Thesis for the Degree of Doctor of Philosophy

Error Analysis and Smoothing Properties of Discretized Deterministic and Stochastic Parabolic Problems

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Abstract

In this thesis we consider smoothing properties and approximation of time derivatives for parabolic equations and error estimates for stochastic parabolic partial differential equations approximated by the finite element method.

In the first two papers, we study smoothing properties and approximation of the time derivative in time discretization schemes with constant and variable time steps for an abstract homogeneous linear parabolic problem. The time stepping schemes are based on using rational functions $r(z)$ which are $A(\theta)$-stable for suitable $\theta \in [0, \pi/2]$ and satisfy $|r(\infty)| < 1$, and the approximations of the time derivative are based on using difference quotients in time.

In the third paper, we consider the smoothing properties and time derivative approximations in multistep backward difference methods for a nonhomogeneous parabolic equation.

In the fourth paper, as an application of the error estimates for the time derivative developed in the previous three papers, we study a postprocessed finite element method for semilinear parabolic equations.

In the last two papers, we consider the finite element method for both linear and nonlinear stochastic parabolic partial differential equations. Using appropriate nonsmooth data error estimates for deterministic finite element problems, we prove error estimates for both space and time discretization.

Keywords: parabolic, smoothing, time derivative, time discretization, finite element method, stochastic parabolic equation, postprocessing.

This thesis consists of an introduction and the following papers:

- **Paper I:** *Smoothing properties and approximation of time derivatives for parabolic equations: constant time steps*, to appear in IMA J. Numer. Anal.

- **Paper II:** *Smoothing properties and approximation of time derivatives for parabolic equations : variable time steps*, to appear in BIT.

- **Paper III:** *Smoothing properties and approximation of time derivatives in multistep backward difference methods for parabolic equations*, preprint 2003–05, Chalmers Finite Element Center, Chalmers University of Technology.

- **Paper IV:** *Postprocessing the finite element method for semilinear parabolic problems*, preprint 2003–06, Chalmers Finite Element Center, Chalmers University of Technology.

- **Paper V:** *The finite element method for a linear stochastic parabolic partial differential equation driven by additive noise*, preprint 2003–07, Chalmers Finite Element Center, Chalmers University of Technology.

- **Paper VI:** *A finite element method for a nonlinear stochastic parabolic equation*, preprint 2003–08, Chalmers Finite Element Center, Chalmers University of Technology.

Contents

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1. Introduction

This thesis consists of six papers. In papers [I, II] we consider the smoothing properties and approximation of time derivatives in single step methods for homogeneous linear parabolic equations. In paper [III] we consider the smoothing properties and time derivative approximation in multistep backward difference methods for nonhomogeneous linear parabolic equations. As an application of the error estimates for the time derivative, we consider, in paper [IV], the postprocessing of the finite element method for semilinear parabolic equations. In papers [V, VI] we apply nonsmooth data error estimates in the context of the finite element method for stochastic parabolic partial differential equations.

1.1. **Homogeneous linear parabolic equations.** In this subsection we introduce the homogeneous linear parabolic equation

$$u_t + Au = 0, \quad \text{for } t > 0, \quad \text{with } u(0) = v, \tag{1.1}$$

in a Banach space $\mathcal{B}$ with norm $\|\cdot\|$, where $v \in \mathcal{B}$ and A is a closed, densely defined linear operator, with domain of definition $\mathcal{D}(A) \subset \mathcal{B}$. We assume that the resolvent set $\rho(A)$ of A is such that, for some $\delta \in (0, \pi/2)$,

$$\rho(A) \supset \Sigma_\delta = \{z \in \mathbf{C} : \delta \le |\arg z| \le \pi, z \ne 0\} \cup \{0\}, \tag{1.2}$$

and that its resolvent, $R(z; A) = (zI - A)^{-1}$, satisfies

$$\|R(z;A)\| \le M|z|^{-1}, \quad \text{for } z \in \Sigma_\delta,\ z \ne 0, \quad \text{with } M \ge 1. \tag{1.3}$$

In particular, if $\mathcal{B}$ is a separable Hilbert space H and A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, then (1.2) and (1.3) are satisfied with δ arbitrarily close to 0.

It is well known that $-A$ is the infinitesimal generator of a uniformly bounded analytic semigroup $E(t) = e^{-tA}$, $t \ge 0$, which is the solution operator of (1.1), so that $u(t) = E(t)v$. The following smoothing properties hold, with $D_t = \partial/\partial t$, see Pazy [41],

$$\|D_t^j E(t)v\| = \|A^j E(t)v\| \le C_j t^{-j} \|v\|, \quad \text{for } t > 0,\ j \ge 0, \tag{1.4}$$

which shows that the solution is regular for positive time even if the initial data are not.

We are mainly interested in the smoothing properties of discretizations of (1.1). Let U^n be an approximation of the solution $u(t_n) = E(t_n)v$ of (1.1) at time $t_n = nk$, where k is the time step, defined by a single step method,

$$U^n = r(kA)U^{n-1}, \quad \text{for } n \ge 1, \quad \text{with } U^0 = v, \tag{1.5}$$

where $r(z)$ is accurate of order $p \geq 1$, i.e.,

$$r(z) - e^{-z} = O(z^{p+1}), \quad \text{as } z \to 0. \tag{1.6}$$

For example, the backward Euler method, given by $r(z) = 1/(1+z)$, is first order accurate and the Crank-Nicolson method, defined by $r(z) = (1-\frac{1}{2}z)/(1+\frac{1}{2}z)$, is second order. As another example, the method defined by the $(q, q+1)$ subdiagonal Padé approximation $r(z) = p_1(z)/p_2(z)$, where p_1 and p_2 are certain polynomials of degrees q and $q+1$, respectively, is accurate of order $2q+1$.

Stability and error estimates for single step methods have been studied by many authors, see Thomée [46] and references therein.

For the smoothing properties of the time discretization scheme (1.5), if $r(\infty) = 0$, $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \frac{1}{2}\pi]$ and A satisfies (1.2) and (1.3), then we have, see, e.g., Hansbo [27] and Thomée [46, Lemma 7.3],

$$\|A^j U^n\| = \|A^j E_k^n v\| \leq C t_n^{-j} \|v\|, \quad \text{for } t_n \geq t_j,\ v \in \mathcal{B}. \tag{1.7}$$

Hansbo [27] also shows an optimal order error estimate in the nonsmooth data case for the approximation $(-A)U^n \approx (-A)u(t_n) = u_t(t_n)$ of the first order time derivative of the solution of (1.1). More precisely, if $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \frac{1}{2}\pi]$ and $r(\infty) = 0$, then

$$\|(-A)U^n - u_t(t_n)\| \leq C k^p t_n^{-p-1} \|v\|, \quad \text{for } t_n > 0,\ v \in \mathcal{B}. \tag{1.8}$$

It is natural to consider the case when $r(\infty) \neq 0$. We know that (1.8) is not valid when $r(\infty) \neq 0$. In paper [I], we mainly consider the smoothing properties for (1.5) when $|r(\infty)| < 1$. To do this we introduce difference quotients $Q_k^j U^n$, which approximate $D_t^j u(t_n)$ to a certain order. We then obtain smoothing properties for (1.5) when $|r(\infty)| < 1$ and error estimates for the time derivative in case of both smooth and nonsmooth data.

In paper [II], we consider the smoothing properties and time derivative approximations for (1.1) with variable time steps. We introduce the so called *increasing quasi-quasiuniform* assumptions for the time steps. Under these assumptions we show the smoothing properties for discretization of (1.1) and the error estimates for the time derivative, where we approximate the time derivative u_t by first and second order difference quotients.

1.2. **Nonhomogeneous linear parabolic equations.** In this subsection we introduce the nonhomogeneous parabolic equation

$$u_t + Au = f, \quad \text{for } t > 0, \quad \text{with } u(0) = v, \tag{1.9}$$

in Hilbert space H with norm $\|\cdot\|$, where A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, where $v \in H$ and f is a function of t with values in H.

Further we introduce the multistep backward difference operator $\bar{\partial}_p, p \geq 1$ by

$$\bar{\partial}_p U^n = \sum_{j=1}^{p} \frac{k^{j-1}}{j} \bar{\partial}^j U^n, \quad \text{where } \bar{\partial} U^n = (U^n - U^{n-1})/k, \tag{1.10}$$

and define our approximate solution U^n by, with $U^0, \cdots, U^{p-1}$ given,

$$\bar{\partial}_p U^n + AU^n = f^n, \quad \text{for } n \geq p, \quad \text{where } f^n = f(t_n). \tag{1.11}$$

It is well known from the theory for numerical solution of ordinary differential equations, see, e.g., Hairer and Wanner [26], that this method is $A(\theta)$-stable for some $\theta = \theta_p > 0$ when $p \leq 6$. The error estimates for such method has been studied in Bramble, Pasciak, Sammon, and Thomée [6]. It is easy to see that, for any smooth real-valued function u, see Thomée [46, Chapter 10],

$$u_t(t_n) = \bar{\partial}_p u^n + O(k^p), \quad \text{as } k \to 0, \quad \text{with } u^n = u(t_n). \tag{1.12}$$

The theory of stability and error estimates for the approximation of the solution of (1.9) by using multistep methods have been well developed, see Becker [5], Bramble, Pasciak, Sammon, and Thomée [6], Crouzeix [9], Hansbo [28], LeRoux [36], [37], Palencia and Garcia-Archilla [40], Savaré [44], Thomée [46]. The smoothing properties and the approximation of time derivatives in single step methods for homogeneous parabolic problem have been studied in Hansbo [27], [28].

In paper [III] we study the smoothing properties of discretization scheme (1.11). We attempt to extend the results in papers [I, II] for the single step methods to the multistep backward difference methods.

1.3. **Semilinear parabolic equations.** In this subsection we introduce the semilinear parabolic equation

$$\begin{aligned} &u_t - \Delta u = F(u), && \text{in } \Omega, \quad \text{for } t \in (0, T], \\ &u = 0, && \text{on } \partial\Omega, \quad \text{for } t \in (0, T], \\ &u(0) = v, \end{aligned} \tag{1.13}$$

where Ω is a bounded domain in $\mathbf{R}^d$, $d = 1, 2, 3$, with a sufficiently smooth boundary $\partial\Omega$, $u_t = \partial u/\partial t$, Δ is the Laplacian, and $F : \mathbf{R} \to \mathbf{R}$ is a smooth function.

Let $H = L_2(\Omega)$. We define the unbounded operator $A = -\Delta$ on H with domain of definition $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, where, for integer $m \geq 1$, $H^m(\Omega)$ denotes the standard Sobolev space $W_2^m(\Omega)$, and $H_0^1(\Omega) = \{v : v \in H^1(\Omega),\ v|_{\partial\Omega} = 0\}$. Then A is a closed, densely defined, and self-adjoint positive definite operator in H with compact inverse. The initial-boundary value problem (1.13) may then be formulated as the following initial value problem

$$u_t + Au = F(u), \quad \text{for } t \in [0, T], \quad \text{with}\, u(0) = v, \tag{1.14}$$

in the Hilbert space H.

We are interested in the postprocessing of the finite element method for (1.14). A postprocessing technique has been introduced to increase the efficiency of Galerkin methods of spectral type, see Canuto, Hussaini, Quarteroni, and Zang [8], De Frutos, Garcia-Archilla, and Novo [16], De Frutos and Novo [17], [19]. Postprocessed methods yield greater accuracy than standard Galerkin schemes at nearly the same computational cost. In Garcia-Archilla and Titi [24], the postprocessing technique has been extended to the h-version of the finite element method for dissipative partial differential equations. There, the authors prove that the postprocessed method has a higher rate of convergence than the standard finite element method when higher order finite elements, rather than linear finite elements, are used. Error estimates in L_2 and H^1 norms in spatially semidiscrete case are proved. More recently, in De Frutos and Novo [18], the authors show that the postprocessing technique can also be applied to the linear finite elements and the convergence rate can be improved in H^1 norm, but not in L_2 norm. The analysis is restricted to the spatially semidiscrete case.

Let $\{S_h\} = \{S_{h,r}\} \subset H_0^1$ be a family of finite element spaces with the accuracy of order $r \geq 2$, i.e., S_h consists of continuous functions on the closure $\bar{\Omega}$ of Ω, which are polynomials of degree at most $r - 1$ in each triangle of the triangulation of Ω, where h denotes the maximal length of the sides of the triangulation.

The semidiscrete problem corresponding to (1.14) is to find the approximate solution $u_h(t) = u_h(\cdot, t) \in S_h$ for each t, such that,

$$u_{h,t} + A_h u_h = P_h F(u_h), \quad \text{with } u_h(0) = v_h, \tag{1.15}$$

where $v_h \in S_h$, $P_h : L_2 \to S_h$ is the orthogonal projection onto S_h, and $A_h : S_h \to S_h$ is the discrete analogue of A, defined by

$$(A_h \psi, \chi) = A(\psi, \chi), \quad \forall\, \psi, \chi \in S_h. \tag{1.16}$$

Here $A(\cdot, \cdot)$ is the bilinear form on H_0^1 obtained from A.

Error estimates for the semilinear parabolic problem based on finite element methods with various conditions on the nonlinear term have been considered in many papers, see, e.g., Akrivis, Crouzeix, and Makridakis [1], [2], Crouzeix, Thomée, and Wahlbin [10], Elliott and Larsson [21], [22], Helfrich [29], Johnson, Larsson, Thomée, and Wahlbin [30], Thomée [46], Thomée and Wahlbin [47], Wheeler [49]. The long time behavior of finite element solutions was studied by Elliott and Stuart [23], Larsson [32], [33], Larsson and Sanz-Serna [34], [35].

Let us now describe the idea of the postprocessed finite element method proposed by Garcia-Archilla and Titi [24]. Suppose that we want to obtain high order approximation, for instance $O(h^{r+2})$. Then we can use, in every time step, either a family of high order finite element spaces $\tilde{S}_h := S_{h,r+2}$ with the order $r+2$ of accuracy, or a family of finite element spaces $\tilde{S}_h := S_{\tilde{h},r}$ with accuracy of order r, but with finer partition $\mathcal{T}_{\tilde{h}}$ of the domain Ω, such that, $h^{r+2} = \tilde{h}^r$. In [24], another technique, called the *postprocessed finite element method*, is presented, which improves the convergence rate without using a high order finite element space $\tilde{S}_h$ in every time step. Suppose that we are interested in the solution of (1.14) at a given time T. At time T, rewriting (1.14), we have

$$Au(T) = -u_t(T) + F(u(T)). \tag{1.17}$$

Thus, $u(T)$ can be seen as the solution of an elliptic problem whose right hand side is not known but can be approximated. Garcia-Archilla and Titi first compute $u_h(T)$ by (1.15) in the finite element space S_h, then replace $u_t(T)$ by $u_{h,t}(T)$ and solve (or, in practice, approximate) the following linear elliptic problem: find $\tilde{u}(T) \in \mathcal{D}(A)$, such that,

$$A\tilde{u}(T) = -u_{h,t}(T) + F(u_h(T)), \tag{1.18}$$

which is the postprocessing step.

They obtained the following error estimate, with $\ell_h = 1 + \log(T/h^2)$,

$$\|\tilde{u}(T) - u(T)\| \le C(u)\ell_h h^{r+2}, \quad \text{for } r \ge 4, \tag{1.19}$$

where $C(u)$ is some constant depending on u. A similar result holds for $r \ge 3$. The proof is based on superconvergence for elliptic finite element methods in norms of negative order, which is the reason for the restriction $r \ge 3$.

We note that the bound (1.19) is an improvement over the error estimates for the standard Galerkin method, which is $O(h^r)$. In practice $\tilde{u}$ can not be computed exactly, since in general it does not belong to a finite element space. However, one can approximate the solution $\tilde{u}$ of (1.18) by some $\tilde{u}_h$ belonging to a finite element space $\tilde{S}_h$ of approximation order $r+2$ as described above. More precisely, we pose the following semidiscrete problem corresponding to (1.18): find $\tilde{u}_h \in \tilde{S}_h$, such that,

$$\tilde{A}_h \tilde{u}_h(T) = \tilde{P}_h\big(- u_{h,t}(T) + F(u_h(T))\big), \tag{1.20}$$

where $\tilde{P}_h : L_2 \to \tilde{S}_h$ is the L_2 projection onto $\tilde{S}_h$ and $\tilde{A}_h$ is the discrete analogue of A with respect to $\tilde{S}_h$. The standard error estimate reads, see, e.g., Brenner and Scott [7],

$$\|\tilde{u}_h(T) - \tilde{u}(T)\| \le C(u)h^{r+2}. \tag{1.21}$$

Combining (1.19) and (1.21), we have

$$\|\tilde{u}_h(T)-u(T)\| \le \|\tilde{u}_h(T)-\tilde{u}(T)\|+\|\tilde{u}(T)-u(T)\| \le C(u)\ell_h h^{r+2}, \quad \text{for } r \ge 4.$$

In paper [IV], we derive error estimates in the fully discrete case for the postprocessed finite element method applied to (1.14). To do this, we introduce a time stepping method to compute an approximate solution of (1.15) and replace the time derivative in (1.18) by a difference quotient. We then show error estimates for the completely discrete postprocessed method by using error estimates for time derivatives based on our methods in papers [I, II, III].

Our technique of proof is related to but different from the one employed in Garcia-Archilla and Titi [24].

1.4. **Stochastic parabolic equations.** In this subsection we will introduce the linear stochastic parabolic partial differential equation

$$du + Au\,dt = dW, \quad \text{for } 0 < t \le T, \quad \text{with } u(0) = u_0, \tag{1.22}$$

in a Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, where $u(t)$ is an H-valued random process, A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, where $W(t)$ is a Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\ge 0})$ and $u_0 \in H$.

For the sake of simplicity, we shall concentrate on the case $A = -\Delta$, where Δ stands for the Laplacian operator subject to homogeneous Dirichlet boundary conditions, and $H = L_2(\mathcal{D})$, where $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d, d = 1, 2, 3$, with a sufficiently smooth boundary $\partial\mathcal{D}$.

Such equations are common in applications. Many mathematics models in physics, chemistry, biology, population dynamics, neurophysiology, etc., are described by stochastic partial differential equations, see, Da Prato and Zabczyk [14], Walsh [48], etc. The existence, uniqueness, and properties of the solutions of such equations have been well studied, see Curtain and Falb [11], Da Prato [12], Da Prato and Lunardi [13], Da Prato and Zabczyk [14], Dawson [15], Gozzi [25], Peszat and Zabczyk [43], Walsh [48], etc. However, numerical approximation of such equations has not been studied thoroughly.

The equation (1.22) can be written formally as

$$u_t + Au = \frac{dW}{dt} \quad \text{for } 0 < t \le T, \quad \text{with } u(0) = u_0, \tag{1.23}$$

where the derivative $\frac{dW}{dt}$ (the noise) does not exist as a function of t in the usual sense. Thus the equation (1.23) is understood in the integral form.

Let $E(t) = e^{-tA}$, $t \geq 0$. Then (1.23) admits a unique mild solution, see Da Prato and Zabczyk [14, Theorem 5.2, which is given by 5.4],

$$(1.24) \qquad u(t) = E(t)u_0 + \int_0^t E(t-s)\,dW(s) \quad \text{for } 0 < t \leq T,$$

where the integral is understood in the Itô sense. In the following two subsections we review the construction of such a stochastic integral with respect to an H-valued Wiener process or a cylindrical Wiener process. For further information on Wiener processes, stochastic integrals and measures on Hilbert spaces, see Curtain and Falb [11], Da Prato and Zabczyk [14], Dawson [15], Gozzi [25], Kuo [31], Peszat [42], Zabczyk [50], etc.

1.4.1. *The stochastic integral with respect to an H-valued Wiener process.* A family $W(t), t \geq 0$, of H-valued random variables is called a Wiener process on H, if and only if, see [14] and [50],

(*i*) $W(0) = 0$,
(*ii*) for almost all $\omega \in \Omega$, $t \mapsto W(t,\omega)$ is a continuous function,
(*iii*) $W(t)$ has independent increments,
(*iv*) $\mathcal{L}(W(t) - W(s)) = \mathcal{L}(W(t-s)), 0 \leq s \leq t$.

Here $\mathcal{L}(X)$ denotes the law, or the distribution, of the H-valued random variable X, i.e., the probability measure on H defined by

$$\mathcal{L}(X)(A) = \mathbf{P}\{\omega : X(\omega) \in A\}, \quad \text{for any } A \in \mathcal{B}(H),$$

where $\mathcal{B}(H)$ is the Borel σ-algebra of H, i.e., the smallest σ-algebra containing all closed (or open) sets of H.

It turns out that if $W(t)$ is a Wiener process, then, for arbitrary t, $\mathcal{L}(W(t))$ is a Gaussian probability measure on H with the mean 0 and the covariance operator tQ, i.e.,

$$\mathcal{L}(W(t)) = \mathcal{N}(0, tQ),$$

where Q is a linear, self-adjoint, positive definite, bounded operator with finite trace, i.e., $\text{Tr}(Q) < \infty$. In other words, if $W(t)$ is a Wiener process, then, for arbitrary t, $W(t)$ is an H-valued Gaussian random variable with

$$\mathbf{E}(W(t)) = 0,$$

and

$$\text{Cov}(W(t)) = tQ,$$

where for any H-valued random variable X, $\text{Cov}(X) \in L(H)$ is defined by

$$\text{Cov}(X) = \mathbf{E}\big((X - \mathbf{E}X) \otimes (X - \mathbf{E}X)\big).$$

Here, for $a, b \in H$, $a \otimes b$ is the bounded linear operator on H defined by

$$(a \otimes b)(h) = (b, h)a, \quad \forall h \in H.$$

Moreover,

$$\mathbf{E}\big((h, W(t))^2\big) = t(Qh, h), \quad \forall h \in H. \tag{1.25}$$

We then call the above $W(t)$ an H-valued Wiener process with covariance operator $Q, \mathrm{Tr}(Q) < \infty$. Here we emphasize $\mathrm{Tr}(Q) < \infty$ since, otherwise, $W(t)$ will not be H-valued, see below.

Since Q is in trace class which implies that Q is a compact operator on H, there exists an orthonormal basis $\{e_l\}_{l=1}^{\infty}$ and a bounded sequence of positive real numbers $\{\gamma_l\}_{l=1}^{\infty}$, such that

$$Qe_l = \gamma_l e_l, \quad l = 1, 2, \cdots . \tag{1.26}$$

Then, $W(t)$ has the form

$$W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} \beta_l(t) e_l, \tag{1.27}$$

which is convergent in $L_2(\Omega; H)$, where

$$\beta_l(t) = \gamma_l^{-1/2}(e_l, W(t)), \quad l = 1, 2, \cdots , \tag{1.28}$$

are real-valued, independent, Brownian motions on $(\Omega, \mathcal{F}, \mathbf{P})$.

We now recall the definition of the stochastic integral for an operator-valued process. There is a natural class of operator-valued processes, which can be stochastically integrated with respect to an H-valued Wiener process $W(t)$. Denote by $Q^{1/2}(H)$ the image of the operator $Q^{1/2}$ on H. Since Q is a linear, self-adjoint, positive definite, and bounded operator with finite trace, we may define $Q^{1/2}$ and its inverse $Q^{-1/2}$ by the eigenvalues $\{\gamma_l\}$ and eigenfunctions $\{e_l\}$ of Q. More precisely, for arbitrary $\alpha \in \mathbf{R}$, we define the operator $Q^{\alpha} : \mathcal{D}(Q^{\alpha}) \to H$ by

$$Q^{\alpha} v = \sum_{l=1}^{\infty} \gamma_l^{\alpha} (v, e_l) e_l, \quad \text{where } \mathcal{D}(Q^{\alpha}) = \Big\{ v \in H : \sum_{l=1}^{\infty} \gamma_l^{2\alpha} (v, e_l)^2 < \infty \Big\}.$$

It is obvious that $\mathcal{D}(Q^{\alpha}) = H$ for $\alpha \geq 0$, since $\{\gamma_l\}_{l=1}^{\infty}$ is a bounded sequence of positive real numbers. It is easy to show that $Q^{1/2}(H)$ is a separable Hilbert space with inner product

$$(v, w)_{Q^{1/2}(H)} = (Q^{-1/2} v, Q^{-1/2} w)$$

and the induced norm $\|v\|_{Q^{1/2}(H)} = \|Q^{-1/2} v\|$ for any $v \in Q^{1/2}(H)$.

For any Hilbert space H_1, we denote by $L^2_{\mathcal{F}}([0, T]; H_1)$ the separable Hilbert space of all $\mathcal{F}_t$-progressively measurable processes x, with values in H_1, such that

$$\|x\|_{L^2_{\mathcal{F}}([0,T]; H_1)} = \Big(\mathbf{E} \int_0^T \|x(t)\|_{H_1}^2 \, dt \Big)^{1/2} < \infty.$$

Denote by $L(H)$ the space of bounded linear operators on H, and by $L_2^0(Q^{1/2}(H),H)$ the space of all Hilbert-Schmidt operators from $Q^{1/2}(H)$ into H, i.e.,

$$L_2^0(Q^{1/2}(H),H)=\Big\{\psi\in L\big(Q^{1/2}(H),H\big):\sum_{j=1}^{\infty}\|\psi g_j\|^2<\infty\Big\},$$

where $\{g_j\}_{j=1}^{\infty}$ is an arbitrary orthonormal basis of $Q^{1/2}(H)$. Its norm is denoted by

$$\|\psi\|_{L_2^0}=\Big(\sum_{j=1}^{\infty}\|\psi g_j\|^2\Big)^{1/2}.$$

In particular, we can choose $g_j=Q^{1/2}e_j=\gamma_j^{1/2}e_j$, so that

$$\|\psi\|_{L_2^0}=\Big(\sum_{j=1}^{\infty}\gamma_j\|\psi e_j\|^2\Big)^{1/2}.$$

For simplicity, we denote $L_2^0=L_2^0\big(Q^{1/2}(H),H\big)$ below. It is easy to see that $L(H)\subset L_2^0$.

Let us review the construction of the stochastic integral $\int_0^T\psi(t)\,dW(t)$ for a process $\psi(\cdot)\in L^2_{\mathcal{F}}([0,T];L_2^0)$. Suppose first that $\psi(t),t\in[0,T]$, is a simple process, i.e., there exist a sequence $0=t_0<t_1<\cdots<t_n=T$ and a sequence $\xi_0,\xi_1,\cdots,\xi_{n-1}$ of $L(H)$-valued random variables such that

$$\psi(t)=\xi_i,\quad \text{for } t\in(t_i,t_{i+1}],\ i=0,1,\cdots,n-1,\quad \text{with } \psi(0)=0,$$

where ξ_i are $\mathcal{F}_{t_i}$-measurable and $\psi(\cdot)\in L^2_{\mathcal{F}}([0,T];L_2^0)$. We then define the stochastic integral by the formula

$$\int_0^T\psi(t)\,dW(t)=\sum_{i=0}^{n-1}\xi_i\big(W(t_{i+1})-W(t_i)\big).$$

For a general $L(H)$-valued stochastic process $\in L^2_{\mathcal{F}}([0,T];L_2^0)$, there is a sequence of simple processes $\{\psi_n(t)\}$ such that $\psi_n\to\psi$ almost everywhere on $[0,T]\times\Omega$ and

$$\lim_{n\to\infty}\int_0^T\mathbf{E}\|\psi_n(t)-\psi(t)\|^2\,dt=0.$$

Moreover, $\{\int_0^T\psi_n(t)dW(t)\}$ has a unique limit in $L_2(\Omega;H)$. We define the stochastic integral $\int_0^T\psi(t)\,dW(t)$ to be this limit.

We have the following isometry property, see [14, Proposition 4.5].

Lemma 1.1. *Let $W(t)$ be an H-valued Wiener process with covariance operator $Q, \mathrm{Tr}(Q) < \infty$. Let $\psi(t)$ be an $L(H)$-valued process and $\psi(\cdot) \in L^2_{\mathcal{F}}([0,T]; L^0_2)$. Then we have*

$$\mathbf{E}\Big\| \int_0^T \psi(t)\, dW(t) \Big\|^2 = \int_0^T \mathbf{E}\|\psi(t)\|^2_{L^0_2}\, dt. \tag{1.29}$$

1.4.2. *The stochastic integral with respect to a cylindrical Wiener process.* The construction of the stochastic integral for an H-valued Wiener process $W(t)$ above requires that $W(t)$ is H-valued, which implies that Q is a trace class operator. In this subsection we extend the definition of the stochastic integral to the case of a cylindrical Wiener process. Let Q be a linear, self-adjoint, positive definite, bounded operator on H, not necessarily in trace class, but with a bounded sequence of positive eigenvalues $\{\gamma_l\}_{l=1}^{\infty}$ and a corresponding orthonormal basis of eigenvectors $\{e_l\}_{l=1}^{\infty}$ in H. Thus Q is not necessarily compact, for example, $Q = I$. By a cylindrical Wiener process with covariance operator $Q, \mathrm{Tr}(Q) \le \infty$, we mean the series, see Da Prato and Zabczyk [14], Peszat [42], Peszat and Zabczyk [43],

$$W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t), \quad t \ge 0, \tag{1.30}$$

where $\{\beta_l(t)\}$ is a family of real-valued, independent, Brownian motions. In the special case $Q = I$, $W(t)$ is defined by

$$W(t) = \sum_{l=1}^{\infty} e_l \beta_l(t), \quad t \ge 0. \tag{1.31}$$

We observe that (1.30) is divergent in $L_2(\Omega; H)$ if Q is not in trace class, in which case $W(t)$ is not an H-valued process. In fact, for arbitrary $t > 0$,

$$\mathbf{E}\Big\| \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t) \Big\|^2 = \sum_{l=1}^{\infty} \gamma_l \mathbf{E}\beta_l(t)^2 = t \sum_{l=1}^{\infty} \gamma_l = t\,\mathrm{Tr}(Q) = \infty.$$

However, let H_L be an arbitrary Hilbert space such that the embedding of $Q^{1/2}(H)$ into H_L is Hilbert-Schmidt. Then we have the following lemma, see [14, Proposition 4.11].

Lemma 1.2. *The cylindrical Wiener process* (1.30) *defines a H_L-valued Wiener process with some covariance operator Q_L.*

For arbitrary $h \in H$, the process

$$\langle h, W(t) \rangle := \sum_{l=1}^{\infty} \gamma_l^{1/2} (h, e_l) \beta_l(t) \tag{1.32}$$

is a real-valued Brownian motion and

$$\mathbf{E}\big(\langle h_1, W(t)\rangle\langle h_2, W(s)\rangle\big) = \min(t,s)(Qh_1,h_2), \quad \textit{for } h_1,h_2 \in H. \tag{1.33}$$

For any $\psi(\cdot) \in L^2_{\mathcal{F}}([0,T];L^0_2)$, we can define the stochastic integral as follows:

$$\int_0^T \psi(t)\,dW(t) = \sum_{l=1}^{\infty}\int_0^T \psi(t)g_l\,d\langle g_l, W(t)\rangle, \tag{1.34}$$

where $\{g_l\}_{l=1}^{\infty}$ is an arbitrarily orthonormal basis in $Q^{1/2}(H)$, and the integral on the right is the standard Itô integral.

Let us consider three special cases.

(i) If $Q = I$, then we can choose $g_l = e_l$, and hence $\langle g_l, W(t)\rangle = \beta_l(t)$ by (1.32), therefore the stochastic integral is

$$\int_0^T \psi(t)\,dW(t) = \sum_{l=1}^{\infty}\int_0^T \psi(t)e_l\,d\beta_l(t),$$

which is consistent with the definition in Peszat [42];

(ii) If $W(t)$ is a Wiener process with $\mathrm{Tr}(Q) < \infty$, then $Q^{1/2}$ is Hilbert-Schmidt and $H_L = H$, in this case, the stochastic integral defined by (1.34) is consistent with the stochastic integral defined in the subsection 1.4.1. In fact, for simplicity, let $\psi(t) = \psi \in L(H) \subset L^0_2$, be independent of t. Then we have, with $g_l = \gamma_l^{1/2}e_l$,

$$\sum_{l=1}^{\infty}\int_0^T \psi(t)g_l\,d\langle g_l, W(t)\rangle = \sum_{l=1}^{\infty}\psi g_l\beta_l(t) = \psi\sum_{l=1}^{\infty} g_l\beta_l(t) = \psi W(t),$$

which is indeed the stochastic integral $\int_0^T \psi(t)\,dW(t)$ defined in subsection 1.4.1.

(iii) In our work we assume that $\|A^{(\beta-1)/2}\|_{L^0_2} < \infty$ for some $\beta \in [0,1]$, i.e.,

$$\|A^{(\beta-1)/2}\|^2_{L^0_2} = \sum_{l=0}^{\infty}\gamma_l\|A^{(\beta-1)/2}e_l\|^2 < \infty.$$

In this case we have $H_L = \dot{H}^{\beta-1}$. In particular, this guarantees that $W(t)$ is $\dot{H}^{-1}$-valued, which is important for the finite element method.

The isometry property (1.29) also holds in the present case, see Da Prato and Zabczyk [14, Corollary 4.14]. Thus, even if the cylindrical Wiener process $W(t)$ is not H-valued, we can still construct the H-valued stochastic integral $\int_0^T \psi(t)\,dW(t)$.

1.4.3. *Numerical methods for stochastic parabolic equation.* In this subsection we will review some numerical methods for (1.22).

The difficulty of the numerical approximation for such equation is how to approximate the noise. Shardlow [45] considers the finite difference approximation of (1.22) in the one-dimensional case with $H = L_2(0,1)$, $A = -\frac{\partial}{\partial x^2}$ with Dirichlet boundary condition. Let $0 = t_0 < t_1 \cdots < t_N = T$ be a partition of $[0,T]$, $t_n = nk$, $n = 0,1,2,\cdots,N$, where k is the time step. Let $0 = x_0 < x_1 < \cdots < x_J = 1$ be a space partition of $[0,1]$, $x_j = jh$, $j = 0,1,2,\cdots,J$. Denote by $e_j = \sqrt{2}\sin j\pi x$, $j = 1,2,\cdots$, the eigenvectors of $A = -\frac{\partial}{\partial x^2}$ with Dirichlet boundary condition, which forms an orthonormal basis of $L_2(0,1)$. Let P_J denote the operator taking f to its first J Fourier modes

$$P_J f = \sum_{j=1}^{J} (f, e_j) e_j.$$

Then, Shardlow [45] approximates the noise $\frac{dW}{dt}$ over the time step (t_{n-1}, t_n) by

$$\frac{dW}{dt} \approx \frac{1}{k} \int_{t_{n-1}}^{t_n} P_J \, dW(s),$$

which is a $L_2(0,1)$ function, since, for example, in the case $Q = I$,

$$\mathbf{E} \Big\| \frac{1}{k} \int_{t_{n-1}}^{t_n} P_J \, dW(s) \Big\| = \frac{1}{k} \int_{t_{n-1}}^{t_n} \|P_J\|^2_{L_2^0} \, ds = \|P_J\|^2_{L_2^0} = J = h^{-1} < \infty.$$

Then he defines a simple discretization based on the θ-method in time and the standard three point approximation to the Laplacian to obtain the discrete solution $U_j^n, j = 0,1,2,\cdots,J$, on the time $t = t_n$, and show the following error estimates, if $k/h^2 = C$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C h^{(1-\epsilon)/2}, \quad \text{for } \forall\, \epsilon > 0,$$

where $U^n = U^n(x)$ is some continuous function which satisfies $U^n(x_j) = U_j^n$, $j = 0,1,2,\cdots,J$. The result shows that, in the one-dimensional case and in the cylindrical Wiener process case with $Q = I$, the convergence order is almost $O(h^{1/2})$ in the $L_2(\Omega;H)$ norm.

Allen, Novosel, and Zhang [3] consider both finite element and finite difference methods for the same problem studied in [45]. They approximate the space-time white noise $\frac{\partial^2 W}{\partial t \partial x}$ by using piecewise constant functions on a partition $[t_{n-1}, t_n] \times [x_{j-1}, x_j]$, $1 \le n \le N$, $1 \le j \le J$ of $[0,T] \times [0,1]$. More precisely, with $k = t_n - t_{n-1}$ and $h = x_j - x_{j-1}$,

$$\frac{\partial^2 W}{\partial t \partial x} \approx \frac{\partial^2 \hat{W}}{\partial t \partial x} = \frac{1}{kh} \sum_{n=1}^{N} \sum_{j=1}^{J} \eta_{nj} \sqrt{kh}\, \chi_n(t) \chi_j(x),$$

where

$$\chi_n(t) = \begin{cases} 1 & t_{n-1} \le t \le t_n, \\ 0 & \text{otherwise}, \end{cases} \qquad \chi_j(x) = \begin{cases} 1 & x_{j-1} \le x \le x_j, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\eta_{jn} = \frac{1}{kh} \int_{t_{n-1}}^{t_n} \int_{x_{j-1}}^{x_j} dW(t,x) = \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ is the standard real-valued Gaussian random variable and η_{jn} are independent and identically distributed (iid). It is obvious that $\frac{\partial^2 \hat{W}}{\partial t \partial x} \in L_2(0,1)$ for fixed $t \in [0,T]$, $w \in \Omega$. Applying the standard finite element and finite difference methods for the new "simpler" problems, they obtain the approximate solution $U^n \approx u(t_n)$ and the corresponding error estimates. For example, using the backward Euler method, the finite element approximate solution U^n satisfies, with $\ell_k = 1 + \log(T/k)$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C\ell_k(k^{1/4} + h^{1/2}).$$

Du and Zhang [20] also consider the numerical approximation for the equation (1.22) but with some special additive noise. More precisely, the noise takes the form

$$\frac{\partial^2 W}{\partial t \partial x} = \sum_{j=1}^{\infty} \sigma_j(t) \dot{\beta}_j(t) e_j(x),$$

where $\sigma_j(t)$ is a continuous function. $\dot{\beta}_j(t)$ is the derivative of the standard real-valued Wiener process, and $e_j(x) = \sqrt{2} \sin j\pi x$ as above. They approximate the noise by

$$\frac{\partial^2 W}{\partial t \partial x} \approx \frac{\partial^2 \hat{W}}{\partial t \partial x} = \sum_{j=1}^{\infty} \hat{\sigma}_j(t) \Big(\sum_{n=1}^{N} \frac{1}{\sqrt{k}} \beta_{jn} \chi_n(t) \Big) e_j(x),$$

where χ_n is defined as above, and

$$\beta_{jn} = \frac{1}{\sqrt{k}} \int_{t_{n-1}}^{t_n} d\beta_j(t) = \mathcal{N}(0,1),$$

are independent and identically distributed (iid). Replacing $\sigma_j(t)$ by $\hat{\sigma}_j(t)$ we get discretization in the x-direction, and replacing $\dot{\beta}_j(t)$ by $\sum_{n=1}^{N} \frac{1}{\sqrt{k}} \beta_{jn} \chi_n(t)$ we get the discretization in the t-direction. The standard finite element method now can apply to the "simpler" problem and obtain the approximate solution U^n and the corresponding error estimates. For example, in the cylindrical Wiener process case, i.e, $\sigma_j(t) = 1$, using $\hat{\sigma}_j(t) = 1$ for $j \le J$

and $\hat{\sigma}_j(t) = 0$ for $j > J$ to approximate $\sigma_j(t)$, they show the following error estimate for the backward Euler method, with $\ell_k = 1 + \log(T/k)$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C\ell_k(k^{1/4} + h^{1/2}).$$

In the case where $\dot{\beta}_j(t)$ are more regular in time, higher order error estimates may be obtained by using the similar technique in the higher order finite element space.

1.4.4. *Finite element method for stochastic parabolic equation.* Let $S_h \subset H_0^1(\mathcal{D})$ be a family of finite element spaces as defined in section 3. The semidiscrete problem of (1.22) is then to find the process $u_h(t) = u_h(\cdot, t) \in S_h$, such that

$$du_h + A_h u_h\, dt = P_h dW, \quad \text{for } 0 < t \le T, \quad \text{with } u_h(0) = P_h u_0, \tag{1.35}$$

where P_h denotes the L_2-projection onto S_h, and $A_h : S_h \to S_h$ is the discrete analogue of A, defined by

$$(A_h\psi, \chi) = A(\psi, \chi), \quad \forall \psi, \chi \in S_h. \tag{1.36}$$

Here $A(\cdot, \cdot)$ is the bilinear form on $H_0^1(\mathcal{D})$ obtained from the operator A in (1.22).

Let $E_h(t) = e^{-tA_h}$, $t \ge 0$ be the semigroup generated by $-A_h$. Then (1.35) admits a unique mild solution

$$u_h(t) = E_h(t)u_{0h} + \int_0^t E_h(t-s)P_h\, dW(s).$$

Let k be a time step and $t_n = nk$ with $n \ge 1$. We define the following backward Euler method

$$\frac{U^n - U^{n-1}}{k} + A_h U^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} P_h\, dW(t), \quad n \ge 1, \quad U^0 = P_h u_0, \tag{1.37}$$

i.e, we approximate the noise $\frac{dW}{dt}$ by

$$\frac{dW}{dt} \approx \frac{1}{k}\int_{t_{n-1}}^{t_n} P_h\, dW(t) = \frac{1}{k}P_h\big(W(t_n) - W(t_{n-1})\big).$$

We make the assumption that $\|A^{(\beta-1)/2}\|_{L_2^0}\| < \infty$ for some $\beta \in [0,1]$, which guarantees that $W(t)$ is $\dot{H}^{-1}$-valued . Hence $P_h W(t)$ is well defined because P_h can be extended to a bounded linear operator from $\dot{H}^{-1}$ to $\dot{H}^{-1}$.

In paper [V], we obtain error estimates for a linear stochastic parabolic partial differential equation in both semidiscrete and fully discrete cases with respect to strong and weak norms in space. In paper [VI] we extend the results in [V] to a nonlinear stochastic parabolic partial differential equation.

Under certain Lipschitz and growth conditions for nonlinear term, we obtain the similar error estimates as in the linear case.

2. Summary of the appended papers

2.1. Paper I. Smoothing properties and approximation of time derivatives for parabolic equations: constant time steps

In this paper we study smoothing properties and approximation of time derivatives for time discretization schemes with constant time steps for a homogeneous parabolic problem formulated as an abstract initial value problem in a Banach space. The time stepping schemes are based on using rational functions $r(z) \approx e^{-z}$ which are $A(\theta)$-stable for suitable $\theta \in [0, \pi/2]$ and satisfy $|r(\infty)| < 1$, and the approximations of time derivatives are based on using difference quotients in time. Both smooth and nonsmooth data error estimates of optimal order for the approximation of time derivatives are proved. Further, we apply the results to obtain error estimates of time derivatives in the supremum norm for fully discrete methods based on discretizing the spatial variable by a finite element method.

For fixed $j \geq 1$, we introduce the finite difference quotient

$$Q_k^j U^n = \frac{1}{k^j} \sum_{\nu=-m_1}^{m_2} c_\nu U^{n+\nu}, \quad \text{for } n \geq m_1, \tag{2.1}$$

where m_1, m_2 are nonnegative integers, and c_ν are real numbers, and where the operator Q_k^j is an approximation of order $p \geq 1$ to D_t^j, that is, for any smooth real-valued function u,

$$D_t^j u(t_n) = Q_k^j u^n + O(k^p), \quad \text{as } k \to 0, \quad \text{with } u^n = u(t_n).$$

For example, let $j = 1,\ m_1 = 1,\ m_2 = 0$, we have

$$D_t u(t_n) = \frac{1}{k}(-u^{n-1} + u^n) + O(k), \quad \text{as } k \to 0,$$

and let $j = 2,\ m_1 = 1,\ m_2 = 1$, we have

$$D_t^2 u(t_n) = \frac{1}{k^2}(u^{n-1} - 2u^n + u^{n+1}) + O(k^2), \quad \text{as } k \to 0.$$

We show that, if A satisfies (1.2) and (1.3), and $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, and $|r(\infty)| < 1$, then the following smoothing property holds:

$$\|Q_k^j U^n\| \leq C t_n^{-j} \|v\|, \quad \text{for } n \geq m_1,\ t_n > 0,\ v \in \mathcal{B}. \tag{2.2}$$

We also obtain the following smooth data error estimates for time derivative approximation: if A satisfies (1.2) and (1.3), and $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, then

$$\|Q_k^j U^n - D_t^j u(t_n)\| \leq C k^p \|A^{p+j} v\|, \quad \text{for } n \geq m_1,\ v \in \mathcal{D}(A^{p+j}). \tag{2.3}$$

To obtain an optimal order error estimate for nonsmooth data, $A(\theta)$-stability is not sufficient. We find that, if A satisfies (1.2) and (1.3), and $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, and $|r(\infty)| < 1$, then

$$(2.4) \quad \|Q_k^j U^n - D_t^j u(t_n)\| \leq C k^p t_n^{-(p+j)} \|v\|, \quad \text{for } n \geq m_1,\ t_n > 0,\ v \in \mathcal{B}.$$

2.2. **Paper II. Smoothing properties and approximation of time derivatives for parabolic equations: variable time steps**

In this paper we study smoothing properties and approximation of time derivatives for time discretization schemes with variable time steps for a homogeneous parabolic problem formulated as an abstract initial value problem in a Banach space. The time stepping methods are based on using rational functions $r(z) \approx e^{-z}$ which are $A(\theta)$-stable for suitable $\theta \in (0, \pi/2]$ and satisfy $|r(\infty)| < 1$. First and second order approximations of time derivatives based on using difference quotients are considered. Smoothing properties are derived and error estimates are established under the so called *increasing quasi-quasiuniform* assumption on the time steps.

Let $0 = t_0 < t_1 < \cdots < t_n < \cdots$ be a partition of the time axis and let $k_n = t_n - t_{n-1}$, $n \geq 1$, be the variable time steps. An approximate solution $U^n \approx u(t_n) = E(t_n)v$ of (1.1) may be defined by

$$(2.5) \qquad U^n = r(k_n A) U^{n-1}, \quad \text{for } n \geq 1, \quad \text{with } U^0 = v,$$

where $r(z)$ is defined in (1.6).

Stability results in variable time steps have been considered by some authors. For example, Palencia [38] shows that, if A satisfies (1.2) and (1.3), and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, if the time steps $\{k_j\}_{j=1}^{\infty}$ satisfy, with some constant μ,

$$(2.6) \qquad 0 < \mu^{-1} \leq \frac{k_i}{k_j} \leq \mu < \infty, \quad \text{for } i, j \geq 1,$$

then there exists a constant $C(\mu)$ such that the following stability result holds

$$(2.7) \qquad \Big\| \prod_{j=1}^{n} E_{k_j} \Big\| \leq C(\mu), \quad \text{where } E_{kj} = r(k_j A).$$

We observe that the stability bound will depend on the maximum ratio μ between the steps, but not on the steps themselves. In this way, the stability bound does not blow up when the maximum time step goes to zero, as long as μ remains bounded. In particular, a family of quasi-uniform grids with $k_{max} \leq \mu k_{min}$ satisfies the assumption (2.6), where $k_{max} = \max_{1 \leq j \leq n} k_j$, $k_{min} = \min_{1 \leq j \leq n} k_j$. More precisely, Bakaev [4] shows that if A satisfies (1.2) and (1.3), and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, then

$$\Big\| \prod_{j=1}^{n} E_{k_j} \Big\| \leq C \log(1+\mu), \quad \text{where } \mu = k_{max}/k_{min}. \tag{2.8}$$

Palencia [39] further finds that if $|r(\infty)| < 1$, then the stability bound holds without any restriction on the time steps.

In the present paper, we first consider error estimates for (2.5) in both smooth and nonsmooth data cases. We show that if A satisfies (1.2) and (1.3) and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$ and accurate of order $p \geq 1$, then the following smooth data error estimate holds:

$$\|U^n - u(t_n)\| \leq C k_{max}^p \|A^p v\|, \quad \text{for } t_n \geq 0,\ v \in \mathcal{D}(A^p).$$

To obtain error estimates in the nonsmooth data case, we introduce the notion of *increasing quasi-quasiuniform grids* $\mathcal{T}$ in time. Let $\{\mathcal{T}\}$ be a family of partitions of the time axis, $\mathcal{T} = \{t_n : \ 0 = t_0 < t_1 < \cdots < t_n < \cdots\}$. $\{\mathcal{T}\}$ is called a family of *quasi-quasiuniform grids* if there exist positive constants c, C, such that

$$c k_{n+1} \leq k_n \leq C t_n/n, \quad \text{for } n \geq 1. \tag{2.9}$$

Further, if $k_1 \leq k_2 \leq \cdots \leq k_n \leq \cdots$, then we call $\{\mathcal{T}\}$ a family of *increasing quasi-quasiuniform grids.* We note that *increasing quasi-quasiuniform* implies that $k_n \sim k_{n+1}$ and $n k_n \sim t_n$, where $a_n \sim b_n$ means that a_n/b_n is bounded above and below.

For example, if we choose the variable time steps $k_n = n^s k$ for some fixed $s \geq 1$, with $k > 0$, then $t_n = k\big(\sum_{j=1}^n j^s\big)$, and the corresponding family of partitions $\{\mathcal{T}\}$ is a family of *increasing quasi-quasiuniform grids.* In fact, it is obvious that $k_n/k_{n+1} = n^s/(n+1)^s \geq 1/2^s$. Further, since $t_n/k = \sum_{j=1}^n j^s \geq C n^{s+1}$ for some positive constant C, we have $k_n \leq C t_n/n$.

Under these assumptions, we have the following nonsmooth data error estimate: If A satisfies (1.2) and (1.3), and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$ and accurate of order $p \geq 1$, if further $|r(\infty)| < 1$ and $\{\mathcal{T}\}$ is a family of *increasing quasi-quasiuniform grids*, then we have

$$\|U^n - u(t_n)\| \leq C k_n^p t_n^{-p} \|v\|, \quad \text{for } t_n > 0,\ v \in \mathcal{B}.$$

As for the smoothing property, we show that, if $r(\infty) = 0$, and $\{\mathcal{T}\}$ satisfies (2.6), then

$$\Big\| A \prod_{j=1}^{n} E_{k_j} v \Big\| \leq C t_n^{-1} \|v\|, \quad \text{for } t_n > 0,\ v \in \mathcal{B}.$$

As in the constant time step case, see paper [I], the above smoothing property is not true in the case of $r(\infty) \neq 0$. However, if $|r(\infty)| < 1$, then we introduce similar difference quotients as (2.1) but with variable time steps. For simplicity we only consider the following first and second order approximations of time derivative $u_t(t_n)$

$$\bar{\partial}U^n = (U^n - U^{n-1})/k_n, \quad \text{for } n \geq 1,$$

and

$$\bar{\partial}^2 U^n = a_n \bar{\partial} U^n + b_n \bar{\partial} U^{n-1} \quad = a_n (U^n - U^{n-1})/k_n + b_n (U^{n-1} - U^{n-2})/k_{n-1}, \quad \text{for } n \geq 2,$$

where

$$a_n = (2k_n + k_{n-1})/(k_n + k_{n-1}), \quad b_n = -k_n/(k_n + k_{n-1}).$$

In both cases, under the assumption of *increasing quasi-quasiuniform grids*, we obtain a smoothing property and error estimates for time derivative in the nonsmooth data case. We also show a smooth data error estimate without any restrictions on the time steps.

2.3. **Paper III. Smoothing properties and approximation of time derivatives in multistep backward difference methods for parabolic equations**

In this paper we consider the smoothing properties and time derivative approximation in multistep backward difference methods for nonhomogeneous parabolic equations. Smoothing properties and time derivative approximation in single step method for homogeneous parabolic equation have been studied in Hansbo [27], [28], and in papers [I, II]. We extend some of the results in paper [I] to the multistep backward difference method.

We obtain the following smoothing property, i.e., if U^n is the solution of (1.11), then we have, with $f = 0$ and $p \leq 6$,

$$\|\bar{\partial}_p U^n\| \leq C t_n^{-1} \sum_{j=0}^{p-1} \|U^j\|, \quad \text{for } n \geq 2p.$$

It is natural to approximate the time derivative $u_t(t_n)$ by $\bar{\partial}_p U^n$ $(n \geq 2p)$, where $U^n, n \geq p$ is computed by the multistep backward difference method (1.11). Denoting $|v|_s = \|A^{s/2} v\|$ for $s \in \mathbf{R}$, we obtained the following error estimates, with $p \leq 6$,

$$\|\bar{\partial}_p U^n - u_t(t_n)\| \leq C \sum_{j=0}^{p-1} \|A(U^j - u^j)\| + C k^p \int_0^{t_n} \|A u^{(p+1)}(s)\| \, ds, \quad \text{for } n \geq 2p,$$

and, with $G(s) = |u^{(p+1)}(s)|_{-2p-1}^2 + s^{2p+2}|u^{(p+1)}(s)|_1^2 + s^2|u_t(s)|_1^2$,

$$t_n^{2p+2}\|\bar{\partial}_p U^n - u_t(t_n)\|^2 \le C \sum_{j=p}^{2p-1} \big(|U^j - u^j|_{-2p}^2 + k^{2p+2}\|A(U^j - u^j)\|^2\big) + Ck^{2p}\Big(\int_0^{t_n} G(s)\,ds + t_{2p}^3|u_t(t_{2p})|_1^2\Big),$$

When we choose some suitable discrete starting values $U^0, U^1, \cdots, U^{p-1}$, we obtain the following nonsmooth data error estimates, with $f = 0$ and $p \le 6$,

$$\|\bar{\partial}_p U^n - u_t(t_n)\| \le Ck^p t_n^{-p-1}, \quad \text{for } n \ge 2p.$$

2.4. **Paper IV. Postprocessing the finite element method for semilinear parabolic problems**

In this paper we consider postprocessing of the finite element method for semilinear parabolic problems. The postprocessing amounts to solving a linear elliptic problem on a finer grid (or higher-order space) once the time integration on the coarser mesh is completed. The convergence rate is increased at almost no additional computational cost. This procedure was introduced and analyzed in Garcia-Archilla and Titi [24]. We extend the analysis to the fully discrete case and prove error estimates for both space and time discretization. The analysis is based on error estimates for the approximation of time derivatives by difference quotients.

Let $\tilde{u}_h$ and u be the solutions of (1.20) and (1.14), respectively. Under some assumptions for the smoothness of F and u, we prove that, with $\ell_h = 1 + \log(T/h^2)$,

$$\|\tilde{u}_h(T) - u(T)\| \le C\ell_h h^{r+2}, \quad \text{for } r \ge 4. \tag{2.10}$$

A similar result holds for $r \ge 3$. The proof is based on superconvergence for elliptic finite element methods in norms of negative order, which is the reason for the restriction $r \ge 3$.

