THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

On the Pricing of Path-Dependent Options and Related Problems

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Abstract

This thesis considers the pricing of European path-dependent options in a multi-dimensional Black-Scholes model. The thesis focuses mainly on the three different classes of path-dependent options: barrier, Asian, and lookback options.

Chapter 1 gives a brief introduction to the theory of option pricing and describes some path-dependent options. Chapter 2 derives pricing formulas for continuous double barrier options and studies the numerical properties of the formulas obtained. Chapter 3 extends a work by Broadie et al.[1] and determines approximation formulas for the price of some discrete barrier options. Chapter 4 estimates the price of discrete barrier options using lattice random walks. Chapter 4 will also discuss the rate of convergence of lattice methods and Besov spaces. Chapter 5 gives a probabilistic interpretation of the θ-method. The θ-method is a class of finite difference methods for the heat equation. Chapter 6 shows, using the isoperimetric inequality for Wiener measure, that the relative error in the Monte Carlo pricing of some path-dependent options is independent of the dimension. Chapter 7 studies a certain class of sublinear functionals of geometric Brownian motion. The chapter discusses convexity properties for the distribution function, tail probabilities, stochastic ordering, moment inequalities, and Stieltjes moment problem. Chapter 8, which is a joint work together with Jenny Dennemark and Håkan Norekrans, describes the Heath-Jarrow model for dividend paying assets and studies how discrete dividends influence the price of some path-dependent options.

Keywords. option pricing, path-dependent options, Brownian motion, geometric Brownian motion, Wiener functionals, hitting times, random walks, finite difference methods, heat equation, rate of convergence, Monte Carlo method, error estimates, geometric inequalities, stochastic ordering, moment inequalities, moment problem

AMS 2000 Mathematics subject classification. 60E15, 60G50, 60J65, 65C05, 65C50, 65M06, 91B28

[1]Broadie, M., Glasserman, P., & Kou, S.G. (1997) *A Continuity Correction for Discrete Barrier Options* Math. Finance **7**, 325-349

Preface

Acknowledgements

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Dedicated to the memory of my mother.

Introduction

In the beginning of the seventies Fischer Black and Myron Scholes published their now very famous article "The Pricing of Options and Corporate Liabilities". Based on the principle that, on a rational market, there are no possibilities to make sure profits, Black and Scholes derived a theoretical price for the European call option. Since then the popularity as well as the number of options have increased considerably. In particular, path-dependent options have received a notable amount of attention in academic as well as trade literature.

The purpose of this thesis is to discuss the pricing of European path-dependent options within a Black-Scholes framework. Although the goal is to develop the theory from its foundations, this thesis is mainly intended for readers who are familiar with the basics of Brownian motion and stochastic calculus.

In Chapter 1, we give a brief introduction to the theory of option pricing. The chapter describes a mathematical model of a financial market based on Brownian motion and stochastic calculus. In particular, the chapter will show that the pricing of path-dependent options amounts to computing Wiener integrals. Chapter 1 will also describe some common path-dependent options.

Chapter 2 considers the valuation of continuous double barrier options. Continuous barrier options constitute one of those few examples of path-dependent options where it is possible to compute the option price analytically. By using the reflection principle for Brownian motion and the Cameron-Martin theorem, Chapter 2 expresses the price of a continuous double barrier option in terms of series of normal distribution functions. The numerical properties of the formulas obtained will also be studied.

The pricing of discrete barrier options is studied in Chapter 3 and 4. Previous research by Broadie, Glasserman, and Kou in [28] has shown that the Siegmund heavy traffic approximation can be useful in the valuation of some discrete barrier options. The objective in Chapter 3 is to extend the methods by Broadie, Glasserman, and Kou to a larger class of discrete barrier options. Chapter 3 has appeared in *Finance and Stochastic* vol. 7, nr. 2.

Chapter 4 makes use of lattice random walks to design a numerical procedure useful to estimate the price of a discrete barrier option. The main idea is to replace the driving Brownian motion in the underlying asset price with a lattice random walk. This is a well known and frequently applied method in option pricing. In Chapter 4, we improve this approach for discrete barrier options using results from the theory of Besov spaces. Chapter 4 will be published in *Mathematical Finance* vol. 13, no. 4.

Chapter 5 will also consider lattice random walks. The chapter shows that the θ-method, which is an important class of finite difference methods for the heat equation, may in some cases be seen as a lattice random walk. Chapter 5 has appeared in *Statistics and Probability Letters* vol. 62, no. 2.

Another very useful tool in option pricing is the Monte Carlo method. Chapter 6 will discuss the error in the Monte Carlo pricing of some path-dependent options, and then in particular, Asian and lookback options. One of the advantages with the Monte Carlo method is that the convergence rate is independent of the dimension. Chapter 6 will show that this property also holds for the error constant in the convergence rate in the Monte Carlo pricing of some path-dependent options. The results in Chapter 6 will mainly be based on the isoperimetric inequality for Wiener measure and the Rosenthal inequality. The results in Chapter 6 have previously been presented at the *2nd Bachelier conference.*

Chapter 7 studies a class of sublinear functionals of geometric Brownian motion. The class includes, for instance, the integral as well as the maximum of a positive linear combination of geometric Brownian motions. By applying various geometric inequalities in Wiener space, Chapter 7 establishes convexity properties for the distribution functions, tail probabilities, and moment inequalities. The chapter will also discuss stochastic ordering, the Stieltjes moment problem, and present financial applications of these results. Some of the results in Chapter 7 will appear in *Journal of Applied Probability*, vol. 40, no. 4.

The final chapter, Chapter 8, is a joint work together with Jenny Dennemark and Håkan Norekrans. The chapter discusses how the Black-Scholes model can be extended to cover assets that pay discrete dividends and how discrete dividends influence the price of some path-dependent options. Chapter 8 has appeared in *Futures and Options World/Equity Derivatives*, November 2001.

Contents

Chapter 1

An Introduction to the Theory of Option Pricing and Path-Dependent Options

This chapter gives a brief introduction to the theory of option pricing. The chapter describes a mathematical model of a financial market based on Brownian motion and stochastic calculus. The material in this chapter is only meant as an introduction, for a more comprehensive treatment on option pricing the reader may consult Björk [16], Harrison and Pliska [56], or Musiela and Rutkowski [95]. The chapter is concluded with a detailed description of some common path-dependent options.

1.1 The Market

Take as given a probability space $(\Omega, \mathcal{F}, P)$ carrying a standard m-dimensional Brownian motion $\{W_t\}_{t\geq 0}$. The i:th coordinate process of $\{W_t\}_{t\geq 0}$ is denoted $\{W_t^i\}_{t\geq 0}$. Suppose that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmentation under P of the filtration generated by the Brownian motion W.

Now, to define the market, suppose r and T are positive numbers and let for each $i = 1, \ldots, m$ the constants $\eta_i \in \mathbb{R}$ and $\sigma_i > 0$ be fixed. Suppose moreover that c_i, $i = 1, \ldots, m$, are linearly independent vectors in $\mathbb{R}^m$ with $|c_i|_2 = 1$, where $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^m$. A vector in $\mathbb{R}^m$ shall be understood as a column vector. On the market there are $m+1$ securities. The price processes for these securities are governed by the following (stochastic) differential equations, viz.

$$\begin{cases} dB_t = B_t r dt, & 0 \le t \le T, \\ dS_t^i = S_t^i \,(\, \eta_i dt + \sigma_i \, c_i' dW_t \,), & 0 \le t \le T, \; i = 1, \dots, m, \end{cases}$$

where c_i' denotes the transpose of c_i. Suppose $B_0 = 1$ and that S_0^i, $i = 1, \dots, m$, are positive constants. The stochastic differentials shall be understood in the Itô sense.

The quantity B_t shall be interpreted as the price (in some currency) at time t of a *risk free bond*, with r being the associated risk free interest rate. Moreover, S_t^i is the price (in the same currency as for the bond) at time t of a *risky security* that pays no dividends. The security can for instance be a stock, a commodity, or an asset linked to a foreign currency. The constant σ_i is often referred to as the volatility. If the market only consists of one risky security, that is $m = 1$, then the market is called the Black-Scholes market. If $m > 1$, the market is known as the multi-asset Black-Scholes market.

The solutions to the above (stochastic) differential equations are given by

$$B_t = e^{rt} \quad \text{and} \quad S_t^i = S_0^i e^{(\eta_i - \sigma_i^2/2)\, t + \sigma_i c_i' W_t},$$

for all $t \in [0, T]$ and all $i = 1, \dots, m$. Thus, the price of the risky security follows a so called geometric Brownian motion (with drift).

In the sequel it will be assumed that the market is *frictionless*, meaning that the investors on the market are allowed to trade continuously, that there are no transaction costs, and that there are no restrictions against selling short. Selling short means selling borrowed assets.

Now suppose that we expand the market by adding a so called *contingent T-claim*, also known as a *financial derivative* or an *option*. These are assets which are defined in terms of the risky asset and the bond, which in this connection are referred to as the *underlying assets* or the *underlying securities*. We make the following mathematical formalisation.

Definition 1.1. *Suppose T is a positive constant. A* **contingent T-claim** *is an $\mathcal{F}_T$-measurable and positive random variable X.*

The interpretation of this definition is that the contingent T-claim is a contract which specifies that the stochastic amount X of money is to be paid out to the holder of the contract at time T. The time T will be referred to as maturity date or expiration date of the option.

One of the most important contingent claims is the European call option. Fix an integer $i = 1, \dots, m$ and assume that K is positive constant. A European call option with underlying asset S^i, strike price K, and maturity date T is a contract which gives the holder the possibility but not the obligation to buy one share of the asset S^i at time T at the price K. If $S^i_T \leq K$, the contract is worthless at the maturity date. If $S^i_T > K$, the holder can buy one share of the risky security at the price K giving the net profit $S^i_T - K$. Thus, the European call option with underlying S^i is equivalent to a contract giving the holder the amount

$$X = \max(S^i_T - K, 0)$$

at time T.

How much would an investor be willing to pay for a given contingent T-claim X? Remarkably enough, Black and Scholes [17] asserted that there is a unique rational value for the option, independent of the investor's attitude to risk. The next three sections will give an argument leading to this unique price.

1.2 Portfolios

First some definitions that will be frequently used in the sequel.

Below we let $\mathcal{B}(A)$, where $A \in \mathbb{R}$, denote the smallest σ-algebra containing all open subsets of A.

Definition 1.2. *The stochastic process $h : [0,T] \times \Omega \to \mathbb{R}$ is said to be* **progressively measurable** *with respect to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ if, for each $t \in [0,T]$ and $B \in \mathcal{B}(\mathbb{R})$, the set*

$$\{(s, \omega) \,:\, 0 \leq s \leq t,\, \omega \in \Omega,\, h_s(\omega) \in B\}$$

belongs to $\mathcal{B}(\,[0,t]\,) \otimes \mathcal{F}_t$.

Definition 1.3. *Suppose P and $\tilde{P}$ are equivalent probability measures. The class $\mathcal{L}^p(\,[0,T], \tilde{P}\,)$ denotes the set of all $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ progressively measurable processes h such that*

$$E^{\tilde{P}}\Big[\int_0^T |h_t|^p dt\Big] < \infty,$$

where $E^{\tilde{P}}$ stands for expectation with respect to $\tilde{P}$.

The class $\mathcal{L}^p_{loc}([0,T],\tilde{P})$ *contains all* $\{\mathcal{F}_t\}_{0\le t\le T}$ *progressively measurable processes* h *such that*

$$\int_0^T |h_t|^p dt < \infty \quad \tilde{P} - a.s. \tag{1.1}$$

Thus, for all $h \in \mathcal{L}^2_{loc}([0,T],\tilde{P})$ and each $i = 1,\dots,m$ the Itô integral

$$Y_t = \int_0^t h_s\, dW^i_s, \quad 0 \le t \le T,$$

is well defined. Moreover, if $h \in \mathcal{L}^2([0,T],P)$ then $\{Y_t\}_{0\le t\le T}$ is a $(P,\{\mathcal{F}_t\}_{0\le t\le T})$-martingale (see Karatzas et al. [71]).

Referring to the previous section we will next define portfolios.

Definition 1.4. *A* **portfolio** *(or a* **trading strategy***)* ϕ *is a stochastic process* $\phi_t = (\phi^0_t, \phi^1_t, \dots, \phi^m_t)$, $0 \le t \le T$, *where* ϕ^i, $i = 0, 1, \dots, m$ *are progressively measurable with respect to the filtration* $\{\mathcal{F}_t\}_{0\le t\le T}$. *The* **value process** $\{V_t(\phi)\}_{0\le t\le T}$ *corresponding to the portfolio* ϕ *is defined by*

$$V_t(\phi) = \phi^0_t B_t + \sum_{i=1}^m \phi^i_t S^i_t,$$

for all $t \in [0,T]$.

The random variable ϕ^0_t is interpreted as the number of shares of bonds held in the portfolio at time t. Moreover, the random variables ϕ^i_t, $i = 1,\dots,m$ shall be understood as the number of shares of the i:th risky asset held in the portfolio at time t.

Next we will define portfolios where all the changes in the portfolio values are due to capital gains.

Definition 1.5. *Let* $\phi = (\phi^0, \phi^1, \dots, \phi^m_t)$ *be a portfolio such that* $\phi^0 \in \mathcal{L}^1_{loc}([0,T],P)$ *and* $\phi^i S^i \in \mathcal{L}^2_{loc}([0,T],P)$, $i = 1,\dots,m$. *The portfolio* ϕ *is said to be* **self-financing** *if*

$$V_t(\phi) - V_0(\phi) = \int_0^t \phi^0_s dB_s + \sum_{i=1}^m \int_0^t \phi^i_s dS^i_s\,, \tag{1.2}$$

for all $t \in [0,T]$.

To motivate equation (1.2), suppose that all trading occur at discrete times $t = t_k,\ k = 0, 1, \ldots, n$. The gain $G_{t_k} = V_{t_k}(\phi) - V_{t_0}(\phi)$ at time t_k is thus given by the equation

$$G_{t_k} = \sum_{j=0}^{k-1} \phi^0_{t_j} \left(B_{t_{j+1}} - B_{t_j} \right) + \sum_{i=1}^{m} \sum_{j=0}^{k-1} \phi^i_{t_j} \left(S^i_{t_{j+1}} - S^i_{t_j} \right),$$

provided all changes in the portfolios value are due to capital gains. By letting $\max_k(t_{k+1} - t_k)$ go to zero we are lead to equation (1.2).

It is often convenient to work with discounted prices, meaning that the prices are expressed in terms of the bond instead of in terms of the monetary unit. For this reason, introduce *discounted price processes* Z^i and a *discounted value process* $V^Z(\phi)$ by setting

$$Z^i_t = S^i_t / B_t, \quad 0 \leq t \leq T,\ i = 1, \ldots, m,$$

and

$$V^Z_t(\phi) = V_t(\phi)/B_t = \phi^0_t + \sum_{i=1}^{m} \phi^i_t Z^i_t, \quad 0 \leq t \leq T.$$

If ϕ is a trading strategy such that $\phi^0 \in \mathcal{L}^1_{loc}([0,T], P)$ and $\phi^i S^i \in \mathcal{L}^2_{loc}([0,T], P)$ for each $i = 1, \ldots, m$, then ϕ is self-financing if and only if the discounted value process satisfies

$$V^Z_t(\phi) - V^Z_0(\phi) = \sum_{i=1}^{m} \int_0^t \phi^i_s dZ^i_s, \tag{1.3}$$

for all $t \in [0, T]$, see for instance Musiela et al. [95].

1.3 Arbitrage

A fundamental concept underlying the option pricing theory is that of *arbitrage.*

Definition 1.6. *An* **arbitrage opportunity** *or an* **arbitrage portfolio** *is a self-financing portfolio ϕ such that the corresponding value process has the following properties,*

$$V_0(\phi) = 0,\ V_T(\phi) \geq 0, \quad \textit{and} \quad P(\, V_T(\phi) > 0\,) > 0.$$

As can be seen from the definition, an arbitrage portfolio is a risk free way to make money, or in the terminology of Björk, "a deterministic money making machine", see Björk [16]. Of course, a rational market will be free of arbitrage opportunities.

Actually, on our market it is possible to construct arbitrage portfolios, see e.g. Harrison et al. [56]. To get a reliable model of a security market we must therefore exclude such examples. One way to achieve this is to put constraints on the trading with the risky assets. Before we present such a constraint, we will introduce the so called *martingale measure.*

Suppose $b \in \mathbb{R}^m$ has coordinates

$$b_i = \frac{r - \eta_i}{\sigma_i}, \quad i = 1, \ldots, m,$$

and C is a m by m matrix with rows c_i, $i = 1, \ldots, m$. Let λ be the solution to $C\lambda = b$. Furthermore, define a measure Q on $\mathcal{F}_T$ by

$$dQ = \exp\big(-\frac{1}{2}|\lambda|_2^2 T + \lambda' W_T \big) dP,$$

where λ' denotes the transpose of λ. Note in particular that Q and P are equivalent. If

$$W_t^Q = W_t - \lambda t, \quad 0 \leq t \leq T,$$

then the Cameron-Martin theorem (see Karatzas et al. [71] p. 191) yields that $\{W_t^Q\}_{0 \leq t \leq T}$ is a standard m-dimensional Brownian motion with respect to $(Q, \{\mathcal{F}_t\}_{0 \leq t \leq T})$. Moreover, since

$$dZ_t^i = Z_t^i \left((\eta_i - r)dt + \sigma_i c_i' dW_t \right) = Z_t^i \sigma_i c_i' dW_t^Q, \tag{1.4}$$

the discounted price processes Z^i, $i = 1, \ldots, m$, are martingales with respect to $(Q, \{\mathcal{F}_t\}_{0 \leq t \leq T})$. The measure Q is often called the risk-neutral martingale measure of the market or just the martingale measure of the market.

We are now in the position to define a class of trading strategies without any arbitrage portfolio.

Definition 1.7. *A portfolio* $\phi = (\phi^0, \phi^1, \ldots, \phi^m)$ *is called* **admissible** *if the corresponding discounted value process* $\{V_t^Z(\phi)\}_{0 \leq t \leq T}$ *is a* $(Q, \{\mathcal{F}_t\}_{0 \leq t \leq T})$*-martingale.*

For instance, equation (1.4) gives that if $\phi^i Z^i \in \mathcal{L}^2([0, T], Q)$ for each $i = 1, \ldots, m$, then ϕ is admissible.

Theorem 1.1. *There exists no admissible arbitrage portfolio.*

Proof. Suppose that ϕ is a self-financing and admissible portfolio. The process $\{V_t^Z(\phi)\}_{0\leq t\leq T}$ is a Q-martingale and hence,

$$V_0(\phi) = V_0^Z(\phi) = E^Q\big[V_T^Z(\phi)\big].$$

Thus, if $V_0(\phi) = 0$ then $E^Q[V_T(\phi)] = 0$. Since P and Q are equivalent we can conclude that ϕ cannot be an arbitrage opportunity. □

1.4 Theoretical Price

Consider a contingent T-claim X. Suppose that there is an admissible and self-financing portfolio ϕ such that $V_T(\phi) = X$. If the claim is not priced according to the value of the portfolio at any time $t \leq T$, then there is a risk free profit on the extended market consisting of the contingent claim, the risky securities, and the bond. This leads us to the following important definition.

Definition 1.8. *Let X be a contingent T-claim. Suppose there is a self-financing and admissible portfolio ϕ such that*

$$V_T(\phi) = X.$$

The **theoretical price** *$v(t)$ at time $t \leq T$ corresponding to the claim X is defined by $v(t) = V_t(\phi)$. The portfolio ϕ is called a* **hedging** *or* **replicating portfolio** *for the claim X.*

It can be shown, using the Itô representation theorem, that for every $X \in L^1(Q)$ there is a replicating portfolio ϕ, see e.g. Musiela et al. [95]. Moreover, since a replicating portfolio is admissible, we have

$$v(t) = B_t V_t^Z(\phi) = B_t E^Q\big[X/B_T \,|\, \mathcal{F}_t\,\big] = e^{-r(T-t)}E^Q\big[X \,|\, \mathcal{F}_t\,\big].$$

We can summarise this as follows:

Theorem 1.2. *The theoretical price $v(t)$ at time t of a contingent T-claim $X \in L^1(Q)$ is given by*

$$v(t) = e^{-r(T-t)}E^Q\big[X \,|\, \mathcal{F}_t\,\big],$$

where Q is defined by

$$dQ = \exp\big(-\frac{1}{2}|\lambda|_2^2 T + \lambda' W_T\big)dP.$$

The vector λ *is the solution to* $C\lambda = b$ *where* $b_i = \frac{r-\eta_i}{\sigma_i}$, $i = 1, \dots, m$.

Moreover, the Q-dynamics for the price process S^i *is given by*

$$dS_t^i = S_t^i\,(\,rdt + \sigma_i c_i' dW_t^Q\,), \quad 0 \le t \le T,$$

where $\{W_t^Q\}_{0 \le t \le T}$ *is a* $(Q, \{\mathcal{F}_t\}_{0 \le t \le T})$*-Brownian motion.*

1.5 Dividends

So far we have assumed that the the risky securities pay no dividends. The same assumption was made in the paper by Black and Scholes, but it is not difficult to extend the theory to cover dividend paying securities as well.

There are several different ways to model dividends, see for instance Samuelson [109], Heath and Jarrow [57], Musiela and Rutkowski [95], or Bakstein and Wilmott [8]. This section will discuss a model proposed by Samuelson in [109]. In Samuelson's model the dividends are paid out continuously at a rate which is proportional to the asset price. To be more specific, if the random variable D_t^i denotes the total dividend amount paid by the i:th asset during the time interval $[0, t]$, then

$$D_t^i = \int_0^t q_i S_u^i du, \quad 0 \le t \le T,$$

where q_i is a constant. The constant q_i is known as the *dividend rate* or the *dividend yield.*

The model is applicable to options on foreign currencies (see Garman and Kohlhagen [50]) and commodities (see Hull [65]) but not to stocks. The dividends to a stock are most often paid out at discrete times and consequently, the dividends process $\{D_t\}_{0 \le t \le T}$ corresponding to a stock is not continuous. We will come back to discrete dividends in Chapter 8.

In Samuelson's model the previous definition of a self-financing portfolio will no longer be relevant, since an investor in the market now receive dividends. A more appropriate definition would be to say that a trading strategy ϕ is self-financing if the corresponding value process $V(\phi)$, which is defined as before, satisfies

$$V_t(\phi) - V_0(\phi) = \int_0^t \phi_s^0 dB_s + \sum_{i=1}^m \int_0^t \phi_s^i dS_s^i + \sum_{i=1}^m \int_0^t \phi_s^i dD_s^i$$

for all $0 \le t \le T$. Thus, in a self-financing portfolio the only external funds invested in the portfolio come from the dividend payments, which are, on the other hand, used in full.

From this definition one can proceed as in the previous sections. The modified definitions and computations are straightforward and we obtain the following generalisation of Theorem 1.2.

Theorem 1.3. *Suppose that the i:th asset pays dividends at a constant rate q_i. The theoretical price $v(t)$ at time t of a contingent T-claim $X \in L^1(Q)$ is given by*

$$v(t) = e^{-r(T-t)} E^Q\big[X \mid \mathcal{F}_t\big],$$

where Q is defined by

$$dQ = \exp\big(-\frac{1}{2}|\lambda|_2^2 T + \lambda' W_T\big) dP,$$

The vector λ is the solution to $C\lambda = b$ with $b_i = \frac{r - q_i - \eta_i}{\sigma_i}$, $i = 1, \dots, m$.

Moreover, the Q-dynamics for the price process S^i is given by

$$dS_t^i = S_t^i\big((r - q_i)dt + \sigma_i c_i' dW_t^Q\big), \quad 0 \le t \le T,$$

where $\{W_t^Q\}_{0 \le t \le T}$ is a $(Q, \{\mathcal{F}_t\}_{0 \le t \le T})$-Brownian motion.

1.6 The Black-Scholes Formula

This section will compute the price of two important examples of contingent claims, namely the European call option and the European put option. For simplicity, from now on in this section the market only consists of one risky asset. The price process of the risky asset will be denoted $\{S_t\}_{t \ge 0}$ and the associated volatility and dividend yield will be written σ and q, respectively. The call option is already defined in Section 1.1. A European put option with strike price K and maturity date T, where K is a positive constant, entitles the holder to *sell* one share of the risky security S at the expiration date T at the prespecified price K. Thus, at maturity the value of the put option equals

$$\max(K - S_T, 0).$$

Theorem 1.4 below establishes the theoretical values of European call and put options. In what follows, let Φ denote the standard normal distribution function, that is

$$\Phi(x) = \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}, \quad -\infty \le x \le \infty.$$

The pricing formula for call options in Theorem 1.4 is, in the special case $q = 0$, known as the Black-Scholes formula.

Theorem 1.4. *Assume that the underlying asset pays dividends at a constnat rate q. The theoretical value $v_c(t)$ at time $t \le T$ of a European call option with strike price K and time to expiration T is given by*

$$v_c(t) = S_t e^{-q(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

where

$$d_1 = \frac{\ln(S_t/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

The theoretical value $v_p(t)$ at time $t \le T$ of a European put option with strike price K and time to expiration T is given by

$$v_p(t) = Ke^{-r(T-t)}\Phi(-d_2) - S_t e^{-q(T-t)}\Phi(-d_1).$$

Proof. Put $\tau = T - t$. By Theorem 1.3, the Markov property, and the scaling property for Brownian motion,

$$\begin{aligned}
v_c(t) &= e^{-r\tau}E^Q\big[\max(S_T - K, 0) \,|\, \mathcal{F}_t\big] \\
&= e^{-r\tau}\int_{-\infty}^{\infty} \max(S_t e^{(r-q-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}x} - K, 0)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= e^{-r\tau}\int_{-d_2}^{\infty} (S_t e^{(r-q-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}x} - K)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= S_t e^{-q\tau}\int_{-\infty}^{d_2} e^{-\frac{\sigma^2}{2}\tau - \sigma\sqrt{\tau}x - \frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} - Ke^{-r\tau}\Phi(d_2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{-\infty}^{d_2} e^{-\frac{\sigma^2}{2}\tau - \sigma\sqrt{\tau}x - \frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} &= \int_{-\infty}^{d_2} e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2}\frac{dx}{\sqrt{2\pi}} \\
&= \Phi(d_1),
\end{aligned}$$

which establishes the price of a European call option.

Next, the relation $\max(K - S_T, 0) = K - S_T + \max(S_T - K, 0)$ implies

$$\begin{aligned}
v_p(t) &= e^{-r\tau}E^Q\big[\big(K - S_T + \max(S_T - K, 0)\big) \,|\, \mathcal{F}_t\big] \\
&= Ke^{-r\tau} - S_t e^{-q\tau} + v_c(t) \\
&= Ke^{-r\tau}\Phi(-d_2) - S_t e^{-q\tau}\Phi(-d_1),
\end{aligned}$$

since $\Phi(-x) = 1 - \Phi(x)$, and the proof is complete. □

1.7 Path-Dependent Options

This thesis will mainly consider the pricing of three different classes of path-dependent options, namely barrier options, Asian options, and lookback options. This section will define and discuss the purpose of these contracts. With the exception of rebate options associated with knock-out options (cf. Subsection 1.7.1 below), this thesis will only consider European styled options and not any American styled options. An American option can be exercised at any time before the maturity date of the option, in contrast to an European styled option which only can be exercised at the maturity date. For a further discussion about American options, see e.g. Karatzas and Shreve [72].

From now on in this chapter the underlying price process will be denoted $\{S_t\}_{t\geq 0}$. The underlying asset can either be one of the assets S^i, $i = 1, \ldots, m$, defined in Section 1.1, or a positive linear combination of these securities, a so called basket or index. Let moreover the maturity date T and the strike price K be fixed positive constants.

1.7.1 Barrier Options

The term barrier option refers to an option with a payoff depending on whether or not the underlying asset price is above or below a prespecified barrier (or barriers) during a certain period of time. Barrier options is a large class of options and they can be divided into two different kinds of contracts, knock-out and knock-in options. The special feature of a knock-out option is that it will be extinguished ('knocked-out') if the underlying asset price breaches some barrier (or barriers) prior to the expiration date. In contrast to knock-out options, knock-in options will instead come alive as soon as the barrier (or barriers) is crossed.

