= A(n;m+1) + \sum_{1 \leq i_m < \cdots < i_2 < n-2} s_{1-3-2}(n-1; n-1, i_2, \ldots, i_m) + \\
&\qquad + \sum_{1 \leq i_m < \cdots < i_2 < i_1 < n-2} s_{1-3-2}(n-1; i_1, i_2, \ldots, i_m)
\end{aligned}$$

$$= A(n;m+1) + A(n-1;m) + \sum_{1 \leq i_{m-1} < \cdots < i_1 < n-2} s_{1-3-2}(n-2; i_1, \ldots, i_{m-1}),$$

$$= \cdots = A(n;m+1) + A(n-1;m) + \cdots + A(n-m-1;0).$$

Hence, using induction on m, we get

$$\begin{aligned}
A(n;m+1) = \sum_{j \geq 0} (-1)^j \binom{m+1-j}{j} s_{1-3-2}(n-j) \\
- \sum_{d=0}^{m} \sum_{j \geq 0} (-1)^j \binom{m-d+1-j}{j} s_{1-3-2}(n-1-d-j).
\end{aligned}$$

Using the identity $\binom{r}{0} - \binom{r}{1} + \cdots + (-1)^s \binom{r}{s} = \binom{r-1}{s}$, we get

$$\begin{aligned}
A(n;m+1) = \sum_{j \geq 0} (-1)^j \binom{m+1-j}{j} s_{1-3-2}(n-j) \\
- \sum_{d=0}^{m} (-1)^d \binom{m-d}{d} s_{1-3-2}(n-1-d).
\end{aligned}$$

Now using the identity $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$, we get

$$A(n;m+1) = \sum_{j \geq 0} (-1)^j \binom{m+2-j}{j} s_{1-3-2}(n-j),$$

which means that the lemma holds for $m+1$. □

Now we find $N_{13-2}^{k(k-1)\ldots 1}(n)$.

Proposition 8. *Let $k \geq 1$. For all $n \geq 0$,*

$$N_{13-2}^{k(k-1)\dots 1}(n) = C_{n+1-k} + \sum_{d=0}^{k-2} \sum_{j \geq 0} (-1)^j \binom{k+1-d-j}{j} C_{n-d-j}.$$

Proof. Claesson [Claes] proved that the set of permutations that avoid the pattern $13-2$ is the same as the set of permutations that avoid the pattern $1-3-2$, hence

$$N_{13-2}^{k(k-1)\dots 1}(n) = N_{1-3-2}^{k(k-1)\dots 1}(n). \tag{3.9}$$

If the leftmost letter of a permutation avoiding 13-2 and beginning with the pattern $k(k-1)\dots 1$ is n, then, obviously, there are $N_{1-3-2}^{(k-1)(k-2)\dots 1}(n-1)$ such permutations. Otherwise, it is easy to see that there are $A(n;k)$ such permutations. So, by Lemma 1 and the considerations above, also using the fact that the number of $(1-3-2)$-avoiding n-permutations in S_n is C_n, we get

$$N_{13-2}^{k(k-1)\dots 1}(n) = N_{13-2}^{(k-1)(k-2)\dots 1}(n-1) + \sum_{j \geq 0} (-1)^j \binom{k+1-j}{j} C_{n-j}.$$

Moreover, using the definitions and Equation (3.9), we have $N_{13-2}^{1}(n) = s_{1-3-2}(n) = C_n$, for all $n \geq 0$. Hence, by induction on k, the proposition holds. □

Now, let us consider the case of $N_{13-2}^{12\dots k}(n)$.

Proposition 9. *Let $k \geq 1$. For all $n \geq k$, we have*

$$N_{13-2}^{12\dots k}(n) = C_{n+1-k}.$$

Proof. Suppose $\pi = \pi' n \pi''$ is a permutation in $S_n(13-2) = S_n(1-3-2)$ (see (3.9)), such that $\pi_1 < \pi_2 < \dots < \pi_k$. It is easy to see that there exists an m such that

$$\pi = (m+1)(m+2)\dots(m+k-1)\beta n \pi'',$$

where β is a $1-3-2$-avoiding permutation on the letters $m+k, m+k+1, \dots, n-1$, and $\pi'' \in S_m(1-3-2)$. Hence, in terms of generating functions, we get

$$\sum_{n \geq 0} N_{13-2}^{12\dots k}(n) x^n = x^k C^2(x).$$

The rest is easy to check using the identity $xC^2(x) = C(x) - 1$. □

3.6.3 The pattern $23-1$

We first consider beginning with the pattern $p = k(k-1)\dots 1$.

Proposition 10. *For all $k \geq 1$,*

$$E_{23-1}^{k(k-1)\dots 1}(x) = x^{k-1} \left(\sum_{d \geq 0} \frac{x^d}{(1-x)(1-2x)\cdots(1-dx)} - 1 \right).$$

Proof. Let $\pi \in S_n(23-1)$ be a permutation such that $\pi_1 < \pi_2 < \cdots < \pi_k$. Since π avoids $23-1$, we have $\pi_j = j$, for each $j = 1, 2, \dots, k-1$. Hence $\pi = 12\dots(k-1)\pi'$, where π' is a non-empty $23-1$-avoiding permutation in S_{n+1-k}. The rest is easy to get by using [ClaesMans, Prop. 17]. □

Now let us consider beginning with the pattern $p = 12\dots k$.

Proposition 11. *Suppose $k \geq 1$. For all $n \geq k+1$,*

$$N_{23-1}^{12\dots k}(n) = \left(1 + \binom{n-1}{k-1}\right) B_{n-k} + \sum_{j=0}^{n-k-2} \left[\binom{n-1}{k+j} - \binom{n-k-1}{j}\right] B_{n-k-1-j},$$

with $N_{23-1}^{12\dots k}(k) = 1$.

Proof. In [ClaesMans, Lemma 16] proved that for all $2 \leq i \leq n-1$,

$$s_{23-1}(n;i) = \sum_{j=0}^{i-2} \binom{i-2}{j} s_{23-1}(n-2-j),$$

together with $s_{23-1}(n;n) = s_{23-1}(n;1) = s_{23-1}(n-1) = B_{n-1}$.

On the other hand, by the definitions, it is easy to see that

$$N_{23-1}^{12\dots k}(n) = \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} s_{23-1}(n-k+1;i).$$

Hence, using (3.8) and the fact that [Claes] $s_{23-1}(n)$ is given by B_n, we get the desired result. □

3.7 Avoiding a pattern xy-z and beginning with the pattern $(k-1)(k-2)\dots 1k$ or $23\dots k1$

In this section we consider avoidance of one of the patterns $12-3$, $13-2$, $23-1$, $21-3$, $31-2$ and $13-2$ and beginning with the pattern $(k-1)(k-2)\dots 1k$. The case when a permutation begins with the pattern $23\dots k1$ and avoids a pattern $xy-z$ can be obtained then by the complement operation.

3.7.1 Avoiding $12-3$ and beginning with $(k-1)(k-2)\dots 1k$

Proposition 12. *We have*

$$N_{12-3}^{(k-1)(k-2)\dots 1k}(n) = \binom{n-1}{k-1} B_{n-k}.$$

Proof. Suppose $\pi = \pi' n \pi''$ avoids the pattern $12-3$ and begins with the pattern $(k-1)(k-2)\dots 1k$. We have that π' must be decreasing, since otherwise we have an occurrence of the pattern $12-3$ involving the letter n, and π'' must avoid $12-3$. Also, since π begins with $(k-1)\dots 21k$, the length of π' is $k-1$. Hence, by [Claes] (the number of permutations in $S_n(12-3)$ is given by B_n), we have

$$N_{12-3}^{(k-1)(k-2)\dots 1k}(n) = \binom{n-1}{k-1} B_{n-k}.$$

□

3.7.2 Avoiding $13-2$ and beginning with $(k-1)(k-2)\dots 1k$

By [Claes], a permutation π avoids the pattern $13-2$ if and only if π avoids $1-3-2$.

Suppose $\pi = \pi' n \pi''$ is an n-permutation avoiding $1-3-2$ and beginning with $(k-1)(k-2)\dots 1k$. Obviously, π' and π'' avoid $1-3-2$ and each letter of π' is greater than any letter of π'', since otherwise we have an occurrence of the pattern $1-3-2$ involving the letter n. Also, π' begins with the pattern $(k-1)(k-2)\dots 1k$ or $\pi' = (k-1)(k-2)\dots 1$.

By [Knuth], the generating function for the number of permutations that avoid $1-3-2$ is $C(x)$, hence, using the considerations above,

$$G_{13-2}^{(k-1)(k-2)\dots 1k}(x) = xG_{13-2}^{(k-1)(k-2)\dots 1k}(x)C(x) + x^k C(x).$$

Therefore, by (3.1), we get the following.

Proposition 13. *We have*

$$G_{13-2}^{(k-1)(k-2)\dots 1k}(x) = x^k C^2(x).$$

Hence

$$N_{13-2}^{(k-1)(k-2)\dots 1k}(n) = \begin{cases} C_{n-(k-1)}, & \text{if } n \geq k \\ 0, & \text{else.} \end{cases}$$

3.7.3 Avoiding $21-3$ and beginning with $(k-1)(k-2)\dots 1k$

If $k \geq 3$ then, by the definitions, we have $N_{21-3}^{(k-1)(k-2)\dots 1k}(n) = 0$.

If $k = 1$ then, by the definitions and [Claes], we have $N_{21-3}^{1}(n) = B_n$.

Suppose $k = 2$ and $\pi = \pi' n \pi''$ is an n-permutation avoiding the pattern $21-3$ and beginning with the pattern $(k-1)(k-2)\dots 1k = 12$. It is easy to see that π' must be increasing, and the length of π' is at least 1. Thus, using the fact that the number of permutations in $S_n(21-3)$ is given by B_n (see [Claes]), we have

$$N_{21-3}^{(k-1)(k-2)\dots 1k}(n) = \sum_{j=1}^{n-1} \binom{n-1}{j} B_{n-1-j}. \tag{3.10}$$

Since $B_n = \sum_{j=0}^{n-1} \binom{n-1}{j} B_{n-1-j}$, equality (3.10) gives that

$$N_{21-3}^{(k-1)(k-2)\dots 1k}(n) = B_n - B_{n-1}.$$

Thus we have proved the following.

Proposition 14.

$$N_{21-3}^{(k-1)(k-2)\dots 1k}(n) = \begin{cases} 0, & \text{if } k \geq 3 \\ B_n - B_{n-1}, & \text{if } k = 2, \\ B_n, & \text{if } k = 1. \end{cases}$$

3.7.4 Avoiding $23-1$ and beginning with $(k-1)(k-2)\dots 1k$

Proposition 15. *We have that* $N_{23-1}^{(k-1)(k-2)\dots 1k}(n)$ *is given by*

$$\begin{cases} B_{n-k} + \sum_{t=2}^{n-k+2} \binom{t+k-3}{k-2} \sum_{j=0}^{t-2} \binom{t-2}{j} B_{n-k-1-j}, & \text{if } k \geq 3 \\ B_{n-1}, & \text{if } k = 2, \\ B_n, & \text{if } k = 1. \end{cases}$$

Proof. Suppose $k = 2$. We are interested in the permutations $\pi \in S_n(23-1)$ that begin with the pattern 12. It is easy to see that $\pi_1 = 1$, hence $B_{12}^{23-1}(n) = B_{n-1}$ for all $n \geq 2$.

Suppose $k \geq 3$. We recall that $s_{23-1}(n;t)$ is the number of permutations in $S_n(23-1)$ having t as the first letter. By [ClaesMans], $s(n;1) = B_{n-1}$ and for $t \geq 2$, we have

$$s(n;t) = \sum_{j=0}^{t-2} \binom{t-2}{j} B_{n-2-j}.$$

On the other hand, if a permutation $\pi = \pi' 1 t \pi''$ avoids $23-1$ and begins with the pattern $(k-1)(k-2)\dots 1k$, then π' is decreasing of length $k-2$, and using $s(n;t)$, we get

$$N_{23-1}^{(k-1)(k-2)\dots 1k}(n) = B_{n-k} + \sum_{t=2}^{n-k+2} \binom{t+k-3}{k-2} \sum_{j=0}^{t-2} \binom{t-2}{j} B_{n-k-1-j}.$$

□

3.7.5 Avoiding $31-2$ and beginning with $(k-1)(k-2)\dots 1k$

By [Claes], a permutation π avoids the pattern $31-2$ if and only if π avoids the pattern $3-1-2$.

Suppose $\pi = \pi' 1 \pi''$ is an n-permutation avoiding $3-1-2$ and beginning with $(k-1)(k-2)\dots 1k$. Obviously, π' and π'' avoid $3-1-2$ and each letter of π' is smaller than any letter of π'', since otherwise we have an occurrence of the pattern $3-1-2$ involving the letter 1. Also, π' begins with the pattern $(k-1)(k-2)\dots 1k$ or $\pi' = (k-1)(k-2)\dots 2$ and π'' is not empty. So, using the generating function for the number of permutations avoiding the pattern $3-1-2$, which is $C(x)$ ([Knuth]), we get

$$G_{31-2}^{(k-1)(k-2)\dots 1k}(x) = x G_{31-2}^{(k-1)(k-2)\dots 1k}(x) C(x) + x^{k-1}(C(x) - 1).$$

Therefore, using (3.1), we get the following.

Proposition 16. *We have*

$$G_{31-2}^{(k-1)(k-2)\ldots 1k}(x) = \begin{cases} x^k C^3(x), & \text{if } k \geq 2, \\ C(x), & \text{if } k = 1. \end{cases}$$

Hence

$$N_{31-2}^{(k-1)(k-2)\ldots 1k}(n) = \begin{cases} C_{n-k+2} - C_{n-k+1}, & \text{if } k \geq 2, \\ C_n, & \text{if } k = 1. \end{cases}$$

3.7.6 Avoiding $32-1$ and beginning with $(k-1)(k-2)\ldots 1k$

Proposition 17.

$$N_{32-1}^{(k-1)(k-2)\ldots 1k}(n) = \begin{cases} 0, & \text{if } k \geq 4 \\ B_{n-1} - (n-2)B_{n-3}, & \text{if } k = 3 \text{ and } n \geq 3, \\ B_n - (n-1)B_{n-2}, & \text{if } k = 2 \text{ and } n \geq 2, \\ B_n, & \text{if } k = 1. \end{cases}$$

Proof. Using the definitions and [Claes], it is easy to see that the statement is true for $k = 1, 2$ and $k \geq 4$.

Suppose now that $k = 3$ and $\pi = \pi' 1 \pi''$ is an n-permutation avoiding the pattern $32-1$ and beginning with the pattern $(k-1)(k-2)\ldots 1k = 213$. We have that π' must be increasing, since otherwise we have an occurrence of the pattern $32-1$ involving the letter 1, and π'' must avoid $32-1$. Moreover, since π begins with 213, the length of π is 1 and the rightmost letter of π'' is greater than the letter of π'. Also, it is easy to see that the number of permutations in $S_{n-1}(32-1)$ beginning with the pattern 12 is the same as the number of permutations in $S_n(32-1)$ beginning with the pattern 213 (one can see it by placing 1 in the second position). Hence $N_{32-1}^{(k-1)\ldots 21k}(n) = B_{n-1} - (n-2)B_{n-3}$ for all $n \geq 3$. □

3.8 Avoiding a pattern x-yz and beginning with the pattern $(k-1)(k-2)\ldots 1k$ or $23\ldots k1$

In this section we consider avoidance of one of the patterns $1-23$, $1-32$, $2-31$, $2-13$, $3-12$ and $1-32$ and beginning with the pattern $(k-1)(k-2)\ldots 1k$. The case when a permutation begins with the pattern $23\ldots k1$ and avoids a pattern $x-yz$ can be obtained by the complement operation.

Proposition 18. *We have*

$$E_{1-32}^{(k-1)(k-2)\ldots 1k}(x) = \begin{cases} e^{e^x} \int_0^x e^{-e^t} \sum\limits_{n \geq k-1} \frac{t^n}{n!}\, dt, & \text{if } k \geq 2, \\ e^{e^x - 1}, & \text{if } k = 1. \end{cases}$$

Proof. Suppose $k \geq 2$. Let $B_{n,k}$ denote the number of n–permutations that avoid the pattern $1-32$ and begin with the pattern $(k-1)(k-2)\dots 1k$. Suppose $\pi = \sigma 1 \tau$ is such a permutation of length $n+1$. Obviously, the letters of τ must be in increasing order, since otherwise we have an occurrence of the pattern $1-32$ in π starting from the letter 1. If $|\sigma| = i$, then we can choose the letters of σ in $\binom{n}{i}$ ways. Since the letters of τ are in increasing order, they do not affect σ and thus there are $B_{i,k}$ possibilities to choose σ. Also, if $n \geq k-1$, then 1 can be in the $(k-1)$th position, and in this case, since π begins with the pattern $(k-1)(k-2)\dots 1k$, it must be that $\pi = (k-1)(k-2)\dots 21k(k+1)\dots(n+1)$. Thus, in the last case we have only one permutation. This leads to the recurrence relation

$$B_{n+1,k} = \sum_{i \geq 0} \binom{n}{i} B_{i,k} + \delta_{n,k},$$

where

$$\delta_{n,k} = \begin{cases} 1, & \text{if } n \geq k-1, \\ 0, & \text{else.} \end{cases}$$

This recurrence relation is identical to the one given in the proof of Proposition 2, so using this proof we get the desired result. □

Proposition 19. *We have*

$$E_{1-23}^{(k-1)(k-2)\dots 1k}(x) = \begin{cases} e^{e^x} \int_0^x \int_0^t \frac{r^{k-2}}{(k-2)!} e^{r-e^t}\, drdt, & \text{if } k \geq 2, \\ e^{e^x - 1}, & \text{if } k = 1. \end{cases}$$

Proof. If $k = 1$, then the statement is true due to Proposition 1.

Suppose $k \geq 2$. Let $B_{n,k}$ denote the number of n-permutations that avoid the pattern $1-23$ and begin with the pattern $(k-1)(k-2)\dots 1k$. Suppose $\pi = \sigma 1 \tau$ is such a permutation of length $n+1$. Obviously, the letters of τ must be in decreasing order since otherwise we have an occurrence of the pattern $1-23$ in π starting from the letter 1. If $|\sigma| = i$, then we can choose the letters of σ in $\binom{n}{i}$ ways. Since the letters of τ are in the decreasing order, they do not affect σ and thus there are $B_{i,k}$ possibilities to choose σ. Besides, if $n \geq k-1$, then 1 can be in the $(k-1)$th position, and in this case, since π begins with the pattern $(k-1)(k-2)\dots 1k$ and τ is decreasing, it must be that the kth letter of π is $(n+1)$ and there are $\binom{n-1}{k-2}$ ways to choose the letters of σ and then write them in decreasing order. Thus,

$$B_{n+1,k} = \sum_{i \geq 0} \binom{n}{i} B_{i,k} + \binom{n-1}{k-2}.$$

Multiplying both sides of the equality with $x^n/n!$ and summing over n, we get the differential equation

$$\frac{d}{dx} E_{1-23}^{(k-1)(k-2)\dots 1k}(x) = E_{1-23}^{(k-1)(k-2)\dots 1k} e^x + \sum_{n \geq 0} \binom{n-1}{k-2} \frac{x^n}{n!},$$

with the initial condition $E_{1-23}^{(k-1)(k-2)\dots 1k}(0) = 0$. If $F(x)$ denotes the last term, then it is easy to see that $F'(x) = \frac{x^{k-2}}{(k-2)!}e^x$, and thus

$$F(x) = \int_0^x \frac{t^{k-2}}{(k-2)!}e^t\, dt.$$

Now, the solution to the equation above is given by

$$E_{1-23}^{(k-1)(k-2)\dots 1k}(x) = e^{e^x} \int_0^x e^{-e^t} F(t)\, dt = e^{e^x} \int_0^x \int_0^t \frac{r^{k-2}}{(k-2)!} e^{r-e^t}\, drdt. \tag{3.11}$$

For example, if $k = 2$, then $(k-1)(k-2)\dots 1k = 12$ and (3.11) gives

$$E_{1-23}^{12} = e^{e^x} \int_0^x e^{-e^t}(e^t - 1)\, dt,$$

which is a particular case of Proposition 3, since the number of n-permutations that avoid the pattern $3-21$ and begin with the pattern 21 is equal to the number of n-permutations that avoid the pattern $1-23$ and begin with the pattern 12 by applying the complement. □

Proposition 20. *We have*

$$G_{2-13}^{(k-1)(k-2)\dots 1k}(x) = \begin{cases} 0, & \text{if } k \geq 3 \\ x^2C^3(x), & \text{if } k = 2 \\ C(x), & \text{if } k = 1. \end{cases}$$

Hence

$$N_{2-13}^{(k-1)(k-2)\dots 1k}(n) = \begin{cases} 0, & \text{if } k \geq 3 \\ C_{n-1} - C_{n-2}, & \text{if } k = 2 \\ C_n, & \text{if } k = 1. \end{cases}$$

Proof. For the case $k = 1$, see Proposition 4. If $k \geq 3$, then the statement is true, since in this case the pattern $(k-1)(k-2)\dots 1k$ does not avoid $2-13$.

Suppose now that $k = 2$. Using the reverse, we see that beginning with the pattern 12 and avoiding $2-13$ is equivalent to ending with the pattern 21 and avoiding $31-2$, which by [Claes] is equivalent to ending with the pattern 21 and avoiding the pattern $3-1-2$.

Let $\pi = \pi' 1 \pi''$ be an n-permutation avoiding $3-1-2$ and ending with the pattern 21. Obviously, π' and π'' avoid $3-1-2$ and each letter of π' is less than any letter of π'', since otherwise we have an occurrence of $3-1-2$ involving the letter 1. Also, π'' ends with the pattern 21 or $|\pi''| = 1$. So, using the generating function for the number of permutations avoiding $3-1-2$, which is $C(x)$ ([Knuth]), we have

$$G_{2-13}^{12}(x) = xG_{2-13}^{12}(x)C(x) + x(C(x) - 1).$$

Therefore, using (3.1), we get the desired result. □

Proposition 21. *We have*

$$G_{2-31}^{(k-1)(k-2)\dots 1k}(x) = x^k C^2(x).$$

Hence

$$N_{2-31}^{(k-1)(k-2)\dots 1k}(n) = \begin{cases} C_{n-(k-1)}, & \text{if } n \geq k \\ 0, & \text{else.} \end{cases}$$

Proof. Using the reverse, we see that beginning with the pattern $(k-1)(k-2)\dots 1k$ and avoiding the pattern $2-31$ is equivalent to ending with the pattern $k12\dots(k-1)$ and avoiding the pattern $13-2$, which, by [Claes], is equivalent to ending with the pattern $k12\dots(k-1)$ and avoiding the pattern $1-3-2$.

Let $\pi = \pi' n \pi''$ be an n-permutation avoiding the pattern $1-3-2$ and ending with the pattern $k12\dots(k-1)$. Obviously, π' and π'' avoid the pattern $1-3-2$ and each letter of π' is greater than any letter of π'', since otherwise we have an occurrence of the pattern $1-3-2$ involving the letter n. Also, π'' ends with the pattern $k12\dots(k-1)$ or $\pi'' = 12\dots(k-1)$.

Using the reverse operation, the generating function for the number of permutations ending with the pattern $k12\dots(k-1)$ and avoiding $1-3-2$ is equal to $G_{2-31}^{(k-1)(k-2)\dots 1k}(x)$. In terms of generating functions, the considerations above lead to

$$G_{2-31}^{(k-1)(k-2)\dots 1k}(x) = xG_{2-31}^{(k-1)(k-2)\dots 1k}(x)C(x) + x^k C(x).$$

Therefore, by (3.1), we get the desired result. □

Proposition 22. *We have*

$$E_{3-12}^{(k-1)(k-2)\dots 1k}(x) = \begin{cases} (e^{e^x}/(k-1)!)\int_0^x t^{k-1}e^{-e^t+t}\,dt, & \text{if } k \geq 2, \\ e^{e^x-1}, & \text{if } k = 1. \end{cases}$$

Proof. Suppose $k \geq 2$. Let $B_{n,k}$ denote the number of n-permutations that avoid the pattern $3-12$ and begin with a decreasing subword of length k. Let $\pi = \sigma(n+1)\tau$ be such a permutation of length $n+1$. Obviously, the letters of τ must be in decreasing order since otherwise we have an occurrence of $3-12$ in π starting from the letter (n+1). If $|\sigma| = i$ then we can choose the letters of σ in $\binom{n}{i}$ ways. Since the letters of τ are in decreasing order, they do not affect σ and thus there are $B_{i,k}$ possibilities to choose σ. Also, if $|\sigma| = k-1$ and the letters of σ are in decreasing order, we get $\binom{n}{k-1}$ additional ways to choose π. Thus

$$B_{n+1,k} = \sum_{i\geq 0}\binom{n}{i}B_{i,k} + \binom{n}{k-1}.$$

This recurrence relation is identical to the one given in the proof of Proposition 1, and we get the desired result using that proof. □

Proposition 23.

$$E_{3-21}^{(k-1)(k-2)\dots 1k}(n) = \begin{cases} 0, & \textit{if } k \geq 4 \\ (e^{e^x}/(k-1)!) \int_0^x t^{k-1} e^{-e^t+t}\, dt, & \textit{if } k=2 \textit{ or } k=3, \\ e^{e^x-1}, & \textit{if } k=1. \end{cases}$$

Proof. If $k \geq 4$ then the statement is true, since in this case the pattern $(k-1)(k-2)\dots 1k$ does not avoid the pattern $3-21$. In the other cases, we use the same arguments as we have in the proof of Proposition 22. The only difference is that instead of decreasing order in τ, we have increasing order. □

3.9 Conclusions

The goal of our paper is to give a complete description for the numbers of permutations avoiding a pattern of the form $x-yz$ or $xy-z$ and either beginning with one of the patterns $12\dots k$, $k(k-1)\dots 1$, $23\dots k1$, $(k-1)(k-2)\dots 1k$, or ending with one of the patterns $12\dots k$, $k(k-1)\dots 1$, $1k(k-1)\dots 2$, $k12\dots(k-1)$. This description is given in Sections 5–8. However, some of our results can be generalized to beginning with a pattern belonging to Γ_k^{min} or Γ_k^{max}, and thus to the ending with a pattern belonging to Δ_k^{min} or Δ_k^{max} (see Section 3.2 for definitions). An example of such a generalisation is given in Theorem 3 below. This theorem generalizes Propositions 1 and 22 and can be proved by using the same considerations as we do in the proofs of these propositions.

Theorem 3. *Suppose $p_1, p_2 \in \Gamma_k^{min}$ and $p_1 \in S_k(1-23)$, $p_2 \in S_k(1-32)$. Thus, the complements $C(p_1), C(p_2) \in \Gamma_k^{max}$ and $C(p_1) \in S_k(1-23)$, $C(p_2) \in S_k(3-12)$. Then, we have*

$$E_{1-23}^{p_1}(x) = E_{3-21}^{C(p_1)}(x) = E_{1-32}^{p_2}(x) = E_{3-12}^{C(p_2)}(x) =$$

$$\begin{cases} (e^{e^x}/(k-1)!) \int_0^x t^{k-1} e^{-e^t+t}\, dt, & \textit{if } k \geq 2, \\ e^{e^x-1}, & \textit{if } k=1. \end{cases}$$

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Paper IV

On multi-avoidance of generalized patterns

On multi-avoidance of generalized patterns

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Abstract

In [Kit1] Kitaev discussed simultaneous avoidance of two 3-patterns with no internal dashes, that is, where the patterns correspond to contiguous subwords in a permutation. In three essentially different cases, the numbers of such n-permutations are 2^{n-1}, the number of involutions in $\mathcal{S}_n$, and $2E_n$, where E_n is the n-th Euler number. In this paper we give recurrence relations for the remaining three essentially different cases.

To complete the descriptions in [Kit3] and [KitMans], we consider avoidance of a pattern of the form x-y-z (a classical 3-pattern) and beginning or ending with an increasing or decreasing pattern. Moreover, we generalize this problem: we demand that a permutation must avoid a 3-pattern, begin with a certain pattern and end with a certain pattern simultaneously. We find the number of such permutations in case of avoiding an arbitrary generalized 3-pattern and beginning and ending with increasing or decreasing patterns.

4.1 Introduction and Background

Permutation patterns: All permutations in this paper are written as words $\pi = a_1a_2\dots a_n$, where the a_i consist of all the integers $1, 2, \dots, n$. Let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. We say that α *contains* τ if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $(\alpha_{i_1}, \dots, \alpha_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that α *avoids* τ, or is τ-*avoiding*, if such a subsequence does not exist. The set of all τ-avoiding permutations in S_n is denoted by $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that α avoids T if α avoids any $\tau \in T$; the corresponding subset of S_n is denoted by $S_n(T)$.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns τ_1, τ_2. This problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [SchSim]), for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [W]), and for $\tau_1, \tau_2 \in S_4$ (see [B, K] and references therein). Several recent papers [CW, MV1, Kr, MV3, MV2] deal with the case $\tau_1 \in S_3$, $\tau_2 \in S_k$ for various pairs τ_1, τ_2.

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Generalized permutation patterns: In [BabStein] Babson and Steingrímsson introduced *generalized permutation patterns* (*GPs*) where two adjacent letters in a pattern may be required to be adjacent in the permutation. Such an adjacency requirement is indicated by the absence of a dash between the corresponding letters in the pattern. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subword 564, but the pattern 2-3-1 occurs also in the subwords 562 and 563. Note that a classical pattern should, in our notation, have dashes at the beginning and end. Since most of the patterns considered in this paper satisfy this, we suppress these dashes from the notation. Thus, a pattern with no dashes corresponds to a contiguous subword anywhere in a permutation. The motivation for introducing these patterns was the study of Mahonian statistics. A number of results on GPs were obtained by Claesson, Kitaev and Mansour. See for example [Claes], [Kit1, Kit2, Kit3] and [Mans1, Mans2, Mans3].

As in [SchSim], dealing with the classical patterns, one can consider the case when permutations have to avoid two or more generalized patterns simultaneously. A complete solution for the number of permutations avoiding a pair of 3-patterns of type (1,2) or (2,1), that is the patterns having one internal dash, is given in [ClaesMans1]. In [Kit1] Kitaev discussed simultaneous avoidance of two 3-patterns with no internal dashes, that is, where the patterns correspond to contiguous subwords in a permutation. In three essentially different cases, the numbers of such n-permutations are 2^{n-1}, the number of involutions in $\mathcal{S}_n$, and $2E_n$, where E_n is the n-th Euler number. The remaining cases are avoidance of 123 and 231, 213 and 231, 132 and 213. In Section 4.3 we give recurrence relations for these cases.

In Section 4, we consider avoidance of a pattern x-y-z, and beginning or ending with increasing or decreasing pattern. This completes the results made in [KitMans], which concerns the number of permutations that avoid a generalized 3-pattern and begin or end with an increasing or decreasing pattern.

In Sections 5–8, we give enumeration for the number of permutations that avoid a generalazed 3-pattern, begin *and* end with increasing or decreasing patterns. We record our results in terms of either *generating functions*, or *exponential generating functions*, or formulas for the numbers appeared.

In Section 4.9, we discuss possible directions of generalization of the results from Sections 5–8.

4.2 Preliminaries

The *reverse* $R(\pi)$ of a permutation $\pi = a_1a_2\dots a_n$ is the permutation $a_n\dots a_2a_1$. The *complement* $C(\pi)$ is the permutation $b_1b_2\dots b_n$ where $b_i = n+1-a_i$. Also, $R\circ C$ is the composition of R and C. For example, $R(13254) = 45231$, $C(13254) = 53412$ and $R\circ C(13254) = 21435$. We call these bijections of S_n to itself *trivial*, and it is easy to see that for any pattern p the number $A_p(n)$ of permutations avoiding the pattern p is the same as for the patterns $R(p)$, $C(p)$ and $R\circ C(p)$. For example, the number of permutations that avoid the pattern

132 is the same as the number of permutations that avoid the pattern 231. This property holds for sets of patterns as well. If we apply one of the trivial bijections to all patterns of a set G, then we get a set G' for which $A_{G'}(n)$ is equal to $A_G(n)$. For example, the number of permutations avoiding $\{123, 132\}$ equals the number of those avoiding $\{321, 312\}$ because the second set is obtained from the first one by complementing each pattern.

In this paper we denote the nth Catalan number by C_n; the generating function for these numbers by $C(x)$; the nth Bell number by B_n.

Also, $N_p^q(n)$ denotes the number of permutations that avoid the pattern p and begin with the pattern q; $G_p^q(x)$ (respectively, $E_p^q(x)$) denotes the ordinary (respectively, exponential) generating function for the number of such permutations. Besides, $N_p^{q,r}(n)$ denotes the number of permutations that avoid the pattern p, begin with the pattern q and end with the pattern r; $G_p^{q,r}(x)$ (respectively, $E_p^{q,r}(x)$) denotes the ordinary (respectively, exponential) generating function for the number of such permutations.

Recall the following properties of $C(x)$:

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1}{1-xC(x)}. \tag{4.1}$$

4.3 Simultaneous avoidance of two 3-patterns with no dashes

4.3.1 Avoidance of patterns 123 and 231 simultaneously

We first consider the avoidance of the patterns 123 and 231 simultaneously.

Let $a(n; i_1, i_2, \dots, i_m)$ denote the number of permutations $\pi \in S_n(123, 231)$ such that $\pi_1\pi_2\dots\pi_m = i_1i_2\dots i_m$ and let $a(n) = |S_n(123, 231)|$.

By the definitions, we get that $a(n) = \sum_{j=1}^{n} a(n; j)$ and $a(n; n) = a(n-1)$. Hence

$$a(n) = a(n-1) + a(n; 1) + a(n; 2) + \cdots + a(n; n-1). \tag{4.2}$$

Also, by the definitions, for all $1 \le i \le n-1$, we get

$$a(n; i) = \sum_{j=1}^{i-1} a(n; i, j) + \sum_{j=i+1}^{n} a(n; i, j). \tag{4.3}$$

Suppose $\pi \in S_n(123, 231)$ is such that $\pi_1 = i$ and $\pi_2 = j$. If $i > j$ then there is no occurrence of the pattern 123 or 231 that contains π_1, so $a(n; i, j) = a(n-1; j)$. If $i < j$ then since π avoids 123 and 231, we get that $i < \pi_3 < j$, and thus in this case $a(n; i, j) = a(n-2; i) + a(n-2; i+1) + \cdots + a(n-2; j-2)$.

Hence, using (4.2) and (4.3), we get the following theorem.

Proposition 1. *Let $s_n = |S_n(123, 231)|$. For all $n \ge 3$,*

$$s_n = s_{n-1} + s_n(1) + s_n(2) + \cdots + s_n(n-1),$$

where for all $1 \le i \le n$,

$$s_n(i) = \sum_{j=1}^{i-1} s_{n-1}(j) + \sum_{j=i}^{n-2} (n-1-j) s_{n-2}(j),$$

and $s_3(1) = 1$, $s_3(2) = 1$, $s_3(3) = 2$.

Using this theorem, we get quickly the first values of the sequence $|S_n(123, 231)|$ for $n = 0, 1, 2, \ldots, 10$:

n	0	1	2	3	4	5	6	7	8	9	10
$\|S_n(123, 231)\|$	1	1	2	4	11	39	161	784	4368	27260	189540

4.3.2 Avoidance of patterns 132 and 213 simultaneously

We consider avoidance of the patterns 132 and 213 simultaneously.

Let $b(n; i_1, i_2, \ldots, i_m)$ denote the number of permutations $\pi \in S_n(132, 213)$ such that $\pi_1 \pi_2 \ldots \pi_m = i_1 i_2 \ldots i_m$ and let $b(n) = |S_n(132, 213)|$.

Suppose $\pi \in S_n(132, 213)$ is such that $\pi_1 = i$ and $\pi_2 = j$. If $i > j$ then, since π avoids 213, we get $\pi_3 \le i - 1$. Thus

$$b(n; i, j) = \sum_{k=1,\ k \ne j}^{i-1} b(n-1; j, k). \tag{4.4}$$

If $i < j$ then, since π avoids 132, we get $\pi_3 \le i - 1$ or $\pi_3 \ge j + 1$. Thus

$$b(n; i, j) = \sum_{k=1}^{i-1} b(n-1; j-1, k) + \sum_{k=j}^{n-1} b(n-1; j-1, k). \tag{4.5}$$

Using (4.4) and (4.5), we get the following theorem.

Proposition 2. *Let* $s_n = |S_n(132, 213)|$. *Then* $s_n = \sum_{i,j=1}^{n} s(n; i, j)$ *with*
$s(n; i, i) = 0$ *for all* $n, i \ge 1$;
$s(n; i, j) = \sum_{k=1}^{i-1} s(n-1; j, k)$ *if* $i > j$;
$s(n; i, j) = \sum_{k=1}^{i-1} s(n-1; j-1, k) + \sum_{k=j}^{n-1} s(n-1; j-1, k)$ *if* $i < j$;
and $s(2; 1, 2) = s(2; 2, 1) = 1$, $s(2; 1, 1) = s(2; 1, 1) = 0$.

