Thesis for the Degree of Doctor of Philosophy

Boundary behaviour of eigenfunctions for the hyperbolic Laplacian

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Abstract

Let $P(z,\varphi)$ denote the Poisson kernel in the unit disc. Poisson extensions of the type $P_\lambda f(z) = \int_{\mathbb{T}} P(z,\varphi)^{\lambda+1/2} f(\varphi)\, d\varphi$, where $f \in L^1(\mathbb{T})$ and $\lambda \in \mathbb{C}$, are then eigenfunctions to the hyperbolic Laplace operator in the unit disc. In the context of boundary behaviour, $P_0 f(z)$ exhibits unique properties.

We investigate the boundary convergence properties of the normalised operator, $P_0 f(z)/P_0 1(z)$, for boundary functions f in some function spaces. For each space, we characterise the so-called natural approach regions along which one has almost everywhere convergence to the boundary function, for any boundary function in that space. This is done, mostly, via estimates of the associated maximal function.

The function spaces we consider are $L^{p,\infty}$ (weak L^p) and Orlicz spaces which are either close to L^p or L^∞. We also give a new proof of known results for L^p, $1 \le p \le \infty$.

Finally, we deal with a problem on the lack of tangential convergence for bounded harmonic functions in the unit disc. We give a new proof of a result due to Aikawa.

Keywords: Square root of the Poisson kernel, approach regions, almost everywhere convergence, maximal functions, harmonic functions, Fatou theorem.

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The thesis consists of a summary and the following four papers:

[MB1] M. Brundin, *Approach regions for the square root of the Poisson kernel and weak L^p boundary functions*, Revised version of Preprint 1999:56, Göteborg University and Chalmers University of Technology, 1999.

[MB2] M. Brundin, *Approach regions for L^p potentials with respect to the square root of the Poisson kernel*, Revised version of Preprint 2001:55, Göteborg University and Chalmers University of Technology, 2001.

[MB3] M. Brundin, *Approach regions for the square root of the Poisson kernel and boundary functions in certain Orlicz spaces*, Revised version of Preprint 2001:59, Göteborg University and Chalmers University of Technology, 2001.

[MB4] M. Brundin, *On a theorem of Aikawa.*

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BOUNDARY BEHAVIOUR OF EIGENFUNCTIONS FOR THE HYPERBOLIC LAPLACIAN

MARTIN BRUNDIN

1. Introduction

This thesis deals mainly with the boundary behaviour of solutions to a specific partial differential equation. We shall content ourselves in this introductory paper with a discussion of the relevant harmonic analysis on the unit disc, where our differential equation is defined. In Section 5, however, we show how some of the concepts treated can be carried over to a different setting (the half space).

We shall be concerned with pointwise, almost everywhere, convergence. The solutions to our differential equation will be given by Poisson-like integral extensions of the boundary functions. More precisely, the integral kernel is given by the square root of the Poisson kernel and possesses unique properties relative to other powers. The associated extensions are eigenfunctions of the hyperbolic Laplacian, at the bottom of the positive spectrum. To recover the boundary values, the extensions must be normalised.

It is a well-known fact that solutions to boundary value problems behave more and more dramatically the closer one gets to the boundary. *A priori*, it is often not even clear in which sense the boundary conditions should be interpreted. Of course, in some sense, the solution should be "equal to the prescribed boundary values on the boundary", but that statement is not precise. It will be clear that if we approach the boundary, the unit circle, too close to the tangential direction, then almost everywhere convergence of the extension to the boundary function will fail. The question we wish to answer is, somewhat vaguely, the following:

Given a space A of integrable functions defined on the unit circle, how tangential can our approach to the boundary be in order to guarantee a.e.

convergence of the extension to the boundary function, for any boundary function in A?

A few comments are in order. The notion of tangency to the boundary will be measured by so-called approach regions, which will depend on the space A, beside the integral kernel. It is to my knowledge impossible to give an answer to the question above for all A. Instead, we shall consider more or less explicit examples of A. The examples we cover are $A = L^p$ for $1 \leq p \leq \infty$, $A = L^{p,\infty}$ (weak L^p) for $1 < p < \infty$ and $A = L^\Phi$ (Orlicz spaces) for certain classes of functions Φ. These results are covered in the papers [MB2], [MB1] and [MB3], respectively.

The paper [MB4] deals with a classical problem concerning the lack of convergence of bounded harmonic functions in the unit disc. We give a modified proof of a result by Aikawa, which in turn is a considerably sharpened version of a theorem of Littlewood (see below).

In the following sections we give an outline of the underlying theory and our results.

2. The Poisson kernel and harmonic functions in the unit disc

Let U denote the unit disc in $\mathbb{C}$, i.e.

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Then $\partial U \cong \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \cong (-\pi, \pi]$. Whenever convenient, we identify $\mathbb{T}$ with the interval $(-\pi, \pi]$.

The Dirichlet problem is the following: Given a function $f \in L^1(\mathbb{T})$, find a function u which is harmonic in U and such that $u = f$ on $\mathbb{T}$. As we shall see below, this question makes sense if $f \in C(\mathbb{T})$. If we only assume that $f \in L^1(\mathbb{T})$, this is a typical example where one has to be very careful with the meaning of the condition $u = f$ on $\mathbb{T}$ (see the results of Fatou and Littlewood below).

Let $P(z, \beta)$ be the Poisson kernel in U,

$$P(z, \beta) = \frac{1}{2\pi} \cdot \frac{1 - |z|^2}{|z - e^{i\beta}|^2},$$

where $z \in U$ and $\beta \in \mathbb{T}$. It is readily checked that $P(z, \beta)$ is the real part of the holomorphic function

$$u(z) = \frac{1}{2\pi} \cdot \frac{e^{i\beta} + z}{e^{i\beta} - z},$$

so that $P(\,\cdot\,, \beta)$ is harmonic in U.

The Poisson integral (or extension) Pf of $f \in L^1(\mathbb{T})$ is defined, for $z \in U$, by

$$Pf(z) = \int_{\mathbb{T}} P(z, \beta) f(\beta)\, d\beta.$$

Note that, if we write $z = (1-t)e^{i\theta}$, then

$$Pf(z) = K_t * f(\theta),$$

where the convolution is taken in $\mathbb{T}$ and

$$K_t(\varphi) = \frac{1}{2\pi} \cdot \frac{t(2-t)}{|(1-t)e^{i\varphi} - 1|^2}.$$

For positive functions f and g, we say that $f \lesssim g$ if $f \leq cg$ for some constant $c > 0$. If $f \lesssim g$ and $g \lesssim f$, we say that $f \sim g$. For later use, we note that

$$K_t(\varphi) \sim L_t(\varphi) = \frac{t}{(t + |\varphi|)^2}.$$

The Poisson extension Pf defines a harmonic function in U. Moreover, we have the following classical result (solution to the continuous Dirichlet problem):

Theorem (Schwarz, [10]). *If $f \in C(\mathbb{T})$, then $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in U$.*

A natural question is what happens when the boundary function f is less regular, e.g. when $f \in L^p(\mathbb{T})$. First of all, of course, the best thing one can hope for is convergence at, at most, almost every boundary point (i.e., convergence fails on at most a set of measure zero). However, it turns out that a.e. convergence may very well fail if the approach to the boundary is "too tangential". To guarantee a.e. convergence, one has to approach the boundary with some care, in the sense of staying within certain *approach regions.*

Definition 1. For any function $h : \mathbb{R}_+ \to \mathbb{R}_+$ we define the (natural) approach region, determined by h at $\theta \in \mathbb{T}$, by

$$\mathcal{A}_h(\theta) = \{z \in U : |\arg z - \theta| \leq h(1 - |z|)\}.$$

If $h(t) \sim t$, as $t \to 0$, we say that $\mathcal{A}_h(\theta)$ is a nontangential cone.

There are also other kinds of approach regions. Maybe the most interesting are those of so-called Nagel-Stein type, being given by means of a "cone condition" and a "cross-section condition". We shall not consider such approach regions, but will instead focus only on those given in Definition 1.

Theorem (Fatou, [6]). *Let $h(t) = O(t)$. Then, for all $f \in L^1(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$.*

In this case, to relate to what we said earlier, the condition $u = f$ on $\mathbb{T}$ should be interpreted as a nontangential limit.

Let us sketch a proof of Fatou's result:

Proof. To keep the proof as simple as possible, assume that $h(t) = t$. As we shall see later, in Section 3, it is now sufficient to see that the maximal operator given by

$$Mf(z) = \sup_{|z|>1/2, |\arg z - \theta| < t} |Pf(z)|,$$

is of weak type $(1,1)$. Note that

$$Mf(z) \lesssim \sup_{t<1/2, |\eta|<t} \tau_\eta L_t * |f|(\theta),$$

where τ_η denotes translation, i.e. $\tau_\eta F(\theta) = F(\theta - \eta)$ for any function F. Since $|\eta| < t$, it is easily seen that

$$\tau_\eta L_t(\varphi) \sim L_t(\varphi).$$

Now, since $\|L_t\|_1 \lesssim 1$ uniformly in t, it follows by standard results (see [14], §2.1) that

$$Mf(z) \lesssim M_{HL} f(\theta),$$

where M_{HL} denotes the ordinary Hardy-Littlewood maximal operator, and the weak type estimate follows, as desired. □

Littlewood [7] proved that Fatou's theorem, in a certain sense, is sharp:

Theorem (Littlewood, [7]). *Let $\gamma_0 \subset U \cup \{1\}$ be a simple closed curve, having a common tangent with the circle at the point 1. Let γ_θ be the rotation of γ_0 by the angle θ. Then there exists a bounded harmonic function f in U with the property that, for a.e. $\theta \in \mathbb{T}$, the limit of f along γ_θ does not exist.*

Littlewood's proof was not elementary. He used a result of Khintchine concerning the rapidity of the approximation of almost all numbers by rationals. Zygmund [15] gave two new proofs, one of which was elementary. The other, which was considerably shorter, used properties of Blaschke products.

Since then, Littlewood's result has been generalised in a number of directions. Aikawa [1] and [2] sharpened the result considerably. A discrete analogue was given by Di Biase, [5]. In the last paper [MB4], we present a new proof of Aikawa's result: If the function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\mathcal{A}_h(\theta)$ is a tangential approach region (i.e. $h(t)/t \to \infty$ as $t \to 0^+$), there exists a bounded harmonic function in U which fails to have a boundary limit along $\mathcal{A}_h(\theta)$ for *any* $\theta \in \mathbb{T}$.

For further results on Fatou type theorems and related topics, the book [4] by Di Biase is recommended.

3. Poisson extensions with respect to powers of the Poisson kernel

For $z = x + iy$ define the hyperbolic Laplacian by

$$L_z = \frac{1}{4}(1 - |z|^2)^2(\partial_x^2 + \partial_y^2).$$

Then the λ-Poisson integral

$$u(z) = P_\lambda f(z) = \int_{\mathbb{T}} P(z,\beta)^{\lambda+1/2} f(\beta)\, d\beta, \text{ for } \lambda \in \mathbb{C},$$

defines a solution of the equation

$$L_z u = (\lambda^2 - 1/4)u.$$

In representation theory of the group $SL(2,\mathbb{R})$, one uses the powers $P(z,\beta)^{i\alpha+1/2}$, $\alpha \in \mathbb{R}$, of the Poisson kernel.

From now on we shall deal only with real powers, greater than or equal to $1/2$, of the Poisson kernel, i.e. $\lambda \geq 0$.

It is readily checked that

$$P_\lambda 1(z) \sim (1-|z|)^{1/2-\lambda}$$

as $|z| \to 1$ if $\lambda > 0$, and that

$$P_0 1(z) \sim (1-|z|)^{1/2} \log \frac{1}{1-|z|},$$

as $|z| \to 1$. To get boundary convergence we have to normalise P_λ, since $P_\lambda 1(z)$ does not converge to 1. If one considers normalised $\lambda-$Poisson integrals for $\lambda > 0$, i.e. $\mathcal{P}_\lambda f(z) = P_\lambda f(z)/P_\lambda 1(z)$, the convergence properties are the same as for the ordinary Poisson integral. This is because the kernels essentially behave in the same way. However, it turns out that the operator $\mathcal{P}_0$ has unique properties in the context of boundary behaviour of corresponding extensions. A somewhat vague explanation is that this is due to the logarithmic factor in $\mathcal{P}_0$, which is absent in $\mathcal{P}_\lambda$ for $\lambda > 0$.

If $f \in C(\mathbb{T})$ then $\mathcal{P}_0 f(z) \to f(\theta)$ unrestrictedly as $z \to e^{i\theta}$ for all $\theta \in \mathbb{T}$, just as in the case of the Poisson integral itself. This is because $\mathcal{P}_0$ is a convolution operator, behaving like an approximate identity.

Theorem (Sjögren, [11]). *Let $f \in L^1(\mathbb{T})$. For a.e. $\theta \in \mathbb{T}$ one has that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, where $h(t) = O(t \log 1/t)$ as $t \to 0$.*

This result was generalised to L^p, $1 \leq p < \infty$, by Rönning [9]:

Theorem (Rönning, [9]). *Let $1 \leq p < \infty$ be given and let $f \in L^p(\mathbb{T})$. For a.e. $\theta \in \mathbb{T}$ one has that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, where $h(t) = O(t(\log 1/t)^p)$ as $t \to 0$.*

Rönning also proved that Sjögren's result is the best possible, when the approach regions are given by Definition 1 and h is increasing, and that in his own theorem for L^p, the exponent p in $h(t) = O(t(\log 1/t)^p)$ cannot be improved.

The method used to prove these theorems was weak type estimates for the corresponding maximal operators. The continuous functions, for which convergence is known to hold, are dense in L^p, so the results follow by approximation.

The case of $f \in L^\infty$ was (thought to be, see below) a deeper question, basically because the continuous functions do not form a dense subset. However, using a result by Bellow and Jones [3], Sjögren [12] managed to determine the approach regions:

Theorem (Sjögren, [12]). *The following conditions are equivalent for any increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any $f \in L^\infty(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $h(t) = O(t^{1-\varepsilon})$ *as* $t \to 0$ *for any* $\varepsilon > 0$.

The content of paper [MB1] is the following result for $L^{p,\infty}$ (weak L^p):

Theorem. (Brundin, [MB1]). *Let $1 < p < \infty$ be given. Then the following conditions are equivalent for any function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any $f \in L^{p,\infty}(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $\sum_{k=0}^{\infty} \sigma_k < \infty$, *where* $\sigma_k = \sup_{2^{-2^k} \le s \le 2^{-2^{k-1}}} \frac{h(s)}{s(\log(1/s))^p}$.

Clearly, (*ii*) is slightly stronger than the condition $h(t) = O(t(\log 1/t)^p)$ appearing in Rönning's L^p result. The proof of the $L^{p,\infty}$ result above follows the same lines as Sjögren's proof for L^∞, in the sense that it relies on a "Banach principle for $L^{p,\infty}$" which is established in the paper.

In paper [MB2] we give a new proof for the L^p case, $1 \le p \le \infty$. It is considerably shorter and more straightforward than the earlier proofs. Also, the L^∞ case is proved without using the Banach principle. The key observation is that one part of the kernel, which previously was thought to be "hard", actually is more or less trivial. In the last section of paper [MB2], the L^∞ case is generalised to higher dimensions (polydiscs).

Paper [MB3], which contains what should be considered our main results, deals with specific classes of Orlicz spaces. The point is to get an insight in the difference between the approach regions for L^p (finite p) and L^∞ (note that the approach regions for L^p are optimal, whereas no optimal approach region exists for L^∞).

Orlicz spaces generalise L^p spaces. One simply replaces the condition $\int |f|^p < \infty$ by

$$\int \Phi(|f|) < \infty,$$

for some "reasonable" function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. Here Φ should be increasing and convex, and $\Phi(0) = \Phi'(0) = 0$. The space defined depends only on the behaviour of $\Phi(x)$ for large x.

In general, the integral condition defining our Orlicz space does not give a linear space of functions. But with a few modifications, which we omit here, we actually get a linear space.

The first class of Orlicz functions Φ treated is denoted by ∇. It consists basically of functions Φ for which $M(x) = \log(\Phi'(x))$ grows at least polynomially as $x \to \infty$. The precise growth condition imposed is given by

$$\liminf_{x\to\infty} \frac{M(2x)}{M(x)} = m_0 > 1.$$

This implies that Φ itself grows at least exponentially at infinity, i.e. we are in some sense closer to L^∞ than to L^p. A typical example is $\Phi(x) \sim e^x$ for large x.

The other class we consider is denoted Δ. It consists basically of functions Φ whose growth at infinity is controlled above and below by power functions (polynomials). Here, the precise growth condition is given by

$$\frac{x\Phi''(x)}{\Phi'(x)} \sim 1,$$

uniformly for $x > x_0$ (some $x_0 \geq 0$). Δ contains, for example, functions of growth $\Phi(x) \sim x^p(\log(1+|x|))^\alpha$ at infinity, for any $p > 1$ and $\alpha \geq 0$. The Orlicz spaces related to Δ are closer to L^p than to L^∞.

The following two theorems are proved:

Theorem. (Brundin, [MB3]). *Let $\Phi \in \nabla$ be given. Then the following conditions are equivalent for any function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any* $f \in L^\Phi$ *one has for almost all* $\theta \in \mathbb{T}$ *that* $\mathcal{P}_0 f(z) \to f(\theta)$ *a.e. as* $z \to e^{i\theta}$ *and* $z \in \mathcal{A}_h(\theta)$.

(*ii*) $\dfrac{M\left(C\frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} \to \infty$ *as* $t \to 0$ *for all* $C > 0$, *where* $g(t) = h(t)/t$.

An example would be $\Phi(x) \sim e^{x^\alpha}$, for $\alpha > 0$, i.e. $M(x) \sim x^\alpha$. It is easily seen that here condition (*ii*) is equivalent with

$$\log g(t) = o\left((\log 1/t)^{\alpha/(\alpha+1)}\right),$$

so that, expressed in a somewhat unorthodox way,

$$h(t) = t \exp\left(o\left((\log 1/t)^{\alpha/(\alpha+1)}\right)\right).$$

Clearly, no optimal approach region exists.

Theorem. (Brundin, [MB3]). *Let* $\Phi \in \Delta$ *be given. Then the following conditions are equivalent for any function* $h : \mathbb{R}_+ \to \mathbb{R}_+$*:*

(*i*) *For any* $f \in L^\Phi$ *one has for almost all* $\theta \in \mathbb{T}$ *that* $\mathcal{P}_0 f(z) \to f(\theta)$ *a.e. as* $z \to e^{i\theta}$ *and* $z \in \mathcal{A}_h(\theta)$.
(*ii*) $h(t) = O(t\Phi(\log 1/t))$, *as* $t \to 0$.

The natural example here is $\Phi(x) = x^p$, $p > 1$. Condition (*ii*) is then equivalent with Rönning's L^p condition.

The key proposition to prove these results could be thought of as an Orlicz space substitute for Hölder's inequality. It is formulated and proved in [MB3].

It is worth noting that optimal approach regions exist in the case the boundary functions are in L^p, $1 \le p < \infty$, and in L^Φ, where $\Phi \in \Delta$. For $L^{p,\infty}$, L^∞ and L^Φ, where $\Phi \in \nabla$, the conditions on h for a.e. convergence do not yield an optimal h. Given an admissible approach region, in these cases, one can always find an essentially larger region which is also admissible. Why is there a difference? It is reasonable to believe that the difference has to do with the fact that the "norms" in the latter spaces are *not* given by simple integral conditions.

4. Almost everywhere convergence and maximal operators

In this section we shall discuss the concept of almost everywhere convergence and how it is related to maximal operators.

Let $M = M(\mathbb{T})$ denote the set of Lebesgue measurable functions on $\mathbb{T}$. Assume that we are given a sequence of sublinear operators $S_n : A(\mathbb{T}) \to M$, where $A(\mathbb{T})$ is some normed subspace of $L^1(\mathbb{T})$ (e.g. $A(\mathbb{T}) = L^p(\mathbb{T})$). We say that $S_n f$ converges almost everywhere (w.r.t. Lebesgue measure m) if $S_n f(\theta)$ converges for a.e. $\theta \in \mathbb{T}$. This is equivalent to

$$m\left(E_\lambda\right) = 0,$$

for all $\lambda > 0$, where

$$E_\lambda(f) = \left\{\theta \in \mathbb{T} : \limsup_{n,m\to\infty} |S_n f(\theta) - S_m f(\theta)| > \lambda\right\}.$$

Define

$$S^* f(\theta) = \sup_{n\geq 1} |S_n f(\theta)|$$

and let

$$E^*_\lambda(f) = \left\{\theta \in \mathbb{T} : (S^* f)(\theta) > \lambda\right\}.$$

S^* is referred to as a *maximal operator*. Somewhat vaguely, one could say that maximal operators are obtained by replacing limits by suprema of the modulus.

Note that $E_\lambda(f) \subset E^*_{\lambda/2}(f)$. Now, assume that $g \in A(\mathbb{T})$ is some function for which $S_n g \to g$ a.e. as $n \to \infty$. Then $E_\lambda(f) = E_\lambda(f - g) \subset E^*_{\lambda/2}(f - g)$. Thus, it follows that

$$m(E_\lambda(f)) \leq m(E^*_{\lambda/2}(f - g)).$$

We are interested in proving a.e. convergence for all functions $f \in A(\mathbb{T})$, where $A(\mathbb{T})$ is equipped with a norm which we denote by $\|\cdot\|_A$.

In order to deduce that $m\left(E_\lambda(f)\right) = 0$ for all $\lambda > 0$, when $f \in A(\mathbb{T})$, it now suffices to have some weak continuity of $S^* : A(\mathbb{T}) \to M$ at 0, and to be able to approximate any f in the norm $\|\cdot\|_A$ with a "good" function g. We sum up this discussion in a theorem:

Theorem. *Let $A(\mathbb{T}) \subset L^1(\mathbb{T})$ be a function space, equipped with a norm $\|\cdot\|_A$. Assume that the following two conditions hold:*

(*i*) $S^* : A(\mathbb{T}) \to M(\mathbb{T})$ *is weakly continuous at* 0*, i.e.* $m\,(E^*_\lambda(f)) = C(\lambda)o(1)$ *as* $\|f\|_A \to 0$ *for all* $f \in A(\mathbb{T})$ *and some function* $C : \mathbb{R}_+ \to \mathbb{R}_+$*.*

(*ii*) *There exists a set* $D(\mathbb{T}) \subset A(\mathbb{T})$*, dense in* $A(\mathbb{T})$*, such that, for all* $g \in D(\mathbb{T})$*,* $S_n g(\theta) \to g(\theta)$ *a.e. as* $n \to \infty$*.*

Then, for all $f \in A(\mathbb{T})$*,* $S_n f(\theta) \to f(\theta)$ *a.e. as* $n \to \infty$*.*

To be specific, if $A(\mathbb{T}) = L^p(\mathbb{T})$, part (*i*) follows if one for example establishes a weak type (p,p) estimate for S^*. In our case, later on, the continuous (or bounded) functions on $\mathbb{T}$ will serve as the set $D(\mathbb{T})$.

It should be pointed out that our results concern *families* of operators S_t, $t \in (0,1)$, and not sequences. However, the difference is slight and the above reasoning works just as well for families (as $t \to 0$) as for sequences (as $n \to \infty$).