In the fully discrete case, we define the following backward Euler method to compute the approximate solution $U^n \in S_h$, with $\bar{\partial} U^n = (U^n - U^{n-1})/k$,

$$\bar{\partial} U^n + A_h U^n = P_h F(U^n), \quad n \ge 1, \quad \text{with } U^0 = v_h. \tag{2.11}$$

It is natural to approximate $u_{h,t}(T)$, $T = t_n$ in (1.18) by $\bar{\partial} U^n$. We therefore define the following postprocessing step in the fully discrete case: find $\tilde{u}(T) \in \mathcal{D}(A)$, such that

$$A\tilde{u}(T) = -\bar{\partial} U^n + F(U^n). \tag{2.12}$$

The finite element solution of the elliptic problem (2.12) with respect to $\tilde{S}_h$ is to find $\tilde{u}_h(T) \in \tilde{S}_h$, such that,

$$\tilde{A}_h \tilde{u}_h(T) = \tilde{P}_h(-\bar{\partial} U^n + F(U^n)). \tag{2.13}$$

Let $\hat{U}^n$ be the solution of

$$\bar{\partial}\hat{U}^n + A_h \hat{U}^n = P_h F(u^n), \quad n \geq 1, \quad \text{with } \hat{U}^0 = v_h.$$

Our main result in this paper is the following: let $\tilde{u}_h$ and u be the solutions of (2.13) and (1.14), respectively, then we have, with $\ell_k = 1 + \log(T/k)$,

$$\begin{aligned}\|\tilde{u}_h(T) - u(T)\| \leq\, & C\big(\|\bar{\partial}\hat{U}^1 - u_t(t_1)\| + |\bar{\partial}\hat{U}^1 - u_t(t_1)|_{-2}\big) \\ & + C(1+\ell_k)(h^{r+2} + k), \quad \text{for } r \geq 4,\ n \geq 2.\end{aligned}$$

2.5. **Paper V. The finite element method for a linear stochastic parabolic partial differential equation driven by additive noise**

In this paper we consider the finite element method for a stochastic parabolic partial differential equation forced by additive space-time noise in the multi-dimensional case. Optimal strong convergence estimates in the L_2 and $\dot{H}^{-1}$ norms with respect to spatial variable are obtained. The proof is based on appropriate nonsmooth data error estimates for the corresponding deterministic parabolic problem.

Let $u_h(t)$ and $u(t)$ be the solutions of (1.35) and (1.22), respectively. Under the condition $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, we obtain the following error estimates in the semidiscrete case,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \leq Ch^{\beta}\Big(\|u_0\|_{L_2(\Omega;\dot{H}^{\beta})} + \|A^{(\beta-1)/2}\|_{L_2^0}\Big),$$

and, with $\ell_h = \log(t/h^2)$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \leq Ch^{\beta+1}\Big(\|u_0\|_{L_2(\Omega;\dot{H}^{\beta})} + \ell_h\|A^{(\beta-1)/2}\|_{L_2^0}\Big).$$

We also consider the error estimates in the fully discrete case. Let U^n and $u(t_n)$ be the solutions of (1.35) and (1.22), respectively. Under the condition $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, we obtain the following error estimates in the fully discrete case,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \leq C(k^{\beta/2} + h^{\beta})\Big(\|u_0\|_{L_2(\Omega;\dot{H}^{\beta})} + \|A^{(\beta-1)/2}\|_{L_2^0}\Big),$$

and, with $\ell_k = \log(t_n/k)$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;\dot{H}^{-1})} \leq C(k^{(\beta+1)/2} + h^{\beta+1})\Big(\|u_0\|_{L_2(\Omega;\dot{H}^{\beta})} + \ell_k\|A^{(\beta-1)/2}\|_{L_2^0}\Big).$$

2.6. **Paper VI. A finite element method for a nonlinear stochastic parabolic equation**

In this paper we consider finite element approximation for nonlinear stochastic parabolic partial differential equations. This paper is the continuation of paper [IV], where the linear case is treated. Under certain global Lipschitz and growth conditions for the nonlinear term, we obtain the optimal error estimates in both semidiscrete and fully discrete cases with respect to strong and weak norms in spatial variable. The proof is based on the nonsmooth data error estimates for deterministic linear homogeneous parabolic problem.

Consider the following nonlinear stochastic parabolic partial differential equation

$$du + Au\,dt = \sigma(u)dW, \quad \text{for}\, 0 < t \le T, \quad \text{with}\; u(0) = u_0, \tag{2.14}$$

in the Hilbert space $H = L_2(\mathcal{D})$, where $A = -\Delta$ and W are as in paper [V].

Assume that $\sigma : H \to L_2^0$ satisfies the following global Lipschitz and growth conditions,

(i) $\quad \|\sigma(x) - \sigma(y)\|_{L_2^0} \le C\|x - y\|, \quad \forall x, y \in H,$
(ii) $\quad \|\sigma(x)\|_{L_2^0} \le C\|x\|, \quad \forall x \in H.$

Then (2.14) admits a unique mild solution which has the form, see Da Prato and Zabczyk [14, Chapter 7],

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\sigma(u(s))\,dW(s),$$

and we have

$$\sup_{t\in[0,T]} \mathbf{E}\|u(t)\|^2 \le C(1 + \mathbf{E}\|u_0\|^2).$$

Note that if $\mathrm{Tr}(Q) < \infty$, then the identity mapping $\sigma(u) = I$ does not satisfies the condition (ii). In order to cover this important case, we introduce a modified version of (ii), i.e.,

(ii') $\quad \|A^{(\beta-1)/2}\sigma(x)\|_{L_2^0} \le C\|x\|, \quad \text{for}\; \beta \in [0,1], \quad \forall x \in H.$

We see that (ii) is the special case $\beta = 1$ of (ii').

The semidiscrete problem of (1.22) is to find the process $u_h(t) = u_h(\cdot, t) \in S_h$, such that

$$du_h + A_h u_h\,dt = P_h\sigma(u_h)dW, \quad \text{for}\; 0 < t \le T, \quad \text{with}\; u_h(0) = P_h u_0, \tag{2.15}$$

where P_h and $A_h : S_h \to S_h$ are defined as before.

Let $E_h(t) = e^{-tA_h}$, $t \ge 0$. Then (2.15) admits a unique mild solution

$$u_h(t) = E_h(t)u_{0h} + \int_0^t E_h(t-s)P_h\sigma(u_h)\,dW(s).$$

Under the assumptions (i) and (ii'), we obtain the following error estimates in the semidiscrete case for $t \in [0, T]$, with $C = C(T)$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le Ch^\beta\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0\le s\le T} \mathbf{E}\|u(s)\|_{L_2(\Omega,H)}\Big).$$

A similar result holds in the fully discrete case.

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Paper I

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Smoothing properties and approximation of time derivatives for parabolic equations: constant time steps

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Abstract.

We study smoothing properties and approximation of time derivatives for time discretization schemes with constant time steps for a homogeneous parabolic problem formulated as an abstract initial value problem in a Banach space. The time stepping schemes are based on using rational functions $r(z) \approx e^{-z}$ which are $A(\theta)$-stable for suitable $\theta \in [0, \pi/2]$ and satisfy $|r(\infty)| < 1$, and the approximations of time derivatives are based on using difference quotients in time. Both smooth and nonsmooth data error estimates of optimal order for the approximation of time derivatives are proved. Further, we apply the results to obtain error estimates of time derivatives in the supremum norm for fully discrete methods based on discretizing the spatial variable by a finite element method.

AMS subject classification: 65M12, 65M15, 65M60, 65J10.

Key words: Banach space, parabolic, smoothing, time derivative, single step time stepping methods, fully discrete schemes, error estimates, finite element methods.

1 Introduction

In this paper, we consider single step time stepping methods for the following homogeneous linear parabolic problem

$$u_t + Au = 0 \quad \text{for } t > 0, \quad \text{with } u(0) = v, \tag{1.1}$$

in a Banach space $\mathcal{B}$. We first study the smoothing properties of the time stepping methods, then we consider approximations of time derivatives based on difference quotients of the approximate solutions of (1.1). Both smooth and nonsmooth data error estimates of the approximations of time derivatives are obtained. As an application we show error estimates in the supremum norm for fully discrete methods based on finite element methods in a spatial domain $\Omega \subset \mathbf{R}^N, N \geq 2$.

We assume that A is a closed, densely defined linear operator defined in $\mathcal{D}(A) \subset \mathcal{B}$, that the resolvent set $\rho(A)$ of A is such that, for some $\delta \in (0, \pi/2)$,

$$\rho(A) \supset \Sigma_\delta = \{z \in \mathbf{C} : \delta \leq |\arg z| \leq \pi, z \neq 0\} \cup \{0\}, \tag{1.2}$$

and that its resolvent, $R(z;A) = (zI - A)^{-1}$, satisfies

$$\|R(z;A)\| \le M|z|^{-1}, \quad \text{for } z \in \Sigma_\delta,\ z \neq 0, \quad \text{with } M \ge 1. \tag{1.3}$$

Throughout this paper $\|\cdot\|$ denotes both the norm in $\mathcal{B}$ and the norm of bounded linear operators on $\mathcal{B}$.

We assume that $0 \in \rho(A)$ for simplicity, but this is not essential. In the case of $0 \notin \rho(A)$ we can add a multiple $\delta' u$ of u to (1.1), thus replacing the operator A by $A + \delta' I$ for some positive number $\delta' > 0$.

Let $-A$ is the infinitesimal generator of a uniformly bounded analytic semigroup $E(t) = e^{-tA}$, $t \ge 0$, which is the solution operator of (1.1), so that $u(t) = E(t)v$. It may be represented as

$$E(t) = \frac{1}{2\pi i} \int_\Gamma e^{-zt} R(z;A)\, dz,$$

where $\Gamma = \{\, z : |\arg z| = \psi\}$ with $\psi \in (\,\delta, \pi/2\,)$ and $\mathrm{Im}z$ decreasing. In particular the smoothing properties of analytic semigroups are valid, see, e.g., Pazy [13]. More precisely, let $j \ge 0$ be fixed, if $v \in \mathcal{B}$, we have, with $D_t = \partial/\partial t$,

$$\|D_t^j E(t)v\| = \|A^j E(t) v\| \le C_j t^{-j} \|v\|, \quad \text{for } t > 0, \tag{1.4}$$

which shows that the solution is regular for positive time even if the initial data are not.

Let U^n be an approximation of the solution $u(t_n) = E(t_n)v$ of (1.1) at time $t_n = nk$, where k is the time step, defined by a single step method,

$$U^n = E_k U^{n-1} \quad \text{for } n \ge 1, \quad \text{with } U^0 = v, \tag{1.5}$$

where $E_k = r(kA)$, and where the rational function $r(z)$ has no poles on $\sigma(kA)$. We may thus write $U^n = E_k^n v$.

We say that the time discretization scheme is accurate of order p, with $p \ge 1$, if

$$r(z) - e^{-z} = O(z^{p+1}), \quad \text{as } z \to 0. \tag{1.6}$$

For example, the backward Euler method given by $r(z) = 1/(1+z)$ is first order accurate and the Crank-Nicolson method, defined by $r(z) = (1 - \frac{1}{2}z)/(1 + \frac{1}{2}z)$, is second order. As another example, the method defined by the $(q, q+1)$ subdiagonal Padé approximation $r(z) = p_1(z)/p_2(z)$, where p_1 and p_2 are certain polynomials of degrees q and $q+1$, respectively, is accurate of order $2q+1$.

Stability and error estimates for single step methods have been studied by many authors, see, e.g., Bakaev [2] [3], Palencia [11] [12], Thomée [16] and references therein. For instance, if A satisfies (1.2) and (1.3), and $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, i.e., $|r(z)| \le 1$ for $|\arg z| \le \theta$, and (1.6) holds, then we have

$$\|U^n - u(t_n)\| = \|E_k^n v - E(t_n)v\| \le C k^p \|A^p v\|, \quad \text{for } v \in \mathcal{D}(A^p), \tag{1.7}$$

see, e.g., Larsson, Thomée, and Wahlbin [9] and Crouzeix, Larsson, Piskarev, and Thomée [7]. Due to the assumption $v \in \mathcal{D}(A^p)$, we refer to this as a smooth data error estimate.

To obtain optimal order error estimates for nonsmooth initial data, $A(\theta)$-stability of the scheme is not sufficient. However, if we require in addition that $|r(\infty)| < 1$, then the following nonsmooth data result is valid:

$$(1.8) \qquad \|U^n - u(t_n)\| = \|E_k^n v - E(t_n)v\| \le Ck^p t_n^{-p}\|v\|, \quad \text{for } t_n > 0,\ v \in \mathcal{B},$$

see, e.g., Larsson, Thomée, and Wahlbin [9] and Crouzeix, Larsson, Piskarev, and Thomée [7]. The condition $|r(\infty)| < 1$ ensures that oscillating components of the error are efficiently damped.

Let us recall some results about the smoothing properties of the time discretization schemes (1.5). When $\mathcal{B}$ is a Hilbert space $\mathcal{H}$ and A a linear, selfadjoint, positive definite, unbounded operator, the following smoothing property holds for $A(0)$-stable time discretization schemes with $r(\infty) = 0$: for each $j \ge 0$ there is C such that

$$(1.9) \qquad \|A^j U^n\| = \|A^j E_k^n v\| \le C t_n^{-j}\|v\|, \quad \text{for } t_n \ge t_j,\ v \in \mathcal{H},$$

see, e.g., Thomée [16, Lemma 7.3]. Hansbo [8] extends this result to Banach space, and shows that, if A satisfies (1.2) and (1.3), and $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \frac{1}{2}\pi]$ and $r(\infty) = 0$, then (1.9) holds. Hansbo [8] also shows an optimal order error estimate in the nonsmooth data case for the approximation $(-A)U^n \approx (-A)u(t_n) = u_t(t_n)$ of the first order time derivative of the solution of (1.1). More precisely, if $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \frac{1}{2}\pi]$ and $r(\infty) = 0$, then

$$(1.10) \qquad \|(-A)U^n - u_t(t_n)\| \le Ck^p t_n^{-p-1}\|v\|, \quad \text{for } t_n > 0,\ v \in \mathcal{B}.$$

However, we observe in Section 3 that the smoothing property (1.9) is not valid when $r(\infty) \ne 0$. Therefore it is natural to investigate the smoothing properties of (1.5) when $r(\infty) \ne 0$. If $|r(\infty)| = 1$, the discrete method (1.5) is not smoothing. However, if such a method of order $p \ge 2$ is combined with a few steps of a smoothing method of order $p-1$, then we have nonsmooth data error estimate of order p. For instance, if one uses the Crank-Nicolson method combined with two steps of the backward Euler method, then a second order nonsmooth data error estimate holds. This analysis is carried out in Hilbert space by Luskin and Rannacher [10] and Rannacher [14]. Hansbo [8] extends the results to Banach space.

In the present paper we shall consider the case $|r(\infty)| < 1$. For fixed $j \ge 1$ we introduce the finite difference quotient,

$$(1.11) \qquad Q_k^j U^n = \frac{1}{k^j}\sum_{\nu=-m_1}^{m_2} c_\nu U^{n+\nu}, \quad \text{for } n \ge m_1,$$

where m_1, m_2 are nonnegative integers, and c_ν are real numbers, and where the operator Q_k^j is an approximation of order $p \ge 1$ to D_t^j, that is, for any smooth real-valued function u,

$$D_t^j u(t_n) = Q_k^j u^n + O(k^p), \quad \text{as } k \to 0, \quad \text{with } u^n = u(t_n). \tag{1.12}$$

For example, Let $j = 1, m_1 = 1, m_2 = 0$, we have

$$D_t u(t_n) = \frac{1}{k}(-u^{n-1} + u^n) + O(k), \quad \text{as } k \to 0.$$

Let $j = 2, m_1 = 1, m_2 = 1$, we have

$$D_t^2 u(t_n) = \frac{1}{k^2}(u^{n-1} - 2u^n + u^{n+1}) + O(k^2), \quad \text{as } k \to 0.$$

In Theorem 2.5, we show that, if A satisfies (1.2) and (1.3), and $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, and $|r(\infty)| < 1$, then the following smoothing property holds:

$$\|Q_k^j U^n\| \le C t_n^{-j} \|v\|, \quad \text{for } n \ge m_1,\ t_n > 0,\ v \in \mathcal{B}. \tag{1.13}$$

Now let us turn to error estimates for approximations of time derivatives of the form (1.11). We show, in Theorem 2.1, the following smooth data error estimate for an $A(\theta)$-stable discretization scheme, i.e., we have

$$\|Q_k^j U^n - D_t^j u(t_n)\| \le C k^p \|A^{p+j} v\|, \quad \text{for } n \ge m_1,\ v \in \mathcal{D}(A^{p+j}). \tag{1.14}$$

To obtain an optimal order error estimate for nonsmooth data, $A(\theta)$-stability is not sufficient. Baker, Bramble and Thomée [5] show the following nonsmooth data error estimate in Hilbert space $\mathcal{H}$ by using a spectral method: if $|r(\lambda)| \le 1$ for $\lambda \ge 0$, and $|r(\infty)| < 1$, then

$$\|Q_k^j U^n - D_t^j u(t_n)\| \le C k^p t_n^{-(p+j)} \|v\|, \quad \text{for } n \ge m_1,\ t_n > 0,\ v \in \mathcal{H}. \tag{1.15}$$

We extend in Theorem 2.6 this result to Banach space, that is, (1.15) also holds under the assumptions of Theorem 2.5.

The above results are proceeded under the assumption that the time step is constant. Some results for variable steps are proceeded in Yan [18].

Let us now discuss some properties of the coefficients c_ν in (1.11). With $u(t) = e^t$ in (1.12) we have at $t_n = 0$

$$k^j = P(e^k) + O(k^{p+j}), \quad \text{as } k \to 0, \quad \text{where } P(x) = \sum_{\nu=-m_1}^{m_2} c_\nu x^\nu. \tag{1.16}$$

Using Taylor expansion of $e^{\nu k}$ at $k = 0$, we therefore easily find that (1.12) is equivalent to

$$P(e^z) - z^j = O(z^{p+j}), \quad \text{as } z \to 0, \tag{1.17}$$

where z is allowed to be complex-valued. For later use we note that (1.11) has the form

$$Q_k^j U^n = k^{-j} P(E_k) E_k^n v. \tag{1.18}$$

The paper is organized as follows. In Section 2, we show the smoothing properties of the abstract time stepping method, and give the optimal order error estimates of the approximation of the time derivatives for both smooth and nonsmooth data. In Section 3, we apply the results obtained in Section 2 to a fully discrete scheme. In Section 4, we give some numerical examples to illustrate our theoretical results.

By C and c we denote positive constants independent of the functions and parameters concerned, but not necessarily the same at different occurrences. When necessary for clarity we distinguish constants by subscripts.

2 Smoothing Properties and Error Estimates in Banach Space

In this section, we discuss smoothing properties of time stepping methods in the general Banach space situation and show smooth and nonsmooth data error estimates in the approximation $Q_k^j U^n$ of $D_t^j u(t_n)$ in the case of constant time steps, where U^n is defined by (1.5) and $u(t_n)$ is the exact solution of (1.1).

We first show that (1.9) is not valid for a scheme with $r(\infty) \neq 0$. In fact, if $\mathcal{B}$ is a separable Hilbert space $\mathcal{H}$ and A is a linear, selfadjoint, positive definite, unbounded operator, we have, by spectral representation,

$$t_n \|AE_k^n\| = t_n \|Ar(kA)^n\| = \sup_{\lambda \in \sigma(kA)} |n\lambda r(\lambda)^n| = \infty, \quad \text{for fixed } n \geq 1,$$

which implies that (1.9) does not hold for $j = 1$. Similar arguments work for any $j > 1$.

As an example of a scheme with $r(\infty) \neq 0$, we consider the θ-method:

$$r(\lambda) = \frac{1-(1-\theta)\lambda}{1+\theta\lambda}, \quad \text{with } \frac{1}{2} < \theta < 1. \tag{2.1}$$

Here we have $|r(\lambda)| < 1$ for $\lambda > 0$, and $r(\infty) = (\theta - 1)/\theta \neq 0$. It is easy to check that $r(\lambda)$ is accurate of order $p = 1$.

Another example is the so called Calahan scheme defined by

$$r(\lambda) = 1 - \frac{\lambda}{1+b\lambda} - \frac{\sqrt{3}}{6}\Big(\frac{\lambda}{1+b\lambda}\Big)^2, \text{ with } b = \frac{1}{2}\Big(1 + \frac{\sqrt{3}}{3}\Big). \tag{2.2}$$

One can show that $|r(\lambda)| < 1$ for $\lambda > 0$, since $r(\lambda)$ is a decreasing function on $(0, \infty)$ and

$$r(\infty) = 1 - \frac{1}{b} - \frac{\sqrt{3}}{6}\frac{1}{b^2} = 1 - \sqrt{3} > -1.$$

A simple calculation shows that this scheme is accurate of order $p = 3$.

Before we study the smoothing properties of the discrete method (1.5), we will show an error estimate for the approximation (1.11) of the time derivative $D_t^j u(t_n)$ in the case that the initial data, and hence the solution of (1.1), are smooth. Recall the error estimate (1.7), which shows that for $v \in \mathcal{D}(A^p)$, the error $U^n - u(t_n)$ has the optimal order of accuracy. Similarly we find in the following theorem that if $v \in \mathcal{D}(A^{p+j})$, then the error estimate for the approximation of $D_t^j u(t_n)$ has the optimal order of accuracy.

THEOREM 2.1. *Let $u(t_n)$ and U^n be the solutions of* (1.1) *and* (1.5). *Assume that A satisfies* (1.2) *and* (1.3), *and that $r(z)$ is accurate of order $p \geq 1$ and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$. Let $j \geq 1$ and assume that Q_k^j, defined in* (1.11), *is an approximation of D_t^j, which is accurate of order p. Then there is a constant C such that*

$$\|Q_k^j U^n - D_t^j u(t_n)\| \leq C k^p \|A^{p+j} v\|, \quad \text{for } n \geq m_1,\ v \in \mathcal{D}(A^{p+j}).$$

To prove this theorem we need the following lemmas, which we quote from Thomée [16, Lemmas 8.1, 8.3].

LEMMA 2.2. *Assume that* (1.2) *and* (1.3) *hold and let $r(z)$ be a rational function which is bounded for $|\arg z| \leq \psi$, $|z| \geq \epsilon > 0$, where $\psi \in (\delta, \pi/2)$, and for $|z| \geq R$ with some positive number R. If $\epsilon > 0$ is so small that $\{z : |z| \leq \epsilon\} \subset \rho(A)$, then we have*

$$r(A) = r(\infty) I + \frac{1}{2\pi i} \int_{\gamma_\epsilon \cup \Gamma_\epsilon^R \cup \gamma^R} r(z) R(z; A)\, dz,$$

where $\gamma_\epsilon = \{z : |z| = \epsilon, |\arg z| \leq \psi\}$, $\Gamma_\epsilon^R = \{z : |\arg z| = \psi, \epsilon \leq |z| \leq R\}$, and $\gamma^R = \{z : |z| = R, \psi \leq |\arg z| \leq \pi\}$, and with the closed path of integration oriented in the negative (clock-wise) sense.

LEMMA 2.3. *Assume that* (1.2) *and* (1.3) *hold, let $\psi \in (\delta, \pi/2)$, and j be any integer. Then we have for $\epsilon > 0$ sufficiently small*

$$A^j E(t) = \frac{1}{2\pi i} \int_{\gamma_\epsilon \cup \Gamma_\epsilon} e^{-zt} z^j R(z; A)\, dz,$$

where $\gamma_\epsilon = \{z : |z| = \epsilon, |\arg z| \leq \psi\}$ and $\Gamma_\epsilon = \{z : |\arg z| = \psi, |z| \geq \epsilon\}$, and where $\mathrm{Im} z$ is decreasing along $\gamma_\epsilon \cup \Gamma_\epsilon$. When $j \geq 0$, we may take $\epsilon = 0$.

PROOF OF THEOREM 2.1. We have

$$Q_k^j U^n - D_t^j u(t_n) = k^{-j} (P(r(kA)) r(kA)^n - (-kA)^j e^{-nkA}),$$

where $P(x)$ is defined by (1.16). With

$$G_n(z) = P(r(z)) r(z)^n - (-z)^j e^{-nz}, \tag{2.3}$$

our result will follow from

$$\|G_n(kA)(kA)^{-(p+j)}\| \leq C.$$

Note that with A also kA satisfies (1.2) and (1.3) since, for $z \in \Sigma_\delta$,

$$\|R(z; kA)\| = \|k^{-1}(zk^{-1} I - A)^{-1}\| \leq k^{-1} M |zk^{-1}|^{-1} = M |z|^{-1}.$$

Therefore it suffices to show

$$\|G_n(A)A^{-(p+j)}\| \le C, \tag{2.4}$$

which we will prove now. Let $\bar{r}(z) = P(r(z))r(z)^n z^{-(p+j)}$. Since $n \ge m_1$ and $r(z)$ is $A(\theta)$-stable, we find that $\bar{r}(z)$ is bounded for $|\arg z| \le \psi$, $|z| \ge \epsilon > 0$, with some $\psi \in (\delta, \theta)$. Further $\bar{r}(z)$ is also bounded for $|z| \ge R$ with R sufficiently large since $\bar{r}(\infty) = 0$. Thus, applying Lemma 2.2 to the rational function $\bar{r}(z) = P(r(z))r(z)^n z^{-(p+j)}$, we have

$$P(r(A))r(A)^n A^{-(p+j)} = \frac{1}{2\pi i}\int_{\gamma_\epsilon \cup \Gamma_\epsilon^R \cup \gamma^R} P(r(z))r(z)^n z^{-(p+j)} R(z;A)\,dz.$$

Since the integrand is of order $O(|z|^{-p-j-1})$ for large z, we may let R tend to ∞. Using also Lemma 2.3 we conclude

$$G_n(A)A^{-(p+j)} = \frac{1}{2\pi i}\int_{\gamma_\epsilon \cup \Gamma_\epsilon} G_n(z) z^{-(p+j)} R(z;A)\,dz. \tag{2.5}$$

We shall show that

$$G_n(z) = O(z^{p+j}), \quad \text{as } z \to 0, \quad \text{with } |\arg z| \le \psi. \tag{2.6}$$

Assuming this and combining this with the fact that $0 \in \rho(A)$, we have that the integrand in (2.5) is bounded on the small domain with boundary $\gamma_\epsilon \cup \Gamma_0^\epsilon$, so that we may let $\epsilon \to 0$. It follows that

$$G_n(A)A^{-(p+j)} = \frac{1}{2\pi i}\int_{\Gamma} G_n(z) z^{-(p+j)} R(z;A)\,dz,$$

where $\Gamma = \{z : |\arg z| = \psi\}$. We now estimate the above integral. Again using (2.6) and the fact that $0 \in \rho(A)$, we find, for η small enough,

$$\|G_n(z)R(z,A)\| \le C|z|^{p+j}, \quad \text{for } |z| \le \eta, \quad |\arg z| = \psi.$$

Further, noting that $P(r(z))r(z)^n$ is bounded on Γ, since $n \ge m_1$ and $r(z)$ is $A(\theta)$-stable, we have, using (1.3) and (2.3) as well as the boundedness of e^{-nz} on Γ,

$$\|G_n(z)R(z,A)\| \le M(C + |z|^j)|z|^{-1}, \quad \text{for } |z| \ge \eta,\ |\arg z| = \psi.$$

Thus

$$\|G_n(A)A^{-(p+j)}\| \le C\int_0^\eta \rho^{p+j}\rho^{-(p+j)}\,d\rho + M\int_\eta^\infty (C + \rho^j)\rho^{-(p+j+1)}\,d\rho \le C.$$

It remains to prove (2.6). Since $r(z) = e^{-z} + O(z^2)$ as $z \to 0$, we have that, for $\tilde{\eta} > 0$ small enough,

$$|r(z)| \le e^{-c|z|}, \quad \text{for } |z| \le \tilde{\eta}, \quad |\arg z| \le \psi, \quad \text{with } 0 < c < 1. \tag{2.7}$$

Thus, using also (1.6),

$$(2.8)\quad |r(z)^n - e^{-nz}| = \Big|(r(z)-e^{-z})\sum_{j=0}^{n-1} r(z)^{n-1-j}e^{-jz}\Big| \le Cn|z|^{p+1}e^{-c(n-1)|z|} \le C|z|^p, \quad \text{for } |z|\le\tilde\eta,\ |\arg z|\le\psi.$$

Further, with $\tilde\eta$ possibly further restricted,

$$(2.9)\qquad |P(r(z)) - (-z)^j| \le C|z|^{p+j}, \quad \text{for } |z|\le\tilde\eta.$$

In fact, by Taylor's formula, we have

$$P(r(z)) - P(e^{-z}) = \sum_{l=1}^{j-1}\frac{P^{(l)}(e^{-z})}{l!}(r(z)-e^{-z})^l + (r(z)-e^{-z})^j\int_0^1 \frac{(1-s)^{j-1}}{(j-1)!}P^{(j)}(r(z)+s(r(z)-e^{-z}))\,ds.$$

Since $P(e^{-z})$ is an analytic function of z and $P(e^{-z}) = O(z^j)$ as $z\to 0$ by (1.17), we have $P^{(l)}(e^{-z}) = O(z^{j-l})$, $0\le l\le j$ as $z\to 0$. Moreover, since $r(z)\to 1$, $e^{-z}\to 1$ as $z\to 0$, it is easy to see that there exist constants $c_1>0$, $c_2>0$ and small $\tilde\eta$ such that $c_1 \le |r(z)+s(r(z)-e^{-z})| \le c_2$ for $|z|\le\tilde\eta$, $0\le s\le 1$, which implies that $|P^{(j)}(r(z)+s(r(z)-e^{-z}))|\le C$ for $|z|\le\tilde\eta$, $0\le s\le 1$, since $P(x)$ has the form $P(x)=\sum_{\nu=-m_1}^{m_2} c_\nu x^\nu$. Thus, using also (1.6) we get

$$|P(r(z)) - P(e^{-z})| = \sum_{l=1}^{j-1} O(z^{j-l})O(z^{l(p+1)}) + O(z^{j(p+1)}) = O(z^{p+j}), \quad \text{as } z\to 0.$$

Combining this with (1.17) shows (2.9).

Thus, by (2.8) and (2.9),

$$|G_n(z)| = \Big|(P(r(z)) - (-z)^j)r(z)^n + (-z)^j(r(z)^n - e^{-nz})\Big| \le C|z|^{p+j}, \quad \text{for } |z|\le\tilde\eta, \quad |\arg z|\le\psi,$$

which is (2.6). □

We next prove a smoothing property of an $A(\theta)$-stable discretization scheme with $|r(\infty)|<1$. Before doing this, we show that $Q_k^jU^n$ defined by (1.11) can be expressed as a linear combination of the backward difference quotients $\bar\partial^jU^{n+\mu}$ for some integers μ.

LEMMA 2.4. *Let $j\ge 1$ and $Q_k^jU^n$ be defined by* (1.11). *Then there exist constants α_μ, $-m_1+j\le\mu\le m_2$, such that*

$$Q_k^jU^n = \sum_{\mu=-m_1+j}^{m_2} \alpha_\mu\bar\partial^jU^{n+\mu}, \quad \textit{where} \quad \bar\partial U^n = (U^n - U^{n-1})/k.$$

PROOF. With $P(x) = \sum_{\nu=-m_1}^{m_2} c_\nu x^\nu$, we have

$$k^j Q_k^j U^n = \sum_{\nu=-m_1}^{m_2} c_\nu E_k^\nu U^n = P(E_k)U^n, \quad \text{where } E_k = r(kA),$$

i.e., the difference operator is associated with the rational function $P(x)$. Similarly the operator $k^j \bar\partial^j U^{n+\mu}$ corresponds to the rational function $\tilde P(x) = x^\mu(1-x^{-1})^j$, since

$$k^j \bar\partial^j U^{n+\mu} = (1-E_k^{-1})^j E_k^{n+\mu} v = (I-E_k^{-1})^j E_k^\mu U^n = \tilde P(E_k)U^n.$$

Thus we only need to show that there exist α_μ such that

$$P(x) = \sum_{\mu=-m_1+j}^{m_2} \alpha_\mu \tilde P(x) = (1-x^{-1})^j \sum_{\mu=-m_1+j}^{m_2} \alpha_\mu x^\mu. \tag{2.10}$$

But by (1.17) we find $P^{(l)}(1) = 0$ for $0 \le l \le j-1$, which implies that $P(x)$, and hence $x^{m_1}P(x)$ contains the factor $(x-1)^j$, that is, there exists a polynomial $\bar P(x)$ of degree m_1+m_2-j such that $x^{m_1}P(x) = (x-1)^j \bar P(x)$. Denoting $\bar P(x) = \sum_{k=0}^{m_1+m_2-j} \beta_k x^k$ for some constants β_k, we get that there exist constants α_μ such that

$$x^{m_1}P(x) = (x-1)^j \sum_{k=0}^{m_1+m_2-j} \beta_k x^k = (x-1)^j \sum_{\mu=-m_1+j}^{m_2} \alpha_\mu x^{\mu+m_1-j},$$

which shows (2.10). □

THEOREM 2.5. *Let U^n be the solution of* (1.5). *Assume that* (1.2) *and* (1.3) *hold, and $r(z)$ is accurate of order $p \ge 1$ and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, and $|r(\infty)| < 1$. Let $j \ge 1$ and assume that Q_k^j, defined in* (1.11), *is an approximation of D_t^j, which is accurate of order p. Then there is a constant C such that*

$$\|Q_k^j U^n\| \le C t_n^{-j} \|v\|, \quad \text{for } n \ge m_1,\ t_n > 0,\ v \in \mathcal{B}.$$

PROOF. By Lemma 2.4, it suffices to show

$$\|\bar\partial^j U^n\| \le C t_n^{-j} \|v\|, \quad \text{for } n \ge j. \tag{2.11}$$

In fact, this implies, for $n \ge m_1,\ t_n > 0$,

$$\|Q_k^j U^n\| = \Big\| \sum_{\mu=-m_1+j}^{m_2} \alpha_\mu \bar\partial^j U^{n+\mu} \Big\| \le C \sum_{\mu=-m_1+j}^{m_2} t_{n+\mu}^{-j} \|v\|$$
$$\le C t_{n-m_1+j}^{-j} \|v\| \le C t_n^{-j} \|v\|.$$

We know show (2.11). Noting that $\bar\partial^j U^n = k^{-j} \tilde P(E_k) E_k^n v$ for $\tilde P(x) = (1-x^{-1})^j = x^{-j}(x-1)^j$, we need to show

$$\|\tilde P(r(A)) r(A)^n\| = \|r(A)^{n-j}(r(A)-1)^j\| \le C n^{-j}, \quad \text{for } n \ge j.$$

As in the proof of Theorem 2.1 this then also holds with A replaced by kA, and thus shows (2.11).

Since $n \geq j$ and $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$ and $|r(\infty)| < 1$, which implies that $r(z)^{n-j}(r(z)-1)^j$ is bounded for $|\arg z| \leq \psi$, $|z| \geq \epsilon$ with some $\psi \in (\delta, \theta)$ and arbitrary $\epsilon > 0$, and also bounded for $|z| \geq R$ with R sufficiently large, we therefore have, by Lemma 2.2,

$$\begin{aligned} r(A)^{n-j}(r(A)-1)^j =& r(\infty)^{n-j}(r(\infty)-1)^j I \\ &+ \frac{1}{2\pi i}\int_{\gamma_\epsilon \cup \Gamma_\epsilon^R \cup \gamma^R} r(z)^{n-j}(r(z)-1)^j R(z;A)\,dz. \end{aligned}$$

By $|r(\infty)| < 1$, we have, for fixed $R \geq 1$ large enough,

$$|r(z)| \leq e^{-c}, \quad \text{for } |z| \geq R. \tag{2.12}$$

Clearly then $|r(\infty)| \leq e^{-c}$, so that

$$|r(\infty)^{n-j}(r(\infty)-1)^j| \leq Ce^{-cn} \leq Cn^{-j}, \quad \text{for } n \geq 1.$$

To bound the integrals over the three components of the path of integration, we have, by (2.12),

$$\left\| \frac{1}{2\pi i}\int_{\gamma^R} r(z)^{n-j}(r(z)-1)^j R(z;A)\,dz \right\| \leq Ce^{-cn}\int_{\gamma^R} \frac{|dz|}{|z|} \leq Cn^{-j}, \quad \text{for } n \geq 1.$$

For the other two components of the path of integration, since $r(z)$ is $A(\theta)$-stable and $0 \in \rho(A)$ and accurate of order $p \geq 1$, which imply that the integrand is bounded on the small domain with boundary $\gamma_\epsilon \cup \Gamma_0^\epsilon$, we may let ϵ tend to 0. Thus it suffices to bound the integral over Γ_0^R. But by $A(\theta)$-stability and the maximum-principle we have $|r(z)| < 1$ for $|\arg z| < \theta$, $z \neq 0$. In particular, $|r(z)| < 1$ on the compact set $\{z : \tilde{\eta} \leq |z| \leq R, \ |\arg z| \leq \psi\}$, which means that the inequality (2.7) also holds for $|z| \leq R$, $|\arg z| \leq \psi$ with c sufficiently small. Thus, we have, noting that $r(z) - 1 = O(z)$ as $z \to 0$,

$$\left\| \frac{1}{2\pi i}\int_{\Gamma_0^R} r(z)^{n-j}(r(z)-1)^j R(z;A)\,dz \right\| \leq C\int_0^R e^{-cn\rho}\rho^{j-1}\,d\rho \leq Cn^{-j}.$$

Together these estimates complete the proof. □

We remark that if $|r(\infty)| = 1$ with $r(\infty) \neq 1$, then the conclusion of Theorem 2.5 is not valid. For example, let us consider the Crank-Nicolson scheme, with $r(\infty) = -1$. Assume that A is a linear selfadjoint, positive definite, unbounded operator with compact inverse in Hilbert space $\mathcal{H}$, and A has eigenvalues $\{\lambda_j\}_{j=1}^\infty$ and a corresponding basis of orthonormal eigenvectors $\{\varphi_j\}_{j=1}^\infty$. Then, with $v = \varphi_j$, we have, noting that $r(\infty) = -1$,

$$\begin{aligned} t_n\|\bar{\partial}U^n\| &= n\|r(kA)^{n-1}(r(kA)-1)v\| \\ &= n|r(k\lambda_j)^{n-1}(r(k\lambda_j)-1)| \to 2n, \quad \text{as } j \to \infty, \end{aligned}$$

which implies that there does not exist a constant C such that

$$t_n\|\bar{\partial}U^n\| \le C\|v\|, \quad \text{for } n \ge 1, \quad v \in \mathcal{H}.$$

However, if $r(\infty) = 1$, then the conclusion of Theorem 2.5 holds in special cases: Let us consider the $(2,2)$ Padé scheme,

$$r(\lambda) = \frac{1 - \frac{1}{2}\lambda + \frac{1}{12}\lambda^2}{1 + \frac{1}{2}\lambda + \frac{1}{12}\lambda^2}, \quad \text{where } r(\infty) = 1. \tag{2.13}$$

We show that in this case $t_n\|\bar{\partial}U^n\| \le C\|v\|$. In fact, for this it suffices to show

$$|nr(\lambda)^{n-1}(r(\lambda) - 1)| \le C, \quad \text{for } \lambda > 0. \tag{2.14}$$

For small λ this follows directly from the fact that $|r(\lambda)| \le e^{-c\lambda}$, $|r(\lambda) - 1| \le C\lambda$ for $0 \le \lambda \le \lambda_0$ and it remains to consider large λ. Noting that $|r(\lambda)| \le e^{-c\lambda^{-1}}$ with some constant c and $|r(\lambda) - 1| \le C\lambda^{-1}$ for $\lambda > \lambda_0$, see, e.g., Thomée [16, Lemma 8.2], we have

$$|nr(\lambda)^{n-1}(r(\lambda) - 1)| \le C(n\lambda^{-1})e^{-c(n-1)\lambda^{-1}} \le C,$$

which shows (2.14).

Our next result is an error estimate in the nonsmooth data case. The estimate has optimal order of accuracy for t_n bounded away from zero, but contains a negative power of t_n. Comparing with the error estimate (1.8), we find that t_n^{-p} is replaced by t_n^{-p-j} in our theorem. The proof in the Hilbert space case can be found in Baker, Bramble, and Thomée [5]. Here we extend the result to Banach space.

THEOREM 2.6. *Let $u(t_n)$ and U^n be the solutions of* (1.1) *and* (1.5). *Assume that* (1.2) *and* (1.3) *hold, and $r(z)$ is accurate of order $p \ge 1$ and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, and $|r(\infty)| < 1$. Let $j \ge 1$ and assume that Q_k^j, defined in* (1.11), *is an approximation of D_t^j, which is accurate of order p. Then there is a constant C such that*

$$\|Q_k^j U^n - D_t^j u(t_n)\| \le Ck^p t_n^{-(p+j)}\|v\|, \quad \text{for } n \ge m_1,\ t_n > 0,\ v \in \mathcal{B}.$$

To prove this theorem, we need the following lemma, which we quote from Thomée [16, Lemma 8.5].

LEMMA 2.7. *Assume that the rational function $r(z)$ is $A(\theta)$-stable with $\theta \le \pi/2$, and that $|r(\infty)| < 1$. Then for any $\psi \in (0, \theta)$ and $R > 0$ there are positive C and c such that, with $\kappa = r(\infty)$,*

$$|r(z)^n - \kappa^n| \le C|z|^{-1}e^{-cn}, \quad \text{for } |z| \ge R,\ |\arg z| \le \psi,\ n \ge 1.$$

PROOF OF THEOREM 2.6. We need to show, with $G_n(z)$ given by (2.3),

$$\|G_n(A)\| \le Cn^{-(p+j)}.$$

We set $\kappa = r(\infty)$ and

$$\tilde{G}_n(z) = G_n(z) - P(\kappa)\kappa^n z/(1+z).$$

Obviously $\tilde{G}_n(\infty) = 0$. Since $|\kappa| < 1$, we have $|\kappa| \le e^{-c}$ for some $c > 0$. Noting that $\|A(I+A)^{-1}\| \le 2M$, we have, since $n \ge m_1,\ n \ge 1$,

$$\begin{aligned}\|P(\kappa)\kappa^n A(I+A)^{-1}\| \le 2M|P(\kappa)\kappa^n| &\le 2M\Big|\sum_{\nu=-m_1}^{m_2} c_\nu e^{-c(n+\nu)}\Big| \\ &\le Ce^{-cn} \le Cn^{-(p+j)}.\end{aligned}$$

It remains to show the same bound for the operator norm of $\tilde{G}_n(A)$. We may now use Lemmas 2.2 and 2.3 to see that, with $\psi \in (\delta, \theta)$,

$$\tilde{G}_n(A) = \frac{1}{2\pi i}\int_{\gamma_\epsilon \cup \Gamma_\epsilon} \tilde{G}_n(z) R(z;A)\,dz.$$

Since $n \ge m_1$ and $r(z)$ is $A(\theta)$-stable and $0 \in \rho(A)$, the integrand is bounded on the small domain with boundary $\gamma_\epsilon \cup \Gamma_0^\epsilon$, so that we may let ϵ tend to 0. We therefore have, with $\Gamma = \{\, z : \ |\arg z| = \psi \,\}$,

$$\tilde{G}_n(A) = \frac{1}{2\pi i}\int_{\Gamma} \tilde{G}_n(z) R(z;A)\,dz.$$

We write

$$\tilde{G}_n(z) = (P(r(z)) - (-z)^{-j})r(z)^n + (-z)^j(r(z)^n - e^{-nz})) - P(\kappa)\kappa^n z/(1+z).$$

Using the estimates (2.7), (2.8), (2.9) and $|1/(1+z)| \le 1$ for $\mathrm{Re}\, z \ge 0$ and the boundedness of $R(z;A)$, we have, for $z \in \Gamma,\ |z| \le 1,\ n \ge 1$,

$$\|\tilde{G}_n(z)R(z,A)\| \le \Big(C|z|^{p+j}e^{-cn|z|} + |z|^j(Cn|z|^{p+1}e^{-cn|z|})\Big) + C\kappa^n \le Cn^{-p-j}.$$

Further, we rewrite

$$\tilde{G}_n(z) = \Big(P(r(z))r(z)^n - P(\kappa)\kappa^n\Big) + P(\kappa)\kappa^n/(1+z) - (-z)^j e^{-nz}.$$

By Lemma 2.7 we have, for $z \in \Gamma,\ |z| \ge 1$ and $n \ge m_1$,

$$\begin{aligned}|P(r(z))r(z)^n - P(\kappa)\kappa^n| &= \Big|\sum_{\nu=-m_1}^{m_2} c_\nu \Big(r(z)^{n+\nu} - \kappa^{n+\nu}\Big)\Big| \\ &\le C|z|^{-1}\sum_{\nu=-m_1}^{m_2} |c_\nu| e^{-c(n+\nu)} \\ &\le C|z|^{-1}e^{-c(n-m_1)} \le C|z|^{-1}e^{-cn}.\end{aligned}$$

Thus, since $|1+z| \ge |z|$ for $\mathrm{Re} z \ge 0$, we get, for $z \in \Gamma$, $|z| \ge 1$, $n \ge m_1$ and $n \ge 1$,

$$\begin{aligned}\|\tilde{G}_n(z)R(z,A)\| &\le \Big(C|z|^{-1}e^{-cn} + \kappa^n|z|^{-1}\Big)|z|^{-1} + C|z|^{-p-1}|z|^{p+j}e^{-cn|z|} \\ &\le Cn^{-p-j}(|z|^{-2} + |z|^{-p-1}) \le Cn^{-p-j}|z|^{-2}.\end{aligned}$$

We therefore obtain

$$\|\tilde{G}_n(A)\| \le \int_0^1 Cn^{-p-j}\,d\rho + \int_1^\infty Cn^{-p-j}\rho^{-2}\,d\rho \le Cn^{-p-j}.$$

Together these estimates complete the proof. □

3 Fully Discrete Schemes

In this section we study fully discrete schemes of the initial boundary value problem

$$\left\{\begin{array}{l} u_t = \Delta u \quad \text{in } \Omega, \quad \text{for } t > 0, \\ u = 0 \quad \text{on } \partial\Omega, \quad \text{for } t > 0, \quad u(\cdot,0) = v \quad \text{in } \Omega, \end{array}\right. \tag{3.1}$$

where Ω is a bounded domain in $\mathbf{R}^N, N \ge 2$ with smooth boundary $\partial\Omega$.

Let $L_p(\Omega)$ denote the usual real Lebesgue spaces with norms

$$\|v\|_{L_p(\Omega)} = \left\{\begin{array}{l} \Big(\int_\Omega |v(x)|^p\,dx\Big)^{\frac{1}{p}}, \ 1 \le p < \infty, \\ \operatorname{ess\,sup}\{|v(x)| : x \in \Omega\}, \ p = \infty, \end{array}\right. \tag{3.2}$$

and let k be a nonnegative integer and let $W_p^k(\Omega)$ be the standard real Sobolev spaces with norms $\|\cdot\|_{W_p^k(\Omega)}$ defined by

$$\|v\|_{W_p^k(\Omega)} = \Big(\sum_{|\alpha|\le k} \|D^\alpha v\|^p_{L_p(\Omega)}\Big)^{\frac{1}{p}},$$

for $1 \le p \le \infty$ with the usual modification in the case $p = \infty$. In the case $p = 2$, we set $H^k(\Omega) = W_2^k(\Omega)$, which is a Hilbert space with the inner product

$$(v,w)_{H^k(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha v D^\alpha w\,dx.$$

Moreover we denote $H_0^k(\Omega) = \{v : v \in H^k(\Omega),\ v|_{\partial\Omega} = 0\}$ and $\dot{W}_p^r(\Omega) = W_p^r \cap H_0^1$ for $1 \le r \le \infty$.

In (3.1), we consider $A = -\Delta$ with $\mathcal{D}(A) = \dot{W}_p^2(\Omega) = W_p^2 \cap H_0^1$. It is known that Δ generates a strongly continuous and analytic semigroup $E(t) = e^{t\Delta}$ in L_p for $1 < p < \infty$, but for $p = \infty$ the strong continuity at $t = 0$ is lost. Nevertheless, the corresponding stability and smoothing estimates are valid in $L_p,\ 2 \le p \le \infty$, i.e.,

$$\|E(t)v\|_{L_p} + \|E'(t)\|_{L_p} \le Ct^{-1}\|v\|_{L_p}, \quad \text{for } v \in L_p,\ 2 \le p \le \infty, \tag{3.3}$$

see, e.g., Thomée [16, Chapter 5] for more details.

We assume that Ω is approximated by a quasi-uniform family of finite element meshes τ_h such that the union of the elements determines a domain Ω_h with boundary nodes on $\partial\Omega$. For simplicity we assume that Ω is convex and let S_h be the space of continuous functions that are linear on each element and vanish outside Ω_h, so that $S_h \subset H_0^1$. We define the discrete Laplacian Δ_h by

$$(\Delta_h\psi, \chi) = -(\nabla\psi, \nabla\chi), \quad \forall\psi, \chi \in S_h. \tag{3.4}$$

The spatially semidiscrete problem is then to find $u_h :\ [0, \infty) \to S_h$, such that

$$u_{h,t} = \Delta_h u_h, \quad \text{for } t > 0, \quad \text{with } u_h(0) = v_h, \tag{3.5}$$

where $v_h \in S_h$ is some approximation of v. Let P_h denote the orthogonal projection of v onto S_h with respect to the inner product to the L_2 norm, i.e.,

$$(P_h v, \chi) = (v, \chi), \quad \forall\chi \in S_h,\ \text{for } v \in L_2(\Omega). \tag{3.6}$$

We also need the so called elliptic or Ritz projection R_h onto S_h as the orthogonal projection with respect to the inner product $(\nabla v, \nabla w)$ so that

$$(\nabla R_h v, \nabla\chi) = (\nabla v, \nabla\chi), \quad \forall\chi \in S_h,\ \text{for } v \in H_0^1(\Omega). \tag{3.7}$$

Note that

$$\Delta_h R_h = P_h\Delta, \tag{3.8}$$

which we need in the proof of our theorems.

We now apply our above time stepping procedure (1.5) to this semidiscrete equation (3.5). This defines the fully discrete approximation $U^n \in S_h$ of $u(t_n)$ recursively by

$$U^n = E_{kh}U^{n-1}, \quad \text{for } n \ge 1, \quad \text{where } E_{kh} = r(-k\Delta_h), \quad \text{with } U^0 = v_h. \tag{3.9}$$

We shall derive L_∞ error estimates for the approximations $Q_k^j U^n$ of the time derivatives $D_t^j u(t_n)$ of the solution of (3.1), where U^n is defined by (3.9). We first show some L_∞ error estimates in the spatially semidiscrete case. We begin with an error estimate in the nonsmooth data case.

THEOREM 3.1. *Let $u(t)$ and $u_h(t)$ be the solutions of* (3.1) *and* (3.5) *and $j \ge 0$. If $v \in L_\infty$ and $v_h = P_h v$, then we have*

$$\|D_t^j u_h(t) - D_t^j u(t)\|_{L_\infty} \le Ch^2\ell_h^2 t^{-j-1}\|v\|_{L_\infty}, \quad \textit{where } \ell_h = \ln(1/h).$$

The proof of the result depends on the following lemmas. The first lemma concerns error bounds for the L_2 and Ritz projections in maximum-norm.

LEMMA 3.2. *Let $u(t)$ be the solution of* (3.1) *and $j \geq 0$. Then, we have, for $\rho = (R_h - I)u$ and $\eta = (P_h - I)u$,*

$$t^{j+1}\Big(\|\rho^{(j)}(t)\|_{L_\infty} + \ell_h\|\eta^{(j)}(t)\|_{L_\infty}\Big) \leq Ch^2\ell_h^2\|v\|_{L_\infty}. \tag{3.10}$$

PROOF. With $I_h : C(\bar\Omega) \to S_h$ the standard Lagrange interpolation operator, we have, see, e.g., Brenner and Scott [6],

$$\|I_h u - u\|_{L_\infty} \leq Ch^{2-2/s}\|u\|_{W_s^2}, \quad \text{for } 2 \leq s \leq \infty,\ u \in \dot W_s^2 = W_s^2 \cap H_0^1.$$

Since $\rho^{(j)} = (R_h - I)D_t^j u = (R_h - I)(I - I_h)D_t^j u$, using the logarithmic maximum-norm stability of R_h, i.e., $\|R_h u\|_{L_\infty} \leq C\ell_h\|u\|_{L_\infty}$, see, e.g., Schatz and Wahlbin [15], we have,

$$\|\rho^{(j)}\|_{L_\infty} \leq C\ell_h\|(I - I_h)D_t^j u\|_{L_\infty} \leq C\ell_h h^{2-2/s}\|D_t^j u\|_{W_s^2}.$$

By the Agmon-Douglis-Nirenberg [1] regularity estimate

$$\|u\|_{W_s^2} \leq Cs\|\Delta u\|_{L_s}, \quad \text{for } 2 \leq s < \infty,\ u \in \dot W_s^2,$$

we hence obtain, using also the smoothing property (3.3),

$$\begin{aligned}\|\rho^{(j)}(t)\|_{L_\infty} &\leq Ch^{2-2/s}\ell_h s\|\Delta D_t^j u(t)\|_{L_s}\\ &\leq Ch^{2-2/s}\ell_h s t^{-j-1}\|v\|_{L_s} \leq Ch^{2-2/s}\ell_h s t^{-j-1}\|v\|_{L_\infty}.\end{aligned}$$

With $s = \ell_h$ this shows the bound in (3.10) for $\rho^{(j)}(t)$. The proof of the bound for $\eta^{(j)}(t)$ is analogous, with one less factor ℓ_h because P_h is bounded in L_∞, see Thomée [16, Lemma 5.7]. □

We also need the following lemma which shows that the discrete solution operator $E_h(t) = e^{t\Delta_h}$ is stable in the L_∞ norm and has a smoothing property, see, e.g., Thomée and Wahlbin [17].

LEMMA 3.3. *Let $E_h(t)$ be the solution operator of* (3.5)*. Then*

$$\|E_h(t)v_h\|_{L_\infty} + t\|E_h'(t)v_h\|_{L_\infty} \leq C\|v_h\|_{L_\infty}, \quad \text{for } t > 0. \tag{3.11}$$

PROOF OF THEOREM 3.1. We write $u_h - u = (u_h - P_h u) + (P_h u - u) = \zeta + \eta$. Here $\eta^{(j)}$ is bounded as desired by Lemma 3.2 and it remains to bound $\zeta^{(j)} = D_t^j\zeta = D_t^j(u_h - P_h u)$. We will show that for each $j \geq 0$ there is C, which maybe depend on j, such that

$$\sup_{0\leq s\leq t}\,(s^{j+1}\|\zeta^{(j)}(s)\|_{L_\infty}) \leq Ch^2\ell_h^2\|v\|_{L_\infty}. \tag{3.12}$$

The case $j = 0$ can be found in Thomée [16, Theorem 5.4]. Assuming now that the result is already shown with j replaced by $j - 1$. Since

$$\zeta_t - \Delta_h\zeta = -\Delta_h P_h\rho,$$

we find

(3.13)

$$\begin{aligned}(t^{j+1}\zeta^{(j)})_t - \Delta_h(t^{j+1}\zeta^{(j)}) &= (j+1)t^j\zeta^{(j)} + t^{j+1}(\zeta_t^{(j)} - \Delta_h\zeta^{(j)}) \\ &= (j+1)t^j(\Delta_h\zeta^{(j-1)} - \Delta_h P_h\rho^{(j-1)}) - t^{j+1}\Delta_h P_h\rho^{(j)}.\end{aligned}$$

Thus, by Duhamel's principle, we have, noting that $E_h(t-s)\Delta_h = E_h'(t-s)$,

$$\begin{aligned}t^{j+1}\zeta^{(j)}(t) = \int_0^t E_h'(t-s)\Big((j+1)s^j\zeta^{(j-1)}(s) \\ - (j+1)s^j P_h\rho^{(j-1)}(s) - s^{j+1}P_h\rho^{(j)}(s)\Big)ds = I + II + III.\end{aligned}$$

For II, we write

$$II = -\Big(\int_0^{t/2} + \int_{t/2}^t\Big)E_h'(t-s)(j+1)s^j P_h\rho^{(j-1)}(s)\,ds = II_1 + II_2.$$

Here, using Lemmas 3.2 and 3.3,

$$\|II_1\|_{L_\infty} \le C\int_0^{t/2}(t-s)^{-1}s^j\|\rho^{(j-1)}(s)\|_{L_\infty}\,ds \le Ch^2\ell_h^2\|v\|_{L_\infty}.$$

Further, after integration by parts,

$$\begin{aligned}II_2 =& \Big[E_h(t-s)(j+1)s^j P_h\rho^{(j-1)}(s)\Big]_{t/2}^t \\ &- \int_{t/2}^t E_h(t-s)(j+1)P_h\Big(js^{j-1}\rho^{(j-1)}(s) + s^j\rho^{(j)}(s)\Big)\,ds,\end{aligned}$$

and thus by Lemmas 3.2 and 3.3, we get

$$\|II_2\|_{L_\infty} \le Ch^2\ell_h^2\|v\|_{L_\infty}.$$

Therefore $\|II\|_{L_\infty} \le Ch^2\ell_h^2\|v\|_{L_\infty}$. Following the estimate of II, we can also show $\|III\|_{L_\infty} \le Ch^2\ell_h^2\|v\|_{L_\infty}$.