Suppose M is a closed subset of $[0, T]$ and $H : M \to (0, \infty)$ is a continuous function. The set M will be called the monitoring dates or the monitoring time and H will be referred to as the barrier. If $M = [0, T]$ then the barrier is called *continuous* and if M is a finite set then the barrier is said to be *discrete*. We may add that in practice most barriers are constant over time, that is $H(t) = H$ for all $t \in M$. Next, let κ be a constant equal to -1 or 1 and introduce the random variable

$$\tau_H^\kappa = \inf\{t \in M \,:\, \kappa S_t \leq \kappa H(t)\}.$$

We use the convention $\inf \emptyset = \infty$.

To begin with we will define knock-out options. Let $\chi = 1$ or -1 and consider

$$X = \max\left(\chi(S_T - K), 0\right)1_{\{\tau_H^\kappa > T\}}.$$

If $\kappa = -1$, then X is the payoff at time T of a *down-and-out call* or *put* ($\chi = 1$ or -1, respectively). If $\kappa = 1$, then X is the payoff at time T of an *up-and-out call* or *put* ($\chi = 1$ or -1, respectively).

A knock-out option may also have two barriers, in this case the contract is called a double barrier knock-out option. To define these claims, set

$$\tau_{H_1,H_2} = \inf\{t \in M \, : \, S_t \leq H_1(t) \text{ or } S_t \geq H_2(t)\}$$

where H_1 and H_2 are two continuous functions on M such that $H_1 < H_2$. The holder of *double barrier knock-out call/put* ($\chi = 1/-1$) will recieve, at time T, the amount

$$X = \max\left(\chi(S_T - K), 0\right)1_{\{\tau_{H_1,H_2} > T\}}.$$

Knock-out options may in some cases be combined with so called rebate options. The purpose with these contracts is to compensate for the loss that occurs when the knock-out option is 'knocked out'. A rebate option will pay to its owner a fixed amount R at the same time the barrier (or barriers) is (are) breached, provided that this event appears before the maturity date T of the knock-out option. For instance, the rebate option associated with a down-and-out call or put option pays at time $\tau_H^{(-1)}$ the amount

$$X = R1_{\{\tau_H^{(-1)} \leq T\}}.$$

Rebate options are also known as American binary options, see Hull [65] for a further discussion about binary options. Note that since the payoff date depends on the underlying asset, this contract does not fit into the theory developed in this chapter. In Section 2.3 we will discuss a solution to this problem.

Next we will define knock-in options. As already mentioned, these contracts will come alive as soon as the underlying asset has passed the barrier or barriers. The payoff at time T of a *down-and-in call* or *put* option ($\chi = 1$ or -1, respectively) is given by

$$X = \max\left(\chi(S_T - K), 0\right)1_{\{\tau_H^{(-1)} \leq T\}}.$$

By replacing $\tau_H^{(-1)}$ with $\tau_H^{(1)}$ in the expression for X we obtain the payoff of an *up-and-in call* or *put* option. The payoff of a double barrier knock-in call or put option is now obvious. Knock-in option may also be combined with a rebate option. For instance, the rebate option associated with a down-and-in call or put option pays at time T the amount

$$X = R1_{\{\tau_H^{(-1)} > T\}}.$$

So far we have described the most traded barrier options. There are other examples of barrier options, for instance the *Parisian barrier option*. This contract is based on the age of the excursion of the underlying price process beyond a given barrier. The Parisian down-and-out call for instance, will expire without value if the underlying price process goes below the barrier and stays continuously below the barrier for a time interval longer than a specified delay. Parisian contracts were introduced in Chesney, Jeanblanc-Picque, and Yor [33]. Parisian barrier options are one example of so called soft barrier options, for a further discsussion about other options in this category, see Linetsky [87].

Barrier options have a relatively long history. Already in 1973, Merton derived the theoretical price of a down-and-out call option, see Merton [94]. Nowadays barrier options are frequently occuring on the market. Its popularity depends mainly on the fact that knock-out and knock-in call/put options are cheaper than the corresponding contracts without any barriers. If an investor finds it unlikely that the underlying asset will fall below a certain price level, it is natural to buy a knock-out option with the barrier at that same level. The difference in price between the knock-out option and the ordinary option can be substantial. Thus, using barrier options, investors can avoid paying for the scenarios they feel are unlikely.

However, these benefits may imply a risk. Barrier options can be very sensitive to price changes of the underlying asset. For instance, consider an up-and-out call option and suppose that the underlying price is just beneath the barrier and that the option is close to maturity. If a small short term price spike occurs, the option will expire without value. On the other hand, if the asset price remains constant until maturity the contract can become very valuable. Thus, investing in barrier options is sometimes combined with a large risk. For a more comprehensive treatment about this topic we refer to Linetsky [87] and the references therein.

1.7.2 Asian Options

Asian options are contracts with payoffs that depend on the average of the underlying instrument over some prespecified time. Asian options are also known as average options.

If μ is a positive and bounded Borel measure on $[0, T]$ then the payoff at time T of an *Asian call/put option with a fixed strike price* $(\chi = 1/-1)$ is given by

$$X = \max\big(\chi(\int_0^T S_t \mu(dt) - K), 0\big).$$

For most traded Asian options the measure μ is a positive linear combination of Dirac measures, the claim is then called a *discrete* Asian option.

If $\alpha > 0$ and

$$X = \max\big(\chi(\int_0^T S_t \mu(dt) - \alpha S_T), 0\big),$$

then X is the payoff at time T of an *Asian call/put option with a floating strike price* $(\chi = 1/-1)$. Moreover, if $u \in (0, T)$ then

$$X = \max\Big(\chi(\int_{(u,T]} S_t \mu(dt) - \int_{[0,u]} S_t \mu(dt)), 0\Big)$$

is the payoff at time T of a *forward start Asian call/put option* $(\chi = 1/-1)$. The option may also be referred to as an *Asian in call/put option.*

Asian options have become increasingly popular and then, in particular, on the commodity market and the foreign exchange market. According to Yor [122] p. 2 over 95 % of options on oil and oil spreads are Asian. The Asian option has a number of attractive features. Asian options provide for the buyer a cost-efficient way of hedging cash and asset flows over extended periods. Moreover, Asian options are in comparison with ordinary options not so sensitive to price manipulations near the maturity date. Asian-styled contracts are thus of special interest for thinly traded assets. For a further discussion on Asian options the reader is recommended Yor [122].

1.7.3 Lookback Options

A lookback option depends on the maximum of the underlying asset over some prespecified time. Let M be a closed subset of $[0, T]$. The buyer of a *standard lookback call/put* $(\chi = 1/-1)$ will recieve at time T the amount

$$X = \max\big(\max_{t \in M} \chi(S_t - S_T), 0\big)$$

The *call on the maximum* or *put on the minimum* ($\chi = 1$ or $\chi = -1$, respectively) will give to its holder at time T the amount

$$X = \max(\max_{t \in M} \chi(S_t - K), 0),$$

where K is a fixed positive constant.

To the best of our knowledge, lookback options was first studied in Goldman, Sosin, and Gatto [54] and Goldman, Sosin, and Shepp [55]. The idea behind lookback options is to solve the investors dilemma of market entrance and/or market exit. Loosely speaking, they are meant to realize the investors dream of "buying at the lowest" and "selling at the highest". However, lookback options are quite expensive and they have therefore never become especially popular on the market.

Chapter 2

Pricing Double Barrier Options with Error Control

To the best of our knowledge, pricing formulas for continuous European double barrier options was first established in Kunitomo and Ikeda [78]. By using the Levy formula (cf. Kunitomo et al. [78]), Kunitomo et al. expressed the price of a double barrier option as series of normal distribution functions. Subsequently several authors have obtained other formulas and methods to value continuous double barrier options. Hui [62], [63] solved the pricing problem using separation of variables and Pelsser [98] derived the value with the aid of contour integration. Both obtained formulas which describe the price of a double barrier option as Fourier sine series. Geman and Yor [51] computed the Laplace transform of the price as a function of time to maturity. They invert the Laplace transform numerically to obtain the option prices.

The numerical characteristics for the two different *series* solutions have been compared in Hui, Lo, and Yuen [64]. They recommend that one should use the Kunitomo-Ikeda pricing formulas since cancellation errors can appear in the Fourier series which may lead to substantial errors in the resulting theoretical values.

The purpose of this chapter is to derive the Kunitomo-Ikeda pricing formulas and investigate the numerical properties of these formulas. In particular, this chapter will derive error estimates for the truncation error that appears when the infinite series are approximated with a partial sum. A similar investigation of the numerical properties of Kunitomo and Ikeda's pricing formulas have independently been carried out by Lou [91] and Schroeder [110].

The remainder of this chapter is structured as follows. Section 2.1 computes analytical expressions for certain distributions which involve stopping times associated with a Brownian motion. The section will also discuss numerical characteristics of the formulas obtained. Section 3 considers the pricing of double barrier options with zero rebate and Section 4 deals with the valuation of rebate options.

2.1 Distributions Involving First Hitting Times

Take as given a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ carrying a standard one-dimensional Brownian motion $\{W_t\}_{t\geq 0}$. If $b \in \mathbb{R}$, the first hitting time of b, hereafter denoted ρ_b, is defined by

$$\rho_b = \inf\{t \geq 0 \,:\, W_t = b\}.$$

Introduce moreover a collection of probability measures $\{P^\theta\}_{\theta\in\mathbb{R}}$ on $\mathcal{F}_1$ according to

$$dP^\theta = e^{-\frac{1}{2}\theta^2+\theta W_1} dP.$$

The Cameron-Martin theorem yields that the stochastic process W^θ, given by

$$W_t^\theta = W_t - \theta t, \quad 0 \leq t \leq 1,$$

is a Brownian motion with respect to $(P^\theta, \{\mathcal{F}_t\}_{0\leq t\leq 1})$.

The objective of the first part of this section is to determine an analytical expression of the distribution function

$$G_+(a, b_1, b_2; \theta) = P^\theta(\, W_1 \leq a,\, \rho_{b_2} < \rho_{b_1},\, \rho_{b_2} \leq 1\,),$$

where $b_1 < 0 < b_2$, $a \leq b_2$, and $\theta \in \mathbb{R}$. The key result is next lemma, the proof of which is based on an idea described in Andersson [4].

Lemma 2.1. *Suppose $b_1 < 0 < b_2$, $a \leq b_2$, and $\theta \in \mathbb{R}$. Set $\rho^{(0)} = 0$, $\varrho^{(0)} = 0$, and define recursively, for $n \geq 1$,*

$$\rho^{(n)} = \inf\{t > \varrho^{(n-1)} \,:\, W_t = b_2\}$$

and

$$\varrho^{(n)} = \inf\{t > \rho^{(n-1)} \,:\, W_t = b_1\}.$$

Then, for any $n \geq 1$,

$$\begin{aligned}G_+(a,&b_1,b_2;\theta)\\&=\sum_{i=1}^{n}\big(P^\theta(\,W_1\le a,\,\rho^{(2i-1)}\le 1\,)-P^\theta(\,W_1\le a,\,\rho^{(2i)}\le 1\,)\big)\\&\quad+P^\theta(\,W_1\le a,\,\rho^{(2n+1)}\le 1,\,\rho^{(1)}<\varrho^{(1)}\,).\end{aligned}$$

Proof. Suppose that $A = \{\rho^{(1)} < \varrho^{(1)}\}$ and $B_n = \{\rho^{(n)} \leq 1\}$. Note that for all $\omega \in A^c$ we have $\rho^{(1)}(\omega) = \rho^{(2)}(\omega)$, which implies $\varrho^{(2)}(\omega) = \varrho^{(3)}(\omega)$, which in turn gives $\rho^{(3)}(\omega) = \rho^{(4)}(\omega)$ and so forth. Hence, by induction on n it can be shown that for all $\omega \in A^c$ we have $\rho^{(2n-1)}(\omega) = \rho^{(2n)}(\omega)$ for every $n \geq 1$, and, accordingly from this

$$1_{B_{2n-1}}1_{A^c} = 1_{B_{2n}}1_{A^c} \tag{2.1}$$

for every $n \geq 1$. By a similar argument it follows that for all $\omega \in A$ and each $n \geq 1$ it holds $\rho^{(2n)}(\omega) = \rho^{(2n+1)}(\omega)$ and thus,

$$1_{B_{2n}}1_A = 1_{B_{2n+1}}1_A \tag{2.2}$$

for every $n \geq 1$. Next observe that for any given sets C_1 and C_2 we have

$$1_{C_1}1_{C_2} = 1_{C_1} - 1_{C_1}1_{C_2^c}. \tag{2.3}$$

Successive applications of equations (2.1), (2.2), and (2.3) yield

$$\begin{aligned}1_{B_1}1_A &= 1_{B_1} - 1_{B_1}1_{A^c}\\&= 1_{B_1} - 1_{B_2}1_{A^c}\\&= 1_{B_1} - 1_{B_2} + 1_{B_2}1_A\\&= 1_{B_1} - 1_{B_2} + 1_{B_3}1_A\\&\;\vdots\\&= 1_{B_1} - 1_{B_2} + 1_{B_3} - \ldots - 1_{B_{2n}} + 1_{B_{2n+1}}1_A\end{aligned}$$

for any $n \geq 1$. By integrating both sides over the set $\{W_1 \leq a\}$ with respect to the measure P^θ we get the desired result. □

Next step is to determine analytical expressions of the terms in the sum in the expression of G_+. To this end, let Φ denote the standard normal distribution function and $\{U_t\}_{t\geq 0}$ be the Brownian semi-group, i.e.

$$(U_t f)(x) = E\big[f(x + W_t)\big], \quad x \in \mathbb{R},$$

where $f : \mathbb{R} \to \mathbb{R}$ is bounded and Borel measurable. Recall that if ρ is a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\rho \leq t$, where t is a fixed positive number, then the strong Markov property for Brownian motion tells us

$$E[f(W_t) \mid \mathcal{F}_\rho](\omega) = (U_{t-\rho(\omega)}f)(W_{\rho(\omega)}(\omega)).$$

The proof of the next lemma follows that in Karatzas et al. [71], p. 95 and p. 98.

Lemma 2.2. *Let $\rho^{(i)}$ be defined as in Lemma 2.1 and suppose $a \leq b_2$. For any $i \geq 1$ it holds*

$$P(\, W_1 \leq a,\, \rho^{(2i-1)} \leq 1\,) = \Phi(a - 2\alpha_i) \tag{2.4}$$

and

$$P(\, W_1 \leq a,\, \rho^{(2i)} \leq 1\,) = \Phi(a - 2\beta_i), \tag{2.5}$$

where $\alpha_i = i(b_2 - b_1) + b_1$ and $\beta_i = i(b_2 - b_1)$.

Proof. Firstly, fix a positive number $t \leq 1$ and note that the symmetry of Brownian motion implies

$$\begin{aligned}\big(U_{1-t}1_{(-\infty,a]}\big)(b) &= P(\, b + W_{1-t} \leq a\,)\\ &= P(\, b + W_{1-t} \geq 2b - a\,)\\ &= \big(U_{1-t}1_{[2b-a,\infty)}\big)(b)\end{aligned} \tag{2.6}$$

for any real numbers a and b.

The stopping time $\bar{\rho} = \rho^{(2i-1)} \wedge 1$ is obviously bounded for any $i \geq 1$. The strong Markov property in combination with equation (2.6) now implies for $\omega \in \{\bar{\rho} < 1\}$,

$$\begin{aligned}E\big[1_{\{W_1 \leq a\}} \mid \mathcal{F}_{\bar{\rho}}\big](\omega) &= \big(U_{1-\bar{\rho}(\omega)}1_{(-\infty,a]}\big)\big(W_{\bar{\rho}(\omega)}(\omega)\big)\\ &= \big(U_{1-\bar{\rho}(\omega)}1_{(-\infty,a]}\big)(b_2)\\ &= \big(U_{1-\bar{\rho}(\omega)}1_{[2b_2-a,\infty)}\big)(b_2)\\ &= \big(U_{1-\bar{\rho}(\omega)}1_{[2b_2-a,\infty)}\big)\big(W_{\bar{\rho}(\omega)}(\omega)\big)\\ &= E\big[1_{\{W_1 \geq 2b_2-a\}} \mid \mathcal{F}_{\bar{\rho}}\big](\omega).\end{aligned}$$

By integrating over $\{\bar{\rho} < 1\} = \{\rho^{(2i-1)} < 1\}$ it follows

$$P(\, W_1 \le a,\, \rho^{(2i-1)} < 1\,) = P(\, W_1 \ge 2b_2 - a,\, \rho^{(2i-1)} < 1\,).$$

It is evident that $P(\,\rho^{(2i-1)} = 1\,) = 0$ and hence,

$$P(\, W_1 \le a,\, \rho^{(2i-1)} \le 1\,) = P(\, W_1 \ge 2b_2 - a,\, \rho^{(2i-1)} \le 1\,).$$

Since $b_2 \le 2b_2 - a$ we find

$$\{\, W_1 \ge 2b_2 - a,\, \rho^{(2i-1)} \le 1\,\} = \{\, W_1 \ge 2b_2 - a,\, \varrho^{(2i-2)} \le 1\,\}$$

so that

$$P(\, W_1 \le a,\, \rho^{(2i-1)} \le 1\,) = P(\, W_1 \ge 2b_2 - a,\, \varrho^{(2i-2)} \le 1\,). \tag{2.7}$$

The relation $2b_2 - a \ge b_1$, the symmetry of Brownian motion, and equation (2.7) give

$$\begin{aligned} &P(\, W_1 \ge 2b_2 - a, \varrho^{(2i-2)} \le 1\,) \\ &\qquad\qquad = P(\, W_1 \le 2(b_1 - b_2) + a,\, \rho^{(2i-3)} \le 1\,) \end{aligned} \tag{2.8}$$

for $i \ge 2$. Equation (2.4) now follows by induction on i.

By replacing $\rho^{(2i-1)}$ by $\rho^{(2i)}$ and $\varrho^{(2i-2)}$ by $\varrho^{(2i-1)}$ in equations (2.7) and (2.8) we get equation (2.5). □

Next we will extend Lemma 2.1 to the case $\theta \ne 0$.

Lemma 2.3. *Let $\rho^{(i)}$ be defined as in Lemma 2.1 and suppose $a \le b_2$. For any $i \ge 1$ and any $\theta \in \mathbb{R}$ it holds*

$$P^\theta(\, W_1 \le a,\, \rho^{(2i-1)} \le 1\,) = e^{2\theta\alpha_i}\Phi(a - 2\alpha_i - \theta)$$

and

$$P^\theta(\, W_1 \le a,\, \rho^{(2i)} \le 1\,) = e^{2\theta\beta_i}\Phi(a - 2\beta_i - \theta),$$

where α_i and β_i are defined as in Lemma 2.2.

Proof. Fix an integer $j \ge 1$. Observe that

$$\begin{aligned} P^\theta(\, W_1 \le a,\, \rho^{(j)} \le 1\,) =& E\big[\exp(-\frac{1}{2}\theta^2 + \theta W_1) 1_{\{W_1 \le a,\, \rho^{(j)} \le 1\}}\big] \\ =& \int_{-\infty}^{a} \exp(-\frac{1}{2}\theta^2 + \theta x) P(W_1 \in dx,\, \rho^{(j)} \le 1). \end{aligned}$$

Define

$$\gamma = \begin{cases} \alpha_i, \ i = (j+1)/2, & \text{if } j \text{ is an odd number}, \\ \beta_i, \ i = j/2, & \text{if } j \text{ is an even number}. \end{cases}$$

Lemma 2.2 yields

$$\frac{d}{dx} P(\, W_1 \leq x, \, \rho^{(j)} \leq 1\,) = e^{-\frac{1}{2}(x-2\gamma)^2} \frac{1}{\sqrt{2\pi}},$$

so that

$$\begin{aligned} P^\theta(\, W_1 \leq a, \, \rho^{(j)} \leq 1\,) &= \int_{-\infty}^{a} \exp\big(-\frac{1}{2}\theta^2 + \theta x - \frac{1}{2}(x-2\gamma)^2\big)\frac{dx}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{a} \exp\big(2\theta\gamma - \frac{1}{2}(x-2\gamma-\theta)^2\big)\frac{dx}{\sqrt{2\pi}} \\ &= e^{2\theta\gamma}\, \Phi(a - 2\gamma - \theta), \end{aligned}$$

which proves Lemma 2.3. □

Next we will focus on the remainder term

$$P^\theta(\, W_1 \leq a, \, \rho^{(2n+1)} \leq 1, \, \rho^{(1)} < \varrho^{(1)}\,)$$

in the expression of G_+. Observe that

$$\begin{aligned} P^\theta(\, W_1 \leq a, \, \rho^{(2n+1)} \leq 1, \, \rho^{(1)} < \varrho^{(1)}\,) &\leq P^\theta(\, W_1 \leq a, \, \rho^{(2n)} \leq 1\,) \\ &= e^{2\theta\beta_n}\Phi(a - 2\beta_n - \theta), \end{aligned}$$

according to Lemma 2.3. The well known inequality

$$\Phi(x) \leq -\frac{1}{x} e^{-x^2/2}, \quad x < 0,$$

implies that the remainder term is bounded by

$$\begin{aligned} &P^\theta(\, W_1 \leq a, \rho^{(2n+1)} \leq 1, \, \rho^{(1)} < \varrho^{(1)}\,) \\ &\qquad \leq \frac{-1}{a - 2\beta_n - \theta} \exp\big(2\theta\beta_n - \frac{(a-2\beta_n-\theta)^2}{2}\big) \end{aligned}$$

provided $a < 2\beta_n + \theta$, and hence, given $\delta < 2$,

$$P^\theta(W_1 \leq a, \, \rho_1^{(2n+1)} \leq 1, \, \rho^{(1)} < \varrho^{(1)}) = o(e^{-n^\delta})$$

as n tends to infinity. This result in combination with Lemmas 2.1 and 2.3 imply Proposition 2.1 below.

Proposition 2.1. *Suppose $b_1 < 0 < b_2$, $a \le b_2$ and let $\alpha_i = i(b_2 - b_1) + b_1$ and $\beta_i = i(b_2 - b_1)$. If $\delta < 2$ then*

$$G_+(a, b_1, b_2; \theta) = \sum_{i=1}^{n} \left(e^{2\alpha_i\theta}\Phi(a - 2\alpha_i - \theta) - e^{2\beta_i\theta}\Phi(a - 2\beta_i - \theta) \right) + R_{n+1},$$

where $R_{n+1} = o(e^{-n^\delta})$, $n \to \infty$, or more precisely

$$|R_{n+1}| \le e^{2\theta\beta_n}\Phi(a - 2\beta_n - \theta).$$

Proposition 2.1 shows how one can control the truncation error, that is, the error appearing as the infinite series in the expression of G_+ is approximated with a partial sum. Indeed, set

$$p_i = e^{2\alpha_i\theta}\,\Phi(a - 2\alpha_i - \theta), \quad q_i = e^{2\beta_i\theta}\,\Phi(a - 2\beta_i - \theta),$$

and define

$$\tilde{G}_+^{(\epsilon)}(a, b_1, b_2; \theta) = \sum_{\{i\,:\,q_i \ge \epsilon\}} (p_i - q_i), \tag{2.9}$$

for any $\epsilon > 0$. Proposition 2.1 now yields

$$\left| \, G_+(a, b_1, b_2; \theta) - \tilde{G}_+^{(\epsilon)}(a, b_1, b_2; \theta) \, \right| < \epsilon.$$

Of course, in practice there is one additional error source besides the truncation error. The standard normal distribution function must be evaluated numerically. However, there are efficient methods with very high accuracy to compute the normal distribution function, see e.g. Cody [35] and thus, the approximation error of the normal distribution function is negligible compared to the truncation error.

The intention with the remaining part of this section is to introduce and determine certain distribution functions that will be useful in the sequel. First recall that for any $\theta \in \mathbb{R}$,

$$G_+(a, b_1, b_2; \theta) = P^\theta(\, W_1 \le a,\ \rho_{b_2} < \rho_{b_1}, \rho_{b_2} \le 1\,)$$

provided $b_1 < 0 < b_2$ and $a \le b_2$. Now, let for any $\theta \in \mathbb{R}$,

$$G_-(a, b_1, b_2; \theta) = P^\theta(\, W_1 \ge a,\ \rho_{b_1} < \rho_{b_2},\ \rho_{b_1} \le 1\,) \tag{2.10}$$

where $b_1 < 0 < b_2$ and $a \geq b_1$. Furthermore, given $b_1 < 0 < b_2$, set

$$\begin{aligned} I_+(b_1, b_2; \theta) &= P^\theta(\, \rho_{b_2} < \rho_{b_1},\, \rho_{b_2} \leq 1\,), \\ I_-(b_1, b_2; \theta) &= P^\theta(\, \rho_{b_1} < \rho_{b_2},\, \rho_{b_1} \leq 1\,), \end{aligned} \tag{2.11}$$

and, if moreover $b_1 \leq a_1 \leq a_2 \leq b_2$,

$$J(a_1, a_2, b_1, b_2; \theta) = P^\theta(\, a_1 < W_1 \leq a_2,\, \rho_{b_1} \wedge \rho_{b_2} > 1\,). \tag{2.12}$$

The next result shows that the above functions can be expressed in terms of G_+.

Proposition 2.2. *Let the functions G_-, I_+, I_-, and J be defined as in equations* (2.10)-(2.12). *Then*

$$\begin{aligned} G_-(a, b_1, b_2; \theta) &= G_+(-a, -b_1, -b_2; -\theta)\,, \\ I_+(b_1, b_2; \theta) &= \Phi(\theta - b_2) \\ &\quad + G_+(b_2, b_1, b_2; \theta) - G_-(b_2, b_1, b_2; \theta)\,, \\ I_-(b_1, b_2; \theta) &= I_+(-b_1, -b_2; -\theta), \end{aligned}$$

and

$$\begin{aligned} J(a_1, a_2, b_1, b_2; \theta) &= \Phi(a_2 - \theta) - \Phi(a_1 - \theta) \\ &\quad - G_+(a_2, b_1, b_2; \theta) + G_+(a_1, b_1, b_2; \theta) \\ &\quad - G_-(a_2, b_1, b_2; \theta) + G_-(a_1, b_1, b_2; \theta). \end{aligned}$$

Proof. The expression for G_- follows at once from the symmetry of Brownian motion.

Below, let $\rho_1 = \rho_{b_1}$ and $\rho_2 = \rho_{b_2}$. To prove the second equation in Proposition 2.2, note that

$$\begin{aligned} I_+(b_1, b_2; \theta) =& P^\theta(\, W_1 \leq b_2,\, \rho_2 < \rho_1,\, \rho_2 \leq 1\,) \\ &+ P^\theta(\, W_1 > b_2,\, \rho_2 < \rho_1,\, \rho_2 \leq 1\,) \\ =& G_+(b_2, b_1, b_2; \theta) \\ &+ P^\theta(\, W_1 > b_2,\, \rho_2 < \rho_1\,), \end{aligned}$$

since $\{W_1 > b_2\} \subset \{\rho_2 \leq 1\}$. It is obvious that $P^\theta(\, W_1 > b_1,\, \rho_1 = \rho_2) = 0$ and, accordingly from this,

$$\begin{aligned} P^\theta(\, W_1 > b_2,\, \rho_2 < \rho_1\,) =& P^\theta(\, W_1 > b_2\,) \\ &- P^\theta(\, W_1 > b_2,\, \rho_1 < \rho_2\,) \\ =& P^\theta(\, W_1 > b_2\,) \\ &- P^\theta(\, W_1 > b_2,\, \rho_1 < \rho_2,\, \rho_1 \leq 1\,) \end{aligned}$$

since $\{\rho_1 < \rho_2\} \cap \{W_1 > b_2\} \subset \{\rho_1 \leq 1\}$, and the statement about I_+ in Proposition 2.2 is established. The expression for I_- follows from symmetry.