Using this theorem, we get

n	0	1	2	3	4	5	6	7	8	9	10
$\|S_n(132, 213)\|$	1	1	2	4	11	37	149	705	3814	23199	156940

4.3.3 Avoidance of the patterns 213 and 231 simultaneously

We now consider avoidance of the patterns 213 and 231 simultaneously. This case is equivalent to avoidance of the patterns 132 and 312 by applying the reverse operation.

Let $c(n; i_1, i_2, \ldots, i_m)$ denote the number of permutations $\pi \in S_n(132, 312)$ such that $\pi_1\pi_2 \ldots \pi_m = i_1 i_2 \ldots i_m$ and let $c(n) = |S_n(132, 312)|$. We proceed as in the previous case. For $n \geq i > j \geq 1$, we have

$$c(n; i, j) = \sum_{k=1}^{j-1} c(n-1; j, k) + \sum_{k=i}^{n-1} c(n-1; j, k). \tag{4.6}$$

For $1 \leq i < j \leq n$, we have

$$c(n; i, j) = \sum_{k=1}^{i-1} c(n-1; j-1, k) + \sum_{k=j}^{n-1} c(n-1; j-1, k). \tag{4.7}$$

Using (4.6) and (4.7), we get the following theorem.

Proposition 3. *Let* $s_n = |S_n(132, 312)|$. *Then* $s_n = \sum_{i,j=1}^{n} s(n; i, j)$ *with*
$s(n; i, i) = 0$ *for all* $n, i \geq 1$;
$s(n; i, j) = \sum_{k=1}^{j-1} s(n-1; j, k) + \sum_{k=i}^{n-1} s(n-1; j, k)$ *if* $i > j$;
$s(n; i, j) = \sum_{k=1}^{i-1} s(n-1; j-1, k) + \sum_{k=j}^{n-1} s(n-1; j-1, k)$ *if* $i < j$;
and $s(2; 1, 2) = s(2; 2, 1) = 1$, $s(2; 1, 1) = s(2; 1, 1) = 0$.

Using this theorem, we get

n	0	1	2	3	4	5	6	7	8	9	10
$\|S_n(132, 312)\|$	1	1	2	4	10	30	108	454	2186	11840	71254

4.4 Avoiding a pattern x-y-z and beginning or ending with certain patterns

Recall the definitions of $G_q^p(x)$, $N_q^p(n)$, $C(x)$ and C_n in Section 4.2.

Proposition 4. *We have*

$$G_{1\text{-}3\text{-}2}^{12\ldots k}(x) = x^k C^2(x).$$

Proof. Suppose $\pi = \pi' n \pi'' \in S_n(1\text{-}3\text{-}2)$ is such that $\pi_1 < \pi_2 < \cdots < \pi_k$ and $\pi_j = n$. It is easy to see that π avoids 1-3-2 if and only if π' is a 1-3-2-avoiding permutation on the letters $n-j+1, n-j+2, \ldots, n$, and $\pi'' \in S_{n-j}(1\text{-}3\text{-}2)$. If we now consider two cases, namely $j = k$ and $j \geq k+1$, we get

$$G_{1\text{-}3\text{-}2}^{12\ldots k}(x) = x^k C(x) + x G_{1\text{-}3\text{-}2}^{12\ldots k}(x) C(x).$$

Thus, $G_{1\text{-}3\text{-}2}^{12\ldots k}(x) = x^k C(x)/(1 - xC(x))$ and, using (4.1), we get the desired result. □

Proposition 5. *We have*

$$G^{k(k-1)\dots 1}_{1\text{-}3\text{-}2}(x) = x^k C^{k+1}(x).$$

Proof. Suppose $\pi = \pi' n \pi'' \in S_n(1\text{-}3\text{-}2)$ is such that $\pi_1 > \pi_2 > \cdots > \pi_k$ and $\pi_j = n$. It is easy to see that π avoids 1-3-2 if and only if π' is a 1-3-2-avoiding permutation on the letters $n-j+1, n-j+2, \dots, n$, and $\pi'' \in S_{n-j}(1\text{-}3\text{-}2)$. If we consider separately the cases $j = 1$ and $j \geq 2$, we get

$$G^{k(k-1)\dots 1}_{1\text{-}3\text{-}2}(x) = xG^{(k-1)(k-2)\dots 1}_{1\text{-}3\text{-}2}(x) + xG^{k(k-1)\dots 1}_{1\text{-}3\text{-}2}(x)C(x).$$

Hence,

$$G^{k(k-1)\dots 1}_{1\text{-}3\text{-}2}(x) = xG^{(k-1)(k-2)\dots 1}_{1\text{-}3\text{-}2}(x)/(1 - xC(x))$$

and, using (4.1), we get $G^{k(k-1)\dots 1}_{1\text{-}3\text{-}2}(x) = xC(x)G^{(k-1)(k-2)\dots 1}_{1\text{-}3\text{-}2}(x)$. By induction on k, using the fact that $G^{1}_{1\text{-}3\text{-}2}(x) = C(x) - 1 = xC^2(x)$, we get the desired result. □

Proposition 6. *We have*

$$G^{12\dots k}_{2\text{-}1\text{-}3}(x) = x^k C^{k+1}(x).$$

Proof. One can use the same considerations as we have in the proof of Proposition 5, by considering a permutation $\pi = \pi' 1 \pi'' \in S_n(2\text{-}1\text{-}3)$ such that $\pi_1 < \pi_2 < \cdots < \pi_k$ and $\pi_j = 1$. □

Proposition 7. *We have*

$$G^{k(k-1)\dots 1}_{2\text{-}1\text{-}3}(x) = x^k C^2(x).$$

Proof. One can use the same considerations as we have in the proof of Proposition 4, by considering a permutation $\pi = \pi' 1 \pi'' \in S_n(2\text{-}1\text{-}3)$ such that $\pi_1 > \pi_2 > \cdots > \pi_k$ and $\pi_j = 1$. □

Let $s_n(i_1, \dots, i_m)$ denote the number of permutations $\pi \in S_n(1\text{-}2\text{-}3)$ such that $\pi_1\pi_2\dots\pi_m = i_1 i_2 \dots i_m$. It is easy to see that

$$s_n(n) = s_n(n-1) = C_{n-1}, \tag{4.8}$$

and

$$s_n(t) = s_n(t, n) + \sum_{j=1}^{t-1} s_n(t, j) = s_{n-1}(t) + \sum_{j=1}^{t-1} s_{n-1}(j). \tag{4.9}$$

Now, (4.8) and (4.9) with induction on t give

$$s_n(n-t) = \sum_{j=0}^{t} (-1)^j \binom{t-j}{j} C_{n-j} \tag{4.10}$$

Let us prove the following proposition.

Proposition 8. *We have*

$$G^{12\dots k}_{1\text{-}2\text{-}3}(x) = \begin{cases} 0, & \text{if } k \geq 3, \\ x^2C^2(x), & \text{if } k = 2, \\ xC^2(x), & \text{if } k = 1\ . \end{cases}$$

Proof. For $k \geq 3$, the statement is obviously true. If $k = 1$ then $G^{1}_{1\text{-}2\text{-}3}(x) = C(x) - 1 = xC^2(x)$.

Suppose now that $k = 2$. From the definitions, for all $n \geq 2$, we have

$$N^{12}_{1\text{-}2\text{-}3}(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} s_n(i,j).$$

In this formula, j can only be equal to n, since otherwise we have an occurrence of the pattern 1-2-3. Using this fact with (4.8) and (4.9), we get for $n \geq 2$,

$$N^{12}_{1\text{-}2\text{-}3}(n) = \sum_{i=1}^{n-1} s_n(i,n) = \sum_{i=1}^{n-1} s_{n-1}(i) = C_{n-1}.$$

Hence, $G^{12}_{1\text{-}2\text{-}3}(x) = x(C(x) - 1) = x^2C^2(x)$. □

Proposition 9. *We have*

$$N^{k(k-1)\dots 1}_{1\text{-}2\text{-}3}(n) = \sum_{t=1}^{n+1-k} \binom{n-t}{k-1} \sum_{j=0}^{n-t} (-1)^j \binom{n-t-j}{j} C_{n-t-j}.$$

Proof. From the definitions, we have

$$N^{k(k-1)\dots 1}_{1\text{-}2\text{-}3}(n) = \sum_{i_1=k}^{n} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} s_n(i_1,\dots,i_k) = \sum_{t=1}^{n+1-k} \binom{n-t}{k-1} s_n(t).$$

Using (4.10), we get

$$N^{k(k-1)\dots 1}_{1\text{-}2\text{-}3}(n) = \sum_{t=1}^{n+1-k} \binom{n-t}{k-1} \sum_{j=0}^{n-t} (-1)^j \binom{n-t-j}{j} C_{n-t-j}.$$

□

4.5 Avoiding a pattern x-y-z, beginning and ending with certain patterns simultaneously

Recall the definitions of $G^{p,r}_q(x)$ and $N^{p,r}_q(n)$ in Section 4.2.

Proposition 10. *We have*

(i) $G^{12\ldots k,12\ldots\ell}_{1\text{-}3\text{-}2}(x) = x^{k+\ell-1}C^{\ell+1}(x) + \frac{x^m - x^{k+\ell-1}}{1-x}$.

(ii) $G^{12\ldots k,\ell(\ell-1)\ldots 1}_{1\text{-}3\text{-}2}(x) = x^{k+\ell-1}C^2(x)$.

(iii) $G^{k(k-1)\ldots 1,\ell(\ell-1)\ldots 1}_{1\text{-}3\text{-}2}(x) = x^{k+\ell-1}C^{k+1}(x) + \frac{x^m - x^{k+\ell-1}}{1-x}$, *where* $m = \max(k,\ell)$.

(iv) *the generating function* $G_{1\text{-}3\text{-}2}(x,y,z) = \sum_{k,\ell\geq 0} G^{k(k-1)\ldots 1,12\ldots\ell}_{1\text{-}3\text{-}2}(x)y^k z^\ell$ *for the sequence* $\{G^{k(k-1)\ldots 1,12\ldots\ell}_{1\text{-}3\text{-}2}(x)\}_{k,\ell\geq 0}$ (*where* k *and* ℓ *go through all natural numbers*) *is*

$$\frac{1}{1-x(y+z)}\left(x(y+z+yz) + \frac{C(x)-1}{(1-xyC(x))(1-xzC(x))}\right).$$

Proof.

Beginning with $12\ldots k$ and ending with $\ell(\ell-1)\ldots 1$: Suppose $\pi = \pi' n \pi'' \in S_n(1\text{-}3\text{-}2)$ is such that $\pi_1 < \pi_2 < \cdots < \pi_k$, $\pi_n < \pi_{n-1} < \cdots < \pi_{n-\ell+1}$ and $\pi_j = n$. It is easy to see that π avoids 1-3-2 if and only if π' is a 1-3-2-avoiding permutation on the letters $n-j+1, n-j+2, \ldots, n$, and $\pi'' \in S_{n-j}(1\text{-}3\text{-}2)$. We now consider three cases, namely $j = k$, $k+1 \leq j \leq n-\ell$ and $j = n-\ell+1$. In terms of generating functions, we have

$$G^{12\ldots k,\ell(\ell-1)\ldots 1}_{1\text{-}3\text{-}2}(x) = x^k G^{\ell(\ell-1)\ldots 1}_{2\text{-}1\text{-}3}(x) + xG^{12\ldots k}_{1\text{-}3\text{-}2}(x)G^{\ell(\ell-1)\ldots 1}_{2\text{-}1\text{-}3}(x) + x^\ell G^{12\ldots k}_{1\text{-}3\text{-}2}(x) + x^{k+\ell-1},$$

where we observed that to avoid 1-3-2 and end with $\ell(\ell-1)\ldots 1$ is the same as to avoid 2-1-3 and begin with $\ell(\ell-1)\ldots 1$ by applying the reverse and complement operations. Also, we added the term $x^{k+\ell-1}$, since when $j = k = n-\ell+1$, we have one "good" $(k+\ell-1)$-permutation, which is not counted by our three cases.

From Propositions 4 and 7, we have that

$$G^{12\ldots k}_{1\text{-}3\text{-}2}(x) = x^k C^2(x) \text{ and } G^{\ell(\ell-1)\ldots 1}_{2\text{-}1\text{-}3}(x) = x^\ell C^2(x).$$

Thus, using the fact that $xC^2(x) = C(x) - 1$, we get

$$\begin{aligned} G^{12\ldots k,\ell(\ell-1)\ldots 1}_{1\text{-}3\text{-}2}(x) &= x^{k+\ell}C^2(x)(2 + xC^2(x)) + x^{k+\ell-1} \\ &= x^{k+\ell-1}(C(x)-1)(C(x)+1) + x^{k+\ell-1} = x^{k+\ell-1}C^2(x). \end{aligned}$$

Beginning with $12\ldots k$ and ending with $12\ldots\ell$: Suppose $\pi = \pi' n \pi'' \in S_n(1\text{-}3\text{-}2)$ is such that $\pi_1 < \pi_2 < \cdots < \pi_k$, $\pi_n > \pi_{n-1} > \cdots > \pi_{n-\ell+1}$ and $\pi_j = n$. As above, π avoids 1-3-2 if and only if π' is a 1-3-2-avoiding permutation on the letters $n-j+1, n-j+2, \ldots, n$, and $\pi'' \in S_{n-j}(1\text{-}3\text{-}2)$. We consider the cases $j = k$, $k+1 \leq j \leq n-\ell$ and $j = n$. In terms of generating functions, the first approximation for the function $G^{12\ldots k,12\ldots\ell}_{1\text{-}3\text{-}2}(x)$ is

$$G^{12\ldots k,12\ldots\ell}_{1\text{-}3\text{-}2}(x) \approx x^k G^{12\ldots\ell}_{2\text{-}1\text{-}3}(x) + xG^{12\ldots k}_{1\text{-}3\text{-}2}(x)G^{12\ldots\ell}_{2\text{-}1\text{-}3}(x) + xG^{12\ldots k,12\ldots(\ell-1)}_{1\text{-}3\text{-}2}(x),$$

where we observed that to avoid 1-3-2 and end with $12\ldots\ell$ is the same as to avoid 2-1-3 and begin with $12\ldots\ell$ by applying the reverse and complement operations. We use the sign "$\approx$" because there are some "good" permutations, which are not counted by our considerations. We discuss them below.

From Propositions 4 and 6, we have that $G^{12\dots k}_{1\text{-}3\text{-}2}(x) = x^k C^2(x)$ and $G^{12\dots \ell}_{2\text{-}1\text{-}3}(x) = x^\ell C^{\ell+1}(x)$. Thus, using the fact that $xC^2(x) = C(x) - 1$ and $G^{12\dots k,1}_{1\text{-}3\text{-}2}(x) = G^{12\dots k}_{1\text{-}3\text{-}2}(x) = x^k C^2(x)$ (Proposition 4), we get

$$\begin{aligned}
&G^{12\dots k,12\dots\ell}_{1\text{-}3\text{-}2}(x)\\
&\quad\approx x^{k+\ell}C^{\ell+1}(x) + x^{k+\ell+1}C^{\ell+3}(x) + xG^{12\dots k,12\dots(\ell-1)}_{1\text{-}3\text{-}2}(x)\\
&\quad= x^{k+\ell}C^{\ell+2}(x) + xG^{12\dots k,12\dots(\ell-1)}_{1\text{-}3\text{-}2}(x)\\
&\quad= x^{k+\ell}C^{\ell+2}(x) + x^{k+\ell}C^{\ell+1}(x) + x^2G^{12\dots k,12\dots(\ell-2)}_{1\text{-}3\text{-}2}(x)\\
&\quad= \dots = x^{k+\ell}C^4(x)(C^{\ell-2}(x) + C^{\ell-3}(x) + \dots + 1) + x^{k+\ell-1}C^2(x)\\
&\quad= x^{k+\ell-1}(C(x)-1)C^2(x)\tfrac{1-C^{\ell-1}(x)}{1-C(x)} + x^{k+\ell-1}C^2(x) = x^{k+\ell-1}C^{\ell+1}(x).
\end{aligned}$$

To complete the proof of this case, we observe that in our considerations above, we do not count increasing permutations of length $m = \max(k,\ell), m+1, \dots, k+\ell-2$, which satisfy all our restrictions. We did not count them because the k-beginning and ℓ-ending in these permutations overlap in more than one letter. So, to get the desired result, we need to add the term

$$x^m + x^{m+1} + \ldots + x^{k+\ell-2} = (x^m - x^{k+\ell-1})/(1-x)$$

to the approximate value of $G^{12\dots k,12\dots\ell}_{1\text{-}3\text{-}2}(x)$. For example, expanding $G^{12,123}_{1\text{-}3\text{-}2}(x)$, we have, in particular, that there are 2002 10-permutations that avoid 1-3-2, begin with the pattern 12 and end with the pattern 123.

Beginning with $k(k-1)\dots1$ and ending with $\ell(\ell-1)\dots1$: If $\ell = 1$ then, by Proposition 5, $G^{k(k-1)\dots1,1}_{1\text{-}3\text{-}2}(x) = x^k C^{k+1}(x)$. Suppose $\ell \geq 2$, and $\pi = \pi'1\pi'' \in S_n(1\text{-}3\text{-}2)$ is such that $\pi_1 > \pi_2 > \dots > \pi_k$, $\pi_n < \pi_{n-1} < \dots < \pi_{n-\ell+1}$ and $\pi_j = 1$. Obviously, π'' is the empty word, since otherwise we have an occurrence of the pattern 1-3-2 starting from the letter 1. Thus, the first approximation for the function $G^{k(k-1)\dots1,\ell(\ell-1)\dots1}_{1\text{-}3\text{-}2}$ is

$$G^{k(k-1)\dots1,\ell(\ell-1)\dots1}_{1\text{-}3\text{-}2}(x) \approx xG^{k(k-1)\dots1,(\ell-1)(\ell-2)\dots1}_{1\text{-}3\text{-}2}(x) = \dots = x^{k+\ell-1}C^{k+1}(x).$$

Like in the previous case, we did not count decreasing permutations of length $m = max(k,\ell), m+1, \dots, k+\ell-2$, which satisfy all our restrictions. Thus, to get the desired result, we add the term $(x^m - x^{k+\ell-1})/(1-x)$ to the approximate value of $G^{k(k-1)\dots1,\ell(\ell-1)\dots1}_{1\text{-}3\text{-}2}(x)$.

Beginning with $k(k-1)\dots1$ and ending with $12\dots\ell$: Suppose $\pi = \pi'n\pi'' \in S_n(1\text{-}3\text{-}2)$. Any letter of π' is greater than any letter of π'', since otherwise we have an occurrence of the pattern 1-3-2 in π containing the letter

n which is forbidden. Also, π' and π'' avoid 1-3-2. If π begins with $k(k-1)\dots 1$, ends with $12\dots\ell$ and π' and π'' are not empty, then π' must begin with $k(k-1)\dots 1$ and π'' must end with $12\dots\ell$. If π' is empty then π'' must begin with $(k-1)(k-2)\dots 1$ and end with $12\dots\ell$. If π'' is empty then π' must begin with $k(k-1)\dots 1$ and end with $12\dots(\ell-1)$. In terms of generating functions, the discussion above leads to the following:

$$G_{1\text{-}3\text{-}2}^{k(k-1)\dots 1,12\dots\ell}(x) \approx$$
$$xG_{1\text{-}3\text{-}2}^{k(k-1)\dots 1}(x)G_{2\text{-}1\text{-}3}^{12\dots\ell}(x) + xG_{1\text{-}3\text{-}2}^{(k-1)\dots 1,12\dots\ell}(x) + xG_{1\text{-}3\text{-}2}^{k(k-1)\dots 1,12\dots(\ell-1)}(x),$$

where we observed that to avoid 1-3-2 and end with $12\dots\ell$ is the same as to avoid 2-1-3 and begin with $12\dots\ell$. However, to put the sign "=" instead of "$\approx$", we have to correct the right-hand side of the recurrence relation by observing that when either $k=1$ and $\ell=0$, or $k=0$ and $\ell=1$, or $k=1$ and $\ell=1$, the formula do not count the permutation $\pi=1$ which satisfies all the conditions needed. Thus, if we make correction of the right-hand side, then multiply both parts of the obtained equality by $x^k y^\ell$ and sum over all natural k and ℓ we get (recall the definition of $G_{1\text{-}3\text{-}2}(x,y,z)$ in the statement of the theorem):

$$G_{1\text{-}3\text{-}2}(x,y,z) = x\sum_{k,\ell\geq 0} G_{1\text{-}3\text{-}2}^{k(k-1)\dots 1}(x)G_{2\text{-}1\text{-}3}^{12\dots\ell}(x)y^k z^\ell + x(y+z)G_{1\text{-}3\text{-}2}(x,y,z) + x(y+z+yz).$$

From Propositions 5 and 6, $G_{1\text{-}3\text{-}2}^{k(k-1)\dots 1}(x)G_{2\text{-}1\text{-}3}^{12\dots\ell}(x) = x^{k+\ell}C^{k+\ell+2}(x)$, and thus

$$\begin{aligned} G_{1\text{-}3\text{-}2}(x,y,z) &= \tfrac{1}{1-x(y+z)}\left(x(y+z+yz) + \sum_{k,\ell\geq 0} x^{k+\ell}C^{k+\ell+2}(x)y^k z^\ell\right) \\ &= \tfrac{1}{1-x(y+z)}\left(x(y+z+yz) + zC^2(z)\sum_{k\geq 0}(xyC(x))^k\sum_{\ell\geq 0}(xzC(x))^\ell\right) \\ &= \tfrac{1}{1-x(y+z)}\left(x(y+z+yz) + \tfrac{C(x)-1}{(1-xyC(x))(1-xzC(x))}\right), \end{aligned}$$

where we used that $xC^2(x) = C(x)-1$. □

Proposition 11. *We have*

(i) $G_{2\text{-}1\text{-}3}^{12\dots k,12\dots\ell}(x) = x^{k+\ell-1}C^{k+1}(x) + \frac{x^m - x^{k+\ell-1}}{1-x}$.

(ii) $G_{2-1-3}^{k(k-1)\dots 1,12\dots\ell}(x) = x^{k+\ell-1}C^2(x)$.

(iii) $G_{2\text{-}1\text{-}3}^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}(x) = x^{k+\ell-1}C^{\ell+1}(x) + \frac{x^m - x^{k+\ell-1}}{1-x}$, *where* $m = \max(k,\ell)$.

(iv) *the generating function* $G_{2\text{-}1\text{-}3}(x,y,z) = \sum_{k,\ell\geq 0} G_{2\text{-}1\text{-}3}^{12\dots k,\ell(\ell-1)\dots 1}(x)y^k z^\ell$ *for the sequence* $\{G_{2\text{-}1\text{-}3}^{12\dots k,\ell(\ell-1)\dots 1}(x)\}_{k,\ell\geq 0}$ *(where k and ℓ go through all natural numbers) is*

$$\frac{1}{1-x(y+z)}\left(x(y+z+yz) + \frac{C(x)-1}{(1-xyC(x))(1-xzC(x))}\right).$$

Proof. We apply the reverse and complement operations and then use the results of Proposition 10. For example, to avoid 2-1-3, begin with $12\dots k$ and end with $12\dots \ell$ is the same as to avoid 1-3-2, begin with $12\dots \ell$ and end with $12\dots k$. □

Let $h_n^{k,\ell}(t;s)$ denote the number of 1-2-3-avoiding n-permutations such that $\pi_k = t$, $\pi_{n-\ell+1} = s$, $\pi_1 > \pi_2 > \cdots > \pi_k$, and $\pi_{n-\ell+1} > \pi_{n-\ell+2} > \cdots > \pi_n$. Also, we define $g_n(i_1, i_2, \dots, i_m; b)$ to be the number of 1-2-3-avoiding n-permutations such that $\pi_1\pi_2\cdots\pi_m = i_1 i_2 \dots i_m$ and $\pi_n = b$. We need the following two lemmas to prove Proposition 12.

Lemma 1. *For all* $n \geq 2$*,*

$$g_n(a;b) = \begin{cases} 0, & 2 \leq a+1 < b \leq n, \\ \binom{n-2}{a-1}, & 1 \leq a \leq n-1, \\ \sum_{j=0}^{n-a}(-1)^j\binom{n-a-j}{j}\left(\sum_{i=0}^{b-1}(-1)^i\binom{b-1-i}{i}C_{n-2-j-i}\right), & 1 \leq b < a \leq n. \end{cases}$$

Proof. By definitions we have

(1) $g_n(a;b) = 0$ for all $2 \leq a+1 < b \leq n$;

(2) $g_n(a;a+1) = g_n(a,1;a+1) + \dots + g_n(a,a-1;a+1) + g_n(a,a+2;a+1) + \dots + g_n(a,n-1;a+1) + g_n(a,n;a+1)$. Using the fact that no there exists a permutation $\pi \in S_n$(1-2-3) such that $\pi_1 = a$, $\pi_2 \leq a-2$, and $\pi_n = a+1$ we get

$$g_n(a;a+1) = g_n(a,a-1;a+1) + g_n(a,a+2;a+1) + \dots + g_n(a,n;a+1).$$

Using the fact that no there exists a permutation $\pi \in S_n$(1-2-3) such that $\pi_1 = a$ and $a \leq \pi_2 \leq n-1$ we get $g_n(a;a+1) = g_n(a,a-1;a+1) + g_n(a,n;a+1)$. On the other hand, it is easy to see that $g_n(a,a-1;a+1) = g_{n-1}(a-1;a)$ and $g_n(a,n;a+1) = g_{n-1}(a;a+1)$. Hence,

$$g_n(a;a+1) = g_{n-1}(a-1;a) + g_{n-1}(a;a+1).$$

Using induction we get that $g_n(a;a+1) = \binom{n-2}{a-1}$ for all $n \geq 2$ and $1 \leq a \leq n-1$.

(3) Similarly as (2) we have for all $a > b$,

$$g_n(a;b) = g_{n-1}(b-1;b) + g_{n-1}(b+1;b) + g_{n-1}(b+2;b) + \cdots + g_{n-1}(a;b).$$

Using Equation (4.10) we get

$$g_n(a;1) = g_n(a;2) = s_{n-1}(a-1) = \sum_{j=0}^{n-a}(-1)^j\binom{n-a-j}{j}C_{n-2-j}.$$

Using induction on b, we get

$$g_n(a;b) = \sum_{j=0}^{n-a}(-1)^j\binom{n-a-j}{j}\left(\sum_{i=0}^{b-1}(-1)^i\binom{b-1-i}{i}C_{n-2-j-i}\right).$$

□

Lemma 2. *We have*

$$h_n^{k,\ell}(t;s)=\begin{cases}\binom{n-t}{k-1}\binom{s-1}{\ell-1}g_{t-(\ell-1);s-(\ell-1)}(n+2-k-\ell), & \text{if } 1\le s<t\le n;\\ h_n^{k,\ell}(t+1;t), & \text{if } s=t+1;\\ h_{n-1}^{k,\ell-1}(t;s-1)+h_{n-1}^{k-1;\ell}(t;s-1), & \text{if } 2\le t+1<s\le n.\end{cases}$$

Proof. (1) Let $n\ge t>s\ge 1$; so by definitions we get

$$h_n^{k,\ell}(t;s)=\binom{n-t}{k-1}\binom{s-1}{\ell-1}g_{t-(\ell-1);s-(\ell-1)}(n-(k-1)-(\ell-1)).$$

(2) Let $s=t+1$; so it is easy to see $h_n^{k,\ell}(t;t+1)=h_n^{k,\ell}(t+1;t)$;

(3) Let $2\le t+1<s\le n$. Let π any permutations in $S_n(1\text{-}2\text{-}3)$ such that $\pi_k=t$ and $\pi_{n+1-\ell}=s$ where $\pi_1>\cdots>\pi_k$ and $\pi_{n+1-\ell}>\cdots>\pi_n$; so there two possibilities either $\pi_{n+2-\ell}=s-1$ or $\pi_j=s-1$ where $j\le k-1$. In this first case we get that there exist $h_{n-1}^{k,\ell-1}(t;s-1)$ permutations, and in the second case we have that there exist $h_{n-1}^{k-1;\ell}(t;s-1)$ permutations. (we extend the number $h_n^{k,l}(a;b)$ as 0 for any $\ell\le 0$ or $k\le 0$. □

We recall that the Kronecker delta $\delta_{n,k}$ is defined to be

$$\delta_{n,k}=\begin{cases}1, & \text{if } n=k,\\ 0, & \text{else.}\end{cases}$$

Proposition 12. *We have*

(i) $G_{1\text{-}2\text{-}3}^{12\ldots k,12\ldots\ell}(x)=\begin{cases}0, & \text{if } k\ge 3 \text{ or } \ell\ge 3\\ xC^2(x), & \text{if } k=1 \text{ and } \ell=1\end{cases}$,

$N_{1\text{-}2\text{-}3}^{12,12}(n)=\begin{cases}0, & \text{if } n=3\\ C_{n-2}, & \text{else}\end{cases}$, *and*

$N_{1\text{-}2\text{-}3}^{12,1}(n)=N_{1-2-3}^{1,12}(n)=C_{n-1}$.

(ii) $N_{1\text{-}2\text{-}3}^{k(k-1)\ldots 1,12\ldots\ell}(n)=$

$$\begin{cases}0, & \text{if } \ell\ge 3,\\ \sum\limits_{t=1}^{n-k}\binom{n-t-1}{k-1}\sum\limits_{j=0}^{n-t-1}(-1)^j\binom{n-t-j-1}{j}C_{n-t-j-1}+(k-1)\delta_{n,k+1}, & \text{if } \ell=2,\\ \sum\limits_{t=1}^{n+1-k}\binom{n-t}{k-1}\sum\limits_{j=0}^{n-t}(-1)^j\binom{n-t-j}{j}C_{n-t-j}, & \text{if } \ell=1.\end{cases}$$

(iii) $N_{1\text{-}2\text{-}3}^{12\ldots k,\ell(\ell-1)\ldots 1}(n)=$

$$\begin{cases}0, & \text{if } k\ge 3,\\ \sum\limits_{t=1}^{n-\ell}\binom{n-t-1}{\ell-1}\sum\limits_{j=0}^{n-t-1}(-1)^j\binom{n-t-j-1}{j}C_{n-t-j-1}+(\ell-1)\delta_{n,\ell+1}, & \text{if } k=2,\\ \sum\limits_{t=1}^{n+1-\ell}\binom{n-t}{\ell-1}\sum\limits_{j=0}^{n-t}(-1)^j\binom{n-t-j}{j}C_{n-t-j}, & \text{if } k=1.\end{cases}$$

(iv) $N_{1-2-3}^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}(x)=\sum_{t=1}^{n-k+1}\sum_{s=\ell}^{n}h_n^{k,\ell}(t;s)$, *where* $h_n^{k,\ell}(t;s)$ *is given in Lemma 2.*

Proof. **Beginning with** $12\dots k$ **and ending with** $12\dots\ell$**:** If $k\geq 3$ or $\ell\geq 3$, the statement is obvious, since in this case $12\dots k$ or $12\dots\ell$ does not avoid the pattern 1-2-3. If $k=1$ or $\ell=1$, we get the statement from Proposition 8 (in the first of these cases we apply the reverse and complement operations). Suppose now that $k=2$, $\ell=2$, and an n-permutation π avoids 1-2-3, begins with the pattern 12 and ends with the pattern 12. The letter n must be next to the leftmost letter, since otherwise two leftmost letters and n form the pattern 1-2-3. Also, the letter 1 must be next to the rightmost letter, since otherwise 1 and two rightmost letters form the pattern 1-2-3. It is easy to see now that there are C_{n-2} possibilities to choose π, since we can take any 1-2-3-avoiding permutation on the letters $\{2,3,\dots,n-1\}$ (there are C_{n-2} such permutations), then let the letters n and 1 be in the second and $(n-1)$-st positions respectively. These considerations are fail only when $n=3$, since in this case the second and $(n-1)$-st positions coincide. However, in this case we obviously have no permutations with the good properties.

Beginning with $k(k-1)\dots 1$ **and ending with** $12\dots\ell$**:** The statement is true for $\ell\geq 3$, since in this case $12\dots\ell$ does not avoid 1-2-3. For the case $\ell=1$ we use Proposition 9. Suppose now that $\ell=2$, and an n-permutation π avoids 1-2-3, begins with the pattern $k(k-1)\dots 1$ and ends with the pattern 12. The letter 1 must be next to the rightmost letter, since otherwise 1 and two rightmost letters form the pattern 1-2-3. So, to form π we can take any $(n-1)$-permutation on the letters $\{2,3,\dots,n\}$ that avoid 1-2-3 and begin with the pattern $k(k-1)\dots 1$ (the number of such permutations is given by Proposition 9), and then let the letter 1 be in the $(n-1)$-st position. Also, we observe that in the case $n=k+1$ we have $k-1$ extra permutations, which are obtained from the $(n-1)$-permutations having the $k-1$ leftmost letters in decreasing order and two rightmost letters in increasing order.

Beginning with $12\dots k$ **and ending with** $\ell(\ell-1)\dots 1$**:** By the reverse and complement operations, to avoid 1-2-3, begin with the pattern $12\dots k$ and end with the pattern $\ell(\ell-1)\dots 1$ is the same as to avoid 1-2-3, begin with the pattern $\ell(\ell-1)\dots 1$ and end with the pattern $12\dots k$, so we can apply the results of the previous case.

Beginning with $k(k-1)\dots 1$ **and ending with** $\ell(\ell-1)\dots 1$**:** The statement is straitforward to prove. □

4.6 Avoiding a pattern xyz, beginning and ending with certain patterns simultaneously

Recall the definitions of $E_q^{p,r}(x)$ in Section 4.2.

Proposition 13. *We have*

(i) $E_{213}^{12\dots k,12\dots \ell}(x) = \begin{cases} E_{132}^{12\dots \ell}(x), & \text{if } k=1 \\ E_{213}^{12\dots k}(x), & \text{if } \ell=1 \end{cases}$, *where* $E_{132}^{12\dots \ell}(x)$ *and* $E_{213}^{12\dots k}(x)$ *are given in* [Kit3, Theorem 6] *and* [Kit3, Theorem 10] *respectively. For* $k,\ell \geq 2$, $E_{213}^{12\dots k,12\dots \ell}(x)$ *satisfies*

$$\frac{\partial}{\partial x} E_{213}^{12\dots k,12\dots \ell}(x) = E_{213}^{12\dots k,12\dots(\ell-1)}(x) + \left(E_{213}^{12\dots k,12}(x) + \frac{x^{k-1}}{(k-1)!}\right) E_{132}^{12\dots \ell}(x).$$

(ii) $E_{213}^{12\dots k,\ell(\ell-1)\dots 1}(x) = \begin{cases} E_{132}^{\ell(\ell-1)\dots 1}(x), & \text{if } k=1 \\ E_{213}^{12\dots k}(x), & \text{if } \ell=1 \end{cases}$, *where* $E_{132}^{\ell(\ell-1)\dots 1}(x)$ *and* $E_{213}^{12\dots k}(x)$ *are given in* [Kit3, Theorem 7] *and* [Kit3, Theorem 10] *respectively. For* $k,\ell \geq 2$, $E_{213}^{12\dots k,\ell(\ell-1)\dots 1}(x)$ *satisfies*

$$\frac{\partial}{\partial x} E_{213}^{12\dots k,\ell(\ell-1)\dots 1}(x) = \frac{x^{\ell-1}}{(\ell-1)!} E_{213}^{12\dots k}(x) +$$

$$\left(E_{213}^{12\dots k,12}(x) + \frac{x^{k-1}}{(k-1)!}\right) E_{132}^{\ell(\ell-1)\dots 1}(x) + \binom{k+\ell-2}{k-1} \frac{x^{k+\ell-2}}{(k+\ell-2)!}.$$

(iii) $E_{213}^{k(k-1)\dots 1,12\dots \ell}(x) = \begin{cases} E_{132}^{12\dots \ell}(x), & \text{if } k=1 \\ E_{213}^{k(k-1)\dots 1}(x), & \text{if } \ell=1 \end{cases}$, *where* $E_{132}^{12\dots \ell}(x)$ *and* $E_{213}^{k(k-1)\dots 1}(x)$ *are given in* [Kit3, Theorem 6] *and* [Kit3, Theorem 11] *respectively. For* $k,\ell \geq 2$, $E_{213}^{k(k-1)\dots 1,12\dots \ell}(x)$ *satisfies*

$$\frac{\partial}{\partial x} E_{213}^{k(k-1)\dots 1,12\dots \ell}(x) = E_{213}^{(k-1)\dots 1,12\dots \ell}(x) +$$

$$E_{213}^{k(k-1)\dots 1,12}(x) E_{132}^{12\dots \ell}(x) + E_{213}^{k(k-1)\dots 1,12\dots(\ell-1)}(x).$$

(iv) $E_{213}^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}(x) = \begin{cases} E_{132}^{\ell(\ell-1)\dots 1}(x), & \text{if } k=1 \\ E_{213}^{k(k-1)\dots 1}(x), & \text{if } \ell=1 \end{cases}$, *where* $E_{132}^{\ell(\ell-1)\dots 1}(x)$ *and* $E_{213}^{k(k-1)\dots 1}(x)$ *are given in* [Kit3, Theorem 7] *and* [Kit3, Theorem 11] *respectively. For* $k,\ell \geq 2$, $E_{213}^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}(x)$ *satisfies*

$$\frac{\partial}{\partial x} E_{213}^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}(x) = E_{213}^{(k-1)\dots 1,\ell(\ell-1)\dots 1}(x) +$$

$$\left(E_{132}^{\ell(\ell-1)\dots 1}(x) + \frac{x^{\ell-1}}{(\ell-1)!}\right) E_{213}^{k(k-1)\dots 1,12}(x).$$

Proof.