A natural question is what one loses by studying the maximal operator instead of the sequence itself. Remarkably enough, as was proved by Stein [13] and by Nikishin [8], in a multitude of cases one does not lose anything. Continuity of the the maximal operator is quite simply often (without going into any details) equivalent with a.e. convergence.

5. An example

In this section, we prove a result for fractional Poisson extensions of L^p boundary functions in the half space. Thus, the setting but also the methods that we shall use are a bit different from those in the papers [MB1], [MB2] and [MB3]. I acknowledge the help received from Yoshihiro Mizuta, who came up with the idea and a brief sketch of the proof.

Let $P_t(x)$ denote the Poisson kernel in the half space

$$\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} \ : x \in \mathbb{R}^n \text{ and } t > 0\},$$

that is

$$P_t(x) = c_n \cdot \frac{t}{(t^2 + |x|^2)^{(n+1)/2}},$$

where c_n is the constant determined by

$$\int_{\mathbb{R}^n} P_t(x)\,dx = 1.$$

We fix an open, nonempty and bounded set $\Omega \subset \mathbb{R}^n$. In the unit disc we consider the square root of the Poisson kernel, but in higher dimensions it is the $\frac{n}{n+1}$:th power of the Poisson kernel that exhibits special properties. Therefore, for $f \in L^p(\mathbb{R}^n)$, let

$$(P_0 f)(x,t) = \int_{\mathbb{R}^n} P_t(x-y)^{\frac{n}{n+1}} f(y)\,dy.$$

We normalise the extension, with respect to Ω, by

$$(\mathcal{P}_0 f)(x,t) = \frac{(P_0 f)(x,t)}{(P_0 \chi_\Omega)(x,t)}.$$

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be given and let $x_0 \in \Omega$. We define the natural approach region at x_0, determined by h, to be

$$\mathcal{A}_h(x_0) = \{(x,t) \in \mathbb{R}^{n+1}_+ \;:\; \sqrt{|x-x_0|^2 + t^2} < h(t)\}.$$

We define

$$A_p(f,r,x) = \left(\frac{1}{r^n} \int_{B(x,r)} |f(y)|^p \,dy \right)^{1/p},$$

and

$$L_f^{(p)}(\Omega) = \{x \in \Omega : A_p(f - f(x), r, x) \to 0 \text{ as } r \to 0\}.$$

Note that, if $f \in L^p(\mathbb{R}^n)$, then $|\Omega \setminus L_f^{(p)}(\Omega)| = 0$ (a.e. point is a Lebesgue point).

Theorem. *Let $1 \le p < \infty$ be given and assume that $h(t) = O(t(\log 1/t)^{p/n})$ as $t \to 0^+$. Furthermore, let $f \in L^p(\mathbb{R}^n)$ be given. Then, for any $x_0 \in L_f^{(p)}(\Omega)$ (in particular, for a.e. $x_0 \in \Omega$) one has that $(\mathcal{P}_0 f)(x,t) \to f(x_0)$ as $(x,t) \to (x_0, 0)$ along $\mathcal{A}_h(x_0)$.*

Proof. We shall prove the result directly, i.e. without using estimates of maximal operators.

As $(x,t) \to (x_0, 0) \in \Omega \times \{0\}$, it is easy to see that

$$(P_0 \chi_\Omega)(x,t) \sim t^{\frac{n}{n+1}} \log 1/t.$$

Now, let $f \in L^p(\mathbb{R}^n)$ and $x_0 \in L_f^{(p)}(\Omega)$ be given. We may, without loss of generality, assume that $f(x_0) = 0$. Furthermore, we assume that $(x,t) \in \mathcal{A}_h(x_0)$. For short, let $r = \sqrt{|x-x_0|^2 + t^2}$. We write

$$
\begin{aligned}
(P_0 f)(x,t) &= \int_{B(x_0,2r)} P_t(x-y)^{\frac{n}{n+1}} f(y)\,dy \\
&\quad + \int_{B(x_0,2r)^c} P_t(x-y)^{\frac{n}{n+1}} f(y)\,dy \\
&= I_1(x,t) + I_2(x,t).
\end{aligned}
$$

By using Hölder's inequality, we obtain

$$
\begin{aligned}
|I_1(x,t)| &\lesssim t^{\frac{n}{n+1}} \left(\int_{|y-x_0|<2r} \frac{dy}{(t+|x-y|)^{nq}} \right)^{1/q} \cdot \left(\int_{|y-x_0|<2r} |f(y)|^p\,dy \right)^{1/p} \\
&\lesssim r^{n/p} \cdot t^{\frac{n}{n+1}} \left(\int_{|x-y|<3r} \frac{dy}{(t+|x-y|)^{nq}} \right)^{1/q} \cdot A_p(f,2r,x_0) \\
&\lesssim (r/t)^{n/p} \cdot t^{\frac{n}{n+1}} \cdot A_p(f,2r,x_0).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|I_2(x,t)| &\lesssim t^{\frac{n}{n+1}} \sum_{k=1}^{\infty} \int_{B(x_0,2^{k+1}r)\setminus B(x_0,2^k r)} \frac{1}{(t+|x_0-y|)^n} |f(y)|\,dy \\
&\lesssim t^{\frac{n}{n+1}} \sum_{k=1}^{\infty} (2^k r)^{-n} \int_{B(x_0,2^{k+1}r)} |f(y)|\,dy \\
&\lesssim t^{\frac{n}{n+1}} \sum_{k=1}^{\infty} A_1(f,2^{k+1}r,x_0).
\end{aligned}
$$

We now note that

$$
\begin{aligned}
A_1(f,2^{k+1}r,x_0) &\lesssim \frac{1}{2^k r} \int_{2^{k+1}r}^{2^{k+2}r} A_1(f,s,x_0)\,ds \\
&\lesssim \int_{2^{k+1}r}^{2^{k+2}r} \frac{A_1(f,s,x_0)}{s}\,ds.
\end{aligned}
$$

Invoking this in the estimate above, we obtain

$$\begin{aligned}|I_2(x,t)| &\lesssim t^{\frac{n}{n+1}} \sum_{k=1}^{\infty} \int_{2^{k+1}r}^{2^{k+2}r} \frac{A_1(f,s,x_0)}{s}\, ds \\ &\lesssim t^{\frac{n}{n+1}} \int_r^{\infty} \frac{A_1(f,s,x_0)}{s}\, ds \\ &\lesssim t^{\frac{n}{n+1}} \int_t^{\infty} \frac{A_1(f,s,x_0)}{s}\, ds.\end{aligned}$$

Thus, it follows that

$$\begin{aligned}|(\mathcal{P}_0 f)(x,t)| &\lesssim \frac{1}{t^{n/(n+1)} \log 1/t}(|I_1(x,t)| + |I_2(x,t)|) \\ &\lesssim \frac{1}{\log 1/t} \cdot [(r/t)^{n/p} \cdot A_p(f,2r,x_0) + \int_t^{\infty} \frac{A_1(f,s,x_0)}{s}\, ds].\end{aligned}$$

Now, using the fact that $r < h(t) \lesssim t(\log 1/t)^{p/n}$, we get

$$|(\mathcal{P}_0 f)(x,t)| \lesssim A_p(f,2r,x_0) + \frac{1}{\log 1/t} \int_t^{\infty} s^{-1} A_1(f,s,x_0)\, ds.$$

It is clear that

$$\int_t^{\infty} \frac{A_1(f,s,x_0)}{s}\, ds$$

is a convergent integral, since

$$\begin{aligned}\frac{A_1(f,s,x_0)}{s} &\lesssim s^{-1} s^{-n} s^{n/q} \|f\|_p \\ &\lesssim s^{-1-n/p} \|f\|_p,\end{aligned}$$

by Hölder's inequality.

Now, as $t \to 0$ we also have $r \to 0$. Since $f(x_0) = 0$ and since we have assumed that $x_0 \in L_f^{(p)}(\Omega)$ (and thus that $x_0 \in L_f^{(1)}(\Omega)$), it follows that

$$(\mathcal{P}_0 f)(x,t) \to 0 = f(x_0),$$

as $(x,t) \to (x_0, 0)$ along $\mathcal{A}_h(x_0)$. This concludes the proof. □

6. Open questions

6.1. **The unit disc.** A more complete picture of the convergence results for the "square root operator" in the unit disc would be desirable. The best one could hope for is a unified convergence theorem, for all function spaces (of some particular but general kind), where the convergence condition is given in terms of the norm on the space. This is probably a very hard problem, and most likely even impossible. However, more partial results would be interesting in their own right, to complete the picture. For instance, results for $\mathrm{BMO}(\mathbb{T})$ and for classes of Orlicz spaces, between ∇ and Δ, would be interesting. A typical example is given by the function $\Phi(x) \sim e^{(\log x)^p}$, where $p > 1$. Attempts have been made to characterise the approach regions for spaces related to such functions, but without success.

6.2. **Higher dimensions.** Results for polydiscs have been obtained by both Sjögren and Rönning, for L^p boundary functions. A natural questions is what happens for Orlicz spaces, weak L^p and so on. The results are of "restricted convergence" type, i.e. the speed with which one approaches the boundary should be approximately the same in all the discs. Whether or not this is necessary is not known. The Russian mathematicians Katkovskaya and Krotov claim that they have proved that a certain maximal operator is of strong type (p,p), which immediately would yield unrestricted convergence. However, the result has not been published. Another natural generalisation is to replace the unit disc with a symmetric space. Results have been obtained for rank 1 spaces, but higher rank generalisations are still a relatively unexplored field.

6.3. **Littlewood type theorems.** It would be nice to replace the negative results "not a.e. convergence" with "everywhere divergence". This is done in the paper [MB4] for the ordinary Poisson integral and bounded boundary functions. An attempt was made to transfer the same machinery to the square root case, but it failed. In this sense, the normalised square root operator behaves completely differently from the ordinary Poisson integral. A new approach is necessary.

6.4. **Weakly regular boundary functions.** One could increase the regularity of the boundary functions (e.g. by transforming L^p in some suitable way) and sharpen the convergence. The natural thing here is to replace Lebesgue measure with some capacity or Hausdorff measure, and obtain corresponding quasi everywhere results, which are stronger than almost everywhere.

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APPROACH REGIONS FOR THE SQUARE ROOT OF THE POISSON KERNEL AND WEAK L^p BOUNDARY FUNCTIONS.

MARTIN BRUNDIN

ABSTRACT. Let $P_0 f(z) = \int_{\mathbb{T}} \sqrt{P(z,\beta)} f(e^{i\beta})\, d\beta$ for $f \in L^1(\mathbb{T})$, where $P(z,\beta)$ is the Poisson kernel in the unit disc. In this paper we consider the convergence properties of the normalised operator $P_0 f / P_0 1$. We give a characterisation of the natural approach regions along which one has almost everywhere convergence for weak L^p boundary functions, $1 < p < \infty$.

1. INTRODUCTION

This paper is divided into four main sections. In this one, Introduction, we introduce the problem and discuss related results. The essence of the second section, Preliminaries, is to prove a converse Banach principle for $L^{p,\infty}$, needed to prove the main result, Theorem 2, which is done in Section 3. Section 4 concludes the paper with a brief discussion of open questions.

Let $P(z,\beta)$ be the standard Poisson kernel in the unit disc U,

$$P(z,\beta) = \frac{1}{2\pi} \cdot \frac{1 - |z|^2}{|z - e^{i\beta}|^2}$$

where $z \in U$ and $\beta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \cong \partial U$. Note that

$$P(z,\beta) = \frac{1}{2\pi} \cdot \mathrm{Re}\left(\frac{e^{i\beta} + z}{e^{i\beta} - z}\right),$$

so the mapping $z \mapsto P(z,\beta)$, being the real part of a holomorphic function, is harmonic.

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In what follows, we shall be concerned with functions defined on $\mathbb{T}$. In that context, whenever convenient, we identify $\mathbb{T}$ with the interval $(-\pi, \pi]$, e.g. we write $f(\theta)$ instead of $f(e^{i\theta})$.

Now, let

$$Pf(z) = \int_{\mathbb{T}} P(z, \beta) f(\beta)\, d\beta,$$

the Poisson integral of $f \in L^1(\mathbb{T})$. Then, if $f \in C(\mathbb{T}) \subset L^1(\mathbb{T})$, $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$. This was shown in 1872 by Schwarz [7], and it is considered to be a well known result today.

For any function $h : \mathbb{R}_+ \to \mathbb{R}_+$ let

$$\mathcal{A}_h(\theta) = \{z \in U : |\arg z - \theta| \leq h(1 - |z|)\}. \tag{1}$$

We refer to $\mathcal{A}_h(\theta)$ as the (natural) approach region determined by h at $\theta \in \mathbb{T}$. This is the only form of approach regions that we will be concerned with, throughout the thesis. However, we point out that there are other approach regions, defined in different manners. The Nagel-Stein approach regions are examples of this (see [5]). The word "region" is usually used only when the set (region) in question is open. However, we shall use it in a wider sense, with no openness assumptions.

Now, if we only assume that $f \in L^1(\mathbb{T})$, the convergence properties are different than in the case of continuous functions. If $h(t) = \alpha t$, $\alpha > 0$, then $Pf(z) \to f(\theta)$ a.e. as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, i.e. the convergence is non-tangential. This is proved by showing that the corresponding maximal operator is dominated by the Hardy-Littlewood maximal operator, which is of weak type $(1, 1)$. Then the result follows via approximation with continuous functions. This result was first shown by Fatou [3] in 1906. Littlewood [4] proved that the theorem, in a certain sense, is sharp.

For a more complete treatise on the concepts and theorems mentioned so far, see Di Biase [2].

For $z = x + iy$ let

$$L_z = \frac{1}{4}(1 - |z|^2)^2(\partial_x^2 + \partial_y^2),$$

the hyperbolic Laplacian. Then

$$u(z) = P_\lambda f(z) = \int_{\mathbb{T}} P(z, \beta)^{\lambda + 1/2} f(\beta)\, d\beta, \text{ for } \lambda \geq 0,$$

defines a solution of the equation

$$L_z u = (\lambda^2 - 1/4)u.$$

The powers $P(z,\beta)^{i\alpha+1/2}$ of the Poisson kernel are used in connection with representation theory of the group $SL(2,\mathbb{R})$.

One can show that

$$P_\lambda 1(z) \sim (1-|z|)^{1/2-\lambda}$$

if $\lambda > 0$, and that

$$P_0 1(z) \sim (1-|z|)^{1/2} \log \frac{1}{1-|z|},$$

where $f \sim g$ means that there exists a constant $c > 0$ such that $c^{-1} \leq f/g \leq c$. This thesis is concerned with convergence properties of the square root of the Poisson kernel ($\lambda = 0$) and boundary functions $f \in L^{p,\infty}$ (weak L^p). To get boundary convergence we have to normalise P_0, since $P_0 1(z)$ does not converge to 1:

$$\mathcal{P}_0 f(z) = \frac{P_0 f(z)}{P_0 1(z)}.$$

We point out that if one considers normalised $\lambda-$Poisson integrals for $\lambda > 0$, i.e. $\mathcal{P}_\lambda f(z) = P_\lambda f(z)/P_\lambda 1(z)$, the convergence properties are the same as for the ordinary Poisson integral. This is because the kernels behave essentially in the same way.

If $f \in C(\mathbb{T})$ then $\mathcal{P}_0 f(z) \to f(\theta)$ unrestrictedly as $z \to e^{i\theta}$ for all $\theta \in \mathbb{T}$, just as in the case of the Poisson integral itself. This is because $\mathcal{P}_0$ is a convolution operator with a kernel being an approximate identity in $\mathbb{T}$. Moreover, convergence results are known for $f \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$. The case $p = 1$ was solved by Sjögren [8]:

Theorem (Sjögren, 1984). *Let $f \in L^1(\mathbb{T})$. For a.e. $\theta \in \mathbb{T}$ one has $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, where $h(t) = O(t \log 1/t)$ as $t \to 0$.*

This result was generalised to L^p, $1 \leq p < \infty$, by Rönning [6]:

Theorem (Rönning, 1992). *Let $1 \leq p < \infty$ be given and let $f \in L^p(\mathbb{T})$. For a.e. $\theta \in \mathbb{T}$ one has $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, where $h(t) = O(t(\log 1/t)^p)$ as $t \to 0$.*

Rönning also proved that Sjögren's result is best possible, when the approach regions are given in the form (1) and h is increasing, and that in his own theorem for L^p, the exponent p in $h(t) \lesssim t(\log 1/t)^p$ cannot be improved.

The method used in the proof of Rönning's result was a weak type estimate for the corresponding maximal operator. The continuous functions, for which convergence is known to hold, are dense in L^p, and a standard approximation argument together with the weak type estimate then proves the theorem.

The case of $f \in L^\infty$ turned out to be different. Since the continuous functions are not dense in this space, the weak type estimate approach would be inadequate. However, using a result by Bellow and Jones [1], Sjögren [9] managed to determine the approach regions:

Theorem (Sjögren, 1997). *The following are equivalent for any increasing function* $h : \mathbb{R}_+ \to \mathbb{R}_+$*:*

(*i*) *For any* $f \in L^\infty(\mathbb{T})$ *one has for almost all* $\theta \in \mathbb{T}$ *that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $h(t) = O(t^{1-\varepsilon})$ *as* $t \to 0$ *for any* $\varepsilon > 0$.

Actually, the assumption that h should be increasing is never invoked in the proof. Thus, this result determines *all* admissible approach regions for L^∞, when given in the form (1). Note also that these approach regions are strictly wider than the ones in the case of finite p (as anticipated, since $L^\infty \subset L^p$ for all $p \geq 1$).

Basically, the Bellow-Jones result for L^∞ states that a.e. convergence is equivalent to continuity of the maximal operator at 0, when restricted to the unit ball in L^∞, in the topology of convergence in measure. Thus, what Sjögren had to show was that if $\|f\|_\infty \leq 1$ then for all $\varepsilon > 0$ and all $\kappa > 0$ there exists $\delta > 0$ such that

$$\|f\|_1 < \delta \Rightarrow |\{\theta \in \mathbb{T} : Mf(\theta) > \varepsilon\}| < \kappa,$$

where M denotes the relevant maximal operator. (It is easy to see that, in the unit ball in L^∞, the topology of convergence in measure is equivalent with the L^1-topology.) This led to a kind of optimisation problem, where the constraint basically was $\|f\|_\infty \leq 1$.

As in the case of L^∞, the continuous functions are not dense in $L^{p,\infty}$. To solve this, we shall extend the Bellow-Jones result to cover functions in $L^{p,\infty}$, and by doing so we may adopt the approach used by Sjögren. The significant difference, of course, is that $L^{p,\infty}$ contains significantly "wilder" functions than L^∞ does.

Our main result, Theorem 2, shows that a convergence result similar to Sjögren's holds in case of $L^{p,\infty}$ boundary functions. We shall prove that $\mathcal{A}_h(\theta)$ is an admissible approach region for almost every $\theta \in \mathbb{T}$ if, and only if,

$$\sum_{k=0}^{\infty} c_k 2^{(1-p)k} < \infty, \tag{2}$$

where $c_k = \sup_{2^{k-1} \leq s \leq 2^k} h(2^{-s}) 2^s s^{-1}$.

We conclude this section with a couple of equivalent ways of writing the series condition (2).

First of all, for $2^{k-1} \leq s \leq 2^k$ we have $2^{(1-p)k} \sim s^{1-p}$. In other words, if we substitute s^{1-p} for $2^{(1-p)k}$ we may move it inside the supremum defining c_k. In that way we get an equivalent condition for the admissible approach regions:

$$\sum_{k=0}^{\infty} d_k < \infty,$$

where

$$d_k = \sup_{2^{k-1} \leq s \leq 2^k} h(2^{-s}) 2^s s^{-p}.$$

Secondly, if we let $\sigma = 2^{-s}$ we may rewrite d_k as

$$d_k \sim \sup_{2^{-2^k} \leq \sigma \leq 2^{-2^{k-1}}} \frac{h(\sigma)}{\sigma (\log 1/\sigma)^p}.$$

With this at hand it is easy to see that if we let $h(t) = t(\log 1/t)^q$ we have a.e. convergence if, and only if, $q < p$. The interesting feature here is the strict inequality, since it reveals that we do not have convergence along

Rönning's L^p-regions. Note that convergence, when $q < p$, follows directly from the inclusion $L^{p,\infty} \subset L^q$ together with Rönning's result.

Finally, note that, as in the case of L^∞ boundary functions, there is no best possible (i.e. largest) approach region in case of boundary functions in $L^{p,\infty}$.

2. Preliminaries

We denote weak $L^p(\mathbb{T})$ by $L^{p,\infty}$, $1 \leq p < \infty$, with quasi-norm

$$[f]_p = \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{1/p},$$

where $\lambda_f(\alpha) = |\{x : |f(x)| > \alpha\}|$, $|\cdot|$ denoting Lebesgue measure. It follows that on $\mathbb{T}$, endowed with Lebesgue measure, we have the inclusions

$$L^p \subset L^{p,\infty} \subset L^q$$

for $1 \leq q < p < \infty$. However, $L^{1,\infty} \not\subset L^1$. That is, $L^{1,\infty}$ contains functions which are not integrable, and thus we assume that $1 < p < \infty$ in what follows.

Note also that $L^\infty \subset L^{p,\infty}$. This means that we can expect smaller approach regions for $L^{p,\infty}$ than for L^∞.

We point out that for $f, g \in L^{p,\infty}$ we have $[f+g]_p \leq 2([f]_p + [g]_p)$. The constant 2 cannot be replaced by 1, so the ordinary triangle inequality fails.

Let $B_{p,\infty} = \{f \in L^{p,\infty} : [f]_p \leq 1\}$, the unit ball in $L^{p,\infty}$, and let M denote the set of all measurable functions on $\mathbb{T}$. Endow $B_{p,\infty}$ and M with the topology of convergence in measure, given by the metric

$$d(f,g) = \int_{\mathbb{T}} \frac{|f(\beta) - g(\beta)|}{1 + |f(\beta) - g(\beta)|} \, d\beta,$$

$f, g \in M$. The metric d is induced by the "norm" ρ defined by

$$\rho(f) = \int_{\mathbb{T}} \frac{|f(\beta)|}{1 + |f(\beta)|} \, d\beta,$$

$f \in M$ (ρ is not a norm, since it fails to be homogeneous, but we still refer to it in this way in lack of better terminology).