It remains to determine J. Observe that

$$\begin{aligned} J(a_1, a_2, b_1, b_2; \theta) =& P^\theta(\, a_1 < W_1 \leq a_2\,) \\ &- P^\theta(\, a_1 < W_1 \leq a_2,\ \rho_1 \wedge \rho_2 \leq 1\,) \\ =& P^\theta(\, W_1 \leq a_2) - P^\theta(W_1 \leq a_1\,) \\ &- P^\theta(\, W_1 \leq a_2,\ \rho_1 \wedge \rho_2 \leq 1\,) \\ &+ P^\theta(\, W_1 \leq a_1,\ \rho_1 \wedge \rho_2 \leq 1\,). \end{aligned}$$

The expression for J is now a consequence of the relation

$$\begin{aligned} P^\theta(\, W_1 \leq a,\ \rho_1 \wedge \rho_2 \leq 1\,) =& P^\theta(\, W_1 \leq a,\ \rho_2 < \rho_1,\ \rho_2 \leq 1) \\ &+ P^\theta(\, W_1 \leq a,\ \rho_1 < \rho_2,\ \rho_1 \leq 1\,), \end{aligned}$$

valid for any number a. □

Next we will introduce the "truncated" counterparts to G_-, I_+, I_-, and J. Let $\tilde{G}_+^{(\epsilon)}$ be defined as in equation (2.9) and set for $b_1 < 0 < b_2$ and $a \geq b_1$,

$$\tilde{G}_-^{(\epsilon)}(a, b_1, b_1; \theta) = \tilde{G}_+^{(\epsilon)}(-a, -b_1, -b_1; -\theta). \tag{2.13}$$

Furthermore, given $b_1 < 0 < b_2$, put

$$\begin{aligned} \tilde{I}_+^{(\epsilon)}(b_1, b_2; \theta) &= \Phi(\theta - b_2) \\ &\quad + \tilde{G}_+^{(\epsilon)}(b_2, b_1, b_2; \theta) - \tilde{G}_-^{(\epsilon)}(b_2, b_1, b_2; \theta), \qquad (2.14) \\ \tilde{I}_-^{(\epsilon)}(b_1, b_2; \theta) &= \tilde{I}_+^{(\epsilon)}(-b_1, -b_2; -\theta), \end{aligned}$$

and, if in addition $b_1 \leq a_1 \leq a_2 \leq b_2$,

$$\begin{aligned} \tilde{J}^{(\epsilon)}(a_1, a_2, b_1, b_2; \theta) =& \Phi(a_2 - \theta) - \Phi(a_1 - \theta) \\ &- \tilde{G}_+^{(\epsilon)}(a_2, b_1, b_2; \theta) + \tilde{G}_+^{(\epsilon)}(a_1, b_1, b_2; \theta) \\ &- \tilde{G}_-^{(\epsilon)}(a_2, b_1, b_2; \theta) + \tilde{G}_-^{(\epsilon)}(a_1, b_1, b_2; \theta). \end{aligned} \tag{2.15}$$

Since $|\tilde{G}_+^{(\epsilon)} - G_+| < \epsilon$, Proposition 2.2 now gives the error estimates

$$\begin{aligned} \big|\, \tilde{I}_+^{(\epsilon)}(b_1, b_2; \theta) - I_+(b_1, b_2; \theta) \,\big| &< 2\epsilon, \\ \big|\, \tilde{I}_-^{(\epsilon)}(b_1, b_2; \theta) - I_-(b_1, b_2; \theta) \,\big| &< 2\epsilon, \end{aligned} \tag{2.16}$$

and

$$\left|\, \tilde{J}^{(\epsilon)}(a_1, a_2, b_1, b_2; \theta) - J(a_1, a_2, b_1, b_2; \theta) \,\right| < 4\epsilon. \qquad (2.17)$$

In the remaining sections it will be shown that the value of a double barrier options with rebate can be expressed as a linear combination of the functions I_+, I_-, and J.

2.2 Pricing Double Barrier Options with Zero Rebate

The objective of this section is to calculate the theoretical value of a double barrier option with zero rebate. Rebate options will be treated in the next section.

Assume from now on that the price of the underlying asset $\{S_t\}_{t\geq 0}$ evolves under the risk-neutral martingale measure Q according to

$$dS_t = S_t\big(\,(r-q)dt + \sigma dW_t^Q\,\big), \quad t \geq 0,$$

where the risk free rate r, the dividend yield q, and the volatility σ are fixed positive constants and W^Q is a standard one-dimensional Q-Brownian motion. Moreover, let the constants K, T, H_1, and H_2 denote strike price, maturity date, lower barrier, and upper barrier, respectively. Finally, set

$$\tau = \inf\{t \geq 0 \,:\, S_t = H_1 \text{ or } S_t = H_2\}$$

and recall that the payoff at time T of a continuous double-barrier knock-out call/put option is given by

$$\max\big(\chi(S_T - K), 0\big)1_{\{\tau > T\}}$$

with $\chi = 1/-1$. The payoff of a continuous double-barrier knock-in call/put is obtained by replacing the event $\{\,\tau > T\,\}$ by its complement.

Firstly, note that it is sufficient to price knock-out options. The value of the corresponding knock-in options then follow by the fact that the sum of two otherwise identical knock-out and knock-in call (put) options is a plain call (put) option. Moreover, it suffices to value the options at time $t = 0$. The price of the contract at time $0 < t \leq T$ then follows by replacing T by $T - t$ and S_0 by S_t, provided that the barriers have not been reached during the time-interval $[0, t]$. In this case the knock-out option is evidently worthless.

Theorem 2.1 below establishes the value of a knock-out double barrier option at time $t = 0$. The proof of Theorem 2.1 is based on a technique sometimes referred to as change of numeraire, see Musiela et al. [95].

Theorem 2.1. *Set*

$$\psi(x) = \frac{\ln(x/S_0)}{\sigma\sqrt{T}}, \quad x > 0,$$

and define $c = \psi(K)$, $d_1 = \psi(H_1)$, *and* $d_2 = \psi(H_2)$. *Suppose moreover*

$$\theta_0 = \frac{(r - q - \sigma^2/2)\sqrt{T}}{\sigma} \quad \text{and} \quad \theta_1 = \theta_0 + \sigma\sqrt{T}.$$

If $K < H_2$, *the theoretical value* v_{koc} *at time* $t = 0$ *of a double-barrier knock-out call option is given by*

$$\begin{aligned} v_{koc} =& S_0 e^{-qT} J(\max(c, d_1), d_2, d_1, d_2; \theta_1) \\ & - K e^{-rT} J(\max(c, d_1), d_2, d_1, d_2; \theta_0)\,, \end{aligned}$$

where J *is defined as in equation* (2.12). *If* $K \geq H_2$, $v_{koc} = 0$.

If $K > H_1$, *the theoretical value* v_{kop} *at time* $t = 0$ *of a double barrier knock-out put option equals*

$$\begin{aligned} v_{kop} =& K e^{-rT} J(d_1, \min(c, d_2), d_1, d_2; \theta_0) \\ & - S_0 e^{-qT} J(d_1, \min(c, d_2), d_1, d_2; \theta_1)\,. \end{aligned}$$

If $K \leq H_1$, $v_{kop} = 0$.

Proof. The theoretical value v at time $t = 0$ of a double barrier knock-out call/put ($\chi = 1/-1$) is given by

$$\begin{aligned} v =& e^{-rT} E^Q \big[\max\big(\chi(S_T - K), 0\big) 1_{\{\tau > T\}} \big] \\ =& \chi S_0 e^{-qT} E^Q \big[e^{-(\sigma^2/2)\,T + \sigma W_T^Q} 1_{\{\chi S_T \geq \chi K,\, \tau > T\}} \big] \\ & - \chi K e^{-rT} E^Q \big[1_{\{\chi S_T \geq \chi K,\, \tau > T\}} \big]. \end{aligned}$$

Define

$$d\tilde{Q} = e^{-(\sigma^2/2)\,T + \sigma W_T^Q} dQ,$$

so that

$$v = \chi S_0 e^{-qT}\tilde{Q}(\,\chi S_T \geq \chi K,\, \tau > T\,) \\ - \chi K e^{-rT} Q(\,\chi S_T \geq \chi K,\, \tau > T\,). \tag{2.18}$$

Set $k = \ln(K/S_0)$, $h_1 = \ln(H_1/S_0)$, and $h_2 = \ln(H_2/S_0)$. Moreover, suppose $\nu \in \mathbb{R}$ and put

$$\theta = \frac{\nu\sqrt{T}}{\sigma}.$$

The scaling property for Brownian motion and the Cameron-Martin theorem give

$$\begin{aligned} &Q\big(\,\chi(\nu T + \sigma W_T^Q) \geq \chi k,\, \min_{0\leq t\leq T}(\nu t + \sigma W_t^Q) > h_1,\, \max_{0\leq t\leq T}(\nu t + \sigma W_t^Q) < h_2\,\big) \\ &\quad = P\big(\,\chi(\theta + W_1) \geq \chi c,\, \min_{0\leq t\leq 1}(\theta t + W_t) > d_1 \,\max_{0\leq t\leq 1}(\theta t + W_t) < d_2\,\big) \\ &\quad = P^{\theta}(\,\chi W_1 \geq \chi c,\, \rho_{d_1} \wedge \rho_{d_2} > 1\,), \end{aligned} \tag{2.19}$$

where c, d_1, and d_2 are defined as in Theorem 2.1.

If $\chi = 1$ ($\chi = -1$), assume, as we may, that $K < H_2$ ($K > H_1$). Let θ_0 and θ_1 be defined as in Theorem 2.1. By substituting ν by $r - q - \sigma^2/2$ in equation (2.19) it follows

$$\begin{aligned} Q(\,\chi S_T \geq \chi K,\, \tau > T\,) &= P^{\theta_0}(\chi W_T \geq \chi c,\, \rho_{d_1} \wedge \rho_{d_2} > 1) \\ &= \begin{cases} J(\max(c, d_1), d_2, d_1, d_2; \theta_0), & \text{if } \chi = 1, \\ J(d_1, \min(c, d_2), d_1, d_2; \theta_0), & \text{if } \chi = -1. \end{cases} \end{aligned}$$

The Cameron-Martin theorem gives that $\{\,W_t^Q - \sigma t\,\}_{0\leq t\leq T}$ is a Brownian motion with respect to $(\tilde{Q}, \{\mathcal{F}_t\}_{0\leq t\leq T})$. Thus, by setting $\nu = r - q + \sigma^2/2$ in equation (2.19),

$$\begin{aligned} \tilde{Q}(\,\chi S_T \geq \chi K,\, \tau > T\,) &= P^{\theta_1}(\chi W_T \geq \chi c,\, \rho_{d_1} \wedge \rho_{d_2} > 1) \\ &= \begin{cases} J(\max(c, d_1), d_2, d_1, d_2; \theta_1), & \text{if } \chi = 1, \\ J(d_1, \min(c, d_2), d_1, d_2; \theta_1), & \text{if } \chi = -1. \end{cases} \end{aligned}$$

Theorem 2.1 now follows by equation (2.18). $\square$

Define the "truncated" price $\tilde{v}_{koc}^{(\epsilon)}$ of a double barrier knock-out call as

$$\begin{aligned}\tilde{v}_{koc}^{(\epsilon)} =& S_0 e^{-qT} \tilde{J}^{(\epsilon)}(\max(c, d_1), d_2, d_1, d_2; \theta_1) \\ &- K e^{-rT} \tilde{J}^{(\epsilon)}(\max(c, d_1), d_2, d_1, d_2; \theta_0),\end{aligned}$$

where c, d_1, d_2, θ_0, and θ_1 are defined as in Theorem 2.1 and where the function $\tilde{J}_2^{(\epsilon)}$ is defined as in equation (2.15). Then, according to equation (2.17),

$$|\, v_{koc} - \tilde{v}_{koc}^{(\epsilon)} \,| < 4(S_0 e^{-qT} + K e^{-rT})\, \epsilon.$$

The same error estimate can also be obtained for the truncated price of an double barrier put option, defined in analogy with the truncated price of the corresponding call option.

2.3 Pricing Rebate Options

The final section of this chapter computes the theoretical value of the rebate options associated with a double barrier option. Firstly, recall that

$$\tau = \inf\{t \geq 0 \,:\, S_t = H_1 \text{ or } S_t = H_2\}.$$

The payoff at the maturity date T of the rebate option to a double barrier knock-in option is given by

$$R\,1_{\{\,\tau > T\,\}},$$

where R is a strictly positive constant. The theoretical value v_{kir} at time $t = 0$ of this option equals therefore

$$\begin{aligned} v_{kir} &= e^{-rT} E^Q \left[R 1_{\{\,\tau > T\,\}} \right] \\ &= R e^{-rT} Q(\,\tau > T\,). \end{aligned}$$

Let θ_0, d_1, and d_2 be defined as in Theorem 2.1. The scaling property for Brownian motion and the Cameron-Martin theorem imply for any $t \geq 0$

$$Q(\,\tau > t\,) = P^{\theta_0}(\rho_{d_1} \wedge \rho_{d_2} > t/T) \tag{2.20}$$

(cf. the proof of Theorem 2.1). Thus,

$$\begin{aligned} v_{kir} &= R e^{-rT} P^{\theta_0}(\,\rho_{d_1} \wedge \rho_{d_2} > 1) \\ &= R e^{-rT} J(d_1, d_2, d_1, d_2; \theta_0). \end{aligned}$$

We now turn our attention to the rebate option associated with a double barrier knock-out option. The payoff of this rebate option equal

$$R1_{\{\tau \le T\}}.$$

The payment will occur at the time the barrier is hit, provided then, of course, that this event occurs before time T. Since the payoff date is random, this contract does not fit in the theory described in the previous chapter. However, from a economical point of view the rebate option is equivalent with a claim that pays at time T the amount

$$e^{r(T-\tau)}R1_{\{\tau \le T\}}.$$

That is, if any the barriers are breached before time T then the money paid out by the rebate option is invested in bonds. Thus, the theoretical value v_{kor} at time $t = 0$ of a rebate option associated with a double barrier knock-out option equals

$$v_{kor} = e^{-rT}E^Q\big[e^{r(T-\tau)}R\,1_{\{\tau \le T\}}\big] = E^Q\big[e^{-r\tau}R\,1_{\{\tau \le T\}}\big].$$

To compute this value, set $\rho_1 = \rho_{d_1}$, $\rho_2 = \rho_{d_2}$, and

$$\bar{\rho}_{1,2} = \rho_1 \wedge \rho_2 \wedge 1.$$

In view of equation (2.20) and since $P(\tau = T) = 0$ we find

$$v_{kor} = E^{\theta_0}\big[Re^{-rT\bar{\rho}_{1,2}}\,1_{\{\bar{\rho}_{1,2}<1\}}\big].$$

It is obvious that

$$\begin{aligned} E^{\theta_0}\big[Re^{-rT\bar{\rho}_{1,2}}\,1_{\{\bar{\rho}_{1,2}<1\}}\big] =& E^{\theta_0}\big[Re^{-rT\bar{\rho}_{1,2}}\,1_{\{\rho_1<\rho_2,\,\rho_1<1\}}\big] \\ &+ E^{\theta_0}\big[Re^{-rT\bar{\rho}_{1,2}}\,1_{\{\rho_2<\rho_1,\,\rho_2<1\}}\big]. \end{aligned}$$

Suppose $A = \{\rho_1 < \rho_2,\, \rho_1 < 1\}$. By the definition of P^{θ_0},

$$E^{\theta_0}\big[\exp(-r\,T\bar{\rho}_{1,2})\,1_A\big] = E\big[\exp(-\frac{1}{2}\theta_0^2 + \theta_0 W_1 - r\,T\bar{\rho}_{1,2}\,)\,1_A\,\big],$$

where $E = E^0$.

Recall that if $\{Z_t\}_{t\ge 0}$ is a $(P, \{\mathcal{F}\}_{t\ge 0})$-martingale, ρ is a stopping time such that $\rho \le 1$, and X is a bounded $\mathcal{F}_\rho$-measurable random variable, then the optional sampling theorem implies

$$E\big[XZ_1\big] = E\big[XZ_\rho\big]. \tag{2.21}$$

Since the random variable $\exp(-r\,T\bar{\rho}_{1,2})\,1_A$ is $\mathcal{F}_\rho$-measurable and bounded and $\bar{\rho}_{1,2} \leq 1$, equation (2.21) yields

$$E\big[\exp(-\frac{1}{2}\theta_0^2 + \theta_0 W_1 - r\,T\bar{\rho}_{1,2}\,)\,1_A\,\big]$$
$$= E\big[\exp(-\,\frac{1}{2}\theta_0^2\bar{\rho}_{1,2} + \theta_0 W_{\bar{\rho}_{1,2}} - rT\bar{\rho}_{1,2})\,1_A\,\big]$$
$$= E\big[\exp\big((\theta_0 - \theta_2)W_{\bar{\rho}_{1,2}} - \frac{1}{2}\theta_2^2\bar{\rho}_{1,2} + \theta_2 W_{\bar{\rho}_{1,2}}\big)\,1_A\,\big],$$

where $\theta_2 = \sqrt{\theta_0^2 + 2rT}$. Note that if $\omega \in A$ then $W_{\bar{\rho}_{1,2(\omega)}}(\omega) = d_1$ and, accordingly from this,

$$E\big[\exp\big((\theta_0 - \theta_2)W_{\bar{\rho}_{1,2}} - \frac{1}{2}\theta_2^2\bar{\rho}_{1,2} + \theta_2 W_{\bar{\rho}_{1,2}}\big)\,1_A\,\big]$$
$$= \exp\big((\theta_0 - \theta_2)d_1\,\big)\,E\big[\exp(-\frac{1}{2}\theta_2^2\bar{\rho}_{1,2} + \theta_2 W_{\bar{\rho}_{1,2}}\big)\,1_A\,\big].$$

Moreover, according to equation (2.21),

$$E\big[\exp(-\frac{1}{2}\theta_2^2\bar{\rho}_{1,2} + \theta_2 W_{\bar{\rho}_{1,2}}\big)\,1_A\,\big] = P^{\theta_2}(A)$$
$$= I_-(d_1, d_2; \theta_2).$$

To sum up,

$$E^{\theta_0}\big[\exp(-rT\bar{\rho}_{1,2})\,1_{\{\rho_1<\rho_2,\,\rho_1<1\}}\big] = \exp\big((\theta_0 - \theta_2)d_1\big)\,I_-(d_1, d_2; \theta_2).$$

In a similar way it can be shown that

$$E^{\theta_0}\big[\exp(-rT\bar{\rho}_{1,2})\,1_{\{\rho_2<\rho_1,\,\rho_2<1\}}\big] = \exp\big((\theta_0 - \theta_2)d_2\big)\,I_+(d_1, d_2; \theta_2).$$

We have arrived at the following theorem.

Theorem 2.2. *Let d_1, d_2, and θ_0 be defined as in Theorem 2.1 and suppose that J is defined as in equation* (2.12)*. The theoretical value v_{kir} at time $t = 0$ of a rebate option to a double barrier knock-in option is given by*

$$v_{kir} = Re^{-rT}\,J(d_1, d_2, d_1, d_2; \theta_0).$$

Let I_+ and I_- be defined as in equation (2.11). *The theoretical value v_{kor} at time $t = 0$ of a rebate option to a double-barrier knock-out option equals*

$$v_{kor} = Re^{(\theta_0 - \theta_2)\, d_1} I_-(d_1, d_2; \theta_2) \\ + Re^{(\theta_0 - \theta_2)\, d_2} I_+(d_1, d_2; \theta_2),$$

where $\theta_2 = \sqrt{\theta_0^2 + 2rT}$.

The price of the contracts at time $0 < t \leq T$ follows by replacing T by $T - t$ and S_0 by S_t, provided that the barriers have not been reached during the time-interval $[0, t]$. In this case, the values of the rebate options are obvious.

Now, put

$$\tilde{v}_{kir}^{(\epsilon)} = Re^{-rT}\, \tilde{J}^{(\epsilon)}(d_1, d_2, d_1, d_2; \theta_0)$$

and

$$\tilde{v}_{kor}^{(\epsilon)} = Re^{(\theta_0 - \theta_2)\, d_1}\, \tilde{I}_-^{(\epsilon)}(d_1, d_2; \theta_2) \\ + Re^{(\theta_0 - \theta_2)\, d_2}\, \tilde{I}_+^{(\epsilon)}(d_1, d_2; \theta_2),$$

where d_1, d_2, θ_0, and θ_2 are defined as in Theorem 2.2 and where $\tilde{I}_+^{(\epsilon)}$, $\tilde{I}_-^{(\epsilon)}$, and $\tilde{J}^{(\epsilon)}$ are given by the equations (2.14) and (2.15). We now have, according to the equations (2.16) and (2.17),

$$|v_{kir} - \tilde{v}_{kir}^{(\epsilon)}| < 4Re^{-rT}\epsilon$$

and

$$|v_{kor} - \tilde{v}_{kor}^{(\epsilon)}| < 2R\left(e^{(\theta_0 - \theta_2)\, d_1} + e^{(\theta_0 - \theta_2)\, d_2}\right)\epsilon,$$

which give the promised error estimates.

Chapter 3

Extension of the Corrected Barrier Approximation by Broadie, Glasserman, and Kou

In contrast to continuous barrier options the price of a discrete barrier option does not in general possess a closed form price formula. The price can be expressed in terms of the multivariate normal distribution. Here the dimension of the relevant multivariate normal distribution is equal to the number of price fixing dates, which, in most cases, is too large for numerical evaluation.

Other methods to price discrete barrier options have been discussed in the literature. Procedures based on so called lattice methods have been investigated by, among many others, Ahn, Figlewski, and Gao [1] and Broadie, Glasserman, and Kou [29]. These methods will be considered in greater detail in Chapters 4 and 5. In Boyle, Broadie, and Glasserman [25] Monte Carlo methods were employed to price discrete barrier options. Another technique which has given remarkably good results was first proposed by Chuang, [34], and, independently, by Broadie, Glasserman, and Kou [28]. They suggested that one should use a result from sequential analysis and queue theory, namely "Siegmund's corrected heavy traffic approximation".

Chuang *only* suggested the possibility of using Siegmund's result to price discrete barrier options. However, Chuang never pursued the idea. Broadie et al. derived pricing formulas for some discrete (single) barrier options, but not all.

The purpose of this chapter is to continue the work initiated by Broadie et al. and, by using the Siegmund heavy traffic approximation, estimate the price of some discrete barrier options omitted in Broadie et al.. The work presented in this chapter has similarities with an independent paper by Kou, see [75]. His results will be discussed in the next section.

This chapter is structured as follows. Section 3.1 presents the results by Broadie et al. as well as our main result about single barrier options. The latter result will be proved in Sections 3.2 and 3.3. Moreover, Section 3.3 discusses Siegmund's corrected heavy traffic approximation. Section 3.4 presents some numerical examples. Finally, Section 3.5 treats the valuation of discrete double barrier options.

3.1 Barrier Corrections for Single Barrier Options

We assume throughout this chapter that the price of the underlying asset $\{S_t\}_{t\geq 0}$ evolves under the risk-neutral martingale measure Q according to

$$dS_t = S_t\big((r-q)dt + \sigma dW_t^Q \big), \quad t \geq 0,$$

where $\{W_t^Q\}_{t\geq 0}$ is a standard one-dimensional Brownian motion with respect to Q and where the risk free rate r, the dividend yield q, the volatility σ, and the initial price S_0 are positive constants. Moreover, let the constants K and T denote the strike price and maturity date, respectively. Suppose that the monitoring dates M are equally spaced in time, i.e.

$$M = \{\Delta t,\, 2\Delta t, \dots ,\, m\Delta t\}, \quad \Delta t = T/m,$$

where m is the number of monitoring times. For single barrier options the barrier level will be denoted H, where H is a positive number, and for double barrier options the barrier levels will be written H_1 and H_2, where H_1, H_2 are positive numbers with $H_1 < H_2$.

The most naive approach to approximate the value of a discrete barrier option would be to ignore the fact that the barrier is discrete and price the option as a continuous barrier option with the same barrier. For continuous barrier options there are known formulas, see e.g. Rich [102] and Chapter 2 in this thesis. However, numerical examples show that this method can lead to substantial mispricings, see e.g. Broadie et al. [28], even in the case of daily monitoring. In a paper by Broadie et al. [28] it was shown that the simple approximation discussed above can be improved for some single barrier options just by shifting the barrier. Theorem 3.1 below is taken from [28]. For the definitions of the various barrier options, see Section 1.7.1.

Theorem 3.1. *Let $v^{(m)}(H)$ be the price of a discretely monitored up-and-out/in put or down-and-out/in call at time $t = 0$ with barrier H and monitoring dates M. Let $v(H)$ be the price at time $t = 0$ of the corresponding continuously monitored barrier option. Then, as $m \to \infty$,*

$$v^{(m)}(H) = v(He^{\pm\beta\sigma\sqrt{T/m}}) + o(\frac{1}{\sqrt{m}})$$

where $+$ applies if $H > S_0$,$-$ applies if $H < S_0$, and $\beta = -\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$, with ζ the Riemann zeta function.

Numerical results in the same paper indicate that the shift of the barrier gives surprisingly good approximations for moderate to large values of m if the initial asset price is not too close to the barrier.

The proof by Broadie et al. of Theorem 3.1 is based on certain results by Siegmund and, to some extend, Yuh, see [111] and [113]. This chapter will apply the same results to find estimates of the price of discrete down-and-out/in put, up-and-out/in call, and double barrier options. Note that none of these contracts are included in Theorem 3.1. This section will consider single barrier options. We will come back to double barrier options in Section 3.5.

Before stating the main result in this section we will introduce some definitions. Suppose that Φ denotes the standard normal distribution function and let the function F_+ be defined by

$$F_+(a, b; \theta) = \Phi(a - \theta) - e^{2b\theta}\Phi(a - 2b - \theta)$$

where $a \leq b$, $b > 0$, and $\theta \in \mathbb{R}$. Moreover, if $a \geq b$, $b < 0$, and $\theta \in \mathbb{R}$ then

$$F_-(a, b; \theta) = F_+(-a, -b; -\theta).$$

Theorem 3.2. *Suppose that*

$$\psi(x) = \frac{\ln(x/S_0)}{\sigma\sqrt{T}}, \quad x > 0,$$

and let $c = \psi(K)$ and $d = \psi(H)$. Set moreover

$$\theta_0 = \frac{(r - q - \sigma^2/2)\sqrt{T}}{\sigma} \quad \textit{and} \quad \theta_1 = \theta_0 + \sigma\sqrt{T}.$$

Let $v_{uoc}^{(m)}$ denote the theoretical value at time $t = 0$ of a discrete up-and-out call with monitoring dates M. If $K < H$ then, as $m \to \infty$,

$$\begin{aligned} v_{uoc}^{(m)} =& S_0 e^{-qT}\big(F_+(d, d+\beta/\sqrt{m};\theta_1) - F_+(c, d+\beta/\sqrt{m};\theta_1)\big) \\ &- Ke^{-rT}\big(F_+(d, d+\beta/\sqrt{m};\theta_0) - F_+(c, d+\beta/\sqrt{m};\theta_0)\big) \\ &+ o(\frac{1}{\sqrt{m}}), \end{aligned}$$

where β is defined as in Theorem 3.1. If $K \geq H$, $v_{uoc}^{(m)} = 0$.

The theoretical value $v_{dop}^{(m)}$ at time $t = 0$ of a discrete down-and-out put with monitoring dates M equals

$$\begin{aligned} v_{dop}^{(m)} =& Ke^{-rT}\big(F_-(d, d-\beta/\sqrt{m};\theta_0) - F_-(c, d-\beta/\sqrt{m};\theta_0)\big) \\ &- S_0 e^{-qT}\big(F_-(d, d-\beta/\sqrt{m};\theta_1) - F_-(c, d-\beta/\sqrt{m};\theta_1)\big) \\ &+ o(\frac{1}{\sqrt{m}}), \quad m \to \infty, \end{aligned}$$

provided $K > H$. If $K \leq H$, $v_{dop}^{(m)} = 0$.

Remarkably enough, one will *not* get the above approximations by simply shifting the barrier as in Theorem 3.1. The pricing formulas for the corresponding continuous barrier options are, of course, obtained by sending $m \to \infty$ in Theorem 3.2.

The value of the corresponding knock-in options now follow by the fact that the sum of two otherwise identical knock-in and knock-out call (put) options is a plain call (put) option.

Before we proceed and prove Theorem 3.2 we will make some comments about the results in Kou [75]. In that paper it is shown that Theorem 3.1 actually can be extended to cover down-and-out/in puts and up-and-out/in calls as well. However, numerical examples that will be presented in Section 3.4 indicate that the approximation formulas in Theorem 3.2 in most cases yield better results.