Now we turn to I, and write, with $a > 1$,

$$I = \Big(\int_0^{t/a} + \int_{t/a}^t\Big)E_h'(t-s)(j+1)s^j\zeta^{(j-1)}(s)\,ds = I_1 + I_2.$$

Here, using Lemma 3.3 and the induction assumption, we have

$$\|I_1\|_{L_\infty} \le C\int_0^{t/a}(t-s)^{-1}s^j\|\zeta^{(j-1)}(s)\|_{L_\infty}\,ds \le C\ln(\frac{a}{a-1})h^2\ell_h^2\|v\|_{L_\infty}.$$

Further, after integration by parts,

$$\begin{aligned}I_2 =& -\Big[E_h(t-s)(j+1)s^j\zeta^{(j-1)}(s)\Big]_{t/a}^t \\ &+ \int_{t/a}^t E_h(t-s)(j+1)s^{-1}(js^j\zeta^{(j-1)}(s) + s^{j+1}\zeta^{(j)}(s))\,ds,\end{aligned}$$

and thus by Lemma 3.3 and the induction assumption, we have

$$\|E_h(t-s)(j+1)s^j\zeta^{(j-1)}(s)\|_{L_\infty} \le Ch^2\ell_h^2\|v\|_{L_\infty}, \quad \text{for } s = t, t/a,$$

and

$$\Big\| \int_{t/a}^{t} E_h(t-s)(j+1)s^{-1}(js^j\zeta^{(j-1)}(s))\,ds\Big\|_{L_\infty} \le C\ln(a)h^2\ell_h^2\|v\|_{L_\infty}.$$

Thus

$$\|I_2\|_{L_\infty} \le (C + C\ln(a))h^2\ell_h^2\|v\|_{L_\infty} + C\ln(a)\sup_{0\le s\le t}\|s^{j+1}\zeta^{(j)}\|_{L_\infty}.$$

Therefore, with $C_a = C + C\ln(a) + C\ln(\frac{a}{a-1})$,

$$\|I\|_{L_\infty} \le C_a h^2\ell_h^2\|v\|_{L_\infty} + C\ln(a)\sup_{0\le s\le t}\|s^{j+1}\zeta^{(j)}(s)\|_{L_\infty}.$$

By (3.13), we get

$$\sup_{0\le s\le t}\|s^{j+1}\zeta^{(j)}(s)\|_{L_\infty} \le C_a h^2\ell_h^2\|v\|_{L_\infty} + C\ln(a)\sup_{0\le s\le t}\|s^{j+1}\zeta^{(j)}(s)\|_{L_\infty}.$$

Choosing a such that $C\ln(a) \le 1/2$, we obtain (3.12). The proof is complete. □

We now turn to an error estimate in the smooth data case.

THEOREM 3.4. *Let $u(t)$ and $u_h(t)$ be the solutions of* (3.1) *and* (3.5) *and $j \ge 0$. If $v \in \dot W_\infty^{2j+2}$, then we have*

$$(3.14) \qquad \|D_t^j u_h(t) - D_t^j u(t)\|_{L_\infty} \le Ch^2\ell_h^2\|v\|_{W_\infty^{2j+2}} + C\|\Delta_h^j v_h - R_h\Delta^j v\|_{L_\infty}.$$

The proof will depend on the following:

LEMMA 3.5. *Let $u(t)$ be the solution of* (3.1) *and $j \ge 0$. Then we have, for $\rho = R_h u - u$,*

$$\|\rho^{(j)}(t)\|_{L_\infty} + t\|\rho^{(j+1)}(t)\|_{L_\infty} \le Ch^2\ell_h^2\|v\|_{W_\infty^{2j+2}}, \quad \textit{for } v \in \dot W_\infty^{2j+2}.$$

PROOF. The case $j = 0$ can be found in Thomée [16, Lemma 5.6]. Hence

$$\|\rho^{(j)}(t)\|_{L_\infty} + t\|\rho^{(j+1)}(t)\|_{L_\infty} \le Ch^2\ell_h^2\|D_t^j u(0)\|_{W_\infty^2} \le Ch^2\ell_h^2\|v\|_{W_\infty^{2j+2}},$$

which completes the proof. □

PROOF OF THEOREM 3.4. First we assume $v_h = T_h^{j+1}(-\Delta)^{j+1}v$, where $T_h = (-\Delta_h)^{-1}$, which implies that $\Delta_h^j v_h = R_h\Delta^j v$. In this case, we want to show

$$\|D_t^j u_h(t) - D_t^j u(t)\|_{L_\infty} \le Ch^2\ell_h^2\|v\|_{W_\infty^{2j+2}},$$

which we will do now.

We write, with $\theta = u_h - R_h u$ and $\rho = R_h u - u$,

$$D_t^j u_h(t) - D_t^j u(t) = \theta^{(j)}(t) + \rho^{(j)}(t).$$

Here $\rho^{(j)}(t)$ is bounded as desired by Lemma 3.5. To estimate $\theta^{(j)}(t)$ we write, since $\theta^{(j)}(0) = \Delta_h^j v_h - R_h \Delta^j v = 0$,

$$\theta^{(j)}(t) = -\Big(\int_0^{t/2} + \int_{t/2}^t \Big) E_h(t-s) P_h \rho_t^{(j)}(s)\, ds = I + II.$$

Here by Lemmas 3.3 and 3.5, noting that $\rho_t^{(j)}(s) = \rho^{(j+1)}(s)$,

$$\|II\|_{L_\infty} \le C \int_{t/2}^t \|\rho_t^{(j)}(s)\|_{L_\infty}\, ds \le C h^2 \ell_h^2 \|v\|_{W_\infty^{2j+2}}.$$

For I we integrate by parts to obtain

$$I = -\Big[E_h(t-s) P_h \rho^{(j)}(s) \Big]_0^{t/2} - \int_0^{t/2} E_h'(t-s) P_h \rho^{(j)}(s)\, ds.$$

Using Lemmas 3.3 and 3.5 we have

$$\|E_h(t-s) P_h \rho_t^{(j)}(s)\|_{L_\infty} \le C \|\rho_t^{(j)}(s)\|_{L_\infty} \le C h^2 \ell_h^2 \|v\|_{W_\infty^{2j+2}}, \quad \text{for } s = 0, t/2,$$

and

$$\begin{aligned} \Big\| \int_0^{t/2} E_h'(t-s) P_h \rho^{(j)}(s)\, ds \Big\|_{L_\infty} &\le C \int_0^{t/2} (t-s)^{-1} \|\rho^{(j)}(s)\|_{L_\infty}\, ds \\ &\le C h^2 \ell_h^2 \|v\|_{W_\infty^{2j+2}}, \end{aligned}$$

which shows (3.14) for present choice of v_h.

It remains to consider the contribution to the semidiscrete solution of the initial data $v_h - T_h^{j+1}(-\Delta)^{j+1} v$. We have by the above proof

$$\|D_t^j E_h(t)(T_h^{j+1}(-\Delta)^{j+1} v) - D_t^j u(t)\|_{L_\infty} \le C h^2 \ell_h^2 \|v\|_{W_\infty^{2j+2}}.$$

On the other hand, by the stability of $E_h(t)$,

$$\|D_t^j E_h(t)(v_h - T_h^{j+1}(-\Delta)^{j+1} v)\|_{L_\infty} \le C \|\Delta_h^j v_h - R_h \Delta^j v\|_{L_\infty}.$$

Together these estimates complete the proof of (3.14). □

Now we consider the error estimates for the fully discrete scheme (3.9). In order to apply the results in Section 2, we have to consider the appropriate bound for the resolvent $R(z, -\Delta_h)$. To do this, we quote the following lemma from Bakaev, Thomée, and Wahlbin [4, Theorem 1.1]

LEMMA 3.6. *For any $\delta \in (0, \pi/2)$ there exists a constant C such that*

$$\|R(z, -\Delta_h) f\|_{L_\infty} \le C |z|^{-1} \|f\|_{L_\infty}, \quad \text{for } z \in \Sigma_\delta.$$

By using this lemma, we see that $A = -\Delta_h$ satisfies (1.2) and (1.3), hence we can apply the results in Section 2 for the case $A = -\Delta_h$, $\mathcal{B} = S_h$ with respect to L_∞ norm.

We first combine Theorem 3.1, for the error estimate in the semidiscrete case, with Theorem 2.6, applied to the semidiscrete equation (3.5), to obtain the following error estimate in the nonsmooth data case.

THEOREM 3.7. *Let $u(t_n)$ and U^n be the solutions of* (3.1) *and* (3.9). *Assume that $r(z)$ is accurate of order $p \geq 1$ and $A(\theta)$-stable with $\theta \in (0, \pi/2]$ and $|r(\infty)| < 1$. Let $j \geq 1$ and assume that Q_k^j, defined in* (1.11), *is an approximation of D_t^j, which is accurate of order p. Then, there is a constant C such that, if $v \in L_\infty$ and $v_h = P_h v$, we have, for $n \geq m_1$, $t_n > 0$,*

$$\|Q_k^j U^n - D_t^j u(t_n)\|_{L_\infty} \leq C(h^2 \ell_h^2 t_n^{-j-1} + k^p t_n^{-p-j})\|v\|_{L_\infty}.$$

We now show an error estimate in the smooth data case.

THEOREM 3.8. *Let $u(t_n)$ and U^n be the solutions of* (3.1) *and* (3.9). *Assume that $r(z)$ is accurate of order $p \geq 1$ and $A(\theta)$-stable with $\theta \in (0, \pi/2]$. Let $j \geq 1$ and assume that Q_k^j, defined in* (1.11), *is an approximation of D_t^j, which is accurate of order p. Then, there is a constant C such that, if $v \in L_\infty$ and $v_h = P_h v$, we have, for $n \geq m_1$,*

$$\|Q_k^j U^n - D_t^j u(t_n)\|_{L_\infty} \leq C(h^2 \ell_h^2 \|v\|_{W_\infty^{2+2j}} + k^p \|v\|_{W_\infty^{2p+2j}}) + C\|\Delta_h^j v_h - \Delta^j v\|_{L_\infty}.$$

In order to prove the theorem, we need the following lemma.

LEMMA 3.9. *Assume that $r(z)$ is $A(\theta)$-stable with $\theta \in (0, \pi/2]$ and accurate of order $p \geq 1$. Let $j \geq 1$ and let $\tilde{G}_{n,s} = G_n(-k\Delta_h) T_h^s$, where G_n is defined by* (2.3) *and $T_h = (-\Delta_h)^{-1}$. Then we have*

$$\|\tilde{G}_{n,l+j} w\|_{L_\infty} \leq C k^{l+j} \|w\|_{L_\infty}, \quad \text{for } 0 \leq l \leq p,\ n \geq m_1.$$

PROOF. Using Lemma 3.6, we obtain by Theorem 2.1, for $n \geq m_1$,

(3.15)
$$\|\tilde{G}_{n,l+j} w\|_{L_\infty} = \|G_n(-k\Delta_h) T_h^{l+j} w\|_{L_\infty} \leq C k^{l+j} \|w\|_{L_\infty}, \quad \text{for } 0 \leq l \leq p.$$

Note that if $r(z)$ is accurate of p it is also accurate of order l with $1 \leq l \leq p$, which shows (3.15) for $1 \leq l \leq p$. The case $l = 0$ follows by the direct proof as in the case $l = p$. □

PROOF OF THEOREM 3.8. By Theorem 3.4, Lemma 3.5 and the estimate

$$\|\Delta_h^j v_h - R_h \Delta^j v\|_{L_\infty} \leq \|\Delta_h^j v_h - \Delta^j v\|_{L_\infty} + \|(R_h - I)\Delta^j v\|_{L_\infty},$$

we only need to show

(3.16)
$$\|Q_k^j U^n - D_t^j u_h(t_n)\|_{L_\infty} \leq C(h^2 \ell_h^2 \|v\|_{W_\infty^{2+2j}} + k^p \|v\|_{W_\infty^{2p+2j}}) + C\|\Delta_h^j v_h - \Delta^j v\|_{L_\infty}.$$

Assuming first that $v_h = T_h^j(-\Delta)^j v$, we have

$$Q_k^j U^n - D_t^j u_h(t_n) = k^{-j}\tilde{G}_{n,j}(-\Delta)^j v.$$

Following Thomée [16, Theorem 8.6], we choose $\tilde{v}_k$, such that, with C independent of s,

$$\|(-\Delta)^j(v-\tilde{v}_k)\|_{L_\infty} \le Ck^p\|\Delta^{p+j}v\|_{L_\infty} \le Ck^p\|v\|_{W_\infty^{2p+2j}}, \tag{3.17}$$

$$\|(-\Delta)^{p+j}\tilde{v}_k\|_{L_\infty} \le C\|\Delta^{p+j}v\|_{L_\infty} \le C\|v\|_{W_\infty^{2p+2j}}, \tag{3.18}$$

$$k^l\|(-\Delta)^{l+j}\tilde{v}_k\|_{W_s^2} \le Cs\|\Delta^j v\|_{W_s^2}, \quad \text{for } 0 \le l \le p-1,\ 2 \le s < \infty. \tag{3.19}$$

Applying now the identity

$$v = \sum_{l=0}^{p-1} T_h^l(T-T_h)(-\Delta)^{l+1}v + T_h^p(-\Delta)^p v,$$

to $(-\Delta)^j\tilde{v}_k$, we have

(3.20)

$$\begin{aligned}\tilde{G}_{n,j}(-\Delta)^j\tilde{v}_k &= G_n(-k\Delta_h)T_h^j(-\Delta)^j\tilde{v}_k \\ &= \sum_{l=0}^{p-1}\tilde{G}_{n,l+j}(T-T_h)(-\Delta)^{l+j+1}\tilde{v}_k + \tilde{G}_{n,p+j}(-\Delta)^{p+j}\tilde{v}_k.\end{aligned}$$

By Lemma 3.9, we have, since $(T-T_h)(-\Delta) = I - R_h$,

$$\|\tilde{G}_{n,l+j}(T-T_h)(-\Delta)^{l+j+1}\tilde{v}_k\|_{L_\infty} \le Ck^{l+j}\|(I-R_h)(-\Delta)^{l+j}\tilde{v}_k\|_{L_\infty}.$$

Using the following bound for the Ritz projection in maximum-norm, see, e.g., Thomée [16, Lemma 5.6],

$$\|(R_h - I)v\|_{L_\infty} \le Ch^{2-2/s}\ell_h\|v\|_{W_s^2}, \quad \text{for } 2 \le s < \infty,$$

and (3.19), choosing $s = \ell_h$, we therefore obtain

$$\begin{aligned}\|\tilde{G}_{n,l+j}(T-T_h)(-\Delta)^{l+j+1}\tilde{v}_k\|_{L_\infty} &\le Ck^{l+j}h^{2-2/s}\ell_h\|\Delta^{l+j}\tilde{v}_k\|_{W_s^2} \\ &\le Ck^j s h^{2-2/s}\ell_h\|\Delta^j v\|_{W_s^2} \\ &\le Ck^j h^2\ell_h^2\|v\|_{W_\infty^{2+2j}}, \quad \text{for } 0 \le l \le p-1.\end{aligned}$$

For the case $l = p$ we have by (3.18),

$$\|\tilde{G}_{n,p+j}(-\Delta)^{p+j}\tilde{v}_k\|_{L_\infty} \le Ck^{p+j}\|\Delta^{p+j}\tilde{v}_k\|_{L_\infty} \le Ck^{p+j}\|v\|_{W_\infty^{2p+2j}}.$$

Together these estimates imply

$$\|\tilde{G}_{n,j}(-\Delta)^j \tilde{v}_k\|_{L_\infty} \le C k^j (h^2 \ell_h^2 \|v\|_{W_\infty^{2+2j}} + k^p \|v\|_{W_\infty^{2p+2j}}).$$

By Lemma 3.9 and (3.17), we have

$$\begin{aligned}\|\tilde{G}_{n,j}(-\Delta)^j(v-\tilde{v}_k)\|_{L_\infty} &= \|G_n(-k\Delta_h)T_h^j(-\Delta)^j(v-\tilde{v}_k)\|_{L_\infty} \\ &\le C k^j \|(-\Delta)^j(v-\tilde{v}_k)\|_{L_\infty} \le C k^{p+j}\|v\|_{W_\infty^{2p+2j}}.\end{aligned}$$

We conclude that

$$\begin{aligned}\|Q_k^j U^n - D_t^j u_h(t_n)\|_{L_\infty} &= \|k^{-j}\tilde{G}_{n,j}(-\Delta)^j v\|_{L_\infty} \\ &\le C(h^2\ell_h^2\|v\|_{W_\infty^{2+2j}} + k^p\|v\|_{W_\infty^{2p+2j}}),\end{aligned}$$

which shows (3.16) for present choice of v_h.

It remains to consider the contribution to the fully discrete solution of $v_h - T_h^j(-\Delta)^j v$. Since

$$Q_k^j E_{kh}^n (v_h - T_h^j(-\Delta)^j)v = P(r(-k\Delta_h))r(-k\Delta_h)^n(-k\Delta_h)^{-j}v, \tag{3.21}$$

it suffices to show

$$\|P(r(-k\Delta_h))r(-k\Delta_h)^n(-k\Delta_h)^{-j}\|_{L_\infty} \le C, \tag{3.22}$$

where $\|\cdot\|_{L_\infty}$ denotes the operator norm. In fact,

$$P(r(-\Delta_h))r(-\Delta_h)^n(-\Delta_h)^{-j} = \frac{1}{2\pi i}\int_\Gamma P(r(z))r(z)^n z^{-j} R(z,-\Delta_h)\,dz.$$

Since $0 \in \rho(-\Delta_h)$, $P(r(z)) = O(z^j)$ as $z \to 0$, there exists small $\eta > 0$, such that $\|R(z,-\Delta_h)\|_{L_\infty} \le C$ and $|P(r(z))z^{-j}| \le C$ for $|z| \le \eta$. Thus, we have, noting $r(z)$ is bounded on Γ and $n \ge m_1$,

$$\Big\|\int_\Gamma P(r(z))r(z)^n z^{-j} R(z,-\Delta_h)\,dz\Big\|_{L_\infty} \le \int_0^\eta d\rho + \int_\eta^\infty \frac{d\rho}{\rho^{j+1}} \le C,$$

which shows (3.22). The proof is now complete. □

4 Numerical Illustrations

In this section, we show some numerical results illustrating our theoretical analysis. We consider a one-dimensional problem with nonsmooth data,

$$\begin{cases} u_t - u_{xx} = 0, \quad \text{in } [0,1], \quad \text{with } u(0,t) = u(1,t) = 0, \quad \text{for } t > 0, \\ u(x,0) = v(x), \quad \text{in } [0,1], \end{cases} \tag{4.1}$$

where

$$v = \begin{cases} 1, & \text{if } \frac{1}{4} < x < \frac{3}{4}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

We have that $v \in L_\infty$, but $v \notin W^s_\infty$ for any $s > 0$.

The exact solution of (4.1) is

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^n \sin\frac{(2n-1)\pi}{4}(2n-1)^{-1}e^{-((2n-1)\pi)^2 t}\sin(2n-1)\pi x,$$

and the derivative of $u(x,t)$ is

$$u_t(x,t) = 4\pi\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{(2n-1)\pi}{4}(2n-1)e^{-((2n-1)\pi)^2 t}\sin(2n-1)\pi x.$$

We define S_h to be the set of continuous piecewise linear functions on a uniform mesh of size h, which vanish at $x = 0$ and $x = 1$. As explained in Section 3, the semidiscrete problem may be written

$$(4.3) \qquad u_{h,t} = A_h u_h, \quad \text{for } t > 0, \quad \text{with } u_h(0) = P_h v,$$

where A_h is the discrete analogue of $A = -d^2/dx^2$, defined by

$$(A_h\psi, \chi) = \int_0^1 \psi'\chi'\,dx, \quad \forall\, \psi,\, \chi \in S_h.$$

We first compute the approximate solution U^n of (4.1) by applying the time stepping method $U^n = r(kA_h)U^{n-1}$ to the semidiscrete problem (4.3), where $r(\lambda)$ will be specified in our examples below. As mentioned in the introduction, if $r(\infty) = 0$, then $u_{h,t}(t_n)$ can be approximated by $-A_h U^n$ and the error estimates (1.10) holds. In the case of $r(\infty) \neq 0$, we then use $\bar\partial U^n = (U^n - U^{n-1})/k$, which is the special case of (1.11), to approximate $u_t(t_n)$. Theorem 3.7 shows an error estimate for the fully discrete method with nonsmooth data in L_∞ norm. More precisely, if $|r(\infty)| < 1$, we have

$$(4.4) \qquad \|\bar\partial U^n - u_t(t_n)\| \le C t_n^{-2}(k + h^2\ell_h^2)\|v\|_{L_\infty}.$$

For the approximation $-A_h U^n$ of $u_t(t_n)$ when $r(\infty) = 0$, combining (1.10) and Theorem 3.1, we have the same error bound as in (4.4).

In our experiment, we consider the θ-method defined by (2.1) with $\theta = 2/3$, in this case $|r(\infty)| = 1/2$. Since we are mostly interested in the time stepping, we choose h very small and a sequence of moderate k. We thus use $h = 1/200$ fixed, and the time step k is chosen as $1/20, 1/40$ and $1/80$.

Denote $\varepsilon(k) = \varepsilon(k, t_n) = \|U^n - u(t_n)\|_{L_\infty}$, and let $\rho(k_1, k_2) = \varepsilon(k_1)/\varepsilon(k_2)$. Table 1 shows the L_∞ norm of the error of the approximation U^n of $u(t_n)$ at time t_n. From Thomée [16], we know that $\|U^n - u(t_n)\| \le C t_n^{-1}(k + h^2\ell_h^2)\|v\|_{L_\infty}$. Table 1 show the expected $O(k)$ order of convergence. We also see that the error becomes large when t tends to 0.

In Table 2, we show the results of the approximation $\bar\partial U^n$ of $u_t(t_n)$. Here $\varepsilon(k) = \varepsilon(k, t_n) = \|\bar\partial U^n - u_t(t_n)\|_{L_\infty}$, and again $\rho(k_1, k_2) = \varepsilon(k_1)/\varepsilon(k_2)$. The results confirm the expected $O(k)$ order of convergence and the singular behavior of the error as $t \to 0$.

t	$\varepsilon(1/20)$	$\varepsilon(1/40)$	$\varepsilon(1/80)$	$\rho(1/20,1/40)$	$\rho(1/40,1/80)$
0.1	1.669E-01	4.343E-02	6.465E-03	3.84	6.71
0.2	4.794E-02	8.957E-03	4.764E-03	5.35	1.87
0.3	1.537E-02	5.082E-03	2.688E-03	3.02	1.89
0.4	5.498E-03	2.570E-03	1.348E-03	2.13	1.90
0.5	2.214E-03	1.218E-03	6.342E-04	1.81	1.92
0.6	1.020E-03	5.548E-04	2.863E-04	1.83	1.93
0.7	4.572E-04	2.456E-04	1.257E-04	1.86	1.95
0.8	2.009E-04	1.065E-04	5.405E-05	1.88	1.97
0.9	8.693E-05	4.548E-05	2.288E-05	1.91	1.98
1.0	3.717E-05	1.918E-05	9.568E-06	1.93	2.00

Table 1. θ-method, with the approximation U^n of $u(t_n)$ in L_∞ norm.

t	$\varepsilon(1/20)$	$\varepsilon(1/40)$	$\varepsilon(1/80)$	$\rho(1/20,1/40)$	$\rho(1/40,1/80)$
0.1	9.664E+00	4.558E+00	6.097E-01	2.11	7.47
0.2	2.460E+00	3.964E-01	1.020E-01	6.20	3.88
0.3	6.737E-01	9.585E-02	4.741E-02	7.02	2.02
0.4	1.939E-01	4.30E-02	2.123E-02	4.50	2.02
0.5	5.960E-02	1.883E-02	9.270E-03	3.16	2.03
0.6	1.970E-02	8.102E-03	3.969E-03	2.43	2.04
0.7	7.118E-03	3.437E-03	1.674E-03	2.07	2.05
0.8	3.018E-03	1.442E-03	6.983E-04	2.09	2.06
0.9	1.268E-03	5.996E-04	2.884E-04	2.11	2.07
1.0	5.293E-04	2.474E-04	1.182E-04	2.13	2.09

Table 2. θ-method, with the approximation $\bar{\partial}U^n$ of $u_t(t_n)$ in L_∞ norm.

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Paper II

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Smoothing properties and approximation of time derivatives for parabolic equations: variable time steps

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Abstract.

We study smoothing properties and approximation of time derivatives for time discretization schemes with variable time steps for a homogeneous parabolic problem formulated as an abstract initial value problem in a Banach space. The time stepping methods are based on using rational functions $r(z) \approx e^{-z}$ which are $A(\theta)$-stable for suitable $\theta \in (0, \pi/2]$ and satisfy $|r(\infty)| < 1$. First and second order approximations of time derivatives based on using difference quotients are considered. Smoothing properties are derived and error estimates are established under the so called *increasing quasi-quasiuniform* assumption on the time steps.

AMS subject classification: 65M12, 65M15, 65M60, 65J10.

Key words: Banach space, parabolic, smoothing, time derivatives, error estimates, variable time steps.

1 Introduction

Let us consider the following homogeneous linear parabolic problem

$$u_t + Au = 0 \quad \text{for } t > 0, \quad \text{with } u(0) = v, \tag{1.1}$$

where A is a closed, linear operator, with dense domain $\mathcal{D}(A) \subset \mathcal{B}$, where $\mathcal{B}$ is a Banach space with norm $\|\cdot\|$ and $v \in \mathcal{B}$. We shall study time discretization schemes with variable time steps and show error estimates for the approximations of u and u_t.

We assume that $-A$ is the infinitesimal generator of a bounded analytic semigroup $E(t) = e^{-tA}$ and that $0 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A. This is equivalent to saying that there is an angle $\delta \in (0, \pi/2)$ such that

$$\rho(A) \supset \Sigma_\delta = \{z \in \mathbf{C} : \delta \leq |\arg z| \leq \pi, z \neq 0\} \cup \{0\}, \tag{1.2}$$

and that the resolvent, $R(z; A) = (zI - A)^{-1}$, satisfies

$$\|R(z; A)\| \leq M|z|^{-1}, \quad \text{for } z \in \Sigma_\delta, \quad \text{with } M \geq 1, \tag{1.3}$$

where $\|\cdot\|$ denotes the standard norm of bounded linear operators on $\mathcal{B}$.

Under these assumptions $E(t)$ may be represented as

$$E(t) = \frac{1}{2\pi i}\int_\Gamma e^{-zt}R(z;A)\,dz,$$

where $\Gamma = \{\, z : |\arg z| = \psi\}$ with $\psi \in (\,\delta, \pi/2\,)$ and $\mathrm{Im}z$ is decreasing along Γ. Furthermore, the smoothing properties of analytic semigroups are valid. More precisely, see Pazy [11], we have

$$\|D_t^j E(t)v\| = \|A^j E(t)v\| \le Ct^{-j}\|v\|, \quad \text{for } t > 0,\ v \in \mathcal{B}, \tag{1.4}$$

which shows that the solution is regular for positive time even if the initial data are not.

Let $0 = t_0 < t_1 < \cdots < t_n < \cdots$ be a partition of the time axis and let $k_n = t_n - t_{n-1},\ n \ge 1$, be the variable time steps. An approximate solution $U^n \approx u(t_n) = E(t_n)v$ of (1.1) may be defined by

$$U^n = E_{k_n}U^{n-1}, \quad \text{for } n \ge 1, \quad \text{with } U^0 = v, \tag{1.5}$$

where $E_{k_n} = r(k_nA)$ and r is a rational function that satisfies certain conditions. For example, $r(z) = 1/(1-z)$ and $r(z) = (1+z/2)/(1-z/2)$ correspond to the backward Euler and Crank-Nicolson methods, respectively.

We say that r is $A(\theta)$-stable with $\theta \in [0, \pi/2]$ if

$$|r(z)| \le 1, \quad \text{for } |\arg z| \le \theta, \tag{1.6}$$

and accurate of order $p \ge 1$, if

$$r(z) - e^{-z} = O(z^{p+1}), \quad \text{as } z \to 0. \tag{1.7}$$

Let us recall some results for the time stepping method (1.5) with constant time step k. If A satisfies (1.2) and (1.3), and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, then we have the stability estimate, with $E_k = r(kA)$,

$$\|U^n\| = \|E_k^n v\| \le C\|v\|, \quad \text{for } t_n \ge 0,\ v \in \mathcal{B}, \tag{1.8}$$

see, e.g., Crouzeix, Larsson, Piskarev, and Thomée [3] and Palencia [9], [10]. If A satisfies (1.2) and (1.3), and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$ and accurate of order $p \ge 1$, then the following smooth data error estimate holds:

$$\|U^n - u(t_n)\| \le Ck^p\|A^p v\|, \quad \text{for } t_n \ge 0,\ v \in \mathcal{D}(A^p). \tag{1.9}$$

Moreover, if $|r(\infty)| < 1$, then the following nonsmooth data error estimate holds:

$$\|U^n - u(t_n)\| \le Ck^p t_n^{-p}\|v\|, \quad \text{for } t_n > 0,\ v \in \mathcal{B}. \tag{1.10}$$

The condition $|r(\infty)| < 1$ ensures that oscillating components of the error are efficiently damped, see, e.g., Le Roux [8], Larsson, Thomée, and Wahlbin [7], Fujita and Suzuki [5].

Smoothing properties and approximation of time derivatives for (1.5) with constant time step have also been studied by some authors. Let $j \geq 1$ be fixed. If A satisfies (1.2) and (1.3), and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, and with $r(\infty) = 0$, then the following smoothing property holds:

$$\|A^j U^n\| = \|A^j E_k^n v\| \leq C t_n^{-j} \|v\|, \quad \text{for } t_n \geq t_j,\ v \in \mathcal{B}, \tag{1.11}$$

see, e.g., Thomée [12] for the Hilbert space case and Hansbo [6] for the Banach space case. However (1.11) is not true in general when $r(\infty) \neq 0$.

Let us introduce the finite difference quotients,

$$Q_k^j U^n = \frac{1}{k^j} \sum_{\nu=-m_1}^{m_2} c_\nu U^{n+\nu}, \quad \text{for } n \geq m_1, \tag{1.12}$$

where m_1, m_2 are nonnegative integers, and c_ν are real numbers such that the operator Q_k^j is an approximation of order $p \geq 1$ to D_t^j, that is, for any smooth real-valued function u,

$$D_t^j u(t_n) = Q_k^j u^n + O(k^p), \quad \text{as } k \to 0, \quad \text{with } u^n = u(t_n).$$

We then have the following smoothing property and nonsmooth data error estimates: If A satisfies (1.2) and (1.3), and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, and $|r(\infty)| < 1$, then we have

$$\|Q_k^j U^n\| \leq C t_n^{-j} \|v\|, \quad \text{for } n \geq m_1,\ t_n > 0,\ v \in \mathcal{B}, \tag{1.13}$$

and, if further $r(z)$ is accurate of order $p \geq 1$,

$$\|Q_k^j U^n - D_t^j u(t_n)\| \leq C k^p t_n^{-(p+j)} \|v\|, \quad \text{for } n \geq m_1,\ t_n > 0,\ v \in \mathcal{B}, \tag{1.14}$$

see Yan [13]

For the smooth data error estimate, the condition $|r(\infty)| < 1$ is not necessary. In fact, we have, for any $A(\theta)$-stable discretization scheme with $\theta \in (\delta, \pi/2]$,

$$\|Q_k^j U^n - D_t^j u(t_n)\| \leq C k^p \|A^{p+j} v\|, \quad \text{for } n \geq m_1,\ v \in \mathcal{D}(A^{p+j}), \tag{1.15}$$

see, e.g., Baker, Bramble, and Thomée [2] for the Hilbert space case and Yan [13] for the Banach space case.

Now let us mention some results for the variable time steps which are related to the present paper. Stability results have been considered by some authors. For example, Palencia [9] shows that, if A satisfies (1.2) and (1.3), and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, if the time steps $\{k_j\}_{j=1}^{\infty}$ satisfy, with some constant μ,

$$0 < \mu^{-1} \leq \frac{k_i}{k_j} \leq \mu < \infty, \quad \text{for } i, j \geq 1, \tag{1.16}$$

then there exists a constant $C(\mu)$ such that the following stability result holds

$$\Big\| \prod_{j=1}^{n} E_{k_j} \Big\| \leq C(\mu), \quad \text{where } E_{k_j} = r(k_j A). \tag{1.17}$$

We observe that the stability bound will depend on the maximum ratio μ between the steps, but not on the steps themselves. In this way, the stability bound does not blow up when the maximum time step goes to zero, as long as μ remains bounded. In particular, a family of quasi-uniform grids with $k_{max} \le \mu k_{min}$ satisfies the assumption (1.17), where $k_{max} = \max_{1 \le j \le n} k_j$, $k_{min} = \min_{1 \le j \le n} k_j$. More precisely, Bakaev [1] shows that if A satisfies (1.2) and (1.3), and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, then

$$\Big\| \prod_{j=1}^{n} E_{k_j} \Big\| \le C \ln(1+\mu), \quad \text{where } \mu = k_{max}/k_{min}. \tag{1.18}$$

Palencia [10] further finds that if $|r(\infty)| < 1$, then the stability bound holds without any restriction on the time steps.

In the present paper, we first consider error estimates for (1.5) in both smooth and nonsmooth data cases. We show, in Theorem 2.1, that if A satisfies (1.2) and (1.3) and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$ and accurate of order $p \ge 1$, then the following smooth data error estimate holds:

$$\|U^n - u(t_n)\| \le C k_{max}^p \|A^p v\|, \quad \text{for } t_n \ge 0,\ v \in \mathcal{D}(A^p).$$

To obtain error estimates in the nonsmooth data case, we introduce the notion of *increasing quasi-quasiuniform grids* $\mathcal{T}$ in time. Let $\{\mathcal{T}\}$ be a family of partitions of the time axis, $\mathcal{T} = \{t_n : 0 = t_0 < t_1 < \cdots < t_n < \cdots\}$. $\{\mathcal{T}\}$ is called a family of *quasi-quasiuniform grids* if there exist positive constants c, C, such that

$$c k_{n+1} \le k_n \le C t_n / n, \quad \text{for } n \ge 1. \tag{1.19}$$

Further, if $k_1 \le k_2 \le \cdots \le k_n \le \cdots$, then we call $\{\mathcal{T}\}$ a family of *increasing quasi-quasiuniform grids.* We note that *increasing quasi-quasiuniform* implies that $k_n \sim k_{n+1}$ and $n k_n \sim t_n$, where $a_n \sim b_n$ means that a_n / b_n is bounded above and below.

For example, if we choose the variable time steps $k_n = n^s k$ for some fixed $s \ge 1$, with $k > 0$, then $t_n = k\big(\sum_{j=1}^{n} j^s\big)$, and the corresponding family of partitions $\{\mathcal{T}\}$ is a family of *increasing quasi-quasiuniform grids.* In fact, it is obvious that $k_n/k_{n+1} = n^s/(n+1)^s \ge 1/2^s$. Further, since $t_n/k = \sum_{j=1}^{n} j^s \ge C n^{s+1}$ for some positive constant C, we have $k_n \le C t_n / n$.

Under these assumptions we have the following nonsmooth data error estimate: If A satisfies (1.2) and (1.3), and r is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$ and accurate of order $p \ge 1$, if further $|r(\infty)| < 1$ and $\{\mathcal{T}\}$ is a family of *increasing quasi-quasiuniform grids*, then we have

$$\|U^n - u(t_n)\| \le C k_n^p t_n^{-p} \|v\|, \quad \text{for } t_n > 0,\ v \in \mathcal{B}.$$

We note that these two error estimates correspond to (1.9) and (1.10) for constant time step, respectively.

As for the smoothing property, we show that, if $r(\infty) = 0$, and $\{\mathcal{T}\}$ satisfies (1.17), then

$$\Big\| A \prod_{j=1}^{n} E_{k_j} v \Big\| \le C t_n^{-1} \|v\|, \quad \text{for } t_n > 0,\ v \in \mathcal{B}.$$

As in the constant time step case, see Yan [13], the above smoothing property is not true in the case of $r(\infty) \neq 0$. However, if $|r(\infty)| < 1$, then we introduce similar difference quotients as (1.12) with variable time steps. For simplicity we only consider the following first and second order approximations of time derivative $u_t(t_n)$

$$\bar\partial U^n = (U^n - U^{n-1})/k_n, \quad \text{for } n \ge 1, \tag{1.20}$$

and

$$\begin{aligned} \bar\partial^2 U^n &= a_n \bar\partial U^n + b_n \bar\partial U^{n-1} \\ &= a_n (U^n - U^{n-1})/k_n + b_n (U^{n-1} - U^{n-2})/k_{n-1}, \quad \text{for } n \ge 2, \end{aligned} \tag{1.21}$$

where

$$a_n = (2k_n + k_{n-1})/(k_n + k_{n-1}), \quad b_n = -k_n/(k_n + k_{n-1}).$$

In both cases, under the assumption of *increasing quasi-quasiuniform grids*, we obtain a smoothing property and error estimates for time derivative in the nonsmooth data case which are similar to (1.14) and (1.15), respectively. We also show a smooth data error estimate without any restrictions on the time steps.

The paper is organized as follows. In Section 2 we show error estimates for the approximation U^n of u^n in both smooth and nonsmooth data cases. In Section 3 we consider the first order approximation (1.21) of $u_t(t_n)$ and show a smoothing property and error estimates for time derivative. In Section 4 we consider the second order approximation (1.22) and obtain similar results as in Section 3.

By C and c we denote large and small positive constants independent of the functions and parameters concerned, but not necessarily the same at different occurrences. When necessary for clarity we distinguish constants by subscripts.

2 Error Estimates

In this section we will consider error estimates for the approximation U^n defined by (1.5) of the solution $u(t_n)$ of (1.1). Our first result is an error estimate in the smooth data case in which there is no restriction on the time steps k_n.

THEOREM 2.1. *Let U^n and $u(t_n)$ be the solutions of* (1.5) *and* (1.1), *respectively. Assume that A satisfies* (1.2) *and* (1.3), *and that $r(z)$ is accurate of order $p \ge 1$, and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$. Let $k_j, 1 \le j \le n$, be time steps. Then we have*

$$\|U^n - u(t_n)\| \le C k_{max}^p \|A^p v\|, \quad \textit{for } t_n \ge 0,\ v \in \mathcal{D}(A^p),$$

where $k_{max} = \max_{1 \le j \le n} k_j$.

In order to prove Theorem 2.1, we need the following lemmas which are simple consequences of (1.6) and (1.7). The first lemma is quoted from Thomée [12, Lemma 8.2].

LEMMA 2.2. *Assume that $r(z)$ is $A(\theta)$-stable with $\theta \in (0, \pi/2]$, and accurate of order $p \geq 1$. Then for arbitrary $R > 0$ and $\psi \in (0, \theta)$ there is $c > 0$ such that*

$$|r(z)| \leq e^{-c|z|}, \quad \textit{for } |z| \leq R, \ |\arg z| \leq \psi.$$

LEMMA 2.3. *Assume that $r(z)$ is $A(\theta)$-stable with $\theta \in (0, \pi/2]$, and accurate of order $p \geq 1$. Let k_j, $1 \leq j \leq n$, be any positive numbers. Then for arbitrary $R > 0$ and $\psi \in (0, \theta)$ there are $c, C > 0$ such that, with $F_n(z) = \prod_{j=1}^{n} r(k_j z) - e^{-t_n z}$,*

$$|F_n(z)| \leq Cn|k_{max} z|^{p+1} e^{-ct_n|z|}, \quad \textit{for } |k_{max} z| \leq R, \ |\arg z| \leq \psi, \tag{2.1}$$

and

$$|F_n(z)| \leq C|k_{max} z|^p t_n |z| e^{-ct_n|z|}, \quad \textit{for } |k_{max} z| \leq R, \ |\arg z| \leq \psi, \tag{2.2}$$

where $k_{max} = \max_{1 \leq j \leq n} k_j$.

PROOF. Since $r(z)$ is accurate of order $p \geq 1$, there exists a small $\eta > 0$ such that

$$|r(z) - e^{-z}| \leq C|z|^{p+1}, \quad \text{for } |z| \leq \eta.$$

Further, by (1.6), we have, for arbitrary $R > 0$ and $\psi \in (0, \theta)$,

$$|r(z) - e^{-z}| \leq C|z|^{p+1}, \quad \text{for } |z| \leq R, \ |\arg z| \leq \psi. \tag{2.3}$$

We next observe that, if $c \leq \cos\psi$,

$$|e^{-z}| = e^{-\mathrm{Re}\, z} \leq e^{-c|z|}, \quad \text{for } |\arg z| \leq \psi. \tag{2.4}$$

It is easy to show that

$$|F_n(z)| \leq C \sum_{j=1}^{n} (k_j|z|)^{p+1} e^{-c(t_n - k_j)|z|}, \quad \text{for } |k_{max} z| \leq R, \ |\arg z| \leq \psi. \tag{2.5}$$

In fact, using Lemma 2.2, (2.4) and (2.5), we have, for $|k_{max} z| \leq R, \ |\arg z| \leq \psi$,

$$|F_1(z)| = |r(k_1 z) - e^{-k_1 z}| \leq C(k_1|z|)^{p+1} e^{-c(t_1 - k_1)|z|},$$

and

$$\begin{aligned} |F_2(z)| &= |(r(k_1 z) - e^{-k_1 z}) r(k_2 z) + e^{-k_1 z}(r(k_2 z) - e^{-k_2 z})| \\ &\leq C|k_1 z|^{p+1} e^{-c(t_2 - k_1)|z|} + Ce^{-c(t_2 - k_2)|z|} |k_2 z|^{p+1} \\ &= C \sum_{j=1}^{2} (k_j|z|)^{p+1} e^{-c(t_n - k_j)|z|}. \end{aligned}$$

In general, for $n \geq 3$,

$$\begin{aligned} |F_n(z)| &= \Big| \big(r(k_1 z) - e^{-k_1 z}\big) \prod_{j=2}^{n} r(k_j z) + e^{-k_1 z}\big(r(k_2 z) - e^{-k_2 z}\big) \prod_{j=3}^{n} r(k_j z) \\ &\quad + \cdots + \Big(\prod_{j=1}^{n-1} e^{-k_j z}\Big)\big(r(k_n z) - e^{-k_n z}\big)\Big| \\ &\leq C \sum_{j=1}^{n} \big(e^{-ct_{j-1}|z|}(k_j|z|)^{p+1} e^{-c(t_n - t_j)|z|}\big) \\ &= C \sum_{j=1}^{n} (k_j|z|)^{p+1} e^{-c(t_n-k_j)|z|}. \end{aligned}$$

Thus, by (2.6), we get, using $k_j|z| \leq |k_{\max} z| \leq R$, $1 \leq j \leq n$,

$$|F_n(z)| \leq Cn|k_{max}z|^{p+1} e^{-ct_n|z|}, \quad \text{for } |k_{max}z| \leq R, \ |\arg z| \leq \psi,$$

and

$$\begin{aligned} |F_n(z)| &\leq Ce^{-ct_n|z|} \sum_{j=1}^{n} (k_j|z|)^{p+1} \\ &\leq C|k_{max}z|^p t_n |z| e^{-ct_n|z|}, \quad \text{for } |k_{max}z| \leq R, \ |\arg z| \leq \psi. \end{aligned}$$

Together these estimates complete the proof. □

The following lemma gives the Dunford-Taylor spectral representation of a rational function of the operator A when the rational function is bounded in a sector in the right halfplane, see Thomée [12, Lemma 8.1].

LEMMA 2.4. *Assume that* (1.2) *and* (1.3) *hold and let* $r(z)$ *be a rational function which is bounded for* $|\arg z| \leq \psi$, $|z| \geq \epsilon > 0$, *where* $\psi \in (\delta, \pi/2)$, *and for* $|z| \geq R$. *If* $\epsilon > 0$ *is so small that* $\{z : |z| \leq \epsilon\} \subset \rho(A)$, *then we have*

$$r(A) = r(\infty)I + \frac{1}{2\pi i} \int_{\gamma_\epsilon \cup \Gamma_\epsilon^R \cup \gamma^R} r(z) R(z; A)\, dz,$$

where $\gamma_\epsilon = \{z : |z| = \epsilon, |\arg z| \leq \psi\}$, $\Gamma_\epsilon^R = \{z : |\arg z| = \psi, \epsilon \leq |z| \leq R\}$, *and* $\gamma^R = \{z : |z| = R, \psi \leq |\arg z| \leq \pi\}$, *and with the closed path of integration oriented in the negative sense.*

For our error estimates we shall apply the following spectral representation of the semigroup, see Thomée [12, Lemma 8.3].

LEMMA 2.5. *Assume that* (1.2) *and* (1.3) *hold, let* $\psi \in (\delta, \pi/2)$, *and* j *be any integer. Then we have for* $\epsilon > 0$ *sufficiently small*

$$A^j E(t) = \frac{1}{2\pi i} \int_{\gamma_\epsilon \cup \Gamma_\epsilon} e^{-zt} z^j R(z; A)\, dz,$$

where $\gamma_\epsilon = \{z : |z| = \epsilon, |\arg z| \leq \psi\}$ *and* $\Gamma_\epsilon = \{z : |\arg z| = \psi, |z| \geq \epsilon\}$, *and where* $\mathrm{Im} z$ *is decreasing along* $\gamma_\epsilon \cup \Gamma_\epsilon$. *When* $j \geq 0$, *we may take* $\epsilon = 0$.

PROOF OF THEOREM 2.1. Since $U^n - u(t_n) = \prod_{j=1}^n r(k_jA)v - e^{-t_nA}v = F_n(A)v$, we need to show $\|F_n(A)v\| \le Ck_{max}^p\|A^pv\|$, or in operator norm,

$$\|F_n(A)(k_{max}A)^{-p}\| \le C,$$

which we will do now. Let $\bar{r}(z) = \prod_{j=1}^n r(k_jz)(k_{max}z)^{-p}$. Since $r(z)$ is $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, we find that $\bar{r}(z)$ is bounded for $|\arg z| \le \psi$, $|z| \ge \epsilon$ with some $\psi \in (\delta, \theta)$ and any $\epsilon > 0$. Further $\bar{r}(z)$ is also bounded for $|z| \ge R$ with R sufficiently large, since $\bar{r}(\infty) = 0$. Thus, applying Lemma 2.4 to the rational function $\bar{r}(z)$, we have

$$\prod_{j=1}^n r(k_jA)(k_{max}A)^{-p} = \frac{1}{2\pi i}\int_{\gamma_\epsilon \cup \Gamma_\epsilon^R \cup \gamma^R} \prod_{j=1}^n r(k_jz)(k_{max}z)^{-p}R(z;A)\,dz.$$

By (1.3) and (1.6), we know that the integrand is of order $O(z^{-p-1})$ for large z which implies that the integrand has no poles when $|z| \ge R$, so that we may let R tend to ∞. Using also Lemma 2.5 we conclude

$$F_n(A)(k_{max}A)^{-p} = \frac{1}{2\pi i}\int_{\gamma_\epsilon \cup \Gamma_\epsilon} F_n(z)(k_{max}z)^{-p}R(z;A)\,dz.$$

Now by (2.1) we see that $F_n(z) = O(z^{p+1})$ as $z \to 0$. Combining this with (1.3) we have that the integrand is bounded on the small domain with boundary $\gamma_\epsilon \cup \Gamma_0^\epsilon$, so that we may let $\epsilon \to 0$. It follows that, using also (1.3),

$$\|F_n(A)(k_{max}A)^{-p}\| \le C\int_0^\infty (|F_n(\rho e^{i\psi})| + |F_n(\rho e^{-i\psi})|)(k_{max}\rho)^{-p}\,\frac{d\rho}{\rho}.$$

By (2.2), we have, for arbitrary $R > 0$,

$$\int_0^{R/k_{max}} |F_n(\rho e^{\pm i\psi})|(k_{max}\rho)^{-p}\,\frac{d\rho}{\rho} \le C\int_0^{R/k_{max}} e^{-ct_n\rho}t_n\,d\rho \le C.$$

Since $r(z)$ and e^{-tz} are bounded on Γ, where $\Gamma = \{z : |\arg z| = \psi\}$, we find

$$\int_{R/k_{max}}^\infty |F_n(\rho e^{\pm i\psi})|(k_{max}\rho)^{-p}\,\frac{d\rho}{\rho} \le C\int_{R/k_{max}}^\infty (k_{max}\rho)^{-p}\,\frac{d\rho}{\rho} \le C.$$

Together these estimates complete the proof. □

We now show a nonsmooth data error estimate.

THEOREM 2.6. *Let U^n and $u(t_n)$be the solutions of* (1.5) *and* (1.1), *respectively. Assume that A satisfies* (1.2) *and* (1.3), *and that $r(z)$ is accurate of order $p \ge 1$ and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$ and $|r(\infty)| < 1$. Assume further that $\{\mathcal{T}\}$ is a family of increasing quasi-quasiuniform grids. Then there is a constant C such that*

$$\|U^n - u(t_n)\| \le Ck_n^pt_n^{-p}\|v\|, \quad \textit{for } t_n > 0.$$

To prove Theorem 2.6 we need the following lemma.

LEMMA 2.7. *If the rational function $r(z)$ is $A(\theta)$-stable with $\theta \in (0, \pi/2]$ and $|r(\infty)| < 1$, then for any $\psi \in (0, \theta)$ and $R > 0$ there are positive c and C such that, for any sequences $k_1 \le k_2 \le \cdots \le k_n$, with $\kappa = r(\infty)$,*

$$\Big| \prod_{j=1}^{n} r(k_j z) - \kappa^n \Big| \le C|k_1 z|^{-1} e^{-cn}, \quad \text{for } |k_1 z| \ge R, \ |\arg z| \le \psi. \tag{2.6}$$

PROOF. Since $r(z) - \kappa$ vanishes at infinity and $r(z)$ is $A(\theta)$-stable with $\theta \in (0, \pi/2]$, we have, see Thomée [12, Lemma 8.5],

$$|r(z) - \kappa| \le C|z|^{-1}, \quad \text{for } |z| \ge R, \ |\arg z| \le \psi.$$

Further,

$$|r(z)| \le e^{-c}, \quad \text{for } |z| \ge R, \ |\arg z| \le \psi. \tag{2.7}$$

In fact, $|\kappa| < 1$ implies that (2.8) holds for $|z| \le \tilde{R}$ with $\tilde{R}$ sufficiently large. By (1.6) and the maximum-principle we have $|r(z)| < 1$ for $|\arg z| < \theta, \ z \ne 0$. In particular, $|r(z)| < 1$ on the compact set $\{z : R \le |z| \le \tilde{R}, \ |\arg z| \le \psi\}$, which shows (2.8).