Lemma 1. *For $f, g \in B_{p,\infty}$ we have $d(f,g) \leq \|f - g\|_1$. Moreover, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(f,g) < \delta \Rightarrow \|f - g\|_1 < \varepsilon$.*

Proof. The inequality $d(f,g) \le \|f-g\|_1$ is trivially true.
To prove the second statement let $\varepsilon > 0$ be given and let $\varphi = f - g$. For $A > 0$ fixed we have that

$$\{x : |\varphi(x)| > \alpha\} = \{x : \alpha < |\varphi(x)| \le A\} \cup \{x : |\varphi(x)| > A\} \subset$$
$$\subset \{x : |\varphi(x)|/(1 + |\varphi(x)|) > \alpha/(1 + A)\} \cup \{x : |\varphi(x)| > A\}.$$

Since $[\varphi]_p \le 2([f]_p + [g]_p) \le 4$, we get the estimate

$$\begin{aligned}
\int |\varphi(x)|\,dx &= \int_0^\infty \lambda_\varphi(\alpha)\,d\alpha \\
&\le \int_0^A \lambda_\varphi(\alpha)\,d\alpha + C\cdot A^{1-p} \\
&\le \int_0^A \lambda_{\frac{\varphi}{1+|\varphi|}}\Big(\frac{\alpha}{1+A}\Big)\,d\alpha + A\lambda_\varphi(A) + C\cdot A^{1-p} \\
&\le (1+A)\int_0^\infty \lambda_{\frac{\varphi}{1+|\varphi|}}(\alpha)\,d\alpha + C\cdot A^{1-p} \\
&= (1+A)d(f,g) + C\cdot A^{1-p}.
\end{aligned}$$

Now take A such that $C\cdot A^{1-p} < \varepsilon/2$, and then take $\delta = \frac{\varepsilon}{2(1+A)}$. □

Lemma 2. *$C(\mathbb{T})$ is dense in $(B_{p,\infty}, d)$.*

Proof. By Lemma 2, we have that $d(f,g) \le \|f-g\|_1$ for any $f, g \in B_{p,\infty}$. Since $C(\mathbb{T})$ is dense in L^1, the lemma follows. □

Theorem 1 below is a slightly modified version of Theorem 1 in [1]. The main difference is that here L^∞ is replaced by $L^{p,\infty}$.

Theorem 1. *Assume that we are given a sequence of operators $\{S_n\}_{n=1}^\infty$, $S_n : L^{p,\infty} \to M$, such that*

(i) each $S_n : L^{p,\infty} \to M$ is sublinear,

(ii) the maximal operator is well defined, that is $S^*f(x) = \sup_{n\ge1} |S_nf(x)|$ is finite a.e. for $f \in L^{p,\infty}$, and

(iii) $S^* : (B_{p,\infty}, d) \to (M, d)$ is continuous at 0.

Then the set E of elements $f \in B_{p,\infty}$ for which $(S_n f)$ converges a.e. is closed in $(B_{p,\infty}, d)$.

Proof. Let $\overline{E}$ be the closure of E in $(B_{p,\infty}, d)$. It follows that $\overline{E} \subset B_{p,\infty}$. Let $f \in \overline{E}$. We want to show that for all $\lambda > 0$

$$|\{\beta \in \mathbb{T} : \limsup_{m,n} |S_m f(\beta) - S_n f(\beta)| > \lambda\}| = 0.$$

We know that $|S_n v(\beta) - S_m v(\beta)| \leq |S_n v(\beta)| + |S_m v(\beta)| \leq 2S^* v(\beta)$ and hence, for any $g \in E$, we have

$$\begin{aligned} &|\{\beta \in \mathbb{T} : \limsup_{m,n} |S_m f(\beta) - S_n f(\beta)| > \lambda\}| \\ &= |\{\beta \in \mathbb{T} : \limsup_{m,n} |S_m (f-g)(\beta) - S_n (f-g)(\beta)| > \lambda\}| \\ &\leq |\{\beta \in \mathbb{T} : 2S^*(f-g)(\beta) > \lambda\}| \\ &= \left|\left\{\beta \in \mathbb{T} : S^*(\frac{1}{4}f - \frac{1}{4}g)(\beta) > \frac{\lambda}{8}\right\}\right|. \end{aligned}$$

Let $\varepsilon > 0$ be given. For $u \in B_{p,\infty}$ we clearly have

$$\frac{\lambda/8}{1+\lambda/8}|\{\beta \in \mathbb{T} : S^* u(\beta) > \lambda/8\}| \leq \int_{\{\theta \in \mathbb{T} : S^* u(\theta) > \lambda/8\}} \frac{S^* u(\beta)}{1 + S^* u(\beta)}\, d\beta \leq \rho(S^* u).$$

By the continuity of S^* at 0 in $(B_{p,\infty}, d)$ and by Lemma 1, we can choose a $\delta = \delta(\varepsilon, \lambda/8)$ such that $u \in B_{p,\infty}$ and $\|u\|_1 \leq \delta$ implies $\rho(S^* u) \leq \lambda\varepsilon/(\lambda+8)$, and therefore $|\{\beta \in \mathbb{T} : S^* u(\beta) > \lambda/8\}| < \varepsilon$.

By Lemma 1 again we can choose $g \in E$ such that $\|f - g\|_1 \leq \delta$. Then $(f-g)/4 \in B_{p,\infty}$ and of course $\|(f-g)/4\|_1 \leq \delta$. Thus

$$\left|\left\{\beta \in \mathbb{T} : S^*(\frac{1}{4}f - \frac{1}{4}g)(\beta) > \frac{\lambda}{8}\right\}\right| \leq \varepsilon.$$

□

The following trivial consequence of Theorem 1 is what we shall make use of:

Corollary 1. *Assume that, in addition to the hypotheses on $\{S_n\}_{n=1}^{\infty}$ in Theorem 1, we also have that there is a set $D \subset B_{p,\infty}$, dense in $(B_{p,\infty}, d)$, such that $\lim_{n\to\infty} S_n f(x)$ exists a.e. for each $f \in D$.*
Then $\lim_{n\to\infty} S_n f(x)$ exists a.e. for all $f \in L^{p,\infty}$.

Proof. Let E be the set of elements $f \in B_{p,\infty}$ for which $(S_n f)$ converges a.e. By assumption $D \subset E$, $\overline{D} = B_{p,\infty}$, and by Theorem 1 we have $\overline{E} = E$. Thus $E = B_{p,\infty}$. By normalising $g \in L^{p,\infty}$, the corollary follows. □

Lemma 3. *Let $\lambda_0 > 0$ and $1 < p < \infty$ be given. For $g \in B_{p,\infty}$ let $f(x) = g(x)\chi_{\{g>\lambda_0\}}$. Then $\|f\|_1 \leq C\lambda_0^{1-p}$, where C only depends on p.*

Proof. Note that $\lambda_g(\alpha) \leq \alpha^{-p}[g]_p^p \leq \alpha^{-p}$ for all $\alpha > 0$. Consequently $\lambda_f(\alpha) = |\{x : g(x) > \lambda_0\}| \leq \lambda_0^{-p}$ for $0 < \alpha \leq \lambda_0$ and $\lambda_f(\alpha) = \lambda_g(\alpha) \leq \alpha^{-p}$ for $\alpha > \lambda_0$. This yields

$$\begin{aligned} \|f\|_1 &= \int_0^\infty \lambda_f(\alpha)\, d\alpha \\ &= \int_0^{\lambda_0} \lambda_f(\alpha)\, d\alpha + \int_{\lambda_0}^\infty \lambda_f(\alpha)\, d\alpha \\ &\leq \lambda_0^{1-p} + \int_{\lambda_0}^\infty \frac{1}{\alpha^p}\, d\alpha \\ &\leq \frac{p}{p-1}\lambda_0^{1-p}. \end{aligned}$$

□

3. The main theorem

This last part of the thesis is entirely devoted to the proof and some of the consequences of the main result, Theorem 2.

Theorem 2. *Let $1 < p < \infty$ be given. Then the following are equivalent for any function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any $f \in L^{p,\infty}(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $\sum_{k=0}^\infty c_k 2^{(1-p)k} < \infty$ *where* $c_k = \sup_{2^{k-1} \leq s \leq 2^k} h(2^{-s})2^s s^{-1}$.

3.1. **Proof of Theorem 2.** We shall prove that (*ii*) implies (*i*) via Proposition 1 below, and that (*i*) implies (*ii*) via contraposition. First we introduce a suitable notation.

If we write $t = 1 - |z|$, then $z = (1-t)e^{i\theta}$ and

$$\mathcal{P}_0 f(z) = R_t * f(\theta),$$

where the convolution is taken in $\mathbb{T}$ and

$$R_t(\theta) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t(2-t)}}{|(1-t)e^{i\theta}-1|} \frac{1}{P_0 1(1-t)}.$$

Here $\theta \in \mathbb{T} = (-\pi, \pi]$. Since $P_0 1(1-t) \sim \sqrt{t} \log 1/t$, the order of magnitude of R_t is given by

$$R_t(\theta) \sim Q_t(\theta) = \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta|}.$$

Since we are only interested in small t, we might as well assume that $t < 1/2$. Now let τ_η denote the translation (or rotation, rather) $\tau_\eta f(\theta) = f(\theta - \eta)$. Then the convergence condition (i) in Theorem 2 above means

$$\lim_{\substack{t \to 0 \\ |\eta| < h(t)}} \tau_\eta R_t * f(\theta) = f(\theta).$$

The relevant maximal operator for our problem is

$$M_0 f(\theta) = \sup_{\substack{|\arg z - \theta| < h(1-|z|) \\ |z| > 1/2}} |\mathcal{P}_0 f(z)|.$$

Notice that $M_0 f(\theta)$ is dominated by a constant times

$$(3) \qquad Mf(\theta) = \sup_{\substack{|\eta| < h(t) \\ t < 1/2}} \tau_\eta Q_t * |f|(\theta).$$

Proposition 1. *Assume that condition* (ii) *in Theorem 2 holds and let* $\varepsilon > 0$. *Given* $\kappa > 0$ *there exists* $\delta > 0$ *such that for* $f \in B_{p,\infty}$

$$\|f\|_1 < \delta \Rightarrow |\{\theta \in \mathbb{T} : Mf(\theta) > \varepsilon\}| < \kappa.$$

Note that Proposition 1 precisely means that M is continuous at 0 in the topology of convergence in measure, when restricted to $B_{p,\infty}$. We can then apply Corollary 1 to the family of operators $f \mapsto \tau_\eta R_t * f$, $|\eta| < h(t)$, $t \in (0, 1/2)$. This is not a sequence of operators, but Corollary 1 is easily extended to families. Thus, the implication $(ii) \Rightarrow (i)$ in Theorem 2 is a consequence of the proposition.

Proof. (*Proposition 1*) We may assume that $f \geq 0$, without loss of generality. Write

$$Q_t(\theta) = Q_t(\theta)\chi_{\{|\theta| \leq 2h(t)\}} + Q_t(\theta)\chi_{\{|\theta| > 2h(t)\}} = Q_t^1(\theta) + Q_t^2(\theta).$$

By letting

$$M_j f(\theta) = \sup_{\substack{|\eta|<h(t)\\ 0<t<1/2}} \tau_\eta Q_t^j * f(\theta) \ , j \in \{1,2\},$$

we get $Mf \leq M_1 f + M_2 f$ and hence

$$\{Mf > \varepsilon\} \subset \{M_1 f > \varepsilon/2\} \cup \{M_2 f > \varepsilon/2\}.$$

To deal with $M_2 f$ we observe that when $|\eta| < h(t)$

$$\begin{aligned} \tau_\eta Q_t^2(\theta) &\leq \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta - \eta|} \chi_{\{|\theta-\eta|>2h(t)\}} \\ &\leq \frac{2}{\log 1/t} \cdot \frac{1}{t + |\theta|}. \end{aligned}$$

The last expression is a decreasing function of $|\theta|$, whose integral in $\mathbb{T}$ is bounded uniformly in t. It is well known that convolution by such a function is controlled by the Hardy-Littlewood maximal operator M_{HL}, so that $M_2 f \leq C M_{HL} f$. Since M_{HL} is of weak type $(1,1)$, we obtain

$$|\{M_2 f > \varepsilon/2\}| \leq C\varepsilon^{-1} \|f\|_1.$$

Finally, we consider $M_1 f$. If $M_1 f(\theta) > \varepsilon$, there exists $t \in (0,1/2)$ and $|\eta| \leq h(t)$ such that $Q_t^1 * f(\theta - \eta) > \varepsilon$. This means that

$$\int_{|\varphi|<2h(t)} \frac{1}{\log 1/t} \cdot \frac{1}{t + |\varphi|} f(\theta - \eta - \varphi)\, d\varphi > \varepsilon.$$

We decompose the kernel $(t + |\varphi|)^{-1}$ as

$$\begin{aligned} \frac{1}{t+|\varphi|} &= \frac{1}{t+|\varphi|} \chi_{\{\varphi:|\varphi|<t\}}(\varphi) + \sum_{m=1}^{\infty} \frac{1}{t+|\varphi|} \chi_{\{\varphi:2^{m-1}t\leq|\varphi|<2^m t\}}(\varphi) \\ &\leq \frac{2\chi_{(-t,t)}(\varphi)}{t} + \sum_{m=1}^{\infty} \frac{1}{2^{m-1}t} \chi_{\{\varphi:|\varphi|<2^m t\}}(\varphi). \end{aligned}$$

For $m \in \mathbb{N} \cup \{0\}$, define

$$K_t^m(\varphi) = \frac{1}{\log 1/t} \cdot \frac{1}{2^{m-1}t} \chi_{\{\varphi:|\varphi|<2^m t\}}(\varphi),$$

so that

$$\frac{1}{\log 1/t} \cdot \frac{1}{t+|\varphi|} \leq \sum_{m=0}^{\infty} K_t^m(\varphi).$$

We then have

$$M_1 f \leq \sum_{m=0}^{\infty} M^{(m)} f,$$

where

$$M^{(m)} f(\theta) = \sup_{\substack{|\eta|<h(t)\\ 0<t<1/2}} \tau_\eta K_t^m * f(\theta).$$

For some suitable sequence $\{\mu_m\}_0^\infty$ of positive numbers, with $\sum_m \mu_m = 1$, we intend to use the inequality

$$\lambda_{M_1 f}(\varepsilon) \leq \sum_{m=0}^{\infty} \lambda_{M^{(m)} f}(\mu_m \varepsilon), \tag{4}$$

in order to show that M_1 is continuous at 0 in the topology of convergence in measure.

Let $m \in \mathbb{N} \cup \{0\}$ be given and assume that $M^{(m)} f(\theta) > \varepsilon$. Then there exists $t \in (0, 1/2)$ and $|\eta| < h(t)$ such that

$$\int_{|\varphi|<2h(t)} \frac{\chi_{(-2^m t, 2^m t)}(\varphi)}{2^{m-1} t \log 1/t} f(\theta - \eta - \varphi)\, d\varphi > \varepsilon.$$

If we let $A(t) = \min\{2h(t), 2^m t\}$ this yields

$$\int_{-A(t)}^{A(t)} f(\theta - \eta - \varphi)\, d\varphi > \varepsilon 2^{m-1} t \log 1/t,$$

which is equivalent to

$$\int_{[\theta-\eta-A(t), \theta-\eta+A(t)]} f(\varphi)\, d\varphi > \varepsilon 2^{m-1} t \log 1/t. \tag{5}$$

Let $A_\nu = \sup_{2^{-\nu} \leq t < 2^{-\nu+1}} A(t)$, $\nu \geq 2$, and let $A = \sup_{\nu \geq 2} A_\nu$. For $j \in \mathbb{N}$ let i_j be the number of ν such that $2^{-j} A < A_\nu \leq 2^{-j+1} A$. Note that, since $\lim_{k\to\infty} A_k = 0$, we have that i_j is finite for all j. Let $\gamma(0) = 0$ and for $j \geq 1$ let $\gamma(j) = \sum_{k=1}^{j} i_k$.

We write

$$2^{-j} A < A_{\nu_{\gamma(j-1)+1}}, \ldots, A_{\nu_{\gamma(j)}} \leq 2^{-j+1} A,$$

i.e. ν_k, for $\gamma(j-1) + 1 \leq k \leq \gamma(j)$, denotes precisely those ν for which $2^{-j} A < A_\nu \leq 2^{-j+1} A$.

Choose a maximal family of mutually disjoint open intervals I of lengths $2A(t)$, with $t \in [2^{-\nu_1}, 2^{-\nu_1+1}) \cup ... \cup [2^{-\nu_{\gamma(1)}}, 2^{-\nu_{\gamma(1)}+1})$, such that $\int_I f(\varphi)\, d\varphi > \varepsilon 2^{m-1} t \log 1/t$. Denote the union of these intervals by J_1.

We now construct J_j recursively: Given J_k for $k \leq j-1$ choose a maximal family of mutually disjoint open intervals I of lengths $2A(t)$, with

$$t \in [2^{-\nu_{\gamma(j-1)+1}}, 2^{-\nu_{\gamma(j-1)+1}+1}) \cup ... \cup [2^{-\nu_{\gamma(j)}}, 2^{-\nu_{\gamma(j)}+1}),$$

disjoint also with $\cup_{i=1}^{j-1} J_i$, such that $\int_I f(\varphi)\, \varphi > \varepsilon 2^{m-1} t \log 1/t$. Denote the union of these intervals by J_j.

Note that each chosen interval is of length $2A(t)$ for some t. Let N_ν, $\nu \geq 2$, denote the number of chosen intervals with corresponding values of t in the interval $[2^{-\nu}, 2^{-\nu+1})$. Denote their union by I_ν.

Let $\nu' \in \mathbb{N}$ be arbitrary and define $\tilde{f} = f\chi_{\{f > \varepsilon\nu'(\log 2)/16\}}$.

For $2^{-\nu} \leq t < 2^{-\nu+1}$ we have $A(t) \leq 2^m t \leq 2^{m-\nu+1}$, so we get

$$\begin{aligned}
\|(\tilde{f} - f)\chi_{I_\nu}\|_1 &= \int_{\mathbb{T}} f(\varphi)\chi_{\{f \leq \varepsilon\nu'(\log 2)/16\}}(\varphi)\chi_{I_\nu}(\varphi)\, d\varphi \\
&\leq (\varepsilon\nu'(\log 2)/16)|I_\nu| \\
&\leq (\log 2)N_\nu 2^{m-\nu+2}\varepsilon\nu' 2^{-4} \\
&= (\log 2)N_\nu 2^{m-\nu-2}\varepsilon\nu'.
\end{aligned}$$

Furthermore, by (5), we get

$$\begin{aligned}
\|f\chi_{I_\nu}\|_1 &\geq N_\nu \varepsilon 2^{m-1} 2^{-\nu} \log 2^\nu \\
&= (\log 2)N_\nu 2^{m-\nu-1}\varepsilon\nu.
\end{aligned}$$

Combining these two estimates, we get

$$\begin{aligned}
\|\tilde{f}\|_1 &\geq \sum_{\nu\geq\nu'} \|\tilde{f}\chi_{I_\nu}\|_1 \\
&\geq \sum_{\nu\geq\nu'} \Big(\|f\chi_{I_\nu}\|_1 - \|(\tilde{f}-f)\chi_{I_\nu}\|_1\Big) \\
&\geq C\sum_{\nu\geq\nu'} \big(N_\nu \varepsilon 2^{m-\nu-1}\nu - N_\nu 2^{m-\nu-2}\varepsilon\nu'\big) \\
&= C\sum_{\nu\geq\nu'} N_\nu 2^{m-\nu-2}\varepsilon(2\nu-\nu') \\
&\geq C\sum_{\nu\geq\nu'} N_\nu 2^{m-\nu}\varepsilon\nu.
\end{aligned}$$

This together with Lemma 3 gives

$$C\sum_{\nu\geq\nu'} N_\nu 2^{m-\nu}\varepsilon\nu \leq \|\tilde{f}\|_1 \leq C(\varepsilon\nu')^{1-p}. \tag{6}$$

Upon dividing the left- and right-hand sides of (6) by $\varepsilon\nu'$ and estimating in the obvious way, we get

$$\sum_{\nu\geq\nu'} N_\nu 2^{m-\nu} \leq \frac{C}{(\varepsilon\nu')^p} \tag{7}$$

for all $\nu' \in \mathbb{N}$, where the constant C is independent of m.

Let $\tilde{h}_\nu = \sup_{2^{-\nu}\leq s\leq 2^{-\nu+1}} h(s)$ and note that for $2^{-\nu} \leq t < 2^{-\nu+1}$ we have $A(t) \leq 2\tilde{h}_\nu$. Recall that I_ν is of the form

$$I_\nu = \cup_{i\in E(\nu)}(\theta_i - A(t_i), \theta_i + A(t_i)),$$

for some index set $E(\nu)$ and some mapping $i \mapsto (\theta_i, t_i)$. For each interval in I_ν we have

$$(\theta_i - A(t_i), \theta_i + A(t_i)) \subset (\theta_i - 10\tilde{h}_\nu, \theta_i + 10\tilde{h}_\nu).$$

Let

$$\tilde{I}_\nu = \cup_{i\in E(\nu)}(\theta_i - 10\tilde{h}_\nu, \theta_i + 10\tilde{h}_\nu).$$

We claim that $\{\theta \in \mathbb{T} : M^{(m)}f(\theta) > \varepsilon\} \subset \cup_{\nu \geq 2}\tilde{I}_\nu$. To prove this, assume that $M^{(m)}f(\theta) > \varepsilon$. Then there is a $t \in (0, 1/2)$ and $|\eta| < h(t)$ such that (5) holds. Assume that $2^{-\nu} \leq t < 2^{-\nu+1}$. If $\theta \in \tilde{I}_\nu$ we are done. If not, the reason must be that $[\theta - \eta - A(t), \theta - \eta + A(t)]$ intersects with some interval I in J_k, for some k. The point is, however, that I, which is of the form $I = (\theta_{i'} - A(t_{i'}), \theta_{i'} + A(t_{i'}))$ for some i', by maximality must have been chosen *before* the intervals in I_ν. Thus, by construction, when J_k is scaled as above it contains θ.

It follows that

$$|\{M^{(m)}f > \varepsilon\}| \leq C \sum_\nu N_\nu \tilde{h}_\nu, \tag{8}$$

for some positive constant C.

We want to show that the right-hand side of (8) tends to zero as $\|f\|_1 \to 0$. To that end, note first that the assumption $\|f\|_1 < \delta$, by the definition of N_ν and by (5), forces N_ν to be zero for $\nu \leq 2^{k_0}$, where $k_0 = k_0(\delta) \to \infty$ as $\delta \to 0$.

If we let $a_\nu = N_\nu 2^{m-\nu} \nu\varepsilon$ and $b_\nu = 2^{\nu-m}(\nu\varepsilon)^{-1}\tilde{h}_\nu$, we get

$$\sum_{\nu > 2^{k_0}} N_\nu \tilde{h}_\nu = \sum_{\nu > 2^{k_0}} a_\nu b_\nu. \tag{9}$$

The definition of a_ν and condition (7) immediately yield that for $k = k_0 + 1, k_0 + 2, ..$ we have

$$\begin{aligned} \sum_{\nu=2^{k-1}+1}^{2^k} a_\nu &\leq 2^k \varepsilon \sum_{\nu=2^{k-1}+1}^{2^k} N_\nu 2^{m-\nu} \\ &\leq 2^k \varepsilon C (2^{k-1}\varepsilon)^{-p} \\ &= C(2^k \varepsilon)^{1-p}. \end{aligned} \tag{10}$$

Furthermore,

$$
\begin{aligned}
\max_{2^{k-1}+1\le\nu\le 2^k} b_\nu &\le 2^{-m}\varepsilon^{-1} \sup_{2^{k-1}+1\le s\le 2^k}\left(2^s s^{-1}\sup_{2^{-s}\le t\le 2^{-s+1}} h(t)\right)\\
&\le C2^{-m}\varepsilon^{-1}\sup_{2^{k-1}+1\le s\le 2^k}\left(\sup_{2^{-s}\le t\le 2^{-s+1}}\frac{h(t)}{t\log 1/t}\right)\\
&\le C2^{-m}\varepsilon^{-1}\sup_{2^{k-1}+1\le s\le 2^k}\left(\sup_{2^{s-1}\le x\le 2^s}\frac{h(x^{-1})x}{\log x}\right)\qquad (11)\\
&\le C2^{-m}\varepsilon^{-1}\sup_{2^{2^{k-1}}\le x\le 2^{2^k}}\frac{h(x^{-1})x}{\log x}\\
&\le C2^{-m}\varepsilon^{-1}\sup_{2^{k-1}\le t\le 2^k} h(2^{-t})2^t t^{-1}\\
&\le C2^{-m}\varepsilon^{-1}c_k,
\end{aligned}
$$

where c_k is defined in Theorem 2.