3.2 An Exact Expression of the Price

Suppose P is a probability measure, $\{W_t\}_{t\geq 0}$ is a standard one-dimensional Brownian motion with respect to P, and

$$dP^\theta = e^{-\frac{1}{2}\theta^2 + \theta W_1} dP,$$

where $\theta \in \mathbb{R}$. Moreover, put for each $b \neq 0$,

$$\rho_b^{(m)} = \inf\{\, t \in A_m \,:\, \kappa W_t \geq \kappa b\,\},$$

where $A_m = \frac{1}{m}\mathbb{N}_+$ and $\kappa = \text{sign}(b)$. Finally, set

$$\rho_{b_1,b_2}^{(m)} = \rho_{b_1}^{(m)} \wedge \rho_{b_1}^{(m)}, \quad b_1 < 0 < b_2.$$

Note that if $\rho_b = \lim_{m\to\infty} \rho_b^{(m)}$, then Lemma 2.3 implies that the functions F_+ and F_-, defined as in the previous section, may be identified as

$$F_+(a, b; \theta) = P^\theta(\, W_1 \leq a,\, \rho_b > 1), \quad b > 0,\ a \leq b,$$

and

$$F_-(a, b; \theta) = P^\theta(\, W_1 \geq a,\, \rho_b > 1), \quad b < 0,\ a \geq b.$$

The next lemma will be useful to price both discrete single and double barrier options.

Lemma 3.1. *Let the function ψ and the constants c, θ_1, and θ_2 be defined as in Theorem 3.2. Suppose that H_1 and H_2 are real numbers such that $0 < H_1 \leq S_0 \leq H_2$ and set $d_1 = \psi(H_1)$, $d_2 = \psi(H_2)$, and*

$$\tau = \inf\{t \in M \,:\, S_t \leq H_1 \text{ or } S_t \geq H_2\}.$$

If $\chi \in \{-1, 1\}$ then

$$\begin{aligned} e^{-rT} E^Q \big[\max\big(\chi(S_T - K), 0\big) 1_{\{\tau > T\}} \big] \\ =& \chi S_0 e^{-qT} P^{\theta_1}\big(\, \chi W_1 \,\geq\, \chi c,\ \rho_{d_1,d_2}^{(m)} > 1\big) \\ &- \chi K e^{-rT} P^{\theta_0}\big(\, \chi W_1 \,\geq\, \chi c,\ \rho_{d_1,d_2}^{(m)} > 1\big). \end{aligned}$$

Proof. Note that

$$\begin{aligned} e^{-rT} E^Q \big[\max\big(\chi(S_T - K), 0\big) 1_{\{\tau > T\}} \big] \\ =& \chi S_0 e^{-qT} E^Q \big[e^{-(\sigma^2/2)\, T + \sigma W_T^Q} 1_{\{\, \chi S_T \geq \chi K,\, \tau > T\,\}} \big] \\ &- \chi K e^{-rT} E^Q \big[1_{\{\, \chi S_T \geq \chi K,\, \tau > T\,\}} \big]. \end{aligned}$$

Let

$$d\tilde{Q} = e^{-(\sigma^2/2)\, T + \sigma W_T^Q} dQ,$$

so that

$$
\begin{aligned}
e^{-rT}E^{Q}\big[\max\big(\chi(S_T-K),0\big)1_{\{\tau>T\}}\big]& \\
=&\chi S_0 e^{-qT}\tilde{Q}(\chi S_T\geq \chi K,\,\tau>T) \\
&-\chi K e^{-rT}Q(\chi S_T\geq\chi K,\,\tau>T).
\end{aligned}
$$

Set $A_m=\frac{1}{m}\mathbb{N}$, $k=\ln(K/S_0)$, $h_1=\ln(H_1/S_0)$, and $h_2=\ln(H_2/S_0)$. Moreover, suppose $\nu\in\mathbb{R}$ and put

$$\theta=\frac{\nu\sqrt{T}}{\sigma}.$$

The scaling property for Brownian motion and the Cameron-Martin theorem give

$$
\begin{aligned}
&Q\big(\chi(\nu T+\sigma W_T^Q)\geq\chi k,\,\min_{t\in M}(\nu t+\sigma W_t^Q)>h_1,\,\max_{t\in M}(\nu t+\sigma W_t^Q)<h_2\big)\\
&\quad=P\big(\chi(\theta+W_1)\geq\chi c,\,\min_{t\in A_m}(\theta t+W_t)>d_1\,\max_{t\in A_m}(\theta t+W_t)<d_2\big)\\
&\quad=P^{\theta}(\chi W_1\geq\chi c,\,\rho_{d_1,d_2}^{(m)}>1),
\end{aligned}
\tag{3.1}
$$

where c, d_1, and d_2 are defined as in Lemma 3.1.

Suppose θ_0 and θ_1 are defined as in Lemma 3.1. Substitute ν by $r-q-\sigma^2/2$ in equation (3.1) and conclude

$$Q\big(\chi S_T\geq\chi K,\,\tau>T\big)=P^{\theta_0}\big(\chi W_T\geq\chi c,\,\rho_{d_1,d_2}^{(m)}>1\big).$$

The Cameron-Martin theorem gives that $\{W_t^Q-\sigma t\}_{0\leq t\leq T}$ is a Brownian motion with respect to $(\tilde{Q},\{\mathcal{F}_t\}_{0\leq t\leq T})$. Thus, by setting $\nu=r-q+\sigma^2/2$ in equation (3.1),

$$\tilde{Q}\big(\chi S_T\geq\chi K,\,\tau>T\big)=P^{\theta_1}\big(\chi W_T\geq\chi c,\,\rho_{d_1,d_2}^{(m)}>1\big)$$

and the proof is complete. □

Assume $b>0$, $a\leq b$, $\theta\in\mathbb{R}$, and consider

$$F_+^{(m)}(a,b;\theta)=P^{\theta}(W_1\leq a,\,\rho_b^{(m)}>1).$$

The next objective is to show that the price of a discrete up-and-out call and a discrete down-and-out put may be expressed in terms of the function $F_+^{(m)}$.

Let us to begin with consider discrete up-and-out call options. Suppose that $K < H$ and let c, d, θ_0, and θ_1 be defined as in Theorem 3.2. By letting $H_1 \to 0$ and setting $H_2 = H$ and $\chi = 1$ in Lemma 3.1 we get

$$\begin{aligned} v_{uoc}^{(m)} =& S_0 e^{-qT} P^{\theta_1}\big(W_1 \geq c, \rho_d^{(m)} > 1\big) \\ &- K e^{-rT} P^{\theta_0}\big(W_1 \geq c, \rho_d^{(m)} > 1\big). \end{aligned}$$

The definition of $F_+^{(m)}$ implies

$$P^{\theta}\big(W_1 \geq c, \rho_d^{(m)} > 1\big) = F_+^{(m)}(d, d; \theta) - F_+^{(m)}(c, d; \theta),$$

for any $\theta \in \mathbb{R}$, which gives the desired representation;

$$\begin{aligned} v_{uoc}^{(m)} =& S_0 e^{-qT}\big(F_+^{(m)}(d, d; \theta_1) - F_+^{(m)}(c, d; \theta_1)\big) \\ &- K e^{-rT}\big(F_+^{(m)}(d, d; \theta_0) - F_+^{(m)}(c, d; \theta_0)\big). \end{aligned}$$

A similar argument in combination with the symmetry for Brownian motion give that the value of a discrete down-and-out put equals

$$\begin{aligned} v_{dop}^{(m)} =& K e^{-rT}\big(F_+^{(m)}(-d, -d; -\theta_0) - F_+^{(m)}(-c, -d; -\theta_0)\big) \\ &- S_0 e^{-qT}\big(F_+^{(m)}(-d, -d; -\theta_1) - F_+^{(m)}(-c, -d; -\theta_1)\big), \end{aligned}$$

provided $K > H$.

The next section discusses an approximation of the function $F_+^{(m)}$.

3.3 Siegmund's Corrected Heavy Traffic Approximation

The next result is often referred to as *Siegmund's corrected heavy traffic approximation.*

Theorem 3.3. *Suppose $b > 0$, $a \leq b$, $\theta \in \mathbb{R}$, and assume that β is defined as in Theorem 3.1. If*

$$\varrho_b^{(m)} = \inf\big\{n \in \mathbb{N} : \frac{\theta n}{\sqrt{m}} + W_n > b\sqrt{m}\,\big\},$$

then, as $m \to \infty$*,*

$$P\big(\,\theta\sqrt{m}{+}W_m < a\sqrt{m},\; \varrho_b^{(m)} \le m\,\big)$$
$$= e^{2\theta(b+\beta/\sqrt{m})}\,\Phi\big(a - 2(b+\beta/\sqrt{m}) - \theta\big) + o(\frac{1}{\sqrt{m}}).$$

For a proof of Theorem 3.3, see Siegmund [112] pp. 220-224.

First some comments about the constant β. If, for simplicity, $\varrho^{(m)} = \varrho_b^{(m)}$ and $W_n^{\theta,m} = \theta n/\sqrt{m} + W_n$, $n \in \mathbb{N}$, then it can be shown that

$$\beta = \lim_{m\to\infty} E\big[W^{\theta,m}_{\varrho^{(m)}} - b\sqrt{m}\,;\; \varrho^{(m)} < \infty\big],$$

see e.g. Siegmund [112], p. 215. Thus, the constant β may be viewed as an approximation to the average of the amount by which the random walk $\{\theta n/\sqrt{m} + W_n\}_{n\in\mathbb{N}}$ exceeds the boundary $b\sqrt{m}$ the first time the random walk is above the boundary. For further details on so called overshoot random variables the reader may consult Lotov [90] or Siegmund [112].

Now, note that the Cameron-Martin theorem and the scaling property yield

$$P(\,\theta\sqrt{m} + W_m < a\sqrt{m},\; \varrho_b^{(m)} \le m\,) = P^\theta\big(\,W_1 < a,\; \rho_b^{(m)} \le 1\,\big).$$

Therefore, according to Theorem 3.3,

$$\begin{aligned} F_+^{(m)}(a,b;\theta) =& P^\theta\big(\,W_1 < a\,\big) \\ & - P^\theta\big(\,W_1 < a,\; \rho_b^{(m)} \le 1\,\big) \\ =& \Phi(a-\theta) \\ & - e^{2\theta(b+\beta/\sqrt{m})}\Phi(a - 2(b+\beta/\sqrt{m}) - \theta) + o(\frac{1}{\sqrt{m}}) \end{aligned}$$

as $m \to \infty$. By comparing this expression of $F_+^{(m)}$ with F_+ in Section 3.1 we find that

$$F_+^{(m)}(a,b;\theta) = F_+(a, b + \beta/\sqrt{m}; \theta) + o(\frac{1}{\sqrt{m}}), \quad \text{as } m \to \infty. \tag{3.2}$$

Thus, to calculate the probability with respect to P^θ of the event

$$\{\,W_1 \le a,\; \rho_b^{(m)} > 1\,\}$$

using the formula for a continuous barrier, one should first lift the barrier $\beta/\sqrt{m}$ units upwards. This compensates for the fact that when the random walk $\{\theta t + W_t,\ t = 1/m, 2/m, \dots\}$ breaches the barrier, it exceeds it approximately with $\beta/\sqrt{m}$ units.

Equation (3.2) in conjunction with the results in the previous section complete the proof of Theorem 3.2.

3.4 Numerical Examples

Let us first consider discrete up-and-out call options. Table 3.1 compares the option values obtained by different methods. The first column in the table displays the level of the barrier. The values of the other option parameters are in the caption. The second column contains the value of the corresponding continuous barrier option. In the third column we have used the formula in Theorem 3.2, with $o(1/\sqrt{m})$ set to zero.

The values in the fourth column are established by a method proposed in Broadie et al. [28] and Kou [75], that is, lifting the barrier upwards by a factor $\exp(\beta\sigma\sqrt{T/m})$ and then use the formula for the value of a continuous up-and-out call.

The prices in the fifth column are determined by a so called trinomial method presented in Broadie et al. [29] (the errors of the trinomial prices are according to the same article approximately ± 0.001). Finally, in the last three columns we have the relative error measured in percentage for the different approximations.

Note the surprisingly great differences in price between the discrete and the corresponding continuous barrier option. So it is worth to emphasise that one should not neglect the fact that some barrier options are discretely and not continuously monitored. The table also shows that the approximation derived in this chapter yields good results, and that the accuracy of the result is dependent of how close the barrier is to the initial price of the underlying asset. One may as well note that our approximation performs better than the approximation in Broadie et al. [28] and Kou [75].

In Table 3.2 we have varied the number of price fixing dates as well. As is to be expected, the estimate developed in this chapter degrade as the number of monitoring times decreases. In the extreme case with the barrier very close to the initial asset price, the method even performs worse than the approximation in Broadie et al. [28] and Kou [75]. However, one may remark that in the extreme case none of the methods work especially well.

In the final example, presented in Table 3.3, we have examined how the

Table 3.1: Price results for up-and-out call options, varying H. The option parameters are $S_0 = 110$, $K = 100$, $\sigma = 0.3$, $r = 0.10$, $q = 0.0$, $T = 0.2$, and $m = 50$. If one assumes that there are 250 trading days per year, then $m = 50$ corresponds to daily monitoring.

H	Continuous Barrier (1)	Hö (2)	BGK and K (3)	Trinomial Method	Relative error (in percent) (1)	(2)	(3)
155	12.775	12.891	12.905	12.894	0.9	0.0	0.1
150	12.240	12.426	12.448	12.431	1.5	0.0	0.1
145	11.395	11.676	11.707	11.684	2.5	0.1	0.2
140	10.144	10.541	10.581	10.551	3.9	0.1	0.3
135	8.433	8.947	8.994	8.959	5.9	0.1	0.4
130	6.314	6.909	6.959	6.922	8.8	0.2	0.5
125	4.012	4.605	4.649	4.616	13.0	0.2	0.7
120	1.938	2.410	2.442	2.418	19.8	0.3	1.0
115	0.545	0.803	0.819	0.807	32.5	0.5	1.5
112	0.127	0.257	0.264	0.260	51.1	1.2	1.6

other parameters influence the accuracy of the approximation.

It is of course not possible to draw any certain conclusions from just numerical examples. But the results presented here indicate that the approximation gives good results for small values of T/m and if the barrier is not too close to the initial price of the underlying asset.

3.5 Barrier Corrections for Double Barrier Options

The purpose of this section is to determine approximations for the value of discrete double barrier options. Firstly, recall that

$$\rho^{(m)}_{b_1,b_2} = \inf\{\, t \in A_m \,:\, W_t \geq b_2 \text{ or } W_t \leq b_1 \,\}, \quad b_1 < 0 < b_2,$$

where $A_m = \frac{1}{m}\mathbb{N}_+$. Now, define

$$J^{(m)}(a_1, a_2, b_1, b_2; \theta) = P^\theta\big(\, a_1 < W_1 \leq a_2,\ \rho^{(m)}_{b_1,b_2} > 1\,\big)$$

for $b_1 < 0 < b_2$ and $b_1 \leq a_1 \leq a_2 \leq b_2$. Lemma 3.1 gives that if $K < H_2$ then the theoretical value $v^{(m)}_{koc}$ at time $t = 0$ of a discrete double-barrier

Table 3.2: Price results for up-and-out call options, varying H and m. The option parameters are $S_0 = 110,\ K = 100,\ \sigma = 0.3,\ r = 0.10,\ q = 0.0$, and $T = 0.2$.

m	H	Continuous Barrier (1)	Hö (2)	BGK and K (3)	Trinomial Method	Relative error (in percent) (1)	(2)	(3)
	130	6.314	7.124	7.221	7.148	11.7	0.3	1.0
	125	4.012	4.829	4.918	4.851	17.3	0.5	1.4
25	120	1.938	2.600	2.669	2.616	25.9	0.6	1.9
	115	0.545	0.916	0.950	0.925	41.1	0.9	2.8
	112	0.127	0.320	0.336	0.329	61.4	3.0	2.0
	130	6.314	7.837	8.286	7.934	20.4	1.2	4.4
	125	4.012	5.622	6.062	5.721	29.9	1.7	5.9
5	120	1.938	3.326	3.683	3.409	43.1	2.5	8.0
	115	0.545	1.404	1.624	1.481	63.2	5.2	9.6
	112	0.127	0.622	0.751	0.708	82.1	12.3	6.0

Table 3.3: Price results for up-and-out call options, varying K, σ, and T. The option parameters are $S_0 = 110$, $r = 0.1$, and $q = 0.0$ for all panels. Panel A has $K = 100$, $\sigma = 0.3$, $T = 1$, and $m = 250$ (daily monitoring). Panel B has $K = 100$, $\sigma = 0.6$, $T = 0.2$, and $m = 50$ (daily monitoring). Panel C has $K = 90$, $\sigma = 0.6$, $T = 0.2$, and $m = 50$ (daily monitoring)

Panel	H	Continuous Barrier (1)	Hö (2)	BGK and K (3)	Trinomial Method	Relative error (in percent) (1)	(2)	(3)
	155	6.798	7.270	7.290	7.274	6.6	0.1	0.2
A	140	2.916	3.251	3.265	3.254	10.4	0.1	0.3
	125	0.566	0.693	0.699	0.695	18.6	0.2	0.6
	140	3.766	4.516	4.578	4.531	16.9	0.3	1.0
B	130	1.576	2.086	2.130	2.097	24.9	0.5	1.6
	120	0.331	0.541	0.561	0.546	39.4	0.9	2.8
	140	7.171	8.277	8.354	8.296	13.6	0.2	0.7
C	130	3.653	4.550	4.608	4.565	20.0	0.3	0.9
	120	1.110	1.629	1.659	1.637	32.2	0.5	1.4

knock-out call with monitoring dates M equals

$$\begin{aligned} v_{koc}^{(m)} =& S_0 e^{-qT} J^{(m)}(\max(c, d_1), d_2, d_1, d_2; \theta_1) \\ & - K e^{-rT} J^{(m)}(\max(c, d_1), d_2, d_1, d_2; \theta_0), \end{aligned}$$

where the constants $c,\ d_1,\ d_2,\ \theta_0$, and θ_1 are defined as in Theorem 3.2 and Lemma 3.1. Similary, if $K > H_1$ then the theoretical value v_{kop} of a discrete double barrier knock-out put at time $t = 0$ is given by

$$\begin{aligned} v_{kop} =& K e^{-rT} J^{(m)}(d_1, \min(c, d_2), d_1, d_2; \theta_0) \\ & - S_0 e^{-qT} J^{(m)}(d_1, \min(c, d_2), d_1, d_2; \theta_1) \,. \end{aligned}$$

The price of a double barrier knock-in option can be obtained using a similar argument as in Section 3.1. Thus, to estimate the price of a discrete double barrier option it suffices to find an approximation of the function $J^{(m)}$.

Let J be the continuous analogue to $J^{(m)}$, that is, for given $b_1 < 0 < b_2$ and $b_1 \leq a_1 \leq a_2 \leq b_2$ set

$$J(a_1, a_2, b_1, b_2; \theta) = P^{\theta}\big(\, a_1 < W_1 \leq a_2,\ \rho_{b_1, b_2} > 1\big)$$

with

$$\rho_{b_1, b_2} = \lim_{m \to \infty} \rho_{b_1, b_2}^{(m)}.$$

The function J can be computed with aid of Propositions 2.1 and 2.2. Siegmund suggests, see [111] p. 716, that one approximate the function $J^{(m)}$ by computing the function J with the lower barrier replaced $\beta/\sqrt{m}$ units downwards and the upper barrier moved $\beta/\sqrt{m}$ units upwards (cf. equation (3.2)). In other words, Siegmund suggests the following approximation

$$J^{(m)}(a_1, a_2, b_1, b_2; \theta) \approx J(a_1, a_2, b_1 - \beta/\sqrt{m}, b_2 + \beta/\sqrt{m}; \theta).$$

However, in this case there are no estimates of the approximation error but numerical examples presented in Siegmund [111] indicate that the estimation yields good results.

To see how the approximation performs, we will now present some numerical examples. Tables 3.4 and 3.5 show the price of a discrete double barrier knock-out call. The prices in the column called "Trinomial method" are determined by a trinomial method that will be discussed in the next chapter. The error for the prices in "Trinomial method" are approximately ± 0.001.

Table 3.4: Price results for double barrier knock-out calls, varying H_1 and H_2. The option parameters are $S_0 = 100,\ K = 100,\ \sigma = 0.3,\ r = 0.10,\ q = 0.0,\ T = 0.2$, and $m = 50$ (daily monitoring).

		Continuous Barrier	Hö	Trinomial Method	Relative error (in percent)	
H_1	H_2	(1)	(2)	(3)	(1)	(2)
70	130	4.5651	4.7784	4.7842	4.6	0.1
75	125	3.5614	3.8375	3.8446	7.3	0.2
80	120	2.3499	2.6524	2.6601	11.7	0.3
85	115	1.1408	1.4055	1.4120	19.2	0.5
90	110	0.2284	0.3791	0.3826	40.3	0.9
75	110	0.3423	0.4799	0.4841	29.3	0.8
90	125	3.2292	3.6074	3.6143	10.7	0.2

Table 3.5: Price results for double barrier knock-out calls, varying H_1, H_2, and m. The option parameters are $S_0 = 100,\ K = 100,\ \sigma = 0.3,\ r = 0.10,\ q = 0.0$, and $T = 0.2$.

			Continuous Barrier	Hö	Trinomial Method	Relative error (in percent)	
m	H_1	H_2	(1)	(2)	(3)	(1)	(2)
	80	120	2.3499	2.7606	2.7752	15.3	0.5
	85	115	1.1408	1.5052	1.5180	24.9	0.9
25	90	110	0.2284	0.4441	0.4514	49.3	1.6
	75	110	0.3423	0.5445	0.5362	37.1	1.5
	90	125	3.2292	3.7363	3.7491	13.9	0.3
	80	120	2.3499	3.1157	3.1726	25.9	1.8
	85	115	1.1408	1.8563	1.9115	40.3	2.9
5	90	110	0.2284	0.7035	0.7401	69.1	4.9
	75	110	0.3423	0.7570	0.7962	57.0	4.9
	90	125	3.2292	4.1294	4.1724	22.6	1.0

The prices in Tables 3.4 and 3.5 indicate that the accuracy in the pricing formulas for double barrier options is slightly worse than in the single barrier case. However, still the approximation gives good results if the value of T/m is small and if the barriers are not too close to the initial price of the underlying asset.

Chapter 4

Pricing Discrete Barrier Options Using Lattice Random Walks

A widely used numerical procedure in option pricing is the so called binomial method. This method estimates the price of an option by replacing the driving Brownian motion in the underlying price by a simple random walk. The binomial method was introduced by Cox, Ross, and Rubinstein and, independently, by Rendleman and Bartter, see [37] and [101], respectively. Since their seminal work the binomial method has been extended in various directions by, among many others, Boyle [23], Kamrad and Ritchken [70], and Rogers and Stapleton [106]. In particular, the binomial method and some of its generalisations have shown to be very useful in the pricing of discrete barrier options, see for instance Ahn, Figlewski, and Gao [1] and Broadie, Glasserman, and Kou [29]

This chapter has two objectives. The first is to design an efficient numerical procedure to price discrete European barrier options using lattice methods, i.e. the driving Brownian motion in the underlying asset price is replaced by a lattice random walk. The method discussed in this chapter will be able to value all forms of discrete barrier options. We may add that our method can also be applied on other problems that involves the evaluation of Wiener integrals. The method can for instance be useful to determine the Wiener measure of cylinder sets. However, for ease of exposition this chapter will mainly study the pricing of discrete barrier options. For a further discussion about methods to evaluate Wiener integrals, see Steinbauer [115].

The second aim of this chapter is to continue the research by, among many others, Leisen and Reimer [84], Heston and Zhou [58], Diener and Diener [40], and Walsh [121] to characterise the rate of convergence of lattice methods for initial value problems. To this end we will present certain results from the theory of Besov spaces (see Bergh and Löfström [14]). These results will also be useful to construct an efficient procedure to price discrete barrier options.

The chapter is structured as follows. Section 4.1 gives a brief introduction to the lattice method and presents a naive approach to price discrete barrier options. Section 4.2 discusses Besov spaces and the rate of convergence for lattice methods. Section 4.3 returns to the problem of pricing discrete barrier options. Based on the results in Section 4.2 we enhance the method discussed in Section 4.1. Section 4.4 presents some numerical examples. Finally, Section 4.5 concludes this chapter with suggestions for future research.

4.1 Preliminaries

Suppose $\{S_t\}_{t\geq 0}$ is a geometric Brownian motion with drift, that is

$$dS_t = S_t(\,\eta dt + \sigma dW_t\,), \quad t \geq 0,$$

where $\{W_t\}_{t\geq 0}$ is a standard one-dimensional Brownian motion and $\eta \in \mathbb{R}$, $\sigma > 0$, and $S_0 > 0$ are fixed constants. Assume $M = \{t_1, t_2, \ldots, t_{m-1}, t_m\}$ where $0 < t_1 < t_2 \ldots < t_m = T$ and suppose that the functions $H_1 : M \to [0, \infty)$ and $H_2 : M \to (0, \infty]$ satisfies $H_1 < H_2$. This chapter will design a numerical procedure to estimate the expectation

$$v = E\big[\Psi(S_T)\, 1_{\{\, H_1(t) < S_t < H_2(t),\, t\in M\,\}}\big], \tag{4.1}$$

where $\Psi : (0,\infty) \to \mathbb{R}$. By choosing $\Psi(x) = e^{-rT}\max(x-K,0)$ or $\Psi(x) = e^{-rT}\max(K-x,0)$ and $\eta = r-q$ we find that v equals the theoretical price at time $t=0$ of a discrete barrier option with strike price K.

Next, put $\nu = \eta - \sigma^2/2$ and set for each $i = 1, 2, \ldots, m$,

$$a_i = \frac{\ln\big(H_1(t_i)/S_0\big) - \nu t_i}{\sigma} \quad \text{and} \quad b_i = \frac{\ln\big(H_2(t_i)/S_0\big) - \nu t_i}{\sigma}.$$

The functions $\chi_i : \mathbb{R} \to \{0,1\}$, $i = 1, 2, \ldots, m$, will denote the indicator functions of the intervals $I_i = (a_i, b_i)$, that is

$$\chi_i(x) = \begin{cases} 1, & \text{if } x \in I_i, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, suppose that

$$\tau_i = t_i - t_{i-1}, \quad i = 1, \dots, m,$$

with $t_0 = 0$.

Now, let the class of operators $\{U_t\}_{t \geq 0}$ denote the Brownian semi-group, that is

$$(U_t f)(x) = E\big[f(x + W_t)\big], \quad x \in \mathbb{R},$$

where $f : \mathbb{R} \to \mathbb{R}$ satisfies some appropriate integrability conditions, and introduce a function v_{m-1} by setting

$$v_{m-1}(x) = U_{\tau_m}\Big(\Psi\big(S_0 e^{\nu T + \sigma(\cdot)}\big)\chi_m(\cdot)\Big)(x), \quad x \in \mathbb{R}.$$

If we define recursively

$$v_{i-1}(x) = \big(U_{\tau_i}(v_i\, \chi_i)\big)(x), \quad x \in \mathbb{R},$$

for $1 \leq i \leq m-1$, then the Markov property for Brownian motion tells us that the quantity v in equation (4.1) equals $v = v_0(0)$.

We are thereby led to the problem to compute the function

$$x \mapsto (U_\tau f)(x)$$

for a given function f and fixed $\tau > 0$. This problem is closely related to solving the initial value problem for the heat equation, an observation that goes back to Bachelier [6]. Namely, the function $u(x,t) = (U_t f)(x)$, $x \in \mathbb{R}$, $t \in [0, \tau]$, is the solution to the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} & \text{in } \mathbb{R} \times (0, \tau], \\ u|_{t=0} = f & \text{on } \mathbb{R}. \end{cases}$$

See Karatzas et al. [71] for further details. Thus, the function $x \mapsto (U_\tau f)(x)$ equals the solution of the initial value problem for the heat equation at a fixed time τ.