(2.7) is obvious for $n = 1$. When $n \ge 2$, we have, for $|k_1 z| \ge R$, noting that $\kappa \le e^{-c}$ and $k_1 \le k_2 \le \cdots \le k_n$,

$$\begin{aligned} \Big| \prod_{j=1}^{n} r(k_j z) - \kappa^n \Big| &= \Big| (r(k_1 z) - \kappa) \prod_{j=2}^{n} r(k_j z) + \cdots + \kappa^{n-1}(r(k_n z) - \kappa) \Big| \\ &\le C e^{-cn} \sum_{j=1}^{n} |k_j z|^{-1} \le C|k_1 z|^{-1} n e^{-cn} \le C|k_1 z|^{-1} e^{-cn}, \end{aligned}$$

which completes the proof of Lemma 2.7. □

PROOF OF THEOREM 2.6. The case $n = 1$ follows from the constant time step case, see Thomée [12]. We now consider $n \ge 2$. With $F_n(z)$ as in Lemma 2.3, we need to show $\|F_n(A)\| \le C k_n^p t_n^{-p}$. Since $t_n \le n k_n$, it suffices to show

$$\|F_n(A)\| \le C n^{-p}.$$

Set $\tilde{F}_n(z) = F_n(z) - \kappa^n k_n z/(1 + k_n z)$, where $\kappa = r(\infty)$. Since $|\kappa| < 1$, and by the obvious fact that $\|k_n A(I + k_n A)^{-1}\| \le C$, we have

$$\|\kappa^n k_n A(I + k_n A)^{-1}\| \le C|\kappa|^n \le C n^{-p},$$

and it remains to show the same bound for the operator norm of $\tilde{F}_n(A)$. Since $\prod_{j=1}^{n} r(k_j z) - \kappa^n k_n z/(1 + k_n z)$ vanishes at $z = \infty$, we may use Lemmas 2.4 and 2.5 to see that

$$\tilde{F}_n(A) = \frac{1}{2\pi i} \int_\Gamma \tilde{F}_n(z) R(z; A)\, dz,$$

where $\Gamma = \{z : |\arg z| = \psi\}$ for some $\psi \in (\delta, \theta)$. By (1.3), we get

$$\|\tilde{F}_n(A)\| \le C \int_0^\infty |\tilde{F}_n(\rho e^{\pm i\psi})| \,\frac{d\rho}{\rho}.$$

Let R be arbitrary. We will bound the above integral over the intervals $[0, R/k_n] \cup [R/k_n, R/k_1] \cup [R/k_1, \infty)$. We rewrite $\tilde{F}_n(z) = (\prod_{j=1}^n r(k_j z) - \kappa^n) + \kappa^n/(1 + k_n z) - e^{-t_n z}$. Using (2.5) and Lemma 2.7 and $|1 + k_n z| \geq |k_n z|$ for $\mathrm{Re}\, z \geq 0$, we get

$$\int_{R/k_1}^{\infty} |\tilde{F}_n(\rho e^{\pm i\psi})| \frac{d\rho}{\rho} \leq C \int_{R/k_1}^{\infty} \left(e^{-cn}(k_1\rho)^{-1} + |\kappa|^n (k_n\rho)^{-1} + e^{-ct_n\rho}\right) \frac{d\rho}{\rho}.$$

Obviously,

$$\int_{R/k_1}^{\infty} e^{-cn}(k_1\rho)^{-1} \frac{d\rho}{\rho} \leq \int_R^{\infty} e^{-cn} x^{-2}\, dx \leq Cn^{-p},$$

and, using $k_1 \leq k_n$,

$$\int_{R/k_1}^{\infty} |\kappa|^n (k_n\rho)^{-1} \frac{d\rho}{\rho} \leq \int_R^{\infty} |\kappa|^n (k_n k_1^{-1})^{-1} x^{-2}\, dx \leq Cn^{-p},$$

and, using $nk_1 \leq t_n$,

$$\begin{aligned} \int_{R/k_1}^{\infty} e^{-ct_n\rho} \frac{d\rho}{\rho} &\leq C \int_{R/k_1}^{\infty} (t_n\rho)^{-p} \frac{d\rho}{\rho} \\ &\leq C \int_R^{\infty} (t_n k_1^{-1})^{-p} x^{-p-1}\, dx \leq Cn^{-p}. \end{aligned}$$

Thus

$$\int_{R/k_1}^{\infty} |\tilde{F}_n(\rho e^{\pm i\psi})| \frac{d\rho}{\rho} \leq Cn^{-p}, \quad \text{for } n \geq 2. \tag{2.8}$$

Using (2.1) and $|1/(1 + k_n z)| \leq 1$ for $\mathrm{Re}\, z \geq 0$, we have, since $nk_n \sim t_n$,

$$\begin{aligned} \int_0^{R/k_n} |\tilde{F}_n(\rho e^{\pm i\psi})| \frac{d\rho}{\rho} &\leq \int_0^{R/k_n} |F_n(\rho e^{\pm i\psi})| \frac{d\rho}{\rho} + \int_0^{R/k_n} |\kappa|^n k_n\, d\rho \\ &\leq C \int_0^{R/k_n} (k_n\rho)^{p+1} e^{-ct_n\rho} n \frac{d\rho}{\rho} + C|\kappa|^n \\ &\leq C \int_0^R x^p e^{-c(t_n/k_n)x} n\, dx + C|\kappa|^n \leq Cn^{-p}, \quad \text{for } n \geq 2. \end{aligned}$$

It remains to consider the integral over the interval $[R/k_n, R/k_1]$ for $n \geq 2$. By Lemma 2.2 and (2.8) there exist constants c_1 and c_2 such that $|r(z)| \leq e^{-c_1|z|}$ for $|z| \leq R, |\arg z| \leq \psi$, and $|r(z)| \leq e^{-c_2}$ for $|z| \geq R, |\arg z| \leq \psi$, where c_2 can be chosen arbitrarily small. Therefore, assuming that $z \in \Gamma_{R/k_{m+1}}^{R/k_m}$ with some $m : 1 \leq m \leq n-1$ so that $k_j|z| \leq R$ for $j \leq m$, we have

$$\Big|\prod_{j=1}^n r(k_j z)\Big| \leq e^{-c_1 t_m |z|} e^{-c_2(n-m)} \leq e^{-c_2 n}\left(e^{c_2 m} e^{-c_1 t_m |z|}\right), \quad \text{for } n \geq 2.$$

Further, by (1.20),

$$c_1 t_m|z| = c_1(t_m/k_m)(k_m/k_{m+1})(k_{m+1}|z|) \geq c_1 c_0 C_0^{-1} Rm = c_3 m. \tag{2.9}$$

Thus if we choose $c_2 \leq c_3$ and let $c_4 = c_3 - c_2$, we get

$$\Big|\prod_{j=1}^{n} r(k_j z)\Big| \leq e^{-c_2 n} e^{-c_4 m}, \quad \text{if } z \in \Gamma_{R/k_{m+1}}^{R/k_m},\ 1 \leq m \leq n-1. \tag{2.10}$$

We rewrite $\tilde{F}_n(z) = \prod_{j=1}^{n} r(k_j z) - e^{-t_n z} - \kappa^n k_n z/(1 + k_n z)$. Using (2.12) and noting that $\ln(k_{m+1}/k_m) \leq \ln C \leq C$, we get

$$\begin{aligned}\int_{R/k_n}^{R/k_1} \Big|\prod_{j=1}^{n} r(k_j \rho e^{\pm i\psi})\Big| \frac{d\rho}{\rho} &\leq \sum_{m=1}^{n-1} \int_{R/k_{m+1}}^{R/k_m} \Big|\prod_{j=1}^{n} r(k_j \rho e^{\pm i\psi})\Big| \frac{d\rho}{\rho} \\ &\leq \sum_{m=1}^{n-1} \int_{R/k_{m+1}}^{R/k_m} e^{-c_2 n} e^{-c_4 m} \frac{d\rho}{\rho} \leq e^{-c_2 n} \sum_{m=1}^{n-1} \big(e^{-c_4 m} \ln(k_{m+1}/k_m)\big) \\ &\leq C e^{-c_2 n} \Big(\sum_{m=1}^{n-1} e^{-c_4 m}\Big) \leq C e^{-c_2 n} \leq C n^{-p}, \quad \text{for } n \geq 2.\end{aligned} \tag{2.11}$$

Further, using (2.5) and noting that (1.20) implies $t_n \rho \geq c(n k_n)\rho \geq cn$ for $\rho \in [R/k_n, R/k_1]$, we have, since $\ln(k_n/k_1) = \sum_{m=1}^{n-1} \ln(k_{m+1}/k_m) \leq Cn$,

$$\begin{aligned}\int_{R/k_n}^{R/k_1} \Big(|e^{-t_n \rho e^{\pm i\psi}}| + \frac{k_n \rho}{1 + k_n \rho} \kappa^n\Big) \frac{d\rho}{\rho} &\leq \int_{R/k_n}^{R/k_1} (e^{-cn} + \kappa^n) \frac{d\rho}{\rho} \\ &\leq (e^{-cn} + \kappa^n) \ln(k_n/k_1) \leq Cn(e^{-cn} + \kappa^n) \leq C n^{-p}, \quad \text{for } n \geq 2.\end{aligned} \tag{2.12}$$

Hence

$$\int_{R/k_n}^{R/k_1} \Big|\tilde{F}_n(\rho e^{\pm i\psi})\Big| \frac{d\rho}{\rho} \leq C n^{-p}, \quad \text{for } n \geq 2.$$

Together these estimates complete the proof. □

3 Approximation of Time Derivative — First Order

In this section, we shall consider a smoothing property of the time discretization scheme (1.5) and error estimates for the first order approximation of time derivative $u_t(t_n)$

$$\bar{\partial} U^n = (U^n - U^{n-1})/k_n, \quad \text{for } n \geq 1. \tag{3.1}$$

We begin with a smooth data error estimate for the approximation (3.1).

THEOREM 3.1. *Let U^n and $u(t_n)$be the solutions of* (1.5) *and* (1.1), *respectively. Assume that A satisfies* (1.2) *and* (1.3), *and that $r(z)$ is accurate of order*

$p \geq 1$, and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$. Let $k_j, 1 \leq j \leq n$, be increasing. Then we have

$$\|\bar{\partial} U^n - u_t(t_n)\| \leq C k_n \|A^2 v\|, \quad \text{for } t_n > 0. \tag{3.2}$$

PROOF. The case $n = 1$ follows from the result in constant time step case, see Yan [13]. Now we consider the case when $n \geq 2$. Setting

$$G_n(z) = \prod_{j=1}^{n-1} r(k_j z)(r(k_n z) - 1) - (-k_n z) e^{-t_n z},$$

our result will follow from

$$\|G_n(A)(k_n A)^{-2}\| \leq C, \quad \text{for } n \geq 2.$$

Let $\bar{r}(z) = \Big(\prod_{j=1}^{n-1} r(k_j z)(r(k_n z) - 1) \Big) (k_n z)^{-2}$. As in the proof of Theorem 2.1, applying Lemma 2.4 to the rational function $\bar{r}(z)$ and using also Lemma 2.5 we conclude

$$G_n(A)(k_n A)^{-2} = \frac{1}{2\pi i} \int_{\gamma_\epsilon \cup \Gamma_\epsilon} G_n(z)(k_n z)^{-2} R(z; A)\, dz.$$

Since $0 \in \rho(A)$, we have, by (1.3),

$$\|R(z; A)\| \leq C, \quad \text{for } \delta \leq |\arg z| \leq \pi. \tag{3.3}$$

We will show that

$$G_n(z) = O(z^2) \quad \text{as } z \to 0. \tag{3.4}$$

Combining this with (3.3) shows that the integrand is bounded on the small domain with boundary $\gamma_\epsilon \cup \Gamma_0^\epsilon$, so that we may let $\epsilon \to 0$. It follows that, with $\Gamma = \{z : |\arg z| = \psi\}$ for some $\psi \in (\delta, \theta)$,

$$G_n(A)(k_n A)^{-2} = \frac{1}{2\pi i} \int_\Gamma G_n(z)(k_n z)^{-2} R(z; A)\, dz. \tag{3.5}$$

In order to show (3.4), we write

$$G_n(z) = G_n^1(z) + G_n^2(z) + G_n^3(z), \quad \text{for } n \geq 2,$$

where

$$G_n^1(z) = \prod_{j=1}^{n-1} r(k_j z)(r(k_n z) - 1 + k_n z),$$

and, with $F_n(z) = \prod_{j=1}^{n} r(k_j z) - e^{-t_n z}$,

$$G_n^2(z) = -k_n z \prod_{j=1}^{n-1} r(k_j z)(1 - r(k_n z)), \quad G_n^3(z) = -k_n z F_n(z).$$

By (1.7), there exists a small $\eta > 0$ such that

(3.6) $$|r(z)| \le C,\ |r(z) - 1| \le C|z|,\ |r(z) - 1 - z| \le C|z|^2, \quad \text{for } |z| \le \eta.$$

Combining this with (2.2) shows $|G_n(z)| \le C|k_n z|^2$ for $|k_n z| \le \eta$, which is (3.4).

It remains to consider (3.5). Let us first consider the integral over Γ_0^{η/k_n}. By (1.2) and (1.3), $R(z;A)$ is analytic in the domain $\{z : \delta \le |\arg z| \le \pi\}$, and hence $G_n(z)(k_n z)^{-2} R(z;A)$ is analytic in the domain bounded by $\Gamma_0^{\eta/k_n} \cup \gamma^{\eta/k_n}$ (see Lemmas 2.4 and 2.5 for the definition of the curve). We then can replace the path of integration in (3.5) by $\tilde{\Gamma} = \gamma^{\eta/k_n} \cup \Gamma_{\eta/k_n}$. We find, using $|G_n(z)| \le C|k_n z|^2$ for $|k_n z| \le \eta$,

(3.7) $$\Big\| \int_{\gamma^{\eta/k_n}} G_n(z)(k_n z)^{-2} R(z;A)\, dz \Big\| \le \int_{\gamma^{\eta/k_n}} \frac{|dz|}{|z|} = C,$$

and, by the boundedness of $r(z)$ and $e^{-t_n z}$ over Γ,

(3.8)
$$\Big\| \int_{\Gamma_{\eta/k_n}} G_n(z)(k_n z)^{-2} R(z;A)\, dz \Big\| \le C \int_{\eta/k_n}^{\infty} (C + C(k_n\rho))(k_n\rho)^{-2} \frac{d\rho}{\rho} \le C.$$

Together these estimates complete the proof. □

We now turn to smoothing properties of (1.5). Recall from the introduction that the smoothing property (1.11) is not valid if $r(\infty) \ne 0$. However, if $r(\infty) = 0$, the analogue of (1.11) holds also for some special schemes $r(z)$ with no restriction on the time steps, see Eriksson, Johnson, and Larsson [4]. For a general scheme $r(z)$ we have the following smoothing property:

THEOREM 3.2. *Assume that* (1.2) *and* (1.3) *hold, and r(z) is accurate of order* $p \ge 1$ *and* $A(\theta)$*-stable with* $\theta \in (\delta, \pi/2]$, *and that* $r(\infty) = 0$. *Let* $\{k_j\}$ *satisfy* $ck_j \le k_{j+1} \le Ck_j$. *Then there is a constant* C *such that*

(3.9) $$\Big\| A \prod_{j=1}^{n} r(k_j A) v \Big\| \le C t_n^{-1} \|v\|, \quad \textit{for } t_n > 0.$$

PROOF. The case $n = 1$ follows the result in the constant time step case, see Hansbo [6]. Here we consider the case when $n \ge 2$. We shall show that, with $g_n(z) = t_n z \prod_{j=1}^{n} r(k_j z)$,

$$\|g_n(A)\| \le C, \quad \text{for } n \ge 2.$$

Since $r(\infty) = 0$, we have, see Thomée [12, Lemma 7.3],

(3.10) $$|r(z)| \le \frac{1}{1 + c|z|}, \quad \text{for } |\arg z| \le \psi,$$

which implies that $g_n(z)$ is bounded for $|\arg z| \le \psi$ and $g_n(\infty) = 0$. Thus there exists $R > 0$ such that $g_n(z)$ is bounded for $|z| \ge R$. Lemma 2.4 shows that

$$g_n(A) = \frac{1}{2\pi i} \int_{\gamma_\epsilon \cup \Gamma_\epsilon^R \cup \gamma^R} g_n(z) R(z;A)\, dz.$$

Noting that $g_n(z)$ is analytic for $|z| \geq R$, $\psi \leq |\arg z| \leq \pi$, and $g_n(z) = O(z)$ as $z \to 0$, $|\arg z| \leq \psi$, we may let $R \to \infty$ and $\epsilon \to 0$, so that

$$g_n(A) = \frac{1}{2\pi i} \int_\Gamma g_n(z) R(z;A)\, dz,$$

where $\Gamma = \{z : |\arg z| = \psi\}$ for some $\psi \in (\delta, \theta)$. We split the path of integration as $\Gamma = \Gamma_0^{R/t_n} \cup \Gamma_{R/t_n}$. By (2.8), we have

$$\Big\| \int_{\Gamma_0^{R/t_n}} g_n(z) R(z;A)\, dz \Big\| \leq C \int_0^{R/t_n} t_n \rho e^{-ct_n\rho} \frac{d\rho}{\rho} \leq C.$$

We now consider the integral over Γ_{R/t_n}. If $k_{max} \leq t_n/2$, then we have

$$t_n^2 = \sum_{l=1}^n k_l^2 + \sum_{l\neq j} k_l k_j \leq k_{max} t_n + \sum_{l\neq j} k_l k_j \leq t_n^2/2 + \sum_{l\neq j} k_l k_j,$$

so that $\sum_{l\neq j} k_l k_j \geq t_n^2/2$ and hence

$$\prod_{j=1}^n (1 + ck_j\rho) = 1 + c\Big(\sum_{j=1}^n k_j\Big)\rho + c\rho^2\Big(\sum_{l\neq j} k_l k_j\Big) + \cdots \geq ct_n^2\rho^2,$$

which implies that, by (3.10),

$$\begin{aligned} \Big\| \int_{\Gamma_{R/t_n}} g_n(z) R(z;A)\, dz \Big\| &\leq C \int_{R/t_n}^\infty \frac{t_n\rho}{\prod_{j=1}^n (1 + ck_j\rho)} \frac{d\rho}{\rho} \\ &\leq C \int_{R/t_n}^\infty \frac{t_n}{c\rho^2 t_n^2}\, d\rho \leq C. \end{aligned}$$

If $k_{max} \geq t_n/2$, then, assuming that $k_{max} = k_m$ for some m with $1 \leq m \leq n$, and since $n \geq 2$, we have

$$\Big\| \int_{\Gamma_{R/t_n}} g_n(z) R(z;A)\, dz \Big\| \leq C \int_{R/t_n}^\infty \frac{t_n}{(1 + ck_m\rho)^2}\, d\rho \leq C \int_{R/t_n}^\infty \frac{t_n}{(1 + ct_n\rho)^2}\, d\rho \leq C.$$

Together these estimates complete the proof. □

As in Yan [13] for the constant time step case, if $|r(\infty)| < 1$, using difference quotients in time rather than the elliptic operator A in (3.9), we have a following smoothing property:

THEOREM 3.3. *Let U^n be the solution of* (1.5), *respectively. Assume that* (1.2) *and* (1.3) *hold and that the discretization scheme is accurate of order $p \geq 1$, and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, and $|r(\infty)| < 1$. Assume that $\{\mathcal{T}\}$ is a family of increasing quasi-quasiuniform grids. Then there is a constant C such that*

$$\|\bar\partial U^n\| \leq Ct_n^{-1}\|v\|, \quad \text{for } t_n > 0. \tag{3.11}$$

PROOF. The case $n = 1$ follows from the constant time step case, see Yan [13]. We now consider the case when $n \geq 2$. We want to show that, with $\tilde{g}_n(z) = \prod_{j=1}^{n-1} r(k_j z)(r(k_n z) - 1)$,

$$\|\tilde{g}_n(A)\| \leq Cn^{-1}, \quad \text{for } n \geq 2.$$

Since $t_n \leq nk_n$ this implies $\|\tilde{g}_n(A)\| \leq Ck_n t_n^{-1}$.

Since $|r(\infty)| < 1$ we find that $\tilde{g}_n(\infty)$ exists, which implies that there is $\tilde{R} > 0$, such that for fixed n, $\tilde{g}_n(z)$ is bounded for $|z| \geq \tilde{R}$. Further, by (1.6), $\tilde{g}_n(z)$ is bounded for $|z| \geq \epsilon$, $|\arg z| \leq \psi$ with $\psi \in (\delta, \theta)$. Applying Lemma 2.4, we get

$$\tilde{g}_n(A) = \tilde{g}_n(\infty)I + \frac{1}{2\pi i}\int_{\gamma_\epsilon \cup \Gamma_\epsilon^{\tilde{R}} \cup \gamma^{\tilde{R}}} \tilde{g}_n(z)R(z;A)\,dz.$$

Since the integrand is bounded for $|z| \geq \tilde{R}$, we may let $\tilde{R}$ tend to ∞. Moreover, by (3.6), we have $\tilde{g}_n(z) = O(z)$ as $z \to 0$, so that we may let $\epsilon \to 0$. Thus

$$\tilde{g}_n(A) = \tilde{g}_n(\infty)I + \frac{1}{2\pi i}\int_\Gamma \tilde{g}_n(z)R(z;A)\,dz,$$

where $\Gamma = \{z : |\arg z| = \psi\}$ for some $\psi \in (\delta, \theta)$.

Clearly,

$$\|\tilde{g}_n(\infty)I\| \leq |r(\infty)^{n-1}(r(\infty) - 1)| \leq Ce^{-cn} \leq Cn^{-1}.$$

Since $|r(\infty)| < 1$, there exist $R > 0$ and $c > 0$ such that

$$|r(z)| \leq e^{-c}, \quad \text{for } |z| \geq R, \tag{3.12}$$

which shows that the integrand has no poles when $|z| \geq R/k_1$, $\delta \leq |\arg z| \leq \pi$. In fact, using also (3.3), we have

(3.13)

$$\|\tilde{g}_n(z)R(z;A)\| \leq Ce^{-c(n-1)}(e^{-c}+1) \leq Ce^{-cn}, \quad \text{for } |z| \geq R/k_1,\ \delta \leq |\arg z| \leq \pi.$$

Thus we can replace the path of the integration by $\tilde{\Gamma} = \Gamma_0^{R/k_n} \cup \Gamma_{R/k_n}^{R/k_1} \cup \gamma^{R/k_1}$. We have, since $|\tilde{g}_n(z)| \leq Ce^{-cn}$ for $|z| \geq R/k_1$,

$$\left\| \int_{\gamma^{R/k_1}} \tilde{g}_n(z)R(z;A)\,dz \right\| \leq C\int_{\gamma^{R/k_1}} e^{-cn}\frac{|dz|}{|z|} \leq Cn^{-1}.$$

By (1.6) and (3.6), we know that, for arbitrary R,

$$|r(z) - 1| \leq C|z|,\ |r(z) - 1 - z| \leq C|z|^2, \quad \text{for } |z| \leq R,\ |\arg z| \leq \psi. \tag{3.14}$$

Using this, Lemma 2.2 and $t_{n-1}/k_n = (t_{n-1}/k_{n-1})(k_{n-1}/k_n) \geq C(t_{n-1}/k_{n-1}) \geq C(n-1)$, we have

$$\begin{aligned}\left\| \int_{\Gamma_0^{R/k_n}} \tilde{g}_n(z)R(z;A)\,dz \right\| &\leq C\int_0^{R/k_n} e^{-ct_{n-1}\rho}(k_n\rho)\frac{d\rho}{\rho} \\ &\leq C\int_0^R e^{-c(t_{n-1}/k_n)x}\,dx \leq C\int_0^R e^{-cnx}\,dx \leq Cn^{-1}.\end{aligned}$$

Finally, we write

$$\int_{\Gamma^{R/k_1}_{R/k_n}} \tilde g_n(z)R(z;A)\,dz = \Big(\int_{\Gamma^{R/k_{n-1}}_{R/k_n}} + \int_{\Gamma^{R/k_1}_{R/k_{n-1}}}\Big)\tilde g_n(z)R(z;A)\,dz = I + II.$$

If $n = 2$, we have, by Lemma 2.2 and $\ln(k_2/k_1) \le C$,

$$\Big\| \int_{\Gamma^{R/k_1}_{R/k_2}} \tilde g_n(z)R(z;A)\,dz\Big\| \le \int_{R/k_2}^{R/k_1} e^{-ck_1\rho}\,\frac{d\rho}{\rho} \le C \le Cn^{-1}.$$

If $n \ge 3$, using Lemma 2.2 and (1.6) and (2.11) with $m = n-1$, we obtain, for $z \in \Gamma^{R/k_m}_{R/k_{m+1}}$, $1 \le m \le n-2$,

$$|\tilde g_n(z)| \le C\Big|\prod_{j=1}^{n-1} r(k_jz)\Big| \le Ce^{-ct_{n-1}|z|} \le Ce^{-c_3(n-1)},$$

which implies that, since $k_{n-1} \sim k_n$,

$$\|I\| \le C\int_{R/k_n}^{R/k_{n-1}} e^{-c_3(n-1)}\,\frac{d\rho}{\rho} \le Ce^{-cn}\ln(k_n/k_{n-1}) \le Ce^{-cn}.$$

Further, by (1.6) and (2.12),

$$|\tilde g_n(z)| \le \Big|\prod_{j=1}^{n-1} r(k_jz)\Big| \le Ce^{-c_2(n-1)}e^{-c_4m}, \quad \text{for } z \in \Gamma^{R/k_m}_{R/k_{m+1}},\ 1 \le m \le n-2,$$

which shows that, following the proof of (2.13),

$$\|II\| \le C\sum_{m=1}^{n-2}\int_{R/k_{m+1}}^{R/k_m} e^{-c_2(n-1)}e^{-c_4m}\,\frac{d\rho}{\rho} \le Ce^{-cn}.$$

We therefore obtain

$$\Big\| \int_{\Gamma^{R/k_1}_{R/k_n}} \tilde g_n(z)R(z;A)\,dz\Big\| \le Ce^{-cn} \le Cn^{-1}, \quad \text{for } n \ge 3. \tag{3.15}$$

Together these estimates complete the proof. □

Our next result is a nonsmooth data error estimate.

THEOREM 3.4. *Let U^n and $u(t_n)$ be the solutions of* (1.5) *and* (1.1), *respectively. Assume that A satisfies* (1.2) *and* (1.3), *and that $r(z)$ is accurate of order $p \ge 1$ and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$ and $|r(\infty)| < 1$. Assume further that $\{\mathcal{T}\}$ is a family of increasing quasi-quasiuniform grids. Then there is a constant C such that*

$$\|\bar\partial U^n - D_t u(t_n)\| \le Ck_nt_n^{-2}\|v\|, \quad \text{for } t_n > 0.$$

PROOF. The case $n = 1$ follows from the constant time step case, see Yan [13]. Here we consider the case when $n \ge 2$.

With the notation of Theorem 3.1 and since $t_n \le nk_n$ we need to show

$$\|G_n(A)\| \le Cn^{-2}, \quad \text{for } n \ge 2.$$

We set, with $\kappa = r(\infty)$,

$$\tilde{G}_n(z) = G_n(z) - \kappa^{n-1}(\kappa - 1)k_n z/(1 + k_n z). \tag{3.16}$$

For the same reason as in the proof of Theorem 2.6, we have

$$\|\kappa^{n-1}(\kappa-1)k_nA(I+k_nA)^{-1}\| \le C|\kappa|^{n-1} \le Cn^{-2},$$

and

$$\tilde{G}_n(A) = \frac{1}{2\pi i}\int_\Gamma \tilde{G}_n(z)R(z;A)\,dz,$$

where $\Gamma = \{z : |\arg z| = \psi\}$ for some $\psi \in (\delta, \theta)$.

We write

$$\begin{aligned}\tilde{G}_n(z) = &\Big(\prod_{j=1}^{n-1} r(k_jz)(r(k_nz)-1) - \kappa^{n-1}(\kappa-1)\Big)\\ &+ \kappa^{n-1}(\kappa-1)/(1+k_nz) - (-k_nz)e^{-t_nz}.\end{aligned} \tag{3.17}$$

By Lemma 2.7 and $|1+k_nz| \ge |k_nz|$ for $\mathrm{Re}\, z \ge 0$, we have

$$\begin{aligned}\int_{R/k_1}^\infty |\tilde{G}_n(\rho e^{\pm i\psi})|\,\frac{d\rho}{\rho} &\le C\int_{R/k_1}^\infty \big(e^{-cn}(k_1\rho)^{-1} + |\kappa|^n(k_n\rho)^{-1} + (k_n\rho)e^{-ct_n\rho}\big)\,\frac{d\rho}{\rho}\\ &\le Cn^{-2}.\end{aligned}$$

Using $|1/(1+k_nz)| \le 1$ for $\mathrm{Re}\, z \ge 0$, we have, by (3.17), with $G_n^l(z), l = 1,2,3$, as in the proof of Theorem 3.1,

$$\begin{aligned}\int_0^{R/k_n} |\tilde{G}_n(\rho e^{\pm i\psi})|\,\frac{d\rho}{\rho} &\le \int_0^{R/k_n} |G_n(\rho e^{\pm i\psi})|\,\frac{d\rho}{\rho} + C\int_0^{R/k_n} |\kappa|^{n-1}k_n d\rho\\ &\le \sum_{l=1}^3 \int_0^{R/k_n} |G_n^l|\,\frac{d\rho}{\rho} + C|\kappa|^n.\end{aligned}$$

Obviously, we have, by Lemma 2.2 and (3.14) and $t_{n-1}/k_n \ge C(n-1)$,

$$\begin{aligned}\int_0^{R/k_n} \big(|G_n^1| + |G_n^2|\big)\,\frac{d\rho}{\rho} &\le C\int_0^{R/k_n} e^{-ct_{n-1}\rho}(k_n\rho)^2\,\frac{d\rho}{\rho}\\ &\le \int_0^R e^{-c(t_{n-1}/k_n)x}x\,dx \le \int_0^R e^{-c(n-1)x}x\,dx \le Cn^{-2},\end{aligned} \tag{3.18}$$

and, by (2.1) with $p = 1$ and $nk_n \sim t_n$,

$$\begin{aligned}\int_0^{R/k_n} |G_n^3|\,\frac{d\rho}{\rho} &\le C\int_0^{R/k_n} (k_n\rho)(k_n\rho)^2e^{-ct_n\rho}n\,\frac{d\rho}{\rho} = \int_0^R x^2e^{-c(t_n/k_n)x}n\,dx\\ &\le C\int_0^R x^2e^{-cnx}n\,dx \le Cn^{-2}.\end{aligned}$$

Thus, combining this with $|\kappa|^n \le Cn^{-2}$, we get

$$\int_0^{R/k_n} |\tilde{G}_n(\rho e^{\pm i\psi})| \frac{d\rho}{\rho} \le Cn^{-2}, \quad \text{for } n \ge 2.$$

It remains to consider the integral on interval $[R/k_n, R/k_1]$. We rewrite

$$\tilde{G}_n(z) = \prod_{j=1}^{n-1} r(k_j z)(r(k_n z - 1) - (-k_n z)e^{-t_n z} - \frac{k_n z}{1 + k_n z}(\kappa^n - \kappa^{n-1}).$$

We have, since $t_n \rho = (t_n/k_n)k_n\rho \ge Cn$ for $\rho \in [R/k_n, R/k_1]$,

(3.19)

$$\begin{aligned}\int_{R/k_n}^{R/k_1} e^{-t_n\rho e^{\pm i\psi}}(k_n\rho)\frac{d\rho}{\rho} &\le \int_{R/k_n}^{R/k_1} e^{-ct_n\rho}k_n d\rho \le e^{-cn}\int_{R/k_n}^{R/k_1} e^{-\frac{c}{2}t_n\rho}t_n d\rho \\ &\le e^{-cn}\int_0^\infty e^{-\frac{c}{2}x}\,dx \le Ce^{-cn} \le Cn^{-2},\end{aligned}$$

and, since $\ln(k_n/k_1) \le Cn$,

$$\int_{R/k_n}^{R/k_1} \frac{k_n\rho}{1 + k_n\rho}(\kappa^n - \kappa^{n-1})\frac{d\rho}{\rho} \le C\kappa^n \ln(k_n/k_1) \le Cn^{-2}. \tag{3.20}$$

Combining this with (3.15) shows

$$\int_{R/k_n}^{R/k_1} |\tilde{G}_n(\rho e^{\pm i\psi})| \frac{d\rho}{\rho} \le Cn^{-2}, \quad \text{for } n \ge 2.$$

The proof is complete. □

4 Approximation of Time Derivative — Second Order

In this section we shall consider the following second order approximation of $u_t(t_n)$,

$$\begin{aligned}\bar{\partial}_2 U^n &= a_n\bar{\partial}U^n + b_n\bar{\partial}U^{n-1} \\ &= a_n(U^n - U^{n-1})/k_n + b_n(U^{n-1} - U^{n-2})/k_{n-1},\end{aligned} \tag{4.1}$$

where

$$a_n = (2k_n + k_{n-1})/(k_n + k_{n-1}), \quad b_n = -k_n/(k_n + k_{n-1}),$$

and U^n is the discrete solution of (1.1) defined by (1.5). Combining (4.1) and Theorem 3.3, we obtain the following smoothing property.

THEOREM 4.1. *Let U^n be the solution of* (1.5). *Assume that* (1.2) *and* (1.3) *hold and that the discretization scheme is accurate of order $p \ge 1$, and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$, and $|r(\infty)| < 1$. Assume that $\{\mathcal{T}\}$ is a family of increasing quasi-quasiuniform grids. Then there is a constant C such that*

$$\|\bar{\partial}_2 U^n\| \le Ct_n^{-1}\|v\|, \quad \textit{for } n \ge 2.$$

Note that (4.1) can also be written in the form

$$\bar{\partial}_2 U^n = k_n^{-1}\big(c_0 U^n + c_1 U^{n-1} + c_2 U^{n-2}\big), \quad \text{for } n \geq 2, \tag{4.2}$$

where $c_1 = 1 + \gamma_n, c_2 = \gamma_n^2/(1+\gamma_n)$, $c_0 = c_1 + c_2$ and $\gamma_n = k_n/k_{n-1}$.

We shall now consider error estimates for the approximation (4.1). We begin with a smooth data error estimate.

THEOREM 4.2. *Let U^n and $u(t_n)$be the solutions of* (1.5) *and* (1.1), *respectively. Assume that A satisfies* (1.2) *and* (1.3), *and that $r(z)$ is accurate of order $p \geq 2$, and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$. Let $k_j, 1 \leq j \leq n$, be increasing. Then we have*

$$\|\bar{\partial}_2 U^n - D_t u(t_n)\| \leq C k_n^2 \|A^3 v\|, \quad \textit{for } n \geq 2. \tag{4.3}$$

PROOF. With $P(x,y) = c_0 + c_1 y^{-1} + c_2 x^{-1} y^{-1}$ and

$$G_n(z) = \prod_{j=1}^n r(k_j z) P(r(k_{n-1}z), r(k_n z)) - (-k_n z) e^{-t_n z}, \quad \text{for } n \geq 2,$$

we want to prove

$$\|G_n(A)(k_n A)^{-3}\| \leq C, \quad \text{for } n \geq 2.$$

Let $\bar{r}(z) = \big(\prod_{j=1}^n r(k_j z) P(r(k_{n-1}z), r(k_n z))\big)(k_n z)^{-3}$. As in the proof of Theorem 2.1, applying Lemma 2.4 to the rational function $\bar{r}(z)$ and using also Lemma 2.5 we conclude

$$G_n(A)(k_n A)^{-3} = \frac{1}{2\pi i} \int_{\gamma_\epsilon \cup \Gamma_\epsilon} G_n(z)(k_n z)^{-3} R(z;A)\, dz.$$

We now show that

$$G_n(z) = O(z^3), \quad \text{as } z \to 0. \tag{4.4}$$

In fact, we write

$$G_n(z) = G_n^1(z) + G_n^2(z) + G_n^3(z), \quad \text{for } n \geq 2, \tag{4.5}$$

where

$$G_n^1(z) = \prod_{j=1}^n r(k_j z)\Big(P(r(k_{n-1}z), r(k_n z)) - P(e^{-k_{n-1}z}, e^{-k_n z})\Big),$$

and, with $F_n(z) = \prod_{j=1}^n r(k_j z) - e^{-t_n z}$,

$$G_n^2(z) = \prod_{j=1}^n r(k_j z) P(e^{-k_{n-1}z}, e^{-k_n z}) - (-k_n z), \quad G_n^3 = k_n z F_n(z).$$

It is easy to see that there exists a small $\eta > 0$ such that

$$|r(k_j z)| \le C, \quad \text{for } 1 \le j \le n,\ |k_n z| \le \eta, \tag{4.6}$$

and

$$|P(e^{-k_{n-1}z}, e^{-k_n z}) - (-k_n z)| \le C|k_n z|^3, \quad \text{for } |k_n z| \le \eta, \tag{4.7}$$

and

$$|P(r(k_{n-1}z), r(k_n z)) - P(e^{-k_{n-1}z}, e^{-k_n z})| \le C|k_n z|^3, \quad \text{for } |k_n z| \le \eta. \tag{4.8}$$

Combining this with (2.2) shows

$$|G_n(z)| \le C|k_n z|^3, \quad \text{for } |k_n z| \le \eta, \tag{4.9}$$

which is (4.5). We remark that we can not extend (4.8) and (4.9) to $|k_n z| \le R,\ |\arg z| \le \psi$ for arbitrary R and $\psi \in (\delta, \theta)$, since $P(x, y)$ is not a polynomial for variables x, y. Combining (3.3) with (4.10) shows that the integrand is bounded on the small domain with boundary $\gamma_\epsilon \cup \Gamma_0^\epsilon$, so that we may let $\epsilon \to 0$. It follows that, with $\Gamma = \{z : |\arg z| = \psi\}$ for some $\psi \in (\delta, \theta)$,

$$G_n(A)(k_n A)^{-3} = \frac{1}{2\pi i}\int_\Gamma G_n(z)(k_n z)^{-3} R(z; A)\, dz.$$

The remainder of the proof is similar to the proof of Theorem 3.1. The proof is complete. □

We close this section with an error estimate in the nonsmooth data case.

THEOREM 4.3. *Let U^n and $u(t_n)$be the solutions of* (1.5) *and* (1.1), *respectively. Assume that A satisfies* (1.2) *and* (1.3), *and that $r(z)$ is accurate of order $p \ge 2$ and $A(\theta)$-stable with $\theta \in (\delta, \pi/2]$ and $|r(\infty)| < 1$. Assume further that $\{\mathcal{T}\}$ is a family of increasing quasi-quasiuniform grids. Then there is a constant C such that*

$$\|\bar\partial_2 U^n - D_t u(t_n)\| \le C k_n^2 t_n^{-3} \|v\|, \quad \textit{for} \quad n \ge 2. \tag{4.10}$$

PROOF. With the notation of Theorem 4.2 we need to show

$$\|G_n(A)\| \le C n^{-3}, \quad \text{for } n \ge 2.$$

Following the argument in the proof of Theorem 3.4, we set, with $\kappa = r(\infty)$,

$$\tilde G_n(z) = G_n(z) - \kappa^n P(\kappa, \kappa) k_n z/(1 + k_n z), \tag{4.11}$$

and we have

$$\|\kappa^n P(\kappa, \kappa) k_n A (I + k_n A)^{-1}\| \le C|\kappa|^n \le C n^{-3},$$

and

$$\tilde G_n(A) = \frac{1}{2\pi i}\int_\Gamma \tilde G_n(z) R(z; A)\, dz.$$

We write

$$\tilde{G}_n(z) = \Big(\prod_{j=1}^{n} r(k_j z) P(r(k_{n-1}z), r(k_n z)) - \kappa^n P(\kappa,\kappa) \Big) + \kappa^n P(\kappa,\kappa)/(1+k_n z) - (-k_n z)e^{-t_n z}, \quad \text{for } n \ge 2. \tag{4.12}$$

By Lemma 2.7 and $|1+k_n z| \ge |k_n z|$ for $\mathrm{Re}\, z \ge 0$, we have, with η as in the proof of Theorem 4.2,

$$\int_{\eta/k_1}^{\infty} |\tilde{G}_n(\rho e^{\pm i\psi})| \, \frac{d\rho}{\rho} \le C \int_{\eta/k_1}^{\infty} \Big(e^{-cn}(k_1\rho)^{-1} + |\kappa|^n (k_n\rho)^{-1} + (k_n\rho)e^{-t_n\rho} \Big) \frac{d\rho}{\rho} \le Cn^{-3}.$$

Using $|1/(1+k_n z)| \le 1$ for $\mathrm{Re}\, z \ge 0$, we have, by (4.13), with $G_n^l(z), l = 1,2,3,$ as in the proof of Theorem 4.2,

$$\int_0^{\eta/k_n} |\tilde{G}_n(\rho e^{\pm i\psi})| \, \frac{d\rho}{\rho} \le \sum_{l=1}^{3} \int_0^{\eta/k_n} |G_n^l| \, \frac{d\rho}{\rho} + C|\kappa|^n.$$

Obviously, using Lemma 2.2, (4.7), (4.8) and (4.9), we have

$$\int_0^{\eta/k_n} (|G_n^1| + |G_n^2|) \, \frac{d\rho}{\rho} \le Cn^{-3},$$

and, by (2.1) with $p = 2$ and $nk_n \sim t_n$,

$$\int_0^{\eta/k_n} |G_n^3| \, \frac{d\rho}{\rho} \le C \int_0^{\eta/k_n} (k_n\rho)(k_n\rho)^3 e^{-ct_n\rho} n \, \frac{d\rho}{\rho} \le Cn^{-3}.$$

Thus, combining this with $|\kappa|^n \le Cn^{-3}$, we get

$$\int_0^{\eta/k_n} |\tilde{G}_n(\rho e^{\pm i\psi})| \, \frac{d\rho}{\rho} \le Cn^{-3}, \quad \text{for } n \ge 2.$$

It remains to consider the integral on the interval $[\eta/k_n, \eta/k_1]$ for $n \ge 2$. If $n = 2$, we write, by (4.3),

$$\tilde{G}_2(z) = \Big(c_0 r(k_1 z) r(k_2 z) + c_1 r(k_1 z) + c_2 \Big) - (-k_2 z)e^{-t_2 z} - \frac{k_2 z}{1+k_2 z}(c_0\kappa^2 + c_1\kappa + c_2) = I + II + III,$$

where the integrals related to II and III can be bounded by (3.20) and (3.21), respectively. For I, we have

$$\Big\| \int_{\Gamma_{\eta/k_2}^{\eta/k_1}} \Big(c_0 r(k_1 z) r(k_2 z) + c_1 r(k_1 z) + c_2 \Big) R(z;A) \, dz \Big\| \le C \int_{\eta/k_2}^{\eta/k_1} \frac{d\rho}{\rho} \le C \ln(k_2/k_1) \le C \le Cn^{-3}.$$

If $n \geq 3$, we write, by (4.3),

$$\tilde{G}_n(z) = \Big(c_0 \prod_{j=1}^{n} r(k_j z) + c_1 \prod_{j=1}^{n-1} r(k_j z) + c_2 \prod_{j=1}^{n-2} r(k_j z)\Big) - (-k_n z)e^{-t_n z} - \frac{k_n z}{1 + k_n z}(c_0 \kappa^n + c_1 \kappa^{n-1} + c_2 \kappa^{n-2}) = I + II + III.$$

We can consider the case for $n = 3$ as for $n = 2$. If $n \geq 4$, the integrals related to II and III can be bounded by (3.20) and (3.21), respectively. Following the argument in the proof of (3.15), we have

$$\int_{\eta/k_n}^{\eta/k_1} \Big| \prod_{j=1}^{n-2} r(k_j \rho e^{\pm i\psi}) \Big| \frac{d\rho}{\rho} \leq C e^{-cn} \leq C n^{-3}.$$

Using this and the boundedness of $r(k_j z)$ on Γ we obtain the desired bound for the integral related to I.

Together these estimates complete the proof. □

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Paper III

Smoothing properties and approximation of time derivatives in multistep backward difference methods for linear parabolic equations

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Abstract.

In this paper we consider smoothing properties and time derivative approximation in multistep backward difference methods for nonhomogeneous parabolic equations. Smoothing properties and time derivative approximations in single step methods for homogeneous parabolic equations have been studied in Hansbo [5], Yan [12], [13]. We extend the similar results in Yan [12] to the multistep backward difference methods.

AMS subject classification: 65M15, 65N30, 65F10.

Key words: parabolic equations, time derivative, multistep difference method, error estimates.

1 Introduction

In this paper we shall consider the smoothing properties and the approximation of time derivatives in multistep backward difference methods for the following nonhomogeneous linear parabolic equation

$$u_t + Au = f, \quad \text{for } t > 0, \quad \text{with } u(0) = v, \tag{1.1}$$

in a Hilbert space H with norm $\|\cdot\|$, where $u_t = du/dt$ and A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, where $v \in H$ and f is a function of t with values in H.

The theory of stability and error estimates for the approximation of the solution of (1.1) by a multistep method have been well developed, see Becker [1], Bramble, Pasciak, Sammon, and Thomée [2], Crouzeix [3], Hansbo [6], LeRoux [7], [8], Palencia and Garcia-Archilla [9], Savaré [10], Thomée [11], and the references there in. The smoothing properties and the approximation of time derivatives in single step methods for homogeneous parabolic problems have been studied by Hansbo [5], [6], Yan [12], [13].

This paper is related to Yan [12]. Let us first recall the main results in Yan [12]. Consider (1.1) with $f = 0$, i.e.,

$$u_t + Au = 0 \quad \text{for } t > 0, \quad \text{with } u(0) = v. \tag{1.2}$$

Let $U^n, n \geq 1$, be an approximation of the solution $u(t_n)$ of (1.2) at time $t_n = nk$, where k is the time step, defined by a single step method,

$$U^n = r(kA)U^{n-1}, \quad \text{for } n \geq 1, \quad \text{with } U^0 = v, \tag{1.3}$$

where the rational function $r(\lambda)$ is accurate of order $p \geq 1$, i.e.,

$$r(\lambda) - e^{-\lambda} = O(\lambda^{p+1}), \quad \text{as } \lambda \to 0. \tag{1.4}$$

Let $j \geq 1$. Define the following finite difference quotient, with some nonnegative integers m_1, m_2 and real numbers c_ν,

$$Q_k^j U^n = \frac{1}{k^j} \sum_{\nu=-m_1}^{m_2} c_\nu U^{n+\nu}, \quad \text{for } n \geq m_1. \tag{1.5}$$

Assume that Q_k^j is an approximation of order $p \geq 1$ to the time derivative D_t^j, that is, for any smooth real-valued function u,

$$D_t^j u(t_n) = Q_k^j u^n + O(k^p), \quad \text{as } k \to 0, \quad \text{with } u^n = u(t_n). \tag{1.6}$$

We then have the following smooth data error estimates

$$\|Q_k^j U^n - D_t^j u(t_n)\| \leq Ck^p \|A^{p+j} v\|, \quad \text{for } n \geq m_1,\ v \in \mathcal{D}(A^{p+j}). \tag{1.7}$$

Further, if $|r(\infty)| < 1$, then we have the following smoothing properties

$$\|Q_k^j U^n\| \leq C t_n^{-j} \|v\|, \quad \text{for } n \geq m_1,\ t_n > 0,\ v \in H, \tag{1.8}$$

and nonsmooth data error estimates

$$\|Q_k^j U^n - D_t^j u(t_n)\| \leq C k^p t_n^{-(p+j)} \|v\|, \quad \text{for } n \geq m_1,\ t_n > 0,\ v \in H. \tag{1.9}$$

The purpose of this paper is to extend the above results for homogeneous parabolic equation, which is approximated by a single step method, to the nonhomogeneous parabolic equation, which will be approximated by a multistep backward difference method.

We introduce the backward difference operator $\bar{\partial}_p$, $p \geq 1$, by

$$\bar{\partial}_p U^n = \sum_{j=1}^{p} \frac{k^{j-1}}{j} \bar{\partial}^j U^n, \quad \text{where } \bar{\partial} U^n = (U^n - U^{n-1})/k. \tag{1.10}$$

With $U^0, \cdots, U^{p-1}$ given, we define our approximate solution U^n by

$$\bar{\partial}_p U^n + AU^n = f^n, \quad \text{for } n \geq p, \quad \text{where } f^n = f(t_n). \tag{1.11}$$

It is well known from the theory for numerical solution of ordinary differential equations, see, e.g., Hairer and Wanner [4], that this method is $A(\theta)$-stable for some $\theta = \theta_p > 0$ when $p \leq 6$. The error estimates for such method has been studied in Bramble, Pasciak, Sammon, and Thomée [2]. It is easy to see that, for any smooth real-valued function u, see Thomée [11, Chapter 10],

$$u_t(t_n) = \bar{\partial}_p u^n + O(k^p), \quad \text{as } k \to 0, \quad \text{with } u^n = u(t_n). \tag{1.12}$$

In Theorem 2.1 below, we obtain the following smoothing property: if U^n is the solution of (1.11) with $f = 0$, then we have, with $p \le 6$,

$$\|\bar{\partial}_p U^n\| \le C t_n^{-1} \sum_{j=0}^{p-1} \|U^j\|, \quad \text{for } n \ge 2p.$$

It is natural to approximate the time derivative $u_t(t_n)$ of the solution of (1.1) by $\bar{\partial}_p U^n$ ($n \ge 2p$), where U^n, $n \ge p$, is computed by the multistep backward difference method (1.11). In Theorems 3.1 and 3.4, we obtain the following error estimates

$$\|\bar{\partial}_p U^n - u_t(t_n)\| \le C \sum_{j=0}^{p-1} \|A(U^j - u^j)\| + Ck^p \int_0^{t_n} \|Au^{(p+1)}(s)\|\, ds, \quad \text{for } n \ge 2p,$$

and, with $G(s) = |u^{(p+1)}(s)|_{-2p-1}^2 + s^{2p+2}|u^{(p+1)}(s)|_1^2 + s^2|u_t(s)|_1^2$,

$$t_n^{2p+2}\|\bar{\partial}_p U^n - u_t(t_n)\|^2 \le C \sum_{j=p}^{2p-1} \Big(|U^j - u^j|_{-2p}^2 + k^{2p+2}\|A(U^j - u^j)\|^2\Big) + Ck^{2p}\Big(\int_0^{t_n} G(s)\, ds + t_{2p}^3 |u_t(t_{2p})|_1^2\Big),$$

respectively.

When we choose some suitable discrete starting values $U^0, U^1, \cdots, U^{p-1}$, we get the following nonsmooth data error estimates, with $f = 0$ and $p \le 6$,

$$\|\bar{\partial}_p U^n - u_t(t_n)\| \le Ck^p t_n^{-p-1}\|v\|, \quad \text{for } n \ge 2p.$$

By C and c we denote large and small positive constants independent of the functions and parameters concerned, but not necessarily the same at different occurrences. When necessary for clarity we distinguish constants by subscripts.

2 Smoothing properties

In this section we will show the smoothing properties for the multistep backward difference method. Before showing this, we first discuss some properties of the backward difference operator $\bar{\partial}_p$ defined by (1.10). We first note that (1.10) can be written in another form, see, e.g., Yan [12],

$$\bar{\partial}_p U^n = k^{-1} \sum_{\nu=0}^{p} c_\nu U^{n-\nu}, \tag{2.1}$$

where the coefficients c_ν are independent of k. Introducing $P(x) = \sum_{\nu=0}^{p} c_\nu x^\nu$, it is easy to check that (1.12) is equivalent to

$$P(e^{-\lambda}) - \lambda = O(\lambda^{p+1}), \quad \text{as } \lambda \to 0. \tag{2.2}$$

In fact, with $u(t) = e^t$ in (1.12), we have

$$P(e^{-k}) - k = O(k^{p+1}), \quad \text{as } k \to 0,$$

replacing k by λ, we show (2.2). On the other hand, if (2.2) holds, (1.12) follows from Taylor expansion of $\bar{\partial}_p u^n$ at t_n.

For $p = 1$, (1.11) reduces to the backward Euler method

$$(U^n - U^{n-1})/k + AU^n = f^n, \quad \text{for } n \geq 1,$$

and the starting value is $U^0 = v$.

For $p = 2$, we have

$$(\tfrac{3}{2}U^n - 2U^{n-1} + \tfrac{1}{2}U^{n-2})/k + AU^n = f^n, \quad \text{for } n \geq 2,$$

and both U^0 and U^1 are needed to start the procedure.

Bramble, Pasciak, Sammon, and Thomée [2] obtain the following stability result, i.e., with U^n the solution of (1.11),

$$\|U^n\| \leq C \sum_{j=0}^{p-1} \|U^j\| + Ck \sum_{j=p}^{n} \|f^j\|, \quad \text{for } n \geq p. \tag{2.3}$$

In this paper we first show the following smoothing property for the multistep backward difference method.

THEOREM 2.1. *Let $p \leq 6$. Then there is a constant C, independent of the positive definite operator A, such that for the solution U^n of (1.11) with $f = 0$,*

$$\|\bar{\partial}_p U^n\| \leq C t_n^{-1} \sum_{j=0}^{p-1} \|U^j\|, \quad \text{for } n \geq 2p. \tag{2.4}$$

To prove this theorem, we need the following lemma from Thomée [11, Lemma 10.3].

LEMMA 2.2. *The solution of (1.11) may be written, with $g^j = kf^j = kf(t_j)$,*

$$U^n = \sum_{j=p}^{n} \beta_{n-j}(kA) g^j + \sum_{s=0}^{p-1} \beta_{ns}(kA) U^s, \quad \text{for } n \geq p, \tag{2.5}$$

where the $\beta_j(\lambda)$ and $\beta_{ns}(\lambda)$ are defined by, with $\lambda > 0$, $P(\zeta) = \sum_{\nu=0}^{p} c_\nu \zeta^\nu$,

$$\sum_{j=0}^{\infty} \beta_j(\lambda)\zeta^j := (P(\zeta) + \lambda)^{-1}, \qquad \beta_{ns}(\lambda) = \sum_{j=p-s}^{p} \beta_{n-s-j}(\lambda) c_j. \tag{2.6}$$

If $p \le 6$, there are positive constants c, C and λ_0 such that

$$|\beta_j(\lambda)| \le \begin{cases} Ce^{-cj\lambda}, & \textit{for } 0 < \lambda \le \lambda_0, \\ C\lambda^{-1}e^{-cj}, & \textit{for } \lambda \ge \lambda_0. \end{cases} \tag{2.7}$$

Proof of Theorem 2.1. By (2.5) and (2.1), we find that

$$\bar{\partial}_p U^n = k^{-1} \sum_{\nu=0}^{p} c_\nu \sum_{s=0}^{p-1} \beta_{(n-\nu)s}(kA)U^s \equiv \sum_{s=0}^{p-1} \beta'_{ns}(kA)U^s,$$

where obviously we require that $n - \nu \ge p$ $(0 \le \nu \le p)$ which implies $n \ge 2p$, and where $\beta'_{ns}(\lambda)$ are some functions of λ. Since $\bar{\partial}_p U^n$ is linearly dependent on U^s $(0 \le s \le p-1)$, it suffices to consider separately the cases when all terms but one on the right of (2.4) vanish.

We consider the case when $U^l \ne 0$, $0 \le l \le p-1$ and $U^s = 0$, $0 \le s \le p-1$, $s \ne l$.

In the case $0 < l \le p-1$, we need to show

$$\|\bar{\partial}_p U^n\| \le Ct_n^{-1}\|U^l\|. \tag{2.8}$$

By Lemma 2.2, we have

$$\begin{aligned} \bar{\partial}_p U^n &= k^{-1} \sum_{\nu=0}^{p} c_\nu \big(\beta_{(n-\nu)l} U^l\big) = k^{-1} \sum_{\nu=0}^{p} c_\nu \Big(\sum_{j=p-l}^{p} \beta_{n-\nu-l-j}(kA)c_j \Big) U^l \\ &= k^{-1} \sum_{j=p-l}^{p} \Big(\sum_{\nu=0}^{p} c_\nu \beta_{n-\nu-l-j}(kA) \Big) c_j U^l, \quad \text{for } 0 \le l \le p-1. \end{aligned}$$

We also note that

$$\sum_{\nu=0}^{p} c_\nu \beta_{n-\nu-s}(\lambda) = -\lambda \beta_{n-s}(\lambda), \quad \text{for } p < s \le n,\ n - \nu - s \ge 0. \tag{2.9}$$

In fact, if $n - s < p$, (2.9) follows from comparing the coefficients of $\zeta^{\bar{s}}$ of (2.6) for $0 \le \bar{s} \le p$. If $n - s \ge p$, by comparing the coefficients of $\zeta^{\bar{s}}$ of (2.6) for $\bar{s} \ge p$, we get

$$(c_0 + \lambda)\beta_{\bar{s}} + \cdots + c_p \beta_{\bar{s}-p} = 0.$$

Replacing $\bar{s}$ by $n - s$ $(n \ge 2p,\ n - s \ge p)$, we get (2.9).

Thus (2.8) follows from

$$\Big| n\lambda \sum_{j=p-l}^{p} \beta_{n-l-j}(\lambda) \Big| \le C, \quad \text{for } 0 < l \le p-1,$$

which follows from, for fixed l, $0 < l \le p-1$,

$$\Big| n\lambda \sum_{j=p-l}^{p} \beta_{n-l-j}(\lambda) \Big| \le C \sum_{j=p-l}^{p} n\lambda e^{-c(n-l-j)\lambda} \le C, \quad \text{for } 0 \le \lambda \le \lambda_0,$$

and

$$\Big| n\lambda \sum_{j=p-l}^{p} \beta_{n-l-j}(\lambda)\Big| \le C \sum_{j=p-l}^{p} ne^{-c(n-l-j)} \le C, \quad \text{for } \lambda \ge \lambda_0.$$

We now consider the case $l = 0$, we have, by Lemma 2.2,

$$\bar{\partial}_p U^n = k^{-1} \sum_{\nu=0}^{p} c_\nu (\beta_{(n-\nu)0} U^0) = k^{-1} \Big(\sum_{\nu=0}^{p} c_\nu \beta_{n-\nu-p}(kA) \Big) c_p U^0.$$

We will show

$$\frac{n}{s} \Big| \sum_{\nu=0}^{p} c_\nu \beta_{n-\nu-s}(\lambda) \Big| \le C, \quad \text{for } \lambda \in \sigma(kA),\ n \ge 2p. \tag{2.10}$$

Assuming this in the moment, by spectral representation, the desired estimate $\|\bar{\partial}_p U^n\| \le C t_n^{-1} \|U^0\|$ follows.