By (8), (9), (11) and (10), in that order, we get

$$
\begin{aligned}
|\{M^{(m)}f>\varepsilon\}| &\le C\sum_\nu a_\nu b_\nu\\
&\le C\sum_{k=k_0+1}^{\infty}\sum_{\nu=2^{k-1}+1}^{2^k} a_\nu b_\nu\\
&\le C\sum_{k=k_0+1}^{\infty}2^{-m}\varepsilon^{-1}c_k\sum_{\nu=2^{k-1}+1}^{2^k} a_\nu\\
&\le C2^{-m}\varepsilon^{-p}\sum_{k=k_0}^{\infty}c_k 2^{(1-p)k}\\
&= C2^{-m}\varepsilon^{-p}S(k_0),
\end{aligned}
$$

where $S(k_0)=\sum_{k=k_0}^{\infty}c_k 2^{(1-p)k}$.

By invoking (4) with

$$
\mu_m=\frac{2^{-m/(2p)}}{\sum_{i=0}^{\infty}2^{-i/(2p)}},
$$

we have that

$$\begin{aligned} \lambda_{M_1 f}(\varepsilon) &\leq C \cdot S(k_0)\varepsilon^{-p} \sum_{m=0}^{\infty} 2^{-m} \mu_m^{-p} \\ &\leq C\varepsilon^{-p} \cdot S(k_0). \end{aligned}$$

By assumption $S(k_0) \to 0$ as $k_0 \to \infty$ (i.e. as $\delta \to 0$). To sum up, we have shown that

$$\begin{aligned} \lambda_{Mf}(\varepsilon) &\leq \lambda_{M_1 f}(\varepsilon/2) + \lambda_{M_2 f}(\varepsilon/2) \\ &\leq C\varepsilon^{-1}\delta + C\varepsilon^{-p} S(k_0) \\ &\to 0 \end{aligned}$$

as $\delta \to 0$. That concludes the proof of Proposition 1. □

Proof. (Theorem 2) We have already shown that (ii) implies (i), as a consequence of Proposition 1.

To prove that (i) implies (ii), assume that (ii) is false, i.e. that

$$\sum_{k=l}^{\infty} c_k 2^{(1-p)k} = \infty \tag{12}$$

for all l. We shall now construct a function $f \in L^{p,\infty}$ that violates (i).

Let $\varepsilon > 0$ be given. Assume for the moment that

$$\lim_{k \to \infty} c_k 2^{(1-p)k} = 0. \tag{13}$$

Recall that $c_k = \sup_{2^{k-1} \leq s \leq 2^k} h(2^{-s}) 2^s s^{-1}$. Thus, for all $k \in \mathbb{N}$, we can find an $s_k \in [2^{k-1}, 2^k]$, such that

$$c_k/2 < h(2^{-s_k}) 2^{s_k} s_k^{-1} \leq c_k. \tag{14}$$

Let $t_k = 2^{-s_k}$. We shall now construct a subset of $\mathbb{T}$ consisting of a number, n_k, of intervals, each of length t_k and with gaps $h(t_k)$, the first of these intervals starting at $\theta = 0$. Then we do this again, starting from the point where we last stopped, but this time with n_{k+1}, t_{k+1} and $h(t_{k+1})$. We shall proceed in this way and show that sooner or later this process yields a subset whose endpoint exceeds $\theta = \pi$. Then we start over again, i.e. at $\theta = 0$, with the next n_k, t_k and $h(t_k)$.

We shall do this infinitely many times, and along the way we construct the function f, which, as we finally shall show, disproves a.e. convergence.

The construction of the subsets of $\mathbb{T}$ mentioned above is done recursively:

Construction on level l:

Let integers $l < m$ be given. For $l \leq k \leq m$, let $s_k \in [2^{k-1}, 2^k]$ be such that (14) holds.

Let $n_k = [2^{s_k}(s_k\varepsilon)^{-1}(2^k\varepsilon)^{1-p}]$ (integer part) and recall that $t_k = 2^{-s_k}$.

Define τ_k, for $l \leq k \leq m$, recursively as $\tau_l = 0$ and $\tau_{k+1} = \tau_k + n_k(h(t_k)+t_k)$. Now, let

$$E_k^j = [\tau_k + (j-1)(h(t_k) + t_k), \tau_k + (j-1)(h(t_k) + t_k) + t_k], \tag{15}$$

$1 \leq j \leq n_k$ and $l \leq k \leq m$. Note that $|E_k^j| = t_k$ and that the distance between E_k^j and E_k^{j+1} is $h(t_k) = h(2^{-s_k})$. The length (on $\mathbb{T}$) required for this construction is therefore $\sum_{k=l}^{m} n_k(h(t_k) + t_k)$. By (14) we have

$$\begin{aligned} n_k(h(t_k) + t_k) &\leq 2^{s_k}(s_k\varepsilon)^{-1}(2^k\varepsilon)^{1-p}(h(2^{-s_k}) + 2^{-s_k}) \\ &\leq \varepsilon^{-p}2^{(1-p)k}(c_k + s_k^{-1}), \end{aligned} \tag{16}$$

and by (13) it follows that the terms in (16) tend to zero as $k \to \infty$.

Hence, there exists an $L \in \mathbb{N}$ such that for $k \geq L$ we have $n_k(h(t_k)+t_k) < \pi$. We assume from now on that $l \geq L$.

Note that condition (13) implies that $\sum_{k=1}^{\infty} h(2^{-s_k})$ converges. Combining this with (12) and (14), we get

$$\begin{aligned}\sum_{k=l}^{m} n_k(h(t_k)+t_k) &\geq \sum_{k=l}^{m}(2^{s_k}(s_k\varepsilon)^{-1}(2^k\varepsilon)^{1-p}-1)(h(2^{-s_k})+2^{-s_k}) \\ &\geq \varepsilon^{-p}\sum_{k=l}^{m} 2^{(1-p)k}(\frac{c_k}{2}+s_k^{-1})-\sum_{k=l}^{m}(h(2^{-s_k})+2^{-s_k}),\end{aligned}$$

so that

$$\sum_{k=l}^{m} n_k(h(t_k)+t_k) \to \infty \tag{17}$$

as $m \to \infty$. Now, (17) and the fact that $n_k(h(t_k)+t_k) < \pi$ yield the existence of an $m = m(l)$ such that

$$\pi < \sum_{k=l}^{m(l)} n_k(h(2^{-s_k})+2^{-s_k}) < 2\pi. \tag{18}$$

This concludes the construction on level l.

The first time we make this construction we get an $l_1 \in \mathbb{N}$, $l_1 \geq L$, and an $m_1 = m(l_1) \in \mathbb{N}$ such that (18) holds. Let $E_k = \cup_{j=1}^{n_k} E_k^j$, $l_1 \leq k \leq m_1$. Take $f_k(\theta) = s_k\varepsilon\chi_{E_k}(\theta)$ and let

$$f^{(1)}(\theta) = \sum_{k=l_1}^{m_1} f_k(\theta).$$

Given $f^{(j)}$, with corresponding values of l_j and m_j, we now describe how to construct $f^{(j+1)}$. Since $m_j \geq L$, we can make a new construction on level $l_{j+1} = m_j + 1$. This yields an $m_{j+1} = m(l_{j+1})$ and a sequence of functions $\{f_k\}_{k=l_{j+1}}^{m_{j+1}}$. Let $f^{(j+1)}(\theta) = \sum_{k=l_{j+1}}^{m_{j+1}} f_k(\theta)$.

Proceeding inductively, we get a sequence of functions $\{f^{(j)}\}_{j=1}^{\infty}$, $f^{(j)} : \mathbb{T} \to \mathbb{R}_+$. Let

$$\begin{aligned} f(\theta) &= \sum_{j=1}^{\infty} f^{(j)}(\theta) \\ &= \sum_{k=l_1}^{\infty} f_k(\theta). \end{aligned}$$

We shall now see that f is an element in $L^{p,\infty}$, violating condition (i) in Theorem 2. For any $k' \in \mathbb{N}$ we have $\sum_{k \le k'-2} s_k \varepsilon \le \varepsilon \sum_{k \le k'-2} 2^k \le \varepsilon 2^{k'-1} \le \varepsilon s_{k'}$, so that

$$\begin{aligned} \lambda_f(s_{k'}\varepsilon) &\le \sum_{k \ge k'-1} |E_k| \\ &\le \varepsilon^{-p} \sum_{k \ge k'-1} s_k^{-1} 2^{(1-p)k} \\ &\le C\varepsilon^{-p} \sum_{k \ge k'-1} 2^{-pk} \\ &\le C\varepsilon^{-p} 2^{-p(k'-1)} \\ &\le C(s_{k'}\varepsilon)^{-p}. \end{aligned}$$

For small $\alpha > 0$, $\alpha \le C$ say, it is clear that $\alpha^p \lambda_f(\alpha)$ is bounded. If α is large, take $k' \in \mathbb{N}$ such that $2^{k'}\varepsilon \le \alpha < 2^{k'+1}\varepsilon$. Then, by what we have just shown, $\lambda_f(\alpha) \le \lambda_f(2^{k'}\varepsilon) \le C(2^{k'}\varepsilon)^{-p} \le C\alpha^{-p}$. It follows that $f \in L^{p,\infty}$.

Furthermore, we have

$$\begin{aligned} |\{\theta \in \mathbb{T} : f(\theta) \ne 0\}| &\le \sum_{j=1}^{\infty} \sum_{k=l_j}^{m_j} n_k t_k \\ &\le \varepsilon^{-p} \sum_{k=l_1}^{\infty} s_k^{-1} 2^{(1-p)k}, \end{aligned}$$

which can be taken arbitrarily small by just taking l_1 sufficiently large, $|\{\theta \in \mathbb{T} : f(\theta) \ne 0\}| < \pi/2$ say.

Let $\theta \in (0,\pi) \subset \mathbb{T}$ be given. The claim is that there exists a subsequence $\{t_{k_i}\}_{i=1}^{\infty}$ of $\{t_k\}$ such that for each t_{k_i} there is a $z_{k_i} \in \mathcal{A}_h(\theta)$ with $|z_{k_i}| = 1 - t_{k_i}$, and $\mathcal{P}_0 f(z_{k_i}) > \tilde{\varepsilon} > 0$, uniformly in i. When we have shown this we are done, since then it follows that

$$\limsup_{\substack{z \to e^{i\theta} \\ z \in \mathcal{A}_h(\theta)}} \mathcal{P}_0 f(z) > \tilde{\varepsilon}$$

for all $\theta \in (0,\pi)$. But that in turn, compared with $|\{\theta \in \mathbb{T} : f(\theta) = 0\}| > \pi/2$, disproves a.e. convergence.

Consider the construction done on level l_i, $i \in \mathbb{N}$, above. The set obtained there was

$$\cup_{k=l_i}^{m_i} E_k = \cup_{k=l_i}^{m_i} \cup_{j=1}^{n_k} E_k^j.$$

Let $k_i, j_i \in \mathbb{N}$, $l_i \le k_i \le m_i$ and $1 \le j_i \le n_{k_i}$ be such that

$$\text{dist}\,(\theta, E_{k_i}^{j_i}) = \min_{l_i \le k \le m_i} \min_{1 \le j \le n_k} \text{dist}\,(\theta, E_k^j), \tag{19}$$

i.e. $E_{k_i}^{j_i}$ is the interval closest to θ, among all intervals constructed on level l_i. Note that $\text{dist}\,(\theta, E_{k_i}^{j_i}) \le h(t_{k_i})/2$.

By (19) and (15) it follows that there exists η, $|\eta| < h(t_{k_i})/2$, such that either $(\theta - \eta, \theta - \eta + t_{k_i}/2)$ or $(\theta - \eta - t_{k_i}/2, \theta - \eta)$ lies completely within $E_{k_i}^{j_i}$. Assume, without loss of generality, that $(\theta - \eta, \theta - \eta + t_{k_i}/2) \subset E_{k_i}^{j_i}$, and let $E(\theta) = (\theta - \eta, \theta - \eta + t_{k_i}/2)$.

Let $z_{k_i} = (1 - t_{k_i})e^{i(\theta - \eta)}$. Then $|z_{k_i}| = 1 - t_{k_i}$, and trivially $|\theta - (\theta - \eta)| < h(t_{k_i})$, so $z_{k_i} \in \mathcal{A}_h(\theta)$.

We now have

$$\begin{aligned}
\mathcal{P}_0 f(z_{k_i}) &\ge \mathcal{P}_0 f_{k_i}(z_{k_i}) \\
&\ge \mathcal{P}_0 \left(s_{k_i} \varepsilon \chi_{E_{k_i}^{j_i}} \right)(z_{k_i}) \\
&\ge \mathcal{P}_0 \left(s_{k_i} \varepsilon \chi_{E(\theta)} \right)(z_{k_i}) \\
&\ge C s_{k_i} \varepsilon \int_{\mathbb{T}} \frac{\chi_{E(\theta)}(\theta - \eta - \varphi)}{(\log 1/t_{k_i})(t_{k_i} + |\varphi|)}\, d\varphi \\
&= C s_{k_i} (\log 1/t_{k_i})^{-1} \varepsilon \int_{-t_{k_i}/2}^{0} \frac{d\varphi}{t_{k_i} + |\varphi|} \\
&= C\varepsilon.
\end{aligned}$$

This concludes the proof in the case when (13) holds. Note that if (13) does not hold, we should intuitively face an easier task than above, since then the divergence of the series $\sum_{k=0}^{\infty} c_k 2^{(1-p)k}$ is even worse than before, meaning that h is larger. We shall construct a function $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $\theta \in \mathbb{T}$ we have $\mathcal{A}_H(\theta) \subset \mathcal{A}_h(\theta)$, and such that (12) and (13) holds, thus disproving convergence (as above).

To make this precise let $\alpha_k = c_k 2^{(1-p)k}$ and assume that (13) does not hold. That is, we have

$$\alpha_k \not\to 0,$$

as $k \to \infty$. Now, there exists $\kappa > 0$ and a subsequence $\{\alpha_{k_j}\}_{j=j_0}^{\infty} \subset \{\alpha_k\}_{k=1}^{\infty}$, where $j_0 > \kappa^{-1}$, such that $\alpha_{k_j} > \kappa$ for all $j \geq j_0$. Let

$$\beta_i = \begin{cases} 0 \text{ if } i \neq k_j \text{ for all } j \geq j_0 \\ \frac{1}{j} \text{ if } i = k_j \text{ for some } j \geq j_0 \end{cases}$$

It is clear that $\beta_k \leq \alpha_k$ for all $k \in \mathbb{N}$. Furthermore, $\lim_{k\to\infty} \beta_k = 0$ and $\sum_{k=0}^{\infty} \beta_k = \infty$.

For each $k \in \mathbb{N}$ fix an $s_k \in [2^{k-1}, 2^k]$ such that (14) holds. Let

$$H(t) = \begin{cases} \beta_k 2^{(p-1)k} 2^{-s_k - 1} s_k \text{ if } t = 2^{-s_k} \text{ for some } k \in \mathbb{N} \\ 0 \text{ otherwise} \end{cases}.$$

Then $H(2^{-s_k}) \leq \alpha_k 2^{(p-1)k} 2^{-s_k-1} s_k = (c_k/2) 2^{-s_k} s_k \leq h(2^{-s_k})$, the last inequality by (14), so that $H(t) \leq h(t)$ for all (relevant) t. It follows that, for all $\theta \in \mathbb{T}$, we have $\mathcal{A}_H(\theta) \subset \mathcal{A}_h(\theta)$.

However, by construction, H satisfies (12) and (13) so we do not have convergence along $\mathcal{A}_H(\theta)$, and consequently not along $\mathcal{A}_h(\theta)$ either. This concludes the proof of Theorem 2. □

4. Open questions

It is easy to see that the sequence of functions one gets by letting l_1 increase in the definition of f, defined above in order to disprove convergence, also disproves continuity of the maximal operator at 0 in the topology of convergence in measure. It is reasonable to believe that this is not a coincidence. As mentioned in section 1, the Bellow-Jones result [1] for L^∞ basically shows that a.e. convergence is equivalent to continuity of the maximal operator at 0. A similar result for $L^{p,\infty}$ could very well hold and would be interesting in its own right.

To understand better the significant difference between the approach regions for L^p and the ones for L^∞ one could investigate "intermediate spaces". Of course $L^{p,\infty}$, via the inclusions $L^\infty \subset L^{p,\infty} \subset L^q$ for $q < p$, is an example of such a one. Convergence results for boundary functions in BMO($\mathbb{T}$) or suitable Orlicz spaces would certainly be a step along the very same lines.

The author intends to investigate these questions further.

Another generalisation would be to find the Nagel-Stein approach regions for $L^{p,\infty}$. Yet another would be to leave $\mathbb{T}$ and consider general symmetric spaces of rank 1.

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APPROACH REGIONS FOR L^p POTENTIALS WITH RESPECT TO THE SQUARE ROOT OF THE POISSON KERNEL.

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ABSTRACT. If one replaces the Poisson kernel of the unit disc by its square root, then normalised Poisson integrals of L^p boundary functions converge along approach regions wider than the ordinary nontangential cones, as proved by Rönning ($1 \leq p < \infty$) and Sjögren ($p = 1$ and $p = \infty$). In this paper we present new proofs of these results. We also generalise the L^∞ result to higher dimensions.

1. INTRODUCTION

The point of this paper is firstly to present a new and simplified proof for two theorems of almost everywhere convergence type. The advantage of the proof, without being precise, is that it reflects that the convergence results are natural consequences of the norm inequalities that characterise the relevant function spaces (Hölder's inequality for L^p), and corresponding norm estimates of the kernel (associated to the normalised square root of the Poisson kernel operator). In the papers by Rönning, [6], and Sjögren, [9], this correspondence is not obvious (even though, of course, present).

$P(z, \beta)$ will denote the Poisson kernel in the unit disc U,

$$P(z, \beta) = \frac{1}{2\pi} \cdot \frac{1 - |z|^2}{|z - e^{i\beta}|^2}$$

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where $z \in U$ and $\beta \in \partial U \cong \mathbb{R}/2\pi\mathbb{Z} = \mathbb{T} \cong (-\pi, \pi]$.

It is well known that $P(\,\cdot\,, \beta)$ is the real part of a holomorphic function, and thus that it is harmonic.

Let

$$Pf(z) = \int_{\mathbb{T}} P(z, \beta) f(\beta)\, d\beta,$$

the Poisson integral (or extension) of $f \in L^1(\mathbb{T})$. Poisson extensions of continuous boundary functions converge unrestrictedly at the boundary, as the following classical result shows:

Theorem (Schwarz, [7]). *Let $f \in C(\mathbb{T})$. Then $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$, $z \in U$.*

For less regular boundary functions, unrestricted convergence fails (see the result by Littlewood below). One way to control the approach to the boundary is by means of so called (natural) approach regions. For any function $h : \mathbb{R}_+ \to \mathbb{R}_+$ let

$$\mathcal{A}_h(\theta) = \{z \in U : |\arg z - \theta| \leq h(1 - |z|)\}.$$

We refer to $\mathcal{A}_h(\theta)$ as the approach region determined by h at $\theta \in \mathbb{T}$. If $h(t) = \alpha \cdot t$, for some $\alpha > 0$, one refers to $\mathcal{A}_h(\theta)$ as a nontangential cone at $\theta \in \mathbb{T}$. It is natural, but not necessary, to think of h as an increasing function. It should be pointed out that our approach regions certainly have a specific shape. For instance, they are not of Nagel-Stein type.

Theorem (Fatou, [4]). *Let $f \in L^1(\mathbb{T})$. Then, for a.e. $\theta \in \mathbb{T}$, one has that $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, if $h(t) = O(t)$ as $t \to 0$.*

The theorem of Fatou was proved to be best possible, in the following sense:

Theorem (Littlewood, [5]). *Let $\gamma_0 \subset U \cup \{1\}$ be a simple closed curve, having a common tangent with the circle at the point 1. Let γ_θ be the rotation of γ_0 by the angle θ. Then there exists a bounded harmonic function f in U with the property that, for a.e. $\theta \in \mathbb{T}$, the limit of f along γ_θ does not exist.*

Littlewood's result has been generalised in several directions. For instance, with the same assumptions as in Littlewood's theorem, Aikawa [1], proves that convergence can be made to fail at *any* point $\theta \in \mathbb{T}$.

In this paper we treat convergence questions for normalised Poisson integrals with respect to the square root of the Poisson kernel.

If f and g are positive functions we say that $f \lesssim g$ provided that there exists some positive constant C such that $f(x) \leq Cg(x)$. We write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

Let

$$P_0 f(z) = \int_{\mathbb{T}} \sqrt{P(z,\beta)} f(\beta)\, d\beta.$$

To get boundary convergence, it is necessary to normalise P_0, since it is readily checked that, for $|z| > 1/2$,

$$P_0 1(z) \sim \sqrt{1-|z|} \log \frac{1}{1-|z|},$$

which does not tend to 1, anywhere, as $|z| \to 1$. Poisson integrals, with respect to powers greater than or equal to 1/2 of the Poisson kernel, arise naturally as eigenfunctions to the hyperbolic Laplace operator. When one considers boundary convergence properties of the corresponding normalisations, it is only the square root integral extension that exhibits special properties. Normalisation of higher power integrals behave just like the Poisson integral itself, in the context of boundary convergence. In connection with representation theory of the group $SL(2, \mathbb{R})$, one uses certain complex powers of the Poisson kernel.

Denote the normalised operator by $\mathcal{P}_0$, i.e.

$$\mathcal{P}_0 f(z) = \frac{P_0 f(z)}{P_0 1(z)}.$$

Definition 1. If $1 \leq p < \infty$ let

$$S_p = \{h : \mathbb{R}_+ \to \mathbb{R}_+ \ : \ h(t) = O(t(\log 1/t)^p) \text{ as } t \to 0\},$$

and let

$$S_\infty = \{h : \mathbb{R}_+ \to \mathbb{R}_+ \ : \ h(t) = O(t^{1-\varepsilon}) \text{ for all } \varepsilon > 0 \text{ as } t \to 0\}.$$

Note that $S_p \subset S_\infty$.

Several convergence results for $\mathcal{P}_0$ are known, in different settings. We state a few below:

Theorem. *Let $f \in C(\mathbb{T})$. Then, for any $\theta \in \mathbb{T}$, one has that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$.*

This result follows if one just notes that $\mathcal{P}_0$ is a convolution operator with a kernel which behaves like an approximate identity in $\mathbb{T}$. In the next section we give explicit expressions for the kernel.