One well-known approach to estimate the function $x \mapsto (U_\tau f)(x)$ is given by a so called lattice method. That is, the Brownian motion in the definition of the Brownian semi-group is replaced by a lattice random walk. To be more specific, let ζ be a lattice random variable with span 1, i.e. $P(\zeta \in \mathbb{Z}) = 1$, such that

$$E[\,\zeta\,] = 0 \quad \text{and} \quad 0 < \lambda = \mathrm{Var}(\zeta) < \infty.$$

Furthermore, suppose $\zeta_1, \ldots, \zeta_n$ are stochastic independent copies of ζ and define

$$(U_t^{(\zeta,h)} f)(x) = E\big[f(x + h \sum_{k=1}^{n} \zeta_k)\big], \quad \text{with } n = \frac{t}{\lambda h^2},$$

for all $h > 0$, $t \in R_h = \lambda h^2 \mathbb{N}_+$, and each $x \in \mathbb{R}$. According to the central limit theorem the sequence of random variables

$$\big\{h_n \sum_{k=1}^{n} \zeta_k\big\}_{n=1}^{\infty}, \quad \text{with } h_n = \sqrt{\frac{\tau}{n\lambda}},$$

will converge in distribution to W_τ. This gives us reason to believe that $U_\tau^{(\zeta,h_n)} f$ might be a good approximation of $U_\tau f$ for sufficiently large values of n. However, as we will show in Section 4.3, this approach can be improved considerably.

Before we go on and consider the rate of convergence for a lattice method, let us just mention that the operator $U_\tau^{(\zeta,h_n)}$ can also been seen as a stable finite difference operator. Chapter 5 in this thesis will give a more detailed discussion about the relation between stable finite difference operators and lattice methods.

4.2 Rate of Convergence

The continuous mapping theorem (see Durrett [44], p. 87) gives that if f is a bounded function such that the set of all discontinuity points of f has Lebesgue measure 0, then, for each fixed $x \in \mathbb{R}$,

$$\big|\big(U_\tau^{(\zeta,h_n)} f\big)(x) - \big(U_\tau f\big)(x)\big| \to 0 \quad \text{as } n \to \infty,$$

where h_n is defined as above, that is

$$h_n = \sqrt{\frac{\tau}{n\lambda}}.$$

The objective of this section is to state necessary and sufficient conditions on f and ζ such that, given $x \in \mathbb{R}$ and $\tau > 0$,

$$\big|(U_\tau^{(\zeta,h_n)} f)(x) - (U_\tau f)(x)\big| = O(1/n^\alpha), \quad \text{as } n \to \infty,$$

for some $\alpha > 0$.

In the literature there are lots of contributions to the solution of this problem. For instance, Berry [13] and Esseen [46] consider the special case when f is piecewise constant and von Bahr [7] when the initial value is a polynomial. Among many others, Diener and Diener [40] and Leimar and Reimer [84] analyse the binomial tree for initial values that corresponds to the payoff function of a call option. Kreiss, Thomée, and Widlund [77], Heston and Zhou [58], Walsh [121], Butzer, Hahn, and Westphal [30], and Löfström [93] examine, among other things, the dependence between the smoothness of f and the rate of convergence for lattice methods.

Before we comment on these papers any further we will introduce some definitions. A lattice random variable ζ with span 1 and $0 < \lambda = \mathrm{Var}(\zeta) < \infty$ is said to be *consistent* of order μ, where μ is an integer, if

$$E\left[\, e^{i\xi\zeta} \,\right] = E\left[\, e^{i\xi W_\lambda} \,\right] + O(\xi^{\mu+2}), \quad \text{as } \xi \to 0,$$

where i is the imaginary unit and $\xi \in \mathbb{R}$. Furthermore, we will say that ζ is *exactly consistent* of order μ if ζ is consistent of order μ but not consistent of order $\mu + 1$.

It is well known that there is a close connection between the consistency number and the moments of a random variable. On one hand, if ζ has an absolute moment of order $\mu + 1$, i.e. $E[\,|\zeta|^{\mu+1}\,] < \infty$, and ζ is consistent of order μ then $E[\,\zeta^k\,] = E[\,W_\lambda^k\,]$ for all positive integers $k \leq \mu + 1$ (see Durrett [44], p. 101). On the other hand, if ζ has a moment of order $\mu + 2$ and $E[\,\zeta^k\,] = E[\,W_\lambda^k\,]$ for all positive integers $k \leq \mu + 1$ then ζ is consistent of order μ (see Lukacs [92], p. 23).

In connection with a trinomial tree, each fixed increment of the underlying random walk equals 0 or ± 1. Suppose the symmetrical random variable ϑ has (at the most) three possible outcomes. Thus,

$$P(\vartheta = 0) = 1 - \lambda \quad \text{and} \quad P(\vartheta = 1) = P(\vartheta = -1) = \frac{1}{2}\lambda,$$

with $0 < \lambda = \mathrm{Var}(\zeta) \leq 1$. The random variable ϑ can be exactly consistent of order 2 or 4. To see this, note that Taylor's formula imply as $\xi \to 0$,

$$\begin{aligned} E\left[\, e^{i\xi\vartheta} \,\right] - E\left[\, e^{i\xi W_\lambda} \,\right] &= 1 - \lambda + \lambda\cos(\xi) - e^{-\lambda\frac{\xi^2}{2}} \\ &= \frac{\lambda}{8}\Big(\frac{1}{3} - \lambda\Big)\,\xi^4 + \frac{\lambda}{48}\big(\lambda^2 - \frac{1}{15}\big)\,\xi^6 + O(\xi^8). \end{aligned}$$

Hence, the random variable ϑ is exactly consistent of order 4 if

$$\lambda = 1/3.$$

For any other choice of λ in $(0,1]$ the random variable ϑ is exactly consistent of order 2. In particular, a Rademacher random variable, that is, a symmetrical random variable only taking the values ± 1, is exactly consistent of order 2.

Next we will introduce certain Banach spaces known as *Besov spaces* and below denoted by B^s_∞, $s > 0$. The Besov spaces are subspaces of the Banach space C_0, where C_0 denotes the class of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x \to \pm\infty} f(x) = 0$$

equipped with the norm $\|f\|_{C_0} = \max_{x\in\mathbb{R}} |f(x)|$.

The norm in the Besov space B^s_∞, henceforth denoted $\|\cdot\|_{B^s_\infty}$, is given as follows: Set $s = k + \gamma$, where k is a nonnegative integer and $0 < \gamma \leq 1$. If $0 < \gamma < 1$ then the norm will be defined in terms of a so called Hölder condition with exponent γ;

$$\|f\|_{B^{k+\gamma}_\infty} = \|f\|_{C_0} + \sup_{t>0} \frac{1}{t^\gamma} \|f^{(k)}(\cdot + t) - f^{(k)}(\cdot)\|_{C_0} ,$$

where $f^{(k)}$ denotes the k:th derivative of f. If $\gamma = 1$, the norm is defined in terms of a so called Zygmund condition:

$$\|f\|_{B^{k+1}_\infty} = \|f\|_{C_0} + \sup_{t>0} \frac{1}{t} \|f^{(k)}(\cdot + t) - 2f^{(k)}(\cdot) + f^{(k)}(\cdot - t)\|_{C_0} .$$

In the literature there exist many other equivalent definitions of the norm in the Besov space B^s_∞. The definition here is taken from Brenner et al. [26].

If $s_1 < s_2$ then $B^{s_1}_\infty \supset B^{s_2}_\infty$ (see Bergh et al. [14], p. 142). Thus, functions in $B^{s_2}_\infty$ are generally smoother than functions in $B^{s_1}_\infty$. Much more can of course be said about Besov spaces and the interested reader is referred to Bergh et al. [14], Chapters 6 and 7, and the references therein.

In the sequel we will sometimes use the term *local* Zygmund condition. A function f satisfies a local Zygmund condition if for all $c > 0$ and each $x \in \mathbb{R}$ it holds

$$\sup_{0<t<c} \frac{1}{t} \big| f(x+t) - 2f(x) + f(x-t) \big| < \infty.$$

Note that if $f^{(k)} \in C_0$ and $f^{(k)}$ satisfies a local Zygmund condition then $f \in B^{k+1}_\infty$.

We shall introduce yet another Banach space, below denoted A_∞^s, $s>0$. The space A_∞^s is a subspace to C_0 with norm

$$\|f\|_{A_\infty^s} = \|f\|_{C_0} + \sup_{0<h<1} \sup_{t\in R_h} \frac{1}{h^s} \|U_t^{(\zeta,h)} f - U_t f\|_{C_0},$$

where, as previous, $R_h = \lambda h^2 \, \mathbb{N}_+$.

The following striking result is due to Löfström [93].

Theorem 4.1. *Suppose that ζ is exactly consistent of order μ. Then*

$$A_\infty^\varsigma = B_\infty^\varsigma \,, \quad 0<\varsigma\le\mu,$$

with equivalent norms. Moreover, if $f\in C_0$ and

$$\sup_{t\in R_h} \| \, U_t^{(\zeta,h)} f - U_t f \, \|_{C_0} = o\big(h^\mu\big), \quad \text{as } h\to 0,$$

then $f\equiv 0$.

Thus, the convergence rate is closely related to the smoothness of the initial value f and the moments of ζ. In particular, if $f\in B_\infty^s$ and ζ is consistent of order μ, Theorem 4.1 yields that there is for each $\varsigma\le\min(\mu,s)$ a constant C, independent of f and n, such that

$$\| \, U_\tau^{(\zeta,h_n)} f - U_\tau f \, \|_{C_0} \;\le \frac{C}{n^{\varsigma/2}} \|f\|_{B_\infty^\varsigma}, \quad n > \frac{\tau}{\lambda}.$$

Here we recall that

$$h_n = \sqrt{\frac{\tau}{n\lambda}}.$$

Consequently, if $f\in B_\infty^s$, ζ is consistent of order μ, and

$$\alpha = \frac{1}{2}\min(\mu,s),$$

then

$$|\,(U_\tau^{(\zeta,h_n)} f)(x) - (U_\tau f)(x)\,| = O(1/n^\alpha), \quad \text{as } n\to\infty. \tag{4.2}$$

By applying this result to the trinomial tree it follows that if f has a derivative f' belonging to C_0 and satisfying a local Zygmund condition, then

$$|\,(U_\tau^{(\vartheta,h_n)} f)(x) - (U_\tau f)(x)\,| = O(1/n), \quad \text{as } n\to\infty,$$

for any $x \in \mathbb{R}$. If, in addition, the derivative of order three $f^{(3)}$ is in C_0 and satisfies a local Zygmund condition and, furthermore, ϑ is consistent of order 4 then

$$|\,(U_\tau^{(\vartheta,h_n)}f)(x) - (U_\tau f)(x)\,| = O(1/n^2), \quad \text{as } n \to \infty.$$

Butzer et al. [30] and Heston et al. [58] have results related to equation (4.2). However, the results by Butzer et al. and Heston et al. require more local regularity of the initial value compared to equation (4.2).

One can note as well that Theorem 4.1 states that if the uniform error on the whole lattice equals $O(h^\varsigma)$, $0 < \varsigma \leq \mu$, then $f \in B_\infty^\varsigma$. The next proposition, settled in Thomée and Wahlbin [119], shows a similar result for a fixed time.

Proposition 4.1. *Suppose that ζ is consistent of order μ and $1 < \varsigma \leq \mu$. Assume that for a fixed $f \in C_0$ holds*

$$\|U_\tau^{(\zeta,h_n)}f - U_\tau f\|_{C_0} = O(h_n^\varsigma), \quad \textit{as } n \to \infty.$$

Then $f \in B_\infty^{\varsigma-1}$.

Next we will focus on the important special case with initial value $f_K(x) = \max(e^x - K, 0)$, $K > 0$, and the lattice variable $\zeta = \varepsilon$, where ε denotes a symmetrical random variable with only two outcomes ± 1. It is known, see for instance Diener et al. [40], Leisen et al. [84], or Walsh [121], that

$$|(U_\tau^{(\varepsilon,h_n)}f_K)(x) - (U_\tau f_K)(x)| = O(1/n), \quad \text{as } n \to \infty,$$

for any fixed $x \in \mathbb{R}$. Thus, for the payoff function of a call option the binomial method converges point-wise as $O(1/n)$. In Walsh [121] it is shown that this result is the best possible in the sense that there exists a constant C, independent of n, such that

$$(U_\tau^{(\varepsilon,h_n)}f_K)(x) - (U_\tau f_K)(x) \sim C/n,$$

where $a_n \sim b_n$ means $a_n/b_n \to 1$ as n tends to infinity.

Finally, we consider the convergence rate for discontinuous initial values. For this problem the next famous theorem by Berry and Esseen is of great value (see [13] or [46]).

Theorem 4.2. *Let $\{X_k\}_{k=1}^\infty$ be a sequence of i.i.d. random variables with mean 0, variance 1, and finite absolute third moment. There is a constant*

C only depending on the third absolute moment such that the distribution function $F_n(x) = P(\frac{1}{\sqrt{n}}\sum_{k=1}^n X_k \le x)$ satisfies

$$|F_n(x) - \Phi(x)| \le C/\sqrt{n}$$

for any x, where Φ is the standard normal distribution function.

Let χ_A denote the indicator function for the set $A \subseteq \mathbb{R}$ and put $f_a = \chi_{(-\infty,a]}$. Theorem 4.2 gives us

$$|\,(U_n^{(\zeta,h_n)} f_a)(x) - (U_\tau f_a)(x)\,| = O(1/\sqrt{n}), \quad \text{as } n \to \infty,$$

for any x and any ζ such that $E[|\zeta|^3] < \infty$. It is possible to show that the convergence rate cannot be better than $1/\sqrt{n}$ in the special case $\zeta = \varepsilon$ and $x = a$. More precisely, we have

$$|\,(U_\tau^{(\varepsilon,h_{2n})} f_a)(a) - (U_\tau f_a)(a)\,| \sim \frac{1}{2\sqrt{\pi n}},$$

see e.g. Esseen [46] or Durrett [44] p. 126.

We are now in position to deal with the main problem in this chapter.

4.3 Pricing Discrete European Barrier Options Using Lattice Random Walks

This section returns to the problem of estimating the price v of a discrete barrier option. Recall that Section 4.1 showed that $v = v_0(0)$, where the function v_0 was given by the recursion scheme

$$\begin{cases} v_{m-1} = U_{\tau_m}\Big(\Psi_K\big(S_0 e^{\nu T + \sigma(\cdot)}\big)\chi_m(\cdot)\Big), & \\ v_{i-1} = U_{\tau_i}\,(v_i\,\chi_i), & i = 1, \dots, m-1, \end{cases}$$

with $\Psi_K(x) = e^{-rT}\max(x - K, 0)$ or $\Psi_K(x) = e^{-rT}\max(K - x, 0)$ and $\nu = r - q - \sigma^2/2$.

Firstly, the function $v_{m-1}(x)$ can easily be expressed in terms of the standard normal distribution function. So in what follows we consider the function v_{m-1} as known.

As already described in Section 4.1, for each $i = 1, 2, \dots, m-1$ it is natural to estimate the function v_{i-1} by $U_{\tau_i}^{(\zeta,h)}\,(v_i\,\chi_i)$. There is, however, one disadvantage with this approach. Depending on the barriers, the function

$v_i\,\chi_i$ is, for any $i = 1, \ldots, m-1$, discontinuous at the boundary points of the interval I_i. According to the discussion in the preceding section, a discontinuous initial value f may cause a slow convergence of the sequence $\{U_\tau^{(\zeta,h)} f\}$ as h tends to 0. But suppose for a moment that the initial value f can be written as

$$f = \phi - g, \tag{4.3}$$

where g is a function such that $U_\tau\, g$ can easily be evaluated analytically and ϕ is in some sense a smooth function. Then the discussion in Section 4.2 gives us strong reasons to believe that the function

$$U_\tau^{(\zeta,h)} \phi - U_\tau g$$

will be a better estimate of $U_\tau f$ than $U_\tau^{(\zeta,h)} f$. Our next aim is to show how the functions $v_i \chi_i$, $i = 1, \ldots, m-1$, can be decomposed as in equation (4.3).

For the sake of simplicity, assume that I_i, $i = 1, 2, \ldots, m-1$, are bounded intervals, that is $a_i > -\infty$ and $b_i < \infty$. We will return to the special case when some of the intervals are unbounded later on in this section.

Fix $i = 1, 2, \ldots, m-1$ and consider the functions

$$\psi_{a_i}(x) = e^{\gamma_{a_i}(x-a_i)} \sum_{k=0}^{d_i} \frac{\alpha_{i,k}}{k!} (x-a_i)^k \tag{4.4}$$

and

$$\psi_{b_i}(x) = e^{\gamma_{b_i}(x-b_i)} \sum_{k=0}^{d_i} \frac{\beta_{i,k}}{k!} (x-b_i)^k, \tag{4.5}$$

where d_i is a nonnegative integer. The constants γ_{a_i} and γ_{b_i} can be thought of as a positive and a negative number, respectively. However, for the moment we will not put any restrictions on γ_{a_i} or γ_{b_i}. The coefficients $\{\alpha_{i,k}\}_{k=0}^{d_i}$ and $\{\beta_{i,k}\}_{k=0}^{d_i}$ above are chosen such that ψ_{a_i} and ψ_{b_i} equal the (infinitely differentiable) function v_i and its first d_i derivatives at the points a_i and b_i, respectively. Thus,

$$\frac{d^k v_i}{dx^k}(a_i) = \frac{d^k \psi_{a_i}}{dx^k}(a_i) \quad \text{and} \quad \frac{d^k v_i}{dx^k}(b_i) = \frac{d^k \psi_{b_i}}{dx^k}(b_i)$$

for each $k = 0, 1, \ldots, d_i$. In practice, however, we will not (or rather cannot) differentiate the function v_i to estimate the coefficients $\alpha_{i,k}$ or $\beta_{i,k}$. Instead we will use numerical differentiation. This step is described in greater details in Appendix A at the end of this chapter.

Let as previous χ_A denote the indicator function of the interval A and set

$$g_i = \psi_{a_i}\,\chi_{(-\infty,a_i]} + \psi_{b_i}\,\chi_{[b_i,\infty)} \tag{4.6}$$

and

$$\phi_i = v_i\chi_i + g_i.$$

Hence,

$$\phi_i(x) = \begin{cases} \psi_{a_i}(x), & \text{if } x \le a_i, \\ v_i(x), & \text{if } a_i < x < b_i, \\ \psi_{b_i}(x), & \text{if } x \ge b_i. \end{cases}$$

The function ϕ_i is obviously d_i times continuously differentiable on $\mathbb{R}$ and, furthermore, since $\phi_i^{(d_i)}$ is infinitely differentiable on the set $\mathbb{R}\setminus\{a_i, b_i\}$, the function $\phi_i^{(d_i)}$ satisfies a local Zygmund condition. If, in addition, $\gamma_{a_i} > 0$ and $\gamma_{b_i} < 0$ then $\phi_i^{(d_i)} \in C_0$ and thus $\phi_i \in B_\infty^{d_i+1}$. Since $U_{\tau_i} g_i$ can be evaluated using the normal distribution and elementary functions (see Appendix B at the end of this chapter), we have obtained the desired decomposition

$$v_i\chi_i = \phi_i - g_i$$

as in equation (4.3). Moreover, if the lattice random variable $\zeta^{(i)}$ is consistent of order μ_i and $d_i = \mu_i - 1$, then equation (4.2) yields that

$$\left|\,(U_{\tau_i}^{(\zeta^{(i)},h_n)}\phi_i)(x) - (U_{\tau_i}\phi_i)(x)\,\right| = O(1/n^{\frac{1}{2}\mu_i})$$

as $n \to \infty$.

We have arrived at the following algorithm.

Algorithm 4.1. *Suppose that $\{\zeta^{(i)}\}_{i=1}^{m-1}$ is a sequence of lattice random variables with $Var(\zeta^{(i)}) = \lambda_i$, where λ_i satisfy*

$$\tau_i \in \lambda_i h^2 \mathbb{N}_+,$$

for each $i = 1, \dots, m-1$ and some fixed $h > 0$. The theoretical price v of a discrete barrier option is then approximately equal to $\tilde{v}_0(0)$, where the *function $\tilde{v}_0$ is determined by the recursion scheme*

$$\begin{cases} \tilde{v}_{m-1} = U_{\tau_m}\Big(\Psi_K\big(S_0 e^{\nu T+\sigma(\cdot)}\big)\chi_m(\cdot)\Big), \\ \tilde{v}_{i-1} = U_{\tau_i}^{(\zeta^{(i)},h)}\left(\tilde{v}_i\chi_i + g_i\right) - U_{\tau_i} g_i, \qquad i = 1,\dots,m-1. \end{cases} \tag{4.7}$$

Here $\Psi_K(x) = e^{-rT}\max(x-K,0)$ or $\Psi_K(x) = e^{-rT}\max(K-x,0)$ and $\nu = r - q - \sigma^2/2$. The functions g_i, $i = 1,\dots,m-1$, are defined as in equation (4.6) and $U_{\tau_i} g_i$ can be computed using the receipt in Appendix B. For each $i = 1,\dots,m-1$, the coefficients $\{\alpha_{i,k}\}_{k=0}^{d_i}$ and $\{\beta_{i,k}\}_{k=0}^{d_i}$ in the definition of ψ_{a_i} and ψ_{b_i}, cf. equations (4.4)-(4.5), are chosen such that

$$\frac{d^k\psi_{a_i}}{dx^k}(a_i) = \frac{d^k\tilde{v}_i}{dx^k}(a_i) \quad \text{and} \quad \frac{d^k\psi_{b_i}}{dx^k}(b_i) = \frac{d^k\tilde{v}_i}{dx^k}(b_i)$$

for $k = 0,1,\dots d_i$, see Appendix A for further details concerning the estimation of the coefficients $\{\alpha_{i,k}\}_{k=0}^{d_i}$ and $\{\beta_{i,k}\}_{k=0}^{d_i}$.

Let us make some comments about Algorithm 4.1. So far we have assumed that $a_i > -\infty$ and $b_i < \infty$. If $a_i = -\infty$ or $b_i = \infty$ for some i, then we simply let $g_i(x) = \psi_{b_i}(x)\chi_{[b_i,\infty)}$ or $g_i(x) = \psi_{a_i}(x)\chi_{(-\infty,a_i]}$, respectively, in equation (4.6).

Note that the parameter h, i.e. the step size in the vertical direction, in Algorithm 4.1 is constant, that is, independent of i. This restriction is imposed so that the lattice recombines between the monitoring dates. Needless to say, the functions $\tilde{v}_i$, $i = 1,2,\dots,m-1$, in equation (4.7) shall only be calculated at the lattice points.

This section is concluded with some comments about the computional complexity of Algorithm 4.1. Suppose that we have a lattice with step size h in the vertical direction. The number of computations to evaluate $U_{\tau_i} g_i$ at the lattice points between a_i and b_i is of order $O(1/h)$ as $h \to 0$. On the other hand, the number of computations to evaluate the functions

$$U_{\tau_i}^{(\zeta^{(i)},h)}(\tilde{v}_i\chi_i + g_i) \quad \text{or} \quad U_{\tau_i}^{(\zeta^{(i)},h)}(\tilde{v}_i\chi_i)$$

at the lattice points between a_i and b_i is of order $O(1/h^4)$, $h \to 0$. In fact, if the step size is h then the lattice has $n = \tau_i/(\lambda_i h^2)$ number of time steps. To evaluate any of the functions above at each lattice points between a_i and b_i it requires $O(n^2)$, $n \to \infty$, number of computations, that is, $O(1/h^4)$, $h \to 0$, number of computations. In particular, the correction term $U_{\tau_i} g_i$ added to the lattice method in Algorithm 4.1 will not change the computational complexity.

We may add that since the proposed algorithm requires several evaluations of polynomials it is possible to improve the performance of the algorithm by using Horner's scheme, see Dahlquist and Björk [38], pp. 14-15. We may also note that if ζ is a bounded *symmetrical* random variable with $P(\zeta = j) = p_j$, $j = -k, \dots, k$, then $E[f(\zeta)]$ should be computed according to the scheme

$$E[f(\zeta)] = p_0 f(0) + \sum_{j=1}^{k} p_j \big(f(j) + f(-j)\big)$$

and *not*

$$E[f(\zeta)] = \sum_{j=-k}^{k} p_j f(j).$$

The first mentioned procedure requires less multiplications.

4.4 Numerical Examples; Choice of Parameters and Random Variables

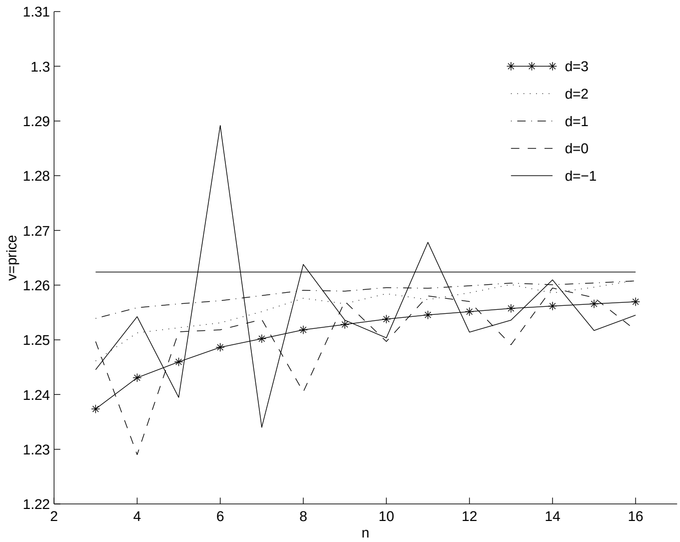
This section presents numerical examples and studies the performance of the algorithm for different choices of random variables $\zeta^{(i)}$ and parameters d_i, γ_{a_i}, and γ_{b_i}, $i = 1, \dots, m-1$. The section is divided into two subsections. Subsection 4.4.1 focuses on trinomial trees and equidistant monitoring times, that is, $\zeta^{(i)} = \vartheta$ (cf. Section (4.2)) and $\tau_i = T/m$ for each $i = 1, \dots, m$. Subsection 4.4.2 discusses other choices of lattice random variables and arbitrary monitoring times.

4.4.1 The Trinomial Tree and Equidistant Monitoring Times

Thus, assume that $\zeta^{(i)} = \vartheta$ (cf. Section 4.2) and $\tau_i = T/m$ for each $i = 1, \dots, m$. Furthermore, let for simplicity the parameters d_i, γ_{a_i}, and γ_{b_i} be independent of i and put $d = d_i$, $\gamma_b = \gamma_{b_i}$, and $\gamma_a = \gamma_{a_i}$ for each $i = 1, \dots, m-1$.

Figure 4.1 presents the value of $\tilde{v}_0(0)$, given by Algorithm 4.1, as a function of n, the number of terms in the random walk between the monitoring dates. In the first example we have picked $\lambda = 2/3$ (i.e. ϑ exactly consistent of order 2) and $\gamma_a = \gamma_b = 0$. The option price is approximately 1.2624 (cf. the straight line in Figure 4.1). Consider first the convergence for the basic trinomial tree (i.e. $g \equiv 0$ in Algorithm 4.1). The corresponding price is

Figure 4.1: Convergence rate for the proposed method with $\zeta = \vartheta$ (trinomial tree), $\lambda = 2/3$, and $\gamma_a = \gamma_b = 0$. The option parameters are $S_0 = 100$, $K = 90$, $H_1 \equiv 80$, $H_2 \equiv 120$, $\sigma = 0.3$, $r = 0.1$, $T = 1$ year, and $m = 50$ (number of monitoring times, corresponds to weekly monitoring). The monitoring times are equally spaced in time.



denoted $d = -1$ in Figure 4.1. We see that the convergence is slow and oscillating. If the proposed algorithm is used with $d = 0$ the convergence is more regular but the rate of convergence seems to be more or less the same. In contrast to these examples, when d is equal to 1, which corresponds to differentiable initial values, the convergence is faster and smoother. When $d = 2$ or 3 the convergence rate does not increase. In fact, it becomes slower.

Before we proceed, let us just mention that this so called 'zigzag convergence' that can be observed in the cases $d = -1$ and $d = 0$ have been analysed more carefully in Diener et al. [40] and Gobet [53].

Figure 4.1 reflects very well the convergence behaviour for the proposed method for all choices of $\lambda \in (0, 1]$ *except* $\lambda = 1/3$, that is, when ϑ is consistent of order 4.