It remains to prove (2.10). In fact, since (2.9), it suffices to show,

$$\frac{n}{s} |\lambda \beta_{n-s}(\lambda)| \le C, \quad \text{for } \lambda \in \sigma(kA), \quad n \ge 2p, \quad p < s \le n, \tag{2.11}$$

which we will now prove. For small $\lambda < \lambda_0$, we have, by (2.7),

$$\frac{n}{s} |\lambda \beta_{n-s}(\lambda)| \le (n\lambda e^{-cn\lambda})(s^{-1} e^{cs\lambda}) \le (n\lambda e^{-cn\lambda}) \max\{p^{-1} e^{cp\lambda}, n^{-1} e^{cn\lambda}\} \le C.$$

For $\lambda \ge \lambda_0$, using again (2.7), we have,

$$\frac{n}{s} |\lambda \beta_{n-s}(\lambda)| \le C(ne^{-cn})(s^{-1} e^{cs}) \le C,$$

which completes the proof of (2.11). Together these estimates complete the proof of Theorem 2.1. □

3 Error estimates

In this section, we will show the error estimates for the approximation $\bar{\partial}_p U^n$ of the time derivative $u_t(t_n)$ in both smooth and nonsmooth data cases. Recall that the error estimate for the approximation U^n of $u(t_n)$ in the smooth data case reads, see Thomée [11, Theorem 10.1],

$$\|U^n - u^n\| \le C \sum_{j=0}^{p-1} \|U^j - u^j\| + Ck^p \int_0^{t_n} \|u^{(p+1)}(s)\|\, ds, \quad \text{for } n \ge p. \tag{3.1}$$

Applying (3.1), we can easily prove the following smooth data error estimate for the time derivative approximation.

THEOREM 3.1. *Let $p \le 6$. Then there is a constant C, independent of the positive definite operator A, such that*

(3.2)

$$\|\bar{\partial}_p U^n - u_t(t_n)\| \le C \sum_{i=0}^{p-1} \|A(U^i - u^i)\| + Ck^p \int_0^{t_n} \|Au^{(p+1)}(s)\|\, ds, \quad \textit{for } n \ge 2p.$$

PROOF. By (1.11) and (1.1), we have

$$\|\bar{\partial}_p U^n - u_t(t_n)\| = \|A(U^n - u(t_n))\|.$$

Applying (3.1) with norm $\|A \cdot \|$, we obtain (3.2). The proof is complete. □

We now turn to nonsmooth data error estimate. Below we will use the norm $|v|_s = (A^s v, v)^{1/2}$, $s \in \mathbf{R}$, defined by

$$|v|_s^2 = \sum_{l=1}^{\infty} \mu_l^s (v, \varphi_l)^2 < \infty, \quad \text{for } s \in \mathbf{R},$$

where $\{\mu_l, \varphi_l\}_{l=1}^{\infty}$ is the eigensystem of the operator A.

We first recall the following stability result, see Thomée [11, Theorem 10.4].

LEMMA 3.2. *Let $p \le 6$ and $s \ge 0$, and let U^n be the solution of* (1.11). *Then we have, with C independent of the positive definite operator A,*

$$t_n^s \|U^n\|^2 + k \sum_{j=p}^{n} t_j^s |U^j|_1^2 \le C \sum_{j=0}^{p-1} (|U^j|_{-s}^2 + k^s \|U^j\|^2) + Ck \sum_{j=p}^{n} (|f^j|_{-s-1}^2 + t_j^s |f^j|_{-1}^2), \quad \textit{for } n \ge p.$$

We need the following generalization of Lemma 3.2.

LEMMA 3.3. *Let $p \le 6$ and $s \ge 0$, and let U^n be the solution of* (1.11). *Assume that $m \ge p$ and $U^{m-p}, \cdots, U^{m-1}$ are given. Then we have, with C independent of the positive definite operator A,*

$$t_n^s \|U^n\|^2 + k \sum_{j=m}^{n} t_j^s |U^j|_1^2 \le C \sum_{j=m-p}^{m-1} (|U^j|_{-s}^2 + k^s \|U^j\|^2) + Ck \sum_{j=m}^{n} (|f^j|_{-s-1}^2 + t_j^s |f^j|_{-1}^2), \quad \textit{for } n \ge m.$$

PROOF. We modify the proof of Lemma 3.2. By eigenfunction expansion, it suffices to show

(3.3)

$$n^s (U^n, \varphi_l)^2 + (k\mu_l) \sum_{j=m}^{n} j^s (U^j, \varphi_l)^2 \le C \sum_{j=m-p}^{m-1} \Big((k\mu_l)^{-s} + 1 \Big) (U^j, \varphi_l)^2 + C \sum_{j=m}^{n} \Big((k\mu_l)^{-s-1} + j^s (k\mu_l)^{-1} \Big) (kf^j, \varphi_l)^2, \quad \text{for } 1 \le l < \infty.$$

By (1.11), we find that, with $1 \le l < \infty$,

$$(c_0 + k\mu_l)(U^n, \varphi_l) + c_1 (U^{n-1}, \varphi_l) + \cdots + c_p (U^{n-p}, \varphi_l) = (kf^n, \varphi_l).$$

We now instead consider the equation, with $\lambda \in \sigma(kA)$, $W^n = W^n(\lambda)$,

$$(c_0 + \lambda)W^n + c_1 W^{n-1} + \cdots + c_p W^{n-p} = F^n, \quad \text{for } n \geq m, \tag{3.4}$$

where $W^{m-p}, \cdots, W^{m-1} \in \mathbf{R}$ are given and $F^l \in \mathbf{R}, (m \leq l \leq n)$ are arbitrary. We shall show

$$\begin{aligned} n^s (W^n)^2 + \lambda \sum_{j=m}^{n} j^s (W^j)^2 \leq C \sum_{j=m-p}^{m-1} (\lambda^{-s} + 1)(W^j)^2 \\ + C \sum_{j=m}^{n} (\lambda^{-s-1} + j^s \lambda^{-1})(F^j)^2. \end{aligned} \tag{3.5}$$

Assuming this and applying this to $W^n = (U^n, \varphi_l)$, $\lambda = k\mu_l$ and $F^n = (kf^n, \varphi_l)$, for fixed $l, 1 \leq l < \infty$, we complete the proof of (3.3).

We now turn to prove (3.5). By linearity it suffices to consider separately the case when $W^{m-l} = 0$, $1 \leq l \leq p$, and then the case when $F^l = 0$ for $l \geq m$.

By Lemma 2.2, we find that

$$n^s |\beta_n| + \lambda \sum_{j=0}^{\infty} j^s |\beta_j| \leq C(1 + \lambda^{-s}), \quad \text{for } n \geq 0. \tag{3.6}$$

In fact, by (2.7), we have, for $0 \leq \lambda \leq \lambda_0$,

$$n^s |\beta_n| \leq C n^s e^{-cn\lambda} \leq C \lambda^{-s},$$

and

$$\lambda \sum_{j=0}^{\infty} j^s |\beta_j| \leq C\lambda \sum_{j=0}^{\infty} j^s e^{-c\lambda j} = C\lambda^{1-s} \sum_{j=0}^{\infty} e^{-\frac{c}{2}\lambda j} \leq C\lambda^{-s}.$$

and for $\lambda \geq \lambda_0$, the left-hand side of (3.6), is less than $Cn^s e^{-cn} + C\sum_{j=0}^{\infty} j^s e^{-cj}$, which is bounded.

We also note that the solutions $W^n (n \geq m)$ of (3.4) satisfy, by (2.5),

$$W^m = \beta_0(\lambda) F^m + \sum_{s=0}^{p-1} \beta_{ps}(\lambda) W^{s+m-p},$$

$$W^{m+1} = \beta_0(\lambda) F^m + \beta_1(\lambda) F^{m+1} + \sum_{s=0}^{p-1} \beta_{(p+1)s}(\lambda) W^{s+m-p},$$

$$\vdots$$

$$W^n = \sum_{j=m}^{n} \beta_{n-j}(\lambda) F^j + \sum_{s=0}^{p-1} \beta_{(p+n-m)s}(\lambda) W^{s+m-p}, \quad n \geq m,$$

or, in general form,

$$W^n = \sum_{j=m}^{n} \beta_{n-j} F^j + \sum_{s=0}^{p-1} \beta_{(p+n-m)s} W^{s+m-p}, \quad \text{for } n \geq m. \tag{3.7}$$

After the above preparations, we now consider the proof of (3.5) in the case when $W^{m-p} = \cdots = W^{m-1} = 0$. We have, by (3.7),

$$W^n = \sum_{j=m}^{n} \beta_{n-j} F^j = \sum_{l=0}^{n-m} \beta_l F^{n-l}, \quad \text{for } n \geq m,$$

so that, using the Schwarz inequality,

$$n^s (W^n)^2 = n^s \Big(\sum_{l=0}^{n-m} \beta_l F^{n-l} \Big)^2 \leq n^s \Big(\sum_{l=0}^{n-m} |\beta_l| \Big) \sum_{l=0}^{n-m} |\beta_l| (F_{n-l})^2.$$

Hence, by (3.6), and noting that $n^s \leq C(l^s + (n-l)^s)$ and $1 \leq (n-l)^s$, we find

$$\begin{aligned} n^s (W^n)^2 &\leq C\lambda^{-1} \sum_{l=0}^{n-m} \Big(l^s |\beta_l| (F^{n-l})^2 + (n-l)^s |\beta_l| (F^{n-l})^2 \Big) \\ &\leq C\lambda^{-1} \sum_{l=0}^{n-m} (\lambda^{-s} + (n-l)^s)(F^{n-l})^2, \end{aligned} \tag{3.8}$$

which is the desired estimate for the first term of the left hand side in (3.5). For the second term in (3.5), we have, by (3.8),

$$\begin{aligned} \lambda \sum_{n=m}^{N} n^s (W^n)^2 &\leq \lambda \sum_{n=m}^{N} \Big(\lambda^{-1} \sum_{j=0}^{n-m} \big(j^s |\beta_j| (F^{n-j})^2 + |\beta_j| (n-j)^s (F^{n-j})^2 \big) \Big) \\ &\leq \sum_{n=m}^{N} \sum_{j=0}^{N-m} \Big(j^s |\beta_j| + n^s |\beta_j| \Big) (F^n)^2 \\ &\leq C\lambda^{-1} \sum_{n=m}^{N} (1 + \lambda^{-s})(F^n)^2 + \lambda^{-1} \sum_{n=m}^{N} n^s (F^n)^2 \\ &\leq C\lambda^{-1} \sum_{n=m}^{N} (\lambda^{-s} + n^s)(F^n)^2, \end{aligned}$$

which completes the proof in the present case.

We next consider the case when $F^j = 0,\ m \leq j \leq n$ and $W^{m-l} \neq 0,\ 1 \leq l \leq p,\ W^{m-\bar{l}} = 0,\ 1 \leq \bar{l} \leq p,\ \bar{l} \neq l$. We begin with the special case $l = p$. By (3.7) with $s = 0$, we have

$$W^n = \beta_{(p+n-m)0}(\lambda) W^{m-p} = \beta_{n-m}(\lambda) c_p W^{m-p},$$

so that, using (2.7) and $n^s \leq C\big((n-m)^s + m^s\big)$,

$$\begin{aligned} n^s (W^n)^2 &\leq C n^s \beta_{n-m}(\lambda)(W^{m-p})^2 \leq C\big(1 + (n-m)^s\big)\beta_{n-m}(\lambda)(W^{m-p})^2 \\ &\leq C(1 + \lambda^{-s})(W^{m-p})^2. \end{aligned}$$

From this we also obtain

$$\lambda \sum_{n=m}^{N} n^s (W^n)^2 \le C\lambda \sum_{n=m}^{N} \Big(1+(n-m)^s\Big)\beta_{n-m}^2(\lambda)(W^{m-p})^2$$
$$\le C(1+\lambda^{-s})(W^{m-p})^2.$$

For the general case $l \ne p$, we have, by (3.7) with $s = p - l$

$$W^n = \beta_{(p+n-m)(p-l)}(\lambda) W^{m-l} = \sum_{j=l}^{p} \beta_{n-m+l-j}(\lambda) c_j W^{m-l},$$

so that, using (2.7) and $n^s \le C\big((n-m+l-j)^s + (m-l+j)^s\big)$,

$$n^s (W^n)^2 \le C n^s \sum_{j=l}^{p} \beta_{n-m+l-j}(\lambda)(W^{m-l})^2$$
$$\le C \sum_{j=l}^{p} \big(1+(n-m+l-j)^s\big)\beta_{n-m+l-j}(\lambda)(W^{m-l})^2$$
$$\le C \sum_{j=l}^{p} (1+\lambda^{-s})(W^{m-l})^2.$$

From this we also obtain

$$\lambda \sum_{n=m}^{N} n^s (W^n)^2 \le C\lambda \sum_{n=m}^{N}\sum_{j=l}^{p} \big(1+(n-m+l-j)^s\big)\beta_{n-m+l-j}^2(\lambda)(W^{m-l})^2$$
$$\le C(1+\lambda^{-s})(W^{m-l})^2.$$

Together these estimates complete the proof. □

Now we are the position to state our error estimate.

THEOREM 3.4. *Let* $p \le 6$ *and let* U^n *and* u *be the solutions of* (1.11) *and* (1.1), *respectively. Then, with* $G(s) = |u^{(p+1)}(s)|_{-2p-1}^2 + s^{2p+2}|u^{(p+1)}(s)|_1^2 + s^2|u_t(s)|_1^2$,

$$t_n^{2p+2}\|\bar\partial_p U^n - u_t(t_n)\|^2 \le C \sum_{j=p}^{2p-1} \big(|U^j - u^j|_{-2p}^2 + k^{2p+2}\|A(U^j-u^j)\|^2\big)$$
$$+ Ck^{2p}\Big(\int_0^{t_n} G(s)\,ds + t_{2p}^3 |u_t(t_{2p})|_1^2\Big),$$

PROOF. The error $\varepsilon^n = \bar\partial_p U^n - u_t(t_n)$ $(n \ge p)$ satisfies

$$\bar\partial_p \varepsilon^n + A\varepsilon^n = -\tau^n, \quad \text{where } \tau^n = A\big(\bar\partial_p u(t_n) - u_t(t_n)\big), \quad \text{for } n \ge 2p.$$

Applying Lemma 3.3 with $s = 2p+2, m = 2p$, we have, for $n \geq 2p$,

$$t_n^{2p+2}\|\varepsilon^n\|^2 \leq C \sum_{j=p}^{2p-1} \big(|\varepsilon^j|_{-2p-2}^2 + k^{2p+2}\|\varepsilon^j\|^2\big) + Ck \sum_{j=2p}^{n} \big(|\tau^j|_{-2p-3}^2 + t_j^{2p+2}|\tau^j|_{-1}^2\big).$$

We now estimate the term $k\sum_{j=2p}^n |\tau^j|_{-2p-3}^2$. We will show that, with any norm $\|\cdot\|$ in H,

$$\|\bar\partial_p u(t_j) - u_t(t_j)\| \leq Ck^{p-1}\int_{t_{j-p}}^{t_j} \|u^{(p+1)}(s)\|\,ds, \quad \text{for } j \geq 2p. \tag{3.9}$$

Assuming this we have

$$|\tau^j|_{-2p-3}^2 \leq Ck^{2p-1}\int_{t_{j-p}}^{t_j} |u^{(p+1)}(s)|_{-2p-1}^2\,ds, \quad \text{for } j \geq 2p.$$

Thus

$$k\sum_{j=2p}^n |\tau^j|_{-2p-3}^2 \leq Ck^{2p}\sum_{j=2p}^n \int_{t_{j-p}}^{t_j} |u^{(p+1)}(s)|_{-2p-1}^2\,ds \leq Ck^{2p}\int_0^{t_n} |u^{(p+1)}(s)|_{-2p-1}^2\,ds.$$

It remains to estimate $k\sum_{j=2p}^n t_j^{2p+2}|\tau^j|_{-1}^2$. If $j \neq 2p$, we have, by (3.9) with norm $\|A^{1/2}\cdot\|$,

$$k\sum_{j=2p+1}^n t_j^{2p+2}|\tau^j|_{-1}^2 \leq Ck^{2p}\sum_{j=2p+1}^n t_j^{2p+2}\int_{t_{j-p}}^{t_j} |u^{(p+1)}(s)|_1^2\,ds.$$

Here we have $t_j \leq cs$ for $s \in [t_{j-p}, t_j]$, $j \geq 2p+1$ which follows from

$$t_j \leq s\frac{t_j}{t_{j-p}} \leq s\frac{t_{2p+1}}{t_{p+1}} \leq cs, \quad \text{for } j \geq 2p+1.$$

Hence

$$k\sum_{j=2p+1}^n t_j^{2p+2}|\tau^j|_{-1}^2 \leq Ck^{2p}\sum_{j=2p+1}^n \int_{t_{j-p}}^{t_j} s^{2p+2}|u^{(p+1)}(s)|_1^2\,ds.$$

For $j = 2p$, we write, since $\sum_{\nu=0}^p c_\nu = 0$,

$$\tau^{2p} = k^{-1}A\Big(\sum_{\nu=0}^p c_\nu u(t_{2p-\nu}) - u_t(t_{2p})\Big) = k^{-1}A\Big(\sum_{\nu=0}^p c_\nu \int_{t_p}^{t_{2p-\nu}} u_t(s)\,ds - u_t(t_{2p})\Big),$$

and we obtain

$$k|\tau^{2p}|_{-1}^2 \le C \int_{t_p}^{t_{2p}} |u_t(s)|_1^2 \, ds + k|u_t(t_{2p})|_1^2,$$

which follows from

$$\begin{aligned} |\tau^{2p}|_{-1}^2 &\le C\Big(k^{-2} \sum_{\nu=0}^{p} \Big| \int_{t_p}^{t_{2p-\nu}} u_t(s)\, ds \Big|_1^2 + |u_t(t_{2p})|_1^2\Big) \\ &\le Ck^{-2} \sum_{\nu=0}^{p} (pk) \int_{t_p}^{t_{2p-\nu}} |u_t(s)|_1^2 \, ds + |u_t(t_{2p})|_1^2 \\ &\le Ck^{-1} \int_{t_p}^{t_{2p}} |u_t(s)|_1^2 \, ds + |u_t(t_{2p})|_1^2. \end{aligned}$$

Thus, we get

$$\begin{aligned} kt_{2p}^{2p+2}|\tau^{2p}|_{-1}^2 &\le Ck^{2p+2}\Big(\int_{t_p}^{t_{2p}} |u_t(s)|_1^2 \, ds + k|u_t(t_{2p})|_1^2\Big) \\ &\le Ck^{2p}\Big(\int_{t_p}^{t_{2p}} s^2|u_t(s)|_1^2 \, ds + t_{2p}^3|u_t(t_{2p})|_1^2\Big). \end{aligned}$$

It remains to estimate (3.9). We write, by Taylor expansion around t_{j-p},

$$\begin{aligned} u(t) &= \sum_{l=0}^{p} \frac{u^{(l)}(t_{j-p})}{l!}(t - t_{j-p})^l + \frac{1}{p!}\int_{t_{j-p}}^{t} (t-s)^p u^{(p+1)}(s)\, ds \\ &\equiv Q(t) + R(t). \end{aligned}$$

By (1.12) and since $Q(t)$ is a polynomial of degree p, we have $\bar{\partial}_p Q(t) - Q_t(t) = 0$. Thus, by (2.1),

$$\bar{\partial}_p u(t_j) - u_t(t_j) = \bar{\partial}_p R(t_j) - R_t(t_j) = k^{-1} \sum_{\nu=0}^{p} c_\nu R(t_{j-\nu}) - R_t(t_j).$$

Noting that

$$\|R(t_{j-\nu})\| \le Ck^p \int_{t_{j-p}}^{t_j} \|u^{(p+1)}(s)\|\, ds, \quad \text{for } 0 \le \nu \le p, \quad j \ge 2p,$$

and

$$\begin{aligned} \|R_t(t_j)\| &= \frac{1}{(p-1)!}\Big\| \int_{t_{j-p}}^{t_j} (t_j - s)^{p-1} u^{(p+1)}(s)\, ds \Big\| \\ &\le Ck^{p-1} \int_{t_{j-p}}^{t_j} \|u^{(p+1)}(s)\|\, ds, \end{aligned}$$

we complete the proof of (3.9).

Together these estimates complete the proof. □

In the homogeneous case, i.e., $f = 0$, we have the following nonsmooth data error estimates.

THEOREM 3.5. *Let $p \le 6$ and let U^n and u be the solutions of* (1.11) *and* (1.1)*, respectively. Assume that $f = 0$ and the discrete initial values satisfy*

$$|U^j - u^j|_{-2p} + k^{p+1}\|A(U^j - u^j)\| \le Ck^p\|v\|, \quad \text{for } p \le j \le 2p-1. \tag{3.10}$$

Then, with C independent of the positive definite operator A,

$$\|\bar{\partial}_p U^n - u_t(t_n)\| \le Ck^p t_n^{-p-1}\|v\|, \quad \text{for } n \ge 2p.$$

PROOF. For the solution u of homogeneous parabolic equation, it is easy to show that

$$\int_0^{t_n} |u^{(p+1)}(s)|_{-2p-1}^2 ds \le C\|v\|^2, \quad \int_0^{t_n} s^{2p+2}|u^{(p+1)}(s)|_1^2 ds \le C\|v\|^2,$$

and $t_{2p}^3|u_t(t_{2p}|_1^2 \le C\|v\|^2$. Applying for Theorem 3.4, we complete the proof. □

4 Error estimates for the starting values

In Theorems 3.4 and 3.5, we see that it is necessary to define starting approximations $\{U^j\}_{j=0}^{p-1}$ such that

$$|U^j - u^j|_{-2p} + k^{p+1}\|A(U^j - u^j)\| = O(k^p), \quad \text{for } p \le j \le 2p-1.$$

In this section we will investigate two simplest cases $p = 1, 2$. The approach can be extended to the general case for $p > 2$, but the proof is more complicated.

In the case of $p = 1$, the approximate solution is defined by the backward Euler method

$$\bar{\partial}_1 U^n + AU^n = f^n, \quad \text{for } n \ge 1, \quad \text{with } U^0 = v, \tag{4.1}$$

or, with $r(\lambda) = 1/(1+\lambda)$,

$$U^n = r(kA)U^{n-1} + kr(kA)f^n, \quad \text{for } n \ge 1, \quad \text{with } U^0 = v.$$

We then have the following lemma.

LEMMA 4.1. *Let U^1 and u be the solutions of* (4.1) *and* (1.1)*, respectively. Then we have*

$$\begin{aligned} &|U^1 - u^1|_{-2} + k^2\|A(U^1 - u^1)\| \\ &\le Ck\|v - A^{-1}f(0)\| + Ck\int_0^k \|A^{-1}f'(\tau)\|\,d\tau + Ck^2\int_0^k \|f'(\tau)\|\,d\tau. \end{aligned} \tag{4.2}$$

In particular, if $f = 0$, then

$$|U^1 - u^1|_{-2} + k^2\|A(U^1 - u^1)\| \le Ck\|v\|. \tag{4.3}$$

PROOF. Noting that $u^1 = e^{-kA}v + \int_0^k e^{-(k-s)A} f(s)\,ds$ and using Taylor's formula, we have

$$\begin{aligned} U^1 - u^1 &= \big(r(kA) - e^{-kA}\big)v + kr(kA)f^1 - \int_0^k e^{-(k-s)A} f(s)\,ds \\ &= \big(r(kA) - e^{-kA}\big)v + kr(kA)\Big(f(0) + \int_0^k f'(\tau)\,d\tau\Big) \\ &\quad - k\int_0^1 e^{-(1-s)kA}\Big(f(0) + \int_0^{ks} f'(\tau)\,d\tau\Big)\,ds \\ &= \big(r(kA) - e^{-kA}\big)v + kb_0(kA)f(0) + kR(f), \end{aligned}$$

where

$$b_0(\lambda) = r(\lambda) - \int_0^1 e^{-(1-s)\lambda}\,ds,$$

and

$$R(f) = r(kA)\int_0^k f'(\tau)\,d\tau - \int_0^1 e^{-(1-s)kA}\int_0^{ks} f'(\tau)\,d\tau\,ds. \tag{4.4}$$

Thus we have, noting that $\lambda b_0(\lambda) = -(r(\lambda) - e^{-\lambda})$,

$$\begin{aligned} A(U^1 - u^1) &= \big(r(kA) - e^{-kA}\big)(Av - f(0)) + kAR(f) \\ &= -\big(r(kA) - e^{-kA}\big)u_t(0) + kAR(f). \end{aligned} \tag{4.5}$$

We first show that

$$k^2\|A(U^1 - u^1)\| \le Ck\|A^{-1}u_t(0)\| + Ck^2\int_0^k \|f'(\tau)\|\,d\tau. \tag{4.6}$$

In fact, by (4.4) and (4.5),

$$\begin{aligned} k^2\|A(U^1 - u^1)\| &\le k\|kA(r(kA) - e^{-kA})A^{-1}u_t(0)\| \\ &\quad + k^2\Big\|kAr(kA)\int_0^k f'(\tau)\,d\tau\Big\| \\ &\quad + k^2\Big\|\int_0^1 kAe^{-(1-s)kA}\int_0^{ks} f'(\tau)\,d\tau\,ds\Big\| \\ &= I + II + III. \end{aligned}$$

For I, we have, since $|\lambda(r(\lambda) - e^{-\lambda})| \le C$ for $0 < \lambda < \infty$,

$$I \le Ck\|A^{-1}u_t(0)\|.$$

For II, we have, since $|\lambda r(\lambda)| \le C$ for $0 < \lambda < \infty$,

$$II \le Ck^2 \int_0^k \|f'(\tau)\|\,d\tau.$$

For III, we have, since $\left|\int_\epsilon^1 \lambda e^{-(1-s)\lambda}\,ds\right| \le C$ for $0 \le \epsilon \le 1$,

$$\begin{aligned} III &= k^2 \Big\| \int_0^k \int_{\tau/k}^1 kAe^{-(1-s)kA} f'(\tau)\,ds\,d\tau \Big\| \\ &\le Ck^2 \Big(\int_0^k \|f'(\tau)\|\,d\tau \Big) \Big\| \int_{\tau/k}^1 kAe^{-(1-s)kA}\,ds \Big\| \\ &\le Ck^2 \int_0^k \|f'(\tau)\|\,d\tau. \end{aligned}$$

Thus we obtain (4.6).

We next show that

$$|A(U^1 - u^1)|_{-4} \le Ck\|A^{-1}u_t(0)\| + Ck \int_0^k \|A^{-1}f'(\tau)\|\,d\tau. \tag{4.7}$$

In fact, by (4.4) and (4.5),

$$\begin{aligned} |A(U^1 - u^1)|_{-4} &\le k\|(kA)^{-1}(r(kA) - e^{-kA})A^{-1}u_t(0)\| \\ &\quad + k\Big\| r(kA) \int_0^k A^{-1}f'(\tau)\,d\tau \Big\| \\ &\quad + k\Big\| \int_0^1 e^{-(1-s)kA} \int_0^{ks} A^{-1}f'(\tau)\,d\tau\,ds \Big\| \\ &= I + II + III. \end{aligned}$$

For I, we have, since $|\lambda^{-1}(r(\lambda) - e^{-\lambda})| \le C$ for $0 < \lambda < \infty$,

$$I \le Ck\|A^{-1}u_t(0)\| \le Ck\|v - A^{-1}f(0)\|.$$

For II, we have, since $|r(\lambda)| \le C$ for $0 < \lambda < \infty$,

$$II \le Ck \int_0^k \|A^{-1}f'(\tau)\|\,d\tau.$$

For III, we have, since $\left|\int_\epsilon^1 e^{-(1-s)\lambda}\,ds\right| \le C$ for $0 \le \epsilon \le 1$,

$$\begin{aligned} III &= k\Big\| \int_0^k \int_{\tau/k}^1 kAe^{-(1-s)kA} A^{-1}f'(\tau)\,ds\,d\tau \Big\| \\ &\le Ck \Big(\int_0^k \|A^{-1}f'(\tau)\|\,d\tau \Big) \Big\| \int_{\tau/k}^1 e^{-(1-s)kA}\,ds \Big\| \\ &\le Ck \int_0^k \|A^{-1}f'(\tau)\|\,d\tau. \end{aligned}$$

Thus we obtain (4.7). □

We now turn to the case $p = 2$. In this case we need two starting values U^0, U^1. We will use the backward Euler method to compute U^1, i.e., the approximation U^n of the solution $u(t_n)$ of (1.1) is defined by

$$\bar\partial_2 U^n + AU^n = f^n \quad \text{for } n \geq 2, \qquad \bar\partial U^1 + AU^1 = f^1, \quad \text{with } U^0 = v. \tag{4.8}$$

We have the following lemma.

LEMMA 4.2. *Let $U^j, j = 2,3$ and u be the solutions of (4.8) and (1.1), respectively. Then we have*

(4.9)
$$|U^j - u^j|_{-4} + k^3\|A(U^j - u^j)\| \leq Ck^2\Big(\|v\| + \|f(0)\| + \int_0^{t_j} \|f'(\tau)\|\, d\tau\Big), \quad j = 2, 3.$$

In particular, if $f = 0$, then

$$|U^j - u^j|_{-4} + k^3\|A(U^j - u^j)\| \leq Ck^2\|v\|, \quad j = 2, 3. \tag{4.10}$$

PROOF. Here we only prove the case $j = 2$, i.e., we will show that

$$|U^2 - u^2|_{-4} + k^3\|A(U^2 - u^2)\| = O(k^2). \tag{4.11}$$

The proof for the case $j = 3$ is similar.

Since $\bar\partial_2 U^2 = k^{-1}\big(\frac{3}{2}U^2 - 2U^1 + \frac{1}{2}U^0\big)$, we may write

$$U^2 = q_1(kA)U^1 + q_2(kA)U^0 + kP(kA)f^2,$$

where

$$q_1(\lambda) = \frac{2}{3/2 + \lambda}, \quad q_2(\lambda) = \frac{-1/2}{3/2 + \lambda}, \quad \text{and} \quad P(\lambda) = \frac{1}{3/2 + \lambda}.$$

Thus, noting that $u^2 = e^{-2kA}v + \int_0^{2k} e^{-(2k-s)A} f(s)\, ds$, we have

$$U^2 - u^2 = q_1(kA)(U^1 - u^1) + q_2(kA)(U^0 - u^0) + E_2.$$

Here

$$
\begin{aligned}
E_2 &= q_1(kA)u^1 + q_2(kA)u^0 + kP(kA)f^2 - u^2 \\
&= q_1(kA)\Big(e^{-kA}v + \int_0^k e^{-(k-s)A} f(s)\,ds\Big) \\
&\quad + q_2(kA)v + kP(kA)f^2 \\
&\quad - \Big(e^{-2kA}v + \int_0^{2k} e^{-(2k-s)A} f(s)\,ds\Big) \\
&= \Big(q_1(kA)e^{-kA} + q_2(kA) - e^{-2k}\Big)v \qquad (4.12)\\
&\quad + kq_1(kA)\int_0^1 e^{-(1-s)kA}\Big(f(0) + \int_0^{ks} f'(\tau)\,d\tau\Big)\,ds \\
&\quad + kP(kA)\Big(f(0) + \int_0^{2k} f'(\tau)\,d\tau\Big) \\
&\quad - 2k\int_0^1 e^{-2(1-s)kA}\Big(f(0) + \int_0^{2ks} f'(\tau)\,d\tau\Big)\,ds \\
&= Q(kA)v + kb_0(kA)f(0) + kR(f),
\end{aligned}
$$

where

$$Q(\lambda) = q_1(\lambda)e^{-\lambda} + q_2(\lambda) - e^{-2\lambda},$$

and

$$b_0(\lambda) = q_1(\lambda)\int_0^1 e^{-(1-s)\lambda}\,ds + P(\lambda) - 2\int_0^1 e^{-2(1-s)\lambda}\,ds,$$

and

$$
\begin{aligned}
R(f) &= q_1(kA)\int_0^1 e^{-(1-s)kA}\Big(\int_0^{ks} f'(\tau)\,d\tau\Big)\,ds \\
&\quad + P(kA)\int_0^{2k} f'(\tau)\,d\tau \\
&\quad - 2\int_0^1 e^{-2(1-s)kA}\Big(\int_0^{2ks} f'(\tau)\,d\tau\Big)\,ds.
\end{aligned}
$$

Thus we have

$$A(U^2 - u^2) = Aq_1(kA)(U^1 - u^1) + AE_2. \qquad (4.13)$$

Let us show that

$$k^3\|A(U^2 - u^2)\| \le Ck^2\Big(\|v\| + k\|f(0)\| + k\int_0^{2k} \|f'(\tau)\|\,d\tau\Big). \qquad (4.14)$$

In fact, by (4.12) and (4.13),

$$
\begin{aligned}
k^3\|A(U^2 - u^2)\| &\le k^3\|Aq_1(kA)(U^1 - u^1)\| + k^3\|AQ(kA)v\| \\
&\quad + k^3\|kAb_0(kA)f(0)\| + k^3\|kAR(f)\| \\
&= I + II + III + IV.
\end{aligned}
$$

We first estimate the terms II, III, and IV, then we turn to the term I.

For II, we have, since $|\lambda Q(\lambda)| < C$ for $0 < \lambda < \infty$,

$$II = k^2\|kAQ(kA)v\| \le Ck^2\|v\|.$$

For III, we have, since $|\lambda b_0(\lambda)| \le C$ for $0 < \lambda < \infty$,

$$III = k^3\|kAb_0(kA)f(0)\| \le Ck^3\|f(0)\|.$$

For IV, we have

$$\begin{aligned} IV &\le Ck^3\Big\|kAq_1(kA)\int_0^k\int_{\tau/k}^1 kAe^{-(1-s)kA}f'(\tau)\,ds\,d\tau\Big\| \\ &\quad + Ck^3\Big\|kAP(kA)\int_0^{2k} f'(\tau)\,d\tau\Big\| \\ &\quad + Ck^3\Big\|kA\int_0^{2k}\int_{\tau/2k}^1 e^{-2(1-s)kA}f'(\tau)\,ds\,d\tau\Big\| \\ &\le Ck^3\int_0^{2k}\|f'(\tau)\|\,d\tau. \end{aligned}$$

Now we turn to I.

$$\begin{aligned} I &= k^3\|Aq_1(kA)(U^1-u^1)\| \le k^3\|Aq_1(kA)(r(kA)-e^{-kA}v\| \\ &\quad + k^3\Big\|Aq_1(kA)\Big(kr(kA) - k\int_0^1 e^{-(1-s)kA}\,ds\Big)f(0)\Big\| \\ &\quad + k^3\Big\|Aq_1(kA)\Big(kr(kA)\int_0^k f'(\tau)\,d\tau - k\int_0^1 e^{-(1-s)kA}\int_0^{ks} f'(\tau)\,d\tau\,ds\Big)\Big\| \\ &= I_1 + I_2 + I_3. \end{aligned}$$

It is easy to show that

$$I_1 \le Ck^2\|v\|, \quad I_2 \le Ck^3\|f(0)\|, \quad \text{and} \quad I_3 \le 2k^3\int_0^k \|f'\|\,d\tau.$$

Thus we get

$$I \le Ck^2\Big(\|v\| + k\|f(0)\| + k\int_0^k \|f'\|\,d\tau\Big).$$

Combining this with the estimates for II, III and IV, we obtain (4.14)

We next show that

$$|A(U^2-u^2)|_{-6} \le Ck^2\Big(\|v\| + \int_0^{2k}\|f'(\tau)\|\,d\tau\Big). \tag{4.15}$$

In fact, by (4.12) and (4.13),

$$\begin{aligned} |A(U^2-u^2)|_{-6} &\le |Aq_1(kA)(U^1-u^1)|_{-6} + |AQ(kA)v|_{-6} \\ &\quad + |kAb_0(kA)f(0)|_{-6} + |kAR(f)|_{-6} \\ &= I' + II' + III' + IV'. \end{aligned} \tag{4.16}$$

We first estimate the terms II', III', and IV', then we turn to the term I'.

For II', we have, since $|\lambda^{-2}Q(\lambda)| < C$ for $0 < \lambda < \infty$,

$$II' = |kAQ(kA)v|_{-6} = k^2\|(kA)^{-2}Q(kA)v\| \le Ck^2\|v\|.$$

For III', we have, since $|\lambda^{-1}b_0(\lambda)| \le C$ for $0 < \lambda < \infty$,

$$III' = |kAb_0(kA)f(0)|_{-6} = k^2\|(kA)^{-1}b_0(kA)A^{-1}\| \le Ck^2\|f(0)\|.$$

For IV', we have

$$\begin{aligned} IV' &\le C\Big\|kA^{-2}q_1(kA)\int_0^k\int_{\tau/k}^1 kAe^{-(1-s)kA}f'(\tau)\,ds\,d\tau\Big\| \\ &\quad + C\Big\|kA^{-2}P(kA)\int_0^{2k} f'(\tau)\,d\tau\Big\| \\ &\quad + C\Big\|kA^{-2}\int_0^{2k}\int_{\tau/2k}^1 e^{-2(1-s)kA}f'(\tau)\,ds\,d\tau\Big\| \\ &\le Ck^2\int_0^{2k}\|f'(\tau)\|\,d\tau. \end{aligned}$$

Now we turn to I'.

$$\begin{aligned} I' &= |Aq_1(kA)(U^1 - u^1)|_{-6} \le |Aq_1(kA)(r(kA) - e^{-kA}v|_{-6} \\ &\quad + \Big|Aq_1(kA)\Big(kr(kA) - k\int_0^1 e^{-(1-s)kA}\,ds\Big)f(0)\Big|_{-6} \\ &\quad + \Big|Aq_1(kA)\Big(kr(kA)\int_0^k f'(\tau)\,d\tau - k\int_0^1 e^{-(1-s)kA}\int_0^{ks} f'(\tau)\,d\tau\,ds\Big)\Big|_{-6} \\ &= I_1' + I_2' + I_3'. \end{aligned}$$

It is easy to show that

$$I_1' \le Ck^2\|v\|, \quad I_2' \le Ck^2\|f(0)\|, \quad \text{and} \quad I_3' \le Ck^2\int_0^k \|f'\|\,d\tau.$$

Thus we get

$$I' \le Ck^2\Big(\|v\| + \|f(0)\| + \int_0^k \|f'\|\,d\tau\Big).$$

Combining this with the estimates for II', III' and IV', we obtain (4.15).

Together these estimates we show (4.11). The proof is complete. □

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Paper IV

Postprocessing the finite element method for semilinear parabolic problems

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Abstract.

In this paper we consider postprocessing of the finite element method for semilinear parabolic problems. The postprocessing amounts to solving a linear elliptic problem on a finer grid (or higher-order space) once the time integration on the coarser mesh is completed. The convergence rate is increased at almost no additional computational cost. This procedure was introduced and analyzed in Garcia-Archilla and Titi [13]. We extend the analysis to the fully discrete case and prove error estimates for both space and time discretization. The analysis is based on error estimates for the approximation of time derivatives by difference quotients.

AMS subject classification: 65M60, 65M15, 65M20.

Key words: time derivative, postprocessing, finite element method, backward Euler method, error estimates, semilinear parabolic problem.

1 Introduction

In this paper we shall consider postprocessing of the finite element method for the semilinear parabolic problem

$$u_t - \Delta u = F(u) \quad \text{in}\, \Omega, \quad \text{for}\, t \in (0,T], \tag{1.1}$$

$$u = 0 \quad on\ \partial\Omega, \quad \text{for}\, t \in (0,T], \quad \text{with}\, u(0) = v,$$

where Ω is a bounded domain in $\mathbf{R}^d$, $d = 1,2,3$, with a sufficiently smooth boundary $\partial\Omega$, $u_t = \partial u/\partial t$, Δ is the Laplacian, and $F : \mathbf{R} \to \mathbf{R}$ is a smooth function.

Let $H = L_2(\Omega)$. We define the unbounded operator $A = -\Delta$ on H with domain of definition $\mathcal{D}(A) = H^2 \cap H_0^1$, where, for integer $m \geq 1$, $H^m = H^m(\Omega)$ denotes the standard Sobolev space $W_2^m(\Omega)$, and $H_0^1 = H_0^1(\Omega) = \{v \in H^1 : v|_{\partial\Omega} = 0\}$. Then A is a closed, densely defined, and self-adjoint positive definite operator in H with compact inverse. The initial-boundary value problem (1.1) may then be formulated as the following initial value problem

$$u_t + Au = F(u), \quad \text{for}\ 0 < t \leq T, \quad \text{with}\, u(0) = v, \tag{1.2}$$

in the Hilbert space H.

Recently, a postprocessing technique has been introduced to increase the efficiency of Galerkin method of spectral type, see Canuto, Hussaini, Quarteroni, and Zang [4], De Frutos, Garcia-Archilla, and Novo [6], De Frutos and Novo [7], [9]. Postprocessed methods yield greater accuracy than standard Galerkin schemes at nearly the same computational cost. In Garcia-Archilla and Titi [13], the postprocessing technique has been extended to the h-version of the finite element method for dissipative partial differential equations. There, the authors prove that the postprocessed method has a higher rate of convergence than the standard finite element method when higher order finite elements, rather than linear finite elements, are used. Error estimates in L_2 and H^1 norms in the spatially semidiscrete case are obtained. More recently, in De Frutos and Novo [8], the authors show that the postprocessing technique can also be applied to linear finite elements and the convergence rate can be improved in H^1 norm, but not in L_2 norm. The analysis is restricted to the spatially semidiscrete case.

The purpose of the present paper is to derive the error estimates in the fully discrete case for the postprocessed finite element method applied to (1.2). To do this, we introduce the time-stepping method to compute the discrete solution of (1.2) and define a difference quotient approximation to time derivative. We then define the postprocessing step in the fully discrete case and show the error estimates for postprocessing method by using the error estimates for time derivatives. For simplicity we only consider the error estimates in L_2 norm. Our technique of proof is related to, but different from, the one employed in Garcia-Archilla and Titi [13].

The paper is organized as follows. In Section 2, we introduce some basic notations and lemmas. In Section 3 we consider error estimates for the postprocessed finite element method in the semidiscrete case. In Section 4, we consider error estimates in the fully discrete case. In Section 5, we consider the starting approximation of time derivatives. Finally, in Section 6, we consider higher order time-stepping in the context of the linear homogeneous problem.

By C_0 we denote positive constant independent of the functions and parameters concerned, but not necessarily the same at different occurrences.

2 Preliminaries

Let $\mathcal{T}$ denote a partition of Ω into disjoint triangles τ such that no vertex of any triangle lies on the interior of a side of another triangle and such that the union of the triangles determine a polygonal domain $\Omega_h \subset \Omega$ with boundary vertices on $\partial\Omega$. Let h denote the maximal length of the sides of the triangulation $\mathcal{T}_h$. We assume that the triangulations are quasiuniform in the sense that the triangles of $\mathcal{T}_h$ are of essentially the same size.

Let r be any nonnegative integer. We denote by $\| \cdot \|_r$ the norm in H^r. Let $\{S_h\} = \{S_{h,r}\} \subset H_0^1$ be a family of finite element spaces with the accuracy of order $r \geq 2$, i.e., S_h consists of continuous functions on the closure $\bar{\Omega}$ of Ω which are polynomials of degree at most $r-1$ in each triangle of $\mathcal{T}_h$ and which vanish outside Ω_h, such that, for small h,

$$\inf_{\chi \in S_h} \{\|v-\chi\| + h\|\nabla(v-\chi)\|\} \le Ch^s\|v\|_s, \quad \text{for } 1 \le s \le r,$$

when $v \in H^s \cap H_0^1$.

The semidiscrete problem of (1.2) is to find the approximate solution $u_h(t) = u_h(\cdot,t) \in S_h$ for each t, such that,

$$u_{h,t} + A_h u_h = P_h F(u_h), \quad \text{with } u_h(0) = v_h, \tag{2.1}$$

where $v_h \in S_h$, $P_h : L_2 \to S_h$ is the L_2 projection onto S_h, and $A_h : S_h \to S_h$ is the discrete analogue of A, defined by

$$(A_h\psi, \chi) = A(\psi, \chi), \quad \forall\, \psi, \chi \in S_h. \tag{2.2}$$

Here $A(\cdot,\cdot) = (\nabla\cdot, \nabla\cdot)$ is the bilinear form on H_0^1 obtained from A.

Error estimates for finite element methods for semilinear parabolic problems with various conditions on the nonlinearity have been considered in many papers, see, e.g., Akrivis, Crouzeix, and Makridakis [1], [2], Crouzeix, Thomée, and Wahlbin [5], Elliott and Larsson [10], [11], Helfrich [14], Johnson, Larsson, Thomée, and Wahlbin [15], Thomée [21], Thomée and Wahlbin [22], Wheeler [23]. The long time behavior of finite element solutions was studied by Elliott and Stuart [12], Larsson [16], [17], Larsson and Sanz-Serna [18], [19].

Let us now describe the idea of the postprocessed finite element method proposed by Garcia-Archilla and Titi [13]. Suppose that we want to obtain high order approximation, for instance $O(h^{r+2})$. Then we can use, in every time step, either a family of high order finite element spaces $\tilde{S}_h := S_{h,r+2}$ with the order $r+2$ of accuracy, or a family of finite element space $\tilde{S}_h := S_{\tilde{h},r}$ with accuracy of order r, but with finer partition $\mathcal{T}_{\tilde{h}}$ of the domain Ω, such that, $h^{r+2} = \tilde{h}^r$. In [13], another technique, called the *postprocessed finite element method*, is presented, which improves the convergence rate without using a high order finite element space $\tilde{S}_h$ in every time step. Suppose that we are interested in the solution of (1.2) at a given time T. At time T, rewriting (1.2), we have

$$Au(T) = -u_t(T) + F(u(T)). \tag{2.3}$$

Thus, $u(T)$ can be seen as the solution of an elliptic problem whose right hand side is not known but can be approximated. Garcia-Archilla and Titi first compute $u_h(T)$ by (2.1) in the finite element space S_h, then replace $u_t(T)$ by $u_{h,t}(T)$ and solve (or, in practice, approximate) the following linear elliptic problem: find $\tilde{u}(T) \in \mathcal{D}(A)$, such that,

$$A\tilde{u}(T) = -u_{h,t}(T) + F(u_h(T)), \tag{2.4}$$

which is the postprocessing step.

They obtained the following error estimate, with $\ell_h = 1 + \log(T/h^2)$,

$$\|\tilde{u}(T) - u(T)\| \le C(u)\ell_h h^{r+2}, \quad \text{for } r \ge 4, \tag{2.5}$$

where $C(u)$ is some constant depending on u. A similar result holds for $r \geq 3$. The proof is based on superconvergence for elliptic finite element methods in norms of negative order, which is the reason for the restriction $r \geq 3$.

We note that the bound (2.5) is an improvement over the error estimates for the standard Galerkin method, which is $O(h^r)$. In practice $\tilde{u}$ can not be computed exactly, since in general it does not belong to a finite element space. However, one can approximate the solution $\tilde{u}$ of (2.4) by some $\tilde{u}_h$ belonging to a finite element space $\tilde{S}_h$ of approximation order $r+2$ as described above. More precisely, we pose the following semidiscrete problem corresponding to (2.4): find $\tilde{u}_h \in \tilde{S}_h$, such that,

$$\tilde{A}_h \tilde{u}_h(T) = \tilde{P}_h\big(-u_{h,t}(T) + F(u_h(T))\big), \tag{2.6}$$

where $\tilde{P}_h : L_2 \to \tilde{S}_h$ is the L_2 projection onto $\tilde{S}_h$ and $\tilde{A}_h$ is the discrete analogue of A with respect to $\tilde{S}_h$. The standard error estimate reads, see, e.g., Brenner and Scott [3],

$$\|\tilde{u}_h(T) - \tilde{u}(T)\| \leq C(u)h^{r+2}. \tag{2.7}$$

Combining (2.5) and (2.7), we have

$$\|\tilde{u}_h(T) - u(T)\| \leq \|\tilde{u}_h(T) - \tilde{u}(T)\| + \|\tilde{u}(T) - u(T)\| \leq C(u)\ell_h h^{r+2}, \quad \text{for } r \geq 4.$$

Let us now introduce norms of negative order. Consider the stationary problem,

$$Au = f. \tag{2.8}$$

The variational form of this problem is to find $u \in H_0^1 = H_0^1(\Omega)$, such that

$$A(u, \varphi) = (f, \varphi), \quad \forall\, \varphi \in H_0^1.$$

The standard Galerkin finite element problem is to find $u_h \in S_h$, such that,

$$A(u_h, \chi) = (f, \chi), \quad \forall\, \chi \in S_h. \tag{2.9}$$

Let $G : L_2 \to H_0^1$ be the exact solution operator of (2.8) and define the approximate solution operator $G_h : L_2 \to S_h$ by $G_h f = u_h$ so that $u_h = G_h f \in S_h$ is the solution of (2.9). We recall that G_h is the selfadjoint, positive semidefinite on L_2 and positive definite on S_h. Further we have, see Thomée [21, Chapter 6],

$$\|(G_h - G)f\| \leq Ch^r \|f\|_{r-2}, \quad \text{for } f \in H^{r-2},\ r \geq 2, \tag{2.10}$$

and

$$\|(G_h - G)f\|_{-2} \leq Ch^{r+2} \|f\|_{r-2}, \quad \text{for } f \in H^{r-2}, \quad r \geq 4. \tag{2.11}$$

Here r is the order of the accuracy of the family $\{S_h\}$, and the negative order norm is defined by

$$\|\cdot\|_{-2} = \sup\Big\{\frac{(v,\phi)}{\|\phi\|_2} : \phi \in H^2\Big\}.$$

We note that $G : L_2 \to H_0^1 \cap H^2$ is the inverse operator of $A : H_0^1 \cap H^2 \to L_2$, i.e., $G = A^{-1}$, and similarly $G_h = A_h^{-1}$ on S_h, where A_h is the discrete Laplacian of A defined by (2.2). Moreover, we will use the following properties, see, Thomée [21, Chapter 2],

$$G_h P_h = G_h \quad \text{and} \quad G_h = R_h G, \tag{2.12}$$

where $R_h : H_0^1 \to S_h$ is the elliptic projection, or Ritz projection, defined by

$$A(R_h u, \chi) = A(u, \chi), \quad \forall\, \chi \in S_h. \tag{2.13}$$

For our analysis it will be convenient to use instead of the negative order norm introduced above, such a norm defined by

$$|v|_{-s} = \|G^{s/2} v\| = (G^s v, v)^{1/2}, \quad \text{for } s \geq 0,$$

we think of this as a norm in L_2.

We introduce also a discrete negative order seminorm on L_2 by

$$|v|_{-s,h} = \|G_h^{s/2} v\| = (G_h^s v, v)^{1/2}, \quad \text{for } s \geq 0;$$

it corresponds to the discrete semi-inner product $(v,w)_{-s,h} = (G_h^s v, w)$, $\forall\, v, w \in L_2$. Since G_h is positive definite on S_h, $|v|_{-s,h}$ and $(v,w)_{-s,h}$ define a norm and an inner product there. We also find that the discrete negative order seminorm is equivalent to the corresponding continuous norm, modulo a small error. More precisely, we have the following bounds, see, e.g., Thomée [21, Lemma 6.3].

LEMMA 2.1. *We have, for* $0 \leq s \leq r$,

$$|v|_{-s,h} \leq C_0(|v|_{-s} + h^s\|v\|), \quad \textit{and} \quad |v|_{-s} \leq C_0(|v|_{-s,h} + h^s\|v\|).$$

We also need Gronwall's lemma.

LEMMA 2.2. *If* a, b *are nonnegative constants and*

$$0 \leq u(t) \leq a + b\int_0^t u(s)\,ds, \quad \textit{for } 0 \leq t \leq T,$$

then we have

$$u(t) \leq a e^{bt}, \quad \textit{for } 0 \leq t \leq T.$$

For the nonlinear operator F, we have the following bounds, see Garcia-Archilla and Titi [13, Lemma 3]. For the sake of completeness, we include the proof, written in our slightly simpler form.

Lemma 2.3. *Let* $u \in H^r(\Omega) \cap H_0^1(\Omega)$, $r \geq 4$, *and* $\chi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. *Assume that* F *is a smooth function. Further assume that* $d \leq 3$ *and* $\|u-\chi\|_{L_\infty} \leq K$ *for some positive number* K. *Then there is a constant* $C = C(\|u\|_r, K)$ *such that*

$$\|F(u) - F(\chi)\| \leq C\|u - \chi\|, \tag{2.14}$$

and

$$|F(u) - F(\chi)|_{-2} \leq C\big(|u - \chi|_{-2} + \|u - \chi\|^2\big). \tag{2.15}$$

Proof. By Taylor's formula, we have, with $\xi = u + \theta(\chi - u)$, $0 \leq \theta \leq 1$,

$$\|F(u) - F(\chi)\| = \|F'(\xi)(u - \chi)\| \leq \|F'(\xi)\|_{L_\infty}\|u - \chi\|.$$

Since $F'(x)$ is bounded in $\{x : |x| \leq \|u\|_{L_\infty} + \|\chi - u\|_{L_\infty}\}$, we have, noting that $\|u\|_{L_\infty} \leq C_0\|u\|_r$ for $r \geq 4$, $d \leq 3$,

$$\|F'(\xi)\|_{L_\infty} \leq C(\|u\|_{L_\infty}, K) \leq C(\|u\|_r, K), \tag{2.16}$$

which shows (2.14).

To prove (2.15), we have, by Taylor's formula,

$$|F(u) - F(\chi)|_{-2} \leq |F'(u)(u - \chi)|_{-2} + \frac{1}{2}|F''(\xi)(u - \chi)^2|_{-2}.$$

We first show

$$|F''(\xi)(u - \chi)^2|_{-2} = \|A^{-1}F''(\xi)(u - \chi)^2\| \leq C\|u - \chi\|^2. \tag{2.17}$$

In fact, by duality, for $\forall\, \phi \in L_2(\Omega)$,

$$\begin{aligned} \big|\big(A^{-1}F''(\xi)(u - \chi)^2, \phi\big)\big| &= \big|\big(F''(\xi)(u - \chi)^2, A^{-1}\phi\big)\big| \\ &\leq \|F''(\xi)\|_{L_\infty}\|(u - \chi)^2\|_{L_1}\|A^{-1}\phi\|_{L_\infty}. \end{aligned}$$

Following (2.16), we have $\|F''(\xi)\|_{L_\infty} \leq C(\|u\|_r, K)$. Further, by Sobolev's inequality and elliptic regularity, we have

$$\|A^{-1}\phi\|_{L_\infty} \leq C_0\|A^{-1}\phi\|_{H^2} \leq C_0\|\phi\|, \quad \text{for } d \leq 3.$$

Thus,

$$\big|\big(A^{-1}F''(\xi)(u - \chi)^2, \phi\big)\big| \leq C\|u - \chi\|^2\|\phi\|, \quad \forall\, \phi \in L_2(\Omega),$$

which implies that (2.17) holds.

Now we show

$$|F'(u)(u - \chi)|_{-2} = \|A^{-1}F'(u)(u - \chi)\| \leq C|u - \chi|_{-2}. \tag{2.18}$$

In fact, by duality, for $\forall\, \phi \in L_2(\Omega)$, noting that $F'(u)A^{-1}\phi \in \mathcal{D}(A)$,

$$\begin{aligned} \big(A^{-1}F'(u)(u - \chi), \phi\big) &= \big(F'(u)(u - \chi), A^{-1}\phi\big) \\ &= \big(A^{-1}(u - \chi), A(F'(u)A^{-1}\phi)\big). \end{aligned}$$

With $A = -\Delta$, we have

$$\begin{aligned}\|A(F'(u)A^{-1}\phi)\| &= \|F'(u)\phi + 2\nabla F'(u)\cdot\nabla(A^{-1}\phi) + (\Delta F'(u))A^{-1}\phi\| \\ &\le \|F'(u)\|_{L_\infty}\|\phi\| + 2\|\nabla F'(u)\|_{L_\infty}\|A^{-1}\phi\|_{H^1} + \|\Delta F'(u)\|_{L_\infty}\|A^{-1}\phi\| \\ &\le C\|F'(u)\|_{W^2_\infty}\|A^{-1}\phi\|_{H^2} \le C(\|u\|_r)\|\phi\|.\end{aligned}$$

Thus we get

$$|(A^{-1}F'(u)(u-\chi),\phi)| \le C(\|u\|_r)\|A^{-1}(u-\chi)\|\,\|\phi\|,$$

which implies (2.18).