Theorem (Sjögren, [8]). *Let $f \in L^1(\mathbb{T})$. Then, for a.e. $\theta \in \mathbb{T}$, one has that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, if $h \in S_1$.*

Theorem (Rönning, [6]). *Let $1 \leq p < \infty$ be given and let $f \in L^p(\mathbb{T})$. Then, for a.e. $\theta \in \mathbb{T}$, one has that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, if $h \in S_p$ (and only if if h is assumed to be monotone).*

The results by Sjögren and Rönning were proved via weak type estimates for the corresponding maximal operators, and approximation with continuous functions.

Theorem (Sjögren, [9]). *The following conditions are equivalent for any increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any $f \in L^\infty(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $h \in S_\infty$.

In his proof, Sjögren never uses the assumption that h should be increasing. Thus, it remains valid for an even larger class of functions h. The proof of this result differs much from the L^p case, since one has to take into account that the continuous functions are not dense in L^∞. Sjögren instead used a result by Bellow and Jones, [2], "A Banach principle for L^∞". Following the same lines, the author proved the following ($L^{p,\infty}$ denotes weak L^p):

Theorem (Brundin, [3]). *Let $1 < p < \infty$ be given. Then the following conditions are equivalent for any function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any $f \in L^{p,\infty}(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $\sum_{k=0}^\infty \sigma_k < \infty$, *where* $\sigma_k = \sup_{2^{-2^k} \leq s \leq 2^{-2^{k-1}}} \frac{h(s)}{s(\log 1/s)^p}$.

In this paper we prove the following theorem, with simpler and different methods than those of Rönning and Sjögren.

Theorem 1. *Let $1 \leq p \leq \infty$ be given and let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be any function. Then the following conditions are equivalent:*

(*i*) *For any* $f \in L^p(\mathbb{T})$ *one has, for almost all* $\theta \in \mathbb{T}$, *that* $\mathcal{P}_0 f(z) \to f(\theta)$ *as* $z \to e^{i\theta}$ *and* $z \in \mathcal{A}_h(\theta)$.

(*ii*) $h \in S_p$.

Obtaining (easily) the result for L^∞ first, we shall use this to treat the L^p case. As in the proofs of Sjögren and Rönning, we decompose the kernel into two parts, one "local" and one "global". The global part is easy. As it turns out here, the local part is also easy. In previous proofs, rather complicated calculations were used to prove that the associated maximal operator is "sufficiently continuous" at 0 (e.g. weak type (p,p) estimates). As it turns out, however, the local part simply does not contribute to convergence and can be treated directly (without estimates of any maximal operator).

Later, we generalise the case $p = \infty$ to higher dimensions (see section "Higher dimensional results for L^∞").

2. The proof of Theorem 1

Before turning to the proof we introduce the notation that we shall use.

Let $t = 1 - |z|$ and $z = (1-t)e^{i\theta}$. Then

$$\mathcal{P}_0 f(z) = R_t * f(\theta),$$

where the convolution is taken in $\mathbb{T}$ and

$$R_t(\theta) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t(2-t)}}{|(1-t)e^{i\theta} - 1|} \frac{1}{P_0 1(1-t)}.$$

Since we are interested only in small values of t, we might as well from now on assume that $t < 1/2$. Then $P_0 1(1-t) \sim \sqrt{t}\log 1/t$, and thus the order of magnitude of R_t is given by

$$R_t(\theta) \sim Q_t(\theta) = \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta|}.$$

Now, let τ_η denote the translation $\tau_\eta f(\theta) = f(\theta - \eta)$. Then the convergence condition (*i*) in Theorem 1 above means

$$\lim_{\substack{t \to 0 \\ |\eta| < h(t)}} \tau_\eta R_t * f(\theta) = f(\theta).$$

Let

$$R_t(\theta) = R_t^1(\theta) + R_t^2(\theta)$$

where

$$R_t^1(\theta) = R_t(\theta)\chi_{\{|\theta|<2h(t)\}},$$

and let Q_t^1 and Q_t^2 be the corresponding cutoffs of the kernel Q_t.

Define

(1) $$Mf(\theta) = \sup_{\substack{|\eta|<h(t)\\ t<1/2}} \tau_\eta Q_t^2 * |f|(\theta).$$

Proposition 1. *Assume that* $1 \le p \le \infty$ *is given and assume that condition* (ii) *in Theorem 1 holds.*

- (a) *For a given* $f \in L^p$ *it holds for a.e.* $\theta \in \mathbb{T}$ *that*

$$\lim_{\substack{t\to 0\\ |\eta|<h(t)}} \tau_\eta Q_t^1 * f(\theta) = 0.$$

- (b) $Mf \lesssim M_{HL}f$, *where* M_{HL} *denotes the ordinary Hardy-Littlewood maximal operator.*

Let us for the moment postpone the proof and instead see how Proposition 1 is used to prove the implication $(ii) \Rightarrow (i)$ in Theorem 1.

Proof. (*Theorem 1,* $(ii) \Rightarrow (i)$) By Proposition 1, part (a), it suffices to prove that, for almost all $\theta \in \mathbb{T}$, one has

(2) $$\lim_{\substack{t\to 0\\ |\eta|<h(t)}} \tau_\eta R_t^2 * f(\theta) = f(\theta).$$

Note that, if $f \in C(\mathbb{T})$, then

$$\lim_{\substack{t\to 0\\ |\eta|<h(t)}} \tau_\eta R_t * f(\theta) = f(\theta).$$

This fact, together with Proposition 1, part (a), and $C(\mathbb{T}) \subset L^p(\mathbb{T})$ gives that (2) must hold for $f \in C(\mathbb{T})$. Hence, to establish (2) for any $f \in L^p$, it suffices to prove that the corresponding maximal operator is of weak type $(1,1)$. But since it is dominated by M, which in turn is dominated by M_{HL} by Proposition 1, part (b), we are done. □

We now proceed with the proof of Proposition 1. The proof of implication $(i) \Rightarrow (ii)$ in Theorem 1 can be found in the end of this section.

Proof. (*Proposition 1*) We start by proving part (b). Since $|\eta| < h(t)$, we have that

$$\begin{aligned} \tau_\eta Q_t^2(\theta) &= \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta - \eta|} \chi_{\{|\theta-\eta|>2h(t)\}} \\ &\lesssim \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta|}, \end{aligned}$$

which is a decreasing function of θ, whose integral in $\mathbb{T}$ is uniformly bounded in t. It is well known that convolution with such a function is controlled by the Hardy-Littlewood maximal operator. Part (b) is thus established.

We proceed now with the proof of part (a), in the case $p = \infty$.

Let $\varepsilon > 0$ be given. We have

$$\begin{aligned} \tau_\eta Q_t^1 * |f|(\theta) &= \frac{1}{\log 1/t} \int_{|\varphi|<2h(t)} \frac{|f(\theta - \eta - \varphi)|}{t + |\varphi|} \, d\varphi \\ &\leq \frac{\|f\|_\infty}{\log 1/t} \int_{|\varphi|<2h(t)} \frac{d\varphi}{t + |\varphi|} \\ &\lesssim \frac{\|f\|_\infty}{\log 1/t} \log\left(h(t)/t\right). \end{aligned}$$

By condition (ii) in Theorem 1, we have that $h(t) \leq Ct^{1-\varepsilon}$, and we get

$$\limsup_{\substack{t \to 0 \\ |\eta|<h(t)}} \tau_\eta Q_t^1 * |f|(\theta) \lesssim \varepsilon \|f\|_\infty,$$

as desired.

Now, assume that $1 \leq p < \infty$ and that $q = p/(p-1)$ (where $q = \infty$ if $p = 1$). Assume also that $f \geq 0$, without loss of generality.

Note, first of all, that

$$\|Q_t\|_q \leq C_q \frac{1}{t^{1/p} \log 1/t} \tag{3}$$

Write $f = f_- + f_R$, where $f_- = f\chi_{\{f\leq R\}} \in L^\infty$, and where $R > 0$ is arbitrary. By (3) and by assumption we have, for $t \in (0, 1/2)$ and $\theta \in \mathbb{T}$, that

$$\begin{aligned}
\tau_\eta Q_t^1 * f_R(\theta) &= \int_{|\varphi|<2h(t)} Q_t(\varphi) f_R(\theta - \varphi - \eta) \\
&\lesssim \frac{1}{t^{1/p}\log 1/t} \cdot \left(\int_{|\varphi+\eta-\theta|\leq 2h(t)} f_R(\varphi)^p \, d\varphi \right)^{1/p} \\
&\lesssim \frac{1}{t^{1/p}\log 1/t} \cdot \left(\int_{|\varphi-\theta|\leq 3h(t)} f_R(\varphi)^p \, d\varphi \right)^{1/p} \\
&\lesssim \left(\frac{h(t)}{t(\log 1/t)^p} \cdot \frac{1}{6h(t)} \int_{|\varphi-\theta|\leq 3h(t)} f_R(\varphi)^p \, d\varphi \right)^{1/p} \\
&\lesssim \left(\frac{1}{6h(t)} \int_{|\varphi-\theta|\leq 3h(t)} f_R(\varphi)^p \, d\varphi \right)^{1/p} .
\end{aligned}$$

For a.e. $\theta \in \mathbb{T}$ (Lebesgue points of f_R^p) we have (using Proposition 1, part (a) for L^∞) that

$$\begin{aligned}
\limsup_{\substack{t\to 0 \\ |\eta|<h(t)}} \tau_\eta Q_t^1 * f(\theta) &\leq \limsup_{\substack{t\to 0 \\ |\eta|<h(t)}} \tau_\eta Q_t^1 * f_-(\theta) + \limsup_{\substack{t\to 0 \\ |\eta|<h(t)}} \tau_\eta Q_t^1 * f_R(\theta) \\
&\leq 0 + C \cdot f_R(\theta).
\end{aligned}$$

By choosing R sufficiently large, we can make $f_R(\theta) = 0$ on a set with measure arbitrarily close to 2π , so part (a) of Proposition 1 is now established also for $1 \leq p < \infty$. □

Proof. (*Proof of the implication* $(i) \Rightarrow (ii)$) We assume here that $1 < p < \infty$, since the results for $p = 1$ and $p = \infty$ are already established by Sjögren[1]. Assume that condition (ii) in Theorem 1 is false. We show that this implies that (i) is false also.

[1] In section "Higher dimensional results for L^∞", we give a proof of the case $p = \infty$ in two dimensions, which is actually just a trivial extension of Sjögrens proof.

Assume that

$$\limsup_{t\to 0} \frac{h(t)}{t(\log 1/t)^p} = \infty, \tag{4}$$

Pick any decreasing sequence $\{t_i\}_1^\infty$, converging to 0, such that

$$1 \leq \frac{h(t_i)}{t_i(\log 1/t_i)^p} \uparrow \infty, \tag{5}$$

as $i \to \infty$. Let

$$f_i(\varphi) = t_i^{1/(p-1)} \log 1/t_i \cdot \left(\frac{1}{t_i + |\varphi|}\right)^{1/(p-1)} \cdot \chi_{\{|\varphi|<h(t_i)\}},$$

Now,

$$\begin{aligned} \|f_i\|_p^p &\lesssim t_i^{p/(p-1)} (\log 1/t_i)^p \int_0^{h(t_i)} \left(\frac{1}{t_i + \varphi}\right)^{p/(p-1)} d\varphi \\ &\lesssim t_i^{p/(p-1)} (\log 1/t_i)^p t_i^{1-p/(p-1)} \\ &= t_i (\log 1/t_i)^p, \end{aligned}$$

where the constant depends only on p. It follows that

$$\frac{h(t_i)}{\|f_i\|_p^p} \geq C(p) \cdot \frac{h(t_i)}{t_i(\log 1/t_i)^p}.$$

By (5) the right hand side tends to ∞ as $i \to \infty$. Thus, by standard techniques, we can pick a subsequence of $\{t_i\}$, with possible repetitions, for simplicity denoted $\{t_i\}$ also, such that

$$\sum_1^\infty h(t_i) = \infty, \tag{6}$$

and

$$\sum_1^\infty \|f_i\|_p^p < \infty. \tag{7}$$

Let $A_1 = h(t_1)$, and for $n \geq 2$ let $A_n = h(t_n) + \sum_{j=1}^{n-1} 2h(t_j)$. By (6) one has that $\lim_{n\to\infty} A_n = \infty$.

Define (on $\mathbb{T}$) $F_j(\varphi) = \tau_{A_j} f_j(\varphi)$, and let

$$F^{(N)}(\varphi) = \sup_{j\geq N} F_j(\varphi).$$

It is clear by construction that any given $\varphi \in \mathbb{T}$ lies in the support of infinitely many F_j:s.

Since $[F^{(N)}(\varphi)]^p = \sup_{j\geq N}[F_j(\varphi)]^p \leq \sum_{j\geq N}[F_j(\varphi)]^p$, it follows that

$$\begin{aligned} \|F^{(N)}\|_p^p &\leq \sum_{j=N}^{\infty} \|F_i\|_p^p \\ &= \sum_{j=N}^{\infty} \|f_i\|_p^p \to 0 \end{aligned}$$

as $N \to \infty$, by (7). Thus, in particular, $F^{(N)} \in L^p$ for any $N \geq 1$.

For $\theta \in \mathbb{T}$ and a given $\xi_0 > 0$ we can, by construction, find $j \in \mathbb{N}$ so that $\theta \in \operatorname{supp}(F_j)$ and so that $t_j \in (0, \xi_0)$. We can then choose η, with $|\eta| < h(t_j)$, so that $\theta - \eta \equiv A_j \mod 2\pi$. It follows that

$$\limsup_{t\to 0,\ |\eta|<h(t)} \mathcal{P}_0 F^{(N)}((1-t)e^{i(\theta-\eta)}) \geq \limsup_{j\to\infty} \mathcal{P}_0 F_j((1-t_j)e^{iA_j}).$$

We have

$$
\begin{aligned}
\mathcal{P}_0 F_j((1-t_j)e^{iA_j}) &\geq \frac{C}{\log 1/t_j} \int_{|\varphi|<h(t_j)} \frac{F_j(A_j-\varphi)}{t_j+|\varphi|}\, d\varphi \\
&= \frac{C}{\log 1/t_j} \int_{|\varphi|<h(t_j)} \frac{f_j(\varphi)}{t_j+|\varphi|}\, d\varphi \\
&= 2Ct_j^{1/(p-1)} \int_0^{h(t_j)} \left(\frac{1}{t_j+\varphi}\right)^{1+1/(p-1)} d\varphi \\
&\geq C_p'' \\
&> 0.
\end{aligned}
$$

To sum up, we have shown that for any $\theta \in \mathbb{T}$ one has

$$\limsup_{t\to 0,\ |\eta|<h(t)} \mathcal{P}_0 F^{(N)}((1-t)e^{i(\theta-\eta)}) \geq C_p'' > 0.$$

Take N so large so that the measure of $\{F^{(N)} > C_p''/2\}$ is small, and a.e. convergence to $F^{(N)}$ is disproved. $\square$

3. Higher dimensional results for L^∞

In this section we prove results for the polydisc U^n, with bounded boundary functions. To simplify, we give the notation and proof for $n = 2$. The generalisation to arbitrary n is clear.

We define the Poisson integral of $f \in L^1(\mathbb{T}^2)$ to be

$$Pf(z_1, z_2) = \int_{\mathbb{T}^2} P(z_1, z_2, \beta_1, \beta_2) f(\beta_1, \beta_2)\, d\beta_1\, d\beta_2,$$

where

$$P(z_1, z_2, \beta_1, \beta_2) = P(z_1, \beta_1) P(z_2, \beta_2).$$

For any functions $h_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, let

$$\mathcal{A}_{h_1,h_2}(\theta_1, \theta_2) = \{(z_1, z_2) \in U^2 : |\arg z_i - \theta_i| \leq h_i(1-|z_i|),\ i = 1, 2\}. \tag{8}$$

We refer to $\mathcal{A}_{h_1,h_2}(\theta_1, \theta_2)$ as the approach region determined by h_1, h_2 at $(\theta_1, \theta_2) \in \mathbb{T}^2$.

Let

$$P_0 f(z_1, z_2) = \int_{\mathbb{T}^2} \sqrt{P(z_1, z_2, \beta_1, \beta_2)}\, f(\beta_1, \beta_2)\, d\beta_1\, d\beta_2,$$

and denote the normalised operator by $\mathcal{P}_0$, i.e.

$$\mathcal{P}_0 f(z_1, z_2) = \frac{P_0 f(z_1, z_2)}{P_0 1(z_1, z_2)}.$$

We shall prove the following theorem:

Theorem 2. *The following conditions are equivalent for any functions $h_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., n$:*

(*i*) *For any $f \in L^\infty(\mathbb{T}^n)$ one has for almost all $(\theta_1, ..., \theta_n) \in \mathbb{T}^n$ that*

$$\mathcal{P}_0 f(z_1, ..., z_n) \to f(\theta_1, ..., \theta_n)$$

as $(z_1, ..., z_n) \to (\theta_1, ..., \theta_n)$ and $(z_1, ..., z_n) \in \mathcal{A}_{h_1,...,h_n}(\theta_1, ..., \theta_n)$.

(*ii*) *$h_i \in S_\infty$, $i = 1, ..., n$. (For S_∞, see Definition 1.)*

4. The proof of Theorem 2

We may assume, without loss of generality, that $\lim_{t\to 0} h_j(t)/t = \infty$, $j = 1, 2$.

We shall begin by proving the implication $(ii) \Rightarrow (i)$ in Theorem 2.

Let $t_j = 1 - |z_j|$ and $z_j = (1 - t_j)e^{i\theta_j}$, $j = 1, 2$. Then

$$\mathcal{P}_0 f(z_1, z_2) = R_{t_1,t_2} * f(\theta_1, \theta_2),$$

where the convolution is taken in $\mathbb{T}^2$ and

$$R_{t_1,t_2}(\theta_1, \theta_2) = \prod_{j=1}^{2} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t_j(2 - t_j)}}{|(1 - t_j)e^{i\theta_j} - 1|} \frac{1}{P_0^{(1)} 1(1 - t_j)},$$

$P_0^{(1)}$ denoting the square root operator in *one* variable.

As before, we are interested only in small values of t_j, so we assume from now on that $t_j < 1/2$, $j = 1, 2$. Then $P_0^{(1)} 1(1 - t) \sim \sqrt{t} \log 1/t$, and thus the order of magnitude of R_{t_1,t_2} is given by

$$\begin{aligned} R_{t_1,t_2}(\theta_1, \theta_2) &\sim Q_{t_1,t_2}(\theta_1, \theta_2) \\ &= \prod_{j=1}^{2} \frac{1}{\log 1/t_j} \cdot \frac{1}{t_j + |\theta_j|}. \end{aligned}$$

Now, let τ_{η_1,η_2} denote the translation $\tau_{\eta_1,\eta_2} f(\theta_1, \theta_2) = f(\theta_1 - \eta_1, \theta_2 - \eta_2)$. Then the convergence condition (i) in Theorem 2 above means

$$\lim_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\, j=1,2}} \tau_{\eta_1,\eta_2} R_{t_1,t_2} * f(\theta_1, \theta_2) = f(\theta_1, \theta_2).$$

We are now ready to prove Theorem 2.

Proof. Assume that condition (ii) holds. We prove that it implies (i).

If we let

$$R_{t_1,t_2}(\theta_1, \theta_2) = R^1_{t_1,t_2}(\theta_1, \theta_2) + R^2_{t_1,t_2}(\theta_1, \theta_2)$$

where

$$R^2_t(\theta_1, \theta_2) = R_{t_1,t_2}(\theta_1, \theta_2)\chi_{\{|\theta_j|\geq 2h_j(t_j),\, j=1,2\}}(\theta_1, \theta_2),$$

we claim that

(9) $$\lim_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\, j=1,2}} \tau_{\eta_1,\eta_2} R^1_{t_1,t_2} * f(\theta_1, \theta_2) = 0$$

and, for almost all $(\theta_1, \theta_2) \in \mathbb{T}^2$,

(10) $$\lim_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\, j=1,2}} \tau_{\eta_1,\eta_2} R^2_{t_1,t_2} * f(\theta_1, \theta_2) = f(\theta_1, \theta_2).$$

To prove (9), it suffices to prove that

$$\limsup_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\, j=1,2}} \tau_{\eta_1,\eta_2} Q^1_{t_1,t_2} * f(\theta_1, \theta_2) = 0,$$

where $Q^1_{t_1,t_2}$ corresponds to Q_{t_1,t_2} as $R^1_{t_1,t_2}$ corresponds to R_{t_1,t_2}. Note that $Q^1_{t_1,t_2}$ is supported in a set where $|\varphi_j| < 2h_j(t_j)$ for $j = 1$ or $j = 2$. Assume, without loss of generality, that $|\varphi_1| < 2h_1(t_1)$ and observe that we then have

$$Q^1_{t_1,t_2}(\varphi_1, \varphi_2) \leq \chi_{\{|\varphi_1|<2h_1(t_1)\}}(\varphi_1, \varphi_2) \prod_{j=1}^{2} \frac{1}{\log 1/t_1} \cdot \frac{1}{t_j + |\varphi_j|}.$$

It follows that

$$\begin{aligned}\tau_{\eta_1,\eta_2} Q^1_{t_1,t_2} * |f|(\theta_1,\theta_2) &\le \|f\|_\infty \int_{\mathbb{T}^2} Q^1_{t_1,t_2}(\varphi_1,\varphi_2)\, d\varphi_1\, d\varphi_2 \\ &= \frac{\|f\|_\infty}{(\log 1/t_1)(\log 1/t_2)} \cdot \\ &\quad \int_{|\varphi_1|<2h_1(t_1)} \frac{d\varphi_1}{t_1+|\varphi_1|} \cdot \int_{\mathbb{T}} \frac{d\varphi_2}{t_2+|\varphi_2|} \\ &\lesssim \frac{\|f\|_\infty}{\log 1/t_1} \log\left(h_1(t_1)/t_1\right).\end{aligned}$$

Let $\varepsilon > 0$ be given. By condition (ii) in Theorem 2, we have that $h_1(t_1) \le Ct_1^{1-\varepsilon}$. Thus,

$$\limsup_{\substack{t_1,t_2\to 0 \\ |\eta_j|<h_j(t_j),\, j=1,2}} \tau_{\eta_1,\eta_2} Q^1_{t_1,t_2} * f(\theta_1,\theta_2) \lesssim \varepsilon \|f\|_\infty,$$

and (9) follows.