The next two figures, Figures 4.2 and 4.3, present the convergence rate for $\lambda = 1$ (the binomial tree) and $\lambda = 1/3$, respectively, with $d = 1, 2, 3$ in both cases. The option parameters are the same as in the previous example. In the special case $\lambda = 1$, Figure 4.2 displays that the convergence pattern is roughly the same as in the case $\lambda = 2/3$. On the other hand, if $\lambda = 1/3$

Figure 4.2: Convergence rate for the proposed method with $\lambda = 1$ (The binomial method). The other parameters are as in Figure 4.1.

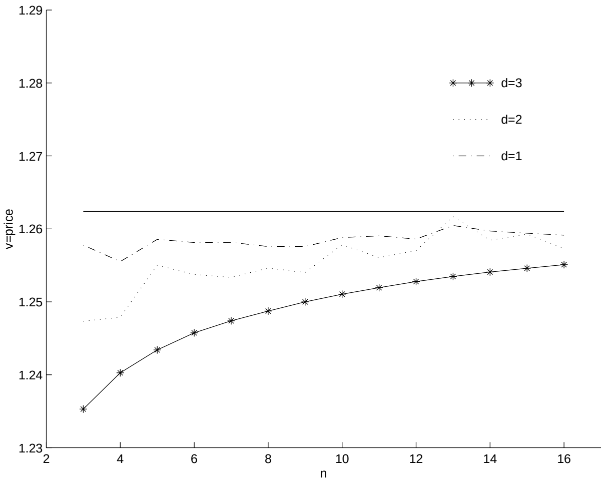


Figure 4.3: Convergence rate for the proposed method with $\lambda = 1/3$. The other parameters are as in Figure 4.1.

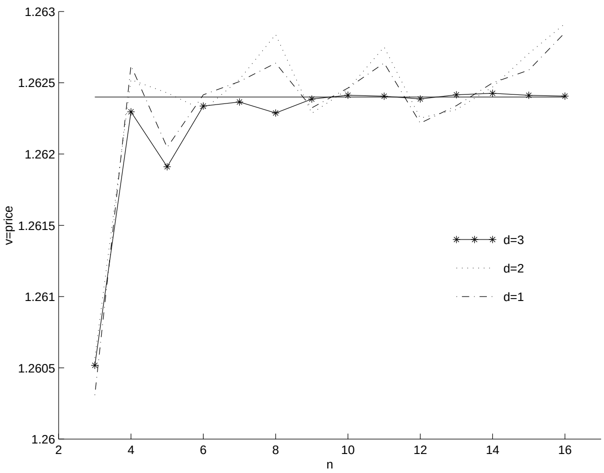


Table 4.1: Convergence rate for the proposed method for different values on λ and d. The other parameters are as in Figure 4.1.

n	$d = 1$, $\lambda = 2/3$	$d = 1$, $\lambda = 1$	$d = 3$, $\lambda = 1/3$
3	1.2539	1.2578	1.2605
4	1.2559	1.2555	1.2623
5	1.2566	1.2586	1.2619
6	1.2572	1.2581	1.2623
7	1.2581	1.2581	1.2624
8	1.2591	1.2576	1.2623
9	1.2589	1.2576	1.2624
10	1.2595	1.2588	1.2624
11	1.2594	1.2590	1.2624

Figure 4.3 shows that the method obtains the best convergence rate with $d = 3$. Table 4.1 collects the prices obtained for different values of λ and d. The table clearly illustrates that the fastest convergence occurs for $d = 3$ and $\lambda = 1/3$. Finally, Figure 4.4 shows how the smoothing of the initial value improves the convergence rate.

The next example, Table 4.2, investigates how the values of γ_a and γ_b influence the error, or rather, if there is a difference in the convergence rate in the two cases $\gamma_a = \gamma_b = 0$ and $\gamma_a > 0$, $\gamma_b < 0$. From a theoretical point of view there is a distinct difference between these cases. If $\gamma_a > 0$ and $\gamma_b < 0$, then the function g (cf. Section 4.3) is bounded, whereas if $\gamma_a = \gamma_b = 0$ then $g(x) = O(x^d)$ as x tends to infinity.

Recall that the density function for the standard normal distribution decreases as $O(e^{-x^2/2})$ as x tends to infinity. Thus we believe that the growth in g has hardly any greater impact on the convergence rate, as the example in Table 4.2 indicates. Unfortunately, we have not been able to prove this (see Section 4.5 for a further discussion). But still, we suggest that Algorithm 4.1 should be used with the parameter values $d = 3$, $\lambda = 1/3$, and $\gamma_a = \gamma_b = 0$. Setting $\gamma_a = \gamma_b = 0$ has one practical advantage, the algorithm is easier to implement.

The next example compares our method with an algorithm developed by Broadie, Glasserman, and Kou in [28]. Their algorithm is designed to estimate the value of a discrete and constant single barrier option. For simplicity, the algorithm by Broadie et al. will henceforth be called the BGK method. The idea behind the BGK method is as follows. A dis-

Figure 4.4: Convergence rate for the proposed method when $\lambda = 1/3$. The other parameters are as in Figure 4.1.

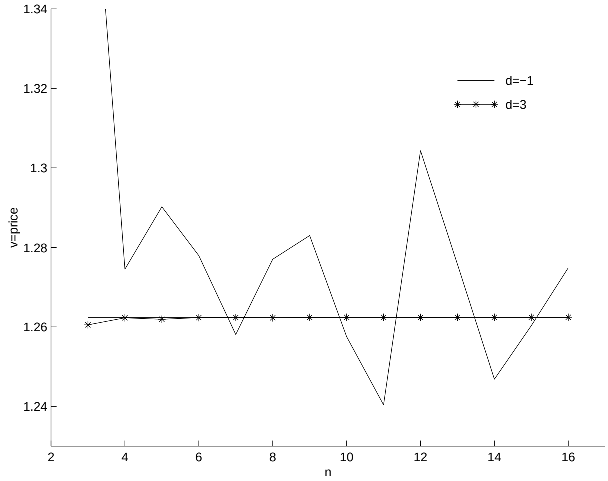


Table 4.2: Convergence rate for the proposed method for different values on γ_a, and γ_b. The value on d, λ, and γ_b are 3, 1/3, and $-\gamma_a$, respectively. The other parameters are as in Figure 4.1.

n	$\gamma_a = 0.0$	$\gamma_a = 0.1$	$\gamma_a = 1$	$\gamma_a = 10$
3	1.2605	1.2605	1.2606	1.2608
4	1.2623	1.2623	1.2623	1.2625
5	1.2619	1.2619	1.2619	1.2620
6	1.2623	1.2623	1.2624	1.2624
7	1.2624	1.2624	1.2624	1.2624
8	1.2623	1.2623	1.2623	1.2623
9	1.2624	1.2624	1.2624	1.2624
10	1.2624	1.2624	1.2624	1.2624
11	1.2624	1.2624	1.2624	1.2624

Table 4.3: The value of a discrete down-and-out call, the option parameters are $S_0 = K = 100$, $H_1 \equiv 95$, $H_2 \equiv \infty$, $T = 0.2$ year, $\sigma = 0.6$, and $r = 0.1$. There are 4 monitoring dates which are equally spaced in time, i.e. the monitoring dates are given by $\{\tau, 2\tau, 3\tau, 4\tau\}$ where $\tau = T/4$. The quantity N denotes the total number of iterations, i.e. $N = 4n$. BGK denotes the method by Broadie, Glasserman, and Kou and Hö stands for the method developed in this chapter. The random variables/parameters in Hö are $\zeta = \vartheta$ (trinomial tree), $\gamma_a = \gamma_b = 0$, $d = 3$, and $\lambda = 1/3$.

N	BGK	BGK (2-pt Extrapol.)	N	Hö
256	9.4969		40	9.4895
504	9.4935	9.4899	60	9.4907
1240	9.4919	9.4907	80	9.4907
2308	9.4912	9.4905	100	9.4906
4524	9.4909	9.4905	120	9.4905
8632	9.4907	9.4905	140	9.4905

crete and constant barrier at place H is replaced by a discrete barrier at place $H\exp(\pm 0.5h_n)$, with $+$ for an upper barrier, $-$ for a lower barrier and where h_n is the step size for the log price. The factor $0.5h_n$ is the expected overshoot of a trinomial random walk. Subsequently the theoretical value is computed using the trinomial method on a mesh with the property that certain nodes on the mesh coincide with the new barrier at place $H\exp(\pm 0.5h_n)$. Numerical experiments in that paper indicate that the convergence rate for the method is $O(1/n)$. In order to increase the convergence rate a Richardson interpolation is used. For further details, see Broadie et al. [28].

Table 4.3 shows results from the different methods. The values in the second and third column are taken from a numerical example in Broadie et al. [28]. The BGK method has been used with as well as without Richardson extrapolation. In the final column we have the values from the method presented in this work. As we can see, the example indicates that the method presented in this paper outperforms the BGK method.

The final example in this subsection computes the value of a discrete *moving* barrier option. The results are presented in Table 4.4. The price of a moving barrier option with continuous barriers can be estimated using a numerical procedure described in Rogers and Zane [107].

Table 4.4: The value of a discrete moving double barrier knock-out call for different number of monitoring dates (monthly, weekly and daily). The monitoring times are equally spaced in time. The option parameters are $S_0 = 95$, $K = 100$, $H_1(t) = 90 + \theta_1 t$, $H_2(t) = 160 + \theta_2 t$, $T = 1$ year, $\sigma = 0.25$, and $r = 0.1$. The random variables and parameters are $\zeta = \vartheta$ (trinomial tree), $\gamma_a = \gamma_b = 0$, $d = 3$, and $\lambda = 1/3$. The corresponding continuous prices are, according to Rogers et al. [107], approximately 4.34 and 2.54, respectively.

m	n	$\theta_1 = -\theta_2 = -5$	$\theta_1 = -\theta_2 = 5$
	10	6.9918	5.0999
12	20	6.9918	5.1001
	30	6.9918	5.1001
	10	5.7039	3.8642
50	15	5.7040	3.8641
	20	5.7040	3.8641
	6	4.9418	3.1920
250	8	4.9420	3.1921
	10	4.9421	3.1920

4.4.2 Arbitrary Monitoring Dates and Other Lattice Random Variables

The previous subsection assumed that the monitoring dates were equally spaced in time. To begin with in this subsection we will investigate how the algorithm works if we drop this assumption. Recall from Algorithm 4.1 that the time between the monitoring times τ_i and the variances λ_i must satisfy the relation

$$\tau_i \in \lambda_i h^2 \mathbb{N}_+$$

for all $i = 1, \ldots, m-1$. Thus, if the monitoring times are arbitrary then the variances λ_i must in certain cases vary with i.

Of course, it is not necessary to assume that the monitoring times are equidistant, as in the previous subsection, to be able to find a λ that satisfies the condition $\tau_i \in \lambda h^2 \mathbb{N}_+$ for each $i = 1, \ldots, m-1$. For instance, if there is a number $\Delta > 0$ such that

$$\tau_i \in \Delta \mathbb{N}_+,$$

for all $i = 1, \ldots, m-1$, and if $h = \sqrt{\Delta}/\sqrt{\lambda n}$, for some $n \in \mathbb{N}_+$ and $\lambda > 0$, then it is evident that $\tau_i \in \lambda h^2 \mathbb{N}_+$ for each $i = 1, \ldots, m-1$.

Table 4.5: The value of a discrete double barrier knock-out call with randomly chosen monitoring dates. The quantity N denotes the total number of iterations, cf. Table 4.3. In the examples above the monitoring dates are given by $M_1 = \{0.06, 0.08, 0.15, 0.35, 0.38, 0.44, 0.45, 0.63, 0.67, 0.69, 0.71, 1.00\}$ $M_2 = \{0.12, 0.25, 0.27, 0.45, 0.48, 0.55, 0.69, 0.72, 0.73, 0.87, 0.89, 1.00\}$, and $M_3 = \{0.19, 0.37, 0.57.0.62, 0.63, 0.73, 0.75, 0, 79, 0.84, 0.90, 0.92, 1.00\}$. The other option parameters are as in Figure 4.1. The proposed algorithm has been used with $\zeta^{(i)} = \iota$, $\gamma_{a_i} = \gamma_{b_i} = 0$, and $d_i = 3$ for each $i = 1, \dots, m-1$.

N	M_1	N	M_2	N	M_3
21	2.6504	26	2.1340	28	2.0236
42	2.3586	52	2.2739	56	2.0219
63	2.3591	78	2.0549	84	2.0216
84	2.3591	104	2.0548	112	2.0216
105	2.3591	130	2.0548	140	2.0216

Now, suppose that we *must* pick λ_i differently depending on i. How should we choose the lattice random walk? Of course, we may apply the trinomial tree for all $\lambda \in (0, 1]$ but this approach has the disadvantage that the random variable ϑ is, unless $\lambda = 1/3$, only consistent of order 2. As we have seen in the previous sections, the proposed algorithm is more efficient if the lattice random variable is consistent of order 4 than of order 2 and hence, it would be of interest to find a class of lattice random variables where the members of this class have different variance but still are consistent of order 4. Next we will construct such a class.

Consider a symmetrical lattice random variable ι that have (at the most) five possible outcomes with

$$P(\iota = 0) = p_0 \quad \text{and} \quad P(\iota = j) = P(\iota = -j) = p_j, \quad j = 1, 2,$$

where $p_0 + 2p_1 + 2p_2 = 1$. A Taylor expansion gives that the random variable ι is exactly consistent of order 4 if

$$p_0 = 1 - \frac{\lambda}{4}(5 - 3\lambda), \quad p_1 = \frac{\lambda}{6}(4 - 3\lambda), \quad \text{and} \quad p_2 = \frac{\lambda}{24}(3\lambda - 1),$$

for any λ such that $1 \le 3\lambda \le 4$.

Table 4.5 presents a numerical example which shows the convergence rate of Algorithm 4.1 applied with $\zeta = \iota$.

Next we will draw attention to certain other random variables that are consistent of order 6. Recall that according to the discussion that precedes

Table 4.6: The value of a discrete double barrier knock-out call. The option parameters are as in Figure 4.1. The proposed algorithm has been used with $\zeta^{(i)} = \kappa$, $\gamma_{a_i} = \gamma_{b_i} = 0$, $\lambda_i = \lambda$, and $d_i = 5$ for each $i = 1, \ldots, m-1$.

n	$\lambda = \frac{1}{5}(5 - \sqrt{10})$	$\lambda = 1$
1	1.2525	1.2613
2	1.2615	1.2624
3	1.2617	1.2624
4	1.2623	1.2624
5	1.2624	1.2624

Algorithm 4.1, we may expect that the convergence rate improves for greater values of consistency number μ_i, provided that $d_i \geq \mu_i - 1$.

For instance, consider the random variable κ, defined by

$$P(\kappa = 0) = p_0, \quad \text{and} \quad P(\kappa = j) = P(\kappa = -j) = p_j, \quad j = 1, 2, 3,$$

with

$$\begin{aligned} p_0 &= 1 - \frac{\lambda}{36}(49 - 42\lambda + 15\lambda^2), \\ p_1 &= \frac{\lambda}{16}(12 - 13\lambda + 5\lambda^2), \\ p_2 &= \frac{\lambda}{40}(-3 + 10\lambda - 5\lambda^2), \\ p_3 &= \frac{\lambda}{720}(4 - 15\lambda + 15\lambda^2), \end{aligned}$$

and

$$(5 - \sqrt{10}) \leq 5\lambda \leq (5 + \sqrt{10}).$$

A Taylor expansion yields that κ is exactly consistent of order 6.

Algorithm 4.1 applied with the random variable κ gives a very fast convergence. For instance, for the option presented in Figure 4.1 the algorithm converges to the price 1.2624 already for $n \geq 2$ if $\lambda = 1$ and $d_i = 5$, see Table 4.6.

For a further discussion on other lattice random variables that can be useful in lattice methods we refer the reader to Alford and Webber [2] and the next chapter in this thesis.

4.5 Conclusions and Suggestions for Future Research

This chapter has designed a numerical procedure to price discrete European barrier options and showed that the convergence rate of lattice methods depends on two factors, namely the smoothness of the initial function and the moments of the terms in the random walk. The pricing of discrete barrier options is equivalent to solving series of initial value problem for the heat equation. The main idea has been to decompose each initial value f_i as a difference

$$f_i = \phi_i - g_i,$$

where ϕ_i is smooth and the expectation of g_i with respect to canonical Gauss measure can be computed explicit. By applying the lattice method to the smooth part ϕ_i we have obtained a numerical procedure that yields fast and accurate results.

However, the research presented in this chapter leaves certain questions unanswered. We will now conclude this section by raising some questions that might be of interest for future research.

- It would be of great value to prove certain modifications of Theorem 4.1. In our application we are perhaps more interested in point-wise estimates of the error rather than estimates in the supremum norm. It seems plausible that the convergence rate for

  $$\left|(U_\tau^{(\zeta,h)} f)(x) - (U_\tau f)(x)\right|$$

  for some fixed x mainly depends on the values of f around some neighbourhood of x. Therefore, it may be possible to derive sharp point-wise bounds for the error in the lattice method without having to assume that the initial value f is in C_0 (for instance).

- It would be of interest to investigate how the derivatives $\tilde{v}_i^{(k)}$ best should be estimated. This problem is also relevant in the estimation of delta and gamma (the first and second derivative of the option price with respect to the underlying asset price).

- Given a positive integer μ, is there a lattice random variable ζ such that ζ is consistent of order μ?

Appendix A

The intention with this appendix is to show how the coefficients $\{\alpha_k\}_{k=0}^d$ in

$$\psi_a(x) = e^{\gamma(x-a)} \sum_{k=0}^{d} \frac{\alpha_k}{k!} (x-a)^k, \quad \gamma, a \in \mathbb{R},$$

can be estimated so that

$$\frac{d^k \psi_a}{dx^k}(a) = \frac{d^k \tilde{v}}{dx^k}(a), \quad k = 0, 1, \ldots, d,$$

where $\tilde{v}$ is a function only known at discrete points, say at $x \in h\mathbb{Z}$ with $h > 0$.

Firstly, the Leibnitz rule implies

$$\frac{d^k \psi_a}{dx^k}(a) = \sum_{j=0}^{k} \binom{k}{j} \gamma^{k-j} \alpha_j, \quad k = 0, 1, \ldots, d.$$

Thus, α_k is defined recursively by

$$\alpha_k = \frac{d^k \tilde{v}}{dx^k}(a) - \sum_{j=0}^{k-1} \binom{k}{j} \gamma^{k-j} \alpha_j, \quad k = 0, 1, \ldots, d.$$

It remains to estimate the derivatives of $\tilde{v}(x)$ at the point $x = a$. A natural approach to this problem is to differentiate an appropriate interpolations polynomial. Suppose that $[r]$ stands for the smallest integer $\geq r$ and $j^* = [a/h]$. In addition, assume $d \leq \delta \in \mathbb{N}$ and let $\theta = [\delta/2]$. If p denotes the (interpolation) polynomial of degree δ which satisfies

$$p\big((j^* + j)h\big) = \tilde{v}\big((j^* + j)h\big), \quad j = -\theta, -\theta + 1, \ldots, -\theta + \delta,$$

then the first d derivatives at the point a can be estimated by

$$\frac{d^k \tilde{v}}{dx^k}(a) \approx \frac{d^k p}{dx^k}(a), \quad k = 0, 1, \ldots, d.$$

In the numerical examples presented in Section 4.4 the above procedure have been used with $\delta = 3$, with exception of Table 4.6 where $\delta = 5$.

Appendix B

Using the following lemma, the functions $U_\tau g_i$, $i = 1, 2, \ldots, m-1$, can be evaluated in an efficient way.

Lemma 4.1. *Suppose* $\varphi(x) = \frac{d}{dx}\Phi(x)$*. If*

$$g = \psi_a\, \chi_{(-\infty,a\,]} + \psi_b\, \chi_{[\,b,\infty)}$$

where ψ_a *and* ψ_b *are defined by*

$$\psi_a(x) = e^{\gamma_a\,(x-a)} \sum_{k=0}^{d} \frac{\alpha_k}{k!}\,(x-a)^k$$

and

$$\psi_b(x) = e^{\gamma_b\,(x-b)} \sum_{k=0}^{d} \frac{\beta_k}{k!}\,(x-b)^k$$

then

$$\begin{aligned}(U_\tau g)(x) = &e^{\gamma_a(x-a)+\gamma_a^2\,\tau/2} \sum_{k=0}^{d} \Big(\frac{\alpha_k^*}{k!}\, M_k\big(\frac{a-x}{\sqrt{\tau}} - \gamma_a\sqrt{\tau}\,\big)\Big) \\ &+ e^{\gamma_b(x-b)+\gamma_b^2\,\tau/2} \sum_{k=0}^{d} \Big(\frac{\beta_k^*}{k!}\,(-1)^k M_k\big(\frac{x-b}{\sqrt{\tau}} + \gamma_b\sqrt{\tau}\,\big)\Big),\end{aligned}$$

where

$$\alpha_k^* = \tau^{k/2} \sum_{i=0}^{d-k} \frac{\alpha_{i+k}}{i!}(x-a+\gamma_a\tau)^i,$$

$$\beta_k^* = \tau^{k/2} \sum_{i=0}^{d-k} \frac{\beta_{i+k}}{i!}(x-b+\gamma_b\tau)^i,$$

and where the functions M_k *are defined recursively by*

$$M_k(y) = \begin{cases} \Phi(y) & \textit{if } k = 0, \\ -\varphi(y) & \textit{if } k = 1, \\ y^{k-1}\, M_1(y) + (k-1)\, M_{k-2}(y) & \textit{if } k = 2, 3, \ldots, d. \end{cases} \tag{4.8}$$

Proof. Fix $x \in \mathbb{R}$. Let α_k^* be defined as above and set $\gamma_a^* = \gamma_a \sqrt{\tau}$. If $\psi_a^*(\xi) = \psi_a(x + \sqrt{\tau}\xi)$ then

$$\psi_a^*(\xi) = e^{\gamma_a(x-a)+\gamma_a^*\xi} \sum_{k=0}^{d} \frac{\alpha_k^*}{k!} (\xi - \gamma_a^*)^k,$$

since, for each $k = 0, 1, \dots, d$,

$$\begin{aligned}\Big(\frac{d^k}{d\xi^k} \sum_{i=0}^{d} \frac{\alpha_i}{i!} (x + \sqrt{\tau}\xi - a)^i \Big)\Big|_{\xi=\gamma_a^*} &= T^{k/2} \sum_{i=0}^{d-k} \frac{\alpha_{i+k}}{i!} (x + \sqrt{\tau}\gamma_a^* - a)^i \\ &= \alpha_k^*.\end{aligned}$$

Let $c_a = (a-x)/\sqrt{\tau}$. The scaling property for Brownian motion and the definition of ψ_a^* give

$$\begin{aligned}\big(U_\tau \left(\psi_a \, \chi_{(-\infty,a]} \right) \big)(x) =& E\big[\psi_a(x + W_\tau) \, \chi_{(-\infty,a]}(x + W_\tau) \big] \\ =& E\big[\psi_a^*(W_1) \, \chi_{(-\infty,c_a]}(W_1) \big] \\ =& e^{\gamma_a(x-a)} \sum_{k=0}^{d} \frac{\alpha_k^*}{k!} \int_{-\infty}^{c_a} e^{\gamma_a^*\xi} \, (\xi - \gamma_a^*)^k \varphi(\xi) d\xi.\end{aligned}$$

Note moreover that

$$\begin{aligned}\int_{-\infty}^{c_a} e^{\gamma_a^*\xi} \, (\xi - \gamma_a^*)^k \varphi(\xi) d\xi &= \int_{-\infty}^{c_a} (\xi - \gamma_a^*)^k \, e^{\gamma_a^{*2}/2} \, \varphi(\xi - \gamma_a^*) d\xi \\ &= e^{\gamma_a^2\tau/2} \int_{-\infty}^{c_a - \gamma_a\sqrt{\tau}} \xi^k \, \varphi(\xi) \, d\xi.\end{aligned}$$

Thus, if

$$M_k(y) = \int_{-\infty}^{y} \xi^k \varphi(\xi) d\xi$$

for each integer $k \geq 0$, then

$$\big(U_\tau \left(\psi_a \, \chi_{(-\infty,a]} \right) \big)(x) = e^{\gamma_a(x-a)+\gamma_a^2\tau/2} \sum_{k=0}^{d} \frac{\alpha_k^*}{k!} M_k(c_a - \gamma_a\sqrt{\tau}).$$

By using a similar argument we get

$$\big(U_\tau \left(\psi_b \, \chi_{[b,\infty)} \right) \big)(x) = e^{\gamma_b(x-b)+\gamma_b^2\tau/2} \sum_{k=0}^{d} \frac{\beta_k^*}{k!} \int_{c_b-\gamma_b\sqrt{\tau}}^{\infty} \xi^k \varphi(\xi) d\xi,$$

where $c_b = (b-x)/\sqrt{\tau}$ and β_k^* is defined as in Lemma 4.1. The symmetry of the normal density yields

$$\big(U_\tau \, (\psi_b \, \chi_{[b,\infty)} \,) \big)(x) = e^{\gamma_b (x-b) + \gamma_b^2 \tau/2} \sum_{k=0}^{d} \frac{\beta_k^*}{k!} (-1)^k \, M_k(\gamma_b \sqrt{\tau} - c_b).$$

It remains to show that the functions M_k, $k = 0, 1, \ldots, d$, satisfy equation (4.8). It is evident that $M_0(y) = \Phi(y)$. Since $\frac{d}{d\xi}\varphi(\xi) = -\xi\varphi(\xi)$ if follows $M_1(y) = -\varphi(y)$. Integration by parts now yields for $k \geq 2$

$$\begin{aligned} M_k(y) &= -\xi^{k-1} \, \varphi(\xi) \Big|_{\xi = y} + (k-1) \int_{-\infty}^{y} \xi^{k-2} \, \varphi(\xi) d\xi \\ &= y^{k-1} \, M_1(y) + (k-1) \, M_{k-2}(y) \, . \end{aligned}$$

□

Chapter 5

Probabilistic Interpretation of the θ-Method

It is well-known that the explicit finite difference method for the heat equation is equivalent to a trinomial tree, see e.g. Heston and Zhou [58]. This chapter studies the θ-method, which is a class of finite difference methods including, for instance, the Crank-Nicolson method, and shows that for some parameter values the θ-method also has a probabilistic interpretation. In particular, the θ-method can for certain parameter values be interpreted as a binomial tree with a independent random time.

5.1 The θ-Method and Its Probabilistic Counterpart

To begin with, consider the initial value problem for the heat equation,

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, & \text{in } (0,T] \times \mathbb{R}, \\ u|_{t=0} = f, & \text{on } \mathbb{R}, \end{cases} \tag{5.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given continuous and bounded function (abbr. $f \in C_b$) and T is a strictly positive constant. There are many various approaches to solve the initial value problem for the heat equation and this section will discuss a finite difference method known as the θ-method.

Let h and λ be positive numbers and set $k = \lambda h^2$. Think of h and k as small increments of the variables x and t, respectively. Moreover, suppose $v_n(x) = u(nk, x)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. The idea behind the θ-method is to approximate the function v_n by $\tilde{v}_n$, where $\tilde{v}_n$ is the solution of the difference equation

$$\begin{cases} \frac{\tilde{v}_{n+1}(x)-\tilde{v}_n(x)}{k} = \frac{1}{2}\Big(\theta \frac{\Delta_h^2 \tilde{v}_{n+1}(x)}{h^2} + (1-\theta)\,\frac{\Delta_h^2 \tilde{v}_n(x)}{h^2}\Big), & x \in \mathbb{R},\ n \in \mathbb{N}, \\ \tilde{v}_0(x) = f(x), & x \in \mathbb{R}. \end{cases} \tag{5.2}$$

Here $0 \le \theta \le 1$ and

$$\Delta_h^2 g(x) = g(x+h) - 2g(x) + g(x-h),$$

for any $g \in C_b$.

Now, introduce for each $t \in k\mathbb{N}_+$ a finite difference operator $V_t^{(\lambda,h)} : C_b \to C_b$, defined for $t = k$ by the operator equation

$$V_k^{(\lambda,h)} - 1 = \frac{1}{2}\lambda\,\Delta_h^2\left(\theta V_k^{(\lambda,h)} + 1 - \theta\right) \tag{5.3}$$

and for $t = nk$, $n \ge 2$, by the semi-group property

$$V_{nk}^{(\lambda,h)} = \left(V_k^{(\lambda,h)}\right)^n.$$

By equation (5.2) it is readily seen that the solution of the θ-method may be written as

$$\tilde{v}_n(x) = \left(V_{nk}^{(\lambda,h)} f\right)(x)$$

for each $n \in \mathbb{N}_+$.