Beginning with $12\dots k$ and ending with $\ell(\ell-1)\dots 1$: The statement is obviously true when $k=1$ and $\ell=1$. Suppose now that $k \geq 2$, $\ell \geq 2$ and an $(n+1)$-permutation π avoids 213, begins with the pattern $12\dots k$ and ends with the pattern $12\dots \ell$. The letter $(n+1)$ can only be in the position k, or in the position i, where $(k+1) \leq i \leq n-\ell+1$, or in the position $n-\ell+2$.

In the first case, we choose the $(k-1)$ leftmost letters in $\binom{n}{k-1}$ ways, rearrange them into the increasing order, and observe, that the letters of π to the right of $(n+1)$ must form an $(n-k+1)$-permutation, that avoids 213 and ends with the pattern $\ell(\ell-1)\dots1$ (the number of such permutations, using the reverse and complement operation, is equal to the number of $(n-k+1)$-permutations that avoid 132 and begin with the pattern $\ell(\ell-1)\dots1$). In the third case, we choose the $(\ell-1)$ rightmost letters in $\binom{n}{\ell-1}$ ways, rearrange them into the decreasing order, and observe, that the letters of π to the right of $(n+1)$ must form an $(n-\ell+1)$-permutation, that avoids 213, begins with the pattern $12\dots k$, and ends with the pattern 12 (if it ends with the pattern 21, the letter $(n+1)$ and two letters immediately to the left of it form the pattern 213). In the second case, we choose the letters of π to the left of $(n+1)$ in $\binom{n}{i-1}$ ways and observe, that these letters must form a $(i-1)$-permutation that avoids 213, begins with the pattern $12\dots k$ and ends with the pattern 12. In the same time, the letters to the right of $(n+1)$ must form an $(n-i+2)$-permutation that avoids 213 and ends with the pattern $\ell(\ell-1)\dots1$. Besides, we observe that if $n=k+\ell-2$, that is $|\pi|=k+\ell-1$, and first k-letters of π are rearranged into the increasing order, whereas the last ℓ letters are rearranged in the decreasing order, we have a number of extra "good" permutations. The number of such permutations is the number of ways of choosing the first $(k-1)$ letters, that is $\binom{k+\ell-2}{k-1}$. This discussion leads to the following:

$$N_{213}^{12\dots k,\ell(\ell-1)\dots1}(n+1)=\binom{n}{k-1}N_{132}^{\ell(\ell-1)\dots1}(n-k+1)+\binom{n}{\ell-1}N_{213}^{12\dots k}(n-\ell+1)$$
$$+\sum_{i=0}^{n}\binom{n}{i}N_{213}^{12\dots k,12}(i)N_{132}^{\ell(\ell-1)\dots1}(n-i)+\binom{k+\ell-2}{k-1}\delta_{n,k+\ell-2},$$

where $\delta_{n,k+\ell-2}$ is the Kronecker delta. We get the desirable result by multiplying both sides of the last equality by $x^n/n!$ and summing over n.

Beginning with $12\dots k$ and ending with $12\dots\ell$: The statement is obviously true when $k=1$ and $\ell=1$. Suppose now that $k\geq 2$, $\ell\geq 2$ and an $(n+1)$-permutation π avoids 213, begins with the pattern $12\dots k$ and ends with the pattern $12\dots\ell$. The letter $(n+1)$ can only be in the position k, or in the position i, where $(k+1)\leq i\leq n-\ell$, or in the $(n+1)$-th position. In the last case, the number of such permutations is obviously $N_{213}^{12\dots k,12\dots\ell-1}(n)$. In the first case, we choose the $(k-1)$ leftmost letters in $\binom{n}{k-1}$ ways, rearrange them into increasing order, and observe, that the letters of π to the right of $(n+1)$ must form an $(n-k+1)$-permutation, that avoids 213 and ends with the pattern $12\dots\ell$ (the number of such permutations, using the reverse and complement operation, is equal to the number of $(n-k+1)$-permutations that avoid 132 and begin with the pattern $12\dots\ell$). In the second case, we choose the letters of π to the left of $(n+1)$ in $\binom{n}{i-1}$ ways and observe, that these letters must form a $(i-1)$-permutation that avoids 213, begins with the pattern $12\dots k$ and ends with the pattern 12 (if it ends with the pattern 21, the letter $(n+1)$ and two letters immediately to the left of it form the pattern 213). In the same time, the letters to the right of $(n+1)$ must form an $(n-i+2)$-permutation that avoids

213 and ends with the pattern $12\dots\ell$. This discussion leads to the following:

$$N_{213}^{12\dots k,12\dots\ell}(n+1) = N_{213}^{12\dots k,12\dots\ell-1}(n)$$
$$+\sum_{i=0}^{n}\binom{n}{i}N_{213}^{12\dots k,12}(i)N_{132}^{12\dots\ell}(n-i)+\binom{n}{k-1}N_{132}^{12\dots\ell}(n-k+1).$$

We get the desirable result by multiplying both sides of the last equality by $x^n/n!$ and summing over n.

Beginning with $k(k-1)\dots 1$ and ending with $12\dots\ell$ or with $\ell(\ell-1)\dots 1$: We proceed in the same way as we do under considering the previous case. □

Proposition 14. *We have*

(i) $E_{132}^{12\dots k,12\dots\ell}(x) = \begin{cases} E_{213}^{12\dots\ell}(x), & \text{if } k=1 \\ E_{132}^{12\dots k}(x), & \text{if } \ell=1 \end{cases}$, *where* $E_{213}^{12\dots\ell}(x)$ *and* $E_{132}^{12\dots k}(x)$ *are given in* [Kit3, Theorem 10] *and* [Kit3, Theorem 6] *respectively. For* $k,\ell\geq 2$, $E_{132}^{12\dots k,12\dots\ell}(x)$ *satisfies*

$$\frac{\partial}{\partial x}E_{132}^{12\dots k,12\dots\ell}(x) = E_{132}^{12\dots k-1,12\dots\ell}(x)+\left(E_{132}^{12,12\dots\ell}(x)+\frac{x^{\ell-1}}{(\ell-1)!}\right)E_{132}^{12\dots k}(x).$$

(ii) $E_{132}^{12\dots k,\ell(\ell-1)\dots 1}(x) = \begin{cases} E_{132}^{12\dots k}(x), & \text{if } \ell=1 \\ E_{213}^{\ell(\ell-1)\dots 1}(x), & \text{if } k=1 \end{cases}$, *where* $E_{132}^{12\dots k}(x)$ *and* $E_{213}^{\ell(\ell-1)\dots 1}(x)$ *are given in* [Kit3, Theorem 6] *and* [Kit3, Theorem 11] *respectively. For* $k,\ell\geq 2$, $E_{132}^{12\dots k,\ell(\ell-1)\dots 1}(x)$ *satisfies*

$$\frac{\partial}{\partial x}E_{132}^{12\dots k,\ell(\ell-1)\dots 1}(x) = E_{132}^{12\dots k,(\ell-1)\dots 1}(x)+$$
$$E_{132}^{12,\ell(\ell-1)\dots 1}(x)E_{132}^{12\dots k}(x)+E_{132}^{12\dots(k-1),\ell(\ell-1)\dots 1}(x).$$

(iii) $E_{132}^{k(k-1)\dots 1,12\dots\ell}(x) = \begin{cases} E_{213}^{12\dots\ell}(x), & \text{if } k=1 \\ E_{132}^{k(k-1)\dots 1}(x), & \text{if } \ell=1 \end{cases}$, *where* $E_{213}^{12\dots\ell}(x)$ *and* $E_{132}^{k(k-1)\dots 1}(x)$ *are given in* [Kit3, Theorem 10] *and* [Kit3, Theorem 7] *respectively. For* $k,\ell\geq 2$, $E_{132}^{k(k-1)\dots 1,12\dots\ell}(x)$ *satisfies*

$$\frac{\partial}{\partial x}E_{213}^{k(k-1)\dots 1,12\dots\ell}(x) = \frac{x^{k-1}}{(k-1)!}E_{213}^{12\dots\ell}(x)$$
$$+\left(E_{132}^{12,12\dots\ell}(x)+\frac{x^{\ell-1}}{(\ell-1)!}\right)E_{132}^{k(k-1)\dots 1}(x)+\binom{k+\ell-2}{\ell-1}\frac{x^{k+\ell-2}}{(k+\ell-2)!}.$$

(iv) $E_{132}^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}(x) = \begin{cases} E_{213}^{\ell(\ell-1)\dots 1}(x), & \text{if } k=1 \\ E_{132}^{k(k-1)\dots 1}(x), & \text{if } \ell=1 \end{cases}$, *where* $E_{132}^{k(k-1)\dots 1}(x)$ *and* $E_{213}^{\ell(\ell-1)\dots 1}(x)$ *are given in* [Kit3, Theorem 7] *and* [Kit3, Theorem 11] *respectively. For* $k,\ell\geq 2$, $E_{132}^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}(x)$ *satisfies*

$$\frac{\partial}{\partial x}E_{132}^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}(x) = E_{132}^{(\ell-1)\dots 1,k(k-1)\dots 1}(x)+$$

$$\left(E_{132}^{k(k-1)\dots1}(x) + \frac{x^{k-1}}{(k-1)!}\right) E_{132}^{12,\ell(\ell-1)\dots1}(x).$$

Proof. We apply the reverse and complement operations and then use the results of Proposition 14. For example, to avoid 213, begin with $12\dots k$ and end with $12\dots\ell$ is the same as to avoid 132, begin with $12\dots\ell$ and end with $12\dots k$. □

Proposition 15. *We have*

(i) $E_{123}^{12\dots k,12\dots\ell}(x) =$

$$\begin{cases} 0, & \text{if } k\ge 3 \text{ or } \ell\ge 3,\\ x-\frac{1}{2}-\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)+ & \\ \sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)\left(\frac{\sqrt{3}}{2}\left(e^{x/2}+e^{-x/2}\right)-\sin\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{3}\right)\right), & \text{if } k=2 \text{ and } \ell=2,\\ \frac{\sqrt{3}}{2}e^{x/2}\sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)-1, & \text{if } k=1 \text{ and } \ell=1,\\ \frac{\sqrt{3}}{2}e^{x/2}\sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)-\frac{1}{2}-\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right), & \text{else;}\end{cases}$$

(ii) $E_{123}^{12\dots k,\ell(\ell-1)\dots1}(x) =$

$$\begin{cases} 0, & \text{if } k\ge 3,\\ \Phi_\ell(x)=\frac{e^{x/2}}{(\ell-1)!}\sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)\int_0^x e^{-t/2}t^{\ell-1}\sin\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right)\,dt, & \text{if } k=1,\\ \int_0^x \sec\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{6}\right)\left(\sin\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right)-\frac{\sqrt{3}}{2}e^{-t/2}\right)\left(\Phi_\ell(t)+\frac{t^{\ell-1}}{(\ell-1)!}\right)\,dt, & \text{if } k=2;\end{cases}$$

(iii) $E_{123}^{k(k-1)\dots1,12\dots\ell}(x) =$

$$\begin{cases} 0, & \text{if } \ell\ge 3,\\ \Phi_k(x)=\frac{e^{x/2}}{(k-1)!}\sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)\int_0^x e^{-t/2}t^{k-1}\sin\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right)\,dt, & \text{if } \ell=1,\\ \int_0^x \sec\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{6}\right)\left(\sin\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right)-\frac{\sqrt{3}}{2}e^{-t/2}\right)\left(\Phi_k(t)+\frac{t^{k-1}}{(k-1)!}\right)\,dt, & \text{if } \ell=2;\end{cases}$$

(iv) $$E_{123}^{k(k-1)\dots1,\ell(\ell-1)\dots1}(x) = \begin{cases} E_{123}^{\ell(\ell-1)\dots1}(x), & \text{if } k=1,\\ E_{123}^{k(k-1)\dots1}(x), & \text{if } \ell=1,\\ E_{123}^{k(k-1)\dots1}(x)-E_{123}^{k(k-1)\dots1,12}(x), & \text{if } \ell=2;\end{cases}$$

For $k\ge 2$ and $\ell\ge 3$, $E_{123}^{k(k-1)\dots1,\ell(\ell-1)\dots1}(x)$ satisfies

$$\frac{\partial}{\partial x}E_{123}^{k(k-1)\dots1,\ell(\ell-1)\dots1}(x) = \left(E_{123}^{\ell(\ell-1)\dots1}(x)+\frac{x^{\ell-1}}{(\ell-1)!}\right)E_{123}^{k(k-1)\dots1,21}(x)+E_{123}^{(k-1)\dots1,\ell(\ell-1)\dots1}(x),$$

where $E_{123}^{k(k-1)\dots1}(x)$ is given in [KitMans, Theorem 2]:

$$E_{123}^{k(k-1)\dots1}(x) = \frac{e^{x/2}\int_0^x e^{-t/2}t^{k-1}\sin(\frac{\sqrt{3}}{2}t+\frac{\pi}{6})\,dt}{(k-1)!\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})},$$

and $E_{123}^{k(k-1)\dots1,12}$ is given in this theorem above.

Proof.

Beginning with $k(k-1)\dots 1$ and ending with $12\dots\ell$: If $\ell \geq 3$ then the pattern $12\dots\ell$ does not avoid 123, thus the statement is true. If $\ell = 1$, the statement is true according to [Kit3, Theorem 8] and the observation that if $k = 1$ then this formula gives the expression

$$\frac{\sqrt{3}}{2}e^{x/2}\sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)-1,$$

which is true according to [ElizNoy, Theorem 4.1] and the assumption that the empty permutation does not begin or end with the pattern $p = 1$. So, we need only to consider the case $\ell = 2$. Recall the definitions of $E_q^p(x)$ in Section 4.2.

Let $P_k(n)$ denote the number of n-permutations that avoid the pattern 123, begin with a decreasing subword of length k and end with the pattern 12. Also, let $R_k(n)$ denote the number of n-permutations that avoid the pattern 123 and begin with a decreasing subword of length k. Let $\pi = \pi_1 1 \pi_2$ be an $(n+1)$-permutation that avoids the pattern 123, begins with the pattern $k(k-1)\dots 1$ and ends with the pattern 12. We observe that π_1 avoids 123 and begins with $k(k-1)\dots 1$; π_2 ends with the pattern 12 and $|\pi_2| > 0$ since otherwise π cannot end with the pattern 12; if $|\pi_2| > 1$ then π_2 must begin with the pattern 21 since otherwise we have an occurrence of the pattern 123 beginning from the letter 1. If $|\pi_1| = i$ then the letters of π_1 can be chosen in $\binom{n}{i}$ ways. So, there are at least

$$\sum_{i\geq 0}\binom{n}{i}R_k(i)P_2(n-i)+nR_k(n-1)$$

$(n+1)$-permutations with the good properties, where the first term corresponds to the case $|\pi_2| > 1$ and the second term to the case $|\pi_2| = 1$. By this formula, we do not count the permutations having $|\pi_1| = k-1$, although in this case π begins with the pattern $k(k-1)\dots 1$. So, we can choose the letters of π_1 in $\binom{n}{k-1}$ ways, and according to whether $|\pi| \geq 1$ or $|\pi| = 1$, we have two terms:

$$\binom{n}{k-1}P_2(n-k+1)+k\delta_{n,k},$$

where $\delta_{n,k}$ is the Kronecker delta. Thus,

$$P_k(n+1)=\sum_{i\geq 0}\binom{n}{i}R_k(i)P_2(n-i)+nR_k(n-1)+\binom{n}{k-1}P_2(n-k+1)+k\delta_{n,k}.$$

After multiplying both sides of the last equality with $x^n/n!$ and summing over n, we have

$$\frac{d}{dx}E_{123}^{k(k-1)\dots 1,12}(x)=(E_{123}^{21,12}(x)+x)\left(E_{123}^{k(k-1)\dots 1}(x)+\frac{x^{k-1}}{(k-1)!}\right), \quad (4.11)$$

with the initial condition $E_{123}^{k(k-1)\dots 1,12}(0) = 0$. Since

$$E_{123}^{k(k-1)\dots 1}(x)=E_{123}^{k(k-1)\dots 1,1}(x)=$$

$$\frac{e^{x/2}}{(k-1)!}\sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)\int_0^x e^{-t/2}t^{k-1}\sin\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right)\,dt,$$

to solve (4.11), we only need to know $E_{123}^{21,12}(x)$. To find it, we set $k = 2$ into (4.11) and solve this equation. For an example how to solve such an equation, we refer to [Kit3, Theorem 6]. We get

$$E_{123}^{21,12}(x) = -x+\sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)e^{-x/2}\int_0^x e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{6}\right)\,dt.$$

Now, we put the formula for $E_{123}^{21,12}(x)$ into (4.11) and solve this differential equation to get the desired result.

Beginning with $12\dots k$ and ending with $\ell(\ell-1)\dots 1$: By the reverse and complement operations, to avoid 123, begin with the pattern $12\dots k$ and end with the pattern $\ell(\ell-1)\dots 1$ is the same as to avoid 123, begin with the pattern $\ell(\ell-1)\dots 1$ and end with the pattern $12\dots k$, so we can apply the results of the previous case.

Beginning with $12\dots k$ and ending with $12\dots \ell$: The statement is obvious if $k \geq 3$ or $\ell \geq 3$. If $k = 1$ and $\ell = 1$ then the statement is true according to [ElizNoy, Theorem 4.1] (but we need to subtract 1, since by our assumption the empty permutation does not begin or end with the pattern $p = 1$). If $\ell = 1$ and $k = 2$, the statement is true according [Kit3, Theorem 9]. If $k = 1$ and $\ell = 2$, we apply the reverse and complement operations, and use again [Kit3, Theorem 9]. So, we only need to consider the case $k = 2$ and $\ell = 2$. It is easy to see that

$$E_{123}^{12,12}(x) = E_{123}^{1,12}(x) - E_{123}^{21,12}(x),$$

and from the previous cases

$$E_{123}^{1,12}(x) = \frac{\sqrt{3}}{2}e^{x/2}\sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)-\frac{1}{2}-\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right),$$

and

$$E_{123}^{21,12}(x) = -x+\sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right)\left(\sin\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{3}\right)-\frac{\sqrt{3}}{2}e^{-x/2}\right).$$

Beginning with $k(k-1)\dots 1$ and ending with $\ell(\ell-1)\dots 1$: If $\ell = 1$, the statement is trivial. If $k = 1$, we get the statement by using the reverse and complement operations. For the case $\ell = 2$, we observe that the number of n-permutations that avoid the pattern 123, begin with the pattern $k(k-1)\dots 1$ and end with the pattern 21 is equal to the number of n-permutation that avoid 123 and begin with the pattern $k(k-1)\dots 1$ minus the number of n-permutations that avoid the pattern 123, begin with the pattern $k(k-1)\dots 1$ and end with the pattern 12. Suppose now that $k \geq 2$ and $\ell \geq 3$ and an $(n+1)$-permutation π avoids 123, begins with $k(k-1)\dots 1$ and ends with $\ell(\ell-1)\dots 1$.

It is easy to see that the letter $(n+1)$ can be either in the first position, or in the position i, where $(k+1) \le i \le (n-\ell)$, or in the position $(n-\ell+1)$. In the first of these cases, obviously we have $N_{123}^{(k-1)\dots1,\ell(\ell-1)\dots1}(n)$ permutations. In the second case, we choose the letters of π to the left of $(n+1)$ in $\binom{n}{i-1}$ ways. These letters must form a permutation that avoids 123, begins with the pattern $k(k-1)\dots1$, and ends with the pattern 21 (if the last condition does not hold, the letter $(n+1)$ and two letters to the left of it form a 123-pattern. In the same time, the letters to the right of $(n+1)$ form a permutation that avoids 123 and ends with the pattern $\ell(\ell-1)\dots1$. In the third case, we can choose the letters to the right of $(n+1)$ in $\binom{n}{\ell-1}$ ways, rearrange them into the decreasing order, and form from the letters to the left of $(n+1)$ a permutation that avoids 123, begins with the pattern $k(k-1)\dots1$ and ends with the pattern 21 (by the same reasons as above) in $N_{123}^{k(k-1)\dots1,21}(n-\ell+1)$ ways. Thus,

$$N_{123}^{k(k-1)\dots1,\ell(\ell-1)\dots1}(n+1) = N_{123}^{(k-1)\dots1,\ell(\ell-1)\dots1}(n)$$
$$+\sum_{i=0}^{n}\binom{n}{i}N_{123}^{k(k-1)\dots1,21}(i)N_{123}^{\ell(\ell-1)\dots1}(n-i)+\binom{n}{\ell-1}N_{123}^{k(k-1)\dots1,21}(n-\ell+1),$$

where we observed, that to avoid 123 and end with $\ell(\ell-1)\dots1$ is the same as to avoid 123 and begin with $\ell(\ell-1)\dots1$ using the reverse and complement. Now, we multiply both sides of the equality by $x^n/n!$ and sum over n to get the desirable result. □

4.7 Avoiding a pattern x-yz, beginning and ending with certain patterns simultaneously

Proposition 16. *We have*

(i) $$E_{1\text{-}32}^{12\dots k,1}(x) = E_{1\text{-}32}^{12\dots k}(x) = \begin{cases} e^{e^x}\int_0^x e^{-e^t}\sum_{n\ge k-1}\frac{t^n}{n!}\,dt, & \text{if } k\ge 2 \\ e^{e^x-1}, & \text{if } k=1 \end{cases}.$$

For $\ell \ge 2$, $E_{1\text{-}32}^{12\dots k,12\dots\ell}(x)$ *satisfies*

$$\frac{\partial}{\partial x}E_{1\text{-}32}^{12\dots k,12\dots\ell}(x) = \left(e^x - \sum_{i=0}^{\ell-2}\frac{x^i}{i!}\right)E_{1\text{-}32}^{12\dots k}(x) + e^x x^{max(\ell,k)-1}.$$

(ii) $E_{1\text{-}32}^{12\dots k,\ell(\ell-1)\dots1}(x)$ *satisfies*

$$\frac{\partial^{\ell-1}}{\partial x^{\ell-1}}E_{1\text{-}32}^{12\dots k,\ell(\ell-1)\dots1}(x) = \begin{cases} e^{e^x}\int_0^x e^{-e^t}\sum_{n\ge k-1}\frac{t^n}{n!}\,dt, & \text{if } k\ge 2, \\ e^{e^x-1}, & \text{if } k=1. \end{cases}$$

(iii)

$$E_{1\text{-}32}^{k(k-1)\dots1,1}(x) = E_{1\text{-}32}^{k(k-1)\dots1}(x) =$$

$$\begin{cases} (e^{e^x}/(k-1)!)\int_0^x t^{k-1}e^{-e^t+t}\,dt, & \text{if } k \geq 2 \\ e^{e^x-1}, & \text{if } k=1 \end{cases}$$

For $\ell \geq 2$, $E_{1\text{-}32}^{k(k-1)\dots1,12\dots\ell}(x)$ *satisfies*

$$\frac{\partial}{\partial x}E_{1\text{-}32}^{k(k-1)\dots1,12\dots\ell}(x) = \left(e^x - \sum_{i=0}^{\ell-2}\frac{x^i}{i!}\right)E_{1\text{-}32}^{k(k-1)\dots1}(x) + \left(e^x - \sum_{i=0}^{\ell-2}\frac{x^i}{i!}\right)\frac{x^{k-1}}{(k-1)!}.$$

(iv) $E_{1\text{-}32}^{k(k-1)\dots1,\ell(\ell-1)\dots1}(x)$ *satisfies*

$$\frac{\partial^{\ell-1}}{\partial x^{\ell-1}}\left(E_{1\text{-}32}^{k(k-1)\dots1,\ell(\ell-1)\dots1}(x) - \frac{x^{max(k,\ell)} - x^{k+\ell-1}}{1-x}\right) =$$

$$\begin{cases} \frac{e^{e^x}}{(k-1)!}\int_0^x t^{k-1}e^{-e^t+t}\,dt, & \text{if } k \geq 2, \\ e^{e^x-1}, & \text{if } k=1. \end{cases}$$

Proof.

Beginning with $12\dots k$ **and ending with** $\ell(\ell-1)\dots1$**:** If $\ell = 1$ then the result follows from [KitMans, Proposition 5], since to avoid 1-32 and begin with $12\dots k$ is the same as to avoid 3-12 and begin with $k(k-1)\dots1$. Suppose now that $\ell \geq 2$ and a permutation π avoids the pattern 1-32, begins with the pattern $12\dots k$ and ends with the pattern $\ell(\ell-1)\dots1$. Since $\ell \geq 2$, we have that the letter 1 must be in the rightmost position since otherwise, this letter and two rightmost letters of π form the pattern 1-32, which is forbidden. Thus,

$$N_{1\text{-}32}^{12\dots k,\ell(\ell-1)\dots1}(n) = N_{1\text{-}32}^{12\dots k,(\ell-1)(\ell-2)\dots1}(n-1) = \cdots = N_{1\text{-}32}^{12\dots k,1}(n-\ell+1).$$

Multiplying both sides of the equality $N_{1\text{-}32}^{12\dots k,\ell(\ell-1)\dots1}(n) = N_{1\text{-}32}^{12\dots k,1}(n-\ell+1)$ by $x^{n-\ell+1}/(n-\ell+1)!$ and summing over n, we get

$$\frac{\partial^{\ell-1}}{\partial x^{\ell-1}}E_{1\text{-}32}^{12\dots k,\ell(\ell-1)\dots1}(x) = E_{1\text{-}32}^{12\dots k}(x),$$

where $E_{1\text{-}32}^{12\dots k}(x)$ is given in [KitMans, Proposition 5], since to avoid 1-32 and begin with $12\dots k$ is the same as to avoid 3-12 and begin with $k(k-1)\dots1$.

Beginning with $k(k-1)\dots1$ **and ending with** $\ell(\ell-1)\dots1$**:** We use the same arguments as those given under consideration of the previous case, but instead of [KitMans, Proposition 5] we use [KitMans, Proposition 4]. However, we observe, that when we use the argument

$$N_{1\text{-}32}^{k(k-1)\dots1,\ell(\ell-1)\dots1}(n) = N_{1\text{-}32}^{k(k-1)\dots1,(\ell-1)(\ell-2)\dots1}(n-1) = \cdots = N_{1\text{-}32}^{k(k-1)\dots1,1}(n-\ell+1)$$

for $k, \ell \geq 2$, we do not count the decreasing permutations of length $max(k,\ell)$, $max(k,\ell)+1, \dots, k+\ell-2$, since in this case, the patterns $k(k-1)\dots1$ and $\ell(\ell-1)\dots1$ overlap in more than one letter, which causes the observation. So, we need to consider additionally the term

$$x^{max(k,\ell)} + x^{max(k,\ell)+1} + \cdots + x^{k+\ell-2} = \frac{x^{max(k,\ell)} - x^{k+\ell-1}}{1-x},$$

which vanishes if $k = 1$ or $\ell = 1$.

Beginning with $12\dots k$ **and ending with** $12\dots\ell$**:** The only interesting case here is the case $k \geq 2$ and $\ell \geq 2$. Using the reverse and complement, instead of considering avoiding 1-32, beginning with $12\dots k$ and ending with $12\dots\ell$, we consider avoiding 1-32, beginning with $12\dots\ell$ and ending with $12\dots k$. Suppose an n-permutation π satisfies all the conditions. We observe, that the letter n can be in the position i, where $\ell \leq i \leq n-k$. Also, n can be in the rightmost position if $n \geq max(\ell, k)$. In any case, the letters of π to the left of n must be in the increasing order, since otherwise we have an occurrence of the pattern 21-3. This means that in the second case we have the only one permutation. In the first case, the letters of π to the right of n must avoid 21-3 and end with the pattern $12\dots k$. The number of such permutations, using the reverse and complement, is given by $N_{1\text{-}32}^{12\dots k}(n-i)$. Thus, for $n \geq max(\ell, k)$,

$$N_{21\text{-}3}^{12\dots\ell,12\dots k}(n) = \sum_{i=\ell}^{n-k} \binom{n-1}{i-1} N_{1\text{-}32}^{12\dots k}(n-i) + 1.$$

This gives

$$N_{21\text{-}3}^{12\dots\ell,12\dots k}(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} N_{1\text{-}32}^{12\dots k}(n-i) - \sum_{i=1}^{\ell-1} \binom{n-1}{i-1} N_{1\text{-}32}^{12\dots k}(n-i) + 1,$$

which leads to the desirable result after multiplying both sides of the last equality by $x^n/n!$ and summing over n.

Beginning with $k(k-1)\dots 1$ **and ending with** $12\dots\ell$**:** The only interesting case here is the case $k \geq 2$ and $\ell \geq 2$. Using the reverse and complement, instead of considering avoiding 1-32, beginning with $k(k-1)\dots 1$ and ending with $12\dots\ell$, we consider avoiding 1-32, beginning with $12\dots\ell$ and ending with $k(k-1)\dots 1$. Suppose an n-permutation π satisfies all the conditions. We observe, that the letter n can only be in the position i, where $\ell \leq i \leq n-k$, or in position $(n-k+1)$ (in the case $n \geq k+\ell-1$). In the first case, it is easy to see that the letters of π to the left of n must be in the increasing order, and the letters of π to the right of n must avoid 21-3 and end with the pattern $k(k-1)\dots 1$. Using the reverse and complement, the total number of permutations counted in the first case is $\sum_{i=\ell}^{n-k} \binom{n-1}{i-1} N_{1\text{-}32}^{k(k-1)\dots 1}(n-i)$. In the second case, the letters to the left of n are in the increasing order, whereas the letters to the right of n are in decreasing order. The number of such permutations is $\binom{n-1}{k-1}$, which is the number of ways to choose the last $k-1$ letters. Thus,

$$N_{21\text{-}3}^{12\dots\ell,k(k-1)\dots 1}(n) = \sum_{i=\ell}^{n-k} \binom{n-1}{i-1} N_{1\text{-}32}^{k(k-1)\dots 1}(n-i) + \binom{n-1}{k-1}.$$

Multiplying both parts of the equality by $x^{n-1}/(n-1)!$ and summing over n, we get

$$\frac{\partial}{\partial x}E_{21\text{-}3}^{12\ldots\ell,k(k-1)\ldots1}(x)=\sum_{n\geq k+\ell}\binom{n-1}{k-1}\frac{x^{n-1}}{(n-1)!}$$
$$+\sum_{n\geq0}\left(\sum_{i=1}^{n-1}\binom{n-1}{i-1}N_{1\text{-}32}^{k(k-1)\ldots1}(n-i)-\sum_{i=1}^{\ell-1}\binom{n-1}{i-1}N_{1\text{-}32}^{k(k-1)\ldots1}(n-i)\right)\frac{x^{n-1}}{(n-1)!},$$

which leads to the desirable result. □

Proposition 17. *We have*

(i) $G_{2\text{-}13}^{12\ldots k,12\ldots\ell}(x)=x^{k+\ell-1}C^{k+1}(x)+\frac{x^m-x^{k+\ell-1}}{1-x}$.

(ii) $G_{2\text{-}13}^{k(k-1)\ldots1,12\ldots\ell}(x)=x^{k+\ell-1}C^2(x)$.

(iii) $G_{2\text{-}13}^{k(k-1)\ldots1,\ell(\ell-1)\ldots1}(x)=x^{k+\ell-1}C^{\ell+1}(x)+\frac{x^m-x^{k+\ell-1}}{1-x}$, *where* $m=\max(k,\ell)$.

(iv) *the generating function* $G_{2\text{-}13}(x,y,z)=\sum_{k,\ell\geq0}G_{2\text{-}13}^{12\ldots k,\ell(\ell-1)\ldots1}(x)y^kz^\ell$ *for the sequence* $\{G_{2\text{-}13}^{12\ldots k,\ell(\ell-1)\ldots1}(x)\}_{k,\ell\geq0}$ *(where* k *and* ℓ *go through all natural numbers) is*

$$\frac{1}{1-x(y+z)}\left(x(y+z+yz)+\frac{C(x)-1}{(1-xyC(x))(1-xzC(x))}\right).$$

Proof. By [Claes], to avoid the pattern 2-13 is the same as to avoid the pattern 2-1-3. Thus we can apply the results of Proposition 11. □

Proposition 18. *We have*

$$\text{(i) } E_{1\text{-}23}^{12\ldots k,12\ldots\ell}(x)=\begin{cases}0, & \text{if } k\geq3 \text{ or } \ell\geq3,\\ E_{1\text{-}23}^{12\ldots k}(x), & \text{if } \ell=1,\\ E_{12\text{-}3}^{12\ldots\ell}(x), & \text{if } k=1,\\ \int_0^x tE_{12\text{-}3}^{12}(t)\,dt+\frac{x^2}{2!}, & \text{if } k=2 \text{ and } \ell=2,\end{cases}$$

where $E_{12\text{-}3}^{12\ldots k}(x)$ *and* $E_{1\text{-}23}^{12\ldots k}(x)$ *are given by* [KitMans, Proposition 10] *and* [KitMans, Proposition 6] *respectively:*

$$E_{12\text{-}3}^{12\ldots k}(x)=\begin{cases}0, & \text{if } k\geq3,\\ x^2\sum_{j=0}^{k}(1-jx)^{-1}\sum_{d\geq0}\frac{x^d}{(1-x)(1-2x)\ldots(1-dx)}, & \text{if } k=2,\\ \sum_{d\geq0}\frac{x^d}{(1-x)(1-2x)\ldots(1-dx)}, & \text{if } k=1;\end{cases}$$

$$E_{1\text{-}23}^{12\ldots k}(x)=E_{3-21}^{k(k-1)\ldots1}(x)=\begin{cases}0, & \text{if } k\geq3,\\ e^{e^x}\int_0^x e^{-e^t}(e^t-1)\,dt, & \text{if } k=2,\\ e^{e^x-1}, & \text{if } k=1.\end{cases}$$

(ii) $N_{1\text{-}23}^{12\ldots k,\ell(\ell-1)\ldots 1}(n) =$

$$\begin{cases} 0, & \textit{if } k \geq 3,\\ 0, & \textit{if } k = 2 \textit{ and } n \leq \ell,\\ 1 + N_{1\text{-}23}^{12,(\ell-1)(\ell-2)\ldots 1}(n-1) + \sum_{j=\ell+1}^{n-2} \binom{n-1}{j-1} N_{1\text{-}23}^{12}(n-j), & \textit{if } k = 2 \textit{ and } n \geq \ell+1,\\ N_{12\text{-}3}^{\ell(\ell-1)\ldots 1}(n), & \textit{if } k = 1, \end{cases}$$

where the numbers $N_{12\text{-}3}^{\ell(\ell-1)\ldots 1}(n)$ *are given in* [KitMans, Proposition 9], *and the numbers* $N_{1\text{-}23}^{12}(n)$ *are given by expending the exponential generating functions in* [KitMans, Proposition 6].

(iii)

$$E_{1\text{-}23}^{k(k-1)\ldots 1,12\ldots \ell}(x) =$$

$$\begin{cases} 0, & \textit{if } \ell \geq 3\\ \frac{1}{(k-1)!}\int_0^x \int_0^t tm^{k-1} e^{e^t - e^m + m}\, dmdt + \frac{kx^{k+1}}{(k+1)!}, & \textit{if } \ell = 2\\ (e^{e^x}/(k-1)!)\int_0^x t^{k-1} e^{-e^t + t}\, dt, & \textit{if } \ell = 1 \end{cases},$$

where $E_{1\text{-}23}^{k(k-1)\ldots 1,1}(n) = E_{1\text{-}23}^{k(k-1)\ldots 1}(n)$ *is given by* [KitMans, Proposition 4], *and* $N_{1\text{-}23}^{1,\ell(\ell-1)\ldots 1}(n) = N_{12\text{-}3}^{\ell(\ell-1)\ldots 1}(n)$ *is given by* [KitMans, Proposition 9]*;*

(iv) *For* $k \geq 2$ *and* $\ell \geq 2$, $E_{1\text{-}23}^{k(k-1)\ldots 1,\ell(\ell-1)\ldots 1}(x)$ *satisfies*

$$\frac{\partial}{\partial x} E_{1\text{-}23}^{k(k-1)\ldots 1,\ell(\ell-1)\ldots 1}(x) = E_{1\text{-}23}^{k(k-1)\ldots 1,(\ell-1)\ldots 1}(x) + \left(e^x - \sum_{i=0}^{\ell-1} \frac{x^i}{i!}\right)\left(E_{1\text{-}23}^{k(k-1)\ldots 1}(x) + \frac{x^k}{(k-1)!}\right).$$

Proof.