To prove (10), it now suffices to prove that the maximal operator M, defined by

$$Mf(\theta) = \limsup_{\substack{t_1,t_2\to 0 \\ |\eta_j|<h_j(t_j),\, j=1,2}} \tau_{\eta_1,\eta_2} Q^2_{t_1,t_2} * |f|(\theta_1,\theta_2),$$

is dominated by a strong type (p,p) operator, for some $p \ge 1$. Then convergence follows by standard arguments, since the continuous functions, for which unrestricted convergence holds for $R^2_{t_1,t_2}$, form a dense subset of L^p. Since $|\eta_j| < h_j(t_j)$, $j = 1,2$, we have that

$$\begin{aligned}\tau_{\eta_1,\eta_2} Q^2_{t_1,t_2}(\theta_1,\theta_2) &= \prod_{j=1}^{2} \frac{1}{\log 1/t_j} \cdot \frac{1}{t_j + |\theta_j - \eta_j|} \chi_{\{|\theta_j-\eta_j|\ge 2h_j(t_j)\}} \\ &\lesssim \prod_{j=1}^{2} \frac{1}{\log 1/t_j} \cdot \frac{1}{t_j+|\theta_j|}.\end{aligned}$$

Each factor in the above product is a decreasing function of $|\theta_j|$ whose integral in $\mathbb{T}$ is bounded uniformly in t_j. Convolution (in one variable) with such a function is dominated by the Hardy-Littlewood maximal operator, as is well known. Since, for example, $L^\infty \subset L^2$ and since the Hardy-Littlewood maximal operator is of strong type $(2,2)$, we have that

$$\begin{aligned}\tau_{\eta_1,\eta_2} Q^2_{t_1,t_2} * |f|(\theta_1,\theta_2) &\leq \frac{1}{\log 1/t_2} \cdot \int_{\mathbb{T}} \frac{1}{t_2+|\varphi_2|} M^{(1)}_{HL} f(\theta_1, \theta_2-\varphi_2)\, d\varphi_2, \\ &\leq M^{(2)}_{HL} M^{(1)}_{HL} f(\theta_1,\theta_2)\end{aligned}$$

where $M^{(j)}_{HL}$ denotes the ordinary (one-dimensional) Hardy-Littlewood maximal operator in variable j. But, since $M^{(2)}_{HL} M^{(1)}_{HL}$ is of strong type $(2,2)$ (weak type is sufficient), we are done.

It remains to prove that (i) implies (ii). The method is similar to that of Sjögren. Assume that (ii) is false. Without loss of generality, we may assume that there exists $\varepsilon > 0$ and a sequence $s_k \to 0$, such that $h_1(s_k)/s_k^{1-\varepsilon} \to \infty$. We may also assume that

$$\sum_{k=1}^{\infty} \frac{s_k^{1-\varepsilon}}{h_1(s_k)} < \infty.$$

Let $E_k \subset \mathbb{T}$ be the union of at most $C/h_1(s_k)$ intervals of length $s_k^{1-\varepsilon}$, chosen such that the distance from E_k to any point in $\mathbb{T}$ is at most $h_1(s_k)$. If $\theta_1 \in \partial E_k$, it is clear that

$$\begin{aligned}\mathcal{P}_0 \chi_{E_k \times \mathbb{T}} \Big((1-s_k)e^{i\theta_1}, (1-t)e^{i\theta_2} \Big) &\geq \frac{C}{(\log 1/s_k)(\log 1/t)} \cdot \\ &\quad \int_0^{s_k^{1-\varepsilon}} \frac{d\varphi_1}{s_k+\varphi_1} \cdot \int_{\mathbb{T}} \frac{d\varphi_2}{t+|\varphi_2|} \\ &\geq C\varepsilon.\end{aligned}$$

Thus, for any $(\theta_1,\theta_2) \in \mathbb{T}^2$ we have

$$\sup_{|\eta_j|<h_j(t_j),\, j=1,2} \mathcal{P}_0 \chi_{E_k \times \mathbb{T}} \Big((1-s_k)e^{i(\theta_1-\eta_1)}, (1-t)e^{i(\theta_2-\eta_2)} \Big) \geq C\varepsilon.$$

Now, since $|E_k| \lesssim s_k^{1-\varepsilon}/h_1(s_k)$, we can choose k_0 so large that the measure of $E = \cup_{k \geq k_0} E_k$ is arbitrarily small. But clearly

$$\limsup_{\substack{t_1,t_2 \to 0 \\ |\eta_j|<h_j(t_j),\, j=1,2}} \mathcal{P}_0 \chi_{E \times \mathbb{T}} \Big((1-t_1)e^{i(\theta_1-\eta_1)}, (1-t_2)e^{i(\theta_2-\eta_2)} \Big) \geq C\varepsilon$$

for each $(\theta_1,\theta_2) \in \mathbb{T}^2$. We have shown that a.e. convergence to $\chi_{E\times\mathbb{T}}$ along the region defined by h_1 and h_2 fails. This completes the proof. □

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APPROACH REGIONS FOR THE SQUARE ROOT OF THE POISSON KERNEL AND BOUNDARY FUNCTIONS IN CERTAIN ORLICZ SPACES.

MARTIN BRUNDIN

ABSTRACT. If the Poisson integral of the unit disc is replaced by its square root, it is known that normalised Poisson integrals of L^p and weak L^p boundary functions converge along approach regions wider than the ordinary nontangential cones, as proved by Rönning and the author, respectively. In this paper we characterise the approach regions for boundary functions in two general classes of Orlicz spaces. The first of these classes contains spaces L^Φ, having the property $L^\infty \subset L^\Phi \subset L^p$, $1 \leq p < \infty$. The second contains spaces L^Φ that resemble L^p spaces.

1. INTRODUCTION

Let $P(z,\varphi)$ be the standard Poisson kernel in the unit disc U,

$$P(z,\varphi) = \frac{1}{2\pi} \cdot \frac{1-|z|^2}{|z-e^{i\varphi}|^2}$$

where $z \in U$ and $\varphi \in \partial U = \mathbb{T} \cong (-\pi, \pi]$.

Let

$$Pf(z) = \int_{\mathbb{T}} P(z,\varphi) f(\varphi)\, d\varphi,$$

the Poisson integral of $f \in C(\mathbb{T})$. Then $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$, as was first shown by Schwarz [12].

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For any function $h : \mathbb{R}_+ \to \mathbb{R}_+$ let

$$(1) \qquad \mathcal{A}_h(\theta) = \{z \in U : |\arg z - \theta| \leq h(1 - |z|)\}.$$

We refer to $\mathcal{A}_h(\theta)$ as the (natural) approach region determined by h at $\theta \in \mathbb{T}$. Note that, even though we use the word "region", we have not imposed any openness assumptions on $\mathcal{A}_h(\theta)$. It is natural, but not necessary, to think of h as an increasing and continuous function, with $h(t) \to 0$ as $t \to 0$. Later, we shall let $z \in U$ approach the boundary ($z \to e^{i\theta}$) within $\mathcal{A}_h(\theta)$. We may think of the function h as a parameter that measures the maximal admissible tangency a curve along which z approaches the boundary may have.

If we only assume that $f \in L^1(\mathbb{T})$, the convergence properties are different than in the case of continuous functions. Fatou [7] proved in 1906 that if $h(t) = \alpha t$, $\alpha > 0$, then $Pf(z) \to f(\theta)$ a.e. as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, i.e. the convergence is non-tangential. To prove this, one establishes a weak type $(1, 1)$ estimate for the corresponding maximal operator. The result then follows via standard techniques. Littlewood [8] proved that the theorem, in a certain sense, is best possible:

Theorem (Littlewood, [8]). *Let $\gamma_0 \subset U \cup \{1\}$ be a simple closed curve, having a common tangent with the circle at the point 1. Let γ_θ be the rotation of γ_0 by the angle θ. Then there exists a bounded harmonic function f in U with the property that, for a.e. $\theta \in \mathbb{T}$, the limit of f along γ_θ does not exist.*

Littlewood's result has been generalised, in different directions. For example, given a curve $\gamma_0 \subset U \cup \{1\}$ that touches $\mathbb{T}$ tangentially at the point 1, Aikawa [1] constructs a bounded harmonic function f in U such that, for any point $\theta \in \mathbb{T}$, the limit $\lim_{z \to e^{i\theta}} f(z)$ does not exist along the curve γ_θ, where γ_θ is the rotation of γ_0 by the angle θ.

It is worth noting that one could consider more general approach regions, not necessarily given in the form (1). This is done, for instance, in [9] by Nagel and Stein. The essence of that paper is to prove that, whereas tangential *curves* are not good for convergence (Littlewood), tangential *sequences* may be.

For a more complete treatise on the theorems and the general theory mentioned so far, see [6].

For $z = x + iy$ let

$$L_z = \frac{1}{4}(1 - |z|^2)^2(\partial_x^2 + \partial_y^2),$$

the hyperbolic Laplacian. Then

$$u(z) = P_\lambda f(z) = \int_{\mathbb{T}} P(z, \varphi)^{\lambda+1/2} f(\varphi)\, d\varphi,$$

for $\lambda \geq 0$, defines a solution of the equation

$$L_z u = (\lambda^2 - 1/4)u.$$

In connection with representation theory of the group $SL(2, \mathbb{R})$, one uses the powers $P(z, \varphi)^{i\alpha+1/2}$, $\alpha \in \mathbb{R}$, of the Poisson kernel.

We shall use the notation $f \lesssim g$, for positive functions f and g, if there exists a constant $C > 0$ such that $f \leq Cg$ at all points, and we write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

Since

$$P_0 1(z) \sim (1 - |z|)^{1/2} \log \frac{1}{1 - |z|},$$

as $|z| \to 1$, one sees that the one has to normalise P_0 in order to get boundary convergence ($P_0 1(z)$ does not converge to 1). Thus, the operator that we shall be concerned with is defined by

$$\mathcal{P}_0 f(z) = \frac{P_0 f(z)}{P_0 1(z)}.$$

For $\lambda > 0$ one has that

$$P_\lambda 1(z) \sim (1 - |z|)^{1/2-\lambda},$$

and if one considers normalised $\lambda-$Poisson integrals for $\lambda > 0$, i.e. $\mathcal{P}_\lambda f(z) = P_\lambda f(z)/P_\lambda 1(z)$, the convergence properties are the same as for the ordinary Poisson integral. This is because the kernels essentially behave in the same way.

We summarise the known convergence results in the following table. It should be read from left to right as "For all $f \in$[**Function space**] one has for almost all $\theta \in \mathbb{T}$ that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$ [**Conv.**]

[$\mathcal{A}_h(\theta)$ **determined by**]." In the table it is assumed that $1 \le p < \infty$ and $1 < p_1 < \infty$, and

$$\sigma_k = \sup_{2^{-2^k} \le s \le 2^{-2^{k-1}}} \frac{h(s)}{s(\log 1/s)^{p_1}}.$$

By $L^{p,\infty}$ we mean weak L^p (standard notation).

Function space	**Conv.**	$\mathcal{A}_h(\theta)$ **determined by**	**Ref.**
$C(\mathbb{T})$	if	$h(t) = +\infty$	–
$L^1(\mathbb{T})$	iff	$\limsup_{t\to 0} \frac{h(t)}{t \log 1/t} < \infty$	[13]
$L^p(\mathbb{T})$	iff	$\limsup_{t\to 0} \frac{h(t)}{t(\log 1/t)^p} < \infty$	[11], [4]
$L^\infty(\mathbb{T})$	iff	$\limsup_{t\to 0} \frac{h(t)}{t^{1-\varepsilon}} = 0 \ \forall \varepsilon > 0$	[14]
$L^{p_1,\infty}(\mathbb{T})$	iff	$\sum_{k\ge 0} \sigma_k < \infty$	[3]

A few comments are in order. First of all, the convergence for continuous functions is at all points, not only almost every point. This is because $\mathcal{P}_0$ is a convolution operator with a kernel which behaves like an approximate identity in $\mathbb{T}$.

The results for $L^p(\mathbb{T})$, for finite values of p, are proved via weak type (p,p) estimates for the corresponding maximal operators. To do this, in [11], Rönning uses a quite technical machinery. In [4], a significantly easier proof is given (relying basically only on Hölder's inequality), and the sharpness of the result is proved (without the assumption that h should be monotone, which Rönning assumed). Actually, it is proved that $M_0 f \le (M_{HL} f^p)^{1/p}$, where

$$M_0 f(\theta) = \sup_{\substack{|\arg z - \theta| < h(1-|z|) \\ |z| > 1/2}} |\mathcal{P}_0 f(z)|,$$

the relevant maximal operator, and M_{HL} is the classical Hardy-Littlewood maximal operator.

In $L^p(\mathbb{T})$ one concludes the proofs with a standard approximation argument with continuous functions, for which convergence is known to hold. However, this is not an option in the case of boundary functions in $L^\infty(\mathbb{T})$, since the continuous functions are not dense in this space. The result by Sjögren,

[14], is therefore deeper in its nature. It relies on a theorem of Bellow and Jones, [2], "A Banach principle for L^∞". Basically, the Bellow-Jones result for L^∞ states that a.e. convergence is equivalent to continuity of the maximal operator at 0, when restricted to the unit ball in L^∞, in the topology of convergence in measure. Actually, what Sjögren had to show was that for all $\varepsilon > 0$ and all $\kappa > 0$ there exists $\delta > 0$ such that

$$\|f\|_1 < \delta \Rightarrow |\{\theta \in \mathbb{T} : M_0 f(\theta) > \varepsilon\}| < \kappa,$$

for any function f in the unit ball of L^∞, where M_0 is the maximal operator defined above. (It is easy to see that, in the unit ball in L^∞, the topology of convergence in measure is equivalent with the L^1-topology.)

In [3], the author used a method similar to Sjögren's to determine the approach regions for boundary functions in $L^{p,\infty}$ (weak L^p), $1 < p < \infty$. It relied on a Banach principle for $L^{p,\infty}$, proved in the paper.

The author has also, with essential help and an original idea from professor Mizuta, Hiroshima University, established a result for the corresponding "square root operator" in the half space $\mathbb{R}^{n+1}_+$ with boundary functions $f \in L^p(G)$, where $G \subset \mathbb{R}^n$ is nonempty, bounded and open. For this result, see [5].

To understand better the significant difference in approach regions for L^p and L^∞ we consider, in this paper, two distinct classes of Orlicz spaces L^Φ. Firstly, Orlicz spaces where $\log \Phi$ grows at least as some positive power, thus possessing the property that $L^\infty \subset L^\Phi \subset L^p$ for any $p \geq 1$. Secondly, Orlicz spaces that resemble L^p spaces. As a special case, with $\Phi(x) = x^p$, $L^\Phi = L^p$. To make this more precise, we shall now define these two classes of functions, ∇ and Δ, from which we then define corresponding Orlicz spaces:

Definition 1. Let $\Phi : [0, \infty) \to [0, \infty)$ be a strictly increasing C^2-function with $\Phi(0) = 0$ and define $M(x) = \log \Phi'(x)$. Then, Φ is said to satisfy the ∇ *condition*, denoted $\Phi \in \nabla$, if the following conditions hold:

(*i*) $M'(x) > 0$ for all $x \in (0, \infty)$.
(*ii*) $M((0, \infty)) = \mathbb{R}$.
(*iii*) $\displaystyle \liminf_{x \to \infty} \frac{M(2x)}{M(x)} = m_0 > 1$ (possibly $m_0 = \infty$).

We note immediately that the conditions in Definition 1 imply that, for sufficiently small $\alpha > 0$, one has

$$\lim_{x\to\infty} \frac{M(x)}{x^\alpha} = \infty. \tag{2}$$

The space L^Φ, $\Phi \in \nabla$, that we shall define below (Definition 3) does not depend on the behaviour of Φ close to 0. Thus, without loss of generality, we impose one further convenient assumption on M:

$$\int_0^1 xM'(x)\,dx < \infty. \tag{3}$$

Definition 2. A function $\Phi : [0,\infty) \to [0,\infty)$ is said to satisfy the Δ *condition*, denoted $\Phi \in \Delta$, if the following conditions hold:

(*i*) $\Phi \in C^2(0,\infty)$ with $\Phi''(x) > 0$ for $x > 0$.
(*ii*) $\lim_{x\to 0} \Phi(x) = \lim_{x\to 0} \Phi'(x) = 0$.
(*iii*) $\dfrac{x\phi'(x)}{\phi(x)} \sim 1$, uniformly for $x > x_0$ for some $x_0 \geq 0$, where $\phi(x) = \Phi'(x)$.

Definition 3. For $\Phi \in \nabla$ we define

$$L^\Phi = \{f \in L^1(\mathbb{T}) : \Phi(c|f|) \in L^1(\mathbb{T}) \text{ for some } c > 0\}.$$

Definition 4. Let $\Phi \in \Delta$. For $f \in L^1(\mathbb{T})$ define $\|f\|_\Phi = \|\Phi(|f|)\|_1$ and let

$$L^\Phi = \{f \in L^1(\mathbb{T}) : \|f\|_\Phi < \infty\}.$$

It is readily checked that L^Φ is a vector space, regardless of if $\Phi \in \nabla$ or $\Phi \in \Delta$. For further reading on Orlicz spaces, we refer to [10].

In this paper we shall prove the following two theorems:

Theorem 1. *Let $\Phi \in \nabla$ be given. Then, the following conditions are equivalent for any function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any $f \in L^\Phi$ one has for almost all $\theta \in \mathbb{T}$ that $\mathcal{P}_0 f(z) \to f(\theta)$ a.e. as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$.*
(*ii*) $\dfrac{M\left(C\frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} \to \infty$ *as $t \to 0$ for all $C > 0$, where $g(t) = h(t)/t$.*

Theorem 2. *Let $\Phi \in \Delta$ be given. Then the following conditions are equivalent for any function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any* $f \in L^\Phi$ *one has for almost all* $\theta \in \mathbb{T}$ *that* $\mathcal{P}_0 f(z) \to f(\theta)$ *a.e. as* $z \to e^{i\theta}$ *and* $z \in \mathcal{A}_h(\theta)$.

(*ii*) $\limsup_{t\to 0} \dfrac{g(t)}{\Phi(\log 1/t)} < \infty$, *where* $g(t) = h(t)/t$.

We conclude this section with some examples of $\Phi \in \nabla$ and $\Phi \in \Delta$, indicating what condition (*ii*) in the theorems reduces to in these cases.

Let $L_1(x) = \log x$ and, for $n \geq 2$, let $L_n(x) = L_{n-1}(\log x)$.

The convergence condition (*ii*) in Theorem 1 and Theorem 2 only takes large arguments of M and Φ into account, respectively. Thus, it is clearly sufficient to know the order of magnitude of $M(x)$ and $\Phi(x)$ as $x \to \infty$.

Example 1 ($\Phi \in \nabla$)**.** Our first example is $M(x) \sim x^p$, $p > 0$, as $x \to \infty$. This example covers all spaces L^Φ, where $\Phi(x) \sim x^\alpha \exp[x^p]$ as $x \to \infty$, $\alpha \in \mathbb{R}$ and $p > 0$.

Since $M(x) \sim x^p$ as $x \to \infty$, we may (in this context) assume that $M(x) = x^p$. We now have

$$\frac{M\left(C\frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} = C^p \left(\frac{(\log 1/t)^{\frac{p}{p+1}}}{\log g(t)}\right)^{p+1}.$$

Clearly, this expression tends to ∞ (for all $C > 0$) if and only if

$$\frac{\log g(t)}{(\log 1/t)^{\frac{p}{p+1}}} \to 0,$$

as $t \to 0$. Note that the convergence is independent of $\alpha > 0$.

Obviously, there is no optimal approach region. Specific examples of admissible functions h determining $\mathcal{A}_h(\theta)$ are $h(t) = t\exp[C(\log 1/t)^s (L_n(1/t))^{s'}]$, for $0 < s < p/(p+1)$, $n \geq 2$ and arbitrary $C, s' > 0$. □

Example 2 ($\Phi \in \nabla$)**.** In this example we assume that $M(x) \sim \exp[x^p]$, $p > 0$, as $x \to \infty$. As above, we may assume that we have equality, i.e. $M(x) = \exp[x^p]$. We get

$$\begin{aligned}\frac{M\left(C\frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} &= \exp\left[\left(C\frac{\log 1/t}{\log g(t)}\right)^p - L_2(g(t))\right] \\ &= \exp\left[L_2(g(t))\left(\left(C\frac{\log 1/t}{L_2(g(t))^{1/p}\log g(t)}\right)^p - 1\right)\right].\end{aligned}$$

Clearly this expression tends to ∞ as $t \to 0$, for *all* $C > 0$, if and only if

$$\frac{\log 1/t}{L_2(g(t))^{1/p} \log g(t)} \to \infty$$

as $t \to 0$.

Again, there is no optimal approach region. Specific examples of admissible functions h determining $\mathcal{A}_h(\theta)$ are

$$h(t) = t \exp\left[\frac{\log 1/t}{L_n(1/t)^\alpha L_2(1/t)^{1/p}}\right],$$

where $\alpha \in (0, 1)$ if $n = 1$ and $\alpha > 0$ if $n \geq 2$. □

Example 3 ($\Phi \in \Delta$)**.** The natural example here is $\Phi(x) = x^p$, $p > 1$, which obviously gives $L^\Phi = L^p$. It is easily seen that we, in this case, recover the convergence result by Rönning. More generally, if $\Phi \in \Delta$, we have convergence along approach regions specified by $h(t) = Ct\Phi(\log 1/t)$, but not along any essentially wider approach regions. This should be compared to the result in Theorem 1, where in general no largest possible approach region exists. □

2. Preliminaries, $\Phi \in \nabla$

In this section we assume that $\Phi \in \nabla$, without further notice. For $c, \beta > 0$ define $\phi_{\beta,c}(x) = \beta \exp[M(cx)]$. Furthermore, let

- $\Phi_{\beta,c}(x) = \int_0^x \phi_{\beta,c}(y)\, dy.$
- $\psi_{\beta,c}(y) = (\phi_{\beta,c})^{-1}(y).$
- $\Psi_{\beta,c}(y) = \int_0^y \psi_{\beta,c}(t)\, dt.$

For abbreviation, if $\beta = c = 1$, we write ϕ, Φ, ψ and Ψ instead of $\phi_{1,1}, \Phi_{1,1}, \psi_{1,1}$ and $\Psi_{1,1}$, respectively.

Note that, if $\beta = c = 1$, this definition is in agreement with Definition 1, where $M(x) = \log \Phi'(x)$. The pair $(\Phi_{\beta,c}, \Psi_{\beta,c})$ is referred to as a *complementary pair.*

We shall make use of the following standard inequality:

Proposition (Young's inequality). *Let* $(\Phi_{\beta,c}, \Psi_{\beta,c})$ *be a complementary pair. Then*

$$xy \leq \Phi_{\beta,c}(x) + \Psi_{\beta,c}(y),$$

for any positive numbers x *and* y*. Equality holds if and only if* $x = \psi_{\beta,c}(y)$.

Lemma 1. *If* $f \in L^\Phi$ *then* $\|f\|_1 \leq 2\pi\Phi_{1,c}^{-1}(\|\Phi_{1,c}(|f|)\|_1/(2\pi))$.

Proof. Φ is convex, so the result is just a restatement of Jensen's inequality. □

For the concluding approximation argument, in the proof of Theorem 1, we need

Lemma 2. *Assume that* $f \in L^\Phi(\mathbb{T})$*, i.e. assume that* $\|\Phi_{1,c}(|f|)\|_1 < \infty$ *for some* $c > 0$*. Then, for* $\varepsilon > 0$ *given, there exists* $g \in L^\infty(\mathbb{T})$ *such that* $\|\Phi_{1,c}(|f - g|)\|_1 < \varepsilon$.