Together these estimates complete the proof. □

REMARK 2.1. *In our application of Lemma 2.3, we will choose u to be the solution of* (1.2) *and χ to be the corresponding finite element approximation solution u_h. It is obvious that u_h and u satisfy the assumptions of the Lemma 2.3. For instance $\|u_h - u\|_\infty \le K$ can be achieved by using the inverse inequality provided we know that the L_2 error estimates for $u_h - u$ is $O(h^r)$, see Thomée [21, Chapter* 14*].*

3 Semidiscrete approximation

In this section we will consider the error estimates for the *postprocessed finite element method* for the semilinear parabolic problem (1.2) in the semidiscrete case. The main theorem in this section is the following:

THEOREM 3.1. *Let $r \ge 4$ and S_h and $\tilde{S}_h$ be the finite element spaces of order r and $r+2$, respectively, as described in Section* 1*. Let $\tilde{u}_h$ and u be the solutions of* (2.6) *and* (1.2)*, respectively. Assume that F satisfies $\|F(u)\|_r \le C_0$ in addition to the assumptions in Lemma 2.3. Let u_h be the solution of* (2.1)*. Assume that $v_h = R_h v$ and*

$$\sup_{s\in[0,T]} \|u_h(s) - u(s)\|_{L_\infty} \le K,$$

and

$$\sup_{s\in[0,T]} \big(\|u(s)\|_r + \|u_t(s)\|_r + \|u_{tt}(s)\|_r\big) \le M,$$

for some positive numbers K, M, T. Then there is a constant $C = C(K,M,T)$ such that, with $\ell_h = 1 + \log(T/h^2)$,

$$\|\tilde{u}_h(T) - u(T)\| \le C\ell_h h^{r+2}. \tag{3.1}$$

As we mentioned in Section 2, Garcia-Archilla and Titi [13] has proved the similar results: if $\tilde{u}(T)$ and u are the solutions of (2.4) and (1.2), then

$$\|\tilde{u}(T) - u(T)\| \le C(u)\ell_h h^{r+2}, \quad \text{for } r \ge 4, \tag{3.2}$$

where $C(u)$ is some constant depending on u.

For the comparison, let us recall the idea of their proof. By (2.4) and (1.2), it follows that

$$\begin{aligned}\|\tilde{u}(T)-u(T)\| &= \|A^{-1}(-u_{h,t}(T)+F(u_h(T)))-A^{-1}(-u_t(T)+F(u(T)))\|\\ &\le |u_{h,t}(T)-u_t(T)|_{-2}+|F(u_h(T))-F(u(T))|_{-2}.\end{aligned}$$

Lemma 2.3 with $u=u(T)$ and $\chi=u_h(T)$ implies that

$$|F(u_h(T))-F(u(T)|_{-2}\le C(\|u\|_r,K)\big(|u_h(T)-u(T)|_{-2}+\|u_h(T)-u(T)\|^2\big).$$

Introducing elliptic projection R_h defined by (2.13), it follows that

$$\begin{aligned}\|\tilde{u}(T)-u(T)\| &\le |\rho_t(T)|_{-2}+|\theta_t(T)|_{-2}\\ &\quad + C(\|u\|_r,K)\big(|\rho(T)|_{-2}+|\theta(T)|_{-2}+\|\rho(T)\|^2+\|\theta(T)\|^2\big),\end{aligned} \tag{3.3}$$

where $\rho=R_hu-u$, $\theta=u_h-R_hu$, $\rho_t=R_hu_t-u_t$, and $\theta_t=u_{h,t}-R_hu_t$.

The desired bounds of $|\rho|_{-l}, l=0,2$, and $\|\rho_t\|_{-2}$ are well known, see, e.g., Thomée [21, Chapter 6]. The task is to estimate $|\theta|_{-l}$ for $l=0,2$, and $|\theta_t|_{-2}$. To do this, consider the following equation

$$\theta_t+A_h\theta=P_h(-\rho_t+F(u_h)-F(u)),\quad \text{with } \theta(0)=v_h-R_hv=0. \tag{3.4}$$

By Duhamel's principle, it follows, with $E_h(t)=e^{-tA_h}$,

$$\theta(T)=\int_0^T E_h(T-s)P_h\big(-\rho_t(T)+F(u_h(T))-F(u(T))\big)\,ds. \tag{3.5}$$

The desired bounds of $\|\theta\|_{-l}$ for $l=0,2$ can be easily proved by using Lemma 2.2 and the stability of $E_h(t)$. To estimate $|\theta_t|_{-2}$, they note that, by the second part of Lemma 2.1 with $s=2$,

$$|\theta_t|_{-2}=C_0(h^2\|\theta_t\|+|\theta_t|_{-2,h}).$$

By (3.4), we have

$$|\theta_t|_{-2,h}\le\|\theta\|+|P_h(-\rho_t+F(u_h)-F(u))|_{-2,h},$$

and, noting that $\|A_h\theta\|\le Ch^{-2}\|\theta\|$,

$$\begin{aligned}\|\theta_t\| &\le \|A_h\theta\|+\|P_h(-\rho_t+F(u_h)-F(u))\|\\ &\le Ch^{-2}\|\theta\|+\|P_h(-\rho_t+F(u_h)-F(u))\|.\end{aligned}$$

Hence

$$\begin{aligned}|\theta_t|_{-2} &\le C_0\|\theta\|+C_0h^2\|P_h(-\rho_t+F(u_h)-F(u))\|\\ &\quad + C_0|P_h(-\rho_t+F(u_h)-F(u))|_{-2,h}.\end{aligned} \tag{3.6}$$

The desired bounds for the last two terms in the right hand side of (3.6) follow from Lemmas 2.1 and 2.3, and the estimates for $|\rho_t|_{-l}$ and $|u_h - u|_{-l}$, $l = 0, 2$. Further they show that θ has the superconvergence property, i.e,

$$\|\theta\| \leq C(u)\ell_h h^{r+2}, \quad \text{for } r \geq 4.$$

Together these estimates completes the proof of (3.2).

We note that the logarithmic factor ℓ_h appears in the superconvergent estimate of θ.

We now return to Theorem 3.1 and state the idea of the proof in present paper. In Theorem 3.1, we consider the error bounds for $\|\tilde{u}_h - u\|$, not only for $\|\tilde{u} - u\|$. To prove Theorem 3.1, it suffices to show the bounds of $|u_h - u|_{-l}$ and $|u_{h,t} - u_t|_{-l}$ for $l = 0, 2$. We first split

$$u_h - u = (u_h - \hat{u}_h) + (\hat{u}_h - u) = \eta + e, \tag{3.7}$$

where $\hat{u}_h$ satisfies

$$\hat{u}_{h,t} + A_h \hat{u}_h = P_h F(u), \quad \hat{u}_h(0) = v_h. \tag{3.8}$$

Since u satisfies

$$u_t + Au = F(u), \quad u(0) = v, \tag{3.9}$$

the desired bounds of $e = \hat{u}_h - u$ and e_t follow from the error estimates for the linear parabolic problem because the right hand side of (3.8) is independent of $\hat{u}_h$. In other words we only need to consider the nonlinear term F when we show the bounds of $\eta = u_h - \hat{u}_h$ and η_t. Note that η satisfies

$$\eta_t + A_h \eta = P_h(F(u_h) - F(u)), \quad \eta(0) = 0. \tag{3.10}$$

By Duhamel's principle, we have

$$\eta(T) = \int_0^T E_h(T-s) P_h\big(F(u_h(s)) - F(u(s))\big)\, ds. \tag{3.11}$$

We obtain the desired bounds for $|\eta|_{-l}, l = 0, 2$, by using Lemmas 2.2 and 2.3 as above for showing the bounds of $|\theta|_{-l}, l = 0, 2$, in [13]. For $|\eta_t|_{-l}, l = 0, 2$, we have two ways to consider the bounds. One way is to use the superconvergence property of η, which can be proved as we mentioned above for proving the superconvergence property of θ in [13]. Another way is to work with the following equality

$$\begin{aligned} \eta_t(T) = & P_h(F(u_h(T)) - F(u(T))) \\ & - \int_0^T A_h E_h(T-s) P_h\big(F(u_h(s)) - F(u(s))\big)\, ds, \end{aligned} \tag{3.12}$$

which follows from (3.10) and (3.11).

Below we will use the second way to estimate $|\eta_t|_{-l}$ for $l = 0, 2$. This is the main difference between our proof and the proof in Garcia-Archilla and Titi [13]. We will extend this idea to the fully discrete case in Section 4.

We remark that since $\eta(0) = 0$, we don't need to consider the term $E_h(T)\eta(0)$ in (3.11). This observation is very useful in the fully discrete case.

LEMMA 3.2. *Let u_h and u be the solutions of* (2.1) *and* (1.2), *respectively. Assume that F satisfies the assumptions in Lemma 2.3. Further assume that $v_h = R_h v$ and*

$$\sup_{0\le s\le T} \|u_h(s) - u(s)\|_{L_\infty} \le K, \tag{3.13}$$

and

$$\sup_{0\le s\le T} \big(\|u(s)\|_r + \|u_t(s)\|_r\big) \le M_1, \tag{3.14}$$

for some positive numbers K, M_1, T. Then there is a constant $C = C(K, M_1, T)$ such that

$$\sup_{0\le t\le T} \|u_h(t) - u(t)\| \le Ch^r \quad \text{for } r \ge 2, \tag{3.15}$$

and

$$\sup_{0\le t\le T} |u_h(t) - u(t)|_{-2} \le Ch^{r+2}, \quad \text{for } r \ge 4. \tag{3.16}$$

PROOF. The error estimate (3.15) is well known, see Wheeler [23] and Thomée [21], where it is proved by splitting $u_h - u = \theta + \rho$, where $\theta = u_h - R_h u, \rho = R_h u - u$. Here we will show (3.15) by splitting $u_h - u = \eta + e$, where η, e are defined by (3.7). We will use this idea in subsequent lemmas for the proof of the error estimate for time derivative approximation and later in the proof of the error estimates in fully discrete case.

For $e = \hat{u}_h - u$, we have, by the standard error estimates for linear parabolic problem in semidiscrete case, see, e.g., Thomée [21, Lemma 1.3],

$$\|e(t)\| \le \|\hat{u}_h(0) - u(0)\| + C_0 h^r\Big(\|v\|_r + \int_0^t \|u_t\|_r\, ds\Big), \quad \text{for } r \ge 2. \tag{3.17}$$

Note that $\hat{u}_h(0) = v_h = R_h v = R_h u(0)$, we therefore have

$$\|e(t)\| \le C(M_1, T)h^r, \quad \text{for } r \ge 2,\ 0 \le t \le T. \tag{3.18}$$

For $\eta = u_h - \hat{u}_h$, we have, by (3.11) and the stability of $E_h(t)$,

$$\|\eta(t)\| \le \int_0^t \|F(u_h(s)) - F(u(s))\|\, ds.$$

By Lemma 2.3, we have

$$\|\eta(t)\| \le C(K, M_1)\int_0^t \|u_h - u\|\, ds \le C(K, M_1)\Big(\int_0^t \|\eta\|\, ds + \int_0^t \|e\|\, ds\Big).$$

Further, by Lemma 2.2 and (3.18),

$$\|\eta(t)\| \le C(K,M_1)\int_0^t \|e\|\,ds \le C(K,M_1,T)h^r, \quad \text{for } r\ge 2,\ 0\le t\le T, \tag{3.19}$$

which shows (3.15).

Now we turn to (3.16). By Thomée [21, Theorem 6.2], we have, since $v_h = R_h v$,

$$|e(t)|_{-2} \le C_0 h^{r+2}\Big(\|v\|_r + \int_0^t \|u_t\|_r\,ds\Big), \quad \text{for } r\ge 4. \tag{3.20}$$

To estimate $|\eta|_{-2}$, we first note that, by Lemma 2.1,

$$|\eta|_{-2} \le C_0(h^2\|\eta\| + |\eta|_{-2,h}) = C_0(h^2\|\eta\| + \|G_h\eta\|). \tag{3.21}$$

Here $G_h\eta$ satisfies, by (3.10),

$$G_h\eta_t + A_hG_h\eta = G_hP_h(F(u_h)-F(u)), \quad G_h\eta(0)=0,$$

which implies, by Duhamel's principle,

$$G_h\eta(t) = \int_0^t E_h(t-s)G_hP_h\big(F(u_h)-F(u)\big)\,ds.$$

Note that, by Lemmas 2.1 and 2.3, and (3.15), (3.20),

$$\begin{aligned}\|G_hP_h(F(u_h)-F(u))\| &= |F(u_h)-F(u)|_{-2,h}\\ &\le C_0\big(h^2\|F(u_h)-F(u)\| + |F(u_h)-F(u)|_{-2}\big)\\ &\le C(\|u\|_r,K)\big(h^2\|u_h-u\| + \|u_h-u\|^2 + |u_h-u|_{-2}\big)\\ &\le C(K,M_1,T)\big(h^{r+2} + |\eta|_{-2}\big).\end{aligned}$$

Hence, by stability of $E_h(t)$,

$$\|G_h\eta(t)\| \le C(K,M_1,T)\Big(h^{r+2} + \int_0^t |\eta|_{-2}\,ds\Big).$$

Combining this with (3.21), (3.19), and using Lemma 2.2, we get

$$|\eta(t)|_{-2} \le C(K,M_1,T)h^{r+2}, \quad \text{for } 0\le t\le T. \tag{3.22}$$

Together these estimates complete the proof. □

Next lemma is the error estimates for time derivative of the solution of (1.2).

LEMMA 3.3. *Let u_h and u be the solutions of* (2.1) *and* (1.2), *respectively. Assume that F satisfies the assumptions in Lemma 2.3. Further assume that $v_h = R_h v$ and*

$$\sup_{0\le s\le T} \|u_h(s)-u(s)\|_{L_\infty} \le K, \tag{3.23}$$

and

$$\sup_{0\le s\le T}\big(\|u(s)\|_r+\|u_t(s)\|_r+\|u_{tt}(s)\|_r\big)\le M_2, \tag{3.24}$$

for some positive numbers K, M_2, T. Then there is a constant $C=C(K,M_2,T)$ such that, with $\ell_h=1+\log(T/h^2)$,

$$\sup_{0\le t\le T}\|u_{h,t}(t)-u_t(t)\|\le C\ell_h h^r, \tag{3.25}$$

and

$$\sup_{0\le t\le T}|u_{h,t}(t)-u_t(t)|_{-2}\le C\ell_h h^{r+2}. \tag{3.26}$$

Proof. We write

$$u_{h,t}-u_t=(u_{h,t}-\hat u_{h,t})+(\hat u_{h,t}-u_t)=\eta_t+e_t.$$

Following the proofs of Theorems 1.3 and 6.2 in Thomée [21] for the error estimate $|e|_{-l},\ l=0,2$, we can show the following error estimates for $|e_t|_{-l},\ l=0,2$, that is,

$$\|e_t(t)\|\le\|\hat u_{h,t}(0)-u_t(0)\|+C_0h^r\Big(\|u_t(0)\|_r+\int_0^t\|u_{tt}\|_r\,ds\Big),$$

and

$$|e_t(t)|_{-2}\le|\hat u_{h,t}(0)-u_t(0)|_{-2}+C_0h^{r+2}\Big(\|u_t(0)\|_r+\int_0^t\|u_{tt}\|_r\,ds\Big).$$

We observe that, by (3.8), and noting that $\hat u_h(0)=R_hu(0)$,

$$\begin{aligned}\hat u_{h,t}(0)&=-A_h\hat u_h(0)+P_hF(u(0))=-A_hR_hu(0)+P_hF(u(0))\\&=P_h\big(Au(0)+F(u(0))\big)=P_hu_t(0).\end{aligned}$$

We therefore have, by the error bounds for the L_2 projection,

$$\|u_{h,t}(0)-u_t(0)\|=\|(P_h-I)u_t(0)\|\le C_0h^r\|u_t(0)\|_r,$$

and

$$|u_{h,t}(0)-u_t(0)|_{-2}\le C_0h^{r+2}\|u_t(0)\|_r.$$

Thus, we get

$$\|e_t(t)\|\le C_0h^r\Big(\|u_t(0)\|_r+\int_0^t\|u_{tt}\|_r\,ds\Big)\le C(M_2,T)h^r, \tag{3.27}$$

and, similarly,

$$|e_t(t)|_{-2}\le C(M_2,T)h^{r+2}. \tag{3.28}$$

We now turn to $|\eta_t|_{-l}$, $l = 0, 2$. Using the fact $\|A_h E_h(t)\| \le C_0(t + h^2)^{-1}$, see Schatz, Thomée, and Wahlbin [20], we have

$$\int_0^t \|A_h E_h(t-s)\|\, ds \le C_0(1 + \log(T/h^2)) \le C_0 \ell_h. \tag{3.29}$$

Thus, by (3.12), (3.15), and Lemma 2.3,

(3.30)

$$\begin{aligned}\|\eta_t(t)\| &\le \|P_h(F(u_h(t)) - F(u(t)))\| \\ &\quad + \int_0^t \big\|A_h E_h(t-s) P_h\big(F(u_h(s)) - F(u(s))\big)\big\|\, ds \\ &\le C(K, M_2, T)(1 + \ell_h) \sup_{0\le s\le T} \|u_h(s) - u(s)\| \le C(K, M_2, T)\ell_h h^r,\end{aligned}$$

For $|\eta_t(t)|_{-2}$, we have, by (3.12),

$$\begin{aligned}|\eta_t(t)|_{-2} &\le |P_h(F(u_h(t)) - F(u(t)))|_{-2} \\ &\quad + \int_0^t \big|A_h E_h(t-s) P_h\big(F(u_h(t)) - F(u(t))\big)\big|_{-2}\, ds.\end{aligned}$$

Here, by Lemmas 2.1 and 2.3, and (3.15), (3.16),

$$\begin{aligned}|P_h(F(u_h) - F(u))|_{-2} &\le C_0\big(h^2\|P_h(F(u_h) - F(u))\| + \|G_h P_h(F(u_h) - F(u))\|\big) \\ &\le C(\|u\|_r, K)\big(h^2\|u_h - u\| + \|u_h - u\|^2 + |u_h - u|_{-2}\big), \\ &\le C(K, M_2, T)h^{r+2}.\end{aligned}$$

Thus, by (3.29),

$$|\eta_t(t)|_{-2} \le C(K, M_2, T)\ell_h h^{r+2}.$$

Together these estimates complete the proof. □

PROOF OF THEOREM 3.1. Combining (2.3) and (2.4), we have, with $\tilde G_h = \tilde A_h^{-1}$,

$$\begin{aligned}\tilde u_h(T) - u(T) &= \tilde G_h \tilde P_h(-u_{h,t} + F(u_h)) - G(-u_t + F(u)) \\ &= (\tilde G_h \tilde P_h - G)\big(-u_{h,t} + F(u_h) + u_t - F(u)\big) \\ &\quad - (\tilde G_h \tilde P_h - G)(u_t - F(u)) \\ &\quad + G(-u_{h,t} + F(u_h) + u_t - F(u))\end{aligned}$$

Thus, by Lemmas 2.3, 3.2 and 3.3, we get, noting that $\|(\tilde G_h \tilde P_h - G)f\| \le Ch^s\|f\|_{s-2}$ for $0 \le s \le r+2$,

$$\begin{aligned}\|\tilde u_h(T) - u(T)\| &\le C_0 h^2\big(\|u_{h,t} - u_t\| + \|F(u_h) - F(u)\|\big) \\ &\quad + C_0 h^{r+2}\big(\|u_t\|_r + \|F(u)\|_r\big) \\ &\quad + |u_{h,t} - u_t|_{-2} + |F(u_h) - F(u)|_{-2} \\ &\le C(K, M, T)\ell_h h^{r+2}.\end{aligned}$$

The proof is complete. □

4 Completely discrete approximation

In this section we will consider the postprocessed finite element method for (1.2) in the fully discrete case.

We use the similar technique developed in Section 3 to derive the error estimates in fully discrete case. Let $t_n = nk$, k time step. We define the following backward Euler method, with $\bar{\partial} U^n = (U^n - U^{n-1})/k$,

$$\bar{\partial} U^n + A_h U^n = P_h F(U^n), \quad n \geq 1, \quad \text{with } U^0 = v_h, \tag{4.1}$$

It is natural to approximate $u_{h,t}(T)$, $T = t_n$ in (2.4) by $\bar{\partial} U^n$ for fixed n. The postprocessing step in the fully discrete case is to find $\tilde{u}(T) \in \mathcal{D}(A)$, such that

$$A\tilde{u}(T) = -\bar{\partial} U^n + F(U^n). \tag{4.2}$$

The semidiscrete problem of (4.2) is to find $\tilde{u}_h(T) \in \tilde{S}_h$, such that,

$$\tilde{A}_h \tilde{u}_h(T) = \tilde{P}_h(-\bar{\partial} U^n + F(U^n)). \tag{4.3}$$

Let $\hat{U}^n$ be the solution of

$$\bar{\partial} \hat{U}^n + A_h \hat{U}^n = P_h F(u^n), \quad n \geq 1, \quad \text{with } \hat{U}^0 = v_h. \tag{4.4}$$

We have the following theorem.

THEOREM 4.1. *Let $r \geq 4$ and S_h and $\tilde{S}_h$ be the finite element spaces of order r and $r+2$, respectively, as described in Section 1. Let $\tilde{u}_h$ and u be the solutions of (4.3) and (1.2), respectively. Assume that F satisfies $\|F(u^n)\|_r \leq C_0$ in addition to the assumptions in Lemma 2.3. Let $T = t_n$ be a fixed time. Let U^n be the solution of (4.1). Assume that $v_h = R_h v$ and*

$$\sup_{0 \leq t_n \leq T} \|U^n - u(t_n)\|_{L_\infty} \leq K,$$

and

$$\sup_{0 \leq s \leq T} \big(\|u(s)\|_r + \|u_t(s)\|_r + \|u_{tt}(s)\| + |u_{tt}(s)|_{-2} + \|Au_{tt}(s)\| \big) \leq M, \tag{4.5}$$

for some positive numbers K, M, T. Then there is a constant $C = C(K, M, T)$ such that, with $\ell_k = 1 + \log(T/k)$,

$$\|\tilde{u}_h(T) - u(T)\| \leq C_0 \big(\|\bar{\partial} \hat{U}^1 - u_t(t_1)\| + |\bar{\partial} \hat{U}^1 - u_t(t_1)|_{-2} \big) + C\ell_k (h^{r+2} + k).$$

We now state a lemma for the error estimate of the approximation U^n of $u(t_n)$ in the L_2 norm.

LEMMA 4.2. *Let U^n and u be the solutions of (4.1) and (1.2), respectively. Assume that F satisfies the assumptions in Lemma 2.3. Further assume that $v_h = R_h v$, and*

$$\sup_{0 \leq t_n \leq T} \|U^n - u(t_n)\|_{L_\infty} \leq K, \tag{4.6}$$

and

$$\sup_{0\le s\le T}\big(\|u(s)\|_r+\|u_t(s)\|_r+\|u_{tt}(s)\|+|u_{tt}(s)|_{-2}\big)\le M_3, \tag{4.7}$$

for some positive numbers K, M_3, T. Then there is a constant $C=C(K,M_3,T)$ such that

$$\sup_{0\le t_n\le T}\|U^n-u(t_n)\|\le C(h^r+k), \tag{4.8}$$

and

$$\sup_{0\le t_n\le T}|U^n-u(t_n)|_{-2}\le C(h^{r+2}+k). \tag{4.9}$$

PROOF. We split

$$U^n-u(t_n)=(U^n-\hat U^n)-(\hat U^n-u(t_n))=\eta^n+e^n,$$

where $\hat U^n$ is defined by (4.4).

For $e^n=\hat U^n-u(t_n)$, we have, by the standard error estimates for linear parabolic problems, see, e.g., Thomée [21, Theorem 1.5],

(4.10)

$$\begin{aligned}\|e^n\|&\le C_0\|R_hv-v\|+C_0h^r\Big(\|v\|_r+\int_0^{t_n}\|u_t\|_r\,ds\Big)+C_0k\int_0^{t_n}\|u_{tt}(s)\|\,ds\\&\le C(M_3,T)(h^r+k).\end{aligned}$$

For $\eta^n=U^n-\hat U^n$, noting that, by (4.4) and (4.1),

$$\begin{cases}\bar\partial\eta^n+A_h\eta^n=P_h(F(U^n)-F(u^n)), & \text{for } n\ge 1,\\ \eta^0=0,\end{cases} \tag{4.11}$$

we have, by Lemma 2.3, with $r(\lambda)=1/(1+\lambda)$,

$$\begin{aligned}\|\eta^n\|&\le k\sum_{j=1}^n\|r(kA_h)^{n-j+1}\|\,\|P_h(F(U^j)-F(u^j))\|\\&\le C_0k\sum_{j=1}^n\|F(U^j)-F(u^j)\|\le C(K,M_3)\Big(k\sum_{j=1}^n\|\eta^j\|+k\sum_{j=1}^n\|e^j\|\Big).\end{aligned}$$

Further, by the discrete Gronwall's lemma, and (4.10), we have

$$\|\eta^n\|\le C(K,M_3,T)(h^r+k),$$

which shows (4.8).

Now we turn to (4.9). Following the proof of (4.10), we can show that,

$$|e^n|_{-2} \le C_0|R_h v - v|_{-2} + C_0 h^{r+2}\Big(\|v\|_r + \int_0^t \|u_t\|_r\,ds\Big) + C_0 k \int_0^{t_n} |u_{tt}(s)|_{-2}\,ds \le C(M_3,T)(h^{r+2}+k). \tag{4.12}$$

To estimate $|\eta^n|_{-2}$, we first note that, by Lemma 2.1,

$$|\eta^n|_{-2} \le C_0(h^2\|\eta^n\| + \|G_h\eta^n\|). \tag{4.13}$$

Here $G_h\eta^n$ satisfies, by (4.11),

$$\begin{cases} \bar\partial(G_h\eta^n) + A_h(G_h\eta^n) = G_hP_h(F(U^n)-F(u^n)), & \text{for } n \ge 1,\\ \eta^0 = 0, \end{cases} \tag{4.14}$$

which implies

$$G_h\eta^n = k\sum_{j=1}^n r(kA_h)^{n-j+1}G_hP_h(F(U^j)-F(u^j)).$$

Note that, by Lemmas 2.1 and 2.3,

$$\begin{aligned}\|G_hP_h(F(U^j)-F(u^j))\| &= |F(U^j)-F(u^j)|_{-2,h}\\ &\le C(\|u\|_r,K)\big(h^2\|U^j-u^j\| + \|U^j-u^j\|^2 + |U^j-u^j|_{-2}\big).\end{aligned}$$

Hence, by the stability of $r(\lambda)$,

$$\|G_h\eta^n\| \le C(K,M_3)\Big(k\sum_{j=1}^n|\eta^j|_{-2} + h^2k\sum_{j=1}^n\|U^j-u^j\| + k\sum_{j=1}^n(\|U^j-u^j\|^2 + |e^j|_{-2})\Big).$$

Combining this with (4.13) and using the discrete Gronwall's lemma, we get, by (4.8) and (4.12),

$$|\eta^n|_{-2} \le C(K,M_3,T)(h^{r+2}+k). \tag{4.15}$$

Together these estimates complete the proof. □

We also need the following lemma for the error estimate of the approximation $\bar\partial U^n$ of $u_t(t_n)$.

LEMMA 4.3. *Let U^n and u be the solutions of* (4.1) *and* (1.2), *respectively. Assume that F satisfies the assumptions in Lemma 2.3. Further assume that $v_h = R_h v$ and*

$$\sup_{0\le t_n\le T}\|U^n - u(t_n)\|_{L_\infty} \le K, \tag{4.16}$$

and

$$\sup_{0\le s\le T}\big(\|u(s)\|_r+\|u_t(s)\|_r+\|u_{tt}(s)\|_r+\|u_{tt}(s)\|+\|Au_{tt}(s)\|\big)\le M_4, \tag{4.17}$$

for some positive numbers K, M_4, T. Then there is a constant $C = C(K, M_4, T)$ such that, with $\ell_k = 1+\log(T/k)$,

$$\sup_{k\le t_n\le T}\|\bar\partial U^n-u_t(t_n)\|\le C_0\|\bar\partial\hat U^1-u_t(t_1)\|+C\ell_k(h^r+k), \tag{4.18}$$

and

$$\sup_{k\le t_n\le T}|\bar\partial U^n-u_t(t_n)|_{-2}\le C_0|\bar\partial\hat U^1-u_t(t_1)|_{-2}+C\ell_k(h^{r+2}+k). \tag{4.19}$$

PROOF. We use the same notation as in Lemma 4.2 and write

$$\begin{aligned}\bar\partial U^n-u_t(t_n)&=(\bar\partial U^n-\bar\partial\hat U^n)+(\bar\partial\hat U^n-u_t(t_n))\\&=\bar\partial\eta^n+\big(\bar\partial\hat U^n-u_t(t_n)\big).\end{aligned}$$

We first show

$$\begin{aligned}\|\bar\partial\hat U^n-u_t(t_n)\|&\le C_0\|\bar\partial\hat U^1-u_t(t_1)\|+C_0h^r\Big(\|u_t(0)\|_r+\int_0^{t_n}\|u_{tt}\|_r\,ds\Big)\\&\quad+C_0k\int_0^{t_n}\|Au_{tt}(s)\|\,ds\\&\le C_0\|\bar\partial\hat U^1-u_t(t_1)\|+C(M_4,T)(h^r+k).\end{aligned} \tag{4.20}$$

To show (4.20), we write

$$\bar\partial\hat U^n-u_t(t_n)=(\bar\partial\hat U^n-R_hu_t(t_n))+(R_hu_t(t_n)-u_t(t_n))=\theta^n+\rho^n.$$

In the standard way ρ^n is bounded as desired, and it remains to consider $\theta^n\in S_h$. We have

$$\bar\partial\theta^n+A_h\theta^n=P_h\omega^n,\quad \text{for } n\ge 2,$$

where

$$\omega^n=(R_h-I)\bar\partial u_t(t_n)+A(\bar\partial u^n-u_t^n)=\sigma^n+\tau^n.$$

By stability estimate, see, e.g., Thomée [21, Theorem 10.2],

$$\|\theta^n\|\le C_0\|\theta^1\|+C_0k\sum_{j=2}^n\|\sigma^j\|+C_0k\sum_{j=2}^n\|\tau^j\|,\quad \text{for } n\ge 2. \tag{4.21}$$

We have

$$k\|\sigma^n\|\le C_0h^r\int_{t_{n-1}}^{t_n}\|u_{tt}\|_r\,ds,$$

and

$$k\|\tau^n\| \le C_0 k \|A(\bar\partial u^n - u_t^n)\| \le C_0 k \int_{t_{n-1}}^{t_n} \|Au_{tt}(s)\|\, ds.$$

Together with $\|\theta^1\| \le \|\bar\partial U^1 - u_t^1\| + \|\rho^1\|$, with the obvious bounds for $\|\rho^1\|$, this completes the proof of (4.20).

For $\|\bar\partial\eta^n\|$, we have, by (4.11),

$$\bar\partial\eta^n = P_h(F(U^n) - F(u^n)) - k\sum_{j=1}^{n} A_h r(kA_h)^{n-j+1} P_h(F(U^n) - F(u^n)). \tag{4.23}$$

Using the following smoothing property

$$k\sum_{j=1}^{n} \|A_h r(kA_h)^{n-j+1}\| \le C_0 \ell_k, \tag{4.24}$$

which follows from

$$\begin{aligned} k\sum_{j=1}^{n} \|A_h r(kA_h)^{n-j+1}\| &\le C_0 k \sum_{j=1}^{n} t_{n-j+1}^{-1} = C_0\Big(1 + \sum_{j=1}^{n-1} t_{n-j+1}^{-1}\Big) \\ &\le C_0\Big(1 + \int_{t_1}^{t_n} \frac{1}{s}\, ds\Big) \le C_0(1 + \log(t_n/k)) \le C_0 \ell_k, \end{aligned}$$

we have, by Lemma 2.3, and (4.8),

$$\begin{aligned} \|\bar\partial\eta^n\| &\le C(K, M_4)\big(\|U^n - u^n\| + \ell_k \max_{1\le j\le n} \|U^j - u^j\|\big) \\ &\le C(K, M_4, T)\ell_k (h^r + k). \end{aligned} \tag{4.25}$$

Together these estimates complete the proof of (4.18).

Now we turn to estimate (4.19). Following the proof of (4.20), we can show

(4.26)

$$\begin{aligned} |\bar\partial\hat U^n - u_t(t_n)|_{-2} &\le C_0 |\bar\partial \hat U^1 - u_t(t_1)|_{-2} + C_0 h^{r+2}\Big(\|u_t(0)\|_r + \int_0^{t_n} \|u_{tt}\|_r\, ds\Big) \\ &\quad + C_0 k \int_0^{t_n} \|u_{tt}(s)\|\, ds, \\ &\le C_0 |\bar\partial\hat U^1 - u_t(t_1)|_{-2} + C(M_4, T)(h^{r+2} + k). \end{aligned}$$

For $|\bar\partial\eta^n|_{-2}$, we have, using (4.22), and by Lemmas 2.1 and 2.3,

$$|\bar\partial\eta^n|_{-2} \le C(K, M_4, T)\ell_k \max_{1\le j\le n} \big(h^2\|U^j - u^j\| + \|U^j - u^j\|^2 + |U^j - u^j|_{-2}\big).$$

Thus, by (4.8) and (4.9),

$$|\bar\partial\eta^n|_{-2} \le C(K, M_4, T)\ell_k (h^{r+2} + k). \tag{4.27}$$

Together these estimates complete the proof. □

PROOF OF THEOREM 4.1. Combining (2.3) and (4.3), we have, with $\tilde{G}_h = \tilde{A}_h^{-1}$,

$$\begin{aligned}\tilde{u}_h(T) - u(T) &= \tilde{G}_h\tilde{P}_h(-\bar{\partial}U^n + F(U^n)) - G(-u_t(t_n) + F(u^n)) \\ &= (\tilde{G}_h\tilde{P}_h - G)\big(-\bar{\partial}U^n + F(U^n) + u_t(t_n) - F(u^n)\big) \\ &\quad - (\tilde{G}_h\tilde{P}_h - G)\big(u_t(t_n) - F(u^n)\big) \\ &\quad + G\big(-\bar{\partial}U^n + F(U^n) + u_t(t_n) - F(u^n)\big).\end{aligned}$$

Thus, we get, noting that $\|(\tilde{G}_h\tilde{P}_h - G)f\| \le Ch^s\|f\|_{s-2}$ for $0 \le s \le r+2$,

$$\begin{aligned}\|\tilde{u}_h(T) - u(T)\| \le\ & C_0h^2\big(\|\bar{\partial}U^n - u_t(t_n)\| + \|F(U^n) - F(u^n)\|\big) \\ &+ C_0h^{r+2}\|u_t(t_n) - F(u^n)\|_r \\ &+ |\bar{\partial}U^n - u_t(t_n)|_{-2} + |F(U^n) - F(u^n)|_{-2}.\end{aligned}$$

Combining this with Lemmas 2.3, 4.2, and 4.3, we complete the proof. □

5 Error estimate for the starting approximation

In this section we will consider the error estimate for the starting approximation of the time derivative $|\bar{\partial}\hat{U}^1 - u_t(t_1)|_{-s},\ s = 0, 2$, which appears in Theorem 4.1, where u and $\hat{U}^1$ satisfy

(5.1) $$u_t + Au = F(u), \quad \text{with } u(0) = v,$$

and

(5.2) $$\bar{\partial}\hat{U}^1 + A_h\hat{U}^1 = P_hF(u^1), \quad \text{with } \hat{U}^0 = v_h = R_hv,$$

respectively.

The semidiscrete problem of (5.1) is to find $\hat{u}_h \in S_h$ such that,

(5.3) $$\hat{u}_{h,t} + A_h\hat{u}_h = P_hF(u), \quad \text{with } \hat{u}_h(0) = R_hv.$$

We observe that we use $F(u^1)$ in (5.2), thus $|\bar{\partial}\hat{U}^1 - u_t(t_1)|_{-s},\ s = 0, 2$, can be bounded by the standard technique for nonhomogeneous linear parabolic problems. We have the following theorem:

THEOREM 5.1. *Let $\hat{U}^1$ and u be the solutions of* (5.2) *and* (5.1), *respectively. Assume that F is continuously differentiable and*

$$\|Au_t(0)\| + \|u_t(0)\|_r + \max_{0\le\tau\le k}\big(\|F'(u(\tau))u_t(\tau)\| + \|u_{tt}(\tau)\|_r\big) \le M_0,$$

for some positive number M_0. Then there is a constant $C = C(M_0)$ such that

(5.4) $$\|\bar{\partial}\hat{U}^1 - u_t(t_1)\| \le C(h^r + k),$$

and

$$|\bar\partial \hat U^1 - u_t(t_1)|_{-2} \le C(h^{r+2} + k). \tag{5.5}$$

PROOF. We first show (5.4). We write

$$\bar\partial \hat U^1 - u_t(t_1) = \big(\bar\partial \hat U^1 - \hat u_{h,t}(t_1)\big) + \big(\hat u_{h,t}(t_1) - u_t(t_1)\big).$$

By (3.27), we have

$$\|\hat u_{h,t}(t_1) - u_t(t_1)\| \le C_0 h^r \Big(\|u_t(0)\|_r + \int_0^{t_1} \|u_{tt}(s)\|_r \, ds\Big). \tag{5.6}$$

For $\bar\partial \hat U^1 - \hat u_{h,t}(t_1)$, we have, by (5.2) and (5.3),

$$\bar\partial \hat U^1 - \hat u_{h,t}(t_1) = A_h(\hat U^1 - \hat u_h^1).$$

Here, by Taylor's formula, with $r(\lambda) = 1/(1+\lambda)$, $E_h(t) = e^{-tA_h}$,

$$\begin{aligned}
\hat U^1 - \hat u_h^1 =& \big(r(kA_h) - E_h(t_1)\big) R_h v + k r(kA_h) P_h F(u^1) \\
&- \int_0^{t_1} E_h(t_1 - s) P_h F(u(s))\, ds \\
=& \big(r(kA_h) - E_h(t_1)\big) R_h v \\
&+ k r(kA_h)\big(P_h F(u(0)) + \int_0^k P_h F'(u(\tau)) u_t(\tau)\, d\tau\big) \\
&- k \int_0^1 e^{-(1-s)kA_h} \big(P_h F(u(0)) + \int_0^{ks} P_h F'(u(\tau)) u_t(\tau)\, d\tau\big)\, ds \\
=& \big(r(kA_h) - E_h(t_1)\big) R_h v + k b_0(kA_h) P_h F(u(0)) + kR(F),
\end{aligned}$$

where

$$b_0(\lambda) = r(\lambda) - \int_0^1 e^{-(1-s)\lambda}\, ds,$$

and

$$\begin{aligned}
R(F) = & \, r(kA_h) \int_0^k P_h F'(u(\tau)) u_t(\tau)\, d\tau \\
& - \int_0^1 e^{-(1-s)kA_h} \int_0^{ks} P_h F'(u(\tau)) u_t(\tau)\, d\tau\, ds.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\bar\partial \hat U^1 - \hat u_{h,t}(t_1) =& \big(r(kA_h) - E_h(t_1)\big) A_h R_h v \\
&+ kA_h b_0(kA_h) P_h F(u(0)) + kA_h R(F).
\end{aligned}$$

Noting that $A_h R_h = P_h A$ and $\lambda b_0(\lambda) = -\big(r(\lambda) - e^{-\lambda}\big)$, we get

$$\begin{aligned}
(5.7) \qquad \bar\partial \hat U^1 - \hat u_{h,t}(t_1) &= \big(r(kA_h) - E_h(t_1)\big)P_h\big(Av - F(u(0))\big) + kA_hR(F) \\
&= \big(r(kA_h) - E_h(t_1)\big)P_h u_t(0) + kA_hR(F) \\
&= I + II.
\end{aligned}$$

For I, we have, by the error estimate for homogeneous parabolic problems,

$$\begin{aligned}
\|I\| &\le \big\|\big(r(kA_h) - E_h(t_1)\big)(P_h - R_h)u_t(0)\big\| + \big\|\big(r(kA_h) - E_h(t_1)R_hu_t(0)\big\| \\
&\le \|(P_h - R_h)u_t(0)\| + C_0k\|A_hR_hu_t(0)\| \\
&\le C_0h^r\|u_t(0)\|_r + C_0k\|Au_t(0)\|.
\end{aligned}$$

For II, we write

$$\begin{aligned}
II &= kA_hr(kA_h)\int_0^k P_hF'(u(\tau))u_t(\tau)\,d\tau \\
&\quad - kA_h\int_0^1 e^{-(1-s)kA_h}\int_0^{ks} P_hF'(u(\tau))u_t(\tau)\,d\tau\,ds \\
&= II_1 + II_2.
\end{aligned}$$

We have, noting that $|\lambda r(\lambda)| \le 1,\ \|P_h\| \le 1$,

$$\begin{aligned}
\|II_1\| &\le \|kA_hr(kA_h)\|\int_0^k \|P_hF'(u(\tau))u_t(\tau)\|\,d\tau \\
&\le k\max_{0\le\tau\le k}\|F'(u(\tau))u_t(\tau)\|,
\end{aligned}$$

and, by exchanging the integral order and noting that $\int_\epsilon^1 \lambda e^{-(1-s)\lambda}\,ds \le 1$, for $0 \le \epsilon \le 1$,

$$\begin{aligned}
\|II_2\| &= \Big\|kA_h\int_0^k P_hF'(u(\tau))u_t(\tau)\int_{\tau/k}^1 e^{-(1-s)kA_h}\,dsd\tau\Big\| \\
&\le k\max_{0\le\tau\le k}\|F'(u(\tau))u_t(\tau)\|\,\Big\|kA_h\int_{\tau/k}^1 e^{-(1-s)kA_h}\,ds\Big\| \\
&\le k\max_{0\le\tau\le k}\|F'(u(\tau))u_t(\tau)\|.
\end{aligned}$$

Together these estimates show

$$(5.8) \quad \|\bar\partial\hat U^1 - \hat u_{h,t}(t)\| \le C_0\big(h^r\|u_t(0)\|_r + k\|Au_t(0)\| + k\max_{0\le\tau\le k}\|F'(u(\tau))u_t(\tau)\|\big).$$

Combining this with (5.6) shows (5.4).

We now turn to (5.5). We again write

$$\bar\partial\hat U^1 - u_t(t_1) = \big(\bar\partial\hat U^1 - \hat u_{h,t}(t_1)\big) + \big(\hat u_{h,t}(t_1) - u_t(t_1)\big).$$

The desired bound for $|\hat{u}_{h,t}(t_1) - u_t(t_1)|_{-2}$ follows from (3.28).

For $\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1)$, we have, by Lemma 2.1,

$$|\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1)|_{-2} \leq C_0\big(h^2\|\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1)\| + |\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1)|_{-2,h}\big).$$

Thus, by (5.7),

$$|\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1)|_{-2,h} \leq |I|_{-2,h} + |II|_{-2,h}.$$

For $|I|_{-2,h}$, we have, by the error estimate for homogeneous parabolic problems, see [24],

$$|I|_{-2,h} = |(r(kA_h) - E_h(t_1))P_h u_t(0)|_{-2,h} \leq C_0\big(h^{r+2}\|u_t(0)\|_r + k\|Au_t(0)\|\big).$$

For $|II|_{-2,h}$, we have, noting that $|r(\lambda)| \leq 1$, $\int_\epsilon^1 e^{-(1-s)\lambda}\,ds \leq 1$ for $0 \leq \epsilon \leq 1$,

$$\begin{aligned} |II|_{-2,h} \leq & \int_0^k \|kr(kA_h)\|\,\|P_h F'(u(\tau))u_t(\tau)\|\,d\tau \\ & + \left\|k\int_0^k P_h F'(u(\tau))u_t(\tau)\int_{\tau/k}^1 e^{-(1-s)kA_h}\,ds\,d\tau\right\| \\ \leq & k^2 \max_{0\leq\tau\leq k}\|F'(u(\tau))u_t(\tau)\|. \end{aligned}$$

Hence we get

$$|\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1)|_{-2,h} \leq C_0\big(h^{r+2}\|u_t(0)\|_r + k\|Au_t(0)\| + k^2\max_{0\leq\tau\leq k}\|F'(u(\tau))u_t(\tau)\|\big).$$

Combining this with (5.8) shows (5.5).

Together these estimates complete the proof of the theorem. □

6 High order time-stepping

The postprocessing requires very accurate time-stepping in order to match the high order spatial approximation. It would be natural then to use a time-stepping method of higher order than the backward Euler method of Section 4. However, we have not been able to analyze such methods except in the case of linear homogeneous problems, where we can apply the analysis of time derivative approximation from [24].

In this section we consider the linear homogeneous parabolic problem

$$u_t + Au = 0, \quad \text{for } t > 0, \quad \text{with } u(0) = v. \tag{6.1}$$

We define the following time-stepping method

$$U^n = r(kA_h)U^{n-1}, \quad U^0 = v_h, \tag{6.2}$$

where $r(\lambda)$ is a rational function and accurate of order $p \geq 1$, i.e.,

$$r(\lambda) - e^{-\lambda} = O(\lambda^{p+1}), \quad \lambda \to 0.$$

For example, if $r(\lambda) = 1/(1+\lambda)$, we have $(1 + kA_h)U^n = U^{n-1}$, which is the backward Euler method. If $r(\lambda) = (1-\lambda/2)/(1+\lambda/2)$, we have $(1+\frac{1}{2}kA_h)U^n = (1-\frac{1}{2}kA_h)U^{n-1}$ which is the Crank-Nicolson method.

Further we define the quotient Q_kU^n to approximate the time derivative $u_{h,t}(t_n)$, with positive integers m_1, m_2, and real numbers c_ν,

$$Q_kU^n = k^{-1}\sum_{\nu=-m_1}^{m_2} c_\nu U^{n+\nu}, \quad \text{for } n \geq m_1, \tag{6.3}$$

We assume that the operator Q_k satisfies, for any smooth function u,

$$Q_ku^n - u_t(t_n) = O(k^p), \quad k \to 0. \tag{6.4}$$

For example,

$$Q_ku^n = \bar{\partial}u^n = (u^n - u^{n-1})/k, \quad \text{for } n \geq 1,$$

and

$$Q_ku^n = (\frac{3}{2}u^n - 2u^{n-1} + \frac{1}{2}u^{n-1})/k, \quad \text{for } n \geq 2,$$

satisfy

$$\bar{\partial}u^n - u_t(t_n) = O(k), \quad k \to 0,$$

and

$$(\tfrac{3}{2}u^n - 2u^{n-1} + \tfrac{1}{2}u^{n-1})/k - u_t(t_n) = O(k^2), \quad k \to 0,$$

respectively.

The postprocessing step in fully discrete case is to find $\tilde{u}(T) \in S_h$, $T = t_n$, such that

$$A\tilde{u}(T) = -Q_kU^n. \tag{6.5}$$

The finite element solution of the elliptic problem (6.5) with respect to $\tilde{S}_h$ is to find $\tilde{u}_h(T) \in \tilde{S}_h$, such that,

$$\tilde{A}_h\tilde{u}_h(T) = \tilde{P}_h(-Q_kU^n). \tag{6.6}$$

Our main theorem in this section is the following:

THEOREM 6.1. *Let $r \geq 4$ and S_h and $\tilde{S}_h$ be the finite element spaces of order r and $r+2$, respectively, as described in Section* 1. *Let $\tilde{u}_h$ and u be the solutions of* (6.6) *and* (6.1), *respectively. Let $T = t_n$ be a fixed time. Then we have, if $v_h = R_hv$,*

$$\|\tilde{u}_h(T) - u(T)\| \leq C_0\big(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)} + h^{r+2}\|u_t(T)\|_r\big), \quad \textit{for } r \geq 4.$$

Recalling the proof of Theorem 4.1, we note that Theorem 6.1 follows once we have proved appropriate estimates of $\|Q_kU^n - u_t(t_n)\|$ and $|Q_kU^n - u_t(t_n)|_{-2}$ which are given in the following two lemmas.

LEMMA 6.2. *Let U^n and u be the solutions of* (6.2) *and* (6.1), *respectively. Assume that $|r(\lambda)| < 1$ for $\lambda > 0$. Then, we have, if $v_h = R_h v$,*

$$\|Q_k U^n - u_t(t_n)\| \le C_0(h^r |v|_{r+2} + k^p |v|_{2(p+1)}).$$

Lemma 6.2 was proved in [24].

LEMMA 6.3. *Let U^n and u be the solutions of* (6.2) *and* (6.1), *respectively. Assume that $|r(\lambda)| < 1$ for $\lambda > 0$. Then, we have, if $v_h = R_h v$,*

$$|Q_k U^n - u_t(t_n)|_{-2} \le C_0\big(h^{r+2} |v|_{r+2} + k^p |v|_{2(p+1)}\big).$$

Let us first prove the following error estimate for the approximation U^n of $u(t_n)$ in negative order norm. We do not need it here but it serves as a guide for the proof of Lemma 6.3. We remark that we choose $v_h = P_h v$, not $R_h v$.

LEMMA 6.4. *Let U^n and u be the solutions of* (6.2) *and* (6.1), *respectively. Assume that $|r(\lambda)| < 1$ for $\lambda > 0$. Then, we have, if $v_h = P_h v$,*

$$|U^n - u(t_n)|_{-2} \le C\big(h^{r+2}|v|_r + k^p |v|_{2p}\big).$$

PROOF. By Thomée [21, Theorem 6.2], we have

$$|u_h(t) - u(t)|_{-2} \le C_0 h^{r+2} |v|_r.$$

Therefore it suffices to show

$$|U^n - u_h(t_n)|_{-2} = C_0\big(h^{r+2}|v|_r + k^p|v|_{2p}\big), \tag{6.7}$$

which we will prove now.

By Lemma 2.1, we have

$$|U^n - u_h(t_n)|_{-2} \le C_0(h^2 \|U^n - u_h(t_n)\| + |U^n - u_h(t_n)|_{-2,h}).$$

We first estimate $|U^n - u_h(t_n)|_{-2,h} = \|G_h(U^n - u_h(t_n)\|$. We write

$$U^n - u_h(t_n) = \big(r(kA_h)^n - e^{-knA_h}\big)P_h v = F_n(kA_h)P_h v,$$

where $F_n(\lambda) = r(\lambda)^n - e^{-n\lambda}$. We need to show

$$\|G_h F_n(kA_h) P_h v\| \le C\big(h^{r+2}|v|_r + k^p |v|_{2p}\big).$$

To do this we set

$$v_k = \sum_{k\lambda_l \le 1} (v, \varphi_l)\varphi_l,$$

where φ_l and λ_l are the eigenfunctions and eigenvalues of the operator A. Then $v_k \in \dot{H}^s$ for each $s \ge 0$. Further, by the definition of the norm in $\dot{H}^s$, we find easily

$$\|v - v_k\| \le k^p |v|_{2p}, \tag{6.8}$$

$$|v_k|_{2p} \le |v|_{2p}, \tag{6.9}$$

and

$$|v_k|_{r+2j} \le k^{-j}|v|_r, \quad \text{for } 0 \le j \le p-1. \tag{6.10}$$

Applying now the identity

$$v = \sum_{j=0}^{p-1} G_h^j (G - G_h) A^{j+1} v + G_h^p A^p v, \quad \text{for } v \in \dot H^{2p}, \quad \text{where } G_h^0 = I, \tag{6.11}$$

to v_k, we have, with $F_n = F_n(kA_h)P_h$,

$$G_h F_n v_k = \sum_{j=0}^{p-1} G_h F_n G_h^j (G - G_h) A^{j+1} v_k + G_h F_n G_h^p A^p v_k. \tag{6.12}$$

It is easy to show that, see, e.g., Thomée [21, Lemma 7.2],

$$\|F_n(kA_h)P_h G_h^j\| \le C_0 k^j \quad \text{for } 0 \le j \le p,\ n \ge 0. \tag{6.13}$$

Thus, by (6.9) and noting the boundedness of G_h,

$$\begin{aligned} \|G_h F_n G_h^p A^p v_k\| &\le \|F_n G_h^p A^p v_k\| \le C_0 k^p \|A^p v_k\| \\ &\le C_0 k^p |v_k|_{2p} \le C_0 k^p |v|_{2p}. \end{aligned}$$

Further, by (6.10), (6.13), and using (2.11), and noting that $P_h G_h^j = G_h^j$, with $0 \le j \le p-1$,

$$\begin{aligned} &\|G_h F_n G_h^j (G - G_h) A^{j+1} v_k\| \le C_0 k^j \|G_h (G - G_h) A^{j+1} v_k\| \\ &\le C_0 k^j h^2 \big\|(G - G_h)(A^{j+1} v_k)\big\| + C_0 k^j \big|(G - G_h)(A^{j+1} v_k)\big|_{-2} \\ &\le C_0 k^j h^{r+2} \|A^{j+1} v_k\|_{r-2} \le C_0 k^j h^{r+2} |v_k|_{r+2j} \le C_0 h^{r+2} |v|_r. \end{aligned}$$

Together these estimates imply

$$\|G_h F_n v_k\| \le C_0 \big(h^{r+2}|v|_r + k^p |v|_{2p}\big).$$

Since obviously, by (6.8), the boundedness of G_h and stability, we get

$$\|G_h F_n (v - v_k)\| \le \|F_n (v - v_k)\| \le 2\|(v - v_k)\| \le C_0 k^p |v|_{2p},$$

so that

$$\big\|G_h (U^n - u_h(t_n))\big\| = \|G_h F_n v\| \le C_0 \big(h^{r+2}|v|_r + k^p |v|_{2p}\big).$$

By Thomée [21, Theorem 7.8], we have

$$\|U^n - u_h(t_n)\| \le C_0 \big(h^r |v|_r + k^p |v|_{2p}\big), \quad t_n \ge 0.$$

Thus,

$$\left|U^n - u_h(t_n)\right|_{-2} \le \left\|(G - G_h)(U^n - u_h(t_n))\right\| + \left\|G_h(U^n - u_h(t_n))\right\| \\ \le C_0\big(h^{r+2}|v|_r + k^p|v|_{2p}\big).$$

Together these estimates complete the proof. □

Now we turn to the proof of Lemma 6.3. The idea of the proof is similar to the one used in Lemma 6.4

PROOF OF THEOREM 6.3. By Thomée [21, Theorem 6.4], we have

$$\left|u_{h,t}(t) - u_t(t)\right|_{-2} \le Ch^{r+2}|v|_{r+2}.$$

Therefore it suffices to show

$$\left|Q_kU^n - u_{h,t}(t_n)\right|_{-2} \le C\big(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)}\big), \tag{6.14}$$

which we will prove now.