If $\theta = 0$, the operators $V^{(\lambda,h)}$ are called explicit. Each equation of an explicit operator gives the unknown $\tilde{v}_{n+1}(x)$ directly in terms (finitely many) of the known quantities $\tilde{v}_n(x + jh)$. If $0 < \theta \le 1$, one must solve a set of linear equations to obtain $\tilde{v}_{n+1}(x)$, and the operator $V^{(\lambda,h)}$ is called implicit. The important special case $\theta = 1/2$ is often referred to as the Crank-Nicolson method.

The θ-method is not stable for all values of $\lambda > 0$ and $0 \le \theta \le 1$. Stability means that the collection of operators

$$\{V_t^{(\lambda,h)} : 0 < h \le h_0,\ t \in \lambda h^2 \mathbb{N}_+,\ t \le T\},$$

where h_0, T, and λ are fixed positive numbers, is uniformly bounded with respect to the (operator) norm in C_b, i.e. there is a constant C such that

$$\|V_t^{(\lambda,h)}\|_{C_b} \le C$$

uniformly for all $0 < h \leq h_0$ and all $t \in \lambda h^2\mathbb{N}_+$ such that $t \leq T$. Here $\|f\|_{C_b} = \sup_{x\in\mathbb{R}} |f(x)|$. Stability is a necessary and sufficient condition for uniform convergence in connection with the θ-method. It can be shown that the θ-method is stable if and only if

$$\begin{aligned} &\lambda \leq \frac{1}{1-2\theta}, && \text{if } 0 \leq \theta < 1/2, \\ &\text{no restriction}, && \text{if } 1/2 \leq \theta \leq 1. \end{aligned}$$

For a further discussion about the θ-method and other finite difference methods, see Atkinson and Weimin [5] or Richtmyer and Morton [103].

In certain cases it is possible to give a probabilistic interpretation of the θ-method. To see this, assume that $U_t^{(\zeta,h)}$ is defined as in Chapter 4. That is, let ζ be a lattice random variable with span 1, expectation zero, and variance λ. Furthermore, suppose $\zeta_1, \ldots, \zeta_n$ are independent stochastic copies of ζ and set

$$(U_t^{(\zeta,h)}f)(x) = E\big[f(x + h\sum_{j=1}^{n}\zeta_j)\big],$$

with $t \in k\mathbb{N}_+$ and $n = t/k$. We recall that $k = \lambda h^2$. The operator $U_t^{(\zeta,h)}$ is called a *lattice method* or a *lattice tree*. In particular, if ε is a Rademacher random variable, i.e. $P(\varepsilon = -1) = P(\varepsilon = 1) = 1/2$, then $U_t^{(\varepsilon,h)}$ is known as a *binomial tree*. The aim in this section is to prove that for some values of θ and λ there is a ζ such that

$$U_t^{(\zeta,h)}f = V_t^{(\lambda,h)}f \tag{5.4}$$

for all $f \in C_b$, $h > 0$, and all $t \in k\mathbb{N}_+$ (abbr. $U^{(\zeta,\cdot)} = V^{(\lambda,\cdot)}$).

To prove that $U^{(\zeta,\cdot)} = V^{(\lambda,\cdot)}$, note that it is sufficient to assume that $t = k$ since

$$U_{nk}^{(\zeta,h)} = \big(U_k^{(\zeta,h)}\big)^n.$$

Moreover, if $\theta = 0$ and $\lambda \leq 1$ then it is obvious that equation (5.4) follows by defining ζ as

$$P(\zeta = 0) = 1 - \lambda \quad \text{and} \quad P(\zeta = -1) = P(\zeta = 1) = \frac{\lambda}{2}.$$

Alternatively, $\zeta = \sum_{j=1}^{N}\varepsilon_j$ where the random variable N is independent of ε_1 with

$$P(N = 0) = 1 - \lambda \quad \text{and} \quad P(N = 1) = \frac{\lambda}{2}.$$

We adopt the convention that $\sum_{j=1}^{0} = 0$.

Next we will consider the case $0 < \theta \leq 1$. An approximation argument gives that it is only necessary to show that equation (5.4) holds for all $f \in C_b$ such that f is integrable with respect to the Lebesgue measure. Let $\hat{v}$ be the Fourier transform of the function $x \mapsto (V_k^{(\lambda,h)} f)(x)$, that is

$$\hat{v}(\xi) = \int_{-\infty}^{\infty} (V_k^{(\lambda,h)} f)(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{R}.$$

Moreover, if g is an integrable function then $\hat{g}$ will denote the Fourier transform of g. The function $\Delta_h^2 g$ possesses the Fourier transform $2\big(\cos(h\xi) - 1\big)\hat{g}(\xi)$ and thus, according to equation (5.3),

$$\hat{v}(\xi) - \hat{f}(\xi) = \lambda\big(\cos(h\xi) - 1\big)\big(\theta\hat{v}(\xi) + (1-\theta)\hat{f}(\xi)\big).$$

This may equivalently be written as

$$\big(1 + \lambda\theta(1 - \cos(h\xi))\big)\, \hat{v}(\xi) = \big(1 - \lambda(1-\theta)(1 - \cos(h\xi))\big)\, \hat{f}(\xi)$$

and therefore

$$\int_{-\infty}^{\infty} (V_k^{(\lambda,h)} f)(x) e^{-i\xi x} dx = \phi(h\xi)\hat{f}(\xi),$$

where

$$\phi(\xi) = \frac{1 - \lambda(1-\theta)(1 - \cos\xi)}{1 + \lambda\theta(1 - \cos\xi)}.$$

The function ϕ is often referred to as the symbol or the characteristic polynomial of the θ-method.

The Fourier transform of $x \mapsto (U_k^{(\zeta,h)} f)(x)$ equals

$$\int_{-\infty}^{\infty} (U_k^{(\zeta,h)} f)(x) e^{-i\xi x} dx = E\big[e^{ih\xi\zeta}\big]\hat{f}(\xi).$$

Hence, $U^{(\zeta,\cdot)} = V^{(\lambda,\cdot)}$ if and only if ϕ is a characteristic function of a lattice random variable with span 1 and variance λ. To examine under what condition this is true, put $\alpha = \lambda\theta/(1+\lambda\theta)$ and note that

$$\phi(\xi) = \frac{1 - \lambda(1-\theta)(1 - \cos\xi)}{(1+\lambda\theta)(1 - \alpha\cos\xi)}.$$

A Taylor expansion now yields for each $\xi \in \mathbb{R}$,

$$\begin{aligned}\phi(\xi) &= \frac{1-\lambda(1-\theta)(1-\cos\xi)}{1+\lambda\theta}\sum_{j=0}^{\infty}\alpha^j\cos^j\xi \\ &= \sum_{j=0}^{\infty} b_j \cos^j \xi\end{aligned} \tag{5.5}$$

where

$$b_j = \begin{cases} p, & \text{if } j = 0, \\ (1-p)\,q(1-q)^{j-1}, & \text{if } j \geq 1, \end{cases}$$

with

$$p = 1 - \frac{\lambda}{1+\lambda\theta} \quad \text{and} \quad q = \frac{1}{1+\lambda\theta}.$$

If λ and θ is chosen such that $p \geq 0$, then $b_j \geq 0$ for all j. Moreover, since $\phi(0) = 1$ we have $\sum_{j=0}^{\infty} b_j = 1$. Recall that $\xi \mapsto \cos\xi$ is the characteristic function of a Rademacher random variable ε. Consequently, if $p \geq 0$ then

$$\phi(\xi) = E\big[e^{i\xi\sum_{j=1}^{N}\varepsilon_j}\big],$$

where $\{\varepsilon_j\}_{j=1}^{\infty}$ is a sequence of independent stochastic copies of ε and N is a random variable independent of $\{\varepsilon_j\}_{j=1}^{\infty}$ with $P(N = j) = b_j$, $j \geq 0$. In particular, ϕ is a characteristic function of a lattice random variable with span 1. In addition, since $\phi'(0) = 0$ and $\phi''(0) = -\lambda$ the random variable $\sum_{j=1}^{N}\varepsilon_j$ has expectation zero and variance λ.

Theorem 5.1. *Suppose $0 \leq \theta \leq 1$. There exists a random variable N, independent of $\{\varepsilon_j\}_{j=1}^{\infty}$, such that the random variable*

$$\zeta = \sum_{j=1}^{N}\varepsilon_j \tag{5.6}$$

satisfies

$$U^{(\zeta,\cdot)} = V^{(\lambda,\cdot)}$$

if and only if $\lambda > 0$ is chosen such that

$$\begin{aligned} &\lambda \leq \frac{1}{1-\theta}, && \text{if } 0 \leq \theta < 1, \\ &\textit{no restriction}, && \text{if } \theta = 1. \end{aligned}$$

Proof. We have already proven that the conditions on λ are sufficient but it remains one additional argument to prove that the conditions are necessary. To be more precise, we need to show that the only choice of parameters β_j, $j \in \mathbb{N}$, satisfying

$$\phi(\xi) = \sum_{j=0}^{\infty} \beta_j \cos^j \xi, \quad \text{for all } \xi \in \mathbb{R},$$

are $\beta_j = b_j$, $j \in \mathbb{N}$. By substituting $x = \cos\xi$ this follows immediately from the theory of analytic functions. □

Binomial trees with independent random times have been studied previously in the financial literature, see Rogers and Stapleton [106].

This chapter is concluded by stating necessary and sufficient conditions for the existence of a lattice random variable ζ such that $V^{(\lambda,\cdot)} = U^{(\zeta,\cdot)}$. We have already shown that if $\theta = 0$, then $\lambda \leq 1$ is equivalent to $U^{(\zeta,\cdot)} = V^{(\lambda,\cdot)}$ for some ζ.

In what follows, suppose $0 < \theta \leq 1$. A Fourier expansion of ϕ yields

$$\phi(\xi) = \frac{1}{2\pi} c_0 + \frac{1}{\pi} \sum_{n=1}^{\infty} c_n \cos n\xi,$$

where

$$c_n = \int_0^{2\pi} \phi(\xi) \cos n\xi \, d\xi. \tag{5.7}$$

Recall that $\phi(0) = 1$, and thus, the function ϕ is a characteristic function of a lattice random variable ζ if and only if $c_n \geq 0$ for all $n \geq 0$ and, in that case, ζ is symmetric with $P(\zeta = n) = c_n/2\pi$, $n \geq 0$.

We claim that $c_n \geq 0$ for all $n \geq 1$ and each $\lambda > 0$, $0 < \theta \leq 1$. In fact, equation (5.5) yields

$$c_n = \sum_{m=0}^{\infty} b_m \int_0^{2\pi} \cos^m \xi \cos n\xi \, d\xi.$$

In addition,

$$\begin{aligned} \cos^m \xi &= E\left[e^{i\xi \sum_{l=1}^m \varepsilon_l}\right] \\ &= \sum_{l=0}^{m} a_{l,m} \cos l\xi, \end{aligned}$$

where $a_{0,m} = P(\sum_{j=0}^m \varepsilon_j = 0)$ and $a_{l,m} = 2P(\sum_{j=0}^m \varepsilon_j = l)$ for $1 \leq l \leq m$. Thus, by orthogonality follows for all $n \geq 1$

$$
\begin{aligned}
c_n =& \sum_{m=0}^{\infty} \sum_{l=0}^{m} b_m a_{l,m} \int_0^{2\pi} \cos l\xi \cos n\xi \, d\xi \\
=& \pi \sum_{m=n}^{\infty} b_m a_{n,m}.
\end{aligned}
$$

Recall that $b_m \geq 0$ for all $m \geq 1$ and hence, it is enough to find conditions on λ and θ such that $c_0 \geq 0$.

By substituting $x = \tan \xi/2$ in equation (5.7) follows

$$c_0 = \int_{-\infty}^{\infty} \frac{1 + (1 - 2\lambda + 2\lambda\theta)\, x^2}{1 + (1 + 2\lambda\theta)\, x^2} \frac{2dx}{1 + x^2},$$

which, after some elementary calculus, turns out to be

$$c_0 = \frac{2\pi}{\theta} \left(\theta - 1 + \sqrt{\frac{1}{1 + 2\theta\lambda}} \right).$$

To sum up, we have shown

Theorem 5.2. *Suppose $0 \leq \theta \leq 1$. There exists a lattice random variable ζ such that*

$$U^{(\zeta,\cdot)} = V^{(\lambda,\cdot)}$$

if and only if $\lambda > 0$ is chosen such that

$$
\begin{aligned}
&\lambda \leq \frac{1}{1-\theta} \frac{2-\theta}{2(1-\theta)}, && \text{if } 0 \leq \theta < 1, \\
&\textit{no restriction}, && \text{if } \theta = 1.
\end{aligned}
$$

It is obvious that the collection $\{U_t^{(\zeta,h)} : 0 < h \leq h_0,\, t \in \lambda h^2 \mathbb{N}_+,\, t \leq T\}$ is stable, to be more precise, for any ζ, $h > 0$, and $t \in \lambda h^2 \mathbb{N}_+$ we have

$$\|U_t^{(\zeta,h)}\|_{C_b} \leq 1.$$

Thus, the θ-method provide us with an example showing that the class of all stable finite difference methods is strictly larger than the class of all lattice methods. In addition, it shows that the class of all lattice methods with symmetrical lattice random variables is strictly larger than the class of all binomial trees with independent random times.

Chapter 6

On the Error in the Monte Carlo Pricing of Some Familiar Path-Dependent Options

The application of the Monte Carlo method to option pricing was first presented in Boyle [23] and it has proved to be an extremely useful tool for the valuation of contingent claims. The popularity of the Monte Carlo method in finance depends mainly on the fact that it provides a robust and simple method for performing integration. For example, the convergence rate of the Monte Carlo method is of order $O(n^{-1/2})$, where n is the number of simulations, independently of the dimension of the integral. The Monte Carlo method is therefore in some cases the only viable method for a large number of high-dimensional problems in finance.

Previous work on the Monte Carlo pricing primarily focuses on different so called variance reduction techniques and on methods to price American contracts. Kemna and Vorst [73] consider the technique of control variates in the pricing of Asian options. Barraquand [10] exploits the idea of quadratic sampling. Glasserman, Heidelberger, and Shahabuddin [52] study importance sampling and stratification for the pricing of path-dependent options. Monte Carlo techniques to price American options is discussed in Rogers [104]. Fournié et al. [48], [49] make use of Malliavin calculus to improve the performance of the Monte Carlo pricing. These articles are just a small part of the research about the Monte Carlo method in option pricing, for a more complete discussion about the Monte Carlo method in option pricing, see Boyle, Broadie, and Glasserman [25] or Lai and Spanier [79].

The main purpose of this chapter is to derive error estimates for the crude Monte Carlo pricing of European options, with particular emphasis on path-dependent options. We must underline that we will only consider the *crude* Monte Carlo technique. A discussion about the error in the so called quasi Monte Carlo method can be found in e.g. Caflish [31].

This chapter presents generalisations of some previous results by Borell in [21], where, among other things, he investigates the relative error in the Monte Carlo pricing of simple European options in a multi-asset Black-Scholes market. A simple option is an option that only depends on the underlying asset prices at the maturity date of the option. In particular, Borell has shown that for some simple options not only the convergence rate but also the *constant* in $O(n^{-1/2})$ is independent of the dimension. To be more specific, the constant is only dependent on the highest volatility amongst the underlying assets, time to maturity, and degree of confidence interval. This chapter will show various extensions of Borell's result for a large number of European styled path-dependent contracts.

The structure of this chapter is as follows. Section 6.1 gives some general results about the error in the Monte Carlo method. The main tool in this section is the Rosenthal and the Hoeffding inequalities. Section 6.2 compares the moments between path-dependent call (put) options and plain vanilla call (put) options in the Black-Scholes multi-asset market. The results in this section are based on some geometric inequalities in Wiener space, and then, in particular, the isoperimetric inequality for Wiener measure. The final section, Section 6.3, combines the results in the preceding sections and gives an upper bound for the error in the Monte Carlo method for an important class of path-dependent options.

6.1 Error Estimates for the Monte Carlo Method

From now in this section, assume that $(\Omega, \mathcal{F}, P)$ is a given probability space and X is a random variable in $L^p(\Omega, \mathcal{F}, P)$ with $p \geq 2$. In addition, let $X_1, \ldots, X_n$ be stochastically independent observations on X and set

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Here $\bar{X}_n$ is called the Monte Carlo estimator of $\alpha = E[X]$. In what follows we abbreviate

$$\| \cdot \|_p = \| \cdot \|_{L^p(P)},$$

if $2 \leq p < \infty$, and

$$l(X) = \operatorname{ess\,sup} X - \operatorname{ess\,inf} X.$$

Finally, if $\alpha \neq 0$ put

$$D_p(X) = \frac{\|X - \alpha\|_p}{|\alpha|}, \quad 2 \leq p < \infty,$$

and

$$D_\infty(X) = \frac{l(X - \alpha)}{|\alpha|} = \frac{l(X)}{|\alpha|}.$$

If R_n denotes the relative error after n simulations, i.e.

$$R_n = \left| \frac{\bar{X}_n - \alpha}{\alpha} \right|, \quad \alpha \neq 0,$$

the Chebychev inequality gives, for any $0 < \epsilon < 1$,

$$P\left(R_n \leq \frac{C_\epsilon(X)}{\sqrt{n}} \right) \geq 1 - \epsilon, \tag{6.1}$$

where

$$C_\epsilon(X) = \frac{D_2(X)}{\sqrt{\epsilon}}. \tag{6.2}$$

Thus, equation (6.1) yields that the convergence rate of the relative error in the Monte Carlo estimation is, with probability $1 - \epsilon$, of order $O(n^{-1/2})$ with a error constant that is bounded by $C_\epsilon(X)$.

By applying the central limit theorem it is easily seen that the convergence rate $O(n^{-1/2})$ is the best possible in the sense that if $\delta_n \to 0^+$ as $n \to \infty$ then

$$P\left(R_n \leq \frac{\delta_n}{n^{1/2}} \right) \to 0, \quad \text{as } n \to \infty.$$

However, in certain cases it may be possible to improve the constant $C_\epsilon(X)$ in equation (6.2). The purpose of the remaining part of this section is to find better estimates of the error constant in the Monte Carlo method.

First we recall a heuristic and well known argument which shows that it is plausible to improve the constant $C_\epsilon(X)$, provided n is sufficiently large as will be the case in most Monte Carlo simulations. In fact, if

$$\xi_i = \frac{X_i - \alpha}{\|X_i - \alpha\|_2}, \quad i = 1, 2, \dots, \tag{6.3}$$

the central limit theorem gives that the random variable

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i$$

is approximately distributed as a normal random variable with mean 0 and variance 1. Thus, for any $\lambda > 0$,

$$\begin{aligned} P\Big(R_n \le \lambda\Big) &= P\left(\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i\Big| \le \frac{\lambda\sqrt{n}}{D_2(X)}\right) \\ &\approx 2\Phi\left(\frac{\lambda\sqrt{n}}{D_2(X)}\right) - 1, \end{aligned} \tag{6.4}$$

where Φ denote the standard normal distribution function.

Suppose Φ^{-1} is the inverse of Φ. By setting

$$\lambda = \frac{D_2(X)}{\sqrt{n}}\Phi^{-1}\Big(1 - \frac{\epsilon}{2}\Big), \quad 0 < \epsilon < 1,$$

in equation (6.4), it follows

$$P\left(R_n \le \frac{\tilde{C}_\epsilon(X)}{\sqrt{n}}\right) \approx 1 - \epsilon,$$

where

$$\tilde{C}_\epsilon(X) = \sqrt{\epsilon}\,\Phi^{-1}\Big(1 - \frac{\epsilon}{2}\Big)C_\epsilon(X)$$

with $C_\epsilon(X)$ defined as in equation (6.2). If ϵ is close to zero then it is evident that the constant $\tilde{C}_\epsilon(X)$ will be considerably much smaller than $C_\epsilon(X)$.

In order to make the above argument more precise we will recall some classical inequalities for random walks. The next theorem is known as the Rosenthal inequality.

Theorem 6.1. *Let ζ and ζ' denote independent Poisson random variables with parameter $1/2$ and let Γ be the gamma function. Suppose $X \in L^p(P)$, $2 \le p < \infty$, is a symmetric random variable. Then*

$$\|\sum_{i=1}^{n} X_i\|_p \le N_p \max\Big(\sqrt{n}\|X\|_2,\, n^{1/p}\|X\|_p\Big),$$

where

$$N_p = \begin{cases} \left(1 + \sqrt{\frac{2^p}{\pi}}\,\Gamma\left(\frac{p+1}{2}\right)\right)^{\frac{1}{p}}, & 2 < p < 4, \\ \|\zeta - \zeta'\|_p, & p \geq 4. \end{cases}$$

A proof of Theorem 6.1 with $N_p = 2^p$ can be found in Rosenthal [108]. The value of the constant N_p given in Theorem 6.1 is the best possible. For $p \geq 4$ the optimal constant was found by Utev [120] and for $2 < p < 4$ it was derived independently by Figiel et al. [47] and Ibragimov and Sharakhmetov [66]. Table 6.1 below shows an upper bound for the value of the constant N_p for various values on p. The table will be useful in the sequel.

Table 6.1: An upper bound for the value of the constant N_p in the Rosenthal inequality.

p	3	4	5	6	7	8	9	10
N_p ($\leq$)	1.37	1.42	1.60	1.78	1.95	2.11	2.26	2.41

By a standard result in probability theory, if X and X' are i.i.d. random variables in $L^p(P)$ with $E[X] = 0$ then $\|X\|_p \leq \|X - X'\|_p$ (cf. p. 263 in Loeve [89]). Since the random variable $X - X'$ is symmetric the Rosenthal inequality implies

Corollary 6.1. *If $X \in L^p(P)$, $2 \leq p < \infty$, and $\alpha = E[X]$ then*

$$\|\sum_{i=1}^{n}(X_i - \alpha)\|_p \leq 2N_p \max\left(\sqrt{n}\|X - \alpha\|_2,\, n^{1/p}\|X - \alpha\|_p\right),$$

where N_p is defined as in Theorem 6.1.

The next theorem is often referred to as the Hoeffding inequality.

Theorem 6.2. *If X is bounded and $\alpha = E[X]$ then*

$$P\left(\left|\sum_{i=1}^{n}(X_i - \alpha)\right| \geq n\lambda\right) \leq 2\exp\left(-\frac{2n\lambda^2}{l(X)^2}\right)$$

for every $\lambda > 0$.

For a proof of Theorem 6.2, see Hoeffding [60]. The bound in Theorem 6.2 is not the best possible, see Talagrand [118] for a further discussion.

More results on tail probabilities and moment estimations for sums of independent random variables can be found in Petrov [99].

We can now formulate the main result in this section.

Theorem 6.3. *Suppose $X \in L^p(P)$, $2 \le p \le \infty$, and $\alpha = E[X] \neq 0$. Moreover, assume that the constant N_p is defined as in Theorem 6.1. If R_n denotes the relative error after n simulations, i.e.*

$$R_n = \left| \frac{\bar{X}_n - \alpha}{\alpha} \right|,$$

then

$$P\left(R_n \le \frac{C_\epsilon^*(X)}{\sqrt{n}} \right) \ge 1 - \epsilon, \quad 0 < \epsilon < 1,$$

if $C_\epsilon^*(X)$ is any of the numbers

$$C_\epsilon(X) = \frac{1}{\sqrt{\epsilon}} D_2(X),$$

$$C_\epsilon^{(r)}(X) = \frac{2N_p}{\epsilon^{1/p}} \max\left(D_2(X),\, n^{\frac{1}{p}-\frac{1}{2}} D_p(X) \right), \quad 2 < p < \infty,$$

or

$$C_\epsilon^{(h)}(X) = \sqrt{\frac{\ln(2/\epsilon)}{2}} D_\infty(X).$$

Proof. The special case $C_\epsilon^*(X) = C_\epsilon(X)$ has already been shown. To prove the remaining part of Theorem 6.3, let ξ_i be defined as in equation (6.3) and observe that

$$R_n = D_2(X) \left| \frac{1}{n} \sum_{i=1}^n \xi_i \right|.$$

First we will prove the case

$$C_\epsilon^*(X) = C_\epsilon^{(r)}(X).$$

Suppose $\lambda > 0$, the Chebychev inequality yields

$$P\left(\left| \frac{1}{n} \sum_{i=1}^n \xi_i \right| \ge \lambda \right) \le \frac{\| \sum_{i=1}^n \xi_i \|_p^p}{\lambda^p n^p}$$

which in combination with Corollary 6.1 imply

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i\right| \geq \lambda\right) \leq \frac{(2N_p)^p}{\lambda^p n^{p/2}} \max\left(1,\, n^{\frac{1}{p}-\frac{1}{2}}\|\xi_i\|_p\right)^p.$$

Note that $\|\xi_i\|_p = D_p(X)/D_2(X)$. Set

$$\lambda = \frac{2N_p}{\sqrt{n}\epsilon^{1/p}} \max\left(1,\, n^{\frac{1}{p}-\frac{1}{2}}\frac{D_p(X)}{D_2(X)}\right)$$

and we are done.

Next we will consider the case

$$C_\epsilon^*(X) = C_\epsilon^{(h)}(X).$$

Theorem 6.2 gives

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i\right| \geq \lambda\right) \leq 2\exp\left(-\frac{2n\lambda^2}{l(\xi_i)^2}\right)$$

for any $\lambda > 0$. In particular, if we choose

$$\lambda = \sqrt{\frac{\ln(2/\epsilon)}{2n}}\, l(\xi_i),$$

the proof is complete. □

Observe that $C_\epsilon^{(r)}(X)$ is dependent on the number of simulations n. However, if n is sufficiently large then

$$\max\left(D_2(X),\, n^{\frac{1}{p}-\frac{1}{2}}\, D_p(X)\right) = D_2(X)$$

and thus, if n is sufficiently large then the constant $C_\epsilon^{(r)}(X)$ is independent of n.

Section 6.3 will compare the values of the constants $C_\epsilon(X)$ and $C_\epsilon^{(r)}(X)$ in some special cases.

6.2 Comparison of Moments

This section will compare the moments between some familiar path-dependent contracts and plain call or put options.

From now on the sample space $\Omega = C_0([0,T];\mathbb{R}^m)$ consists of all functions $\omega = (\omega_1, \omega_2, \dots, \omega_m)$ such that, for each $i = 1, \dots, m$, the function $\omega_i : [0,T] \to \mathbb{R}$ is continuous and $\omega_i(0) = 0$. The space Ω is equipped with the norm $\|\cdot\|_{C_0}$, defined by

$$\|\omega\|_{C_0} = \max_{i=1,\dots,m} \max_{0 \le t \le T} |\omega_i(t)|.$$

The probability measure will henceforth be denoted Q, where Q is the Wiener measure on Ω. Defining

$$W_t^Q(\omega) = \omega(t), \quad 0 \le t \le T, \ \omega \in \Omega,$$

the process $\{W_t^Q\}_{0 \le t \le T}$ is a standard m-dimensional Brownian motion with respect to Q. A vector in $\mathbb{R}^m$ is interpreted as a column vector.

If $x \in \mathbb{R}^m$ then x_i will denote the i:th coordinate of x. Furthermore, we adopt the conventions that

$$e^x = (e^{x_1}, e^{x_2}, \dots, e^{x_m}), \quad x \in \mathbb{R}^m,$$

$$xy = (x_1 y_1, x_2 y_2, \dots, x_m y_m), \quad x, y \in \mathbb{R}^m,$$

$$(x\omega)(t) = x\omega(t), \quad x \in \mathbb{R}^m, \ \omega \in \Omega, \ 0 \le t \le T,$$

and

$$(e^\omega)(t) = e^{\omega(t)}, \quad \omega \in \Omega, \ 0 \le t \le T.$$

Next we will define a certain class $\mathcal{L}$ of functionals Ψ on the space $C([0,T];\,(0,\infty)^m)$. We will say that $\Psi \in \mathcal{L}$ if $\Psi > 0$ and for any fixed $x \in (0,\infty)^m$ the map

$$\omega \mapsto \ln \Psi(xe^\omega)$$

is Ω-Lipschitz with constant 1, i.e.

$$|\ln \Psi(xe^{\omega+\tilde{\omega}}) - \ln \Psi(xe^\omega)| \le \|\tilde{\omega}\|_{C_0}$$

for all $\omega, \tilde{\omega} \in \Omega$. Clearly, this definition equivalently means that $\Psi > 0$ and

$$\frac{\Psi(xe^{\omega+\tilde{\omega}})}{\Psi(xe^\omega)} \le e^{\|\tilde{\omega}\|_{C_0}}$$

for all $\omega, \tilde{\omega} \in \Omega$.