Proof. Let $g(x) = f(x)\chi_{\{|f|<R\}}$ for sufficiently large $R > 0$. □

Next, we prove an elementary lemma:

Lemma 3. *Assume that* $\{a_k\}$ *and* $\{b_k\}$ *are two sequences of positive numbers, such that* $\lim_{k\to\infty} a_k = 0$ *and such that*

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \infty.$$

Then there exists subsequences $\{a_{k_i}\}$ *and* $\{b_{k_i}\}$ *and a sequence* $\{N_i\} \subset \mathbb{N}$ *such that*

$$\sum_i N_i a_{k_i} = \infty,$$

and

$$\sum_i N_i b_{k_i} < \infty.$$

Proof. For $i \in \mathbb{N}$ choose $k_i \uparrow \infty$ such that $a_{k_i}/b_{k_i} > 2^i$ and $a_{k_i} < 1$. Now, choose $N_i \in \mathbb{N}$ such that $1 \leq N_i a_{k_i} < 2$. Then $\sum_i N_i a_{k_i} \geq \sum_i 1$ and $\sum_i N_i b_{k_i} \lesssim \sum_i 2^{-i}$. □

The following proposition is a key observation, solving an extremal problem.

Proposition 1. *Let a, c and ε be given positive numbers. Let $g \in L^{\Psi}$ be a nonnegative function, not identically 0, supported in $[-a, a]$. Then there exists a nonnegative and measurable function $\tilde{f}$, supported in $[-a, a]$ and satisfying $\int_{\mathbb{T}} \tilde{f}(\varphi)g(\varphi)\,d\varphi = \varepsilon$, such that, for all nonnegative functions f such that $\int_{\mathbb{T}} f(\varphi)g(\varphi)\,d\varphi \geq \varepsilon$, one has that*

$$\int_{|\varphi|<a} \Phi_{1,c}(f(\varphi))\,d\varphi \geq \int_{|\varphi|<a} \Phi_{1,c}(\tilde{f}(\varphi))\,d\varphi.$$

Moreover, $\tilde{f}(\varphi) = \psi_{\beta,c}(g(\varphi))$, where $\beta > 0$ is the unique number determined by $\int_{|\varphi|<a} \psi_{\beta,c}(g(\varphi))g(\varphi)\,d\varphi = \varepsilon$.

Proof. By the Young inequality we have, for any $\beta > 0$, that

$$\int_{|\varphi|<a} f(\varphi)g(\varphi)\,d\varphi \leq \int_{|\varphi|<a} \Phi_{\beta,c}(f(\varphi))\,d\varphi + \int_{|\varphi|<a} \Psi_{\beta,c}(g(\varphi))\,d\varphi,$$

where equality holds if and only if $f(\varphi) = \tilde{f}(\varphi) = \psi_{\beta,c}(g(\varphi))$. Choose $\beta > 0$ (uniquely) such that

$$\int_{|\varphi|<a} \tilde{f}(\varphi)g(\varphi)\,d\varphi = \varepsilon.$$

For an arbitrary nonnegative function f with $\displaystyle\int_{\mathbb{T}} f(\varphi)g(\varphi)\,d\varphi \geq \varepsilon$, we then have

$$\begin{aligned}
\int_{|\varphi|<a} \Phi_{\beta,c}(f(\varphi))\,d\varphi &\geq \int_{|\varphi|<a} f(\varphi)g(\varphi)\,d\varphi - \int_{|\varphi|<a} \Psi_{\beta,c}(g(\varphi))\,d\varphi \\
&\geq \varepsilon - \int_{|\varphi|<a} \Psi_{\beta,c}(g(\varphi))\,d\varphi \\
&= \int_{|\varphi|<a} \Phi_{\beta,c}(\tilde{f}(\varphi))\,d\varphi,
\end{aligned}$$

which is equivalent to

$$\int_{|\varphi|<a} \Phi_{1,c}(f(\varphi))\,d\varphi \geq \int_{|\varphi|<a} \Phi_{1,c}(\tilde{f}(\varphi))\,d\varphi,$$

as desired. ∎

3. The proof of Theorem 1

Throughout this section we assume that $g(t) = h(t)/t \to \infty$ as $t \to 0$, without loss of generality.

Before turning to the proofs of the two implications, we introduce a suitable notation. If we write $t = 1 - |z|$ and $z = (1-t)e^{i\theta}$, then

$$\mathcal{P}_0 f(z) = R_t * f(\theta),$$

where the convolution is taken in $\mathbb{T}$ and

$$R_t(\theta) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t(2-t)}}{|(1-t)e^{i\theta} - 1|} \frac{1}{P_0 1(1-t)}.$$

Here $\theta \in \mathbb{T} \cong (-\pi, \pi]$, as before. We are interested only in small values of t, so we might as well assume from now on that $t < 1/2$. Since $P_0 1(1-t) \sim \sqrt{t} \log 1/t$, the order of magnitude of R_t is given by

$$R_t(\theta) \sim Q_t(\theta) = \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta|}.$$

Now let τ_η denote the translation $\tau_\eta f(\theta) = f(\theta - \eta)$. Then the convergence condition (i) in Theorem 1 above means

$$\lim_{\substack{t \to 0 \\ |\eta| < h(t)}} \tau_\eta R_t * f(\theta) = f(\theta).$$

The relevant maximal operator for our problem is

$$M_0 f(\theta) = \sup_{\substack{|\arg z - \theta| < h(1-|z|) \\ |z| > 1/2}} |\mathcal{P}_0 f(z)|.$$

Notice that $M_0 f(\theta)$ is dominated by a constant times

$$Mf(\theta) = \sup_{\substack{|\eta| < h(t) \\ t < 1/2}} \tau_\eta Q_t * |f|(\theta). \tag{4}$$

3.1. **Proof of** $(ii) \Rightarrow (i)$**.**

Proof. Let $f \in L^\Phi$ and $\varepsilon > 0$ be given.

We may assume that $f \geq 0$, without loss of generality. Write

$$\begin{aligned} Q_t(\theta) &= Q_t(\theta)\chi_{\{|\theta|\leq 2h(t)\}} + Q_t(\theta)\chi_{\{|\theta|>2h(t)\}} \\ &= Q_t^1(\theta) + Q_t^2(\theta). \end{aligned}$$

By letting

$$M_j f(\theta) = \sup_{\substack{|\eta|<h(t) \\ 0<t<1/2}} \tau_\eta Q_t^j * f(\theta),$$

$j \in \{1, 2\}$, we get $Mf \leq M_1 f + M_2 f$ and hence

$$\{Mf > 2\varepsilon\} \subset \{M_1 f > \varepsilon\} \cup \{M_2 f > \varepsilon\}.$$

To deal with $M_2 f$, we observe that when $|\eta| < h(t)$

$$\begin{aligned} \tau_\eta Q_t^2(\theta) &= \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta - \eta|} \chi_{\{|\theta-\eta|>2h(t)\}} \\ &\leq \frac{2}{\log 1/t} \cdot \frac{1}{t + |\theta|}. \end{aligned}$$

The last expression is a decreasing function of $|\theta|$, whose integral in $\mathbb{T}$ is bounded uniformly in t. It is well known that convolution by such a function is controlled by the Hardy-Littlewood maximal operator M_{HL}, so that $M_2 f \leq C M_{HL} f$. Since M_{HL} is of weak type $(1, 1)$, we obtain

$$|\{M_2 f > \varepsilon\}| \leq C\varepsilon^{-1}\|f\|_1.$$

By invoking Lemma 1, we get

$$|\{M_2 f > \varepsilon\}| \leq \frac{C \cdot \Phi_{1,c}^{-1}\left(\|\Phi_{1,c}(f)\|_1/(2\pi)\right)}{\varepsilon}. \tag{5}$$

Let us now turn our attention to M_1. Assume that $M_1 f(\theta) > \varepsilon$. Then there exists $t \in (0, 1/2)$ and $|\eta| < h(t)$ such that

$$\frac{1}{\log 1/t} \int_{|\varphi|<2h(t)} \frac{f(\theta - \eta - \varphi)}{t + |\varphi|}\, d\varphi > \varepsilon.$$

It follows then, by Proposition 1, that

$$(6)\qquad \int_{|\varphi|<2h(t)} \Phi_{1,c}(f(\theta-\eta-\varphi))\,d\varphi \geq \int_{|\varphi|<2h(t)} \Phi_{1,c}(\psi_{\beta,c}\left(\frac{1}{t+|\varphi|}\right))\,d\varphi,$$

where β is chosen such that

$$(7)\qquad \int_{|\varphi|<2h(t)} \psi_{\beta,c}\left(\frac{1}{t+|\varphi|}\right)\cdot\frac{1}{t+|\varphi|}\,d\varphi = \varepsilon\log 1/t.$$

We shall now use (7) to get an estimate of the size of β. We have

$$\begin{aligned}
\varepsilon\log 1/t &= \int_{|\varphi|<2h(t)} \psi_{\beta,c}\left(\frac{1}{t+|\varphi|}\right)\cdot\frac{1}{t+|\varphi|}\,d\varphi \\
&= 2\int_{\frac{1}{t+2h(t)}}^{\frac{1}{t}} \frac{\psi_{\beta,c}(y)}{y}\,dy \\
&\leq 2\psi_{\beta,c}(1/t)\cdot\log(1+2g(t)) \\
&\leq C\psi_{\beta,c}(1/t)\cdot\log g(t),
\end{aligned}$$

so that

$$(8)\qquad \frac{1}{\beta} \geq t\phi_{1,c}\left(C_\varepsilon\frac{\log 1/t}{\log g(t)}\right).$$

Now, let $B(s) = \Phi_{1,c}(\psi_{1,c}(s))$. Then it is clear that B is increasing and $\lim_{s\to\infty} B(s) = \infty$. For convenience, let I_t denote the interval $[-2h(t), 2h(t)]$. We have

$$\begin{aligned}
\|\Phi_{1,c}(\psi_{\beta,c}(\frac{1}{t+|\varphi|}))\|_{L^1(I_t)} &= \int_{I_t} B\left(\frac{1}{\beta(t+|\varphi|)}\right)d\varphi \\
&\geq 4h(t)B\left(\frac{1}{\beta(t+2h(t))}\right) \\
&\geq 4h(t)B\left(\frac{1}{3\beta h(t)}\right).
\end{aligned}$$

We may now invoke (8) to get

$$\begin{aligned}
\|\Phi_{1,c}(\psi_{\beta,c}(\frac{1}{t+|\varphi|}))\|_{L^1(I_t)} &\geq 4h(t)B\left(\frac{t\phi_{1,c}\left(C_\varepsilon\frac{\log 1/t}{\log g(t)}\right)}{3h(t)}\right) \\
&\geq 4h(t)B\left(C\exp\left[M\left(C_\varepsilon\frac{\log 1/t}{\log g(t)}\right)-\log g(t)\right]\right) \\
&\geq C(\varepsilon)h(t),
\end{aligned}$$

by condition (ii) in Theorem 1. Thus, we have

$$\frac{h(t)}{\|\Phi_{1,c}(\psi_{\beta,c}(\frac{1}{t+|\varphi|}))\|_{L^1(I_t)}} \leq C,$$

which gives, by (6),

$$\begin{aligned}
h(t) &\leq C\int_{I_t}\Phi_{1,c}(\tilde{f}(\varphi))\,d\varphi \\
&\leq C\int_{I_t}\Phi_{1,c}(f(\theta-\eta-\varphi))\,d\varphi.
\end{aligned}$$

To sum up, we have shown that for each θ with $M_1f(\theta)>\varepsilon$ there exists a t such that the interval $J(\theta)=[\theta-3h(t),\theta+3h(t)]$ has the property

$$\int_{J(\theta)}\Phi_{1,c}(f(\varphi))\,d\varphi \geq Ch(t).$$

A covering argument now yields a sequence (θ_i,t_i) with $M_1f(\theta_i)>\varepsilon$ such that the corresponding intervals $J(\theta_i)$ are disjoint, and such that the union of the scaled intervals $J'(\theta_i)=[\theta_i-10h(t_i),\theta_i+10h(t_i)]$ covers the set $\{M_1f>\varepsilon\}$. In particular we have

$$\begin{aligned}
\|\Phi_{1,c}(f)\|_1 &\geq \sum_i\int_{J(\theta_i)}\Phi_{1,c}(f(\varphi))\,d\varphi \\
&\geq C\sum_i h(t_i).
\end{aligned}$$

Thus,

$$\begin{aligned} |\{M_1 f > \varepsilon\}| &\leq \sum_i |J'(\theta_i)| \\ &\leq C \sum_i h(t_i) \\ &\leq C\|\Phi_{1,c}(f)\|_1. \end{aligned}$$

It follows, from the above estimate and from (5), that

$$|\{Mf > 2\varepsilon\}| \leq C_1(\varepsilon)\|\Phi_{1,c}(f)\|_1 + C_2(\varepsilon)\Phi_{1,c}^{-1}\left(\|\Phi_{1,c}(f)\|_1/(2\pi)\right).$$

For each $\varepsilon > 0$ the right hand side tends to 0 with $\|\Phi_{1,c}(f)\|_1$. By Lemma 2 we are done (approximation with bounded functions).

□

3.2. **Proof of** $(i) \Rightarrow (ii)$**.**

Proof. Assume that condition (ii) in Theorem 1 is false. We show that this implies that (i) is false also.

Assume that, for some $C_0 > 0$,

$$\liminf_{t\to 0} \frac{M\left(C_0 \frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} = A < \infty.$$

The claim now is that we may assume that

$$\liminf_{t\to 0} \frac{M\left(C_0 \frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} = A \in (1/4, 1/2). \tag{9}$$

To see that we may assume that $A < 1/2$ we note that, by the conditions we have on M, there is a number $m \in (0,1)$ such that $M(x) \leq mM(2x)$ for sufficiently large x. Thus we have

$$\liminf_{t\to 0} \frac{M\left(2^{-N} C_0 \frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} \leq m^N \liminf_{t\to 0} \frac{M\left(C_0 \frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} = m^N A.$$

By choosing $N = N(A)$ large enough, we can make $m^N A < 1/2$. Thus, we can assume from now on that $A < 1/2$.

To see that we may assume that $A > 1/4$, note that if for some $t > 0$ we have

$$\frac{M\left(C_0 \frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} \le 1/4,$$

then we can clearly make $g(t)$ smaller so that the quotient above is greater than $1/4$, say, and still smaller than $1/2$. Then the corresponding approach region for the new function g (at any $\theta \in \mathbb{T}$) is a subset of the original one, and it suffices to disprove convergence in the new one.

Pick a decreasing sequence $\{t_i\}_1^\infty$, converging to 0, such that

$$\frac{M\left(C_0 \frac{\log 1/t_i}{\log g(t_i)}\right)}{\log g(t_i)} \to A, \tag{10}$$

as $i \to \infty$. For convenience, let $s_i = C_0 \dfrac{\log 1/t_i}{\log g(t_i)}$. We may assume that $\{t_i\}_1^\infty$ is chosen such that

$$1/4 \le \frac{M(s_i)}{\log g(t_i)} \le 1/2, \tag{11}$$

for all $i \in \mathbb{N}$.

Let

$$f_i(\varphi) = \psi_{\beta_i,1}\left(\frac{1}{t_i + |\varphi|}\right) \cdot \chi_{\{|\varphi|<h(t_i)\}},$$

where $\beta_i^{-1} = t_i \phi(s_i)$.

Note that $\Phi(x) \leq x \cdot \phi(x)$, so that $\Phi(\psi_{\beta,1}(x)) \leq \frac{x}{\beta} \cdot \psi_{\beta,1}(x) = \frac{x}{\beta} \cdot \psi(x/\beta)$, and thus

$$\begin{aligned} \|\Phi(f_i)\|_1 &\leq 2 \int_0^{h(t_i)} \frac{\psi\left(\frac{1}{\beta_i(t_i+\varphi)}\right)}{\beta_i(t_i+\varphi)}\, d\varphi \\ &= \frac{2}{\beta_i} \int_{\frac{1}{\beta_i(t_i+h(t_i))}}^{\frac{1}{\beta_i t_i}} \frac{\psi(y)}{y}\, dy \\ &\leq 2t_i \cdot \frac{1}{\beta_i t_i} \int_0^{\frac{1}{\beta_i t_i}} \frac{\psi(y)}{y}\, dy \\ &= 2t_i \cdot \phi(s_i) \int_0^{\phi(s_i)} \frac{\psi(y)}{y}\, dy. \end{aligned}$$

At this stage we make a change of variables, $y = \phi(x)$, and use (3) to get

$$\begin{aligned} \|\Phi(f_i)\|_1 &\leq 2t_i \cdot \phi(s_i) \int_0^{s_i} xM'(x)\, dx \\ &\leq 2t_i \cdot \phi(s_i) \left(\int_0^1 xM'(x)\, dx + \int_1^{s_i} xM'(x)\, dx \right) \\ &\leq 2t_i \cdot \phi(s_i) \left(C + s_i \int_1^{s_i} M'(x)\, dx \right) \\ &\leq 2t_i \cdot \phi(s_i)\, (C + s_i M(s_i)) \\ &\leq Ct_i \cdot \phi(s_i) \cdot s_i M(s_i) \end{aligned}$$

Now, using the above estimate, we get

$$\begin{aligned} \frac{h(t_i)}{\|\Phi(f_i)\|_1} &\geq C \frac{h(t_i)}{t_i \cdot \phi(s_i) \cdot s_i M(s_i)} \\ &\geq \frac{C}{\log 1/t_i} \exp\left[\log g(t_i) - M(s_i)\right] \\ &\geq \frac{Cg(t_i)^{1/2}}{\log 1/t_i}, \end{aligned}$$

the last two inequalities by (11). For all $t > 0$ sufficiently small, we have that

$$1/2 \geq \frac{M\left(C_0 \frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} \geq C_0^{\alpha} \frac{(\log 1/t)^{\alpha}}{(\log g(t))^{1+\alpha}},$$

for some sufficiently small $\alpha > 0$, by (11) and (2).

It follows that

$$\frac{h(t_i)}{\|\Phi(f_i)\|_1} \to \infty, \tag{12}$$

as $i \to \infty$.

It follows from (12), by Lemma 3, that we can pick a subsequence of $\{t_i\}$, with possible repetitions, for simplicity denoted $\{t_i\}$ also, such that

$$\sum_1^\infty h(t_i) = \infty, \tag{13}$$

and

$$\sum_1^\infty \|\Phi(f_i)\|_1 < \infty. \tag{14}$$

We shall now proceed with the construction of a function that disproves boundary convergence a.e. The idea is to distribute mass on $\mathbb{T}$ over and over again, sufficient to make the relevant Poisson integral larger than some positive constant, at all points in $\mathbb{T}$, and at the same time being able to make the function arbitrarily close to 0 on a set with positive measure.

Let $A_1 = h(t_1)$, and for $n \geq 2$ let $A_n = h(t_n) + \sum_{j=1}^{n-1} 2h(t_j)$. By (13) one has that $\lim_{n\to\infty} A_n = \infty$.

Define (on $\mathbb{T}$) $F_j(\varphi) = \tau_{A_j} f_j(\varphi)$, and let

$$F^{(N)}(\varphi) = \sup_{j \geq N} F_j(\varphi).$$

It is clear by construction that any given $\varphi \in \mathbb{T}$ lies in the support of infinitely many F_j:s.

Pointwise one obviously has that

$$\Phi(F^{(N)}(\varphi)) \leq \sum_{j=N}^\infty \Phi(F_j(\varphi)),$$

so that

$$
\begin{aligned}
\|\Phi(F^{(N)})\|_1 &\leq \sum_{j=N}^{\infty} \|\Phi(F_j)\|_1 \\
&= \sum_{j=N}^{\infty} \|\Phi(f_j)\|_1 \to 0
\end{aligned}
$$

as $N \to \infty$, by (14). Thus, in particular, $F^{(N)} \in L^{\Phi}$ for any $N \geq 1$.

For $\theta \in \mathbb{T}$ and a given $\xi_0 > 0$ we can, by construction, find $j \in \mathbb{N}$ so that $\theta \in \operatorname{supp}(F_j)$ and so that $t_j \in (0, \xi_0)$. We can then choose η, with $|\eta| < h(t_j)$, so that $\theta - \eta \equiv A_j \mod 2\pi$. It follows that

$$
\limsup_{t \to 0,\ |\eta| < h(t)} \mathcal{P}_0 F^{(N)}((1-t)e^{i(\theta-\eta)}) \geq \limsup_{j \to \infty} \mathcal{P}_0 F_j((1-t_j)e^{iA_j}).
$$

We shall now conclude the proof by proving that the right hand side above is always greater than some positive constant.

We have

$$
\begin{aligned}
\mathcal{P}_0 F_j((1-t_j)e^{iA_j}) &\geq \frac{C}{\log 1/t_j} \int_{|\varphi| < h(t_j)} \frac{F_j(A_j - \varphi)}{t_j + |\varphi|}\, d\varphi \\
&= \frac{C}{\log 1/t_j} \int_{|\varphi| < h(t_j)} \frac{f_j(\varphi)}{t_j + |\varphi|}\, d\varphi \\
&= \frac{C}{\log 1/t_j} \int_0^{h(t_j)} \frac{\psi\left(\frac{1}{\beta_j(t_j+\varphi)}\right)}{t_j + \varphi}\, d\varphi \\
&= \frac{C}{\log 1/t_j} \int_{\frac{1}{\beta_j(t_j + h(t_j))}}^{\frac{1}{\beta_j t_j}} \frac{\psi(y)}{y}\, dy \\
&\geq \frac{C}{\log 1/t_j} \int_1^{\phi(s_j)} \frac{\psi(y)}{y}\, dy.
\end{aligned}
$$

In the last inequality, the lower limit $\frac{1}{\beta_j(t_j+h(t_j))}$ can be replaced by 1, since by (11) we have

$$\begin{aligned} \beta_j(t_j + h(t_j)) &\geq \beta_j h(t_j) \\ &= \exp\left[\log g(t_j) - M(s_j)\right] \\ &\geq \exp\left[(\log g(t_j))/2\right] \\ &\to \infty, \end{aligned}$$

as $j \to \infty$.

We continue the estimate by making the change of variables $y = \phi(x)$, and we get

$$\begin{aligned} \mathcal{P}_0 F_j((1-t_j)e^{iA_j}) &\geq \frac{C}{\log 1/t_j} \int_{\psi(1)}^{s_j} \frac{x\phi'(x)}{\phi(x)}\, dx \\ &= \frac{C}{\log 1/t_j} \int_{\psi(1)}^{s_j} xM'(x)\, dx \\ &\geq \frac{C}{\log 1/t_j} \int_{s_j/2}^{s_j} xM'(x)\, dx \\ &\geq \frac{Cs_j}{\log 1/t_j}(M(s_j) - M(s_j/2)). \end{aligned}$$

At this point we note that, by Definition 1 (*iii*), we have $M(s_j) - M(s_j/2) \geq CM(s_j)$ for some positive constant C (depending only on m_0). We may now, finally, continue the estimate to get the desired conclusion. We have

$$\begin{aligned} \mathcal{P}_0 F_j((1-t_j)e^{iA_j}) &\geq \frac{Cs_j M(s_j)}{\log 1/t_j} \\ &= \frac{CM(s_j)}{\log g(t_j)} \\ &\geq C_1, \end{aligned}$$

the last inequality by (11).

To sum up, we have shown that for any $\theta \in \mathbb{T}$ one has

$$\limsup_{t \to 0,\ |\eta|<h(t)} \mathcal{P}_0 F^{(N)}((1-t)e^{i(\theta-\eta)}) \geq C_1. \tag{15}$$

Take N so large so that $\lambda_{F^{(N)}}(C_1/2) < \pi$, say, and a.e. convergence is disproved. $\square$

4. The proof of Theorem 2

In this section we assume that $\Phi \in \Delta$, without further notice. We use basically the same notation as we did in the proof of Theorem 1, and we shall carry out only those calculations that differ from that proof. Remember that the parameter c should have the value 1 when applying the other proof to this. The results from Section 2 are easily seen to remain true for $\Phi \in \Delta$ (again with $c = 1$).