Beginning with $k(k-1)\ldots 1$ and ending with $12\ldots\ell$: If $\ell \geq 3$ then $E_{1\text{-}23}^{k(k-1)\ldots 1,12\ldots\ell}(x) = 0$, since in this case the pattern $12\ldots\ell$ does not avoid 1-23. If $\ell = 1$ then we use [KitMans, Proposition 4], since in this case the only restrictions to the permutations are avoiding 1-23 and beginning with the pattern $k(k-1)\ldots 1$. Suppose now that $\ell = 2$ and an $(n+1)$-permutation π avoids 1-23, begins with $k(k-1)\ldots 1$ and ends with the pattern 12. The letter 1 must be in next to the rightmost position, since otherwise this letter and two rightmost letters form the pattern 1-23. We can choose the rightmost letter of π in n ways, and the letters to the left of 1 must form a 1-23-avoiding permutation that begins with $k(k-1)\ldots 1$. Besides, if $n = k$, and the $k-1$ letters to the right of 1 are in the decreasing order, we get n extra permutations that satisfy our restrictions. Thus,

$$N_{1\text{-}23}^{k(k-1)\ldots 1,12}(n+1) = nN_{1\text{-}23}^{k(k-1)\ldots 1}(n) + n\delta_{n,k},$$

where $\delta_{n,k}$ is the Kronecker delta. Multiplying both sides of the equality by $x^n/n!$ and summing over n we get

$$E_{1\text{-}23}^{k(k-1)\dots 1,12}(x) = \int_0^x tE_{1\text{-}23}^{k(k-1)\dots 1}(t)\ dt + \frac{kx^{k+1}}{(k+1)!}.$$

Using the formula for $E_{1\text{-}23}^{k(k-1)\dots 1}(t)$ in [KitMans, Proposition 4], we get the desirable result.

Beginning with $12\dots k$ **and ending with** $12\dots \ell$**:** The first three cases are easy to prove in the same manner as we do in the proves of previous propositions. The only interesting case is when $k = 2$ and $\ell = 2$. Using the reverse and complement operations, instead of considering avoiding 1-23, beginning with 12 and ending with 12, we consider avoiding 12-3, beginning with 12 and ending with 12, which we find to be more easy. Suppose an $(n+1)$-permutation π satisfies all the restrictions. It is easy to see that $|\pi| \neq 1$ and $|\pi| \neq 3$, as well as if $|\pi| = 2$ (that is $n = 1$) then π must be 12. Suppose $|\pi| \geq 4$. Since π begins with the pattern 12, it is impossible for the letter $(n+1)$ to be somewhere to the right of the second letter of π or to be the leftmost letter. Thus, $(n+1)$ must be in the second position. We can choose the leftmost letter of π in n ways, since any choice of this letter will not lead to an occurrence of the pattern $12-3$ beginning with two leftmost letters. If $\pi = a(n+1)\pi'$ then π' must avoid 12-3 and end with the pattern 12. The number of such permutations, using the reverse and complement, is given by $N_{1\text{-}23}^{12}(n-1)$. Thus,

$$N_{12\text{-}3}^{12,12}(n+1) = nN_{1\text{-}23}^{12}(n-1).$$

Multiplying both sides of the equality by $x^n/n!$ and summing over all n, we get

$$(E_{12\text{-}3}^{12,12}(x))' = xE_{1\text{-}23}^{12}(x) + x,$$

where the term x corresponds to the permutation 12. We have the desirable result by integrating both sides of the last equality.

Beginning with $12\dots k$ **and ending with** $\ell(\ell-1)\dots 1$**:** All the cases but $k = 2$ and $n \geq \ell + 1$ are easy to prove. Let us consider this case. Using the reverse and complement operations, instead of considering avoiding 1-23, beginning with 12 and ending with $\ell(\ell-1)\dots 1$, we consider avoiding 12-3, beginning with $\ell(\ell-1)\dots 1$ and ending with 12, which we find to be more easy. Let an n-permutation π satisfies all the conditions. We observe, that the letter n is either in the first position, or in position j, where $k+1 \leq j \leq n-2$, or in the last position. Obviously, in the first of these cases the number of "good" permutations is given by $N_{12\text{-}3}^{(\ell-1)(\ell-2)\dots 1,12}(n-1)$, which is equivalent to $N_{1\text{-}23}^{12,(\ell-1)(\ell-2)\dots 1}(n-1)$ by using the reverse and complement. In the second case, we choose the letters to the left of n in $\binom{n-1}{j-1}$ ways, rearrange them to the decreasing order (we do it since otherwise we have an occurrence of the pattern 12-3 having the letter n). After that, the letters to the right of n must form a permutation that avoid 12-3 and end with the pattern 12. Using the reverse and complement, there are $N_{1\text{-}23}^{12}(n-j)$ such permutations. So, totally, in the second case there are $\sum_{j=\ell+1}^{n-2}\binom{n-1}{j-1}N^{12}_{1\text{-}23}(n-j)$ permutations. Finally, if n is at the last position, we have the only one such permutation, since the other letters must be in the decreasing order.

Beginning with $k(k-1)\dots 1$ and ending with $\ell(\ell-1)\dots 1$: The only interesting case here is the case $k \geq 2$ and $\ell \geq 2$. Using the reverse and complement operations, instead of considering avoiding 1-23, beginning with $k(k-1)\dots 1$ and ending with $\ell(\ell-1)\dots 1$, we consider avoiding 12-3, beginning with $\ell(\ell-1)\dots 1$ and ending with $k(k-1)\dots 1$, which we find to be more easy. Let an n-permutation π satisfies all the conditions. We observe, that the letter n is either in the first position, or in position j, where $\ell+1 \leq j \leq n-k$, or in the last position $n-k+1$. We proceed as in the previous case to get the following

$$N^{\ell(\ell-1)\dots 1,k(k-1)\dots 1}_{12\text{-}3} = N^{(\ell-1)\dots 1,k(k-1)\dots 1}_{12\text{-}3} + \sum_{i=\ell+1}^{n-k}\binom{n-1}{i-1}N^{k(k-1)\dots 1}_{1\text{-}23}(n-i) + \binom{n-1}{k-1},$$

where three terms in the right-hand side correspond to the three cases described above. We now multiply both sides of the equality by $x^n/n!$, sum over n and observe the following detail. We cannot write instead of $i=\ell+1$ (in the sum above) $i=1$ as we did in most of the cases above, since, for instance, the case $i=1$ do not necessarily make the term of summation equal 0 as it was before. Thus, instead of the factor e^x, we have the factor $\left(e^x - \sum_{i=0}^{\ell-1}\frac{x^i}{i!}\right)$ □

4.8 Avoiding a pattern xy-z, beginning and ending with certain patterns simultaneously

Proposition 19. *We have*

(i) $G^{12\dots k,12\dots\ell}_{13\text{-}2}(x) = x^{k+\ell-1}C^{\ell+1}(x) + \frac{x^m - x^{k+\ell-1}}{1-x}$.

(ii) $G^{12\dots k,\ell(\ell-1)\dots 1}_{13\text{-}2}(x) = x^{k+\ell-1}C^2(x)$.

(iii) $G^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}_{13\text{-}2}(x) = x^{k+\ell-1}C^{k+1}(x) + \frac{x^m - x^{k+\ell-1}}{1-x}$, *where* $m = \max(k,\ell)$.

(iv) *the generating function* $G_{13-2}(x,y,z) = \sum_{k,\ell\geq 0} G^{k(k-1)\dots 1,12\dots\ell}_{13\text{-}2}(z)y^k z^\ell$ *for the sequence* $\{G^{k(k-1)\dots 1,12\dots\ell}_{13\text{-}2}(x)\}_{k,\ell\geq 0}$ *(where k and ℓ go through all natural numbers) is*

$$\frac{1}{1-x(y+z)}\left(x(y+z+yz) + \frac{C(x)-1}{(1-xyC(x))(1-xzC(x))}\right).$$

Proof. We apply the reverse and complement operations and then use the results of Proposition 17. For example, to avoid 2-13, begin with $12\dots k$ and end with $12\dots\ell$ is the same as to avoid 13-2, begin with $12\dots\ell$ and end with $12\dots k$. □

Proposition 20. *We have*

(i) $E^{12\ldots k,1}_{21\text{-}3}(x) = E^{12\ldots k}_{21\text{-}3}(x)$ *is given by* [KitMans, Proposition 14]. *For* $\ell \geq 2$, $E^{12\ldots k,12\ldots \ell}_{21\text{-}3}(x)$ *satisfies*

$$\frac{\partial}{\partial x} E^{12\ldots k,12\ldots\ell}_{21\text{-}3}(x) = \left(e^x - \sum_{i=0}^{k-2} \frac{x^i}{i!}\right) E^{12\ldots\ell}_{1\text{-}32}(x) + e^x x^{max(\ell,k)-1}.$$

where $E^{12\ldots\ell}_{1\text{-}32}(x) = E^{\ell(\ell-1)\ldots 1}_{3\text{-}12}(x)$ *is given by* [KitMans, Proposition 5].

(ii) *For* $\ell \geq 2$, $E^{12\ldots k,\ell(\ell-1)\ldots 1}_{21\text{-}3}(x)$ *satisfies*

$$\frac{\partial}{\partial x} E^{12\ldots k,\ell(\ell-1)\ldots 1}_{21\text{-}3}(x) = \left(e^x - \sum_{i=0}^{k-2} \frac{x^i}{i!}\right) E^{\ell(\ell-1)\ldots 1}_{1\text{-}32}(x) + \left(e^x - \sum_{i=0}^{k-2} \frac{x^i}{i!}\right) \frac{x^{\ell-1}}{(\ell-1)!},$$

where $E^{\ell(\ell-1)\ldots 1}_{1\text{-}32}(x)$ *is given by* [KitMans, Proposition 4].

(iii) $E^{k(k-1)\ldots 1,12\ldots\ell}_{21\text{-}3}(x)$ *satisfies*

$$\frac{\partial^{k-1}}{\partial x^{k-1}} E^{k(k-1)\ldots 1,12\ldots\ell}_{21\text{-}3}(x) = \begin{cases} e^{e^x} \int_0^x e^{-e^t} \sum_{n \geq \ell-1} \frac{t^n}{n!}\, dt, & \text{if } \ell \geq 2, \\ e^{e^x - 1}, & \text{if } \ell = 1. \end{cases}$$

(iv) $E^{k(k-1)\ldots 1,\ell(\ell-1)\ldots 1}_{21\text{-}3}(x)$ *satisfies*

$$\frac{\partial^{k-1}}{\partial x^{k-1}} \left(E^{k(k-1)\ldots 1,\ell(\ell-1)\ldots 1}_{2\text{-}13}(x) - \frac{x^{max(k,\ell)} - x^{k+\ell-1}}{1-x} \right) =$$

$$\begin{cases} \frac{e^{e^x}}{(\ell-1)!} \int_0^x t^{\ell-1} e^{-e^t + t}\, dt, & \text{if } \ell \geq 2, \\ e^{e^x - 1}, & \text{if } \ell = 1. \end{cases}$$

Proof. We apply the reverse and complement operations and then use the results of Proposition 16. For example, to avoid 1-32, begin with $12\ldots k$ and end with $12\ldots\ell$ is the same as to avoid 21-3, begin with $12\ldots\ell$ and end with $12\ldots k$. □

Proposition 21. *We have*

(i) $$E^{12\ldots k,12\ldots\ell}_{12\text{-}3}(x) = \begin{cases} 0, & \text{if } k \geq 3 \text{ or } \ell \geq 3, \\ E^{12\ldots k}_{12\text{-}3}(x), & \text{if } \ell = 1, \\ E^{12\ldots \ell}_{1\text{-}23}(x), & \text{if } k = 1, \\ \int_0^x t E^{12}_{1\text{-}23}(t)\, dt + \frac{x^2}{2!}, & \text{if } k = 2 \text{ and } \ell = 2, \end{cases}$$

where $E^{12\ldots k}_{12\text{-}3}(x)$ *and* $E^{12\ldots k}_{1\text{-}23}(x)$ *are given in Proposition 18.*

(ii) $$E^{12\ldots k,\ell(\ell-1)\ldots 1}_{1\text{-}23}(x) = \begin{cases} 0, & \text{if } k \geq 3, \\ \frac{1}{(\ell-1)!} \int_0^x \int_0^t t m^{\ell-1} e^{e^t - e^m + m}\, dm dt + \frac{\ell x^{\ell+1}}{(\ell+1)!}, & \text{if } k = 2, \\ (e^{e^x}/(\ell-1)!) \int_0^x t^{\ell-1} e^{-e^t + t}\, dt, & \text{if } k = 1; \end{cases}$$

(iv) $N_{12\text{-}3}^{k(k-1)\dots 1,12\dots\ell}(n) =$

$$\begin{cases} 0, & \textit{if } \ell \geq 3, \\ 0, & \textit{if } \ell = 2 \textit{ and} \\ & n \leq k, \\ 1 + N_{12\text{-}3}^{(k-1)(k-2)\dots 1,12}(n-1) + \sum_{j=k+1}^{n-2} \binom{n-1}{j-1} N_{3\text{-}21}^{21}(n-j), & \textit{if } \ell = 2 \textit{ and} \\ & n \geq k+1, \\ N_{12\text{-}3}^{k(k-1)\dots 1}(n), & \textit{if } \ell = 1, \end{cases}$$

where the numbers $N_{12\text{-}3}^{k(k-1)\dots 1}(n)$ *are given in* [KitMans, Proposition 9], *and the numbers* $N_{3\text{-}21}^{21}(n)$ *are given by expending the exponential generating functions in* [KitMans, Proposition 6].

(iv) $N_{12\text{-}3}^{k(k-1)\dots 1,1}(n) = N_{12\text{-}3}^{k(k-1)\dots 1}(n)$ *is given by* [KitMans, Proposition 9], *and* $N_{12\text{-}3}^{1,\ell(\ell-1)\dots 1}(n) = N_{1\text{-}23}^{\ell(\ell-1)\dots 1}(n)$ *is given by* [KitMans, Proposition 4]. *For* $k \geq 2$ *and* $\ell \geq 2$, $E_{12\text{-}3}^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}(x)$ *satisfies*

$$\frac{\partial}{\partial x} E_{12\text{-}3}^{k(k-1)\dots 1,\ell(\ell-1)\dots 1}(x) =$$

$$E_{12\text{-}3}^{(k-1)\dots 1,\ell(\ell-1)\dots 1}(x) + \left(e^x - \sum_{i=0}^{k-1} \frac{x^i}{i!}\right)\left(E_{1-23}^{\ell(\ell-1)\dots 1}(x) + \frac{x^\ell}{(\ell-1)!}\right).$$

Proof. We apply the reverse and complement operations and then use the results of Proposition 18. For example, to avoid 1-23, begin with $12\dots k$ and end with $12\dots\ell$ is the same as to avoid 12-3, begin with $12\dots\ell$ and end with $12\dots k$. □

4.9 Further results

In this section, we propose two directions of generalization of the results from the previous sections. The first one is a consideration of avoiding more than one pattern, beginning with some pattern and ending with another pattern. For example, suppose that v = 12-3, w = 21-3, $p = 12\dots k$, $q = 12\dots\ell$, and $E_{v,w}^{p,q}(x)$ denotes the exponential generating function for the number of permutations that avoid the patterns v and w simultaneously, begin with the pattern p and end with the pattern q. It is easy to see that if $k \geq 3$ or $\ell \geq 3$ then $E_{12\text{-}3,21\text{-}3}^{12\dots k,12\dots\ell}(x) = 0$. For the other k and ℓ, one can prove the following theorem:

Theorem 1. *We have*

(i) $E_{12\text{-}3,21\text{-}3}^{1,1}(x) = e^{x+x^2/2} - 1$.

(ii) $E_{12\text{-}3,21\text{-}3}^{1,12}(x) = e^{x+x^2/2}\left(1 - \int_0^x e^{-t-t^2/2}dt\right) - 1$.

(iii) $E_{12\text{-}3,21\text{-}3}^{12,1}(x) = \int_0^x te^{t+t^2/2}dt$.

(iv) $E_{12\text{-}3,21\text{-}3}^{12,12}(x) = \frac{1}{2}x^2 + \int_0^x \left[e^{t+t^2/2}\left(1 - \int_0^t e^{-r-r^2/2}dr\right) - 1\right] dt$.

The second direction is a consideration of permutations in S_n containing a pattern v exactly r times, beginning with some pattern and ending with another pattern. For example, suppose that v = 12-3, $r = 1$, $p = 1 \dots k$, $q = 1$, and $N^{p;q}_{v;r}(n)$ denotes the number of n-permutations that contain the pattern v exactly r times, begin with the pattern p, and end with the pattern q. It is easy to see that the only interesting case is $1 \leq k \leq 3$, since otherwise $N^{12\dots k,1}_{12\text{-}3;1}(n) = 0$. Moreover, one can prove the following theorem:

Theorem 2. *Let F_n denote the number of n-permutations containing* 12-3 *exactly once. Then, for all $n \geq 3$,*

$$N^{1,1}_{12\text{-}3;1}(n) = F_n N^{12,1}_{12\text{-}3;1}(n) = (n-1)F_{n-1} + (n-2)B_{n-2},$$
$$N^{123,1}_{12\text{-}3;1}(n) = (n-2)B_{n-3},$$

where B_n is the nth Bell number, and F_n is given by [ClaesMans2, Corollarly 13].

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Paper V

Partially Ordered Generalized Patterns

Partially Ordered Generalized Patterns

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Abstract

We introduce partially ordered generalized patterns (POGPs), which further generalize the generalized permutation patterns (GPs) introduced by Babson and Steingrímsson [BabStein]. A POGP p is a GP some of whose letters are incomparable. Thus, in an occurrence of p in a permutation π, two letters that are incomparable in p pose no restrictions on the corresponding letters in π. We describe many relations between POGPs and GPs and give general theorems about the number of permutations avoiding certain classes of POGPs. These theorems have several known results as corollaries but also give many new results. We also give the generating function for the entire distribution of the maximum number of non-overlapping occurrences of a pattern p with no dashes, provided we know the e.g.f. for the number of permutations that avoid p.

5.1 Introduction and Background

All permutations in this paper are written as words $\pi = a_1 a_2 \cdots a_n$, where the a_i consist of all the integers $1, 2, \ldots, n$.

We will be concerned with *patterns* in permutations. A pattern is a word on some alphabet of letters, where some of the letters may be separated by dashes. In our notation, the classical permutation patterns, first studied systematically by Simion and Schmidt [SchSim], are of the form $p = 1-3-2$, the dashes indicating that the letters in a permutation corresponding to an occurrence of p don't have to be adjacent. In the classical case, an occurrence of a pattern p in a permutation π is a subsequence in π (of the same length as the length of p) whose letters are in the same relative order as those in p. For example, the permutation 41352 has only one occurrence of the pattern $1-2-3$, namely the subword 135.

Note that a classical pattern should, in our notation, have dashes at the beginning and end. Since all patterns considered in this paper satisfy this, we suppress these dashes from the notation. Thus, a pattern with no dashes corresponds to a contiguous subword anywhere in a permutation.

In [BabStein] Babson and Steingrímsson introduced *generalized permutation patterns (GPs)* where two adjacent letters in a pattern may be required to be adjacent in the permutation. Such an adjacency requirement is indicated by the absence of a dash between the corresponding letters in the pattern. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern

2-31, namely the subword 564, but the pattern 2-3-1 occurs also in the subwords 562 and 563. The motivation for introducing these patterns in [BabStein] was the study of Mahonian statistics.

A number of interesting results on GPs were obtained by Claesson in [Claes]. Relations to several well studied combinatorial structures, such as set partitions, Dyck paths, Motzkin paths and involutions, were shown there. In [Kit] the present author investigated simultaneous avoidance of two or more 3-letter GPs with no dashes. This work is of particular interest here since avoidance of the patterns considered in this paper has a close connection to simultaneous avoidance of two or more GPs with no dashes. Also important here is the work of Elizalde and Noy [ElizNoy] where they find the distribution of several patterns with no dashes.

In this paper we introduce a further generalization of GPs — namely *partially ordered generalized patterns (POGP)*. A POGP is a GP some of whose letters are incomparable. For instance, if we write $p = 1 - 1'2'$ then we mean that in an occurrence of p in a permutation π the letter corresponding to the 1 in p can be either larger or smaller than the letters corresponding to $1'2'$. Thus, the permutation 13425 has four occurrences of p, namely 134, 125, 325 and 425.

We consider two particular classes of POGPs — *shuffle patterns* and *multi-patterns*. A multi-pattern is of the form $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ and a shuffle pattern is of the form $p = \sigma_0 - a_1 - \sigma_1 - a_2 - \cdots - a_k - \sigma_k$, where for any i and j, the letter a_i is greater than any letter of σ_j and for any $i \neq j$ each letter of σ_i is incomparable with any letter of σ_j These patterns are investigated in Sections 5.4 and 5.5. A corollary to one of our theorems (Theorem 5) about the shuffle patterns is the result of Claesson [Claes, Proposition 2] that the number of n-permutations that avoid the pattern $12 - 3$ is the n-th Bell number.

Let $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ be an arbitrary multi-pattern and let $A_i(x)$ be the exponential generating function (e.g.f.) for the number of permutations that avoid σ_i for each i. In Theorem 11 we find the e.g.f., in terms of the $A_i(x)$, for the number of permutations that avoid p. In particular, this allows us to find the e.g.f. for the entire *distribution* of the maximum number of non-overlapping occurrences of a pattern p with no dashes, if we only know the e.g.f. for the number of permutations that *avoid p*. In many cases, this gives nice generating functions.

We also give alternative proofs, using inclusion-exclusion, of some of the results of Elizalde and Noy [ElizNoy]. Our proofs result in explicit formulas for the e.g.f. in terms of infinite series whereas Elizalde and Noy obtained differential equations for the same e.g.f..

5.2 Definitions and Preliminaries

A *partially ordered generalized pattern (POGP)* is a GP where some of the letters can be incomparable.

Example 1. *The simplest non-trivial example of a POGP that differs from the ordinary GPs is* $p = 1' - 2 - 1''$*, where the second letter is the greatest one and* *the first and the last letters are incomparable to each other. The permutation 3142 has two occurrences of p, namely, the subwords 342 and 142.*

It is easy to see that the number of permutations that avoid p in Example 1 is equal to 2^{n-1}. Indeed, if $\pi = a_1 \dots a_n$ and a_i is the leftmost letter in π that is smaller than its successor, then all letters to the right of a_i must be in increasing order. So any permutation π avoiding p can be written as $\pi_1 1 \pi_2$, where π_1 is decreasing and π_2 is increasing and there are 2^{n-1} ways to pick the permutation π_1, which determines π.

Definition 1. *If the number of permutations in S_n, for each n, that avoid a POGP p is equal to the number of permutations that avoid a POGP q, then p and q are said to be* equivalent *and we write $p \equiv q$ in this case.*

If A_n is the number of n-permutations that avoid a pattern p, then the *exponential generating function*, or *e.g.f.*, of the class of such permutations is

$$A(x) = \sum_{n \geq 0} A_n \frac{x^n}{n!}.$$

We will talk about *bivariate generating functions*, or *b.g.f.*, exclusively as generating functions of the form

$$A(u,x) = \sum_{\pi} u^{p(\pi)} \frac{x^{|\pi|}}{|\pi|!} = \sum_{n,k \geq 0} A_{n,k} u^k \frac{x^n}{n!},$$

were $A_{n,k}$ is the number of n-permutations with k occurrences of the pattern p.

The *reverse* $R(\pi)$ of a permutation $\pi = a_1 a_2 \dots a_n$ is the permutation $a_n a_{n-1} \dots a_1$. The *complement* $C(\pi)$ is the permutation $b_1 b_2 \dots b_n$ where $b_i = n+1-a_i$. Also, $R \circ C$ is the composition of R and C. For example, $R(13254) = 45231$, $C(13254) = 53412$ and $R \circ C(13254) = 21435$. We call these bijections of S_n to itself *trivial*, and it is easy to see that any pattern p is equivalent to the patterns $R(p)$, $C(p)$ and $R \circ C(p)$. For example, the number of permutations that avoid the pattern 132 is the same as the number of permutations that avoid the patterns 231, 312 and 213, respectively.

It is convenient to introduce the following definition.

Definition 2. *Let p be a GP without internal dashes. A permutation π* quasi-avoids *p if π has exactly one occurrence of p and this occurrence consists of the $|p|$ rightmost letters of π.*

For example, the permutation 51342 quasi-avoids the pattern $p = 231$, whereas the permutations 54312 and 45231 do not. Indeed, 54312 ends with 312, which is not an occurrence of the pattern p, and 45231 has an occurrence of p, namely 452, in a forbidden place.

Proposition 1. *Let p be a non-empty GP with no dashes. Let $A(x)$ (resp. $A^\star(x)$) be the e.g.f. for the number of permutations that avoid (resp. quasi-avoid) p. Then*

$$A^\star(x) = (x-1)A(x) + 1.$$

Proof. We first show that

$$A_n^\star = nA_{n-1} - A_n. \tag{5.1}$$

If we consider all $(n-1)$-permutations that avoid p and all possible extending of these permutations to the n-permutations by writing one more letter to the right, then the number of obtained permutations will be nA_{n-1}. Obviously, the set of these permutations is a disjoint union of the set of all n-permutations that avoid p and the set of all n-permutations that quasi-avoid p. Thus we get (5.1). Multiplying both sides of (5.1) with $x^n/n!$ and summing over all natural numbers n, observing that $A_0^\star = 0$, we get the desired result. □

Definition 3. *Suppose* $\{\sigma_0, \sigma_1, \dots, \sigma_k\}$ *is a set of GPs with no dashes and* $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ *where each letter of* σ_i *is incomparable with any letter of* σ_j *whenever* $i \neq j$*. We call such POGPs* multi-patterns.

Definition 4. *Suppose* $\{\sigma_0, \sigma_1, \dots, \sigma_k\}$ *is a set of GPs with no dashes and* $a_1 a_2 \dots a_k$ *is a permutation of* k *letters. We define a* shuffle *pattern to be a pattern of the form*

$$\sigma_0 - a_1 - \sigma_1 - a_2 - \cdots - \sigma_{k-1} - a_k - \sigma_k,$$

where for any i *and* j*, the letter* a_i *is greater than any letter of* σ_j *and for any* $i \neq j$ *each letter of* σ_i *is incomparable with any letter of* σ_j*. We also allow* σ_0 *and* σ_k*, but not the other* σ_i*, to be empty patterns.*

The pattern from Example 1 is an example of a shuffle pattern. It follows from the definitions that we can get a multi-pattern from a shuffle pattern by removing all the a_i.

Let $\mathcal{S}_\infty$ denote the disjoint union of the $\mathcal{S}_n$ for all $n \in \mathbb{N}$. The POGPs (which include the GPs, as well as the classical patterns), can be considered as functions from $\mathcal{S}_\infty$ to $\mathbb{N}$ that count the number of occurrences of the pattern in a permutation in $\mathcal{S}_\infty$. This allows us to write a POGP (as a function) as a linear combination of GPs. For example,

$$1' - 2 - 1'' \;=\; (1 - 3 - 2) \;+\; (2 - 3 - 1),$$

from which, in particular, we see that to avoid $1' - 2 - 1''$ is the same as to avoid simultaneously the patterns $1 - 3 - 2$ and $2 - 3 - 1$. A straightforward argument leads to the following proposition.

Proposition 2. *For any POGP* p *there exists a set* S *of GPs such that a permutation* π *avoids* p *if and only if* π *avoids all the patterns in* S*.*

The following theorem can be easily proved by induction on k:

Theorem 1. *Let* $p_1 = \sigma_0 - a_1 - \sigma_1 - a_2 - \cdots - \sigma_{k-1} - a_k - \sigma_k$ *(resp.* $p_2 = \sigma_0 - \sigma_1 - \cdots - \sigma_k$*) be an arbitrary shuffle pattern (resp. multi-pattern) with*

$|\sigma_i| = \ell_i$ *for all* $i = 0, \ldots, k$*. Then to avoid the pattern* p_1 *(resp.* p_2*) is the same as to avoid*

$$\prod_{i=1}^{k} \binom{\ell_0 + \ell_1 + \cdots + \ell_i}{\ell_i} = \binom{\ell_0 + \ell_1}{\ell_1} \binom{\ell_0 + \ell_1 + \ell_2}{\ell_2} \cdots \binom{\ell_0 + \ell_1 + \cdots + \ell_k}{\ell_k}$$

ordinary GPs.

Example 2. *Let* $p = 1'2' - 3 - 1''$*. That is* $\sigma = 12$ *and* $\tau = 1$*. By Theorem 1, to avoid p is the same as to avoid* $\binom{3}{2} = 3$ *GPs simultaneously, namely* $12 - 4 - 3$, $13 - 4 - 2$ *and* $23 - 4 - 1$.

There is a number of results on the distribution of several classes of patterns with no dashes. These results can be used as building blocks for some of the results in the present paper. The most important of these is the following result by Elizalde and Noy [ElizNoy]:

Theorem 2. *[ElizNoy, Theorem 3.4] Let m and a be positive integers with* $a \le m$*, let* $\sigma = 12 \ldots a\tau(a+1) \in \mathcal{S}_{m+2}$*, where* τ *is any permutation of* $\{a+2, a+3, \ldots, m+2\}$*, and let* $P(u,z)$ *be the b.g.f. for permutations where u marks the number of occurrences of* σ*. Then* $P(u,z) = 1/w(u,z)$*, where w is the solution of*

$$w^{a+1} + (1-u)\frac{z^{m-a+1}}{(m-a+1)!}w' = 0$$

with $w(0) = 1$*,* $w'(0) = -1$ *and* $w^{(k)}(0) = 0$ *for* $2 \le k \le a$*. In particular, the distribution does not depend on* τ*.*

5.3 GPs with no dashes

In order to apply our results in what follows we need to know how many patterns avoid a given ordinary GP with no dashes. We are also interested in different approaches to studying these patterns. The theorems in this section can be proved using an inclusion-exclusion argument similar to the one given in the proof of Theorem 12 and we omit these proofs. This allows us to get explicit formulas for the e.g.f. in terms of infinite series instead of having to solve differential equations as done by Elizalde and Noy [ElizNoy] for the same e.g.f.. However, in particular cases, we use certain differential equations to simplify our series.

Theorem 3. *[GoulJack] Let* $A_k(x)$ *be the e.g.f. for the number of permutations avoiding the pattern* $p = 123 \ldots k$*. Then*

$$A_k(x) = 1/F_k(x),$$

where $F_k(x) = \sum_{i \ge 0} \frac{x^{ki}}{(ki)!} - \sum_{i \ge 0} \frac{x^{ki+1}}{(ki+1)!}$.

For some k it is possible to simplify the function $F_k(x)$ in the theorems above. Indeed, $F_k(x)$ satisfies the differential equation $F_k^{(k)}(x) = F_k(x)$ with the k initial conditions $F_k(0) = 1$, $F'_k(0) = -1$ and $F_k^{(i)}(0) = 0$ for all $i = 2, 3, \ldots, k-1$. For instance, if $k = 4$ then

$$F_4(x) = \frac{1}{2}(\cos x - \sin x + e^{-x}).$$

Theorem 4. *Let k and a be positive integers with $a < k$, let $p = 12\ldots a\tau(a+1) \in \mathcal{S}_{k+1}$, where τ is any permutation of the elements $\{a+2, a+3, \ldots, k+1\}$, and let $A_{k,a}(x)$ be the e.g.f. for the number of permutations that avoid p. Let*

$$F_{k,a}(x) = \sum_{i\geq 1} \frac{(-1)^{i+1} x^{ki+1}}{(ki+1)!} \prod_{j=2}^{i} \binom{jk-a}{k-a}.$$

Then

$$A_{k,a}(x) = 1/(1 - x + F_{k,a}(x)).$$

If $k = 2$ and $a = 1$ in the previous theorem, corresponding to the pattern $p = 132$, then from Theorem 4 the function $F_{2,1}(x)$, which is the same for the patterns p, 231, 312 and 213 because of the trivial bijections, can be written as:

$$F_{2,1}(x) = \sum_{i\geq 1} \frac{(-1)^{i+1} x^{ki+1}}{i!(k!)^i(ki+1)} = x - \int_0^x e^{-t^2/2}\, dt.$$

That is

$$A_{2,1} = \frac{1}{1 - \displaystyle\int_0^x e^{-t^2/2}\, dt},$$

which is a special case of Theorem 4.1 in [ElizNoy].

5.4 The Shuffle Patterns

We recall that according to Definition 4, a shuffle pattern is a pattern of the form $\sigma_0 - a_1 - \sigma_1 - a_2 - \cdots - \sigma_{k-1} - a_k - \sigma_k$, where $\{\sigma_0, \sigma_1, \ldots, \sigma_k\}$ is a set of GPs with no dashes, $a_1 a_2 \ldots a_k$ is a permutation of k letters, for any i and j the letter a_i is greater than any letter of σ_j and for any $i \neq j$ each letter of σ_i is incomparable with any letter of σ_j.

Let us consider a shuffle pattern that in fact is an ordinary generalized pattern. This pattern is $p = \sigma - k$, where σ is an arbitrary pattern with no dashes that is built on elements $1, 2, \ldots, k-1$. So the last element of p is greater than any other element.

Theorem 5. *Let $p = \sigma - k$ and let $A(x)$ (resp. $B(x)$) be the e.g.f. for the number of permutations that avoid σ (resp. p). Then $B(x) = e^{F(x, A(y))}$, where*

$$F(x, A(y)) = \int_0^x A(y)\, dy.$$

Proof. Suppose that $\pi \in \mathcal{S}_{n+1}$ and that π avoids p. Suppose the letter $(n+1)$ is in the i-th position and $\pi = \pi_1(n+1)\pi_2$, where π_1 and π_2 might be empty.

Since π is p-avoiding, π_1 must be σ-avoiding, because otherwise an occurrence of σ in π_1 together with the letter $(n+1)$ gives an occurrence of p in π. But if π_1 is σ-avoiding then there is no interaction between π_1 and π_2, that is, if π_2 is p-avoiding and π_1 is σ-avoiding then π is p-avoiding. To see this it is enough to see that if an occurrence of σ in π contains the letter $(n+1)$, then this occurrence of σ can not lead to an occurrence of $p = \sigma - k$ containing the letter $(n+1)$.

From the above, considering all possible positions of $(n+1)$, we get the recurrence relation

$$B_{n+1} = \sum_i \binom{n}{i} A_i B_{n-i},$$

where B_j (resp. A_j) is the number of j-permutations that avoid p (resp. σ), because we can choose the elements of π_1 in $\binom{n}{i}$ ways.

Multiplying both sides of the equality by $x^n/n!$ we get

$$\frac{B_{n+1}}{n!}x^n = \sum_i \frac{A_i}{i!}x^i \frac{B_{n-i}}{(n-i)!}x^{n-i}.$$

Taking the sum over all natural numbers n leads us to

$$B'(x) = A(x)B(x)$$

where the derivative of B is with respect to x. Since $B(0) = 1$, the solution of the differential equation is $B(x) = e^{F(x,A(y))}$. □

Example 3. *Let $p = 1-2$. Here $\sigma = 1$, so $A(x) = 1$ since $A_n = 0$ for all $n \geq 1$ and $A_0 = 1$. So*

$$B(x) = e^{F(x,1)} = e^x.$$

This corresponds to the fact that for each $n \geq 1$ there is exactly one permutation that avoids the pattern p, namely $\pi = n(n-1)\ldots 1$.

Example 4. *Suppose $p = 12-3$. Here $\sigma = 12$, so $A(x) = e^x$, since there is exactly one permutation that avoids the pattern σ. So*

$$B(x) = \sum_{n\geq 0} \frac{B_n}{n!}x^n = e^{F(x,e^y)} = e^{e^x-1}.$$

According to [Claes, Proposition 2], for all $n \geq 1$, B_n is the n-th Bell number and the e.g.f. for the Bell numbers is e^{e^x-1}.

The table below gives the initial values of B_n for several patterns $p = \sigma - k$. These numbers were obtained by expanding the corresponding $B(x)$. The functions $A(x)$ are taken from the previous section.

pattern	initial values for B_n
132-4	1, 2, 6, 23, 107, 585, 3671, 25986, 204738
123-4	1, 2, 6, 23, 108, 598, 3815, 27532, 221708
1234-5	1, 2, 6, 24, 119, 705, 4853, 38142, 336291
12345-6	1, 2, 6, 24, 120, 719, 5022, 40064, 359400

Theorem 6. *Let p be the shuffle pattern $\sigma - k - \tau$. So k is the greatest letter of the pattern, and each letter of σ is incomparable with any letter of τ. Let $A(x)$, $B(x)$ and $C(x)$ be the e.g.f. for the number of permutations that avoid σ, τ and p respectively. Then $C(x)$ is the solution of the differential equation*

$$C'(x) = (A(x) + B(x))C(x) - A(x)B(x),$$

with $C(0) = 1$.