We first estimate $\left|Q_kU^n - u_{h,t}(t_n)\right|_{-2,h}$. Noting that, with $v_h = R_hv = G_hAv$,

$$Q_kU^n - u_{h,t}(t_n) = k^{-1}\Big(\sum_{\nu=-m_1}^{m_2} c_\nu U^{n+\nu} - (-A_h)e^{-nkA_h}\Big)G_hAv \\ = k^{-1}g_n(kA_h)G_hAv,$$

where $g_n(\lambda) = \sum_{\nu=-m_1}^{m_2} r(\lambda)^{n+\nu} - (-\lambda)e^{-n\lambda}$, we need to show

$$\left\|G_h\big(k^{-1}g_n(kA_h)G_hAv\big)\right\| \le C_0\big(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)}\big).$$

As in the proof of Lemma 6.4, we introduce v_k which satisfies:

$$\|A(v - v_k)\| \le k^p|v|_{2p+2}, \tag{6.15}$$

$$|v_k|_{2(p+1)} \le |v|_{2(p+1)}, \tag{6.16}$$

and

$$|v_k|_{r+2l+2} \le k^{-l}|v|_{r+2}, \quad \text{for } 0 \le l \le p-1. \tag{6.17}$$

Applying now the identity (6.11) to $v = Av_k$, we get

$$G_hg_n(kA_h)G_hAv_k = \sum_{l=0}^{p-1} g_n(kA_h)G_h^{l+1}\big(G_h(G - G_h)A^{l+2}v_k\big) \\ + G_hg_n(kA_h)G_h^{p+1}A^{p+1}v_k.$$

It is easy to show that, see, e.g., [24, Lemma 3.9],

$$\|g_n(kA_h)G_h^{l+1}\| \le C_0k^{l+1}, \quad \text{for } 0 \le l \le p,\ n \ge 0. \tag{6.18}$$

Thus, by (6.16) and noting the boundedness of G_h,

$$\begin{aligned}\|G_h g_n(kA_h)G_h^{p+1}A^{p+1}v_k\| &\le \|g_n(kA_h)G_h^{p+1}A^{p+1}v_k\| \\ &\le C_0k^{p+1}\|A^{p+1}v_k\| \le C_0k^{p+1}|v_k|_{2(p+1)} \le C_0k^{p+1}|v|_{2(p+1)}.\end{aligned}$$

Further, by (6.17), (6.18), and using (2.11), we have, with $0 \le l \le p-1$,

$$\begin{aligned}&\big\|g_n(kA_h)G_h^{l+1}\big(G_h(G-G_h)A^{l+2}v_k\big)\big\| \le C_0k^{l+1}\|G_h(G-G_h)A^{l+2}v_k\| \\ &\le C_0k^{l+1}h^2\|(G-G_h)(A^{l+2}v_k)\| + C_0k^{l+1}h^{r+2}|A^{l+2}v_k|_{r-2} \\ &\le C_0k^{l+1}h^{r+2}\|A^{l+2}v_k\|_{r-2} \le C_0k^{l+1}h^{r+2}|v_k|_{r+2l+2} \le C_0kh^{r+2}|v|_{r+2}.\end{aligned}$$

Together these estimates imply

$$\|G_h g_n(kA_h)G_hAv_k\| \le C_0k(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)}).$$

Since obviously, by (6.15), the boundedness of G_h and stability, we get

$$\begin{aligned}\|G_h g_n(kA_h)G_hA(v-v_k)\| &\le \|g_n(kA_h)G_hA(v-v_k)\| \\ &\le C_0k\|A(v-v_k)\| \le C_0k^{p+1}|v|_{2(p+1)},\end{aligned}$$

we conclude that

$$\begin{aligned}\|G_h(Q_kU^n - u_{h,t}(t_n))\| &= k^{-1}\|G_h g_n(kA_h)G_hAv\| \\ &\le C_0(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)}).\end{aligned}$$

By [24, Theorem 3.8], we have

$$\|Q_kU^n - u_{h,t}(t_n)\| \le C_0(h^r|v|_{r+2} + k^p|v|_{2p}).$$

Thus

$$\begin{aligned}|Q_kU^n - u_{h,t}(t_n)|_{-2} &\le \|(G-G_h)(Q_kU^n - u_{h,t}(t_n))\| \\ &\quad + \|G_h(Q_kU^n - u_{h,t}(t_n))\| \\ &\le C_0(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)}).\end{aligned}$$

Together these estimates complete the proof. □

After the preparations above we now come to the proof of Theorem 6.1.

PROOF OF THEOREM 6.1. Combining (6.6) and (6.1), we get, with $\tilde G_h = \tilde A_h^{-1}$,

$$\begin{aligned}\tilde u_h(T) - u(T) &= \tilde G_h\tilde P_h(-Q_kU^n) - G(-u_t) \\ &= (\tilde G_h\tilde P_h - G)(-Q_kU^n + u_t(t_n)) \\ &\quad - (\tilde G_h\tilde P_h - G)u_t(t_n) + G(Q_kU^n - u_t).\end{aligned}$$

Thus, by Lemmas 6.2 and 6.3, and noting that $\|(\tilde G_h\tilde P_h - G)f\| \le Ch^s\|f\|_{s-2}$, for $0 \le s \le r+2$, we have

$$\begin{aligned}\|\tilde u_h(T) - u(T)\| &\le C_0h^2\|Q_kU^n - u_t(t_n)\| \\ &\quad + C_0h^{r+2}\|u_t(t_n)\|_r + |(Q_kU^n - u_t(t_n))|_{-2} \\ &\le C_0\big(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)} + h^{r+2}\|u_t(t_n)\|_r\big).\end{aligned}$$

Together these estimates complete the proof. □

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Paper V

The finite element method for a linear stochastic parabolic partial differential equation driven by additive noise

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Abstract.

In this paper we consider the finite element method for a stochastic parabolic partial differential equation forced by additive space-time noise in the multi-dimensional case. Optimal strong convergence error estimates in the L_2 and $\dot{H}^{-1}$ norms with respect to spatial variable have been obtained. The proof is based on appropriate nonsmooth data error estimates for the corresponding deterministic parabolic problem.

AMS subject classification: 65M, 60H15, 65C30, 65M65.

Key words: stochastic parabolic partial differential equations, finite element method, backward Euler method, additive noise, Hilbert space.

1 Introduction

In this paper we will study the finite element approximation of the linear stochastic parabolic partial differential equation

$$du + Au\,dt = dW, \quad \text{for } 0 < t \le T, \quad \text{with } u(0) = u_0, \tag{1.1}$$

in a Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, where $u(t)$ is an H-valued random process, A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, where $W(t)$ is a Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $u_0 \in H$.

For the sake of simplicity, we shall concentrate on the case $A = -\Delta$, where Δ stands for the Laplacian operator subject to homogeneous Dirichlet boundary conditions, and $H = L_2(\mathcal{D})$, where $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d, d = 1, 2, 3$, with a sufficiently smooth boundary $\partial\mathcal{D}$.

Such equations are common in applications. Many mathematics models in physics, chemistry, biology, population dynamics, neurophysiology, etc., are described by stochastic partial differential equations, see, Da Prato and Zabczyk [5], Walsh [18], etc. The existence, uniqueness, and properties of the solutions of such equations have been well studied, see Curtain and Falb [2], Da Prato [3], Da Prato and Lunardi [4], Da Prato and Zabczyk [5], Dawson [7], Gozzi [9], Peszat and Zabczyk [14], Walsh [18], etc. However, numerical approximation of such equations has not been studied thoroughly.

The equation (1.1) can be written formly as

$$u_t + Au = \frac{dW}{dt} \quad \text{for}\, 0 < t \leq T, \quad \text{with}\, u(0) = u_0, \tag{1.2}$$

where the derivative $\frac{dW}{dt}$ (noise) does not exist as a function of t in the usual sense. Thus the equation (1.2) is understood in the integral form.

Let $E(t) = e^{-tA},\ t \geq 0$. Then (1.2) admits a unique mild solution, see Da Prato and Zabczyk [5, Theorem 5.2, 5.4],

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\, dW(s) \quad \text{for } 0 < t \leq T, \tag{1.3}$$

where the integral is understood in Itô sense.

We assume that $W(t)$ is a Wiener process with covariance operator Q. This process may be considered in terms of its Fourier series. Suppose that Q has eigenvalues $\gamma_l > 0$ and corresponding eigenfunctions e_l. Then

$$W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t),$$

where β_l, $l = 1, 2, \cdots$, is a sequence of real-valued independent identically distributed Brownian motions.

If Q is in trace class, then $W(t)$ is an H-valued process. If Q is not in trace class, for example $Q = I$, then $W(t)$ does not belong to H, which is called a cylindrical Wiener process, but stochastic integral can be defined with respect to W, when the integral smoothes the noise process sufficiently.

Let $L_2^0 = HS(Q^{1/2}(H), H)$ denote the space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to H, i.e.,

$$L_2^0 = \Big\{ \psi \in L(H) : \sum_{l=1}^{\infty} \|\psi Q^{1/2} e_l\|^2 < \infty \Big\},$$

with norm $\|\psi\|_{L_2^0} = \left(\sum_{l=1}^{\infty} \|\psi Q^{1/2} e_l\|^2 \right)^{1/2}$, where $L(H)$ is the space of bounded operator from H to H.

Let $\mathbf{E}$ denote the expectation. The Itô isometry for a Wiener process of covariance operator Q states that, for an integrand $\psi \in L_2^0$,

$$\mathbf{E} \Big\| \int_0^t \psi(s)\, dW(s) \Big\|^2 = \int_0^t \|\psi(s)\|_{L_2^0}^2\, ds.$$

Let S_h be a family of finite element spaces, where S_h consists of continuous piecewise polynomials of degree ≤ 1 with respect to the triangulation $\mathcal{T}$ of Ω. For simplicity, we always assume that $\{S_h\} \subset H_0^1 = H_0^1(\mathcal{D}) = \{v \in L_2(\mathcal{D}), \nabla v \in$

$L_2(\mathcal{D}), v|_{\partial\mathcal{D}} = 0\}$. The semidiscrete problem of (1.1) is to find the process $u_h(t) = u_h(\cdot, t) \in S_h$, such that

$$du_h + A_h u_h \, dt = P_h dW, \quad \text{for } 0 < t \leq T, \quad \text{with } u_h(0) = P_h u_0, \tag{1.4}$$

where P_h denotes the L_2-projection onto S_h, and $A_h : S_h \to S_h$ is the discrete analogue of A, defined by

$$(A_h \psi, \chi) = A(\psi, \chi), \quad \forall \psi, \chi \in S_h. \tag{1.5}$$

Here $A(\cdot, \cdot)$ is the bilinear form on $H_0^1(\mathcal{D})$ obtained from the operator A in (1.1).

With $E_h(t) = e^{-tA_h}$, $t \geq 0$, then (1.4) admits a unique mild solution

$$u_h(t) = E_h(t) P_h u_0 + \int_0^t E_h(t-s) P_h \, dW(s).$$

Let $\dot{H}^s = \dot{H}^s(\mathcal{D}) = \mathcal{D}(A^{s/2})$ for any $s \in \mathbf{R}$ and denote the norm by $|\cdot|_s = \|A^{s/2} \cdot \|$. For any Hilbert space H_1, we denote $L_2(\Omega; H_1)$ by

$$L_2(\Omega; H_1) = \Big\{ v : \mathbf{E}\|v\|_{H_1}^2 = \int_\Omega \|v(\omega)\|_{H_1}^2 \, d\mathbf{P}(\omega) < \infty \Big\},$$

with the norm $\|v\|_{L_2(\Omega; H_1)} = (\mathbf{E}\|v\|_{H_1}^2)^{1/2}$.

Under the condition $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0, 1]$, we show in Lemma 2.5 that $W(t) \in \dot{H}^{\beta-1} \subset \dot{H}^{-1}$, so that $P_h W(t)$ is well defined, and we obtain, in Theorems 3.2, 3.4, the error estimates in semidiscrete case,

$$\|u_h(t) - u(t)\|_{L_2(\Omega; H)} \leq C h^\beta \Big(\|u_0\|_{L_2(\Omega; \dot{H}^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0} \Big), \tag{1.6}$$

and, with $\ell_h = \log(T/h^2)$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega; \dot{H}^{-1})} \leq C h^{\beta+1} \Big(\|u_0\|_{L_2(\Omega; \dot{H}^\beta)} + \ell_h \|A^{(\beta-1)/2}\|_{L_2^0} \Big). \tag{1.7}$$

We also consider the error estimates in the fully discrete case . Let k be a time step and $t_n = nk$ with $n \geq 1$. We use the backward Euler scheme to approximate $u(t_n)$,

$$\frac{U^n - U^{n-1}}{k} + A_h U^n = \frac{1}{k} \int_{t_{n-1}}^{t_n} P_h \, dW(s), \quad n \geq 1, \quad U^0 = P_h u_0. \tag{1.8}$$

With $r(\lambda) = (1 + \lambda)^{-1}$, we can rewrite (1.8) in the form

$$U^n = r(kA_h) U^{n-1} + \int_{t_{n-1}}^{t_n} r(kA_h) P_h \, dW(s), \quad n \geq 1, \quad U^0 = P_h u_0. \tag{1.9}$$

Under the condition $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0, 1]$, we obtain, in Theorems 4.2, 4.4, the error estimates in the fully discrete case,

$$\|U^n - u(t_n)\|_{L_2(\Omega; H)} \leq C(k^{\beta/2} + h^\beta) \Big(\|u_0\|_{L_2(\Omega; \dot{H}^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0} \Big), \tag{1.10}$$

and, with $\ell_k = \log(t_n/k)$,
(1.11)
$$\|U^n - u(t_n)\|_{L_2(\Omega;\dot H^{-1})} \le C(k^{(\beta+1)/2} + h^{\beta+1})\Big(\|u_0\|_{L_2(\Omega;\dot H^\beta)} + \ell_k\|A^{(\beta-1)/2}\|_{L_2^0}\Big).$$

We briefly recall some previous works on the numerical approximation for (1.1). Allen, Novosel, and Zhang [1] consider both finite element and finite difference methods of (1.1) in the one-dimensional case and in the cylindrical Wiener process case with $Q = I$ and $H = L_2(0,1)$, $A = -\frac{\partial}{\partial x^2}$ with Dirichlet boundary condition. Shardlow [16] also considers the finite difference approximation of (1.1) in the one-dimensional case. Du and Zhang [8] consider the numerical approximation for (1.1) but with some special additive noises. Printems [15] considers the time discretization in more general case in abstract framework based on the θ-method. For the numerical approximation of nonlinear evolution partial differential equation, we mention Davie and Gaines [6], Gyöngy [10], [11], Hausenblas [12], etc.

This paper is organized as follows. In Section 2, we consider the regularity of the solution of (1.1). In Section 3, we consider the error estimates in semidiscrete case. In Section 4, we consider the error estimates in the fully discrete case. Finally in Section 5, we consider how to compute the approximate solution U^n numerically.

By C and c we denote large and small positive constants independent of the functions and parameters concerned, but not necessarily the same at different occurrences. When necessary for clarity we distinguish constants by subscripts.

2 Regularity of the mild solution

In this section we will consider the regularity of the mild solution of (1.1). We have

THEOREM 2.1. *Let $u(t)$ be the mild solution* (1.3) *of* (1.1). *If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$. Then we have, for fixed $t \in [0,T]$,*
(2.1)
$$\|u(t)\|_{L_2(\Omega;\dot H^\beta)} \le C\Big(\|u_0\|_{L_2(\Omega;\dot H^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0}\Big), \quad \textit{for } u_0 \in L_2(\Omega;\dot H^\beta).$$

In particular, if $W(t)$ is an H-valued Wiener process with covariance operator $Q, \mathrm{Tr}(Q) < \infty$, then we have

$$(2.2)\quad \|u(t)\|_{L_2(\Omega;\dot H^1)} \le C\Big(\|u_0\|_{L_2(\Omega;\dot H^1)} + \mathrm{Tr}(Q)^{1/2}\Big), \quad \textit{for } u_0 \in L_2(\Omega;\dot H^1).$$

To prove this theorem, we need some regularity results which are related to the fact that $E(t) = e^{-tA}$ is an analytic semigroup on H. For later use, we collect some results in the next two lemmas, see Thomée [17] or Pazy [13].

LEMMA 2.2. *Let $\alpha, \beta \in \mathbf{R}$ and let $l \ge 0$ be any integer. We have*

$$(2.3)\qquad |D_t^l E(t)v|_\beta \le Ct^{-(\beta-\alpha)/2-l}|v|_\alpha, \quad \textit{for } t>0, \quad 2l+\beta \ge \alpha,$$

and

$$\int_0^t s^\alpha |D_t^l E(s)v|_\beta^2 \, ds \le C|v|_{2l+\beta-\alpha-1}^2, \quad \text{for } t \ge 0, \quad \alpha \ge 0. \tag{2.4}$$

LEMMA 2.3. *For arbitrary $\alpha \ge 0$, $0 \le \beta \le 1$, we have*

$$\|A^\alpha E(t)\| \le Ct^{-\alpha}, \quad \text{for } t > 0, \tag{2.5}$$

and

$$\|A^{-\beta}(I - E(t))\| \le Ct^\beta, \quad \text{for } t \ge 0. \tag{2.6}$$

PROOF OF THEOREM 2.1. Since the mild solution has the form

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\, dW(s).$$

Thus, for arbitrary $\beta \ge 0$, using stability property of $E(t)$ and isometry property,

$$\begin{aligned} \mathbf{E}(|u(t)|_\beta^2) &\le 2\mathbf{E}(|E(t)u_0|_\beta^2) + 2\mathbf{E}\Big\| \int_0^t A^{\beta/2} E(t-s)\, dW(s) \Big\|^2 \\ &= 2\mathbf{E}(|u_0|_\beta^2) + 2\mathbf{E} \int_0^t \|A^{\beta/2} E(t-s)\|_{L_2^0}^2 \, ds. \end{aligned} \tag{2.7}$$

With $\{e_l\}_{l=1}^\infty$ an arbitrary orthonormal basis on H, we have, using Lemma 2.2,

$$\begin{aligned} \int_0^t \|A^{\beta/2} E(t-s)\|_{L_2^0}^2 \, ds &= \sum_{j=1}^\infty \int_0^t \|A^{\beta/2} E(t-s) Q^{1/2} e_l\|^2 \, ds \\ = \sum_{j=1}^\infty \int_0^t |E(s) Q^{1/2} e_l|_\beta^2 \, ds &\le C \sum_{j=1}^\infty |Q^{1/2} e_l|_{\beta-1}^2 = C\|A^{(\beta-1)/2}\|_{L_2^0}^2. \end{aligned}$$

Together with (2.7) this shows (2.1).

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ because

$$\|I\|_{L_2^0}^2 = \sum_{j=1}^\infty \|Q^{1/2} e_j\|^2 = \sum_{j=1}^\infty \gamma_j = \mathrm{Tr}(Q).$$

□

COROLLARY 2.4. *Let $u(t)$ be the solution of* (1.1) *and $A = -\frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$. Assume that $W(t)$ is a cylindrical Wiener process with $Q = I$. Then we have, for every $\beta \in [0, 1/2)$,*

$$\|u(t)\|_{L_2(\Omega;\dot{H}^\beta)} \le C(1 + \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}), \quad \text{for } u_0 \in L_2(\Omega; \dot{H}^\beta).$$

PROOF. By (2.1), it suffices to check that in what case $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$. It is well known that A has eigenvalues $\lambda_j = j^2\pi^2, j = 1, 2, \cdots$, and corresponding eigenfunctions $\varphi_j = \sqrt{2}\sin j\pi x, j = 1, 2, \cdots$, which form an orthonormal basis in $H = L_2(0, 1)$. Thus, we have

$$\|A^{(\beta-1)/2}\|_{L_2^0}^2 = \sum_{j=1}^{\infty} \|A^{(\beta-1)/2}\varphi_j\|^2 = \sum_{j=1}^{\infty} \lambda_j^{\beta-1},$$

which is convergent if $\beta \in [0, 1/2)$. The proof is complete. □

We note that in Theorem 2.1, we require the condition $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for $\beta \in [0, 1]$. The following lemma shows that this condition is equivalent to saying tht $W(t)$ is $\dot{H}^{\beta-1}$-valued. In particular, $W(t) \in \dot{H}^{-1}$, which is important when applying the finite element method.

LEMMA 2.5. *Assume that $W(t)$ is a Wiener process with covariance operator Q. Assume that A and Q have the same eigenvectors. Then the following statements hold.*

(i) If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0, 1]$, then

$$W(t) = \sum_{l=1}^{\infty} Q^{1/2} e_l \beta_l(t), \quad t \geq 0,$$

defines an $\dot{H}^{\beta-1}$-valued Wiener process with covariance operator $\tilde{Q}$, $\mathrm{Tr}(\tilde{Q}) < \infty$. In particular, $\tilde{Q} = Q$ if $\mathrm{Tr}(Q) < \infty$;

(ii) If $W(t) = \sum_{l=1}^{\infty} Q^{1/2} e_l \beta_l(t),\ t \geq 0$, is an $\dot{H}^{\beta-1}$-valued Wiener process with the covariance operator $\tilde{Q}$, $\mathrm{Tr}(\tilde{Q}) < \infty$, then

$$\|A^{(\beta-1)/2}\|_{L_2^0} < \infty, \quad \textit{for some } \beta \in [0, 1].$$

PROOF. We first prove (i). With $\{\gamma_l, e_l\}_{l=1}^{\infty}$ the eigensystem of Q in H, it is easy to show that $g_l = Q^{1/2} e_l = \gamma_l^{1/2} e_l$ is an orthonormal basis of $Q^{1/2}(H)$. In fact,

$$(g_l, g_k)_{Q^{1/2}(H)} = (Q^{-1/2} g_l, Q^{1/2} g_k) = (e_l, e_k) = \delta_{k,l}.$$

Note that

$$\sum_{l=1}^{\infty} |g_l|_{\beta-1}^2 = \sum_{l=1}^{\infty} \|A^{(\beta-1)/2} Q^{1/2} e_l\|^2 = \|A^{(\beta-1)/2}\|_{L_2^0} < \infty,$$

which means that the embedding of $Q^{1/2}(H)$ into $\dot{H}^{\beta-1}$ is Hilbert-Schmidt. By Lemma 4.11 in Da Prato and Zabczyk [5], $W(t)$ defines an $\dot{H}^{\beta-1}$-valued Wiener process with covariance operator $\tilde{Q}$, $\mathrm{Tr}(\tilde{Q}) < \infty$. It is obvious that $\tilde{Q} = Q$ if $\mathrm{Tr}(Q) < \infty$.

We now turn to (ii). Since $W(t) = \sum_{l=1}^{\infty} Q^{1/2} e_l \beta_l(t),\ t \geq 0$, is an $\dot{H}^{\beta-1}$-valued Wiener process with the covariance operator $\tilde{Q}$, $\mathrm{Tr}(\tilde{Q}) < \infty$, we have

$$\mathbf{E}|W(t)|_{\beta-1}^2 < \infty.$$

With $\{\lambda_l, e_l\}_{l=1}^{\infty}$ the eigensystem of A in H, we have

$$\begin{aligned}\mathbf{E}|W(t)|_{\beta-1}^2 &= \mathbf{E}\Big|\sum_{l=1}^{\infty} Q^{1/2} e_l \beta_l(t)\Big|_{\beta-1}^2 \\ &= \mathbf{E}\sum_{l=1}^{\infty} \lambda_l^{\beta-1}\gamma_l\beta_l(t)^2 = t\|A^{(\beta-1)/2}\|_{L_2^0},\end{aligned}$$

which implies that $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for $\beta \in [0,1]$. The proof is complete. □

3 Error estimates in the semidiscrete case

In this section we will consider the error estimates for stochastic partial differential equation in semidiscrete case.

3.1 Error estimates for deterministic problem

In order to prove our error estimates for stochastic partial differential equation, we need some nonsmooth data error estimates for homogeneous deterministic parabolic problem.

Let us first consider the stationary problem

$$-\Delta u = f \text{ in } \mathcal{D}, \quad \text{with } u = 0 \text{ on } \partial\mathcal{D}, \tag{3.1}$$

where $f \in \dot{H}^{-1}$.

The variational form of (3.1) is to find $u \in H_0^1$ such that

$$(\nabla u, \nabla\phi) = \langle f, \phi\rangle, \quad \forall\phi \in H_0^1, \tag{3.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\dot{H}^{-1}$ and H_0^1.

Let $S_h \subset H_0^1$ be the finite element space. The semidiscrete problem of (3.2) is to find $u_h \in S_h$ such that

$$(\nabla u_h, \nabla\chi) = \langle f, \chi\rangle, \quad \forall\chi \in S_h. \tag{3.3}$$

By Lax-Milgram lemma, there exist unique solutions $u \in H_0^1$ and $u_h \in S_h$ such that (3.2) and (3.3) hold. Moreover the following stability result holds:

$$|u|_1 \le C|f|_{-1}, \quad \forall f \in \dot{H}^{-1}. \tag{3.4}$$

The standard error estimates read:

$$\|u_h - u\| \le Ch^s|u|_s, \quad s = 1, 2. \tag{3.5}$$

Let $G : \dot{H}^{-1} \to H_0^1$ denote the exact solution operator of (3.1), i.e., $u = Gf$. We define the linear operator $G_h : \dot{H}^{-1} \to S_h$ by $G_h f = u_h$, so that $u_h = G_h f \in S_h$ is the approximate solution of (3.2). It is easy to see that G_h is selfadjoint, positive semidefinite on H, and positive definite on S_h. Introducing the elliptic projection $R_h : H_0^1 \to S_h$ by

$$(\nabla R_h v, \nabla\chi) = (\nabla v, \nabla\chi), \quad \forall v \in H_0^1.$$

We see that $G_h = R_h G$, and $R_h v$ is the finite element approximation of the solution of the corresponding elliptic problem with exact solution v. By (3.5), we get

$$\|R_h v - v\| \le Ch^s |v|_s, \quad s = 1, 2.$$

Hence, using (3.4) and the elliptic regularity estimate, we have

$$\big\|(G_h - G)f\big\| = \|(R_h - I)Gf\| \le Ch^s |Gf|_s = Ch^s |f|_{s-2}, \quad \text{for } s = 1, 2, \tag{3.6}$$

which we need below.

Let $E_h(t) = e^{-tA_h}$ with $A_h = G_h^{-1}$, and let $E(t) = e^{-tA}$ with $A = G^{-1}$. We then have the following error estimates for deterministic parabolic problem.

LEMMA 3.1. *Let $F_h(t) = E_h(t)P_h - E(t)$. Then*

$$\|F_h v\|_{L_\infty([0,T];H)} \le Ch^\beta |v|_\beta, \quad \textit{for } v \in \dot{H}^\beta,\ 0 \le \beta \le 1, \tag{3.7}$$

and

$$\|F_h v\|_{L_2([0,T];H)} \le Ch^\beta |v|_{\beta-1}, \quad \textit{for } v \in \dot{H}^{\beta-1},\ 0 \le \beta \le 1. \tag{3.8}$$

Further, in the weak norm,

$$\|F_h v\|_{L_\infty([0,T];\dot{H}^{-1})} \le Ch^\beta |v|_{\beta-1}, \quad \textit{for } v \in \dot{H}^{\beta-1},\ 1 \le \beta \le 2, \tag{3.9}$$

and, with $\ell_h = \log(T/h^2)$,

$$\|F_h v\|_{L_2([0,T];\dot{H}^{-1})} \le Ch^\beta \ell_h |v|_{\beta-2}, \quad \textit{for } v \in \dot{H}^{\beta-2},\ 1 \le \beta \le 2. \tag{3.10}$$

PROOF. We denote $u(t) = E(t)v$, $u_h(t) = E_h(t)P_h v$, and $e(t) = u_h(t) - u(t) = F_h(t)v$. We first show (3.7). By the stability properties of the L_2 projection operator P_h and the solution operators $E_h(t)$ and $E(t)$, we have

$$\|e(t)\| = \|E_h(t)P_h v - E(t)v\| \le 2\|v\|, \quad \text{for } t \ge 0,\ v \in H. \tag{3.11}$$

We will show that

$$\|e(t)\| \le Ch|v|_1, \quad \text{for } t \ge 0,\ v \in \dot{H}^1. \tag{3.12}$$

Combining this with interpolation theory, we get (3.7).

To show (3.12), let us consider the error equation

$$G_h e_t + e = \rho, \tag{3.13}$$

where $\rho = (G_h - G)u_t$. We note that $G_h e(0) = 0$ for

$$(G_h e(0), w) = (P_h v - v, G_h w) = 0, \quad \text{for } w \in H, \tag{3.14}$$

since $G_h w \in S_h$.

By the energy method, we can show, see Thomée [17, Lemma 3.3],

$$\|e(t)\| \leq C \sup_{s \leq t} \Big(s\|\rho_t(s)\| + \|\rho(s)\| \Big), \quad t \geq 0.$$

Obviously, by (3.6) and Lemma 2.2,

$$\|\rho(s)\| = \|(G_h - G)u_t\| \leq Ch|u_t|_{-1} \leq Ch|v|_1,$$

and

$$s\|\rho_t(s)\| \leq Chs|u_t(s)|_1 \leq Ch|v|_1.$$

Hence (3.12) follows and therefore we get (3.7).

We next show (3.8). By interpolation theory, it suffices to show that

$$\|e\|_{L_2([0,T];H)} \leq C|v|_{-1}, \tag{3.15}$$

and

$$\|e\|_{L_2([0,T];H)} \leq Ch\|v\|. \tag{3.16}$$

Taking the inner product of (3.13) with e, we get

$$(G_h e_t, e) + (e, e) = (\rho, e).$$

Integrating with respect to t, we get, noting that $G_h e(0) = 0$ and using the inequality $(\rho, e) \leq \frac{1}{2}(\|\rho\|^2 + \|e\|^2)$,

$$(G_h e(T), e(T)) + \int_0^T \|e\|^2 \, dt \leq \int_0^T \|\rho\|^2 \, dt. \tag{3.17}$$

Obviously, by (3.6) and Lemma 2.2,

$$\int_0^T \|\rho\|^2 \, dt \leq \int_0^T \|(G_h - G)u_t\|^2 \, dt \leq Ch^2 \int_0^T |u|_1^2 \, dt \leq Ch^2\|v\|^2, \tag{3.18}$$

which implies that (3.16) holds.

To show (3.15), we note that, by Lemma 2.2 and its discrete counterpart,

$$\int_0^T \|e\|^2 \, dt \leq 2 \int_0^T \Big(\|u_h\|^2 + \|u\|^2 \Big) \, dt \leq 2|v|_{-1,h}^2 + 2|v|_{-1}^2, \tag{3.19}$$

where $|v|_{-1,h}$ is a discrete seminorm defined by

$$|v|_{-1,h} = (G_h v, v)^{1/2} = \|G_h^{1/2} v\|.$$

Since $|v|_{-1} = \sup\{(v, w)/|w|_1 : w \in \dot{H}^1\}$, see Thomée [17, Chapter 6], we thus have, with $w = G_h v, \ v \in \dot{H}^{-1}$,

$$|v|_{-1} = \sup_{w \in \dot{H}^1} \frac{(v, w)}{|w|_1} \geq \frac{(v, G_h v)}{|G_h v|_1} = \frac{(v, G_h v)}{(v, G_h v)^{1/2}} = |v|_{-1,h},$$

since

$$|G_h v|_1^2 = (AG_h v, G_h v) = A(G_h v, G_h v) = (A_h G_h v, G_h v) = (v, G_h v),$$

where $A_h = G_h^{-1}$. Hence by (3.19), we get $\int_0^T \|e\|^2\,dt \le 4|v|_{-1}^2$, which implies that (3.15) holds.

We now turn to (3.9). It suffices to show that

$$|e(t)|_{-1} \le Ch\|v\|, \tag{3.21}$$

and

$$|e(t)|_{-1} \le Ch^2|v|_1. \tag{3.22}$$

By (3.17) and (3.18), we have

$$(G_h e, e) = |e|_{-1,h}^2 \le Ch^2\|v\|^2. \tag{3.23}$$

Using

$$|e|_{-1} \le |e|_{-1,h} + Ch\|e\|, \tag{3.24}$$

which follows from, by (3.6),

$$|e|_{-1}^2 = (G_h e, e) + ((G - G_h)e, e) \le |e|_{-1,h}^2 + Ch^2\|e\|^2.$$

We obtain, by (3.11)

$$|e|_{-1} \le |e|_{-1,h} + Ch\|e\| \le Ch\|v\|,$$

which is (3.21).

By (3.17) and (3.6), we obtain

$$|e(t)|_{-1,h}^2 = (G_h e(t), e(t)) \le \frac{1}{2}\int_0^t \|\rho\|^2\,ds \le Ch^4\int_0^t |u_t|^2\,ds \le Ch^4|v|_1^2.$$

Combining this with (3.12) and (3.24), we get (3.22).

It remains to show (3.10). Integrating (3.13) with respect to t, we have, with $\tilde{e}(t) = \int_0^t e(s)\,ds$, $\tilde{\rho}(t) = \int_0^t \rho(s)\,ds$,

$$G_h e + \tilde{e} = \tilde{\rho}, \quad \tilde{e}(0) = 0. \tag{3.25}$$

Taking the inner product of (3.25) with e, we get, since $e = \tilde{e}_t$,

$$(G_h e, e) + \frac{1}{2}\frac{d}{dt}\|\tilde{e}\|^2 = (\tilde{\rho}, e) = \frac{d}{dt}(\tilde{\rho}, \tilde{e}) - (\rho, \tilde{e}).$$

After integration, we have, noting that $\tilde{e}(0) = 0$,

$$\begin{aligned}\int_0^T |e|_{-1,h}^2\,ds + \frac{1}{2}\|\tilde{e}(T)\|^2 &= \int_0^T (\tilde{\rho}, e)\,ds = \Big[(\tilde{\rho}, \tilde{e})\Big]_0^T - \int_0^T (\rho, \tilde{e})\,ds\\ &\le \|\tilde{\rho}(T)\|\|\tilde{e}(T)\| + \Big(\int_0^T \|\rho\|\,ds\Big)\sup_{0\le s\le T}\|\tilde{e}(s)\|\\ &\le 2\Big(\int_0^T \|\rho\|\,ds\Big)\sup_{0\le s\le T}\|\tilde{e}(s)\|.\end{aligned}$$

By a kick-back argument, we obtain

$$\int_0^T |e|_{-1,h}^2\,ds \le C\Big(\int_0^T \|\rho\|\,ds\Big)^2.$$

Noting that

$$\begin{aligned}\int_0^T \|\rho\|\,ds &= \int_0^{h^2} \|\rho\|\,ds + \int_{h^2}^T \|\rho\|\,ds\\ &\le C\int_0^{h^2} s^{-1/2}|v|_{-1}\,ds + C\int_{h^2}^T h|u|_1\,ds \le Ch\ell_h|v|_{-1},\end{aligned}$$

and, similarly,

$$\begin{aligned}\int_0^T \|\rho\|\,ds &= \int_0^{h^2} \|\rho\|\,ds + \int_{h^2}^T \|\rho\|\,ds\\ &\le Ch^2\|v\| + Ch^2\int_{h^2}^T |u|_2\,ds\\ &\le Ch^2\|v\| + Ch^2\log(T/h^2)\|v\| \le Ch^2\ell_h\|v\|,\end{aligned}$$

we therefore get

$$\int_0^T |e|_{-1,h}^2\,ds \le Ch^2\ell_h^2|v|_{-1}^2,$$

and

$$\int_0^T |e|_{-1,h}^2\,ds \le Ch^4\ell_h^2\|v\|^2.$$

By (3.18), (3.19) and (3.24), we obtain

$$\begin{aligned}\int_0^T |e|_{-1}^2\,ds &\le C\int_0^T |e|_{-1,h}^2\,ds + Ch^2\int_0^T \|e\|^2\,ds\\ &\le Ch^2\ell_h^2|v|_{-1}^2 + Ch^2|v|_{-1}^2 \le Ch^2\ell_h^2|v|_{-1}^2,\end{aligned}$$

and

$$\int_0^T |e|_{-1}^2\,ds \le Ch^4\ell_h^2\|v\|^2 + Ch^4\|v\|^2 \le Ch^4\ell_h^2\|v\|^2.$$

Now (3.10) follows from the interpolation theory. The proof is complete. □

3.2 Strong norm convergence

In this subsection, we will consider the error estimate for (1.1) in semidiscrete case with respect to strong norm. We have

THEOREM 3.2. *Let u_h and u be the solutions of* (1.4) *and* (1.1). *If* $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ *for some* $\beta \in [0,1]$, *then we have, for* $t \ge 0$ *and* $u_0 \in L_2(\Omega;\dot H^\beta)$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le Ch^\beta\Big(\|u_0\|_{L_2(\Omega;\dot H^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0}\Big). \tag{3.26}$$

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, then we have, for $t \geq 0$ and $u_0 \in L_2(\Omega; \dot{H}^1)$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \leq Ch\Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \mathrm{Tr}(Q)^{1/2}\Big). \tag{3.27}$$

PROOF. By definition of the mild solution, we have, with $E(t) = e^{-tA}$,

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\, dW(s),$$

and, with $E_h(t) = e^{-tA_h}$,

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)\, P_h dW(s).$$

Denoting $e(t) = u_h(t) - u(t)$ and $F_h(t) = E_h(t)P_h - E(t)$, we write

$$\begin{aligned} e(t) &= E_h(t)P_h u_0 - E(t)u_0 + \int_0^t \Big(E_h(t-s)P_h - E(t-s)\Big)\, dW(s) \\ &= F_h(t)u_0 + \int_0^t F_h(t-s)\, dW(s) = I + II. \end{aligned}$$

Thus

$$\|e(t)\|_{L_2(\Omega;H)} \leq 2\Big(\|I\|_{L_2(\Omega;H)} + \|II\|_{L_2(\Omega;H)}\Big).$$

For I, we have, by (3.7) with $v = u_0$,

$$\|I\| = \|F_h(t)u_0\| \leq Ch^\beta |u_0|_\beta, \quad \text{for } 0 \leq \beta \leq 1,$$

which implies that $\|I\|_{L_2(\Omega;H)} \leq Ch^\beta \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}$, for $0 \leq \beta \leq 1$.

For II, we have, by the isometry property,

$$\begin{aligned} \|II\|^2_{L_2(\Omega;H)} &= \Big\|\mathbf{E}\int_0^t F_h(t-s)\, dW(s)\Big\|^2 = \int_0^t \|F_h(t-s)\|^2_{L_2^0}\, ds \\ &= \sum_{l=1}^{\infty} \int_0^t \|F_h(t-s)Q^{1/2}e_l\|^2\, ds, \end{aligned}$$

where $\{e_l\}$ is any orthonormal basis in H.

Using (3.8) with $v = Q^{1/2}e_l$, we obtain

$$\begin{aligned} \|II\|^2_{L_2(\Omega;H)} &\leq C\sum_{l=1}^{\infty} h^{2\beta}\|Q^{1/2}e_l\|^2_{\beta-1} = C\sum_{l=1}^{\infty} h^{2\beta}\|A^{(\beta-1)/2}Q^{1/2}e_l\|^2 \\ &= Ch^{2\beta}\|A^{(\beta-1)/2}\|^2_{L_2^0}, \end{aligned}$$

which completes the proof of (3.26).

In particular, if $W(t)$ is a Wiener process with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ in (3.26) and obtain (3.27), since $\|I\|^2_{L_2^0} = \mathrm{Tr}(Q)$. □

COROLLARY 3.3. *Let u_h and u be the solutions of* (1.4) *and* (1.1), *respectively. Assume that $A = -\frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) \subset H_0^1(0,1) \cap H^2(0,1)$. If $W(t)$ is a cylindrical Wiener process with $Q = I$, then we have, for $t \geq 0$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \leq Ch^\beta \big(1 + \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}\big), \quad \text{for } 0 \leq \beta < 1/2.$$

PROOF. The proof is similar to the proof of Corollary 2.4. □

3.3 Weak norm convergence

In this subsection we state our weak norm convergence error estimate.

THEOREM 3.4. *Let u_h and u be the solutions of* (1.4) *and* (1.1). *If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, then we have, for $0 \leq t \leq T$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$, with $\ell_h = \log(T/h^2)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \leq Ch^{\beta+1}\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \ell_h \|A^{(\beta-1)/2}\|_{L_2^0}\Big). \tag{3.28}$$

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, *then we have, for $0 \leq t \leq T$ and $u_0 \in L_2(\Omega; \dot{H}^1)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \leq Ch^2\Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \ell_h \,\mathrm{Tr}(Q)^{1/2}\Big). \tag{3.29}$$

PROOF. Using the same notation as in Theorem 3.2, we have, by (3.9),

$$\|I\|_{L_2(\Omega;\dot{H}^{-1})} \leq Ch^{\beta+1}\|u_0\|_{L_2(\Omega;\dot{H}^\beta)}, \quad \text{for } 0 \leq \beta \leq 1.$$

For II, we have, by the isometry property, and (3.10) with $v = Q^{1/2}e_l$,

$$\begin{aligned}
\|II\|^2_{L_2(\Omega;\dot{H}^{-1})} &= \mathbf{E}\Big|\int_0^t F_h(t-s)\,dW(s)\Big|^2_{-1} = \mathbf{E}\Big\|\int_0^t A^{-1/2}F_h(t-s)\,dW(s)\Big\|^2 \\
&= \int_0^t \|A^{-1/2}F_h(t-s)\|^2_{L_2^0}\,ds \\
&\leq Ch^{2\beta}\ell_h^2 \sum_{l=1}^\infty \|A^{(\beta-1)/2}Q^{1/2}e_l\|^2 \leq Ch^{2(\beta+1)}\ell_h^2\|A^{(\beta-1)/2}\|^2_{L_2^0},
\end{aligned}$$

which completes the proof of (3.28).

In particular, if $W(t)$ is a Wiener process on H with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ in (3.28) and obtain (3.29). □

COROLLARY 3.5. *Let u_h and u be the solutions of* (1.4) *and* (1.1), *respectively. Assume that $A = -\frac{\partial^2}{\partial x^2}$ and $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$. If $W(t)$ is a cylindrical Wiener process with $Q = I$, then we have, for $0 \leq t \leq T$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$, with $\ell_h = \log(T/h^2)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \leq Ch^{\beta+1}(1 + \ell_h\|u_0\|_{L_2(\Omega;\dot{H}^\beta)}), \quad \text{for } 0 \leq \beta < 1/2.$$

4 Error estimates in the fully discrete case

In this section we will consider the error estimates for (1.1) in the fully discrete case.

4.1 Error estimates for deterministic problem

As in the semidiscrete case, in order to prove error estimates for the stochastic partial differential equation in the fully discrete case, we need some error estimates for deterministic parabolic problem.

Let $E_{kh} = r(kA_h)$ and $E(t_n) = e^{-t_n A}$, where $r(\lambda) = 1/(1+\lambda)$ is introduced in (1.9). We have

LEMMA 4.1. *Let* $F_n = E_{kh}^n P_h - E(t_n)$. *Then*

$$\|F_n v\| \leq C(k^{\beta/2} + h^{\beta})|v|_{\beta}, \quad \text{for } v \in \dot{H}^{\beta},\ 0 \leq \beta \leq 1, \tag{4.1}$$

and

$$\Big(k \sum_{j=1}^{n} \|F_j v\|^2\Big)^{1/2} \leq C(k^{\beta/2} + h^{\beta})|v|_{\beta-1}, \quad \text{for } v \in \dot{H}^{\beta-1},\ 0 \leq \beta \leq 1. \tag{4.2}$$

Further, in the weak norm,

$$|F_n v|_{-1} \leq C(k^{\beta/2} + h^{\beta})|v|_{\beta-1}, \quad \text{for } v \in \dot{H}^{\beta-1},\ 1 \leq \beta \leq 2, \tag{4.3}$$

and, with $\ell_k = \log(T/k)$ *where* $T = t_n$,

$$\Big(k \sum_{j=1}^{n} |F_j v|_{-1}^2\Big)^{1/2} \leq C(k^{\beta/2} + h^{\beta})\ell_k |v|_{\beta-2}, \quad \text{for } v \in \dot{H}^{\beta-2},\ 1 \leq \beta \leq 2. \tag{4.4}$$

PROOF. We denote $u(t_n) = u^n = E(t_n)v$, $U^n = E_{kh}^n P_h v$, and $e^n = F_n v$. We first show (4.1). By the stability properties of the L_2 projection operator P_h and the solution operators $E_{kh}(t)$ and $E(t)$, we have

$$\|e^n\| = \|E_{kh}^n P_h v - E(t_n)v\| \leq 2\|v\|, \quad \text{for } t \geq 0,\ v \in H. \tag{4.5}$$

We will show that

$$\|e^n\| \leq C(k^{1/2} + h)|v|_1, \quad \text{for } v \in \dot{H}^1. \tag{4.6}$$

Combining this with interpolation theory, we get (4.1).

To show (4.6), let us consider the error equation, with $\partial_t e^n = (e^n - e^{n-1})/k$,

$$G_h \partial_t e^n + e^n = \rho^n + G_h \tau^n, \tag{4.7}$$

where $\rho^n = (G_h - G)u_t(t_n)$ and $\tau^n = u_t(t_n) - \partial_t u^n$.

By the energy method, we have

$$t_n \|e^n\|^2 \leq t_n \|\rho^n\|^2 + k \sum_{j=1}^{n} \Big(\|\rho^j\|^2 + t_{j-1}^2 \|\partial_t \rho^j\|^2 + \|G_h \tau^j\|^2 + t_{j-1}^2 \|\tau^j\|^2\Big).$$

Here, by (3.6) and Lemma 2.2, we have

$$\|\rho^j\| = \|(G_h - G)u_t(t_j)\| \le Ch|u_t(t_j)|_{-1} \le Ch|v|_1,$$

and

$$\begin{aligned} t_{j-1}\|\partial_t \rho^j\| &= \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j} t_{j-1}\rho_t(s)\,ds\Big\| \le \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j} s\rho_t(s)\,ds\Big\| \\ &\le \sup_{0\le s\le t_n}\|s\rho_t(s)\| \le Ch \sup_{0\le s\le t_n}|su_t(s)|_1 \le Ch|v|_1. \end{aligned}$$

Further, we write

$$\|G_h\tau^j\| = \|(G_h - G)\tau^j\| + \|G\tau^j\|,$$

where, using (3.6) and Lemma 2.2,

$$\|(G_h - G)\tau^j\| \le Ch|\tau^j|_{-1} \le Ch \sup_{0\le s\le t_n}|u_t(s)|_{-1} \le Ch|v|_1.$$

Hence we obtain

$$\|e^n\|^2 \le Ch^2|v|_1^2 + Ckt_n^{-1}\sum_{j=1}^{n}\Big(\|G\tau^j\|^2 + t_{j-1}^2\|\tau^j\|^2\Big).$$

By Taylor's formula, we have

$$\begin{aligned} \|G\tau^j\|^2 &= \Big\|G\frac{1}{k}\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_{tt}(s)\,ds\Big\|^2 = \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_t(s)\,ds\Big\|^2 \\ &\le \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}(s - t_{j-1})^{1/2}k^{1/2}u_t(s)\,ds\Big\|^2 \le \int_{t_{j-1}}^{t_j}(s - t_{j-1})\|u_t(s)\|^2\,ds \\ &\le t_n\int_{t_{j-1}}^{t_j}\|u_t(s)\|^2\,ds, \end{aligned}$$

and

$$\begin{aligned} t_{j-1}^2\|\tau^j\|^2 &= t_{j-1}^2\Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}u_{tt}(s)\,ds\Big\|^2 \le t_{j-1}^2\int_{t_{j-1}}^{t_j}(s - t_{j-1})\|u_{tt}(s)\|^2\,ds \\ &\le t_n\int_{t_{j-1}}^{t_j}s^2\|u_{tt}(s)\|^2\,ds. \end{aligned}$$

Applying Lemma 2.2, we have

$$\begin{aligned} kt_n^{-1}\sum_{j=1}^{n}t_{j-1}^2\|\tau^j\|^2 &\le k\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|u_t(s)\|^2\,ds \\ &= k\int_0^{t_n}\|u_t(s)\|^2\,ds \le Ck|v|_1, \end{aligned}$$

and

$$kt_n^{-1}\sum_{j=1}^n t_{j-1}^2\|\tau^j\|^2 \le k\sum_{j=1}^n \int_{t_{j-1}}^{t_j} s^2\|u_{tt}(s)\|^2\,ds$$
$$= k\int_0^{t_n} s^2\|u_{tt}(s)\|^2\,ds \le Ck|v|_1.$$

Hence (4.6) follows and therefore we get (4.1).

We next show (4.2). By interpolation theory, it suffices to show that

$$\Big(k\sum_{j=1}^n \|F_j v\|^2\Big)^{1/2} \le C|v|_{-1}, \tag{4.8}$$

and

$$\Big(k\sum_{j=1}^n \|F_j v\|^2\Big)^{1/2} \le C(k^{1/2}+h)\|v\|. \tag{4.9}$$

Taking the inner product of (4.7) with e^n, we get

$$(G_h\partial_t e^n, e^n) + (e^n, e^n) = (\rho^n, e^n) + (G_h\tau^n, e^n).$$

By summation on n, using the inequality $(\rho^n, e^n) \le \frac{1}{2}(\|\rho^n\|^2 + \|e^n\|^2)$, and noting that $G_h e^0 = 0$, we have

(4.10)

$$(G_h e_n, e_n) + k\sum_{j=1}^n \|e_j\|^2 \le Ck\sum_{j=1}^n \|\rho_j\|^2 + Ck\sum_{j=1}^n \|G\tau^j\|^2 + Ck\sum_{j=1}^n \|(G_h - G)\tau^j\|^2.$$

Here, using Lemma 2.2, we have, since $\rho^j = \rho(s) + \int_s^{t_j}\rho_t(\tau)\,d\tau$,

(4.11)

$$k\sum_{j=1}^n \|\rho^j\|^2 = k\|\rho\|^2 + \sum_{j=2}^n \int_{t_{j-1}}^{t_j} \|\rho^j\|^2\,ds$$
$$\le k\|\rho\|^2 + 2\sum_{j=2}^n \int_{t_{j-1}}^{t_j} \Big(\|\rho(s)\|^2 + \|\int_s^{t_j}\rho_t(\tau)\,d\tau\|^2\Big)\,ds$$
$$\le k\|\rho\|^2 + 2\int_{t_1}^{t_n} \|\rho(s)\|^2\,ds + 2\sum_{j=2}^n \int_{t_{j-1}}^{t_j} \Big((t_j - s)\int_s^{t_j} \|\rho_t(\tau)\|^2\,d\tau\Big)\,ds$$
$$\le k\|\rho\|^2 + 2\int_{t_1}^{t_n} \|\rho(s)\|^2\,ds + 2k\sum_{j=2}^n \int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{t_j} \tau\|\rho_t(\tau)\|^2\,d\tau$$
$$\le k\|\rho\|^2 + 2\int_{t_1}^{t_n} \|\rho(s)\|^2\,ds + 2k\int_{t_1}^{t_n} \tau\|\rho_t(\tau)\|^2\,d\tau$$
$$\le Ck\|u\|^2 + Ch^2\int_0^{t_n} |u(s)|_1^2\,ds + Ck\int_0^{t_n} \tau\|u_t(\tau)\|^2\,d\tau \le C(k+h^2)\|v\|^2,$$

and, by Taylor's formula,

$$
\begin{aligned}
k\sum_{j=1}^{n}\|(G_h-G)\tau^j\|^2 &\le Ckh^2|\tau^1|_{-1}^2 + Ckh^2\sum_{j=2}^{n}|\tau^j|_{-1}^2 \\
&= Ckh^2\Big|u_t(k)-\frac{1}{k}\int_0^k u_t(\tau)\,d\tau\Big|_{-1}^2 + Ckh^2\sum_{j=2}^{n}\Big|\frac{1}{k}\int_{t_{j-1}}^{t_j}(s-t_{j-1})u_{tt}(s)\,ds\Big|_{-1}^2 \\
&\le Ch^2\Big(k|u_t(k)|_{-1}^2+\int_0^k|u_t(\tau)|_{-1}^2\,d\tau\Big) + Ch^2\sum_{j=2}^{n}\Big|\int_{t_{j-1}}^{t_j}(s-t_{j-1})^{1/2}u_{tt}(s)\,ds\Big|_{-1}^2 \\
&\le Ch^2\|v\|^2 + Ch^2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}k(s-t_{j-1})|u_{tt}(s)|_{-1}^2\,ds \\
&\le Ch^2\|v\|^2 + Ch^2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}s^2|u_{tt}(s)|_{-1}^2\,ds \le Ch^2\|v\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
k\sum_{j=1}^{n}\|G\tau^j\|^2 &= k\sum_{j=1}^{n}\Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}(s-t_{j-1})u_t(s)\,ds\Big\|^2 \\
&\le k\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\|u_t(s)\|^2 \\
&\le Ck\int_0^{t_n}s\|u_t(s)\|^2\,ds \le k\|v\|^2.
\end{aligned}
$$

We therefore obtain

$$
(G_he^n,e^n)^{1/2} + \Big(k\sum_{j=1}^{n}\|e^j\|^2\Big)^{1/2} \le C(k^{1/2}+h)\|v\|, \tag{4.12}
$$

which implies that (4.9) holds.

To show (4.8), we note that,

$$
k\sum_{j=1}^{n}\|e^j\|^2 \le Ck\sum_{j=1}^{n}\|U^j\|^2 + Ck\sum_{j=1}^{n}\|u(t_j)\|^2. \tag{4.13}
$$

Here, we have, following (4.11) with ρ replaced by u,

(4.14)

$$
k\sum_{j=1}^{n}\|u(t_j)\|^2 \le k\|u(t_1)\|^2 + 2\int_{t_1}^{t_n}\|u(s)\|^2\,ds + 2\int_{t_1}^{t_n}s^2\|u_t(s)\|^2\,ds \le C|v|_{-1}^2,
$$

and, by (3.20),

$$
k\sum_{j=1}^{n}\|U^j\|^2 \le C|v|_{-1,h}^2 \le C|v|_{-1}^2,
$$

which imply that (4.8) holds and the proof of (4.2) is complete.

We now turn to (4.3). It suffices to show that

$$|e^n|_{-1} \le C(k^{1/2}+h)\|v\|, \tag{4.15}$$

and

$$|e^n|_{-1} \le C(k+h^2)|v|_1. \tag{4.16}$$

Obviously (4.15) follows by (3.11), (3.24), and (4.12). Note that, by Lemma 2.2,

$$\begin{aligned} k\sum_{j=1}^n \|\rho_j\|^2 &\le C\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \Big(\|\rho(s)\|^2 + \Big\|\int_s^{t_j} \rho_t(\tau)\,d\tau\Big\|^2\Big)\,ds \\ &\le C\int_0^{t_n} \|\rho(s)\|^2\,ds + \sum_{j=1}^n \int_{t_{j-1}}^{t_j} k^2\|\rho_t(\tau)\|^2\,d\tau \\ &\le \int_0^{t_n} \|\rho(s)\|^2 + Ck^2\int_0^{t_n} \|\rho_t(\tau)\|^2\,d\tau \\ &\le Ch^4\int_0^{t_n} |u|_2^2\,ds + Ck^2\int_0^{t_n} \|u_t\|^2\,ds \le C(h^4+k^2)|v|_1^2, \end{aligned}$$

and

$$\begin{aligned} k\sum_{j=1}^n \|(G_h-G)\tau^j\|^2 &\le Ckh^4\sum_{j=1}^n \|\tau^j\|^2 \\ &= Ckh^4\sum_{j=1}^n \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j} (s-t_{j-1})u_{tt}(s)\,ds\Big\|^2 \\ &\le Ch^4\sum_{j=1}^n \int_{t_{j-1}}^{t_j} (s-t_{j-1})^2\|u_{tt}(s)\|^2\,ds \\ &\le Ch^4\int_0^{t_n} s^2\|u_{tt}(s)\|^2\,ds \le Ch^4|v|_1^2, \end{aligned}$$

and

$$\begin{aligned} k\sum_{j=1}^n \|G\tau^j\|^2 &= k\sum_{j=1}^n \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j} (s-t_{j-1})u_t(s)\,ds\Big\|^2 \\ &\le k^{-1}\sum_{j=1}^n \Big(\int_{t_{j-1}}^{t_j} (s-t_{j-1})^2\,ds\Big)\int_{t_{j-1}}^{t_j} \|u_t(s)\|^2\,ds \\ &\le Ck^2\int_0^{t_n} \|u_t(s)\|^2\,ds \le Ck^2|v|_1^2. \end{aligned}$$

Combining these estimates with (4.10), we get (4.16)

It remains to show (4.4). As in the proof of (3.10), it suffices to show

$$\Big(k\sum_{j=1}^{n}|e_j|_{-1}^2\Big)^{1/2} \le C(k+h^2)\ell_k\|v\|, \tag{4.17}$$

and

$$\Big(k\sum_{j=1}^{n}|e_j|_{-1}^2\Big)^{1/2} \le C(k^{1/2}+h)\ell_k|v|_{-1}. \tag{4.18}$$

Let $\tilde{e}^n = k\sum_{j=1}^n e^j$, $\tilde{e}^0 = 0$, and $\partial_t\tilde{e}^n = (\tilde{e}^n - \tilde{e}^{n-1})/k = e^n$ for $n \ge 1$. We have the error equation

$$G_h\partial_t\tilde{e}^n + \tilde{e}^n = \tilde{\rho^n} + G_h\tilde{\tau}^n, \quad \text{for}\, n \ge 1, \tag{4.19}$$

where $\tilde{\tau}^n = k\sum_{j=1}^n \tau^j$, and $\tilde{\rho}^n = k\sum_{j=1}^n \rho^j$, where τ^j and ρ^j are defined as before.