For $\beta > 0$, let $\Phi_\beta(x) = \beta\Phi(x)$. Furthermore, let

- $\phi_\beta(x) = \Phi'_\beta(x)$.
- $\psi_\beta(y) = (\phi_\beta)^{-1}(y)$.
- $\Psi_\beta(y) = \int_0^y \psi_\beta(t)\,dt$.

(Φ_β, Ψ_β) is referred to as a *complementary pair*, as before.

For short, if $\beta = 1$, we write ϕ, Φ, ψ and Ψ instead of ϕ_1, Φ_1, ψ_1 and Ψ_1, respectively.

Lemma 4. *Assume that $\Phi \in \Delta$. Then the following hold, uniformly in (x_0, ∞):*

(i) $\phi(2x) \sim \phi(x)$ *and* $\Phi(2x) \sim \Phi(x)$.

(ii) $\Phi(x) \sim x\phi(x)$.

(iii) $\int_0^x \frac{\psi(y)}{y}\,dy \sim \psi(x)$.

Proof. To prove the first part of (i), note that

$$\begin{aligned} \log \frac{\phi(2x)}{\phi(x)} &= \int_x^{2x} \frac{\phi'(t)}{\phi(t)}\,dt \\ &\sim \int_x^{2x} \frac{dt}{t} \\ &\sim 1, \end{aligned}$$

and the statement follows. If we can establish (ii), then the second part of (i) follows with the same techniques used to prove the first part. We have

$$\begin{aligned} \Phi(x) &= \int_0^x \phi(t)\,dt \\ &\sim \int_0^x t\phi'(t)\,dt \\ &= x\phi(x) - \Phi(x), \end{aligned}$$

and thus $\Phi(x) \sim x\phi(x)$, so (ii) is proved. Statement (iii) is trivial, via the change of coordinates given by $y = \phi(t)$. □

4.1. **Proof of** $(ii) \Rightarrow (i)$**.**

Proof. All we need to prove, according to the proof of Theorem 1, is that

$$\frac{h(t)}{\|\Phi(\psi_\beta(\frac{1}{t+|\varphi|}))\|_{L^1(I_t)}} \leq C. \tag{16}$$

In fact, all we need to do to show this, is to estimate β slightly differently. Here we have

$$\begin{aligned} \varepsilon \log 1/t &= 2\int_{\frac{1}{t+2h(t)}}^{\frac{1}{t}} \frac{\psi_\beta(y)}{y}\,dy \\ &\leq 2\int_0^{\frac{1}{t}} \frac{\psi_\beta(y)}{y}\,dy \\ &\lesssim \psi_\beta(1/t), \end{aligned}$$

the last inequality by Lemma 4, (iii), so that

$$\frac{1}{\beta} \geq t\phi\left(C_\varepsilon \log 1/t\right). \tag{17}$$

Now, let $B(s) = \Phi(\psi(s))$. Then, by Lemma 4, (ii), we have $B(s) \sim s\psi(s)$. For convenience, let I_t denote the interval $[-2h(t), 2h(t)]$. We have

$$\begin{aligned} \|\Phi(\psi_\beta(\frac{1}{t+|\varphi|}))\|_{L^1(I_t)} &= \int_{I_t} B\left(\frac{1}{\beta(t+|\varphi|)}\right) d\varphi \\ &\sim C \int_{I_t} \psi_\beta\left(\frac{1}{t+|\varphi|}\right) \cdot \frac{1}{\beta(t+|\varphi|)}\, d\varphi \\ &= \frac{C\varepsilon \log 1/t}{\beta}, \end{aligned}$$

the last equality by (7). We may now invoke (17) to get

$$\begin{aligned} \|\Phi(\psi_\beta(\frac{1}{t+|\varphi|}))\|_{L^1(I_t)} &\geq C_1(\varepsilon) t (\log 1/t) \phi\left(C_\varepsilon \log 1/t\right) \\ &\geq C_2(\varepsilon) t \Phi\left(C_\varepsilon \log 1/t\right) \\ &\sim C_3(\varepsilon) t \Phi\left(\log 1/t\right), \end{aligned}$$

where we have used Lemma 4, (i) and (ii). Thus, by assumption (ii) in Theorem 2, the desired inequality (16) follows.

□

4.2. **Proof of** $(i) \Rightarrow (ii)$.

Proof. Assume that condition (ii) in Theorem 2 is false. We show that this implies that (i) is false also.

Pick a decreasing sequence $\{t_i\}_1^\infty$, converging to 0, such that

$$\frac{g(t_i)}{\Phi(\log 1/t_i)} \to \infty, \tag{18}$$

as $i \to \infty$. Let $s_i = \log 1/t_i$, and define

$$f_i(\varphi) = \psi_{\beta_i}\left(\frac{1}{t_i + |\varphi|}\right) \cdot \chi_{\{|\varphi|<h(t_i)\}},$$

where $\beta_i^{-1} = t_i \phi(s_i)$.

Using $\Phi(\psi_\beta(x)) \sim \frac{x}{\beta} \cdot \psi_\beta(x)$, we get

$$\begin{aligned}
\|f_i\|_\Phi &\lesssim \int_0^{h(t_i)} \frac{\psi_{\beta_i}\left(\frac{1}{t_i+\varphi}\right)}{\beta_i(t_i+\varphi)}\, d\varphi \\
&= \frac{1}{\beta_i} \int_{\frac{1}{\beta_i(t_i+h(t_i))}}^{\frac{1}{\beta_i t_i}} \frac{\psi(y)}{y}\, dy \\
&\leq t_i \cdot \phi(s_i) \int_0^{\phi(s_i)} \frac{\psi(y)}{y}\, dy \\
&\lesssim t_i \cdot \phi(s_i) s_i \\
&\lesssim t_i \cdot \Phi(s_i).
\end{aligned}$$

Now, using the above estimate, we get

$$\frac{h(t_i)}{\|f_i\|_\Phi} \geq C \frac{g(t_i)}{\Phi(s_i)}.$$

Thus, by (18), we have

$$\frac{h(t_i)}{\|f_i\|_\Phi} \to \infty,$$

as $i \to \infty$.

Copying the proof of Theorem 1, we now see that it suffices to prove that

$$\frac{1}{\log 1/t_j} \int_{\psi(1)}^{s_j} \frac{x\phi'(x)}{\phi(x)}\, dx \geq C,$$

for some constant $C > 0$, to disprove convergence. However, by Definition 2, (iii), we have

$$\frac{1}{\log 1/t_j}\int_{\psi(1)}^{s_j}\frac{x\phi'(x)}{\phi(x)}\,dx \geq \frac{1}{\log 1/t_j}\int_0^{s_j} C_0\,dx = C_0.$$

We are done. □

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ON A THEOREM OF AIKAWA

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Abstract. Let F be a bounded harmonic function in the unit disc U. Then F has a nontangential boundary limit at a.e. point, as is wellknown. However, if C_0 is a tangential curve touching ∂U at 1 and C_θ is its rotation through an angle θ about the origin, Aikawa constructs a bounded harmonic function which, for any $\theta \in \partial U$, fails to have a boundary limit along C_θ. In this paper we present a modified proof of this result.

1. Introduction

We let $P(z, \varphi)$ be the Poisson kernel of the unit disc U, i.e.

$$P(z, \varphi) = \frac{1}{2\pi} \cdot \frac{1 - |z|^2}{|z - e^{i\varphi}|^2},$$

where $z \in U$ and $\varphi \in \partial U \cong \mathbb{T} \cong (-\pi, \pi]$.

From now on we will identify points on the boundary ∂U, whenever convenient, with points in the interval $(-\pi, \pi]$.

The Poisson integral of $f \in L^1(\mathbb{T})$ is defined by

$$Pf(z) = \int_{\mathbb{T}} P(z, \varphi) f(\varphi)\, d\varphi,$$

$z \in U$. It is well known that Pf is a harmonic function.

Several boundary convergence theorems for Poisson integrals have been proved through the years. We state the three most classical below.

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Theorem (Schwarz, [7]). *Let $f \in C(\mathbb{T})$. Then $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$, $z \in U$.*

Theorem (Fatou, [5]). *Let $f \in L^1(\mathbb{T})$. Then, for a.e. $\theta \in \mathbb{T}$, one has that $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $|\arg z - \theta| \leq C \cdot (1 - |z|)$ (i.e., nontangential convergence).*

The theorem of Fatou was proved to be best possible, in the following sense:

Theorem (Littlewood, [6]). *Let $\gamma_0 \subset U \cup \{1\}$ be a simple closed curve, having a common tangent with the circle at the point 1. Let γ_θ be the rotation of γ_0 by the angle θ. Then there exists a bounded harmonic function f in U with the property that, for a.e. $\theta \in \mathbb{T}$, the limit of f along γ_θ does not exist.*

Littlewood's result has been generalised in several directions. In Zygmund's paper, [8], two new proofs appeared, one of which used complex analytic methods (Blaschke products). Aikawa, [1], improved Littlewood's theorem by constructing, with the same assumptions as Littlewood, a bounded harmonic function in U which fails to have a tangential limit at *any* point $\theta \in \mathbb{T}$. In [2], he generalised further what he did in [1]. Di Biase, [4], showed that the notion of nontangential and tangential approach to the boundary also makes sense in a tree[1], and proved a theorem about everywhere divergence for corresponding Poisson extensions, under the assumption that the approach to the boundary is tangential.

For a more thorough treatise on Fatou type theorems, see [3].

In this paper we give a modified proof of the theorem in [1]. It is more or less a translation of the discrete setting in [4] into the continuous setting on U. The main advantage of our proof is that it is more explicit and somewhat more straightforward than that of Aikawa. However, it should be pointed out that we construct the counterexample function F in the same spirit as Aikawa (and Di Biase) did.

Let us now turn to the precise formulation of the theorem.

Throughout this paper, we let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function such that $h(t)/t \to \infty$ as $t \to 0^+$. Let $\mathcal{A}(\theta) = \{z \in U : |\arg z - \theta| \leq h(1 - |z|)\}$. Then $\mathcal{A}(\theta)$ is a tangential approach region at θ. It is natural to think of h as a strictly increasing function with $h(t) \to 0$ as $t \to 0^+$, but we actually

[1]A tree is an infinite connected graph which is locally finite and simply connected.

do not need those assumptions in our proof. We let h^∞ denote the space of bounded harmonic functions in U.

We shall prove the following:

Theorem 1. *There exists a function $F \in h^\infty$ such that*

$$\lim_{z \to e^{i\theta},\, z \in \mathcal{A}(\theta)} F(z)$$

does not exist for any $\theta \in \mathbb{T}$.

2. Estimates of the Poisson extension

Versions similar to the results in this section can be found in [1]. All functions that appear in the lemmas below are (of course) assumed to be measurable.

Lemma 1. *Let $\eta \in \mathbb{T}$ be given and assume that $f(\varphi) = 0$ if $|\varphi - \eta| < s$ for some (small) positive number s. Furthermore, assume that $|f| \leq 1$ on $\mathbb{T}$. Then there exists an absolute constant $\alpha_0 > 0$ such that*

$$\left| Pf(re^{i\eta}) \right| \leq \frac{1}{4},$$

if $1 - \alpha_0 s \leq r < 1$.

Proof. Assume that $1 - \alpha_0 s \leq r < 1$, for some $\alpha_0 \in (0, 2/\pi)$. Then, if $s \leq |\varphi - \eta| \leq \pi$, we have

$$\begin{aligned} |re^{i\eta} - e^{i\varphi}| &\geq |e^{i\eta} - e^{i\varphi}| - (1 - r) \\ &\geq 2\left|\sin\frac{\varphi - \eta}{2}\right| - \alpha_0|\varphi - \eta| \\ &\geq \left(\frac{2}{\pi} - \alpha_0\right)|\varphi - \eta|. \end{aligned}$$

It follows that

$$\begin{aligned} |Pf(re^{i\eta})| &\leq \frac{1 - r^2}{2\pi}\left(\frac{2}{\pi} - \alpha_0\right)^{-2} \int_{|\varphi - \eta| \geq s} \frac{d\varphi}{|\varphi - \eta|^2} \\ &\leq \frac{\alpha_0 s}{\pi}\left(\frac{2}{\pi} - \alpha_0\right)^{-2} \cdot \frac{2}{s} \\ &\leq \frac{1}{4}, \end{aligned}$$

if only α_0 is chosen small enough. □

What we in fact shall make use of is the following "dual" version of Lemma 1:

Lemma 2. *With the same assumptions as in Lemma 1, with the exception that* $f(\varphi) = 1$ *if* $|\varphi - \eta| < s$, *one has that*

$$Pf(re^{i\eta}) \geq \frac{1}{2},$$

if $1 - \alpha_0 s \leq r < 1$.

Proof. Write $f = 1 + 2g$, where $g = (f-1)/2$. Then g satisfies the assumptions in Lemma 1 and Lemma 2 follows immediately. □

Lemma 3. *Let* $0 < s < 1$. *Assume that* $\|f\|_\infty \leq 1$ *and that*

$$\frac{1}{|I|}\int_I |f(\varphi)|\, d\varphi \leq \varepsilon < 1/4,$$

for all intervals $I \subset \mathbb{T}$ *such that* $|I| = 2s$. *Then*

$$\sup_{|z|\leq 1-s} |Pf(z)| \leq C\sqrt{\varepsilon},$$

where $C > 0$ *is an absolute constant.*

Proof. This is exactly Lemma 3 in [1]. We refrain from proving it again. □

3. The proof

The proof of Theorem 1 is based on the following technical lemma. It corresponds directly to Lemma 5 in [4].

Lemma 4. *There exists a sequence* $\{F_k\}_1^\infty \subset h^\infty$, *an increasing sequence* $\{N_k\}_1^\infty \subset \mathbb{N}$ *and a number* $k_0 \in \mathbb{N}$ *such that, for any* $\theta \in \mathbb{T}$, *there exists a sequence* $\{z_k^\theta\}_{k_0}^\infty \subset U$ *with the following properties:*

(i) $z_k^\theta \in \mathcal{A}(\theta)$, $|z_k^\theta| \leq 1 - \pi 2^{-N_{k+1}}$ *and* $z_k^\theta \to e^{i\theta}$ *as* $k \to \infty$.

(ii) $(-1)^k F_k(z_k^\theta) \geq 1/2$.

(iii) $$\limsup_{k\to\infty,\,|z|\leq 1-\pi 2^{-N_{k+1}}} \sum_{j\geq k} |F_{j+1}(z) - F_j(z)| = 0.$$

Moreover, $F_k \to F \in h^\infty$ *pointwise.*

We postpone for the moment the proof of Lemma 4. Instead we start by proving Theorem 1:

Proof. (Theorem 1.) Let $\theta \in \mathbb{T}$ be given. By the second statement in part (i) of Lemma 4, and by part (iii) it follows that

$$\sum_{j=n}^{\infty} |F_{j+1}(z_n^\theta) - F_j(z_n^\theta)| < 1/4,$$

if $n \geq n_0$ for some n_0. Note that

$$F(z_n^\theta) = F_n(z_n^\theta) + \sum_{j=n}^{\infty}(F_{j+1}(z_n^\theta) - F_j(z_n^\theta)) = I + II.$$

If $n \geq n_0$ it follows that $|II| < 1/4$, and by part (ii) of Lemma 4, I oscillates beyond $1/2$ and $-1/2$ respectively. Thus,

$$(-1)^n(F(z_n^\theta) - (-1)^n/4) \geq 0.$$

Since $z_n^\theta \to e^{i\theta}$ within $\mathcal{A}(\theta)$, it follows that $\operatorname{osc}_{\mathcal{A}(\theta)} F(\theta) \geq 1/2$ and convergence at θ is disproved. □

We now conclude the paper by proving Lemma 4:

Proof. (Lemma 4.) Later in the proof we shall define the sequence $\{N_k\}$ more precisely, but for the time being let us just assume that it is a given increasing sequence of natural numbers. If E is a set and x a point, we let $x + E = \{x + e : e \in E\}$. Given $k \in \mathbb{N}$, define $S(k)$ as follows (see Figure 1): Let $I_0(k) = (-\pi, -\pi + 2\pi 2^{-N_k} 2^{-k})$ and $I_j(k) = j \cdot 2\pi 2^{-N_k} + I_0(k)$. We now define

$$S(k) = \bigcup_{j=0}^{2^{N_k}-1} I_j(k).$$

It follows immediately that $|S(k)| = 2\pi 2^{-k}$, so that $\sum_{k \geq 1} |S(k)| < \infty$. By the Borel-Cantelli Lemma we thus have that

(1) $$\left| \limsup_{k \to \infty} S(k) \right| = 0,$$

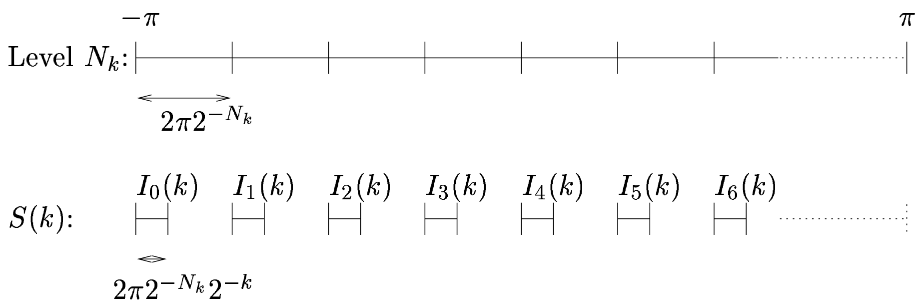

FIGURE 1. $S(k)$ is defined as a the union of the "left dyadic k-descendants" of the N_k:th dyadic decomposition of $\mathbb{T}$.

where

$$\begin{aligned}\limsup_{k\to\infty} S(k) &= \bigcap_{m\geq 0}\bigcup_{k\geq m} S(k)\\ &= \{x : x \in S(k) \text{ for infinitely many } k\}.\end{aligned}$$

If $\omega \in \cup_{k\geq 1} S(k)$ we define $Q(\omega) = \max\{i : \omega \in S(i)\}$. Similarly, if $\omega \in \cup_{k=1}^{n} S(k)$ we define $Q_n(\omega) = \max\{i \leq n : \omega \in S(i)\}$.

For any $\omega \in \mathbb{T}$ let

$$f(\omega) = \begin{cases} (-1)^{Q(\omega)} & \text{if } \omega \in \cup_{k\geq 1} S(k) \text{ and } Q(\omega) < \infty \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_n(\omega) = \begin{cases} (-1)^{Q_n(\omega)} & \text{if } \omega \in \cup_{k=1}^{n} S(k) \\ 0 & \text{otherwise} \end{cases}.$$

The properties of f_n and f that we will need are the following:

$$|f_n(\omega)| \leq 1. \tag{2}$$

$$f_n(\omega) \to f(\omega) \text{ for a.e. } \omega \in \mathbb{T}. \tag{3}$$

$$f_n(\omega) = (-1)^n \text{ if } \omega \in S(n). \tag{4}$$

$$|f_{n+1} - f_n| \leq 2\chi_{S(n+1)} \text{ on } \mathbb{T}. \tag{5}$$

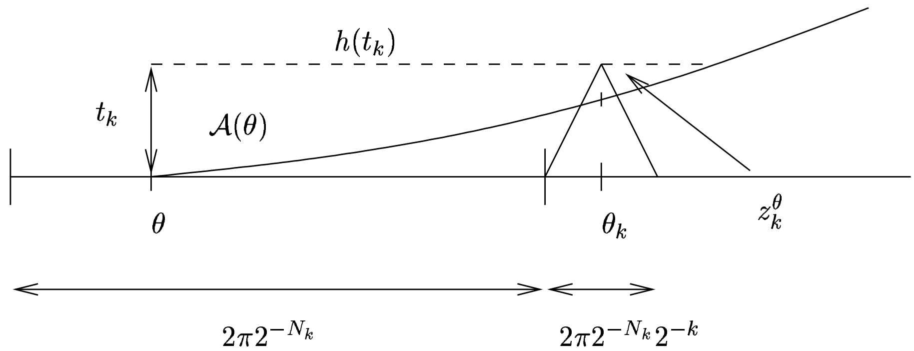

FIGURE 2. N_k is chosen such that $z_k^\theta \in \mathcal{A}(\theta)$, i.e. such that $t_k \geq s_k$.

Equation (3) follows by equation (1). Equation (5) follows if one notes that if $\omega \notin S(n+1)$, then $Q_{n+1}(\omega) = Q_n(\omega)$ and thus $f_{n+1} = f_n$.

We let $F = Pf$ and $F_n = Pf_n$. Then F_n and F are elements in h^∞ and, by the dominated convergence theorem, $F_n \to F$ pointwise.

We shall now construct the sequences $\{N_k\}$ and, for a given $\theta \in \mathbb{T}$, $\{z_k^\theta\}$. See Figure 2 for a geometric picture.

Let $\theta \in \mathbb{T}$ and $k \in \mathbb{N}$ be given. Going clockwise from θ along $\mathbb{T}$, we let θ_k denote the midpoint of the first interval $I_j(k)$ in $S(k)$ that we hit and which does not contain θ. Note that this determines θ_k uniquely. This choice of θ_k yields that $|\theta - \theta_k| \leq 2\pi 2^{-N_k} + 2\pi 2^{-N_k} 2^{-k}/2 = \pi 2^{-N_k}(2 + 2^{-k}) \leq 4\pi 2^{-N_k}$. Define $t_k = \alpha_0 \pi 2^{-N_k} 2^{-k}$, where α_0 is the constant in Lemma 1. Define $z_k^\theta = (1 - t_k)e^{i\theta_k}$. We claim that if N_k is chosen large enough (and increasing), then $z_k^\theta \in \mathcal{A}(\theta)$.

Note that $z_k^\theta \in \mathcal{A}(\theta)$ is equivalent to $|\theta - \theta_k| \leq h(t_k)$. Since $|\theta - \theta_k| \leq 4\pi 2^{-N_k}$, it suffices to prove that $h(t_k) \geq 4\pi 2^{-N_k}$ for sufficiently large choices of N_k, which is equivalent to

$$h(t_k)/t_k \geq \frac{4 \cdot 2^k}{\alpha_0}.$$

But k is fixed and $h(t)/t \to \infty$ as $t \to 0^+$, so the claim follows (since $t_k \to 0$ as $N_k \to \infty$). If necessary, by choosing inductively the sequence $\{N_k\}$ even larger we can obtain also the second claim in Lemma 4, part (i), which is equivalent to $2^{N_{k+1}} \geq 2^{N_k} 2^k \alpha_0^{-1}$.

We have now proved Lemma 4, part (i). Part (ii) follows immediately from equation (4) and Lemma 2.

To prove part (iii), note that we have $|F_{j+1}(z) - F_j(z)| \leq 2(P\chi_{S(j+1)})(z)$, by equation (5). Now, with $s = \pi 2^{-N_{j+1}}$, we apply Lemma 3 to $\chi_{S(j+1)}$, where ε can be taken to be 2^{-j}. Part (iii) follows. □

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