Proof. As before, we consider the symmetric group $\mathcal{S}_{n+1}$ and a permutation $\pi \in \mathcal{S}_{n+1}$ that avoids p. Suppose the letter $(n+1)$ is in the i-th position and $\pi = \pi_1(n+1)\pi_2$, where π_1 and π_2 might be empty.

There are exactly four mutually exclusive possibilities:

1) π_1 does not avoid σ, π_2 does not avoid τ.

2) π_1 avoids σ, π_2 does not avoid τ;

3) π_1 does not avoid σ, π_2 avoids τ;

4) π_1 avoids σ, π_2 avoids τ;

Obviously, the situation 1) is impossible, since an occurrence of σ in π_1 with $(n+1)$ and with an occurrence of τ in π_2 gives us an occurrence of p in π. On the other hand, if p occurs in π then it is easy to see that the letter $(n+1)$ cannot be one of the letters in the occurrences of σ or τ, so all p-avoiding permutations are described by the possibilities 2)–4). We count these permutations in the following way.

In $\binom{n}{i}$ ways we choose first i elements from the letters $1, 2 \dots n$, that is, the elements of π_1. Let A_i, B_i and C_i be the number of i-permutations that avoid σ, τ and p respectively.

If π_1 is σ-avoiding, we let π_2 be any p-avoiding permutation of the remaining $(n - i + 1)$ letters. This accounts for all "good" permutations from the possibilities 2) and 4). There are $\binom{n}{i}A_iC_{n-i}$ such permutations.

If π_2 is τ-avoiding, we let π_1 be any p-avoiding permutation of chosen i letters. This covers all "good" permutations from 3) and 4). There are $\binom{n}{i}B_iC_{n-i}$ such permutations.

But we have counted p-avoiding permutations that correspond to 4) twice, so we must subtract $\binom{n}{i}A_iB_{n-i}$, which is the number of such permutations.

So we have

$$C_{n+1} = \sum_i \binom{n}{i}(A_iC_{n-i} + B_iC_{n-i} - A_iB_{n-i}).$$

Multiplying both sides of the equality by $x^n/n!$ we get

$$\frac{C_{n+1}}{n!}x^n = \sum_i \left(\frac{A_i + B_i}{i!}x^i \frac{C_{n-i}}{(n-i)!}x^{n-i} - \frac{A_i}{i!}x^i \frac{B_{n-i}}{(n-i)!}x^{n-i} \right),$$

so

$$C'(x) = (A(x) + B(x))C(x) - A(x)B(x).$$

□

Example 5. *Let $p = 1'{-}2{-}1''$. That is, $\sigma = 1$ and $\tau = 1$. So $A(x) = B(x) = 1$ and we need to solve the equation*

$$C'(x) = 2C(x) - 1$$

with $C(0) = 1$. The solution of this equation is $C(x) = \frac{1}{2}(e^{2x} + 1)$, so for all $n \geq 1$ we have $C_n = 2^{n-1}$, as in Example 1.

In the table below we record the initial values of C_n for several patterns $p = \sigma - k - \tau$.

σ	τ	**initial values for C_n**
1	12	1, 2, 6, 21, 82, 354, 1671, 8536, 46814
1	132	1, 2, 6, 24, 116, 652, 4178, 30070, 240164
1	123	1, 2, 6, 24, 116, 657, 4260, 31144, 253400
1	1234	1, 2, 6, 24, 120, 715, 4946, 38963, 344350
12	12	1, 2, 6, 24, 114, 608, 3554, 22480, 152546
12	132	1, 2, 6, 24, 120, 710, 4800, 36298, 302780
12	123	1, 2, 6, 24, 120, 710, 4815, 36650, 308778
12	1234	1, 2, 6, 24, 120, 720, 5025, 39926, 355538
123	123	1, 2, 6, 24, 120, 720, 5020, 39790, 352470
123	132	1, 2, 6, 24, 120, 720, 5020, 39755, 351518
132	132	1, 2, 6, 24, 120, 720, 5020, 39720, 350496

Remark 2. *The pattern $p = \sigma - k$ from Theorem 5 is a particular case of the pattern $p = \sigma - k - \tau$ from Theorem 6 when τ is the empty word. The e.g.f. for the number of permutations that avoid the empty word is zero, because no permutation avoids the empty word. So if τ is empty, we can use Theorem 6 to get Theorem 5. Indeed, $B(x) = 0$, and after renaming C with B we get in Theorem 6 exactly the same differential equation as we have in Theorem 5.*

We now give two corollaries to Theorem 6.

Corollary 1. *Suppose we have the shuffle pattern $p = \sigma - k - \tau$. We consider the pattern $\varphi(p) = \varphi_1(\sigma) - k - \varphi_2(\tau)$, where φ_1 and φ_2 are any trivial bijections. Then $p \equiv \varphi(p)$.*

Proof. We just observe that if $A(x)$ (resp. $B(x)$) is the e.g.f. for the number of permutations that avoid σ (resp. τ) then $A(x)$ (resp. $B(x)$) is the e.g.f. for the number of permutations that avoid $\varphi_1(\sigma)$ (resp. $\varphi_2(\tau)$). □

Corollary 2. *We have* $\sigma - k - \tau \equiv \tau - k - \sigma$.

Proof. This follows directly from the differential equation of Theorem 6 ($A(x)$ and $B(x)$ are symmetric in that equation), but we can also obtain this as a corollary to Corollary 1. By Corollary 1, the pattern $\sigma - k - \tau$ is equivalent to the pattern $\sigma - k - R(\tau)$. Reversing the pattern $\sigma - k - R(\tau)$, we obtain the pattern

$$R(\sigma - k - R(\tau)) = R(R(\tau)) - k - R(\sigma) = \tau - k - R(\sigma),$$

which thus is equivalent to $\sigma - k - \tau$. Finally, we use Corollary 1 one more time to get

$$\tau - k - R(\sigma) \equiv \tau - k - R(R(\sigma)) = \tau - k - \sigma.$$

□

5.5 The Multi-Patterns

We recall that according to Definition 3, a multi-pattern is a pattern $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$, where $\{\sigma_0, \sigma_1, \ldots, \sigma_k\}$ is a set of GPs with no dashes and each letter of σ_i is incomparable with any letter of σ_j whenever $i \neq j$.

We first discuss patterns of the type $p = \sigma - \tau$ which are a particular case of the multi-patterns to be treated in this section.

If σ or τ is the empty word then we are dealing with ordinary GPs with no dashes, some of which were investigated in [ElizNoy] and Section 3. The analysis of the case when σ or τ is equal to 1 can also be reduced to the analysis of ordinary GPs. For example, suppose that $\sigma = 1$, that is, $p = 1 - \tau$, and we want to count the number of permutations in $\mathcal{S}_n$ that avoid p. We can choose the leftmost letter of a permutation avoiding p in n ways, then the remainder of the permutation must avoid τ, so we multiply n by the number of permutations in $\mathcal{S}_{n-1}$ that avoid τ. For instance, if $p = 1 - 1'2'$ then the number of permutations in $\mathcal{S}_n$ avoiding p is exactly n.

Theorem 7. *Let* $p = \sigma - \tau$ *and* $q = \varphi_1(\sigma) - \varphi_2(\tau)$, *where* φ_1 *and* φ_2 *are any of the trivial bijections. Then* p *and* q *are equivalent.*

Proof. The theorem is equivalent to the following statement:

Let $p = \sigma - \tau$ and $q = \sigma - \varphi(\tau)$, where φ is a trivial bijection. Then p and q are equivalent.

It is obvious that the statement follows from Theorem 7. Conversely, suppose we have $p = \sigma - \tau$. We observe that any two trivial bijections commute, that is for any trivial bijection ψ, we have $\psi(R(x)) = R(\psi(x))$. This observation, the statement and the fact that $x \equiv R(x)$ give

$$p = \sigma - \tau \equiv \sigma - \varphi_2(\tau) \equiv R(\varphi_2(\tau)) - R(\sigma) \equiv R(\varphi_2(\tau)) - \varphi_1(R(\sigma)) \equiv$$

$$R(\varphi_2(\tau)) - R(\varphi_1(\sigma)) \equiv \varphi_1(\sigma) - \varphi_2(\tau).$$

So to prove the theorem we now prove the statement.

Let $p = \sigma - \tau$ and $q = \sigma - \varphi(\tau)$, where φ is a trivial bijection. Let A_n (resp. B_n) be the number of n-permutations that avoid p (resp. q). We are going to prove that $A_n = B_n$.

Suppose π avoids p and $\pi = \pi_1\sigma'\pi_2$, where $\pi_1\sigma'$ has exactly one occurrence of the pattern σ, namely σ'. Then π_2 must avoid τ, $\varphi(\pi_2)$ must avoid $\varphi(\tau)$ and $\pi_\varphi = \pi_1\sigma'\varphi(\pi_2)$ avoids q. The converse is also true, that is, if π_φ has no occurrences of q then π has no occurrences of p. If π has no occurrences of σ then π has no occurrences of p as well as no occurrences of q. Since any permutation either avoids σ or can be factored as above, we have a bijection between the class of permutations that avoid p and the class of permutations that avoid q. Thus $A_n = B_n$. □

We get the following corollary to Theorem 7:

Corollary 3. *The pattern $\sigma - \tau$ is equivalent to the pattern $\tau - \sigma$.*

Proof. We proceed as in the proof of Corollary 2. From Theorem 7 we have:

$$\sigma - \tau \equiv \sigma - R(\tau) \equiv R(R(\tau)) - R(\sigma) \equiv \tau - R(R(\sigma)) \equiv \tau - \sigma.$$

□

We observe that the presence of the dash in the patterns in Theorem 7 is essential. That is, generally speaking, the pattern $\sigma\tau$ is not equivalent to the pattern $\varphi_1(\sigma)\varphi_2(\tau)$ for any trivial bijections φ_1 and φ_2. For example, there are 66 permutations in $\mathcal{S}_5$ that avoid the pattern $122'1'$ but only 61 that avoid $121'2'$. In Section 6 we investigate the pattern $122'1'$.

Theorem 8 and Corollary 4 generalise Theorem 7 and Corollary 3:

Theorem 8. *Suppose we have multi-patterns $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ and $q = \tau_1 - \tau_2 - \cdots - \tau_k$, where $\tau_1\tau_2 \ldots \tau_k$ is a permutation of $\sigma_1\sigma_2 \ldots \sigma_k$. Then p and q are equivalent.*

Proof. We proceed by induction on k. If $k = 2$ then the statement is true by Corollary 3. Suppose the statement is true for all $k' < k$. Suppose p has exactly k blocks. If a permutation π avoiding p has no occurrences of σ_1 then it obviously avoids both p and q. Otherwise we factor π as $\pi = \pi_1\sigma_1'\pi_2$ where $\pi_1{\sigma_1}'$ has exactly one occurrence of the pattern σ_1, namely ${\sigma_1}'$. Then π_2 must avoid $\sigma_2 - \cdots - \sigma_k$. Moreover it is irrelevant from which letters $\pi_1{\sigma_1}'$ is built and therefore we can apply the inductive hypothesis. We can rearrange $\sigma_2' \ldots \sigma_k'$ of $\sigma_2 \ldots \sigma_k$ in such a way that the blocks in $\tau_1\tau_2 \ldots \tau_k$ corresponding to $\sigma_2, \ldots, \sigma_k$ are arranged in the same order as the τ's. Now we consider separately two cases: $\tau_k \neq \sigma_1$ and $\tau_k = \sigma_1$. In the first case we use the following equivalences:

$$p = \sigma_1 - \sigma_2 - \cdots - \sigma_k \equiv \sigma_1 - {\sigma_2}' - \cdots - \sigma_k' \equiv R(\sigma_k') - \cdots - R(\sigma_2') - R(\sigma_1).$$

For the pattern $R(\sigma'_k) - \cdots - R(\sigma'_2) - R(\sigma_1)$ we use the factorisation of a permutation π avoiding this pattern, where the role of σ_1 is played by $R(\sigma'_k)$. So by the inductive hypothesis we put the pattern $R(\sigma_1)$ in the right place somewhere to the left of $R(\sigma'_2)$ and apply R to get that $p \equiv q$.

In the second case we have:

$$p \equiv R(\sigma'_k) - \cdots - R(\sigma'_2) - R(\sigma_1) \equiv R(\sigma'_k) - \cdots - R(\sigma_1) - R(\sigma'_2) \equiv$$
$$\sigma'_2 - \sigma_1 - \cdots - \sigma'_k \equiv \sigma'_2 - \cdots - \sigma'_k - \sigma_1 = q$$

The first equivalence here is taken from the considerations above; the second one uses the inductive hypothesis; then we use the fact that $R(R(x)) = x$ and apply the inductive hypothesis again. □

Corollary 4. *Suppose we have multi-patterns $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ and $q = \varphi_1(\sigma_1) - \varphi_2(\sigma_2) - \cdots - \varphi_k(\sigma_k)$, where each φ_i is an arbitrary trivial bijection. Then p and q are equivalent.*

Proof. We use induction on k, Theorem 8 and the factorisation of permutations, which is discussed in the proof of Theorem 8. If $k = 2$ then the statement is true by Theorem 7. Suppose the statement is true for all $k' < k$. Then

$$p = \sigma_1 - \sigma_2 - \cdots - \sigma_k \equiv \sigma_1 - \varphi_2(\sigma_2) - \cdots - \varphi_k(\sigma_k) \equiv$$
$$\varphi_2(\sigma_2) - \sigma_1 - \cdots - \varphi_k(\sigma_k) \equiv \varphi_2(\sigma_2) - \varphi_1(\sigma_1) - \cdots - \varphi_k(\sigma_k) \equiv$$
$$\varphi_1(\sigma_1) - \varphi_2(\sigma_2) - \cdots - \varphi_k(\sigma_k) = q,$$

where first we apply the inductive hypothesis then Theorem 8 then the inductive hypothesis and finally Theorem 8 again. □

Theorem 9. *Suppose $p = \sigma - p'$, where p' is an arbitrary POGP, and the letters of σ are incomparable to the letters of p'. Let $C(x)$ (resp. $A(x)$, $B(x)$) be the e.g.f. for the number of permutations that avoid p (resp. σ, p'). Moreover let $A^\star(x)$ be the e.g.f. for the number of permutations that quasi-avoid σ. Then*

$$C(x) = A(x) + B(x)A^\star(x).$$

Proof. Let A_n, B_n, C_n be the number of n-permutations that avoid the patterns σ, p' and p respectively. Also $A^\star_n$ is the number of n-permutations that quasi-avoid σ. If a permutation π avoids σ then it avoids p. Otherwise we find the leftmost occurrence of σ in π. We assume that this occurrence consists of the $|\sigma|$ rightmost letters among the i leftmost letters of π. So the subword of π beginning at the $(i+1)$st letter must avoid p'. From this we conclude

$$C_n = A_n + \sum_{i=|\sigma|}^{n} \binom{n}{i} A^\star_i B_{n-i}.$$

We observe that we can change the lower bound in the sum above to 0, because $A^\star_i = 0$ for $i = 0, 1, \ldots, |\sigma| - 1$. Multiplying both sides by $x^n/n!$ and taking the sum over all n we get the desired result. □

Corollary 5. *Suppose $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ is a multi-pattern where $|\sigma_i| = 2$ for all i, so each σ_i is equal to either 12 or 21. If $B(x)$ is the e.g.f. for the number of permutations that avoid p then*

$$B(x) = \frac{1 - (1 + (x-1)e^x)^k}{1 - x}.$$

Proof. We use Theorem 9, induction on k and the fact that $A(x) = e^x$ and $A^\star(x) = 1 + (x-1)e^x$. □

The following corollary to Corollary 5 can be proved combinatorially.

Theorem 10. *There are $(n-2)2^{n-1} + 2$ permutations in $\mathcal{S}_n$ that avoid the pattern $p = 12 - 1'2'$ or, according to Theorem 7, the pattern $p = 12 - 2'1'$.*

One more corollary to Theorem 9 is the following theorem that is the basis for calculating the number of permutations that avoid a multi-pattern, and therefore is the main result for multi-patterns in this paper.

Theorem 11. *Let $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ be a multi-pattern and let $A_i(x)$ be the e.g.f. for the number of permutations that avoid σ_i. Then the e.g.f. $B(x)$ for the number of permutations that avoid p is*

$$B(x) = \sum_{i=1}^{k} A_i(x) \prod_{j=1}^{i-1} ((x-1)A_j(x) + 1).$$

Proof. We use Theorem 9 and prove by induction on k that

$$B(x) = \sum_{i=1}^{k} A_i(x) \prod_{j=1}^{i-1} A_j^\star(x).$$

Then we use Proposition 1 to get the desired result. □

Remark 3. *One can consider the function $B(x)$ from Theorem 11 as a function in k variables $B(x) = B(A_1(x), A_2(x), \ldots, A_k(x))$. Then, by Theorem 8, this function is symmetric in the variables $A_1(x), A_2(x), \ldots, A_k(x)$. That means that we can rename the variables, which may simplify the calculation of $B(x)$.*

5.6 Patterns of the Form $\sigma\tau$

Theorem 12. *Let $B(x)$ be the e.g.f. for the number of permutations that avoid the pattern $p = 122'1'$. Then*

$$B(x) = \frac{1}{2} + \frac{1}{4}\tan x(1 + e^{2x} + 2e^x \sin x) + \frac{1}{2}e^x \cos x.$$

Proof. Let B_n be the number of n-permutations that avoid p and A_n be the number of n-permutations that avoid p and begin with the pattern 12. Let also $A(x)$ be the e.g.f. for the numbers A_n. We set $B_0 = A_0 = A_1 = 1$. Suppose π is a $(n+1)$-permutation that avoids p. There are three mutually exclusive possibilities:

1) $\pi = (n+1)\pi_2$;

2) $\pi = \pi_1(n+1)$;

3) $\pi = \pi_1(n+1)\pi_2$ and $\pi_1, \pi_2 \neq \varepsilon$.

Obviously, in 1) and 2) the letter $(n+1)$ does not affect the rest of the permutation π, and therefore in each of these cases we have B_n permutations that avoid p. In 3), it is easy to see that if π_1 has more than one letter then π_1 must end with a 21 pattern whereas if π_2 has more than one letter then π_2 must begin with a 12 pattern. The key observation is that the number of n-permutations that avoid p and end with a 21 pattern is the same as the number of n-permutations that avoid p and begin with a 12 pattern. To see this it is enough to apply the reverse function to any n-permutation π that begins with 12-pattern and avoids p and observe that $R(p) = p$, that is, $R(\pi)$ avoids p and ends with a 21 pattern. Obviously this is a bijection. So if $|\pi_1| = i$ then we can choose the letters of π_1 in $\binom{n}{i}$ ways and then choose a permutation π_1 in A_i ways and a permutation π_2 in A_{n-i} ways, since the letters of π_1 and π_2 do not affect each other. From all this we get

$$B_{n+1} = 2B_n + \sum_{i=1}^{n-1} \binom{n}{i} A_i A_{n-i} = 2B_n + \sum_{i=0}^{n} \binom{n}{i} A_i A_{n-i} - 2A_n.$$

We multiply both sides of the last equality by $x^n/n!$ to get

$$B_{n+1}\frac{x^n}{n!} = 2B_n\frac{x^n}{n!} + \sum_{i=0}^{n} \frac{A_i}{i!}x^i \frac{A_{n-i}}{(n-i)!}x^{n-i} - 2A_n\frac{x^n}{n!}.$$

Summing both sides over all natural numbers n we get:

$$B'(x) = 2B(x) + A^2(x) - 2A(x). \tag{5.2}$$

To solve this differential equation with the initial condition $B(0) = 1$, we need to determine $A(x)$. One can observe that if a permutation π avoids p and begins with the pattern 12 then π has the structure $\pi = a_1b_1a_2b_2a_3b_3\cdots$, where $a_i < b_i$ for all i. Moreover, if $b_1 < a_2$ then we must have $a_1 < b_1 < a_2 < b_2 < a_3 < \cdots$ since otherwise we obviously have an occurrence of the pattern p. A first approximation is that $A_n = \binom{n}{2}A_{n-2}$, because we can choose a_1b_1 in π in $\binom{n}{2}$ ways and then pick an arbitrary $(n-2)$-permutation that avoids p and begins with the pattern 12, to be $a_2b_2a_3b_3\ldots$, in A_{n-2} ways. But it is possible that $b_1 < a_2$ in which case $b_1a_2b_2a_3$ can be an occurrence of p in π, and it is an occurrence of p unless $a_2 < b_2 < a_3 < \cdots$. So in order to avoid this we must subtract the number of permutations of the form $abcd\pi'$, where $a < b < c < d$ and π' is any $(n-4)$-permutation that avoids p, from the first approximation of A_n. Thus the second approximation is that $A_n = \binom{n}{2}A_{n-2} - \binom{n}{4}A_{n-4}$. We observe that in the second approximation we do not count the increasing permutation $123\dots n$. Moreover, among the permutations counted by $\binom{n}{4}A_{n-4}$, there are the permutations that begin with 6 increasing letters. Except for the increasing permutation, such permutations are not counted by $\binom{n}{2}A_{n-2}$. We must therefore add the number of such permutations. So the third approximation is that $A_n = \binom{n}{2}A_{n-2} - \binom{n}{4}A_{n-4} + \binom{n}{6}A_{n-6}$ and so on. That is,

$$A_n = \binom{n}{2}A_{n-2} - \binom{n}{4}A_{n-4} + \binom{n}{6}A_{n-6} - \binom{n}{8}A_{n-8} + \cdots = \sum_{i\geq 1}(-1)^{i+1}\binom{n}{2i}A_{n-2i}. \tag{5.3}$$

We observe that if $n = 4k$ or $n = 4k+1$ then we do not count the increasing permutation in our sum. This, together with Equation 5.3, gives us

$$\sum_{i\geq 0}(-1)^i\binom{n}{2i}A_{n-2i} = \begin{cases} 1, & \text{if } n = 4k \text{ or } n = 4k+1, \\ 0, & \text{if } n = 4k+2 \text{ or } n = 4k+3. \end{cases}$$

Multiplying both sides of the equality with $x^n/n!$ and summing over all natural numbers n we get

$$(A_0 + A_1 x + \frac{A_2}{2!}x^2 + \cdots)(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) = \sum_{k=0}^{\infty}\left(\frac{x^{4k}}{(4k)!} + \frac{x^{4k+1}}{(4k+1)!}\right).$$

The left hand side of this equality is equal to $A(x)\cos x$. Let $F(x)$ be the function in the right hand side of the equality. Then it is easy to see that $F(x)$ is the solution to the differential equation $F^{(4)}(x) = F(x)$ with the initial conditions $F(0) = F'(0) = 1$, $F^{(2)}(0) = F^{(3)}(0) = 0$. So $F(x) = \frac{1}{2}(\cos x + \sin x + e^x)$ and

$$A(x) = \frac{1}{2}\left(1 + \tan x + \frac{e^x}{\cos x}\right).$$

Now we solve the differential equation (5.2) and get

$$B(x) = \frac{1}{2} + \frac{1}{4}\tan x(1 + e^{2x} + 2e^x \sin x) + \frac{1}{2}e^x\cos x.$$

□

Remark 4. *The series expansion of $B(x)$ in Theorem 12 begins with*

$$B(x) = 1 + x + x^2 + x^3 + \frac{3}{4}x^4 + \frac{11}{20}x^5 + \frac{7}{20}x^6 + \frac{7}{30}x^7 + \frac{103}{720}x^8 + \cdots.$$

That is, the initial values for B_n are 1, 2, 6, 18, 66, 252, 1176, 5768.

5.7 The Distribution of Non-Overlapping GPs

A descent in a permutation $\pi = a_1a_2\dots a_n$ is an i such that $a_i > a_{i+1}$. The number of descents in a permutation π is denoted des π (and is equivalent to the generalized pattern 21). Any statistic with the same distribution as des is said to be *Eulerian*. The *Eulerian numbers* $A(n,k)$ count permutations in the symmetric group $\mathcal{S}_n$ with k descents and they are the coefficients of the *Eulerian polynomials* $A_n(t)$ defined by $A_n(t) = \sum_{\pi\in\mathcal{S}_n} t^{1+\text{des }\pi}$. The Eulerian polynomials satisfy the identity

$$\sum_{k\geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Two descents i and j *overlap* if $j = i+1$. We define a new statistic, namely the *maximum number of non-overlapping descents*, or MND, in a permutation. For instance, MND(321) = 1 whereas MND(41532) = 2. One can find the distribution of this new statistic by using Corollary 5. This distribution is given in Example 6. However, we prove a more general theorem:

Theorem 13. *Let p be a GP with no dashes. Let $A(x)$ be the e.g.f. for the number of permutations that avoid p. Let $D(x,y) = \sum_{\pi} y^{N(\pi)} \frac{x^{|\pi|}}{|\pi|!}$ where $N(\pi)$ is the maximum number of non-overlapping occurrences of p in π. Then*

$$D(x,y) = \frac{A(x)}{1 - y((x-1)A(x)+1)}.$$

Proof. We fix the natural number k and consider an auxiliary multi-pattern $P_k = p - p - \cdots - p$ with k copies of p. If a permutation avoids P_k then it has at most $k-1$ non-overlapping occurrences of p. From Theorem 11, the e.g.f. $B_k(x)$ for the number of permutations avoiding P_k is equal to $\sum_{i=1}^{k} A(x) \prod_{j=1}^{i-1}((x-1)A(x)+1)$. If we subtract $B_k(x)$ from the e.g.f. $B_{k+1}(x) = \sum_{i=1}^{k+1} A(x) \prod_{j=1}^{i-1}((x-1)A(x)+1)$ for the number of permutations avoiding P_{k+1}, which is obtained by applying Theorem 11 to the pattern P_{k+1}, then we get the e.g.f. $D_k(x)$ for the number of permutations that have exactly k non-overlapping occurrences of the pattern p. So

$$D_k(x) = \sum_n D_{n,k}\frac{x^n}{n!} = B_{k+1}(x) - B_k(x) = A(x)((x-1)A(x)+1)^k.$$

Now

$$D(x,y) = \sum_{n,k\geq 0} D_{n,k}y^k\frac{x^n}{n!} = \sum_k D_k(x)y^k = \frac{A(x)}{1-y((x-1)A(x)+1)}.$$

□

All of the following examples are corollaries to Theorem 13.

Example 6. *If we consider descents then $A(x) = e^x$, hence the distribution of* MND *is given by the formula:*

$$D(x,y) = \frac{e^x}{1 - y(1 + (x-1)e^x)}.$$

Example 7. *Theorems 3 and 13 give the distribution of the maximum number of non-overlapping occurrences of the increasing subword of length k (the pattern $123\dots k$), which is equal to*

$$D(x,y) = \frac{1}{(1-x)y + (1-y)F_k(x)},$$

were $F_k(x) = \sum_{i\geq 0} \frac{x^{ki}}{(ki)!} - \sum_{i\geq 0} \frac{x^{ki+1}}{(ki+1)!}$.

Example 8. *If we consider the maximum number of non-overlapping occurrences of the pattern 132 then the distribution of these numbers is given by the formula*

$$D(x,y) = \frac{1}{1 - yx + (y-1)\int_0^x e^{-t^2/2}\,dt}.$$

Example 9. *The distribution of the maximum number of non-overlapping occurrences of the pattern from Theorem 4 is given by the formula:*

$$D(x,y) = \frac{1}{1 - x + (1-y)F_{k,a}(x)},$$

where $F_{k,a}(x) = \sum_{i\geq 1} \frac{(-1)^{i+1}x^{ki+1}}{(ki+1)!} \prod_{j=2}^{i} \binom{jk-a}{k-a}$.

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Paper VI

Partially ordered generalized patterns and k-ary words

Partially Ordered generalized patterns and k-ary words

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Abstract

Recently, Kitaev [Ki2] introduced partially ordered generalized patterns (POGPs) in the symmetric group, which further generalize the generalized permutation patterns introduced by Babson and Steingrímsson [BS]. A POGP p is a GP some of whose letters are incomparable. In this paper, we study the generating functions (g.f.) for the number of k-ary words avoiding some POGPs. We give analogues, extend and generalize several known results, as well as get some new results. In particular, we give the g.f. for the entire distribution of the maximum number of non-overlapping occurrences of a pattern p with no hyphens (that allowed to have repetition of letters), provided we know the g.f. for the number of k-ary words that avoid p.

6.1 Introduction

Let $[k]^n$ denote the set of all the words of length n over the (totally ordered) alphabet $[k] = \{1, 2, \dots, k\}$. We call these words by n-long k-ary words. A *generalized pattern* τ is a word in $[\ell]^m$ (possibly with hyphens between some letters) that contains each letter from $[\ell]$ (possibly with repetitions). We say that the word $\sigma \in [k]^n$ *contains* a generalized pattern τ, if σ contains a subsequence isomorphic to τ in which the entries corresponding to consecutive entries of τ, which are not separated by a hyphen, must be adjacent. Otherwise, we say that σ *avoids* τ and write $\sigma \in [k]^n(\tau)$. Thus, $[k]^n(\tau)$ denotes the set of all the words in $[k]^n$ that avoid τ. Moreover, if P is a set of generalized patterns then $|[k]^n(P)|$ denotes the set all the words in $[k]^n$ that avoid each pattern from P simultaneously.

Example 1. *A word $\pi = a_1a_2 \dots a_n$ avoids the pattern* 13-2 *if π has no subsequence $a_ia_{i+1}a_j$ with $j > i+1$ and $a_i < a_j < a_{i+1}$. Also, π avoids the pattern* 121 *if it has no subword $a_ia_{i+1}a_{i+2}$ such that $a_i = a_{i+2} < a_{i+1}$.*

Classical patterns are generalized patterns with all possible hyphens (say, 2-1-3-4), in other words, those that place no adjacency requirements on σ. The first case of classical patterns studied was that of permutations avoiding a pattern of length 3 in $\mathcal{S}_3$. Knuth [Knuth] found that, for any $\tau \in \mathcal{S}_3$, $|\mathcal{S}_n(\tau)| = C_n$,

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the nth Catalan number. Later, Simion and Schmidt [SS] determined the number $|\mathcal{S}_n(P)|$ of permutations in $\mathcal{S}_n$ simultaneously avoiding any given set of patterns $P \subseteq \mathcal{S}_3$. Burstein [Bu] extended this to $|[k]^n(P)|$ with $P \subseteq \mathcal{S}_3$. Burstein and Mansour [BM1] considered forbidden patterns with repeated letters. Also, Burstein and Mansour [BM2, BM3] considered forbidden generalized patterns with repeated letters.

Generalized permutation patterns were introduced by Babson and Steingrímsson [BS] with the purpose of the study of Mahonian statistics. Claesson [C] and Claesson and Mansour [CM] considered the number of permutations avoiding one or two generalized patterns with one hyphen. Kitaev [Ki1] examined the number of $|\mathcal{S}_n(P)|$ of permutations in $\mathcal{S}_n$ simultaneously avoiding any set of generalized patterns with no hyphens. Besides, Kitaev [Ki2] introduced a further generalization of the generalized permutation patterns namely *partially ordered generalized patterns.*

In this paper we introduce a further generalization of the generalized patterns namely *partially ordered generalized patterns in words* (*POGPs*), which is an analogue of POGPs in permutations [Ki2]. A POGP is a generalized pattern some of whose letters are incomparable. For example, if we write $\tau = 1\text{-}1'2'$, then we mean that in occurrence of τ in a word $\sigma \in [k]^n$ the letter corresponding to the 1 in τ can be either larger, smaller, or equal to the letters corresponding to $1'2'$. Thus, the word $113425 \in [5]^6$ contains seven occurrence of τ, namely 113, 134 twice, 125 twice, 325, and 425.

Following [Ki2], we consider two particular classes of POGPs – *shuffle patterns* and *multi-patterns*, which allows us to give an analogue for all the main results of [Ki2] for k-ary words. A multi-pattern is of the form $\tau = \tau^0\text{-}\tau^1\text{-}\cdots\text{-}\tau^s$ and a shuffle pattern of the form $\tau = \tau^0\text{-}a_1\text{-}\tau^1\text{-}a_2\text{-}\cdots\text{-}\tau^{s-1}\text{-}a_s\text{-}\tau^s$, where for any i and j, the letter a_i is greater than any letter of τ^j and for any $i \neq j$ each letter of τ^i is incomparable with any letter of τ^j. These patterns are investigated in Sections 6.3 and 6.4.

Let $\tau = \tau^0\text{-}\tau^1\text{-}\cdots\text{-}\tau^s$ be an arbitrary multi-pattern and let $A_{\tau^i}(x;k)$ be the generating function (g.f.) for the number of words in k-letter alphabet that avoid τ^i for each i. In Theorem 6 we find the g.f., in terms of the $A_{\tau^i}(x;k)$, for the number of k-ary words that avoid τ. In particular, this allows us to find the g.f. for the entire *distribution* of the maximum number of non-overlapping occurrences of a pattern τ with no hyphens, if we only know the g.f. for the number of k-ary words that avoid τ. Thus, in order to apply our results in what follows we need to know how many k-ary words avoid a given ordinary generalized pattern with no hyphens. This question was examined, for instance, in [BM1, Sections 2 and 3], [BM2, Section 3] and [BM3, Section 3.3].

6.2 Definitions and Preliminaries

A *partially ordered generalized pattern* (*POGP*) is a generalized pattern where some of the letters can be incomparable.

Example 2. *The simplest non-trivial example of a POGP that differs from* *the ordinary generalized patterns is* $\tau = 1'\text{-}2\text{-}1''$, *where the second letters is the greatest one and the first and the last letters are incomparable to each other. The word* $\sigma = 31421$ *has five occurrences of* τ, *namely* 342, 341, 142, 141, *and* 121.

Let $A_\tau(x;k) = \sum_{n\geq 0} a_\tau(n;k)x^n$ denote the *generating function* (*g.f.*) for the numbers $a_\tau(n;k)$ of words in $[k]^n$ avoiding the pattern τ. For $\tau = 1'\text{-}2\text{-}1''$, we have

$$A_{1'\text{-}2\text{-}1''}(x;k) = \frac{1}{(1-x)^{2k-1}} - \sum_{j=1}^{k-1} \frac{x}{(1-x)^{2j}}. \tag{6.1}$$

Indeed, if $\sigma \in [k]^n$ avoids τ, and σ contains $s > 0$ copies of the letter k, then the letters k appear as leftmost or rightmost letters of σ. If σ contains no k then $\sigma \in [k-1]^n$. So, for all $n \geq 0$, we have

$$a_\tau(n;k) = a_\tau(n;k-1) + 2a_\tau(n-1;k-1) + 3a_\tau(n-2;k-1) + \cdots + (n+1)a_\tau(0;k-1),$$

since there are $(i+1)a_\tau(n-i;k-1)$ possibilities to place i letters k into σ, for $0 \leq i \leq n$. Hence, for all $n \geq 2$,

$$a_\tau(n;k) - 2a_\tau(n-1;k) + a_\tau(n-2;k) = a_\tau(n;k-1),$$

together with $a_\tau(0,k) = 1$ and $a_\tau(1,k) = k$. Multiplying both sides of the recurrence above with x^n and summing over all $n \geq 2$, we get Equation 6.1.

Definition 5. *If the number of words in* $[k]^n$, *for each* n, *that avoid a POGP* τ *is equal to the number of words that avoid a POGP* ϕ, *then* τ *and* ϕ *are said to be* equivalent *and we write* $\tau \equiv \phi$.

The *reverse* $R(\sigma)$ of a word $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ is the word $\sigma_n\ldots\sigma_2\sigma_1$. The *complement* $C(\sigma)$ is the word $\theta = \theta_1\theta_2\ldots\theta_n$ where $\theta_i = k+1-\sigma_i$ for all $i = 1,2,\ldots,n$. For example, if $\sigma = 123331 \in [3]^6$, then $R(\sigma) = 133321$, $C(\sigma) = 321113$, and $R(C(\sigma)) = 311123$. We call these bijections of $[k]^n$ to itself *trivial*. For example, the number of words that avoid the pattern 12-2 is the same as the number of words that avoid the patterns 2-21, 1-12, and 21-1, respectively.

Following [Ki2], it is convenient to introduce the following definition.

Definition 6. *Let* τ *be a generalized pattern without hyphens. A word* σ quasi-avoids τ *if* σ *has exactly one occurrence of* τ *and this occurrence consists of the* $|\tau|$ *rightmost letters of* σ, *where* $|\tau|$ *denotes the number of letters in* τ.

For example, the word 5112234 quasi-avoids the pattern 1123, whereas the words 5223411 and 1123345 do not.