Taking the inner product of (4.19) with $\partial_t\tilde{e}^n$, we get, since $\partial_t\tilde{e}^n = e^n$,

$$\begin{aligned}(G_h\partial_t\tilde{e}^n, \partial_t\tilde{e}^n) + \frac{1}{2}\partial_t(\tilde{e}^n, \tilde{e}^n) + \frac{1}{k}(\partial_t\tilde{e}^n, \partial_t\tilde{e}^n) &= (\tilde{\rho}^n, \partial_t\tilde{e}^n) + (G_h\partial_t\tilde{\tau}^n, \partial_t\tilde{e}^n)\\ = \partial_t(\tilde{\rho}^n, \tilde{e}^n) - (\partial_t\tilde{\rho}^n, \tilde{e}^{n-1}) + \partial_t(G_h\tilde{\tau}^n, \tilde{e}^n) &- (\partial_t(G_h\tilde{\tau}^n), \tilde{e}^{n-1}).\end{aligned}$$

By summation on n, noting that $\tilde{e}^0 = 0$, we have

$$\begin{aligned}&k\sum_{j=1}^{n}(G_h\partial_t\tilde{e}^j, \partial_t\tilde{e}^j) + \frac{1}{2}(\tilde{e}^n, \tilde{e}^n)\\ &\le \|\tilde{\rho}^n\|\|\tilde{e}^n\| + k\sum_{j=1}^{n}|(\rho^n, \tilde{e}^{n-1})| + \|G_h\tilde{\tau}^n\|\|\tilde{e}^n\| + k\sum_{j=1}^{n}|(G_h\tau^n, \tilde{e}^{n-1})|\\ &\le \max_j\|\tilde{e}^j\|\Big(\|\tilde{\rho}^n + k\sum_{j=1}^{n}\|\rho^j\| + k\sum_{j=1}^{n}\|G_h\tau^j\| + \|G_h\tilde{\tau}^n\|\Big).\end{aligned}$$

By a kick-back argument, we obtain

$$\Big(k\sum_{j=1}^{n}(G_h\partial_t\tilde{e}^j, \partial_t\tilde{e}^j)\Big)^{1/2} \le Ck\sum_{j=1}^{n}\|\rho^j\| + Ck\sum_{j=1}^{n}\|(G_h - G)\tau^j\| + Ck\|G\tau^j\|.$$

Here, with $\ell_k = \log(T/k)$ where $T = t_n$, we have

$$\begin{aligned}k\sum_{j=1}^{n}\|\rho^j\| = k\|\rho\| + k\sum_{j=2}^{n}\|\rho^j\| &\le Ck\|v\| + Ck\sum_{j=2}^{n}t_j^{-1}\|v\|\\ &\le Ck\|v\| + Ck\ell_k\|v\| \le Ck\ell_k\|v\|,\end{aligned}$$

and

$$
\begin{aligned}
k\sum_{j=1}^{n}\|(G_h-G)\tau^j\| &\le Ckh^2\sum_{j=1}^{n}\|\tau^j\| = Ckh^2\|\tau^1\| + Ckh^2\sum_{j=2}^{n}\|\tau^j\| \\
&= Ckh^2\Big\|u_t(k)-\partial_t u^1\Big\| + Ch^2\sum_{j=2}^{n}\Big\|\int_{t_{j-1}}^{t_j}(s-t_j)u_{tt}(s)\,ds\Big\| \\
&\le Ch^2\big(\|ku_t(k)\| + \|u(k)\| + \|v\|\big) + Ch^2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\|su_{tt}(s)\|\,ds \\
&\le Ch^2\|v\| + Ch^2\int_{t_1}^{t_n}\|su_{tt}(s)\|\,ds \le Ch^2\ell_k\|v\|,
\end{aligned}
$$

and

$$
k\sum_{j=1}^{n}\|G\tau^j\| = k\|\tau^1\| + k\sum_{j=2}^{n}\|G\tau^j\| \le Ck\ell_k\|v\|,
$$

which imply that (4.17) holds. Similarly we can show (4.18). Hence (4.4) follows. Together these estimates complete the proof. □

4.2 Strong norm convergence

We have the following strong norm convergence result in the fully discrete case.

THEOREM 4.2. *Let U^n and $u(t_n)$ be the solutions of* (1.9) *and* (1.1), *respectively. If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, then we have, for $u_0 \in L_2(\Omega, \dot H^\beta)$,*

$$
\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{\beta/2}+h^\beta)\Big(\|u_0\|_{L_2(\Omega;\dot H^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0}\Big). \tag{4.20}
$$

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, *then we have, for $u_0 \in L_2(\Omega; \dot H^1)$,*

$$
\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{1/2}+h)\Big(\|u_0\|_{L_2(\Omega;\dot H^1)} + \mathrm{Tr}(Q)^{1/2}\Big). \tag{4.21}
$$

PROOF. We have, by (1.9), with $E_{kh}^n = r(kA_h)^n$,

$$
U^n = E_{kh}^n P_h u_0 + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_{kh}^{n-j+1} P_h\, dW(s),
$$

and, by the definition of the mild solution of (1.1), with $E(t) = e^{-tA}$,

$$
u(t_n) = E(t_n)u_0 + \int_0^{t_n} E(t_n - s)\, dW(s).
$$

Denoting $e^n = U^n - u(t_n)$ and $F_n = E_{kh}^n P_h - E(t_n)$, we write

$$\begin{aligned} e^n &= F_n u_0 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} F_{n-j+1}\, dW(s) \\ &\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big(E(t_n - t_{j-1}) - E(t_n - s) \Big)\, dW(s) \\ &= I + II + III. \end{aligned}$$

Thus

$$\|e^n\|_{L_2(\Omega;H)} \le C\Big(\|I\|_{L_2(\Omega;H)} + \|II\|_{L_2(\Omega;H)} + \|III\|_{L_2(\Omega;H)} \Big).$$

For I, we have, by (4.1) with $v = u_0$,

$$\|I\| = \|F_n u_0\| \le C(k^{\beta/2} + h^{\beta})|u_0|_{\beta},$$

which implies that $\|I\|_{L_2(\Omega;H)} \le C(k^{\beta/2} + h^{\beta})\|u_0\|_{L_2(\Omega;\dot{H}^{\beta})}$.

For II, we have, by the isometry property,

$$\begin{aligned} \|II\|^2_{L_2(\Omega;H)} &= \mathbf{E}\Big\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} F_{n-j+1}\, dW(s) \Big\|^2 = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|F_{n-j+1}\|^2_{L_2^0}\, ds \\ &= \sum_{l=1}^{\infty} \Big(k \sum_{j=1}^{n} \|F_{n-j+1} Q^{1/2} e_l\|^2 \Big), \end{aligned}$$

where $\{e_l\}$ is any orthonormal basis in H. Using (4.2) with $v = Q^{1/2} e_l$, we obtain

$$\begin{aligned} \|II\|^2_{L_2(\Omega;H)} &\le C \sum_{l=1}^{\infty} (k^{\beta} + h^{2\beta}) |Q^{1/2} e_l|^2_{\beta-1} \\ &= C \sum_{l=1}^{\infty} (k^{\beta} + h^{2\beta}) \|A^{(\beta-1)/2} Q^{1/2} e_l\|^2 \\ &= C(k^{\beta} + h^{2\beta}) \|A^{(\beta-1)/2}\|^2_{L_2^0}. \end{aligned}$$

For III, we have, by the isometry property,

$$\begin{aligned} \|III\|^2_{L_2(\Omega;H)} &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big\| \Big(E(t_n - t_{j-1}) - E(t_n - s) \Big) \Big\|^2_{L_2^0}\, ds \\ &= \sum_{l=1}^{\infty} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big\| A^{-\beta/2} \Big(E(s - t_{j-1}) - I \Big) A^{\beta/2} E(t_n - s) Q^{1/2} e_l \Big\|^2\, ds. \end{aligned}$$

Using (2.6), and (2.4) with $v = A^{(\beta-1)/2}Q^{1/2}e_l$, we obtain

$$(4.22)\qquad \|III\|^2_{L_2(\Omega;H)} \le Ck^\beta \sum_{l=1}^\infty \int_0^{t_n} \|A^{1/2}E(t_n-s)A^{(\beta-1)/2}Q^{1/2}e_l\|^2\,ds \le Ck^\beta \sum_{l=1}^\infty \|A^{(\beta-1)/2}Q^{1/2}e_l\|^2 = Ck^\beta \|A^{(\beta-1)/2}\|^2_{L_2^0},$$

which completes the proof of (4.20).

In particular, if $W(t)$ is a Wiener process with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ in the proof of (3.26) and obtain (3.27) since $\|I\|_{L_2^0} = \mathrm{Tr}(Q)$. □

COROLLARY 4.3. *Let U^n and $u(t_n)$ be the solutions of (1.9) and (1.1), respectively. Assume that $A = -\frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) \subset H_0^1(0,1) \cap H^2(0,1)$. If $W(t)$ is a cylindrical Wiener process with $Q = I$, then we have, for $u_0 \in L_2(\Omega; \dot{H}^\beta)$,*

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{\beta/2} + h^\beta)(1 + \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}), \quad \textit{for } 0 \le \beta < 1/2.$$

4.3 Weak norm convergence

In this subsection we show the weak norm convergence error estimate.

THEOREM 4.4. *Let U^n and $u(t_n)$ be the solutions of (1.9) and (1.1), respectively. If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, then we have, for $u_0 \in L_2(\Omega; \dot{H}^\beta)$, with $\ell_k = \log(T/k)$ where $T = t_n$,*

$$(4.23)\qquad \|U^n - u(t_n)\|_{L_2(\Omega;\dot{H}^{-1})} \le C(k^{(\beta+1)/2} + h^{\beta+1})\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \ell_k \|A^{(\beta-1)/2}\|_{L_2^0}\Big).$$

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, then we have, for $u_0 \in L_2(\Omega; \dot{H}^1)$,

$$(4.24)\qquad \|U^n - u(t_n)\|_{L_2(\Omega;\dot{H}^{-1})} \le C(k + h^2)\Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \ell_k\, \mathrm{Tr}(Q)^{1/2}\Big).$$

PROOF. Using the same notation as in Theorem 4.2, we have, by (4.3),

$$\|I\|_{L_2(\Omega;\dot{H}^{-1})} \le Ch^{\beta+1}\|u_0\|_{L_2(\Omega;\dot{H}^\beta)}, \quad \text{for } 0 \le \beta \le 1.$$

For II, we have, by the isometry property, and (3.10) with $v = Q^{1/2}e_l$,

$$\begin{aligned}\|II\|^2_{L_2(\Omega;\dot{H}^{-1})} &= \mathbf{E}\Big\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} A^{-1/2} F_{n-j+1}\, dW(s) \Big\|^2 \\ &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|A^{-1/2}F_{n-j+1}\|^2_{L_2^0}\, ds \\ &= \sum_{l=1}^{\infty} \Big(k \sum_{j=1}^{n} \|A^{-1/2}F_{n-j+1}Q^{1/2}e_l\|^2 \Big) \\ &\le C(k^{\beta+1} + h^{2(\beta+1)})\ell_k^2 \sum_{l=1}^{\infty} \|A^{(\beta-1)/2}Q^{1/2}e_l\|^2 \\ &\le C(k^{\beta+1} + h^{2(\beta+1)})\ell_k^2 \|A^{(\beta-1)/2}\|^2_{L_2^0}.\end{aligned}$$

For III, we have, by the isometry property,

$$\|III\|^2_{L_2(\Omega;\dot{H}^{-1})} = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big\| A^{-1/2}\Big(E(t_n - t_{j-1}) - E(t_n - s)\Big)\Big\|^2_{L_2^0} ds$$

$$= \sum_{l=1}^{\infty}\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big\| A^{-(\beta+1)/2}\Big(E(s - t_{j-1}) - I\Big) A^{1/2}E(t_n - s)A^{(\beta-1)/2}Q^{1/2}e_l \Big\|^2 ds.$$

Following the proof of (4.22), we get

$$\|III\|^2_{L_2(\Omega;\dot{H}^{-1})} \le Ck^{\beta}\|A^{(\beta-2)/2}\|^2_{L_2^0},$$

which completes the proof of (4.23).

In particular, if $W(t)$ is a Wiener process, then we can choose $\beta = 1$ in (4.23) and obtain (4.24). □

COROLLARY 4.5. *Let U^n and $u(t_n)$ be the solutions of* (1.9) *and* (1.1), *respectively. Assume that $A = -\frac{\partial^2}{\partial x^2}$ and $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$. If $W(t)$ is a cylindrical Wiener process with $Q = I$, then we have, for $u_0 \in L_2(\Omega;\dot{H}^{\beta})$, with $\ell_k = \log(T/k)$ where $T = t_n$,*

$$\|U^n - u(t_n)\|_{L_2(\Omega;\dot{H}^{-1})} \le C(k^{(\beta+1)/2} + h^{(\beta+1)})\big(1 + \ell_k\|u_0\|_{L_2(\Omega;\dot{H}^{\beta})}\big), \quad \textit{for } 0 \le \beta < 1/2.$$

5 Computational analysis

In this section we consider how to compute the approximate solution U^n of the solution u of (1.1). Recall that the Wiener process $W(t)$ with covariance operator Q has the form, see Da Prato and Zabczyk [5, Chapter 4],

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} e_j \beta_j(t),$$

where $\{\gamma_j, e_j\}_{j=1}^{\infty}$ is eigensystem of Q, and $\{\beta_j(t)\}_{j=1}^{\infty}$ are independently and identically distributed (iid) real-valued Brownian motions. If $\mathrm{Tr}(Q) < \infty$, then $W(t)$ is an H-valued process. In fact

$$\mathbf{E}\|W(t)\|^2 = \mathbf{E}\sum_{j=1}^{\infty} \gamma_j \beta_j(t)^2 = \sum_{j=1}^{\infty} \gamma_j \big(\mathbf{E}\beta_j(t)^2\big) = t\,\mathrm{Tr}(Q) < \infty.$$

If $\mathrm{Tr}(Q) = \infty$, for example $Q = I$, then $W(t)$ is not H-valued.

Let U^n be the approximation in S_h of $u(t)$ at $t = t_n = nk$. The backward Euler method is to find $U^n \in S_h$, s.t., with $\bar{\partial} U^n = (U^n - U^{n-1})/k$, $n \geq 1$, $U^0 = P_h u_0$,

$$(\bar{\partial} U^n, \chi) + (A_h U^n, \chi) = \Big(\frac{1}{k}\int_{t_{n-1}}^{t_n} P_h\, dW(s), \chi\Big), \quad \forall \chi \in S_h, \tag{5.1}$$

where A_h, P_h are defined in the introduction.

If $W(t)$ is H-valued, then $P_h W(t)$ is well-defined. We therefore can write

$$\int_{t_{n-1}}^{t_n} P_h\, dW(s) = P_h\big(W(t_n) - W(t_{n-1})\big) = P_h \sum_{j=1}^{\infty} \gamma_j^{1/2}\big(\beta_j(t_n) - \beta_j(t_{n-1})\big).$$

Here

$$\frac{1}{\sqrt{k}}\Big(\beta_j(t_n) - \beta_j(t_{n-1})\Big) = \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ is the real-valued Gaussian random variable.

Thus the right hand side of (5.1) can be computed by truncating the following series to J terms, i.e.,

$$\begin{aligned}\Big(\frac{1}{k}\int_{t_{n-1}}^{t_n} P_h\, dW(s), \chi\Big) &= \Big(\frac{1}{k}\sum_{j=1}^{\infty} \gamma_j^{1/2} e_j\big(\beta_j(t_n) - \beta_j(t_{n-1})\big), \chi\Big) \\ &= \frac{1}{k}\sum_{j=1}^{\infty} \gamma_j^{1/2}\big(\beta_j(t_n) - \beta_j(t_{n-1})\big)(e_j, \chi) \\ &\approx \frac{1}{k}\sum_{j=1}^{J} \gamma_j^{1/2}\big(\beta_j(t_n) - \beta_j(t_{n-1})\big)(e_j, \chi).\end{aligned} \tag{5.2}$$

If $W(t)$ is not H-valued, then we see that, from Lemma 2.5, $W(t)$ is $\dot{H}^{\beta-1}$-valued with $\beta \in [0,1]$. In this case we may introduce the $\dot{H}^{-1}$-projection $P_h : \dot{H}^{-1} \to S_h$ defined by

$$(P_h v, \chi) = \langle v, \chi\rangle, \quad \forall v \in \dot{H}^{-1},\ \chi \in S_h \subset \dot{H}^1,$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $\dot{H}^{-1}$ and $\dot{H}^1$.

Below we will show that it is sufficient to choose $J = N_h$ in order to achieve the required convergence order. To see this, let us consider the semidiscrete approximation solution u_h of u of (1.1). Recall that the semidiscrete solution u_h satisfies

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)P_h\, dW(s) = E_h(t)P_h u_0 + \sum_{j=1}^{\infty} \int_0^t E_h(t-s)P_h e_j \gamma_j^{1/2}\, d\beta_j(s). \tag{5.3}$$

Truncating the series in the right side of (5.3), we have

$$u_h^J(t) = E_h(t)P_h u_0 + \sum_{j=1}^{J} \int_0^t E_h(t-s)P_h e_j \gamma_j^{1/2}\, d\beta_j(s). \tag{5.4}$$

We then have the following lemma with respect to L_2 norm in space.

LEMMA 5.1. *Let u_h^J and u_h be defined by (5.3) and (5.4), respectively. If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$. Assume that $\{S_h\}$ is defined on a quasi-uniform family of triangulations and let N_h be the dimension of S_h. If $J \geq N_h$, then we have, for $t > 0$,*

$$\|u_h^J(t) - u_h(t)\|_{L_2(\Omega,H)} \leq C h^{\beta} \|A^{(\beta-1)/2}\|_{L_2^0}. \tag{5.5}$$

PROOF. Using the same notation as in the proof of Theorem 3.2, we have, by isometry property,

$$\begin{aligned}
\mathbf{E}\|u_h^J(t) - u_h(t)\|^2 &= \mathbf{E}\Big\| \sum_{j=J+1}^{\infty} \int_0^t E_h(t-s)P_h e_j \gamma_j^{1/2}\, d\beta_j(s)\Big\|^2 \\
&= \sum_{j=J+1}^{\infty} \gamma_j \int_0^t \|E_h(t-s)P_h e_j\|^2\, ds \\
&\leq 2 \sum_{j=J+1}^{\infty} \gamma_j \int_0^t \|E(t-s)e_j\|^2\, ds \\
&\quad + 2 \sum_{j=J+1}^{\infty} \gamma_j \int_0^t \|F_h(t-s)e_j\|^2\, ds \\
&= I + II.
\end{aligned}$$

For I, we have

$$\begin{aligned}
I &= 2 \sum_{j=J+1}^{\infty} \gamma_j \int_0^t e^{-2(t-s)\lambda_j}\, ds \leq \sum_{j=J+1}^{\infty} \gamma_j \lambda_j^{-1} \\
&= \sum_{j=J+1}^{\infty} \lambda_j^{-\beta} \lambda_j^{\beta-1} \gamma_j \leq \lambda_{J+1}^{-\beta} \|A^{(\beta-1)/2}\|_{L_2^0}^2.
\end{aligned}$$

For II, we have, by (3.8),

$$\begin{aligned} II &\le Ch^{2\beta} \sum_{j=J+1}^{\infty} \gamma_j |e_j|_{\beta-1}^2 \le Ch^{2\beta} \sum_{j=1}^{\infty} |Q^{1/2} e_j|_{\beta-1}^2 \\ &= Ch^{2\beta} \|A^{(\beta-1)/2}\|_{L_2^0}^2. \end{aligned}$$

Thus we get

$$\mathbf{E}\|u_h^J(t) - u_h(t)\|^2 \le C(\lambda_{J+1}^{-\beta} + h^{2\beta}) \|A^{(\beta-1)/2}\|_{L_2^0}^2.$$

Hence (5.5) follows from the following obvious facts: with some constant C which may be different in different inequalities,

$$\lambda_{J+1}^{-1} \le CJ^{-2/d} \le CN_h^{-2/d} \le Ch^2,$$

where d is the dimension of the spatial domain $\mathcal{D}$.
□

Under the same assumptions as in Lemma 5.1, we can also show the following results with respect to weak norm in space,

$$\|u_h^J(t) - u_h(t)\|_{L_2(\Omega, \dot{H}^{-1})} \le Ch^{\beta+1} \ell_h \|A^{(\beta-1)/2}\|_{L_2^0}.$$

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Paper VI

A finite element method for a nonlinear stochastic parabolic equation

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Abstract.

In this paper we consider the finite element method for a stochastic parabolic partial differential equation forced by additive space-time noise in the multi-dimensional case. Optimal strong convergence error estimates in the L_2 and $\dot{H}^{-1}$ norms with respect to the spatial variable are obtained. The proof is based on appropriate nonsmooth data error estimates for the corresponding deterministic linear parabolic problem.

AMS subject classification: 60H15, 60H25, 65C20, 60H35.

Key words: stochastic parabolic partial differential equation, nonlinear, finite element method, backward Euler method, Hilbert space.

1 Introduction

In this paper we study the finite element approximation of the nonlinear stochastic parabolic partial differential equation

$$du + Au\,dt = \sigma(u)dW, \quad \text{for } 0 < t \leq T, \quad \text{with } u(0) = u_0, \tag{1.1}$$

in a Hilbert space H, with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, where $u(t)$ is an H-valued random process, A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, σ is a nonlinear operator-valued function defined on H which we will specify later. Here $W(t)$ is a Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$ and $u_0 \in H$.

For the sake of simplicity, we shall concentrate on the case $A = -\Delta$, where Δ stands for the Laplacian operator subject to homogeneous Dirichlet boundary conditions, and $H = L_2(\mathcal{D})$, where $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d, d = 1, 2, 3$, with a sufficiently smooth boundary $\partial\mathcal{D}$.

Such equations are common in applications. Many mathematics models in physics, chemistry, biology, population dynamics, neurophysiology, etc., are described by stochastic partial differential equations, see, Da Prato and Zabczyk [5], Walsh [17], etc. The existence, uniqueness, and properties of the solutions of such equations have been well studied, see Curtain and Falb [1], [2], Da Prato [3], Da Prato and Lunardi [4], Da Prato and Zabczyk [5], Dawson [7], Gozzi [8], Peszat and Zabczyk [13], Walsh [17], etc. However, numerical approximation of such equations has not been studied thoroughly.

This paper is closely related to [18], where we consider the finite element method for a linear stochastic parabolic partial differential equation. As in [18], we assume that $W(t)$ is a Wiener process with covariance operator Q. This process may be considered in terms of its Fourier series. Suppose that Q is a bounded, linear, selfadjoint, positive definite operator on H, with eigenvalues $\gamma_l > 0$ and corresponding eigenfunctions e_l. Let β_l, $l = 1, 2, \cdots$, be a sequence of real-valued independently and identically distributed Brownian motions. Then

$$W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t),$$

is a Wiener process with covariance operator Q.

If Q is in trace class, then $W(t)$ is an H-valued process. If Q is not in trace class, for example, $Q = I$, then $W(t)$ does not belong to H, in which case $W(t)$ is called a cylindrical Wiener process.

Let $L_2^0 = HS(Q^{1/2}(H), H)$ denote the space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to H, i.e.,

$$L_2^0 = \Big\{ \psi \in L(H) : \sum_{l=1}^{\infty} \|\psi Q^{1/2} e_l\|^2 < \infty \Big\},$$

with norm $\|\psi\|_{L_2^0} = \Big(\sum_{l=1}^{\infty} \|\psi Q^{1/2} e_l\|^2 \Big)^{1/2}$, where $L(H)$ is the space of bounded linear operators from H to H.

Let $\mathbf{E}$ denote the expectation. Let $\psi \in L_2^0$. Then $\int_0^t \psi(s)\, dW(s)$ can be defined and have the isometry

$$\mathbf{E}\Big\| \int_0^t \psi(s)\, dW(s) \Big\|^2 = \int_0^t \|\mathbf{E}\psi(s)\|_{L_2^0}^2\, ds. \tag{1.2}$$

Following Da Prato and Zabczyk [5, Chapter 7], we assume that $\sigma : H \to L_2^0$ satisfies the following global Lipschitz and growth conditions,

(i) $\|\sigma(x) - \sigma(y)\|_{L_2^0} \le C\|x - y\|, \quad \forall x, y \in H,$

(ii) $\|\sigma(x)\|_{L_2^0} \le C\|x\|, \quad \forall x \in H.$

Then (1.1) admits a unique mild solution which has the form,

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\sigma(u(s))\, dW(s), \tag{1.3}$$

where $E(t) = e^{-tA}$ is the analytic semigroup generated by $-A$. Moreover

$$\sup_{t\in[0,T]} \mathbf{E}\|u(t)\|^2 \le C(1 + \mathbf{E}\|u_0\|^2).$$

Note that if $\mathrm{Tr}(Q) < \infty$, then the identity mapping $\sigma(u) = I$ does not satisfy the condition (ii). In order to cover this important case, we introduce a modified version of (ii), i.e.,

(ii') $\quad \|A^{(\beta-1)/2}\sigma(x)\|_{L_2^0} \le C\|x\|, \quad$ for some $\beta \in [0,1], \quad \forall x \in H.$
We see that (ii) is the special case $\beta = 1$ of (ii'). If $\sigma(\cdot) = I$, the condition (ii') reduces to $\|A^{(\beta-1)/2}\|_{L_2^0} \le C$ which is the condition used in [18] for the numerical approximation for linear stochastic parabolic partial differential equation.

Numerical methods for equations of the form (1.1), with various assumptions on the nonlinearity σ and the Wiener process $W(t)$, have been studied, for example, by Davie and Gaines [6], Gyöngy [9], [10], Hausenblas [11], Shardlow [15], etc. Our approach is similar to Printems [14], who considers the time discretization in an abstract framework.

In this paper we will consider error estimates for approximations of (1.1) based on the finite element method in space and the backward Euler method in time.

Let S_h be a family of finite element spaces, where S_h consists of continuous piecewise polynomials of degree ≤ 1 with respect to the triangulation $\mathcal{T}_h$ of Ω. For simplicity, we always assume that $\{S_h\} \subset H_0^1 = H_0^1(\mathcal{D}) = \{v \in L_2(\mathcal{D}), \nabla v \in L_2(\mathcal{D}), v|_{\partial\mathcal{D}} = 0\}$. The semidiscrete problem of (1.1) is to find the process $u_h(t) = u_h(\cdot, t) \in S_h$, such that

$$(1.4) \qquad du_h + A_h u_h \, dt = P_h \sigma(u_h) \, dW, \quad \text{for } 0 < t \le T, \quad \text{with } u_h(0) = P_h u_0,$$

where P_h denotes the L_2-projection onto S_h, and $A_h : S_h \to S_h$ is the discrete analogue of A, defined by

$$(1.5) \qquad (A_h\psi, \chi) = A(\psi, \chi), \quad \forall \psi, \chi \in S_h.$$

Here $A(\cdot,\cdot) = (\nabla \cdot, \nabla \cdot)$ is the bilinear form on $H_0^1(\mathcal{D})$ obtained from the operator A.

Let $E_h(t) = e^{-tA_h}$, $t \ge 0$. Then (1.4) admits a unique mild solution

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)P_h\sigma(u_h(s))\, dW(s).$$

Let $\dot{H}^s = \dot{H}^s(\mathcal{D}) = \mathcal{D}(A^{s/2})$ with norm $|v|_s = \|A^{s/2}v\|$ for any $s \in \mathbf{R}$. For any Hilbert space H, we denote

$$L_2(\Omega; H) = \Big\{ v : \mathbf{E}\|v\|_H^2 = \int_\Omega \|v(\omega)\|_H^2 \, d\mathbf{P}(\omega) < \infty \Big\},$$

with norm $\|v\|_{L_2(\Omega;H)} = (\mathbf{E}\|v\|_H^2)^{1/2}$.

Under the assumptions (i) and (ii'), we show, in Theorem 3.2, the following error estimates for $t \in [0, T]$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le C(T)h^\beta \Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0\le s\le T} \mathbf{E}\|u(s)\|_{L_2(\Omega;H)} \Big).$$

We also consider error estimates in the fully discrete case. Let k be a time step and $t_n = nk$ with $n \ge 1$. We define the backward Euler method U^n,

$$(1.7) \quad \frac{U^n - U^{n-1}}{k} + A_h U^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} P_h\sigma(U^n)\, dW(s), \quad n \ge 1, \quad U^0 = P_h u_0.$$

With $r(\lambda) = (1+\lambda)^{-1}$, we can rewrite (1.7) in the form

$$\begin{aligned} U^n &= r(kA_h)U^{n-1} + \int_{t_{n-1}}^{t_n} r(kA_h)P_h\sigma(U^n)\,dW(s), \quad n \geq 1, \\ U^0 &= P_h u_0. \end{aligned} \tag{1.8}$$

Under the assumptions (i) and (ii'), in Theorem 4.2, we have, for $0 \leq \gamma < \beta \leq 1$, $t_n \in [0,T]$,

(1.9)

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \leq C(T)(k^{\gamma/2} + h^{\beta})\Big(\|u_0\|_{L_2(\Omega;\dot H^\beta)} + \sup_{0\leq s\leq T}\|u(s)\|_{L_2(\Omega;H)}\Big).$$

This paper is organized as follows. In Section 2, we consider the regularity of the solution of (1.1). In Section 3, we consider error estimate in semidiscrete case. In Section 4, we consider error estimate in the fully discrete case.

2 Regularity of the mild solution

In this section we will consider the regularity of the mild solution of (1.1). We have the following theorem.

THEOREM 2.1. *Assume that σ satisfies (i) and (ii'). Let $u(t)$ be the mild solution* (1.3) *of* (1.1). *Then we have, for $u_0 \in L_2(\Omega;\dot H^\beta)$,*

$$\|u(t)\|_{L_2(\Omega;\dot H^\beta)} \leq C\Big(\|u_0\|_{L_2(\Omega;\dot H^\beta)} + \sup_{0\leq s\leq t}\|u(s)\|_{L_2(\Omega;H)}\Big). \tag{2.1}$$

In particular, if σ satisfies (i) and (ii), then we have, for $u_0 \in L_2(\Omega;\dot H^1)$,

$$\|u(t)\|_{L_2(\Omega;\dot H^1)} \leq C\Big(\|u_0\|_{L_2(\Omega;\dot H^1)} + \sup_{0\leq s\leq t}\|u(s)\|_{L_2(\Omega;H)}\Big). \tag{2.2}$$

To prove this theorem, we need some regularity results which are related to the fact that $E(t) = e^{-tA}$ is an analytic semigroup on H. For later use, we collect some results in the next two lemmas, see Thomée [16] or Pazy [12].

LEMMA 2.2. *For any $\mu,\nu \in \mathbf{R}$ and $l \geq 0$, there is $C > 0$ such that*

$$|D_t^l E(t)v|_\nu \leq Ct^{-(\nu-\mu)/2-l}|v|_\mu, \quad \text{for } t>0, \quad 2l+\nu \geq \mu, \tag{2.3}$$

and

$$\int_0^t s^\mu |D_t^l E(s)v|_\nu^2\,ds \leq C|v|_{2l+\nu-\mu-1}^2, \quad \text{for } t \geq 0, \quad \mu \geq 0. \tag{2.4}$$

LEMMA 2.3. *For any $\mu \geq 0$, $0 \leq \nu \leq 1$, there is $C > 0$ such that*

$$\|A^\mu E(t)\| \leq Ct^{-\mu}, \quad \text{for } t > 0, \tag{2.5}$$

and

$$\|A^{-\nu}(I - E(t))\| \leq Ct^{\nu}, \quad \text{for } t \geq 0. \tag{2.6}$$

PROOF OF THEOREM 2.1. Recall that the mild solution has the form

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\sigma(u(s))\,dW(s).$$

Thus, for any $\beta \geq 0$, using the stability of $E(t)$ and the isometry (1.2),

$$\begin{aligned}
\mathbf{E}|u(t)|_\beta^2 &\leq 2\mathbf{E}|E(t)u_0|_\beta^2 + 2\mathbf{E}\Big\| \int_0^t A^{\beta/2}E(t-s)\,\sigma(u(s))dW(s)\Big\|^2 \\
&= 2\mathbf{E}|u_0|_\beta^2 + 2\mathbf{E}\int_0^t \|A^{\beta/2}E(t-s)\sigma(u(s))\|_{L_2^0}^2\,ds \\
&= 2\mathbf{E}|u_0|_\beta^2 + 2\mathbf{E}\int_0^t \|A^{1/2}E(t-s)A^{(\beta-1)/2}\sigma(u(s))\|_{L_2^0}^2\,ds.
\end{aligned}$$

By (ii') and Lemma 2.2, we have

$$\begin{aligned}
&\mathbf{E}\int_0^t \|A^{1/2}E(t-s)A^{(\beta-1)/2}\sigma(u(s))\|_{L_2^0}^2\,ds \\
&\leq \Big(\int_0^t \|A^{1/2}E(t-s)\|^2\,ds\Big) \sup_{0\leq s\leq t} \mathbf{E}\|u(s)\|^2 \\
&\leq C \sup_{0\leq s\leq t} \mathbf{E}\|u(s)\|^2.
\end{aligned}$$

Thus we get

$$\mathbf{E}|u(t)|_\beta^2 \leq C(\mathbf{E}|u_0|_\beta^2 + \sup_{0\leq s\leq t} \mathbf{E}\|u(s)\|^2),$$

which implies (2.1) by noting that

$$\Big(\sup_{0\leq s\leq t} \mathbf{E}\|u(s)\|^2\Big)^{1/2} \leq \sup_{0\leq s\leq t} \Big(\mathbf{E}\|u(s)\|^2\Big)^{1/2} = \sup_{0\leq s\leq t} \mathbf{E}\|u(s)\|_{L_2(\Omega;H)}.$$

In particular, if (ii) holds, then $\beta = 1$ and we get (2.2). □

REMARK 2.1. *In Theorem 2.1, if $\sigma(\cdot) = I$, the condition (ii') reduces to $\|A^{(\beta-1)/2}\|_{L_2^0} \leq C$ which is the condition used in [18] for the numerical approximation for linear stochastic parabolic partial differential equation.*

3 Error estimates in the semidiscrete case

In this section we consider error estimates for stochastic partial differential equation in the semidiscrete case. In order to prove our error estimates, we need some nonsmooth data error estimates for the homogeneous deterministic parabolic problem.

Let $E_h(t) = e^{-tA_h}$ and $E(t) = e^{-tA}$. We then have the following error estimates for deterministic parabolic problem, see [18].

LEMMA 3.1. *Let $F_h(t) = E_h(t)P_h - E(t)$. Then*

$$\|F_h v\|_{L_\infty([0,T];H)} \leq Ch^\beta|v|_\beta, \quad \text{for } v \in \dot{H}^\beta,\ 0 \leq \beta \leq 1, \tag{3.1}$$

and

$$\|F_h v\|_{L_2([0,T];H)} \le Ch^\beta |v|_{\beta-1}, \quad \text{for } v \in \dot H^{\beta-1},\ 0 \le \beta \le 1. \tag{3.2}$$

Our main result in this section is the following.

THEOREM 3.2. *Assume that σ satisfies (i) and (ii'). Let u_h and u be the solutions of (1.4) and (1.1), respectively. Then there is $C = C(T)$ such that, for $t \in [0,T]$ and $u_0 \in L_2(\Omega; \dot H^\beta)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le Ch^\beta \Big(\|u_0\|_{L_2(\Omega;\dot H^\beta)} + \sup_{0\le s\le T} \mathbf{E}\|u(s)\|_{L_2(\Omega;H)} \Big). \tag{3.3}$$

In particular, if σ satisfies (i) and (ii), then we have

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le Ch \Big(\|u_0\|_{L_2(\Omega;\dot H^1)} + \sup_{0\le s\le T} \mathbf{E}\|u(s)\|_{L_2(\Omega;H)} \Big). \tag{3.4}$$

PROOF. We have, with $E(t) = e^{-tA}$,

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\sigma(u(s))\, dW(s),$$

and, with $E_h(t) = e^{-tA_h}$,

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)\, P_h \sigma(u_h(s))\, dW(s).$$

Denoting $e(t) = u_h(t) - u(t)$ and $F_h(t) = E_h(t)P_h - E(t)$, we write

$$\begin{aligned} e(t) &= F_h(t)u_0 + \int_0^t F_h(t-s)\sigma(u(s))\, dW(s) \\ &\quad + \int_0^t E_h(t-s)P_h \Big(\sigma(u_h(s) - \sigma(u(s)) \Big)\, dW(s) \\ &= I + II + III. \end{aligned}$$

Thus

$$\|e(t)\|_{L_2(\Omega;H)} \le C \Big(\|I\|_{L_2(\Omega;H)} + \|II\|_{L_2(\Omega;H)} + \|III\|_{L_2(\Omega;H)} \Big).$$

For I, we have, by (3.1) with $v = u_0$,

$$\|I\| = \|F_h(t)u_0\| \le Ch^\beta |u_0|_\beta,$$

which implies that $\|I\|_{L_2(\Omega;H)} \le Ch^\beta \|u_0\|_{L_2(\Omega;\dot H^\beta)}$.

For II, we have, by the isometry (1.2),

$$\begin{aligned}\|II\|^2_{L_2(\Omega;H)} &= \mathbf{E}\Big\|\int_0^t F_h(t-s)\sigma(u(s))\,dW(s)\Big\|^2\\ &= \int_0^t \mathbf{E}\|F_h(t-s)A^{(1-\beta)/2}A^{(\beta-1)/2}\sigma(u(s))\|^2_{L_2^0}\,ds\\ &\le \Big(\int_0^t \|F_h(t-s)A^{(1-\beta)/2}\|^2\,ds\Big)\sup_{0\le s\le t}\mathbf{E}\|A^{(\beta-1)/2}\sigma(u(s))\|^2_{L_2^0}.\end{aligned}$$

We will show that

$$\int_0^t \|F_h(t-s)A^{-(\beta-1)/2}\|^2\,ds \le Ch^{2\beta}. \tag{3.5}$$

Assuming this for the moment, we have, by the growth condition (ii'),

$$\|II\|^2_{L_2(\Omega;H)} \le Ch^{2\beta}\sup_{0\le s\le t}\mathbf{E}\|A^{(\beta-1)/2}\sigma(u(s))\|^2_{L_2^0} \le Ch^{2\beta}\sup_{0\le s\le t}\mathbf{E}\|u(s)\|^2.$$

For III, we have, by the isometry property and the Lipschitz condition (i),

$$\begin{aligned}\|III\|^2_{L_2(\Omega;H)} &= \mathbf{E}\int_0^t \Big\|E_h(t-s)P_h\Big(\sigma(u_h(s)-\sigma(u(s))\Big)\Big\|^2_{L_2^0}\,ds\\ &\le \mathbf{E}\int_0^t \|E_h(t-s)P_h\|^2\|u_h(s)-u(s)\|^2_{L_2^0}\,ds\\ &\le \int_0^t \mathbf{E}\|e(s)\|^2\,ds.\end{aligned}$$

Hence

$$\|e(t)\|^2_{L_2(\Omega;H)} \le Ch^{2\beta}\Big(|u_0|^2_\beta + \sup_{0\le s\le t}\mathbf{E}\|u(s)\|\Big) + C\int_0^t \|e(s)\|^2_{L_2(\Omega;H)}\,ds.$$

Then (3.3) follows from Gronwall's lemma.

It remains to show (3.5). In fact, by the definition of the operator norm and the monotone convergence theorem, we have

$$\begin{aligned}\int_0^t \|F_h(t-s)A^{-(\beta-1)/2}\|^2\,ds &= \int_0^t \sup_{v\ne 0}\frac{\|F_h(t-s)A^{-(\beta-1)/2}v\|^2}{\|v\|^2}\,ds\\ &= \sup_{v\ne 0}\frac{\int_0^t \|F_h(t-s)A^{-(\beta-1)/2}v\|^2\,ds}{\|v\|^2}.\end{aligned}$$

Combining this with (3.2), we show (3.5) and therefore (3.3) holds.

In particular, if (ii) holds then $\beta = 1$ and we obtain (3.4). ∎

4 Error estimates in the fully discrete case

In this section we will consider the error estimates in the fully discrete case. As in the semidiscrete case, we need some error estimates for the deterministic parabolic problem.

Let $E_{kh} = r(kA_h)$ and $E(t) = e^{-tA}$, where $r(\lambda) = 1/(1+\lambda)$ is introduced in (1.8). We have, see [18],

LEMMA 4.1. *Let $F_n = E_{kh}^n P_h - E(t_n)$. Then*

$$\|F_n v\| \le C(k^{\beta/2} + h^\beta)|v|_\beta, \quad \text{for } v \in \dot{H}^\beta,\ 0 \le \beta \le 1, \tag{4.1}$$

and

$$\Big(k \sum_{j=1}^{n} \|F_j v\|^2\Big)^{1/2} \le C(k^{\beta/2} + h^\beta)|v|_{\beta-1}, \quad \text{for } v \in \dot{H}^{\beta-1},\ 0 \le \beta \le 1. \tag{4.2}$$

Our main result in this section is the following.

THEOREM 4.2. *Assume that σ satisfies (i) and (ii′). Let U^n and $u(t_n)$ be the solutions of* (1.8) *and* (1.1), *respectively. Let $0 \le \gamma < \beta$. Then there is $C = C(T)$ such that, for $t_n \in [0,T]$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$,*

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{\gamma/2} + h^\beta)\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0\le s\le T} \|u(s)\|_{L_2(\Omega;H)}\Big) \tag{4.3}$$

In particular, if σ satisfies (i) and (ii), then we have, for $u_0 \in L_2(\Omega; \dot{H}^1)$, and $0 \le \gamma < 1$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{\gamma/2} + h)\Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \sup_{0\le s\le T} \|u(s)\|_{L_2(\Omega;H)}\Big). \tag{4.4}$$

To prove this theorem we need the following regularity result for the solution of (1.1).

LEMMA 4.3. *Assume that (ii′) holds. Let u be the mild solution of* (1.1). *Then we have, for $0 \le \gamma < \beta \le 1$,*

$$\begin{aligned} \mathbf{E}\|u(t_2) - u(t_1)\|^2 &\le C(t_2 - t_1)^\gamma \mathbf{E}|u_0|_\gamma^2 \\ &\quad + C(t_2 - t_1)^\gamma \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2. \end{aligned} \tag{4.5}$$

PROOF. The weak solution of (1.1) has the form, with $E(t) = e^{-tA}$,

$$u(t) = E(t)u_0 + \int_0^T E(t-s)\sigma(u(s))\,dW(s).$$

Thus we have

$$\begin{aligned} u(t_2) - u(t_1) &= \Big(E(t_2)u_0 - E(t_1)u_0\Big) \\ &\quad + \Big(\int_0^{t_2} E(t_2 - s)\sigma(u(s))\,dW(s) - \int_0^{t_1} E(t_1 - s)\sigma(u(s))\,dW(s)\Big), \\ &= I + II. \end{aligned}$$

and therefore

$$\mathbf{E}\|u(t_2)-u(t_1)\|^2 \le 2\mathbf{E}\|I\|^2 + 2\mathbf{E}\|II\|^2.$$

For I, we have, by Lemma 2.3, for $0 \le \gamma \le 2$, with $t_1 \ne 0$,

$$\begin{aligned}\|I\| &= \|E(t_1)A^{-\gamma/2}(E(t_2)-E(t_1))A^{\gamma/2}u_0\| \\ &\le C(t_2-t_1)^{\gamma/2}|u_0|_\gamma,\end{aligned} \tag{4.6}$$

which implies that $\mathbf{E}\|I\|^2 \le C(t_2-t_1)^\gamma \mathbf{E}|u_0|_\gamma^2$.

For II, we have

$$\begin{aligned}II &= \int_0^{t_1} \Big(E(t_2-s)-E(t_1-s)\Big)\sigma(u(s))\,dW(s) \\ &\quad + \int_{t_1}^{t_2} E(t_2-s)\sigma(u(s))\,dW(s) \\ &= II_1 + II_2.\end{aligned}$$

Using (ii'), isometry, and Lemma 2.3, we have, for $0 \le \gamma < \beta \le 1$,

$$\begin{aligned}\mathbf{E}\|II_1\|^2 &= \mathbf{E}\Big\| \int_0^{t_1} \Big(E(t_2-s)-E(t_1-s)\Big)\sigma(u(s))\,dW(s)\Big\|^2 \\ &= \int_0^{t_1} \mathbf{E}\Big\|\Big(E(t_2-s)-E(t_1-s)\Big)A^{(1-\beta)/2}A^{(\beta-1)/2}\sigma(u(s))\Big\|_{L_2^0}^2\,ds \\ &\le \int_0^{t_1} \Big\|A^{(1-\beta)/2}\Big(E(t_2-s)-E(t_1-s)\Big)\Big\|^2\,ds \sup_{0\le s\le t_1}\mathbf{E}\|u(s)\|^2 \\ &= \int_0^{t_1} \Big\|A^{(1-\beta)/2+\gamma/2}E(t_1-s)A^{-\gamma/2}\Big(I-E(t_2-t_1)\Big)\Big\|^2\,ds \sup_{0\le s\le t_1}\mathbf{E}\|u(s)\|^2 \\ &\le C(t_2-t_1)^\gamma\Big(\int_0^{t_1}(t_1-s)^{-(1-\beta)-\gamma}\,ds\Big)\sup_{0\le s\le t_1}\mathbf{E}\|u(s)\|^2 \\ &\le C(t_2-t_1)^\gamma \sup_{0\le s\le t_1}\mathbf{E}\|u(s)\|^2,\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\|II_2\|^2 &= \int_{t_1}^{t_2}\mathbf{E}\|A^{(1-\beta)/2}E(t_2-s)A^{(\beta-1)/2}\sigma(u(s))\|_{L_2^0}^2\,ds \\ &\le C\int_{t_1}^{t_2}\|A^{(1-\beta)/2}E(t_2-s)\|^2\cdot\mathbf{E}\|A^{(\beta-1)/2}\sigma(u(s))\|_{L_2^0}^2\,ds \\ &\le C\Big(\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\,ds\Big)\sup_{t_1\le s\le t_2}\mathbf{E}\|u(s)\|^2 \\ &\le C(t_2-t_1)^\beta\sup_{t_1\le s\le t_2}\mathbf{E}\|u(s)\|^2, \quad \text{for } \beta>0.\end{aligned}$$

Hence we get, for $0 \le \gamma < \beta \le 1$,

$$\mathbf{E}\|II\|^2 \le 2\mathbf{E}\|II_1\|^2 + 2\mathbf{E}\|II_2\|^2 \le C(t_2-t_1)^\gamma \sup_{t_1\le s\le t_2}\mathbf{E}\|u(s)\|^2.$$

Together these estimates complete the proof. □

PROOF OF THEOREM 4.2. We have, by (1.8), with $E_{kh}^n = r(kA_h)^n$,

$$U^n = E_{kh}^n P_h u_0 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} E_{kh}^{n-j+1} P_h \sigma(U^j)\, dW(s),$$

and, by the definition of the mild solution of (1.1), with $E(t) = e^{-tA}$,

$$u(t_n) = E(t_n)u_0 + \int_0^{t_n} E(t_n - s)\sigma(u(s))\, dW(s).$$

Denoting $e^n = U^n - u(t_n)$ and $F_n = E_{kh}^n P_h - E(t_n)$, we write

$$\begin{aligned}
e^n &= F_n u_0 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} r(kA_h)^{n-j+1} P_h \Big(\sigma(U^j) - \sigma(u(t_j))\Big)\, dW(s) \\
&\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} r(kA_h)^{n-j+1} P_h \Big(\sigma(u(t_j)) - \sigma(u(s))\Big)\, dW(s) \\
&\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big(r(kA_h)^{n-j+1} P_h - E(t_n - t_{j-1})\Big)\sigma(u(s))\, dW(s) \\
&\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big(E(t_n - t_{j-1}) - E(t_n - s)\Big)\sigma(u(s))\, dW(s) \\
&= \sum_{j=1}^{5} I_j.
\end{aligned}$$

Thus

$$\|e^n\|_{L_2(\Omega;H)} \le C \sum_{j=1}^{5} \|I_j\|_{L_2(\Omega;H)}.$$

For I_1, we have, by (4.1) with $v = u_0$,

$$\|I_1\| = \|F_n u_0\| \le C(k^{\beta/2} + h^\beta)|u_0|_\beta,$$

which implies that $\|I_1\|_{L_2(\Omega;H)} \le C(k^{\beta/2} + h^\beta)\|u_0\|_{L_2(\Omega;\dot{H}^\beta)}$.

For I_2, we have, by isometry and the stability of $r(\lambda)$ and the Lipschitz condition (i),

$$\begin{aligned}\|I_2\|^2_{L_2(\Omega;H)} &= \mathbf{E}\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} r(kA_h)^{n-j+1}P_h\Big(\sigma(U^j)-\sigma(u(t_j))\Big)\,dW(s)\Big\|^2\\ &= k\sum_{j=1}^{n}\mathbf{E}\Big\|r(kA_h)^{n-j+1}P_h\Big(\sigma(U^j)-\sigma(u(t_j))\Big)\Big\|^2_{L_2^0}\\ &\le k\sum_{j=1}^{n}\|r(kA_h)^{n-j+1}P_h\|^2\,\mathbf{E}\|\sigma(U^j)-\sigma(u(t_j))\|^2_{L_2^0}\\ &\le Ck\sum_{j=1}^{n}\mathbf{E}\|U^j-u(t_j)\|^2 = C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\|e^j\|^2\,ds.\end{aligned}$$

For I_3, we have, by Lemma 4.3, for $0\le\gamma<\beta\le 1$,

$$\begin{aligned}\|I_3\|^2_{L_2(\Omega;H)} &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\Big\|r(kA_h)^{n-j+1}P_h\Big(\sigma(u(t_j))-\sigma(u(s))\Big)\Big\|^2_{L_2^0}\,ds\\ &\le C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\|u(t_j)-u(s)\|^2\,ds\\ &\le C\Big(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(t_j-s)^{\gamma}\,ds\Big)\big(\mathbf{E}|u_0|^2_{\gamma}+\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\big)\\ &\le Ck^{\gamma}\big(\mathbf{E}|u_0|^2_{\gamma}+\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\big).\end{aligned}$$

For I_4, we have

$$\begin{aligned}\|I_4\|^2_{L_2(\Omega;H)} &= \mathbf{E}\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}F_{n-j+1}\sigma(u(s))\,dW(s)\Big\|^2\\ &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\|F_{n-j+1}A^{(1-\beta)/2}A^{(\beta-1)/2}\sigma(u(s))\|^2_{L_2^0}\,ds\\ &\le C\Big(k\sum_{j=1}^{n}\|F_jA^{(1-\beta)/2}\|^2\Big)\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2.\end{aligned}$$

We will show that

$$k\sum_{j=1}^{n}\|F_jA^{(1-\beta)/2}\|^2\le C(k^{\beta}+h^{2\beta}). \tag{4.7}$$

Assuming this for the moment, we get

$$\|I_4\|^2_{L_2(\Omega;H)}\le C(k^{\beta}+h^{2\beta})\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2.$$

For I_5, we have

$$\begin{aligned}\|I_5\|^2_{L_2(\Omega;H)} &= \mathbf{E}\Big\|\sum_{j=1}^n \int_{t_{j-1}}^{t_j} (E(t_n - t_{j-1}) - E(t_n - s))\sigma(u(s))\,dW(s)\Big\|^2 \\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \mathbf{E}\|(E(t_n - t_{j-1}) - E(t_n - s))A^{(1-\beta)/2}A^{(\beta-1)/2}\sigma(u(s))\|^2_{L_2^0}\,ds \\ &\le C\Big(\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|(E(t_n - t_{j-1}) - E(t_n - s))A^{(1-\beta)/2}\|^2\,ds\Big) \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2.\end{aligned}$$

Noting that, by Lemmas 2.2 and 2.3,

$$\begin{aligned}&\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|(E(t_n - t_{j-1}) - E(t_n - s))A^{(1-\beta)/2}\|^2\,ds \\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|A^{1/2}E(t_n - s)A^{-\beta/2}(I - E(s - t_{j-1}))\|^2\,ds \\ &\le Ck^\beta \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|A^{1/2}E(t_n - s)\|^2\,ds \\ &= Ck^\beta \int_0^{t_n} \|A^{1/2}E(s)\|^2\,ds \le Ck^\beta,\end{aligned}$$

we have

$$\|I_5\|^2_{L_2(\Omega;H)} \le Ck^\beta \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2.$$

It remains to show (4.7). In fact, by (4.2),

$$\begin{aligned}k\sum_{j=1}^n \|F_jA^{(1-\beta)/2}\|^2 &= k\sum_{j=1}^n \Big(\sup_{v\ne 0} \frac{\|F_jA^{(1-\beta)/2}v\|}{\|v\|}\Big)^2 \\ &= \sup_{v\ne 0} \frac{k\sum_{j=1}^n \|F_jA^{(1-\beta)/2}v\|^2}{\|v\|^2} \\ &\le \sup_{v\ne 0} \frac{C(k^\beta + h^{2\beta})|A^{(1-\beta)/2}v|^2_{\beta-1}}{\|v\|^2} \le C(k^\beta + h^{2\beta}).\end{aligned}$$

Together these estimates show, for $0 \le \gamma < \beta \le 1$,

$$\begin{aligned}\mathbf{E}\|e^n\|^2 \le C(k^\gamma + h^{2\beta})\mathbf{E}|u_0|^2_\beta + Ck\sum_{j=1}^n \mathbf{E}\|e^j\|^2 \\ + C(k^\gamma + h^{2\beta}) \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2.\end{aligned} \tag{4.8}$$

By the discrete Gronwall lemma, we get

$$\mathbf{E}\|e^n\|^2 \le C(k^\gamma + h^{2\beta})\big(\mathbf{E}|u_0|^2_\beta + \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2\big), \tag{4.9}$$

which implies that,

$$\|e^n\|_{L_2(\Omega;H)} \le C(k^{\gamma/2} + h^\beta)\big(\mathbf{E}|u_0|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0\le s\le t_n} \|u(s)\|_{L_2(\Omega;H)}\big). \tag{4.10}$$

The proof is now complete. □

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