Proposition 1. *Let* τ *be a non-empty generalized pattern with no hyphens. Let* $A^*_\tau(x;k)$ *denote the g.f. for the number of words in* $[k]^n$ *that quasi-avoid* τ. *Then*

$$A^*_\tau(x;k) = (kx-1)A_\tau(x;k) + 1. \tag{6.2}$$

Proof. Using the similar arguments as those in the proof of [Ki2, Proposition 4], we get that, for $n \geq 1$,

$$a^*_\tau(n;k) = ka_\tau(n-1;k) - a_\tau(n;k),$$

where $a^*_\tau(n;k)$ denotes the number of words in $[k]^n$ that quasi-avoid τ. Multiplying both sides of the last equality by x^n and summing over all natural numbers n, we get the desired result. □

Definition 7. *Suppose $\{\tau^0, \tau^1, \ldots, \tau^s\}$ is a set of generalized patterns with no hyphens and*

$$\tau = \tau^0\text{-}\tau^1\text{-}\cdots\text{-}\tau^s,$$

where each letter of τ^i is incomparable with any letter of τ^j whenever $i \neq j$. We call such POGPs multi-patterns.

Definition 8. *Suppose $\{\tau^0, \tau^1, \ldots, \tau^s\}$ is a set of generalized patterns with no hyphens and $a_1 a_2 \ldots a_s$ is a word of s letters. We define a* shuffle *pattern to be a pattern of the form*

$$\tau = \tau^0\text{-}a_1\text{-}\tau^1\text{-}a_2\text{-}\cdots\text{-}\tau^{s-1}\text{-}a_s\text{-}\tau^s,$$

where each letter of τ^i is incomparable with any letter of τ^j whenever $i \neq j$, and the letter a_i is greater than any letter of τ^j for any i and j.

For example, $1'$-2-$1''$ is a shuffle pattern, and $1'$-$1''$ is a multi-patterns. From definitions, we obtain that we can get a multi-pattern from a shuffle pattern by removing all the letters a_i.

There is a connection between multi-avoidance of the generalized patterns and the POGPs. In particular, to avoid $1'$-2-$1''$ is the same as to avoid simultaneously the patterns 1-2-1, 1-3-2, and 2-3-1. A straightforward argument leads to the following proposition.

Proposition 2. *For any POGP τ there exists a set T of generalized patterns such that a word σ avoids τ if and only if σ avoids all the patterns in T.*

For example, if $\tau = 1'2'$-3-$1''$, then to avoid τ is the same to avoid 5 patterns, 12-3-1, 12-3-2, 12-4-3, 13-4-2, and 23-4-1. Moreover, the following proposition holds:

Proposition 3. *Suppose $\tau = \tau_1$-a-τ_2 (resp. $\phi = \phi_1$-ϕ_2) is a shuffle pattern (resp. a multi-pattern) such that $\tau_1,\ \phi_1 \in [r_1]^{\ell_1}$, $\tau_2,\ \phi_2 \in [r_2]^{\ell_2}$ and each letter of $[r_1]$ is incomparable with any letter of $[r_2]$. Also, without lose the generality, suppose $r_1 \geq r_2$. Then to avoid τ (resp. ϕ) is the same as to avoid $\sum_{i=0}^{r_2} \binom{r_1}{i}\binom{r_2}{i}\binom{r_1+r_2-i}{r_1}$ generalazed patterns. In particular, the number of generalized patterns does not depend on the lengths ℓ_1 and ℓ_2.*

Proof. Obviously, to prove the statement, we need to find the number of ways to make a total order on $[r_1] \cup [r_2]$ (the letter a does not play any roll, since it is always the greatest letter). Any total order on $[r_1] \cup [r_2]$ is an alphabet that can consist of $r_1 + r_2 - i$ letters, where i is the number of letters in $[r_2]$ that supposed to coincide with some letters in $[r_1]$. Clearly, $0 \le i \le r_2$ and we can choose coinciding letters in $\binom{r_1}{i}\binom{r_2}{i}$ ways. Now, after choosing the coinciding letters, we can make a total order in $\binom{r_1+r_2-i}{r_1}$ ways, which is given by [Ki2, Theorem 8]. □

6.3 The shuffle patterns

We recall that according to Definition 8, a shuffle pattern is a pattern of the form

$$\tau = \tau^0\text{-}a_1\text{-}\tau^1\text{-}a_2\text{-}\cdots\text{-}\tau^{s-1}\text{-}a_s\text{-}\tau^s,$$

where $\{\tau^0, \tau^1, \ldots, \tau^s\}$ is a set of generalized patterns with no hyphens, $a_1a_2\ldots a_s$ is a word of s letters, for any i and j the letter a_i is greater than any letter of τ^j and for any $i \neq j$ each letter of τ^i is incomparable with any letter of τ^j.

Let us consider the shuffle pattern $\phi = \tau\text{-}\ell\text{-}\tau$, where ℓ is the greatest letter in ϕ and letters each letter in the left τ is incomparable with any letter in the right τ.

Theorem 1. *Let ϕ be the shuffle pattern $\tau\text{-}\ell\text{-}\tau$ described above. Then for all $k \ge \ell$,*

$$A_\phi(x;k) = \frac{1}{(1 - xA_\tau(x;k-1))^2}\Big(A_\phi(x;k-1) - xA_\tau^2(x;k-1)\Big).$$

Proof. We show how to get a recurrence relation on k for $A_\phi(x;k)$, which is the g.f. for the number of words in $[k]^n(\phi)$. Suppose $\sigma \in [k]^n(\phi)$ is such that it contains exactly d copies of the letter k. If $d = 0$ then the g.f. for the number of such words is $A_\phi(x;k-1)$. Assume that $d \ge 1$. Clearly, σ can be written in the following form:

$$\sigma = \sigma^0 k \sigma^1 k \cdots k \sigma^d,$$

where σ^j is a ϕ-avoiding word on $k-1$ letters, for $j = 0, 1, \ldots, d$. There are two possibilities: either σ^j avoids τ for all j, or there exists j_0 such that σ^{j_0} contains τ and for any $j \neq j_0$, the word σ^j avoids τ. In the first case, the number of such words is given by the g.f. $x^d A_\tau^{d+1}(x;k-1)$, whereas in the second case, by $(d+1)x^d A_\tau^d(x;k-1)(A_\phi(x;k-1) - A_\tau(x;k-1))$. In the last expression, the multiple $(d+1)$ is the number of ways to choose j, such that σ^j has an occurrence of τ, and $A_\phi(x;k-1) - A_\tau(x;k-1)$ is the g.f. for the number of words avoiding ϕ and containing τ.

Therefore,

$$A_\phi(x;k) = A_\phi(x;k-1) + \\ \sum_{d\ge1}(d+1)x^d A_\tau^d(x;k-1)A_\phi(x;k-1) - \sum_{d\ge1} d x^d A_\tau^{d+1}(x;k-1),$$

equivalently,

$$A_\phi(x;k) = A_\phi(x;k-1)+$$
$$A_\phi(x;k-1)\frac{2xA_\tau(x;k-1)-x^2A_\tau^2(x;k-1)}{(1-xA_\tau(x;k-1))^2} - \frac{xA_\tau^2(x;k-1)}{(1-xA_\tau(x;k-1))^2}.$$

The rest is easy to check. □

Example 3. *Let $\phi = 1'$-2-$1''$. Here $\tau = 1$, so $A_\tau(x;k) = 1$ for all $k \geq 1$, since only the empty word avoids τ. Hence, according to Theorem 1, we have*

$$A_\phi(x;k) = \frac{A_\phi(x;k-1)-x}{(1-x)^2},$$

which together with $A_\phi(x;1) = \frac{1}{1-x}$ (for any n only the word $\underbrace{11\ldots1}_{n\ times}$ avoids ϕ) gives Equation 6.1.

More generally, we consider a shuffle pattern of the form τ^0-ℓ-τ^1, where ℓ is the greatest element of the pattern.

Theorem 2. *Let ϕ be the shuffle pattern τ-ℓ-ν. Then for all $k \geq \ell$,* $A_\phi(x;k) =$

$$\frac{1}{(1-xA_\tau(x;k-1))(1-xA_\nu(x;k-1))}\Big(A_\phi(x;k-1)-xA_\tau(x;k-1)A_\nu(x;k-1)\Big).$$

Proof. We proceed as in the proof of Theorem 1. Suppose $\sigma \in [k]^n(\phi)$ is such that it contains exactly d copies of the letter k. If $d = 0$ then the g.f. for the number of such words is $A_\phi(x;k-1)$. Assume that $d \geq 1$. Clearly, σ can be written in the following form:

$$\sigma = \sigma^0 k \sigma^1 k \cdots k \sigma^d,$$

where σ^j is a ϕ-avoiding word on $k-1$ letters, for $j = 0,1,\ldots,d$. There are two possibilities: either σ^j avoids τ for all j, or there exists j_0 such that σ^{j_0} contains τ, σ^j avoids τ for all $j = 0,1,\ldots,j_0-1$ and σ^j avoids ν for any $j = j_0+1,\ldots,d$. In the first case, the number of such words is given by the g.f. $x^dA_\tau^{d+1}(x;k-1)$. In the second case, we have

$$x^d\sum_{j=0}^{d} A_\tau^j(x;k-1)A_\nu^{d-j}(x;k-1)(A_\phi(x;k-1)-A_\tau(x;k-1)).$$

Therefore, we get

$$A_\phi(x;k) = A_\phi(x;k-1) + A_\phi(x;k-1)\sum_{d\geq1} x^d \sum_{j=0}^{d} A_\tau^j(x;k-1)A_\nu^{d-j}(x;k-1)$$
$$- \sum_{d\geq1} x^d \sum_{j=1}^{d} A_\tau^j(x;k-1)A_\nu^{d+1-j}(x;k-1),$$

equivalently,

$$A_\phi(x;k) = (A_\phi(x;k-1) - xA_\tau(x;k-1)A_\nu(x;k-1)) \sum_{d\geq 0} x^d \sum_{j=0}^{d} A_\tau^j(x;k-1)A_\nu^{d-j}(x;k-1).$$

Hence, using the identity $\sum_{n\geq 0} x^n \sum_{j=0}^{n} p^j q^{n-j} = \frac{1}{(1-xp)(1-xq)}$ we get the desired result. □

We now give two corollaries to Theorem 2.

Corollary 6. *Let $\phi = \tau^0$-ℓ-τ^1 be a shuffle pattern, and let $f(\phi) = f_1(\tau^0)$-ℓ-$f_2(\tau^1)$, where f_1 and f_2 are any trivial bijections. Then $\phi \equiv f(\phi)$.*

Proof. Using Theorem 2, and the fact that the number of words in $[k]^n$ avoiding τ (resp. ν) and $f_1(\tau)$ (resp. $f_2(\nu)$) have the same generating functions, we get the desired result. □

Corollary 7. *For any shuffle pattern τ-ℓ-ν, we have*

$$\tau\text{-}\ell\text{-}\nu \equiv \nu\text{-}\ell\text{-}\tau.$$

Proof. Corollary 6 yields that the shuffle pattern τ-ℓ-ν is equivalent to the pattern τ-ℓ-$R(\nu)$, which is equivalent to the pattern $R(\tau\text{-}\ell\text{-}R(\nu)) = \nu\text{-}\ell\text{-}R(\tau)$. Finally, we use Corollary 6 one more time to get the desired result. □

6.4 The multi-patterns

We recall that according to Definition 7, a multi-pattern is a pattern of the form $\tau = \tau^0\text{-}\tau^1\text{-}\cdots\text{-}\tau^s$, where $\{\tau^0, \tau^1, \dots, \tau^s\}$ is a set of generalized patterns with no hyphens and each letter of τ^i is incomparable with any letter of τ^j whenever $i \neq j$.

The simplest non-trivial example of a multi-pattern is the multi-pattern $\phi = 1\text{-}1'2'$. To avoid ϕ is the same as to avoid the patterns 1-12, 1-23, 2-12, 2-13, and 3-12 simultaneously. To count the number of words in $[k]^n(1\text{-}1'2')$, we choose the leftmost letter of $\sigma \in [k]^n(1\text{-}1'2')$ in k ways, and observe that all the other letters of σ must be in a non-increasing order. Using [BM1], for all $n \geq 1$, we have

$$|[k]^n(1\text{-}1'2')| = k \cdot \binom{n+k-2}{n-1}.$$

The following theorem is an analogue to [Ki2, Theorem 21].

Theorem 3. *Let $\tau = \tau^0$-τ^1 and $\phi = f_1(\tau^0)$-$f_2(\tau^1)$, where f_1 and f_2 are any of the trivial bijections. Then $\tau \equiv \phi$.*

Proof. First, let us prove that the pattern $\tau = \tau^0\text{-}\tau^1$ is equivalent to the pattern $\phi = \tau^0\text{-}f(\tau^1)$, where f is a trivial bijection. Suppose that $\sigma = \sigma^1\sigma^2\sigma^3 \in [k]^n$ avoids τ and $\sigma^1\sigma^2$ has exactly one occurrence of τ^0, namely σ^2. Then σ^3 must avoid τ^1, so $f(\sigma^3)$ avoids $f(\tau^3)$ and $\sigma_f = \sigma^1\sigma^2 f(\sigma^3)$ avoids ϕ. The converse is also true, if σ_f avoids ϕ then σ avoids τ. Since any word either avoids τ^0 or can be factored as above, we have a bijection between the class of words avoiding τ and the class of words avoiding ϕ. Thus $\tau \equiv \phi$.

Now, we use the considerations above as well as the properties of trivial bijections to get

$$\begin{aligned}\tau \equiv \tau^0\text{-}f_2(\tau^1) \equiv R(\tau^0\text{-}f_2(\tau^1)) \equiv R(f_2(\tau^1))\text{-}R(\tau^0) \equiv \\ R(f_2(\tau^1))\text{-}f_1(R(\tau^0)) \equiv R(f_2(\tau^1))\text{-}R(f_1(\tau^0)) \equiv f_1(\tau^0)\text{-}f_2(\tau^1).\end{aligned}$$

□

Using Theorem 3, we get the following corollary, which is an analogue to [Ki2, Corollary 22].

Corollary 8. *The multi-pattern $\tau^0\text{-}\tau^1$ is equivalent to the multi-pattern $\tau^1\text{-}\tau^0$.*

Proof. From Theorem 3, using the properties of the trivial bijection R, we get

$$\tau^0\text{-}\tau^1 \equiv \tau^0\text{-}R(\tau^1) \equiv R(R(\tau^1))\text{-}R(\tau^0) \equiv \tau^1\text{-}R(R(\tau^0)) \equiv \tau^1\text{-}\tau^0.$$

□

Using induction on s, Corollary 8, and proceeding in the way proposed in [Ki2, Theorem 23], we get

Theorem 4. *Suppose we have multi-patterns $\tau = \tau^0\text{-}\tau^1\text{-}\cdots\text{-}\tau^s$ and $\phi = \phi^0\text{-}\phi^1\text{-}\cdots\text{-}\phi^s$, where $\tau^1\tau^2\cdots\tau^s$ is a permutation of $\phi^1\phi^2\cdots\phi^s$. Then $\tau \equiv \phi$.*

The last theorem is an analogue to [Ki2, Theorem 23]. As a corollary to Theorem 4, using Theorem 3 and the idea of the proof of [Ki2, Corollary 24], we get the following corollary which is an analogue to [Ki2, Corollary 24].

Corollary 9. *Suppose we have multi-patterns $\tau = \tau^0\text{-}\tau^1\text{-}\cdots\text{-}\tau^s$ and $\phi = f_0(\tau^0)\text{-}f_1(\tau^1)\text{-}\cdots\text{-}f_s(\tau^s)$, where f_i is an arbitrary trivial bijection. Then $\tau \equiv \phi$.*

The following theorem is a good auxiliary tool for calculating the g.f. for the number of words that avoid a given POGP. For particular POGPs, it allows to reduce the problem to calculating the g.f. for the number of words that avoid another POGP which is shorter. We recall that $A^*_\tau(x;k)$ is the generating function for the number of words in $[k]^n$ that quasi-avoid the pattern τ.

Theorem 5. *Suppose $\tau = \tau^0\text{-}\phi$, where ϕ is an arbitrary POGP, and the letters of τ^0 are incomparable to the letters of ϕ. Then for all $k \geq 1$, we have*

$$A_\tau(x;k) = A_{\tau^0}(x;k) + A_\phi(x;k)A^*_{\tau^0}(x;k).$$

Proof. Suppose $\sigma = \sigma^1\sigma^2\sigma^3 \in [k]^n$ avoids the pattern τ, where $\sigma^1\sigma^2$ quasi-avoids the pattern τ^0, and σ^2 is the occurrence of τ^0. Clearly, σ^3 must avoid ϕ. To find $A_\tau(x;k)$, we observe that there are two possibilities: either σ avoids τ^0, or σ does not avoid τ^0. In these cases, the g.f. for the number of such words is equal to $A_{\tau^0}(x;k)$ and $A_\phi(x;k)A^*_{\tau^0}(x;k)$ respectively (the second term came from the factorization above). Thus, the statement is true. □

Corollary 10. *Let $\tau = \tau^1\text{-}\tau^2\text{-}\cdots\text{-}\tau^s$ be a multi-pattern such that τ^j is equal to either* 12 *or* 21, *for $j = 1, 2, \ldots, s$. Then*

$$A_\tau(x;k) = \frac{1 - \left(1 + \frac{kx-1}{(1-x)^k}\right)^s}{1 - kx}.$$

Proof. According to [BM2], $A_{12}(x;k) = A_{21}(x;k) = \frac{1}{(1-x)^k}$. Using Theorem 5, Proposition 1 and induction on s, we get the desired result. □

More generally, using Theorem 5 and Proposition 1, we get the following theorem that is the basis for calculating the number of words that avoid a multi-pattern, and therefore is the main result for multi-patterns in this paper.

Theorem 6. *Let $\tau = \tau^1\text{-}\tau^2\text{-}\cdots\text{-}\tau^s$ be a multi-pattern. Then*

$$A_\tau(x;k) = \sum_{j=1}^{s} A_{\tau^j}(x;k) \prod_{i=1}^{j-1} ((kx-1)A_{\tau^i}(x;k) + 1).$$

6.5 The distribution of non-overlapping generalized patterns

A descent in a word $\sigma \in [k]^n$ is an i such that $\sigma_i > \sigma_{i+1}$. Two descents i and j *overlap* if $j = i + 1$. We define a new statistics, namely the *maximum number of non-overlapping descents,* or MND, in a word. For example, MND(33211) = 1 whereas MND(13211143211) = 3. One can find the distribution of this new statistic by using Corollary 10. This distribution is given in Example 4. However, we prove a more general theorem:

Theorem 7. *Let τ be a generalized pattern with no hyphens. Then for all $k \geq 1$,*

$$\sum_{n\geq 0} \sum_{\sigma\in[k]^n} y^{N_\tau(\sigma)} x^n = \frac{A_\tau(x;k)}{1 - y((kx-1)A_\tau(x;k) + 1)},$$

where $N_\tau(\sigma)$ is the maximum number of non-overlapping occurrences of τ in σ.

Proof. We fix the natural number s and consider the multi-pattern $\Phi_s = \tau\text{-}\tau\text{-}\cdots\text{-}\tau$ with s copies of τ. If a word avoids Φ_s then it has at most $s-1$ non-overlapping occurrences of τ. Theorem 6 yields

$$A_{\Phi_s}(x;k) = \sum_{j=1}^{s} A_\tau(x;k) \prod_{i=1}^{j-1} ((kx-1)A_\tau(x;k) + 1).$$

So, the g.f. for the number of words that has exactly s non-overlapping occurrences of the pattern τ is given by

$$A_{\Phi_{s+1}}(x;k) - A_{\Phi_s}(x;k) = A_\tau(x;k)((kx-1)A_\tau(x;k)+1)^s.$$

Hence,

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_\tau(\sigma)}x^n = \sum_{s\geq 0} A_\tau(x;k)((kx-1)A_\tau(x;k)+1)^s =$$

$$\frac{A_\tau(x;k)}{1-y((kx-1)A_\tau(x;k)+1)}.$$

□

All of the following examples are corollaries to Theorem 7.

Example 4. *If we consider descents (the pattern* 12*) then* $A_{12}(x;k) = \frac{1}{(1-x)^k}$ *(see [BM2]), hence the distribution of* MND *is given by the formula:*

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{12}(\sigma)}x^n = \frac{1}{(1-x)^k + y(1-kx-(1-x)^k)}.$$

Example 5. *The distribution of the maximum number of non-overlapping occurrences of the pattern* 122 *is given by the formula:*

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{122}(\sigma)}x^n = \frac{x}{(1-x^2)^k + x - 1 + y(1-kx^2-(1-x^2)^k)},$$

since according to [BM3, Theorem 3.10], $A_{122}(x;k) = \frac{x}{(1-x^2)^k-(1-x)}$.

Example 6. *If we consider the pattern* 212 *then* $A_{212}(x;k) = \left(1 - x\sum_{j=0}^{k-1}\frac{1}{1+jx^2}\right)^{-1}$ *(see [BM3, Theorem 3.12]), hence the distribution of the maximum number of non-overlapping occurrences of the pattern* 212 *is given by the formula:*

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{212}(\sigma)}x^n = \frac{1}{1 - x\sum_{j=0}^{k-1}\frac{1}{1+jx^2} + xy\left(\sum_{j=0}^{k-1}\frac{1}{1+jx^2} - k\right)}.$$

Example 7. *Using [BM3, Theorem 3.13], the distribution of the maximum number of non-overlapping occurrences of the pattern* 123 *is given by the formula:*

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{123}(\sigma)}x^n = \frac{1}{\sum_{j=0}^{k} a_j\binom{k}{j}x^j + y\left(1 - kx - \sum_{j=0}^{k} a_j\binom{k}{j}x^j\right)},$$

where $a_{3m}=1$, $a_{3m+1}=-1$, *and* $a_{3m+2}=0$, *for all* $m\geq 0$.

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Paper VII

Counting the occurrences of generalized patterns in words generated by a morphism

Counting the occurrences of generalized patterns in words generated by a morphism

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Abstract

We count the number of occurrences of certain patterns in given words. We choose these words to be the set of all finite approximations of a sequence generated by a morphism with certain restrictions. The patterns in our considerations are either classical patterns 1-2, 2-1, 1-1-$\cdots$-1, or arbitrary generalized patterns without internal dashes, in which repetitions of letters are allowed. In particular, we find the number of occurrences of the patterns 1-2, 2-1, 12, 21, 123 and 1-1-$\cdots$-1 in the words obtained by iterations of the morphism $1 \to 123$, $2 \to 13$, $3 \to 2$, which is a classical example of a morphism generating a nonrepetitive sequence.

7.1 Introduction and Background

We write permutations as words $\pi = a_1a_2\cdots a_n$, whose letters are distinct and usually consist of the integers $1, 2, \ldots, n$.

An occurrence of a pattern p in a permutation π is "classically" defined as a subsequence in π (of the same length as the length of p) whose letters are in the same relative order as those in p. Formally speaking, for $r \leq n$, we say that a permutation σ in the symmetric group $\mathcal{S}_n$ has an occurrence of the pattern $p \in \mathcal{S}_r$ if there exist $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that $p = \sigma(i_1)\sigma(i_2)\ldots\sigma(i_r)$ in reduced form. The *reduced form* of a permutation σ on a set $\{j_1, j_2, \ldots, j_r\}$, where $j_1 < j_2 < \cdots < j_r$, is a permutation σ_1 obtained by renaming the letters of the permutation σ so that j_i is renamed i for all $i \in \{1, \ldots, r\}$. For example, the reduced form of the permutation 3651 is 2431. The first case of classical patterns studied was that of permutations avoiding a pattern of length 3 in $\mathcal{S}_3$. Knuth [Knuth] found that, for any $\tau \in \mathcal{S}_3$, the number $|\mathcal{S}_n(\tau)|$ of n-permutations avoiding τ is C_n, the nth Catalan number. Later, Simion and Schmidt [SimSch] determined the number $|\mathcal{S}_n(P)|$ of permutations in $\mathcal{S}_n$ simultaneously avoiding any given set of patterns $P \subseteq \mathcal{S}_3$.

In [BabStein] Babson and Steingrímsson introduced *generalised permutation patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a "classical" pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if

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this pattern occurs in the permutation, then the letters in the permutation that correspond to 3 and 1 are adjacent. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563. A motivation for introducing these patterns in [BabStein] was the study of Mahonian statistics. A number of interesting results on generalised patterns were obtained in [Claes]. Relations to several well studied combinatorial structures, such as set partitions, Dyck paths, Motzkin paths and involutions, were shown there.

Burstein [Burstein] considered words instead of permutations. In particular, he found the number $|[k]^n(P)|$ of words of length n in k-letter alphabet that avoid each pattern from a set $P \subseteq \mathcal{S}_3$ simultaneously. Burstein and Mansour [BurMans1] (resp. [BurMans2, BurMans3]) considered forbidden patterns (resp. generalized patterns) with repeated letters.

The most attention, in the papers on classical or generalized patterns, is paid to counting exact formulas and/or generating functions for the number of words or permutations avoiding, or having k occurrences of, certain pattern. In this paper we suggest another problem, namely counting the number of occurrences of a particular pattern τ in given words. We choose these words to be a set of all finite approximations (to be defined below) of a sequence generated by a morphism with certain restrictions. A motivation for such a choice is big interest in studying classes of sequences and words that are defined by iterative schemes [Lothaire, Salomaa]. The pattern τ in our considerations is either a classical pattern from the set $\{1\text{-}2, 2\text{-}1, 1\text{-}1\text{-}\cdots\text{-}1\}$, or an arbitrary generalized pattern without internal dashes, in which repetitions of letters are allowed. In particular, we find that there are $(3 \cdot 4^{n-1} + 2^n)$ occurrences of the pattern 1-2 in the n-th finite approximation of the sequence w defined below, which is a classical example of a nonrepetitive sequence.

Let Σ be an alphabet and $\Sigma^\star$ be the set of all words of Σ. A map $\varphi : \Sigma^\star \to \Sigma^\star$ is called a *morphism*, if we have $\varphi(uv) = \varphi(u)\varphi(v)$ for any $u, v \in \Sigma^\star$. It is easy to see that a morphism φ can be defined by defining $\varphi(i)$ for each $i \in \Sigma$. The set of all rules $i \to \varphi(i)$ is called a *substitution system*. We create words by starting with a letter from the alphabet Σ and iterating the substitution system. Such a substitution system is called a *D0L (Deterministic, with no context Lindenmayer) system* [LindRoz]. D0L systems are classical objects of formal language theory. They are interesting from mathematical point of view [Frid], but also have applications in theoretical biology [Lind]. Let $|X|$ denote the length of a word X, that is the number of letters in X.

Suppose a word $\varphi(a)$ begins with a for some $a \in \Sigma$, and that the length of $\varphi^k(a)$ increases without bound. The symbolic sequence $\lim\limits_{k\to\infty} \varphi^k(a)$ is said to be *generated* by the morphism φ. In particular, $\lim\limits_{k\to\infty} \varphi^k(a)$ is a *fixed point* of φ. However, in this paper we are only interesting in the *finite approximations* of $\lim\limits_{k\to\infty} \varphi^k(a)$, that is in the words $\varphi^k(a)$ for $k = 1, 2, \ldots$.

An example of a sequence generated by a morphism can be the following sequence w. We create words by starting with the letter 1 and iterating the substitution system ϕ_w: $1 \to 123$, $2 \to 13$, $3 \to 2$. Thus, the initial letters of

w are 123132123213.... This sequence was constructed in connection with the problem of constructing a nonrepetitive sequence on a 3-letter alphabet, that is, a sequence that does not contain any subwords of the type $XX = X^2$, where X is any non-empty word over a 3-letter alphabet. The sequence w has that property. The question of the existence of such a sequence, as well as the questions of the existence of sequences avoiding other kinds of repetitions, were studied in algebra [Adian, Justin, Kol], discrete analysis [Carpi, Dekk, Evdok, Ker, Pleas] and in dynamical systems [MorseHedl]. In Examples 1, 4 and 5 we give the number of occurrences of the patterns 1-2, 2-1, 1-1-$\cdots$-1, 12, 123 and 21 in the finite approximations of w.

To proceed further, we need the following definitions. Let $N_{\phi}^{\tau}(n)$ denote the number of occurrences of the pattern τ in a word generated by some morphism ϕ after n iterations. We say that an occurrence of τ is *external* for a pair of words (X, Y), if this occurrence starts in X and ends in Y. Also, an occurrence of τ for a word X is *internal*, if this occurrence starts and ends in this X.

7.2 Patterns 1-2, 2-1 and 1-1-$\cdots$-1

Theorem 1. *Let $\mathcal{A} = \{1, 2, \dots, k\}$ be an alphabet, where $k \geq 2$ and a pattern $\tau \in \{1\text{-}2, 2\text{-}1\}$. Let X_1 begins with the letter 1 and consists of ℓ copies of each letter $i \in \mathcal{A}$ ($\ell \geq 1$). Let a morphism ϕ be such that*

$$1 \to X_1,\ 2 \to X_2,\ 3 \to X_3, \dots, k \to X_k,$$

where we allow X_i to be the empty word ϵ for $i = 2, 3, \dots, k$ (that is, ϕ may be an erasing morphism), $\sum_{i=2}^{k} |X_i| = k \cdot d$, and each letter from $\mathcal{A}$ appears in the word $X_2X_3 \dots X_k$ exactly d times. Besides, let $e_{i,j}$ (resp. e_i) be the number of external occurrences of τ for (X_i, X_j) (resp. (X_i, X_i)), where $i \neq j$. We assume that $e_{i,j} = e_{j,i}$ for all i and j. Let s_i be the number of internal occurrences of τ in X_i. In particular, $s_i = e_i = e_{i,j} = e_{j,i} = 0$, whenever $X_i = \epsilon$; also, $e_i = |X_i| \cdot (|X_i| - 1)/2$, whenever there are no repetitive letters in X_i. Then $N_{\phi}^{\tau}(1) = s_1$ and for $n \geq 2$, $N_{\phi}^{\tau}(n)$ is given by

$$\ell \cdot (d+\ell)^{n-2} \sum_{i=1}^{k} s_i + \binom{\ell \cdot (d+\ell)^{n-2}}{2} \sum_{i=1}^{k} e_i + \ell^2 \cdot (d+\ell)^{2n-4} \sum_{1 \leq i < j \leq k} e_{i,j}.$$

Proof. We assume that $\tau = 1\text{-}2$. All the considerations for this τ remain the same for the case $\tau = 2\text{-}1$.

If $n = 1$ then the statement is trivial.

Suppose $n \geq 2$. Using the fact that $X_1X_2X_3 \dots X_k$, has exactly $d + \ell$ occurrences of each letter i, $i = 1, 2, \dots, k$, one can prove by induction on n, that the word $\phi^n(1)$ is a permutation of $\ell \cdot (d+\ell)^{n-2}$ copies of each word X_i, where $i = 1, 2, \dots, k$. This implies, in particular, that $|\phi^n(1)| = k \cdot \ell \cdot (d+\ell)^{n-1}$.

An occurrence of τ in $\phi^n(1)$ can be either internal, that is when τ occurs inside a word X_i, or external, which means that τ begins in a word X_i and ends in another word X_j. In the first of these cases, since there are $\ell\cdot(d+\ell)^{n-2}$ copies of each X_i, we have $\ell\cdot(d+\ell)^{n-2}\sum_{i=1}^{k} s_i$ possibilities. In the second case, either $i=j$, which gives $\binom{\ell\cdot(d+\ell)^{n-2}}{2}\sum_{i=1}^{k} e_i$ possibilities, or $i\neq j$, in which case there are $\ell\cdot(d+\ell)^{n-2}$ possibilities to choose X_i (resp. X_j) among $\ell\cdot(d+\ell)^{n-2}$ copies of X_i (resp. X_j), and using the fact that $e_{i,j}=e_{j,i}$ (the order in which the words X_i and X_j occur in $\phi^n(1)$ is unimportant), we have $\ell^2\cdot(d+\ell)^{2n-4}\sum_{1\leq i<j\leq k} e_{i,j}$ possibilities. Summing all the possibilities, we finish the proof. □

Let s (resp. e) denote the vector $(s_1, s_2, \ldots, s_k)$ (resp. $(e_1, e_2, \ldots, e_k)$), where s_i and e_j are defined in Theorem 1. All of the following examples are corollaries to Theorem 1.

Example 1. *If we consider the morphism ϕ_w defined in Section 7.1 and the pattern $\tau=$ 1-2 then $d=\ell=1$, $s=(3,1,0)$, $e=(3,1,0)$ and $e_{1,2}=e_{2,1}=2$, $e_{1,3}=e_{3,1}=1$, $e_{2,3}=e_{3,2}=1$. Hence, the number of occurrences of τ is given by $N_{\phi_w}^{1\text{-}2}(1)=3$ and, for $n\geq 2$, $N_{\phi_w}^{1\text{-}2}(n)=(3\cdot 4^{n-1}+2^n)/2$. If $\tau=$ 2-1 then $s=(0,0,0)$, $e=(3,1,0)$ and $e_{1,2}=e_{2,1}=2$, $e_{1,3}=e_{3,1}=1$, $e_{2,3}=e_{3,2}=1$. Hence, $N_{\phi_w}^{2\text{-}1}(1)=0$ and, for $n\geq 2$, $N_{\phi_w}^{2\text{-}1}(n)=(3\cdot 4^{n-1}-2^n)/2$.*

Example 2. *If we consider the morphism ϕ: $1\rightarrow 1324$, $2\rightarrow\epsilon$, $3\rightarrow 14$, and $4\rightarrow 23$ then for the pattern $\tau=$ 1-2, we have $d=\ell=1$, $s=(5,0,1,1)$, $e=(6,0,1,1)$, and $e_{i,j}$, for $i\neq j$, are elements of the matrix*

$$\begin{pmatrix} - & 0 & 3 & 3 \\ 0 & - & 0 & 0 \\ 3 & 0 & - & 2 \\ 3 & 0 & 2 & - \end{pmatrix}.$$

Hence, $N_{\phi}^{1\text{-}2}(1)=5$ and, for $n\geq 2$, $N_{\phi}^{1\text{-}2}(n)=3\cdot 4^{n-1}+11\cdot 2^{n-2}$.

Example 3. *If we consider the morphism ϕ: $1\rightarrow 13542$, $2\rightarrow 423$, $3\rightarrow\epsilon$, $4\rightarrow 5115$, and $5\rightarrow 234$ then for the pattern $\tau=$ 1-2, we have $\ell=1$, $d=2$, $s=(6,1,0,2,3)$, $e=(10,3,0,4,3)$, and $e_{i,j}$, for $i\neq j$, are elements of the matrix*

$$\begin{pmatrix} - & 6 & 0 & 8 & 6 \\ 6 & - & 0 & 6 & 3 \\ 0 & 0 & - & 0 & 0 \\ 8 & 6 & 0 & - & 6 \\ 6 & 3 & 0 & 6 & - \end{pmatrix}.$$

Hence, $N_{\phi}^{1\text{-}2}(1)=6$ and, for $n\geq 2$, $N_{\phi}^{1\text{-}2}(n)=5\cdot 9^{n-1}+2\cdot 3^{n-2}$.

Using the proof of Theorem 1, we have the following.

Theorem 2. *Let a morphism ϕ satisfy all the conditions in the statement of Theorem 1 and the pattern $\tau=\underbrace{\text{1-1-}\cdots\text{-1}}_{r\ \text{times}}$. Then, for $n\geq 2$, the number of* *occurrences of τ in $\phi^n(1)$ is given by $k \cdot \binom{\ell\cdot(d+\ell)^{n-1}}{r}$, whereas for $n = 1$, by $k \cdot \binom{\ell}{r}$.*

Proof. From the proof of Theorem 1, we have that if $n \geq 2$ (resp. $n = 1$) then $\phi^n(1)$ has exactly $\ell \cdot (d+\ell)^{n-1}$ (resp. ℓ) copies of each letter from $\mathcal{A}$. We can choose r of them in $\binom{\ell\cdot(d+\ell)^{n-1}}{r}$ (resp. $\binom{\ell}{r}$) ways to form the pattern τ. The rest is clear. □

The following example is a corollary to Theorem 2.

Example 4. *If we consider the morphism ϕ_w defined in Section 7.1 and the pattern $\tau =$ 1-1-1-1 then $d = \ell = 1$, $r = 4$, hence the number of occurrences of τ in $\phi^n(1)$ is 0, whenever $n = 1$ or $n = 2$, and $3 \cdot \binom{2^{n-1}}{4}$ otherwise.*

7.3 Patterns without internal dashes

In what follows we need to extend the notion of an external occurrence of a pattern. Suppose $W = AXBYC$, where A, X, B, Y and C are some subwords. We say that an occurrence of τ in W is external for a pair of words (X, Y), if this occurrence starts in X, ends in Y and is allowed to have some of its letters in B. For instance, if $W = 12324245$, where $A = 1$, $X = 23$, $B = 2$ and $Y = 424$ then an occurrence of the generalized pattern 213, namely the subword 324 is an external occurrence for (X, Y).

Theorem 3. *Let $\mathcal{A} = \{1, 2, \ldots, k\}$ be an alphabet and a generalized pattern τ has no internal dashes. Let X_1 begins with the letter 1 and consists of ℓ copies of each letter $i \in \mathcal{A}$ ($\ell \geq 1$). Let a morphism ϕ be such that*

$$1 \to X_1,\ 2 \to X_2,\ 3 \to X_3, \ldots, k \to X_k,$$

where we allow X_i to be the empty word ϵ for $i = 2, 3, \ldots, k$ (that is, ϕ may be an erasing morphism), $\sum_{i=2}^{k} |X_i| = k \cdot d$, and each letter from $\mathcal{A}$ appears in the word $X_2X_3 \ldots X_k$ exactly d times. Besides, we assume that there are no external occurrences of τ in $\phi^n(1)$ for the pair (X_i, X_j) for each i and j. Let s_i be the number of internal occurrences of τ in X_i. In particular, $s_i = 0$, whenever $X_i = \epsilon$. Then $N_\phi^\tau(1) = s_1$ and for $n \geq 2$,

$$N_\phi^\tau(n) = \ell \cdot (d+\ell)^{n-2} \sum_{i=1}^{k} s_i.$$

Proof. The theorem is straightforward to prove by observing that for $n \geq 2$, $\phi^n(1)$ has $\ell\cdot(d+\ell)^{n-2}$ occurrences of each word X_i (see the proof of Theorem 1). □

Remark 5. *In order to use Theorem 3, we need to control the absence of external occurrences of a pattern* τ *for given* τ *(without internal dashes) and a morphism* ϕ*. To do this, we need, for any pair* (X_i, X_j)*, to consider all the words* $X_i W X_j$*, where* $|W| < |\tau| - 1$*, and* W *is a permutation of a number of words from the set* $\{X_1, X_2, \dots, X_k\}$*.*

The following examples are corollaries to Theorem 3.

Example 5. *If we consider the morphism* ϕ_w *defined in Section 7.1 and the pattern* $\tau = 12$ *then all the conditions of Theorems 3 hold. In this case* $d = \ell = 1$ *and* $s = (2, 1, 0)$*. Hence, the number of occurrences of the patterns* 12*, that is the number of rises, is given by* $N_{\phi_w}^{12}(1) = 2$ *and, for* $n \geq 2$*,* $N_{\phi_w}^{12}(n) = 3 \cdot 2^{n-2}$*. If* $\tau = 123$ *then we can apply the theorem to get that for* $n \geq 2$*,* $N_{\phi_w}^{123}(n) = 2^{n-2}$*.*

If we want to count the number of occurrences of the pattern $\tau = 21$*, that is the number of descents, then we cannot apply Theorem 1, since for instance, the pair* $(X_1, X_2) = (123, 13)$ *has an external occurrence of* τ*. However, it is obvious that the number of descents in* $\phi^n(1)$ *is equal to* $|\phi^n(1)| - N_{\phi_w}^{12}(1) - 1 = 3 \cdot 2^{n-2} - 1$*.*

Example 6. *If we consider the morphism* ϕ*:* $1 \to 1243$*,* $2 \to 3$*,* $3 \to \epsilon$*, and* $4 \to 124$ *then for the pattern* $\tau = 123$*, all the conditions of Theorems 3 hold. In this case* $d = \ell = 1$*,* $s = (1, 0, 0, 1)$*. Hence, for* $n \geq 1$*,* $N_{\phi}^{123}(n) = 2^{n-1}$*. For* $\tau = 321$ *we cannot apply Theorem 3, since the pair* (X_4, X_1) *has an external occurrence of* τ *(look at* $X_4 X_2 X_1 = 12\mathbf{431}243$*). Consideration of the words* $X_4 X_2$ *and* $X_4 X_1$ *implies that the theorem cannot be apply for the patterns* 132 *and* 231 *respectively. However, we can apply the theorem to the pattern* 213 *to prove that it does not occur in* $\phi^n(1)$ *for any* n*.*

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Paper VIII

The Peano curve and counting occurrences of some patterns

The Peano curve and counting occurrences of some patterns

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Abstract

We introduce *Peano words*, which are words corresponding to finite approximations of the Peano space filling curve. We then find the number of occurrences of certain patterns in these words.

8.1 Introduction and Background

We write permutations as words $\pi = a_1a_2\cdots a_n$, whose letters are distinct and usually consist of the integers $1, 2, \ldots, n$.

An occurrence of a pattern p in a permutation π is "classically" defined as a subsequence in π (of the same length as the length of p) whose letters are in the same relative order as those in p. Formally speaking, for $r \leq n$, we say that a permutation σ in the symmetric group $\mathcal{S}_n$ has an occurrence of the pattern $p \in \mathcal{S}_r$ if there exist $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that $p = \sigma(i_1)\sigma(i_2)\ldots\sigma(i_r)$ in reduced form. The *reduced form* of a permutation σ on a set $\{j_1, j_2, \ldots, j_r\}$, where $j_1 < j_2 < \cdots < j_r$, is a permutation σ_1 obtained by renaming the letters of the permutation σ so that j_i is renamed i for all $i \in \{1, \ldots, r\}$. For example, the reduced form of the permutation 3651 is 2431. The first case of classical patterns studied was that of permutations avoiding a pattern of length 3 in $\mathcal{S}_3$. Knuth [Knuth] found that, for any $\tau \in \mathcal{S}_3$, the number $|\mathcal{S}_n(\tau)|$ of n-permutations avoiding τ is C_n, the nth Catalan number. Later, Simion and Schmidt [SimSch] determined the number $|\mathcal{S}_n(P)|$ of permutations in $\mathcal{S}_n$ simultaneously avoiding any given set of patterns $P \subseteq \mathcal{S}_3$.

In [BabStein] Babson and Steingrímsson introduced *generalised permutation patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a "classical" pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if this pattern occurs in the permutation, then the letters in the permutation that correspond to 3 and 1 are adjacent. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563. A motivation for introducing these patterns in [BabStein] was the study of Mahonian

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statistics. A number of interesting results on generalised patterns were obtained in [Claes]. Relations to several well studied combinatorial structures, such as set partitions, Dyck paths, Motzkin paths and involutions, were shown there.

Burstein [Burstein] considered words instead of permutations. In particular, he found the number $|[k]^n(P)|$ of words of length n in a k-letter alphabet that avoid all patterns from a set $P \subseteq \mathcal{S}_3$ simultaneously. Burstein and Mansour [BurMans1] (resp. [BurMans2, BurMans3]) considered forbidden patterns (resp. generalized patterns) with repeated letters.

The most attention, in the papers on classical or generalized patterns, is paid to finding exact formulas and/or generating functions for the number of words or permutations avoiding, or having k occurrences of, certain patterns. In [KitMans] the present authors suggested another problem, namely counting the number of occurrences of certain patterns in certain words. These words were chosen to be the set of all finite approximations of a sequence generated by a *morphism* with certain restrictions. A motivation for this choice was the interest in studying classes of sequences and words that are defined by iterative schemes [Lothaire, Salomaa].

In the present paper we also study the number of occurrences of certain patterns in certain words. But here we choose these words to be the discrete analogue given by Evdokimov, of subdivision stages from which the *Peano curve* is obtained. We call these words the *Peano words.* The Peano curve was studied by the Italian mathematician Giuseppe Peano in 1890 as an example of a continuous space filling curve. We consider the Peano words and find the number of occurrences of the patterns 12, 21, 1^ℓ, $[x\text{-}y^\ell)$, $(x^\ell\text{-}y]$ and $[x\text{-}y^\ell\text{-}z]$, where $x, y, z \in \{1, 2, 3\}$, $y^\ell = y\text{-}y\text{-}\cdots\text{-}y$ (ℓ times), and "[" in $p = [x - w)$ indicates that in an occurrence of p, the letter corresponding to the x must be the first letter of the word.

8.2 The Peano curve and the Peano words

We follow [GelbOlm] and present a description (of a curve that fills the unit square $S = [0,1]\times[0,1]$) given in 1891 by the German mathematician D. Hilbert.

As indicated in Figure 8.3, the idea is to subdivide S and the unit interval $I = [0,1]$ into 4^n closed subsquares and subintervals, respectively, and to set up a correspondence between subsquares and subintervals so that inclusion relationships are preserved (at each stage of subdivision, if a square corresponds to an interval, then its subsquares correspond to subintervals of that interval).

We now define the continuous mapping f of I onto S: If $x \in I$, then at each stage of subdivision x belongs to *at least* one closed subinterval. Select either one (if there are two) and associate the corresponding square. In this way a decreasing sequence of closed squares is obtained corresponding to a decreasing sequence of closed intervals. This sequence of closed squares has the property that there is exactly one point belonging to all of them. This point is by definition $f(x)$. It can be shown that the point $f(x)$ is well-defined, that is, independent of any choice of intervals containing x; the range of f is S; and f

is continuous.

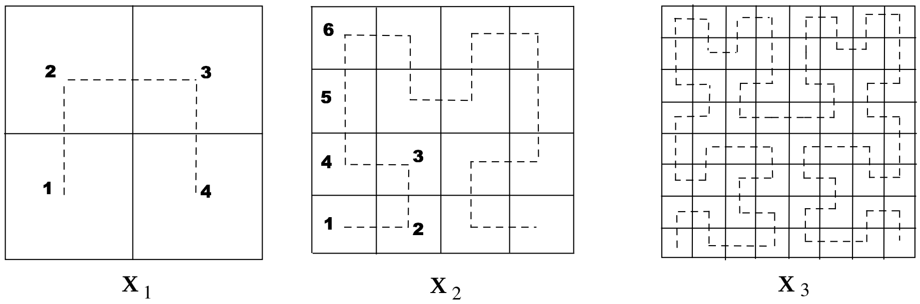

Figure 8.3: the Peano words

The following discrete analogue of the Peano curve was given by Evdokimov [Evdok]. We consider a subdivision stage (an iteration), go through the curve inside S starting in the point 1 (see Figure 8.3), and coding any movement "up" by 1, "right" by 2, "down" by 3, "left" by 4. Thus, we start with the first iteration $X_1 = 123$, the second iteration is $X_2 = 214112321233432$. More generally, it is easy to see that the n-th iteration is given by

$$X_n = \varphi_1(X_{n-1})1X_{n-1}2X_{n-1}3\varphi_2(X_{n-1}),$$

where the function $\varphi_1(A)$ reverses the letters in the word A and makes the substitution corresponding to the permutation 4123, that is, 1 becomes 4 etc. The function φ_2 does the same, except with 4123 replaced by 2341. In this paper, we are interested in the words X_n, for $n = 1, 2, \ldots$, which appear as the subdivision stages of the Peano curve. We call these words the Peano words.

8.3 The main results

It is easy to see that the length of the curve after the n-th iteration is $|X_n| = 4^n - 1$. Moreover, the following lemma holds.

Lemma 1. *The number of occurrences of the letters* 1, 2, 3 *and* 4 *in* X_n *is given by* 4^{n-1}, $4^{n-1} + 2^{n-1} - 1$, 4^{n-1} *and* $4^{n-1} - 2^{n-1}$ *respectively.*

Proof. Suppose d_1^n (resp. d_2^n, d_3^n, d_4^n) denote the number of occurrences of the letter 1 (resp. 2,3,4) in the word X_n. It is easy to see, using the way we construct X_n, that

$$\begin{pmatrix} d_1^n \\ d_2^n \\ d_3^n \\ d_4^n \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} d_1^{n-1} \\ d_2^{n-1} \\ d_3^{n-1} \\ d_4^{n-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Using the diagonalization of the matrix in the identity above, namely the fact that

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} -1/4 & 1/4 & -1/4 & 1/4 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix},$$

we get that the vector $(d_1^n, d_2^n, d_3^n, d_4^n)'$ is equal to the vector

$$(4^{n-1}, 4^{n-1} + 2^{n-1} - 1, 4^{n-1}, 4^{n-1} - 2^{n-1})'.$$

□

As a corollary to Lemma 1 we have the following.

Corollary 11. *The number of occurrences of the pattern* $\tau = \underbrace{1\text{-}1\text{-}\cdots\text{-}1}_{\ell} = 1^\ell$ *in* X_n *is equal to*

$$\binom{4^{n-1} - 2^{n-1}}{\ell} + 2\binom{4^{n-1}}{\ell} + \binom{4^{n-1} + 2^{n-1} - 1}{\ell}.$$

Proof. The number of occurrences of a subsequence $\underbrace{i\text{-}i\text{-}\cdots\text{-}i}_{\ell}$ in X_n, for $i = 1, 2, 3, 4$, is obviously given by $\binom{d_i^n}{\ell}$, where d_i^n is defined and determined in the proof of Lemma 1. The rest is easy to see. □

Definition 9. *Let* $r(A)$ *(resp.* $d(A)$*) denote the number of occurrences of the pattern* 12 *(resp.* 21*), that is the number of rises (resp. descents), in a word* A.

Lemma 2. *Suppose* $A = 1X3$ *and* $B = 2Y2$ *for some words* X *and* Y*. Then* $r(\varphi_1(A)) = d(A) + 1$, $d(\varphi_1(A)) = r(A) - 1$, $r(\varphi_2(B)) = d(B)$ *and* $d(\varphi_2(B)) = r(B)$.

Proof. If $\bar{A}$ and $\bar{B}$ denote the reverses of A and B then $r(\bar{A}) = d(A)$, $d(\bar{A}) = r(A)$, $r(\bar{B}) = d(B)$, and $d(\bar{B}) = r(B)$.

We consider two factorizations of each word $\bar{A}$ and $\bar{B}$. We can write $\bar{A}$ as

$$\bar{A} = 3A_1 \underbrace{1\ldots1}_{i_1} A_2 \underbrace{1\ldots1}_{i_2} A_3 \ldots A_k \underbrace{1\ldots1}_{i_k},$$

where A_i, for $i = 1, 2, \ldots, k$ is a word over the alphabet $\{2, 3, 4\}$, only A_1 can be the empty word ϵ, and $i_j \geq 1$ for $j = 1, 2 \ldots, k$. Also, we can write $\bar{A}$ as

$$\bar{A} = 3A_0' \underbrace{4\ldots4}_{i_1'} A_1' \underbrace{4\ldots4}_{i_2'} A_2' \ldots A_{k-1}' \underbrace{4\ldots4}_{i_k'} A_k' 1,$$

where $A_i^{'}$, for $i = 0, 1, \dots, k$ is a word over the alphabet $\{1, 2, 3\}$, only $A_0^{'}$ and $A_k^{'}$ can be ϵ, and $i_j^{'} \geq 1$ for $j = 1, 2 \dots, k$.

The word $\bar{B}$ can be written as

$$\bar{B} = 2B_0 \underbrace{1 \dots 1}_{j_1} B_1 \underbrace{1 \dots 1}_{j_2} B_2 \dots B_{\ell-1} \underbrace{1 \dots 1}_{j_\ell} B_\ell 2,$$

where B_i, for $i = 0, 1, \dots, \ell$, is a word over the alphabet $\{2, 3, 4\}$, only B_0 and B_ℓ can be ϵ, and $j_i \geq 1$ for $i = 1, 2 \dots, \ell$. Also, $\bar{B}$ can be written as

$$\bar{B} = 2B_0^{'} \underbrace{4 \dots 4}_{j_1^{'}} B_1^{'} \underbrace{4 \dots 4}_{j_2^{'}} B_2^{'} \dots B_{\ell-1}^{'} \underbrace{4 \dots 4}_{j_\ell^{'}} B_\ell^{'} 2,$$

where $B_i^{'}$, for $i = 0, 1, \dots, \ell$, is a word over the alphabet $\{1, 2, 3\}$, only $B_0^{'}$ and $B_\ell^{'}$ can be ϵ, and $j_i^{'} \geq 1$ for $i = 1, 2 \dots, \ell$.

It follows from the definitions that $\varphi_1(A)$ and $\varphi_1(B)$ (resp. $\varphi_2(A)$ and $\varphi_2(B)$) are obtained from $\bar{A}$ and $\bar{B}$ by permuting the letters with the function π_1 (resp. π_2) that acts as the permutation 4123 (resp. 2341).

We now consider the first factorizations of $\bar{A}$ and $\bar{B}$, and the function π_1. It is easy to see that if W is equal to A_i, or B_i, or $3A_1$, or $2B_0$, or $B_\ell 2$, then $r(W) = r(\pi_1(W))$ and $d(W) = d(\pi_1(W))$, since π_1 is an order-preserving function when it acts from the set $\{2, 3, 4\}$ to the set $\{1, 2, 3\}$. From the other hand, occurrences of the rises 12, 13 and 14 (resp. the descents 41, 31 and 21) in $\bar{A}$ and $\bar{B}$, give occurrences of the descents 41, 42 and 43 (resp. the rises 34, 24 and 14) in $\pi_1(\bar{A})$ and $\pi_1(\bar{B})$ respectively. If we now read the first factorizations of $\bar{A}$ and $\bar{B}$ from the left to the right, then the occurrences of the subwords $a1$ alternate with the occurrences of the subwords $1b$, where $a, b \in \{2, 3, 4\}$. Moreover, in the factorization of $\bar{A}$, we begin and end with the subword $a1$ for some $a \in \{2, 3, 4\}$, which gives that $d(A) + 1 = r(\bar{A}) + 1 = r(\pi_1(\bar{A})) = r(\varphi_1(A))$ and $r(A) - 1 = d(\bar{A}) - 1 = d(\pi_1(\bar{A})) = d(\varphi_1(A))$; in the factorization of $\bar{B}$, we begin with the subword $a1$ and end with the subword $1b$ for some $a, b \in \{2, 3, 4\}$, which gives that $d(B) = r(\bar{B}) = r(\pi_1(\bar{B})) = r(\varphi_1(B))$ and $r(B) = d(\bar{B}) = d(\pi_1(\bar{B})) = d(\varphi_1(B))$.

If we consider the second factorizations of $\bar{A}$ and $\bar{B}$, and the function π_2, one can see that if W is equal to $A_i^{'}$, or $B_i^{'}$, or $3A_0^{'}$, or $A_k^{'}1$, or $2B_0^{'}$, or $B_\ell^{'}2$, then $r(W) = r(\pi_2(W))$ and $d(W) = d(\pi_2(W))$, since π_2 is an order-preserving function when it acts from the set $\{1, 2, 3\}$ to the set $\{2, 3, 4\}$. From the other hand, occurrences of the rises 14, 24 and 34 (resp. the descents 41, 42 and 43) in $\bar{A}$ and $\bar{B}$, give occurrences of the descents 21, 31 and 41 (resp. the rise 12, 13 and 14) in $\pi_2(\bar{A})$ and $\pi_2(\bar{B})$ respectively. If we now read the second factorizations of $\bar{A}$ and $\bar{B}$ from the left to the right, then the occurrences of the subwords $a4$ alternate with the occurrences of the subwords $4b$, where $a, b \in \{1, 2, 3\}$. Moreover, in both cases, we begin with the subword $a4$ and end with the subword $4b$ for some $a, b \in \{1, 2, 3\}$, which gives that $d(A) = r(\bar{A}) = r(\pi_2(\bar{A})) = r(\varphi_2(A))$, $r(A) = d(\bar{A}) = d(\pi_2(\bar{A})) = d(\varphi_2(A))$, $d(B) = r(\bar{B}) = r(\pi_2(\bar{B})) = r(\varphi_2(B))$ and $r(B) = d(\bar{B}) = d(\pi_2(\bar{B})) = d(\varphi_2(B))$. □

Theorem 1. *Let r_n (resp. d_n) be the number of occurrences of the pattern* 12 (*resp.* 21) *in X_n. Then for all $k \geq 0$,*

$$\begin{aligned} r_{2k+1} &= \tfrac{2}{5}(4 \cdot 16^k + 1), \\ r_{2k+2} &= \tfrac{2}{5}(16^{k+1} - 1), \\ d_{2k+1} &= \tfrac{8}{5}(16^k - 1), \\ d_{2k+2} &= \tfrac{2}{5}(16^{k+1} - 1). \end{aligned}$$

Proof. Using the properties of φ_1 and φ_2, as well as the way we construct X_n, it is easy to check by induction, that X_{2k+1} and X_{2k+2} can be factorized as follow:

$$X_{2k+1} = \underbrace{1X^{(1)}1}_{\varphi_1(X_{4k})}\,1\,\underbrace{2Y^{(1)}2}_{X_{4k}}\,2\,\underbrace{2Y^{(1)}2}_{X_{4k}}\,3\,\underbrace{3Z^{(1)}3}_{\varphi_2(X_{4k})},$$

$$X_{2k+2} = \underbrace{2X^{(2)}4}_{\varphi_1(X_{4k+1})}\,1\,\underbrace{1Y^{(2)}3}_{X_{4k+1}}\,2\,\underbrace{1Y^{(2)}3}_{X_{4k+1}}\,3\,\underbrace{4Z^{(2)}2}_{\varphi_2(X_{4k+1})},$$

where $X^{(i)}$, $Y^{(i)}$ and $Z^{(i)}$ are some words for $i = 1, 2$.

Suppose we know r_{2k+1} and d_{2k+1} for some k. Since $X_{2k+1} = 1W3$ for some word W, using Lemma 2 and the factorization of the word X_{2k+2}, we can find r_{2k+2} and d_{2k+2}. Indeed, $\varphi_1(X_{4k+1})$ has $d_{2k+1} + 1$ rises and $r_{2k+1} - 1$ descents; $\varphi_2(X_{4k+1})$ has d_{2k+1} rises and r_{2k+1} descents; two subwords X_{2k+1} give $2r_{2k+1}$ rises and $2d_{2k+1}$ descents. Besides, we have some extra rises and descents appeared between different blocks of the decomposition. They are one extra rise between the letter 3 and the subword $\varphi_2(X_{4k+1})$, and 3 extra descents between the subword $\varphi_1(X_{4k+1})$ and the letter 1, the subword X_{4k+1} and the letter 2, the letter 2 and the subword X_{4k+1}. Thus, $r_{2k+2} = 2r_{2k+1} + 2d_{2k+1} + 2$ and $d_{2k+2} = 2r_{2k+1} + 2d_{2k+1} + 2$, which shows, in particular, that for even n, in X_n, the number of rises is equal to the number of descents.

We now analyze the factorization of X_{2k+3}, which is similar to that of X_{2k+1}. Using the fact that $X_{2k+2} = 2W^{'}2$ for some word $W^{'}$ and Lemma 2, we can find r_{2k+3} and d_{2k+3}. Indeed, we can use the similar considerations as above to get $r_{2k+3} = 2r_{2k+2} + 2d_{2k+2} + 2 = 8r_{2k+1} + 8d_{2k+1} + 10$ and $d_{2k+3} = 2r_{2k+2} + 2d_{2k+2} = 8r_{2k+1} + 8d_{2k+1} + 8$. Thus, if x_k denote the vector $(r_{2k+1}, d_{2k+1})^{'}$ then

$$x_{k+1} = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} x_k + \begin{pmatrix} 10 \\ 8 \end{pmatrix},$$

with $x_0 = (2, 0)$, since in $X_1 = 123$, there are two rises and no descents. This recurrence relation, using diagonalization of the matrix in it, leads us to

$$x_k = (\frac{2}{5}(4 \cdot 16^k + 1), \frac{8}{5}(16^k - 1))^{'}.$$

Finally, $r_{2k+2} = d_{2k+2} = 2r_{2k+1} + 2d_{2k+1} + 2 = \frac{2}{5}(16^{k+1} - 1)$. □

Let $N_\tau(W)$ denote the number of occurrences of the pattern τ in the word W.

Using Lemma 1 and the proof of Theorem 1, we can count, for X_n, the number of occurrences of the patterns $\tau_1(x,y) = [x\text{-}y^\ell)$, $\tau_2(x,y) = x^\ell\text{-}y]$ and $\tau_3(x,y,z) = [x\text{-}y^\ell\text{-}z]$, where $x,y,z \in \{1,2,3\}$, $y^\ell = y\text{-}y\text{-}\cdots\text{-}y$ (ℓ times), and "[" in $p = [x - w)$ indicates that in an occurrence of p, the letter corresponding to the x must be the first letter of the word, whereas "]" in $\tau_3(x,y,z)$ indicates that in an occurrence of $\tau_3(x,y,z)$, the letter corresponding to the z must be the last (rightmost) letter of the word.

If we consider, for instance, the pattern $\tau_1(1,2) = [1\text{-}2^\ell)$ then the letter 1 in this pattern must correspond to the leftmost letter of the word X_n. Now if $n = 2k+1$ then from the proof of Theorem 1 $X_n = 1W$ for some word W, which means that to the sequence 2^ℓ there can correspond any subsequence i^ℓ in X_n, where $i = 2,3,4$. Thus, using Lemma 1 and the way we prove Corollary 11, there are $\binom{4^{2k}-2^{2k}}{\ell} + \binom{4^{2k}}{\ell} + \binom{4^{2k}+2^{2k}-1}{\ell}$ occurrences of the pattern $\tau_1(1,2)$ in X_{2k+1}. If $n = 2k+2$ then $X_n = 2W$ for some word W and for the sequence 2^ℓ there correspond any subsequence i^ℓ in X_n, where $i = 3,4$. Thus, $N_{\tau_1(1,2)}(X_{2k+2}) = \binom{4^{2k}-2^{2k}}{\ell} + \binom{4^{2k}}{\ell}$.

In the example above, as well as in the following considerations, we assume ℓ to be greater then 0. If $\ell = 0$ then obviously $N_{\tau_1(x,y)}(X_n) = N_{\tau_2(x,y)}(X_n) = 1$, whereas $N_{\tau_3(x,y,z)}(X_n)$ is equal to 1 if $x < z$ and $n = 2k+1$, or $x = z$ and $n = 2k+2$, and it is equal to 0 otherwise.

When we consider $\tau_3(x,y,z)(X_n)$, we observe that since $X_{2k+2} = 2W2$ for some W, $N_{\tau_3(x,y,z)}(X_{2k+2}) = 0$, whenever $x \neq z$. Also, since $X_{2k+1} = 1W3$ for some W, $N_{\tau_3(x,y,z)}(X_{2k+1}) = 0$, whenever $x \geq z$.

Let us consider the pattern $\tau_3(2,1,3) = [2\text{-}1^\ell\text{-}3]$. As it was mentioned before, $N_{\tau_3(2,1,3)}(X_{2k+2}) = 0$. But, if we consider $X_{2k+1} = 1W3$, then it is easy to see that $N_{\tau_3(2,1,3)}(X_{2k+1}) = 0$, since the leftmost letter of X_{2k+1} is the least letter, which means that it cannot correspond to the letter 2 in the pattern. As one more example, we can consider the pattern $\tau_3(1,1,2) = [1\text{-}1^\ell\text{-}2]$. We are only interested in case $X_n = X_{2k+1}$, since $N_{\tau_3(1,1,2)}(X_{2k+2}) = 0$. The number of occurrences of the pattern is obviously given by the number of ways to choose ℓ letters among $4^{2k} - 1$ letters 1 (totally, there are 4^{2k} letters 1 according to Lemma 1, but we cannot consider the leftmost 1 since it corresponds to the leftmost 1 in the pattern). Thus, $N_{\tau_3(1,1,2)}(X_{2k+1}) = \binom{4^{2k}-1}{\ell}$.

All the other cases of x, y, z in the patterns $\tau_1(x,y)$, $\tau_2(x,y)$ and $\tau_3(x,y,z)$ can be considered in the same way. Let S_1 and S_2 denote the following:

$$S_1 = \binom{4^{2k} - 2^{2k}}{\ell} + \binom{4^{2k}}{\ell} + \binom{4^{2k} + 2^{2k} - 1}{\ell}, \quad S_2 = \binom{4^{2k+1}}{\ell} + \binom{4^{2k+1} - 2^{2k+1}}{\ell}.$$

The tables below give all the results concerning the patterns under consideration, except those triples (x,y,z), for which $N_{\tau_3(x,y,z)}(X_n) = 0$ for all n.

x	y	$N_{\tau_1(x,y)}(X_{2k+1})$	$N_{\tau_2(x,y)}(X_{2k+1})$	$N_{\tau_1(x,y)}(X_{2k+2})$	$N_{\tau_2(x,y)}(X_{2k+2})$
1	1	$\binom{4^{2k}-1}{\ell}$	$\binom{4^{2k}-1}{\ell}$	$\binom{4^{2k+1}+2^{2k+1}-1}{\ell}$	$\binom{4^{2k+1}+2^{2k+1}-1}{\ell}$
1	2	S_1	$\binom{4^{2k}}{\ell}+\binom{4^{2k}+2^{2k}-1}{\ell}$	S_2	$\binom{4^{2k+1}}{\ell}$
2	1	0	$\binom{4^{2k}-2^{2k}}{\ell}$	$\binom{4^{2k+1}}{\ell}$	S_2

x	y	z	$N_{\tau_3(x,y,z)}(X_{2k+1})$	$N_{\tau_3(x,y,z)}(X_{2k+2})$
1	1	1	0	$\binom{4^{2k+1}-2}{\ell}$
1	1	2	$\binom{4^{2k}-1}{\ell}$	0
1	2	1	0	S_2
1	2	2	$\binom{4^{2k}-1}{\ell}$	0
2	1	2	0	$\binom{4^{2k+1}}{\ell}$
1	2	3	$\binom{4^{2k}+2^{2k}-1}{\ell}$	0
1	3	2	$\binom{4^{2k}-2^{2k}}{\ell}$	0

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Paper IX

The sigma-sequence and counting occurrences of some patterns, subsequences and subwords

The sigma-sequence and counting occurrences of some patterns, subsequences and subwords

Sergey Kitaev [1]

Abstract

We consider *sigma-words*, which are words used by Evdokimov in the construction of the sigma-sequence [Evdok]. We then find the number of occurrences of certain patterns, subsequences and subwords in these words.

9.1 Introduction and Background

We write permutations as words $\pi = a_1a_2\cdots a_n$, whose letters are distinct and usually consist of the integers $1, 2, \dots, n$.

An occurrence of a pattern p in a permutation π is "classically" defined as a subsequence in π (of the same length as the length of p) whose letters are in the same relative order as those in p. Formally speaking, for $r \leq n$, we say that a permutation σ in the symmetric group $\mathcal{S}_n$ has an occurrence of the pattern $p \in \mathcal{S}_r$ if there exist $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that $p = \sigma(i_1)\sigma(i_2)\dots\sigma(i_r)$ in reduced form. The *reduced form* of a permutation σ on a set $\{j_1, j_2, \dots, j_r\}$, where $j_1 < j_2 < \cdots < j_r$, is a permutation σ_1 obtained by renaming the letters of the permutation σ so that j_i is renamed i for all $i \in \{1, \dots, r\}$. For example, the reduced form of the permutation 3651 is 2431. The first case of classical patterns studied was that of permutations avoiding a pattern of length 3 in $\mathcal{S}_3$. Knuth [Knuth] found that, for any $\tau \in \mathcal{S}_3$, the number $|\mathcal{S}_n(\tau)|$ of n-permutations avoiding τ is C_n, the nth Catalan number. Later, Simion and Schmidt [SimSch] determined the number $|\mathcal{S}_n(P)|$ of permutations in $\mathcal{S}_n$ simultaneously avoiding any given set of patterns $P \subseteq \mathcal{S}_3$.

In [BabStein] Babson and Steingrímsson introduced *generalised permutation patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a "classical" pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if this pattern occurs in the permutation, then the letters in the permutation that correspond to 3 and 1 are adjacent. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563. A motivation for introducing these patterns in [BabStein] was the study of Mahonian statistics. A number of interesting results on generalised patterns were obtained in [Claes]. Relations to several well studied combinatorial structures, such as set partitions, Dyck paths, Motzkin paths and involutions, were shown there.

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Burstein [Burstein] considered words instead of permutations. In particular, he found the number $|[k]^n(P)|$ of words of length n in a k-letter alphabet that avoid all patterns from a set $P \subseteq \mathcal{S}_3$ simultaneously. Burstein and Mansour [BurMans1] (resp. [BurMans2, BurMans3]) considered forbidden patterns (resp. generalized patterns) with repeated letters.

The most attention, in the papers on classical or generalized patterns, is paid to finding exact formulas and/or generating functions for the number of words or permutations avoiding, or having k occurrences of, certain patterns. In [KitMans1] the authors suggested another problem, namely counting the number of occurrences of certain patterns in certain words. These words were chosen to be the set of all finite approximations of a sequence generated by a *morphism* with certain restrictions. A motivation for this choice was the interest in studying classes of sequences and words that are defined by iterative schemes [Lothaire, Salomaa]. In [KitMans2] the authors also studied the number of occurrences of certain patterns in certain words. But there they choose these words to be the subdivision stages from which the *Peano curve* is obtained. The authors called these words the *Peano words*. The Peano curve was studied by the Italian mathematician Giuseppe Peano in 1890 as an example of a continuous space filling curve.

In the present paper we consider the *sigma-words*, which are words used by Evdokimov in construction of the *sigma-sequence* [Evdok]. Evdokimov used this sequence to construct chains of maximal length in the n-dimensional unit cube. Independent interest to the sigma-sequence appears in connection with the well-known *Dragon curve*, discovered by physicist John E. Heighway and defined as follows: we fold a sheet of paper in half, then fold in half again, and again, etc. and then unfold in such way that each crease created by the folding process is opened out into a 90-degree angle. The "curve" refers to the shape of the partially unfolded paper as seen edge on. If one travels along the curve, some of the creases will represent turns to the left and others turns to the right. Now if 1 indicates a turn to the right, and 2 to the left, and we start travelling along the curve indicating the turns, we get the sigma-sequence [Evdokimov]. In [Kitaev] the sigma-sequence was studied from another point of view. It was proved there that this sequence cannot be defined by iterated morphism.

Since the sigma-sequence w_σ is a sequence in a 2-letter alphabet, we consider patterns in 2-letter alphabets. Moreover, the patterns in a 1-letter alphabet (for example 1-1-1) correspond to two subsequences (for this example, these subsequences are 1-1-1 and 2-2-2), whereas the patterns in a 2-letter alphabet (with at least one letter 2) uniquely determine the subsequences in w_σ that correspond to them, and conversely. For example, an occurrence of the pattern 1-2-1 is an occurrence of the subsequence 1-2-1, whereas an occurrence of the subsequence (subword) 211 is an occurrence of the pattern 211. Thus, any our result for a pattern, can be interpreted in term of subsequences or subwords, depending on the context, and conversely.

In our paper we give either an explicit formula or recurrence relation for the number of occurrences for some classes of patterns, subwords and subsequences in the sigma-words. In particular, Theorem 1, allows to find the number of occurrences of an arbitrary generalized pattern without internal dashes of length ℓ, provided we know four certain numbers that can be easily calculated for the sigma-words C_k, D_k, C_{k+1} and D_{k+1} (to be defined below), where $k = \lceil \log_2 \ell \rceil$. Theorem 2 gives a recurrence relation for counting occurrences of patterns of the form τ_1-τ_2. In Section 9.6 we discuss occurrences of patterns of the form τ_1-τ_2-$\cdots$-τ_k, where the pattern τ_i does not overlap with the patterns τ_{i-1} and τ_{i+1} for $i = 1, 2, \dots, k-1$. Finally, Section 9.7 deals with patterns of the form $[\tau_1$-τ_2-$\cdots$-$\tau_k]$, $[\tau_1$-τ_2-$\cdots$-$\tau_k)$ and $(\tau_1$-τ_2-$\cdots$-$\tau_k]$ in Babson and Steingrímsson notation, which means that we use "[x" in a pattern p to indicate that in an occurrence of p, the letter corresponding to the x must be the first letter of a word under consideration, whereas if we use "y]", we mean that the letter corresponding to y must be the last (rightmost) letter in the word.

9.2 Preliminaries

In [Evdok, Yab], Evdokimov constructed chains of maximal length in the n-dimensional unit cube using the *sigma-sequence*. The sigma-sequence w_σ was defined there by the following inductive scheme:

$$C_1 = 1, \qquad D_1 = 2$$
$$C_{k+1} = C_k 1 D_k, \qquad D_{k+1} = C_k 2 D_k$$
$$k = 1, 2, \dots$$

and $w_\sigma = \lim_{k \to \infty} C_k$. Thus, the initial letters of w_σ are 11211221112212.... We call the words C_k the *sigma words*. The first four values of the sequence $\{C_k\}_{k \geq 1}$ are 1, 112, 1121122, 112112211122122.

In [Kitaev] an equivalent definition of w_σ was given: any natural number $n \neq 0$ can be presented unambiguously as $n = 2^t(4s + \sigma)$, where $\sigma < 4$, and t is the greatest natural number such that 2^t divides n. If n runs through the natural numbers then σ runs through some sequence consisting of 1 and 3. If we substitute 3 by 2 in this sequence, we get w_σ.

In this paper we count occurrences of patterns in the sigma-words, which are particular initial subwords of w_σ. However, the challenging question is to find the number of occurrences of patterns or subwords in an arbitrary initial subword of w_σ, or more generally, in a subword of w_σ starting in the position i and ending in the position j.

It turns out that for counting occurrences of certain patterns or subwords in C_n, one needs to know the number of occurrences of certain patterns in D_n. So, in the most cases, we give results for both C_n and D_n. However, our main purpose is the words C_n for $n \geq 1$, and in some propositions and examples we do not consider D_n.

In what follows, we give initial values for the words C_i and D_i:

$C_1 = 1$ $\qquad D_1 = 2$

$C_2 = 112$ $\qquad D_2 = 122$

$C_3 = 1121122$ $\qquad D_3 = 1122122$

$C_4 = 112112211122122$ $\qquad D_4 = 112112221122122$

$C_5 = 1121122111221221112112221122122$

$D_5 = 1121122111221222112112221122122$

We now give some other definitions.

A *descent* (resp. *rise*) in a word $w = a_1a_2 \dots a_n$ is an i such that $a_i > a_{i+1}$ (resp. $a_i < a_{i+1}$). It follows from the definitions that an occurrence of a descent (resp. rise) is an occurrence of the pattern 21 (resp. 12).

Let c_n^τ (resp. d_n^τ) denote the number of occurrences of the pattern τ in C_n (resp. D_n).

Suppose a word $W = AaB$, where A and B are some words of the same length, and a is a letter. We define the *kernel of order k* for the word W to be the subword consisting of the $k-1$ rightmost letters of A, the letter a, and the $k-1$ leftmost letters of B. We denote it by $\mathcal{K}_k(W)$. For example, $\mathcal{K}_3(111211221) = 12112$. If $|A| < k-1$ then we assume $\mathcal{K}_k(W) = \epsilon$, that is, the kernel in this case is the empty word. Also, $m_k(\tau, W)$ denotes the number of occurrences of the pattern (or the word, or the subsequence depending on the context) τ in $\mathcal{K}_k(W)$.

We denote $x\text{-}x\text{-}\cdots\text{-}x$ (ℓ times) by x^ℓ. Also, $\lceil a \rceil$ denotes the least natural number b such that $a \leq b$.

9.3 Patterns 1-1-$\cdots$-1, 1-2 and 2-1

It is easy to see that $|C_n| = |D_n| = 2^n - 1$. The following lemma gives the number of the letters 1 and 2 in C_n and D_n.

Lemma 1. *The number of 1s (resp. 2s) in C_n is 2^{n-1} (resp. $2^{n-1}-1$). The number of 1s (resp. 2s) in D_n is $2^{n-1}-1$ (resp. 2^{n-1}).*

Proof. It is enough to find the number of 1s c_n and d_n in C_n and D_n respectively, since the number of 2s in C_n and D_n are obviously equal to $|C_n| - c_n$ and $|D_n| - d_n$ respectively.

It is easy to see from the structure of C_n and D_n that

$$\begin{cases} c_n = c_{n-1} + d_{n-1} + 1, \\ d_n = c_{n-1} + d_{n-1}, \end{cases}$$

together with $c_1 = 1$ and $d_1 = 0$. The solution to this recurrence is $c_n = 2^{n-1}$ and $d_n = 2^{n-1} - 1$. □

Proposition 1. *The number occurrences of the subsequence* 1^k *(resp.* 2^k*) in* C_n *is* $\binom{2^{n-1}}{k}$ *(resp.* $\binom{2^{n-1}-1}{k}$*). Thus, the number of occurrences of the pattern* 1^k *in* C_n *is equal to*

$$c_n^{1^k} = \binom{2^{n-1}}{k} + \binom{2^{n-1}-1}{k} = \frac{2^n - k}{2^{n-1}-k}\binom{2^{n-1}-1}{k}.$$

Proof. From Lemma 1, there are 2^{n-1} (resp. $2^{n-1}-1$) occurrences of the letter 1 (resp. 2) in C_n, and thus there are $\binom{2^{n-1}}{k}$ (resp. $\binom{2^{n-1}-1}{k}$) occurrences of the subsequence 1^k (resp. 2^k) there. □

Proposition 2. *We have that for all* $n \geq 2$, $c_n^{1\text{-}2} = d_n^{1\text{-}2} = 2 \cdot 4^{n-2} + (n-2) \cdot 2^{n-2}$, *and* $c_n^{2\text{-}1} = d_n^{2\text{-}1} = 2 \cdot 4^{n-2} - n \cdot 2^{n-2}$.

Proof. Let us first consider the pattern 1-2. An occurrence of this pattern in $C_n = C_{n-1}1D_{n-1}$ is either inside C_{n-1}, or inside D_{n-1}, or the letter 1 is from the word $C_{n-1}1$, whereas the letter 2 is from the word D_{n-1}. Thus

$$c_n^{1\text{-}2} = c_{n-1}^{1\text{-}2} + d_{n-1}^{1\text{-}2} + \{\text{ (the number of 1s in } C_{n-1}) + 1\} \cdot \{\text{ the number of 2s in } D_{n-1}\}.$$

Using the same considerations for $D_n = C_{n-1}2D_{n-1}$, one can get

$$d_n^{1\text{-}2} = c_{n-1}^{1\text{-}2} + d_{n-1}^{1\text{-}2} + \{\text{ the number of 1s in } C_{n-1}\} \cdot \{\text{ (the number of 2s in } D_{n-1}) + 1\}.$$

The number of 1s and 2s in C_{n-1} and D_{n-1} is given in Lemma 1. So,

$$\begin{cases} c_n^{1\text{-}2} = c_{n-1}^{1\text{-}2} + d_{n-1}^{1\text{-}2} + 2^{n-2} \cdot (2^{n-2}+1) \\ d_n^{1\text{-}2} = c_{n-1}^{1\text{-}2} + d_{n-1}^{1\text{-}2} + 2^{n-2} \cdot (2^{n-2}+1) \end{cases} \Leftrightarrow$$

$$\begin{pmatrix} c_n^{1\text{-}2} \\ d_n^{1\text{-}2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_{n-1}^{1\text{-}2} \\ d_{n-1}^{1\text{-}2} \end{pmatrix} + \begin{pmatrix} 2^{n-2} \cdot (2^{n-2}+1) \\ 2^{n-2} \cdot (2^{n-2}+1) \end{pmatrix} \quad (9.1)$$

together with $c_2^{1\text{-}2} = 2$ and $d_2^{1\text{-}2} = 2$. Here, and several times in what follows, we need to solve recurrence relations of the form

$$x_n = Ax_{n-1} + b,$$

where A is a matrix, and x_n, x_{n-1} and b are some vectors, where b sometimes depends on n. We recall from linear algebra that such relations can be solved by diagonalization of the matrix A, that is, by writing $A = VDV^{-1}$, where D is a diagonal matrix consisting of eigenvalues of A, and the columns of V are eigenvectors of A. For example, if A is a 2×2 matrix that consists of 1s, then we use

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

for computing powers of A, and thus for solving the recurrence relations. For the recurrence 9.1, we get that for all $n \geq 2$, $c_n^{1\text{-}2} = d_n^{1\text{-}2} = 2 \cdot 4^{n-2} + (n-2) \cdot 2^{n-2}$.

In the same manner, we can get that for the pattern 2-1,

$$\begin{cases} c_n^{2\text{-}1} = c_{n-1}^{2\text{-}1} + d_{n-1}^{2\text{-}1} + 2^{n-2} \cdot (2^{n-2} - 1), \\ d_n^{2\text{-}1} = c_{n-1}^{2\text{-}1} + d_{n-1}^{2\text{-}1} + 2^{n-2} \cdot (2^{n-2} - 1), \end{cases}$$

together with $c_3^{2\text{-}1} = 2$ and $d_3^{2\text{-}1} = 2$. This gives, that for all $n \geq 2$, $c_n^{2\text{-}1} = d_n^{2\text{-}1} = 2 \cdot 4^{n-2} - n \cdot 2^{n-2}$. □

Proposition 2 shows that asymptotically, the numbers of occurrences of the patterns, or the subsequences, 1-2 and 2-1 in C_n or D_n are equal.

9.4 Patterns without internal dashes

Recall the definitions in Section 9.2.

Theorem 1. *Let $\tau = \tau_1\tau_2 \ldots \tau_\ell$ be an arbitrary generalized pattern without internal dashes that consists of 1s and 2s. Suppose $k = \lceil \log_2 \ell \rceil$, $a = m_\ell(\tau, D_k 1 C_k)$, and $b = m_\ell(\tau, D_k 2 C_k)$. Then for $n > k+1$, we have*

$$c_n^\tau = (a + b + c_{k+1}^\tau + d_{k+1}^\tau) \cdot 2^{n-k-2} - b,$$

$$d_n^\tau = (a + b + c_{k+1}^\tau + d_{k+1}^\tau) \cdot 2^{n-k-2} - a.$$

Proof. Suppose $n > k+1$. In this case, $C_n = C_{n-1} 1 D_{n-1} = W_1 \mathcal{K}_\ell(D_k 1 C_k) W_2$, for some words W_1 and W_2 such that $|W_1| = |W_2|$. Because of the definition of the kernel $\mathcal{K}_\ell(D_k 1 C_k)$, an occurrence of the pattern τ in C_n is in either C_{n-1}, or D_{n-1}, or $\mathcal{K}_k(D_k 1 C_k)$ (from the definitions $|C_{n-1} \cap \mathcal{K}_k(D_k 1 C_k)| = |D_{n-1} \cap \mathcal{K}_k(D_k 1 C_k)| = \ell - 1$ and thus these intersections cannot be an occurrence of τ). So,

$$c_n^\tau = c_{n-1}^\tau + d_{n-1}^\tau + a,$$

whereas in the same way, we can obtain that

$$d_n^\tau = c_{n-1}^\tau + d_{n-1}^\tau + b.$$

By solving these recurrence relations, we get the desirable. □

In particular, Theorem 1 is valid for $\ell = 1$, in which case the number of occurrences of τ in C_n (or D_n) is the number of letters in C_n (or D_n). Indeed, in this case, $k = 0$, $a = b = c_1^1 = d_1^1 = 1$, hence $c_n^1 = d_n^1 = 2^n - 1 = |C_n| = |D_n|$. Also, as a corollary to Theorem 1 we have, that if $a = b = c_{k+1}^\tau = d_{k+1}^\tau = 0$ for some pattern τ, then this pattern never appears in sigma-sequence.

All of the following examples are corollaries to Theorem 1.

Example 1. *Suppose $\tau = 12$. We have that $k = 1$, $a = m_2(12, D_1 1 C_1) = 0$ and $b = m_2(12, D_1 2 C_1) = 0$. Besides, $c_2^{12} = 1$ and $d_2^{12} = 1$. Thus using Theorem 1, for all $n > 2$, $c_n^{12} = 2^{n-2}$. So, the number of rises in C_n is equal to 2^{n-2}, for $n \geq 2$.*

If $\tau = 21$, again $k = 1$, but now $a = m_2(21, D_1 1 C_1) = 1$, $b = m_2(21, D_1 2 C_1) = 1$. Besides, $c_3^{21} = 1$ and $d_3^{21} = 1$. From Theorem 1, for all $n > 3$, $c_n^{21} = 2^{n-2} - 1$, which shows that the number of descents in C_n is one less than the number of rises.

Since in both cases $a = b$, using the recurrences in Theorem 1, we have that $c_n^{12} = d_n^{12} = 2^{n-2}$, whereas $c_n^{21} = d_n^{21} = 2^{n-2} - 1$.

Example 2. *Suppose $\tau = 112$. We have that $k = 2$, $a = m_3(112, D_2 1 C_2) = 0$, and $b = m_3(112, D_2 2 C_2) = 0$. Besides, $c_3^{112} = 2$ and $d_3^{112} = 1$. Now, from Theorem 1, we have that for all $n > 3$, $c_n^{112} = d_n^{112} = 3 \cdot 2^{n-4}$.*

Example 3. *Suppose $\tau = 221$. We have that $k = 2$, $a = m_3(221, D_2 1 C_2) = 1$, and $b = m_3(221, D_2 2 C_2) = 1$. Besides, $c_3^{221} = 0$ and $d_3^{221} = 1$. Now, from Theorem 1, we have that for all $n > 3$, $c_n^{221} = d_n^{221} = 3 \cdot 2^{n-4} - 1$.*

Example 4. *If $\tau = 2212221$ then $k = 3$, $a = m_7(221, D_3 1 C_3) = 0$, $b = m_7(221, D_3 2 C_3) = 1$, $c_4^{2212221} = 0$, and $d_4^{2212221} = 0$. Thus for $n \geq 4$, $c_n^{2212221} = 2^{n-4} - 1$.*

9.5 Patterns of the form τ_1-τ_2

Theorem 2. *Let $p = \tau_1$-τ_2 be a generalized pattern such that $|\tau_1| = k_1$ and $|\tau_2| = k_2$. Suppose $k = \lceil \log_2(k_1 + k_2 - 1) \rceil$. The following denote the number of occurrences of the subwords τ_1 and τ_2 in the kernels:*

$$\begin{array}{ll} a_{\tau_1} = m_{k_1}(\tau_1, D_k 1 C_k) & a_{\tau_2} = m_{k_2}(\tau_2, D_k 1 C_k) \\ b_{\tau_1} = m_{k_1}(\tau_1, D_k 2 C_k) & b_{\tau_2} = m_{k_2}(\tau_2, D_k 2 C_k) \end{array}$$

Also, let r_1^a (resp. r_2^a, r_1^b, r_2^b) denote the number of occurrences of overlapping subwords τ_1 and τ_2 in the word $D_k 1 C_k$ (resp. $D_k 1 C_k$, $D_k 2 C_k$, $D_k 2 C_k$), where $\tau_1 \in \mathcal{K}_{k_1}(D_k 1 C_k)$ and $\tau_2 \in C_k$ (resp. $\tau_1 \in D_k$ and $\tau_2 \in \mathcal{K}_{k_2}(D_k 1 C_k)$, $\tau_1 \in \mathcal{K}_{k_1}(D_k 2 C_k)$ and $\tau_2 \in C_k$, $\tau_1 \in D_k$ and $\tau_2 \in \mathcal{K}_{k_2}(D_k 2 C_k)$).

Besides, we assume that we know $c_n^{\tau_i}$ and $d_n^{\tau_i}$ for $n > n_i$, $i = 1, 2$. Then for $n > \max(k+1, n_1+1, n_2+1)$, c_n^{τ} and d_n^{τ} are given by the following recurrence:

$$\begin{pmatrix} c_n^{\tau} \\ d_n^{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_{n-1}^{\tau} \\ d_{n-1}^{\tau} \end{pmatrix} + \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix},$$

where

$$\alpha_n = (c_{n-1}^{\tau_1} + a_{\tau_1} - r_1^a) d_{n-1}^{\tau_2} + (a_{\tau_2} - r_2^a) c_{n-1}^{\tau_1}$$

and

$$\beta_n = (c_{n-1}^{\tau_1} + b_{\tau_1} - r_1^b) d_{n-1}^{\tau_2} + (b_{\tau_2} - r_2^b) c_{n-1}^{\tau_1}.$$

Proof. Suppose $n > \max(k+1, n_1+1, n_2+1)$. Let us find a recurrence for the number c_n^{τ} (one can use the same considerations for d_n^{τ}).

An occurrence of the pattern τ in $C_n = C_{n-1} 1 D_{n-1}$ is either inside C_{n-1}, or inside D_{n-1}, or begins in C_{n-1} or the letter 1 between C_{n-1} and D_{n-1} and ends in D_{n-1} or the letter 1. The first two cases obviously give c_{n-1} and d_{n-1} occurrences of τ. To count the contribution of the last to cases, we work with words instead of patterns. We do it to take in account the situations when τ_1 or τ_2 consists of copies of only one letter. In this case, we cannot count occurrence of these patterns separately, and then use this information, since, for instance, occurrences of the pattern $\tau_1 = 111$ are subwords 111 and 222 (the last one of these subwords we do not need), whereas occurrences of the pattern $\tau_1 = 222$ are not defined at all (222 is not a pattern).

If an occurrence of τ_1-τ_2 does not entirely belong to C_{n-1} or D_{n-1} then we only have one of the following possibilities:

(a) the subword τ_1 entirely belongs to C_{n-1} and the subword τ_2 entirely belongs to D_{n-1};

(b) the subword τ_1 belongs entirely to C_{n-1} and the subword τ_2 belongs to the kernel $\mathcal{K}_{k_2}(D_k 1 C_k)$, where $k = \lceil \log_2(k_1 + k_2 - 1) \rceil$ is the least number that allow to control, in C_n $(n > k)$, overlapping occurrences of subwords τ_1 and τ_2 where τ_1 is entirely from C_{n-1} and $\tau_2 \in \mathcal{K}_{k_2}(D_k 1 C_k)$;

(c) the subword τ_2 belongs entirely to D_{n-1} and the subword τ_1 belongs to the kernel $\mathcal{K}_{k_1}(D_k 1 C_k)$.

In (a) we obviously have $c^{\tau_1}_{n-1} \cdot d^{\tau_2}_{n-1}$ possibilities.

In (b) we have $c^{\tau_1}_{n-1} \cdot a_{\tau_2} - c^{\tau_1}_{n-1} \cdot r^a_2$ possibilities, since we need to subtract those occurrences of τ_1 and τ_2 that overlap.

Analogically to (b), in (c) we have $d^{\tau_2}_{n-1} \cdot a_{\tau_1} - d^{\tau_2}_{n-1} \cdot r^a_1$ possibilities, which completes the proof. □

Remark 6. *For using Theorem 2, one needs to know c^τ_n and d^τ_n for patterns τ without internal dashes. These numbers could be obtained by using Theorem 1.*

The following corollary to Theorem 2 is straitforward to prove, using the fact that for non-overlapping patterns τ_1 and τ_2, $r^a_1 = r^a_2 = r^b_1 = r^b_2 = 0$.

Corollary 12. *We make the same assumptions as those in Theorem 2. Suppose additionally that the words τ_1 and τ_2 are not overlapping in the following sense: no one suffix of τ_1 is a prefix of τ_2. Then for $n > \max(k+1, n_1+1, n_2+1)$, c^τ_n and d^τ_n are given by the same recurrence as that in Theorem 2 with*

$$\alpha_n = (c^{\tau_1}_{n-1} + a_{\tau_1}) d^{\tau_2}_{n-1} + a_{\tau_2} c^{\tau_1}_{n-1}$$

and

$$\beta_n = (c^{\tau_1}_{n-1} + b_{\tau_1}) d^{\tau_2}_{n-1} + b_{\tau_2} c^{\tau_1}_{n-1}.$$

Remark 7. *Corollary 12 is valid under more weak assumptions, namely we only need the property of non-overlapping of the patterns τ_1 and τ_2 when one of them is in its kernel and the other one is not in its kernel. Example 7 deals with the pattern τ that has overlapping blocks τ_1 and τ_2, but Corollary 12 can be applied. However, from practical point of view, checking the fact if two subwords are non-overlapping is more easy than considering the kernels and checking the non-overlapping of the subwords there.*

Example 5. *Suppose $\tau = 12\text{-}21$. We have that $|\tau_1| = |\tau_2| = 2$. Now, in the statement of Theorem 2 we have that $k = 2$, $a_{\tau_1} = 0$, $a_{\tau_2} = 1$, $b_{\tau_1} = 0$ and $b_{\tau_2} = 1$. Also, since there are no overlapping occurrences of the subwords 12 and 21 in $\mathcal{K}_3(1221112)$ and $\mathcal{K}_3(1222112)$, we have $r_1^a = 0$, $r_2^a = 0$, $r_1^b = 0$ and $r_2^b = 0$. Besides, from example 1, $c_n^{12} = d_n^{12} = 2^{n-2}$ and $c_n^{21} = d_n^{21} = 2^{n-2} - 1$. Thus, $\alpha_n = \beta_n = 4^{n-3}$. Using the fact that $c_3^{12\text{-}21} = 0$ and $d_3^{12\text{-}21} = 1$, this allows us to get an explicit formula for $c_n^{12\text{-}21}$ and $d_n^{12\text{-}21}$ for $n > 3$:*

$$c_n^{12\text{-}21} = d_n^{12\text{-}21} = \frac{1}{2}4^{n-2} - 3 \cdot 2^{n-4}.$$

In particular $c_4^{12\text{-}21} = 5$.

Example 6. *Suppose $\tau = 1\text{-}221$. We have that $|\tau_1| = 1$ and $|\tau_2| = 3$. Moreover, the words τ_1 and τ_2 are not overlapping, hence we can use Corollary 12. We have that $k = 2$, $a_{\tau_1} = 1$, $a_{\tau_2} = 1$, $b_{\tau_1} = 0$ and $b_{\tau_2} = 1$. From example 3, $d_n^{221} = 3 \cdot 2^{n-4} - 1$. Also, the number of occurrences of the letter 1 (the subword $\tau_1 = 1$) is given by Lemma 1: $c_n^1 = 2^{n-1}$. So, $\alpha_n = 6 \cdot 4^{n-4} + 3 \cdot 2^{n-5} - 1$ and $\beta_n = 6 \cdot 4^{n-4}$. One can get now an explicit formula for $c_n^{1\text{-}221}$ and $d_n^{1\text{-}221}$ for $n > 4$:*

$$c_n^{1\text{-}221} = \tfrac{1}{2}4^{n-2} + 27 \cdot 2^{n-5} - n - 7,$$
$$d_n^{1\text{-}221} = \tfrac{1}{2}4^{n-2} + 21 \cdot 2^{n-5} - 8.$$

In particular, $c_5^{1\text{-}221} = 47$.

Example 7. *Suppose $\tau = 112\text{-}21$. We have that $|\tau_1| = k_1 = 3$ and $|\tau_2| = k_2 = 2$. The other parameters in Theorem 2 are $k = 3$, $a_{\tau_1} = 0$, $a_{\tau_2} = 1$, $b_{\tau_1} = 0$, $b_{\tau_2} = 1$, $r_1^a = r_2^a = r_1^b = r_2^b = 0$. From Example 2, for $n \geq 4$, $c_n^{112} = 3 \cdot 2^{n-4}$, and from Example 1, $d_n^{21} = 2^{n-2} - 1$. Thus, in Theorem 2, $\alpha_n = \beta_n = c_{n-1}^{112}(d_{n-1}^{21} + 1) = 3 \cdot 4^{n-4}$. Now, we solve the recurrence relation from the theorem to get, that for $n > 3$*

$$c_n^{112\text{-}21} = d_n^{112\text{-}21} = \frac{3}{2} \cdot 4^{n-3} - 2^{n-4}.$$

9.6 Counting occurrences of $\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k$

In this section we study the number of occurrences of a pattern $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k$, where τ_i are patterns without internal dashes. We say that τ consists of k blocks. We assume that for $i = 1, 2, \ldots, k-1$, the pattern τ_i does not overlap with the patterns τ_{i-1} and τ_{i+1}. In this case we give a recurrence relation for the number of occurrences of τ, provided we know the number of occurrences of certain patterns consisting of less than, or equal to, $k-1$ blocks, as well as $2k$ certain numbers which can be calculated by considering the words $D_\ell 1 C_\ell$ and $C_\ell 2 D_\ell$, where ℓ is the maximum number such that $\ell \leq \max_i \lceil \log_2 |\tau_i| \rceil$. The cases of $k = 1$ and $k = 2$ are studied in the previous sections; they give the bases for our calculations. However, the case of overlapping patterns τ_i is not solved, and it remains as a challenging problem, since an answer to this problem gives the way to count occurrences of an arbitrary generalized pattern, or an arbitrary subsequence, in σ-words.

Theorem 3. *Let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k$ be a generalized pattern such that $|\tau_i| = k_i$ for $i = 1, 2, \ldots, k$. We assume that for $i = 1, 2, \ldots, k-1$, the subword τ_i does not overlap with the subwords τ_{i-1} and τ_{i+1} in the following sense: no one suffix of τ_{i-1} is a prefix of τ_i and no one suffix of τ_i is a prefix of τ_{i+1}.*

Suppose $\ell_i = \lceil \log_2 k_i \rceil$, $\ell = \max_i \ell_i$, and for the subwords τ_i we have $a_i = m_{k_i}(\tau_i, D_{\ell_i} 1 C_{\ell_i})$ and $b_i = m_{k_i}(\tau_i, D_{\ell_i} 2 C_{\ell_i})$, for $i = 1, 2, \ldots, k$.

We assume that we know $c_{n-1}^{\tau_1\text{-}\cdots\text{-}\tau_i}$ and $d_{n-1}^{\tau_{i+1}\text{-}\cdots\text{-}\tau_k}$ for each $1 \le i \le k-1$ and for all $n > n^\star$. Then for all $n > \max(\ell+1, n^\star+1)$, c_n^τ and d_n^τ are given by the following recurrence:

$$\begin{pmatrix} c_n^\tau \\ d_n^\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_{n-1}^\tau \\ d_{n-1}^\tau \end{pmatrix} +$$

$$\sum_{i=1}^{k-1} \begin{pmatrix} c_{n-1}^{\tau_1\text{-}\cdots\text{-}\tau_i} \cdot d_{n-1}^{\tau_{i+1}\text{-}\cdots\text{-}\tau_k} \\ c_{n-1}^{\tau_1\text{-}\cdots\text{-}\tau_i} \cdot d_{n-1}^{\tau_{i+1}\text{-}\cdots\text{-}\tau_k} \end{pmatrix} + \sum_{i=1}^{k} \begin{pmatrix} a_i \cdot c_{n-1}^{\tau_1\text{-}\cdots\text{-}\tau_{i-1}} \cdot d_{n-1}^{\tau_{i+1}\text{-}\cdots\text{-}\tau_k} \\ b_i \cdot c_{n-1}^{\tau_1\text{-}\cdots\text{-}\tau_{i-1}} \cdot d_{n-1}^{\tau_{i+1}\text{-}\cdots\text{-}\tau_k} \end{pmatrix}.$$

Proof. We consider only c_n^τ, since the same arguments can be applied to d_n^τ. We use the considerations similar to those in Theorem 2.

An occurrence of the pattern τ in $C_n = C_{n-1} 1 D_{n-1}$ can be entirely in C_n or D_n. The first term counts such occurrences. Otherwise, we have two possibilities: either the letter 1 between the words C_{n-1} and D_{n-1} does not belong to an occurrence of τ, or it does do it, in which case there exist i (exactly one) such that the subword τ_i occurs in its kernel. The first sum in the statement is obviously responsible for the first of this cases, whereas the second sum is responsible for the second case (in the last case we use the fact that subwords τ_i are not overlapping). □

As a corollary to Theorem 3, we have Corollary 12.

The following example is another corollary to Theorem 3.

Example 8. *Suppose $\tau = 2-1-221$, that is, $\tau_1 = 2$, $\tau_2 = 1$ and $\tau_3 = 221$. So, parameters in Theorem 3 are the following: $k_1 = k_2 = 1$, $k_3 = 3$, $\ell_1 = \ell_2 = 1$, $\ell_3 = 2$, $\ell = 2$. From $D_1 1 C_1 = 211$ we obtain $a_1 = 0$, $a_2 = 1$. From $D_2 1 C_2 = 1221112$ we obtain $a_3 = 1$. From $D_1 2 C_1 = 221$ we get $b_1 = 1$, $b_2 = 0$. From $D_2 2 C_2 = 1222112$ we get $b_3 = 1$. Besides, from Proposition 2, Examples 3 and 6, we have*

$$c_n^{\tau_1\text{-}\tau_2} = c_n^{2-1} = 2 \cdot 4^{n-2} - n \cdot 2^{n-2}, \textit{for } n > 1;$$

$$d_n^{\tau_3} = d_n^{221} = 3 \cdot 2^{n-4} - 1, \textit{for } n > 3;$$

$$d_n^{\tau_2\text{-}\tau_3} = d_n^{1\text{-}221} = \tfrac{1}{2} \cdot 4^{n-2} + 21 \cdot 2^{n-5} - 8, \textit{for } n > 4.$$

Also, the number of occurrences of the subword $\tau_1 = 2$ in C_n is given by Proposition 1: $c_n^{\tau_1} = c_n^2 = 2^{n-1} - 1$. So, the number of occurrences of the pattern τ

in C_n and D_n, for $n > 5$, satisfies the following recurrence relation:

$$\begin{pmatrix} c_n^\tau \\ d_n^\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_{n-1}^\tau \\ d_{n-1}^\tau \end{pmatrix} + \begin{pmatrix} \frac{5}{1024}8^n + \frac{25-3n}{256}4^n - \frac{171}{64}2^n + 9 \\ \frac{5}{1024}8^n + \frac{21-3n}{256}4^n - 2^{n+1} \end{pmatrix},$$

with initial conditions $c_5^\tau = 70$ and $d_5^\tau = 74$.

9.7 Patterns of the form $[\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k]$, $[\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k)$ and $(\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k]$

We recall that according to Babson and Steingrímsson notation for generalized patterns, if we use "[" in a pattern, for example if we write $p = [1\text{-}2)$, we indicate that in an occurrence of p, the letter corresponding to the 1 must be the first letter of a word under consideration, whereas if we write, say, $p = (1\text{-}2]$, then the letter corresponding to 2 must be the last (rightmost) letter of the word.

In the theorems of this section, we assume that we can find the numbers $c_n^{\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k}$ and $d_n^{\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k}$ for any patterns τ_i, $i = 1, 2, \ldots, k$, without internal dashes. For certain special cases, these numbers can be obtained using the theorems of Sections 9.5 and 9.6.

Theorem 4. *Suppose τ_1 and τ_2 are two patterns without internal dashes such that $|\tau_1| = k_1$ and $|\tau_2| = k_2$. Also, suppose $\ell_1 = \log_2(k_1 + 1)$, $\ell_2 = \log_2(k_2 + 1)$ and $\ell = \log_2(k_1 + k_2 + 1)$.*

Let $a(\tau_1, \tau_2)$ be the number of overlapping subwords τ_1 and τ_2 in C_ℓ such that τ_1 consists of the k_1 leftmost letters of C_ℓ; $b(\tau_1, \tau_2)$ is the number of overlapping subwords τ_1 and τ_2 in C_ℓ such that τ_2 consists of the k_2 rightmost letters of C_ℓ.

We assume that we know $c_n^{\tau_i}$ and $d_n^{\tau_i}$ for $i = 1, 2$ and for all $n > n^\star$.

i. For $n \geq \max(\ell_1, n^\star)$,

$$c_n^{[\tau_1\text{-}\tau_2)} = \begin{cases} c_n^{\tau_2} - a(\tau_1, \tau_2), & \text{if } C_{\ell_1} \text{ begins with } \tau_1, \\ 0, & \text{otherwise.} \end{cases}$$

ii. For $n \geq \max(\ell_2, n^\star)$,

$$c_n^{(\tau_1\text{-}\tau_2]} = \begin{cases} c_n^{\tau_1} - b(\tau_1, \tau_2), & \text{if } C_{\ell_2} \text{ ends with } \tau_2, \\ 0, & \text{otherwise.} \end{cases}$$

iii. For $n \geq \ell$,

$$c_n^{[\tau_1\text{-}\tau_2]} = \begin{cases} 1, & \text{if } C_\ell \text{ begins with } \tau_1 \text{ and ends with } \tau_2, \\ 0, & \text{otherwise.} \end{cases}$$

iv. For $n \geq \max(\ell_1, n^\star)$*,*

$$d_n^{[\tau_1\text{-}\tau_2)} = \begin{cases} d_n^{\tau_2} - a(\tau_1, \tau_2), & \text{if } D_{\ell_1} \text{ begins with } \tau_1, \\ 0, & \text{otherwise.} \end{cases}$$

v. For $n \geq \max(\ell_2, n^\star)$*,*

$$d_n^{(\tau_1\text{-}\tau_2]} = \begin{cases} d_n^{\tau_1} - b(\tau_1, \tau_2), & \text{if } D_{\ell_2} \text{ ends with } \tau_2, \\ 0, & \text{otherwise.} \end{cases}$$

vi. For $n \geq \ell$*,*

$$d_n^{[\tau_1\text{-}\tau_2]} = \begin{cases} 1, & \text{if } D_\ell \text{ begins with } \tau_1 \text{ and ends with } \tau_2, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We prove case i, all the other cases are then easy to see.

Clearly, if C_{ℓ_1} does not begin with τ_1 then C_n does not begin with τ_1 for all $n \geq \ell_1$, which means that $c_n^{[\tau_1\text{-}\tau_2)} = 0$ in this case. Otherwise, to count occurrences of the pattern $[\tau_1\text{-}\tau_2)$ is the same as to find the number of occurrences of the pattern τ_2 in C_n and then subtract the number of such occurrences of τ_2 that begin from the i-th letter of C_n, where $1 \leq i \leq k_1$. □

The following two examples are corollaries to Theorem 4.

Example 9. *Suppose we have the patterns* $\sigma_1 = [1122 - 21211)$ *and* $\sigma_2 = (21221 - 12]$*. From Theorem 4,* $c_n^{\sigma_1} = d_n^{\sigma_1} = 0$*, since* C_3 *does not begin with* 1122 *(*$\ell_1 = 3$*). Also,* $c_n^{\sigma_2} = d_n^{\sigma_2} = 0$*, since* C_3 *does not end with* 12 *(*$\ell_2 = 3$*).*

Example 10. *Suppose* $\tau = [112\text{-}21)$*. We have that* $k_1 = 3$*,* $\ell_1 = 2$ *and* C_2 *begins with the subword* 112*. Besides,* $a(112, 21) = 1$ *and, from Example 1,* $c_n^{21} = d_n^{21} = 2^{n-2} - 1$*. Theorem 4 now gives, that for* $n > 3$*, we have* $c_n^{[112\text{-}21)} = c_n^{\tau_2} - a(\tau_1, \tau_2) = 2^{n-2} - 2$*.*

The following theorem is straitforward to prove using the assumptions concerning non-overlapping of certain subwords.

Theorem 5. *Let* $\{\tau_1, \tau_2, \ldots, \tau_k\}$ *be a set of generalized patterns without internal dashes. Suppose* $|\tau_1| = s_1$*,* $|\tau_k| = s_k$*,* $\ell_1 = \log_2(s_1 + 1)$ *and* $\ell_k = \log_2(s_k + 1)$*. Also,* $\ell = \max(\ell_1, \ell_k)$*.*

i. With the assumption that the subword τ_1 *does not overlap with the subword* τ_2*, that is, no one suffix of* τ_1 *is a prefix of* τ_2*, we have*

(a)

$$c_n^{[\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k)} = \begin{cases} c_n^{\tau_2\text{-}\tau_3\text{-}\cdots\text{-}\tau_k}, & \text{if } C_{\ell_1} \text{ begins with } \tau_1, \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$d_n^{[\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k)} = \begin{cases} d_n^{\tau_2\text{-}\tau_3\text{-}\cdots\text{-}\tau_k}, & \textit{if } D_{\ell_1} \textit{ begins with } \tau_1, \\ 0, & \textit{otherwise.} \end{cases}$$

ii. With assumption that the subword τ_{k-1} does not overlap with the subword τ_k, that is, no one suffix of τ_{k-1} is a prefix of τ_k, we have

(a)

$$c_n^{(\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k]} = \begin{cases} c_n^{\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_{k-1}}, & \textit{if } C_{\ell_k} \textit{ ends with } \tau_k, \\ 0, & \textit{otherwise.} \end{cases}$$

(b)

$$d_n^{(\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k]} = \begin{cases} d_n^{\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_{k-1}}, & \textit{if } D_{\ell_k} \textit{ ends with } \tau_k, \\ 0, & \textit{otherwise.} \end{cases}$$

iii. With the assumption that the subword τ_1 does not overlap with the subword τ_2, and the subword τ_{k-1} does not overlap with the subword τ_k, we have

(a)

$$c_n^{[\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k]} = \begin{cases} c_n^{\tau_2\text{-}\tau_3\text{-}\cdots\text{-}\tau_{k-1}}, & \textit{if } C_{\ell} \textit{ begins with } \tau_1 \textit{ and ends with } \tau_k, \\ 0, & \textit{otherwise.} \end{cases}$$

(b)

$$d_n^{[\tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_k]} = \begin{cases} d_n^{\tau_2\text{-}\tau_3\text{-}\cdots\text{-}\tau_{k-1}}, & \textit{if } D_{\ell} \textit{ begins with } \tau_1 \textit{ and ends with } \tau_k, \\ 0, & \textit{otherwise.} \end{cases}$$

The following example is a corollary to Theorem 5.

Example 11. *Suppose $\tau = [112\text{-}1\text{-}221\text{-}22]$. The parameters of Theorem 5 are $k_1 = 3$, $k_2 = 2$, $\ell_1 = 2$, $\ell_2 = 2$, $\ell = 2$. C_3 begins with the subword 112 and ends with the subword 22. Thus by Theorem 5 and Example 6, $c_n^{[112\text{-}1\text{-}221\text{-}22]} = c_n^{1\text{-}221} = \frac{1}{2}4^{n-2} + 27 \cdot 2^{n-5} - n - 7$.*

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