In addition, suppose K is a positive constant and set

$$\mathcal{C}_K = \max(\mathcal{L} - K, 0),$$

that is, $\Psi_K \in \mathcal{C}_K$ if and only if there exists a $\Psi \in \mathcal{L}$ such that $\Psi_K = \max(\Psi - K, 0)$. In particular, $\mathcal{C}_0 = \mathcal{L}$ and if $K \leq L$ then $\mathcal{C}_K \subseteq \mathcal{C}_L$, since the class $\mathcal{L}$ is closed under addition of a positive constant.

We next give some examples of functionals in the classes $\mathcal{C}_K$. Assume that μ is a positive and bounded Borel measure on $[0, T]$. It is evident that the functional

$$\Psi^\mu(xe^\omega) = \int_0^T \sum_{i=1}^m x_i e^{\omega_i(t)} \mu(dt)$$

belongs to the class $\mathcal{L}$. Thus, for any fixed $K \geq 0$, if

$$\Psi_K^\mu(xe^\omega) = \max(\,\Psi^\mu(xe^\omega) - K,\, 0)$$

then $\Psi_K \in \mathcal{C}_K$. It should be emphasised that the measure μ must be positive, otherwise Ψ^μ will not be a member of the class $\mathcal{L}$.

If $M \subseteq [0, T]$ then the functional

$$\Psi^M(xe^\omega) = \sup_{t \in M} \sum_{i=1}^m x_i e^{\omega_i(t)}$$

is included in $\mathcal{L}$. Hence, for fixed $K \geq 0$,

$$\Psi_K^M(xe^\omega) = \max(\,\Psi^M(xe^\omega) - K,\, 0)$$

is a member of $\mathcal{C}_K$.

Other examples of functionals in the class $\mathcal{C}_K$ can be constructed by taking the maximum or minimum of members of $\mathcal{C}_K$. To see this, if $\Psi, \Upsilon \in \mathcal{L}$ then it is evident that

$$\max(\Psi, \Upsilon) \in \mathcal{L} \quad \text{and} \quad \min(\Psi, \Upsilon) \in \mathcal{L}.$$

Thus, if $K \leq L$ and $\Psi_K \in \mathcal{C}_K$, $\Upsilon_L \in \mathcal{C}_L$ then

$$\max(\Psi_K, \Upsilon_L) \in \mathcal{C}_L \quad \text{and} \quad \min(\Psi_K, \Upsilon_L) \in \mathcal{C}_L,$$

since $\mathcal{C}_K \subseteq \mathcal{C}_L$.

Assume in the remaining part of this chapter that the dynamics of the underlying asset price *vector* S is given by

$$dS_t = S_t\,(\,\eta dt + \sigma C dW_t^Q\,), \quad 0 \le t \le T,$$

where C is a non-singular m by m matrix such that each row c_i in C satisfies $|c_i|_2 = 1$, where $|\cdot|_2$ is the Euclidean norm in $\mathbb{R}^m$, and where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m) \in (0,\infty)^m$. Moreover, $\eta \in \mathbb{R}^m$ has coordinates $\eta_i = r - q_i$, $i = 1, \dots, m$, where r and q_i denote the interest rate and the dividend yield of the i:th asset, respectively. Finally, assume that the initial asset prices are positive constants, that is, $S_0 = (S_0^1, \dots, S_0^m) \in (0,\infty)^m$. Finally, suppose that

$$\max_{i=1,\dots,m} \sigma_i = \sigma_m,$$

Note in particular that $t \mapsto S_t(\omega)$ is a member of $C(\,[0,T]\,;\,(0,\infty)^m\,)$ and that $\omega \mapsto \Psi_K(S(\omega))$, $\Psi_K \in \mathcal{C}_K$, is Borel measurable since Ψ_K is continuous. Furthermore, $\Psi_K^\mu(S)$ and $\Psi_K^M(S)$, where Ψ_K^μ and Ψ_K^M are defined as previous, represent the payoff of an Asian basket call option and a call on the maximum of a basket, respectively.

Our main result about path-dependent call options is

Theorem 6.4. *If* $\Psi_K \in \mathcal{C}_K$, $K \ge 0$, *and* κ *is a constant such that*

$$\|\Psi_K(S)\|_1 = \|\max(\kappa S_T^m - K, 0)\|_1,$$

then

$$\|\Psi_K(S)\|_p \le \|\max(\kappa S_T^m - K, 0)\|_p$$

for each $1 \le p < \infty$.

In the special case that the functional Φ_K is merely dependent on the value of $S_T(\omega)$, the result in Theorem 6.4 has already been shown in Borell [21].

The proof of Theorem 6.4 is based on the isoperimetric inequality for Wiener measure. To present this inequality, let $\mathcal{H}$ denote the Cameron-Martin space. Here $\mathcal{H}$ consists of all functions $h = (h_1, h_2, \dots, h_m)$ such that, for each $i = 1, \dots, m$, the function $h_i : [0,T] \to \mathbb{R}$ is absolutely continuous with a square integrable derivative and $h_i(0) = 0$. The space $\mathcal{H}$ is equipped with the norm $\|\cdot\|_{\mathcal{H}}$, defined by

$$\|h\|_{\mathcal{H}} = \Big(\sum_{i=1}^m \int_0^T \big(h_i'(t)\big)^2\,dt\Big)^{\frac{1}{2}}, \quad h \in \mathcal{H}.$$

We can now formulate the isoperimetric inequality for Wiener measure.

Theorem 6.5. *Let O be the set of all $h \in \mathcal{H}$ such that $\|h\|_{\mathcal{H}} \leq 1$. If A is a Borel set in Ω and*

$$Q(A) = \Phi(a),$$

then

$$Q(A + \lambda O) \geq \Phi(a + \lambda)$$

for each real number $\lambda \geq 0$.

Theorem 6.5 is a special case of the celebrated isoperimetric inequality for Gaussian measures, which is due independently to Borell [20] and Sudakov and Tsirelson [116]. In both papers the proof was based on the isoperimetric inequality on the sphere. Ledoux [82] has developed a short and self contained proof based on the Ornstein-Uhlenbeck semigroup. The Gaussian isoperimetric inequality can also be established using stochastic calculus, see Barthe and Maurey [11]. It should be mentioned that the results in Ledoux [82] and Barthe et al. [11] is restricted to finite-dimensional version of Theorem 6.5, that is, the Wiener measure in Theorem 6.5 is replaced by standard Gauss measure in $\mathbb{R}^n$, A by a Borel set in $\mathbb{R}^n$, and O by the Euclidean unit ball in $\mathbb{R}^n$. However, the finite-dimensional version can quite easily be generalised to Theorem 6.5, see Borell [20] or Ledoux [81], pp. 205-209, for details.

By Theorem 6.5 follows

Corollary 6.2. *Assume that $\Psi \in \mathcal{L}$. If, for $a, b > 0$,*

$$Q\big(\,\Psi(S) > b\,\big) = Q\big(\,S_T^m > a\,\big),$$

then

$$Q\big(\,\Psi(S) > \theta b\,\big) \leq Q\big(\,S_T^m > \theta a\,\big)$$

for each real number $\theta \geq 1$.

Proof. Fix $\theta \geq 1$ and let $\lambda \geq 0$ be given by the equation $\theta = \exp(\sigma_m \sqrt{T}\lambda)$. We first prove that

$$Q\big(\,\Psi(S) \leq \theta b\,\big) \geq Q\big(\inf_{\|h\|_{\mathcal{H}} \leq 1} \Psi\big(S(\cdot + \lambda h)\big) \leq b\,\big). \tag{6.5}$$

Firstly, note that the random variable $\inf_{\|h\|_{\mathcal{H}} \leq 1} \Psi\big(S(\cdot + \lambda h)\big)$ is Borel measurable since $\Psi(S(\cdot))$ is continuous and O is a compact subset of Ω, and thus, the infimum can be taken over a dense denumerable subset of O. Now, suppose that $h = (h_1, \dots, h_m) \in \mathcal{H}$ and let $\langle\, , \rangle$ denote the inner product in $\mathbb{R}^m$. Since

$$h(t) = \int_0^t h'(u)du, \quad 0 \le t \le T,$$

the Cauchy-Schwarz inequality gives for all $0 \le t \le T$ and all $j = 1, \dots, m$,

$$\begin{aligned}
\langle c_j, h(t) \rangle &\le \Big(\sum_{i=1}^m \Big(\int_0^t |h_i'(u)| du \Big)^2 \Big)^{\frac{1}{2}} \\
&\le \sqrt{t} \Big(\sum_{i=1}^m \int_0^t \big(h_i'(u)\big)^2 du \Big)^{\frac{1}{2}} \\
&= \sqrt{t} \|h\|_{\mathcal{H}}
\end{aligned}$$

and thus $\|\sigma C h\|_{C_0} \le \sigma_m \sqrt{T} \|h\|_{\mathcal{H}}$. Next, observe that if $\omega \in \Omega$ then

$$\begin{aligned}
\Psi\big(S(\omega)\big) &= \Psi\big(S(\omega + h) e^{-\sigma C h}\big) \\
&\le e^{\|\sigma C h\|_{C_0}} \Psi\big(S(\omega + h)\big) \\
&\le e^{\sigma_m \sqrt{T} \|h\|_{\mathcal{H}}} \Psi\big(S(\omega + h)\big)
\end{aligned}$$

and, consequently,

$$\inf_{\|h\|_{\mathcal{H}} \le 1} \Psi\big(S(\omega + \lambda h)\big) \ge \frac{1}{\theta} \Psi\big(S(\omega)\big),$$

which proves equation (6.5).

Suppose that a and b are picked as in Corollary 6.2. Since

$$Q(\, S_T^m \le s) = \Phi\Big(\frac{\ln(s/S_0^m) - (r - q_m - \sigma_m^2/2)T}{\sigma_m \sqrt{T}} \Big)$$

for any $s > 0$, Theorem 6.5 implies

$$Q\big(\inf_{\|h\|_H \le 1} \Psi\big(S(\cdot + \lambda h)\big) \le b \,\big) \ge Q(\, e^{-\sigma_m \sqrt{T} \lambda} S_T^m \le a).$$

Moreover,

$$Q\big(e^{-\sigma_m \sqrt{T} \lambda} S_T^m \le a \big) = Q\big(\, S_T^m \le \theta a \,\big)$$

and therefore, according to equation (6.5),

$$Q\big(\, \Psi(S) \le \theta b \,\big) \ge Q\big(\, S_T^m \le \theta a \,\big),$$

which gives Corollary 6.2. □

Before proving Theorem 6.4 we will introduce some additional definitions. Suppose $X, Y \in L^1(Q)$. We will write $X \prec Y$ if, for all convex functions $\psi : \mathbb{R} \to \mathbb{R}$,

$$E^Q\big[\psi(X)\big] \leq E^Q\big[\psi(Y)\big], \tag{6.6}$$

where the expectations are finite or $+\infty$. The relation $\prec$ is known as a stochastic ordering or convex dominance.

Suppose $\psi_k(s) = \max(s-k, 0)$, $k, s \in \mathbb{R}$. If $X, Y \in L^1(Q)$ have equal mean and

$$E^Q[\psi_k(X)] \leq E^Q[\psi_k(Y)]$$

for all $k \in \mathbb{R}$, then it follows that $X \prec Y$. In fact, if the relation (6.6) holds for all functions ψ_k, $k \in \mathbb{R}$, then it is also valid for all functions $\tilde{\psi}_k(s) = \max(k-s, 0)$, $k, s \in \mathbb{R}$, since $\psi_k(s) - \tilde{\psi}_k(s) = s - k$ and $E^Q[X] = E^Q[Y]$. An approximation argument now gives that equation (6.6) holds for all convex functions ψ, see Szekli [117] pp. 10-11 for further details.

Corollary 6.3. *Suppose $\Psi \in \mathcal{L}$ and $\varphi : (0, \infty) \to \mathbb{R}$ is a monotonic function such that $\varphi(\Psi(S)) \in L^1(Q)$. In addition, suppose there is a $\kappa > 0$ such that $\varphi(\kappa S_T^m) \in L^1(Q)$ and*

$$E^Q\big[\varphi\big(\Psi(S)\big)\big] = E^Q\big[\varphi\big(\kappa S_T^m\big)\big].$$

Then

$$\varphi\big(\Psi(S)\big) \prec \varphi\big(\kappa S_T^m\big).$$

Proof. Firstly, we may without loss of generality assume that φ is non-decreasing since $\psi : \mathbb{R} \to \mathbb{R}$ is a convex function if and only if $s \mapsto \psi(-s)$ is convex.

Suppose ψ_k is defined as above. For any random variable $X \in L^1(Q)$ one has

$$E^Q\big[\psi_k(X)\big] = \int_k^\infty Q(\, X > s \,)\, ds.$$

Thus, if

$$f(s) = Q\big(\, \varphi(\kappa S_T^m) > s \,\big) - Q\big(\, \varphi(\Psi(S)) > s \,\big)$$

then

$$E^Q\big[\psi_k \circ \varphi(\kappa S_T^m)\big] - E^Q\big[\psi_k \circ \varphi\big(\Psi(S)\big)\big] = \int_k^\infty f(s)\, ds. \tag{6.7}$$

Firstly, if $\tilde{\psi}_k$ is defined as above then

$$\Big| \int_k^\infty f(s)\,ds \Big| \le E^Q\big[\tilde{\psi}_k \circ \varphi(\kappa S_T^m)\big] + E^Q\big[\tilde{\psi}_k \circ \varphi\big(\Psi(S)\big)\big],$$

and therefore, by bounded convergence,

$$\lim_{k \to -\infty} \int_k^\infty f(s)\,ds = 0. \tag{6.8}$$

Next, let

$$s_* = \inf\{\, s > \varphi(0+) \;:\; f(s) > 0 \,\}.$$

If $s_* = \infty$ then equation (6.8) implies $f = 0$ and the integral in equation (6.7) is zero. If $\varphi(0+) < s_* < \infty$ then $f(s_*) \ge 0$, since f is right continuous, and Corollary 6.2 gives that $f(s) \ge 0$ for all $s \ge s_*$. In combination with equation (6.8) this implies that the integral in equation (6.7) is non-negative. Moreover, the same argument shows that $s_* \ne \varphi(0+)$. Thus, the integral in equation (6.7) is non-negative and the proof is complete. ☐

Now, to prove Theorem 6.4, put $\varphi(s) = \max(s - K, 0)$, $\psi(s) = s^p$, $p \ge 1$, and let κ be chosen as in Theorem 6.4. Corollary 6.3 yields

$$E\big[\psi \circ \varphi\big(\Psi(S)\big)\big] \le E\big[\psi \circ \varphi\big(\kappa S_T^m\big)\big],$$

which proves Theorem 6.4.

Next we will state a result similar to Theorem 6.6 for path-dependent *put* options. Let $\mathcal{L}$ be defined as previous and set

$$\mathcal{P}_K = \max(K - \mathcal{L}, 0), \quad K > 0.$$

Theorem 6.6. *If $\Psi_K \in \mathcal{P}_K$, $K > 0$, and κ is a constant such that*

$$\|\Psi_K(S)\|_1 = \|\max(K - \kappa S_T^m, 0)\|_1,$$

then

$$\|\Psi_K(S)\|_p \le \|\max(K - \kappa S_T^m, 0)\|_p$$

for each $1 \le p < \infty$.

Proof. Follows directly by setting $\varphi(s) = \max(K - s, 0)$ in Corollary 6.3. ☐

This section is concluded with an example of a payoff function which satisfies the converse inequality compared to Theorem 6.4 and Theorem 6.6. The example we have chosen is a down-and-out call option. Set

$$\tau = \inf\{t \in M \,:\, S_t^m \leq H(t)\},$$

where M is a closed subset of $[0,T]$ and $H : M \to (0,\infty)$ is continuous. In particular, the random variable τ is Borel measurable. The payoff function of a down-and-out call option written on the m:th asset is defined as

$$\max(S_T^m - K, 0)1_{\{\tau > T\}}$$

with $K \geq 0$.

Proposition 6.1. *If $K \geq 0$ and κ is a constant such that*

$$\|\max(S_T^m - K, 0)1_{\{\tau > T\}}\|_1 = \|\max(\kappa S_T^m - K, 0)\|_1,$$

then

$$\|\max(S_T^m - K, 0)1_{\{\tau > T\}}\|_p \geq \|\max(\kappa S_T^m - K, 0)\|_p$$

for each $1 \leq p < \infty$.

There is a similar result as in Proposition 6.1 for certain other barrier options such as up-and-out put options. The details are omitted here.

To prove Proposition 6.1, the following so called shift inequality will be useful.

Theorem 6.7. *Assume that A is a Borel set in Ω. If $\|h\|_{\mathcal{H}} = 1$ and*

$$Q(A) = \Phi(a),$$

then

$$\Phi(a - \lambda) \leq Q(A + \lambda h) \leq \Phi(a + \lambda)$$

for each $\lambda \geq 0$.

A proof of Theorem 6.7 can be found in Kuelbs and Li [76].

Proof of Proposition 6.1. We will prove that if, for $a, b > 0$,

$$Q\big(S_T^m > a, \tau > T\big) = Q\big(S_T^m > b\big), \tag{6.9}$$

then

$$Q(S_T^m > \theta a, \tau > T) \geq Q(S_T^m > \theta b) \tag{6.10}$$

for each real number $\theta \geq 1$. Proposition 6.1 then follows in the same way as Theorem 6.4 follows from Corollary 6.2.

Suppose a and b are as in equation (6.9) and fix $\theta \geq 1$. In addition, assume that

$$h(t) = c_m \frac{t}{\sqrt{T}}, \quad 0 \leq t \leq T,$$

so that $\langle c_m, h(T) \rangle = \sqrt{T}$ and $\|h\|_{\mathcal{H}} = 1$. If $\lambda \geq 0$ satisfies $\exp\big(\lambda \sigma_m \sqrt{T}\big) = \theta$, then

$$\begin{aligned} Q\big(S_T^m > \theta a,\, \tau > T\big) &= Q\big(e^{-\lambda \sigma_m \langle c_m, h(T) \rangle} S_T^m > a,\, \tau > T\big) \\ &= Q\big(S_T^m(\cdot - \lambda h) > a,\, \tau > T\big) \\ &\geq Q\big(S_T^m(\cdot - \lambda h) > a,\, \tau(\cdot - \lambda h) > T\big) \end{aligned}$$

since $\tau(\omega) \geq \tau(\omega - \lambda h)$ for each $\omega \in \Omega$. Thus, if $A = \{S_T^m > a,\, \tau > T\}$ then

$$Q\big(S_T^m > \theta a,\, \tau > T\big) \geq Q(A + \lambda h).$$

Theorem 6.7 and equation (6.9) imply

$$\begin{aligned} Q(A + \lambda h) &\geq \Phi\Big(-\frac{\ln(b/S_0^m) - (r - q_m - \sigma_m^2/2)T}{\sigma_m \sqrt{T}} - \lambda\Big) \\ &= Q(S_T^m \geq \theta b), \end{aligned}$$

which gives equation (6.10) and the proof is complete. □

6.3 The Error in the Monte Carlo Pricing of Some Familiar Path-Dependent Options

This section shows, using Theorems 6.4 and 6.6, how to obtain an explicit upper bound of $D_p(X)$ for different choices of payoff functions X and thereby establish error bounds for the Monte Carlo estimation of the quantity $E^Q[X]$. To begin with we will consider call options.

6.3.1 Call Options

First we state two lemmas that will be used below.

Lemma 6.1. *Suppose $K \geq 0$. The function*

$$\kappa \mapsto \frac{\|\max(\kappa S_T^m - K, 0)\|_2}{\|\max(\kappa S_T^m - K, 0)\|_1}, \quad \kappa > 0,$$

is non-increasing.

For a proof of Lemma 6.1, see Borell [21].

Lemma 6.2. *Suppose $p \geq 2$. Then*

$$D_p\big(\max(S_T^m - K, 0)\big) \to \infty \quad \textit{as } K \to \infty.$$

Proof. It suffices to prove that

$$\lim_{K\to\infty} \frac{\|\max(S_T^m - K, 0)\|_2^2}{\|\max(S_T^m - K, 0)\|_1^2} = \infty,$$

since

$$D_p(X) \geq D_2(X) = \left(\frac{\|X\|_2^2}{\|X\|_1^2} - 1\right)^{\frac{1}{2}}$$

for any $p \geq 2$ and any non-negative random variable $X \in L^p(Q)$ with $E^Q[X] > 0$.

By differentiating under the integral it follows

$$\frac{\partial}{\partial K} E^Q\big[\max(S_T^m - K, 0)^2\big] = -2E^Q\big[\max(S_T^m - K, 0)\big].$$

The relation $E^Q\big[\max(S_T^m - K, 0)\big] = \int_K^\infty Q(S_T^m > s)ds$ gives

$$\frac{\partial}{\partial K} E^Q\big[\max(S_T^m - K, 0)\big]^2 = -2E^Q\big[\max(S_T^m - K, 0)\big]\, Q(S_T^m > K).$$

The l'Hôpital rule now yields

$$\lim_{K\to\infty} \frac{\|\max(S_T^m - K, 0)\|_2^2}{\|\max(S_T^m - K, 0)\|_1^2} = \lim_{K\to\infty} \frac{1}{Q(S_T^m > K)} = \infty.$$

□

To begin with, consider a call on the maximum written on the m:th asset. That is, a contract with payoff function

$$\Psi_K(S) = \max(\max_{t \in M} S_t^m - K, 0),$$

where $T \in M \subseteq [0, T]$ and $K \geq 0$. Clearly, if $\kappa \leq 1$ then

$$\|\Psi_K(S)\|_1 \geq \|\max(\kappa S_T^m - K, 0)\|_1.$$

Thus, according to Theorem 6.4 and Lemma 6.1,

$$\frac{\|\Psi_K(S)\|_2}{\|\Psi_K(S)\|_1} \leq \frac{\|\max(S_T^m - K, 0)\|_2}{\|\max(S_T^m - K, 0)\|_1}$$

and therefore

$$D_2\big(\Psi_K(S)\big) \leq D_2\big(\max(S_T^m - K, 0)\big) \tag{6.11}$$

because

$$D_2(X) = \left(\frac{\|X\|_2^2}{\|X\|_1^2} - 1\right)^{\frac{1}{2}}$$

for any non-negative random variable $X \in L^2(Q)$ with $E^Q[X] > 0$.

Since the right hand side in equation (6.11) can be evaluated analytically we may easily obtain an upper bound for $D_2(\Psi_K(S))$ in the special case that $\Psi_K(S)$ is the payoff of a call on the maximum.

However, there are some disadvantages with this approach. According to Lemma 6.2 the right hand side in equation (6.11) may be very large if K is large. Moreover, for an arbitrary payoff $\Psi_K(S)$, $\Psi_K \in \mathcal{C}_K$, it may be very difficult to find an upper bound for κ other than zero such that

$$\|\Psi_K(S)\|_1 \geq \|\max(\kappa S_T^m - K, 0)\|_1.$$

In any case, Lemma 6.2 shows there is no uniform bound of $D_p\big(\Psi_K(S)\big)$ for all $\Psi_K \in \mathcal{C}_K$ and all $K > 0$.

Fortunately, there is a way to get around these problems. If $\Psi_K \in \mathcal{C}_K$ then

$$\Psi_K = \max(\Psi - K, 0) = \max(\Psi, K) - K$$

for some $\Psi \in \mathcal{L}$. Thus, to price the option with payoff $X = \Psi_K(S)$ it is enough to estimate the expectation of the random variable

$$Y = \max(\Psi(S), K).$$

It is easy to find an upper bound for $D_p(Y)$. In fact, since $\max(\Psi, K) \in \mathcal{C}_0$ it follows

$$\frac{\|Y\|_p}{\|Y\|_1} \leq \frac{\|S_T^m\|_p}{\|S_T^m\|_1} = e^{\frac{1}{2}(p-1)\sigma_m^2 T}$$

which yields, in combination with the Minkowski inequality in the case $2 < p < \infty$,

$$D_p(Y) \leq \begin{cases} \left(e^{\sigma_m^2 T} - 1\right)^{\frac{1}{2}}, & \text{if } p = 2, \\ e^{\frac{1}{2}(p-1)\sigma_m^2 T} + 1, & \text{if } 2 < p < \infty. \end{cases} \tag{6.12}$$

Remarkably enough, the estimate is only dependent of p, the greatest volatility, and time to maturity.

Next we will consider the relative error for the Monte Carlo simulation of $E^Q[Y]$ and give two numerical examples, see Figures 6.1 and 6.2. The first example will compare the value of the error constants $C_\epsilon(Y)$ and $C_\epsilon^{(r)}(Y)$, given in Theorem 6.3, for different values of ϵ. Recall here that Theorem 6.3, among other things, stated that the upper endpoint of a $100(1-\epsilon)$ % confidence interval for the relative error in the Monte Carlo estimation of $E^Q[Y]$ is bounded by

$$\frac{\min\left(C_\epsilon(Y), C_\epsilon^{(r)}(Y)\right)}{\sqrt{n}},$$

where n is the number of simulations,

$$C_\epsilon(Y) = \frac{D_2(Y)}{\sqrt{\epsilon}}, \quad \text{and} \quad C_\epsilon^{(r)}(Y) = \frac{2N_p}{\epsilon^{1/p}} \max\left(D_2(Y),\, n^{\frac{1}{p}-\frac{1}{2}}\, D_p(Y)\right).$$

If the option parameters are $\sigma_m = 0.3$ and $T = 1$ and if $n \geq 10^4$, then the estimate in equation (6.12) and some calculations give

$$\max\left(D_2(Y),\, n^{\frac{1}{p}-\frac{1}{2}}\, D_p(Y)\right) \leq \left(e^{\sigma_m^2 T} - 1\right)^{\frac{1}{2}},$$

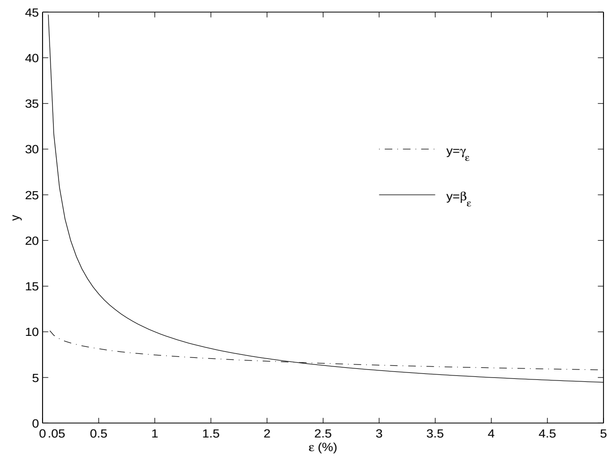
for at least all p such that $4 \leq p \leq 10$. Thus, if $n \geq 10^4$ then

$$C_\epsilon^{(r)}(Y) \leq \frac{2N_p}{\epsilon^{1/p}} \left(e^{\sigma_m^2 T} - 1\right)^{\frac{1}{2}}, \quad 4 \leq p \leq 10.$$

Since the estimate in equation (6.12) also yields that

$$C_\epsilon(Y) \leq \frac{1}{\sqrt{\epsilon}} \left(e^{\sigma_m^2 T} - 1\right)^{\frac{1}{2}}$$

Figure 6.1: The graph of $\epsilon \mapsto \beta_\epsilon$ and $\epsilon \mapsto \gamma_\epsilon$ for $0.05\% \leq \epsilon \leq 5\%$.



it would be of interest to compare the numbers

$$\beta_\epsilon = \frac{1}{\sqrt{\epsilon}} \quad \text{and} \quad \gamma_\epsilon = \min_{p=4,5,\ldots,10} \frac{2N_p}{\epsilon^{1/p}},$$

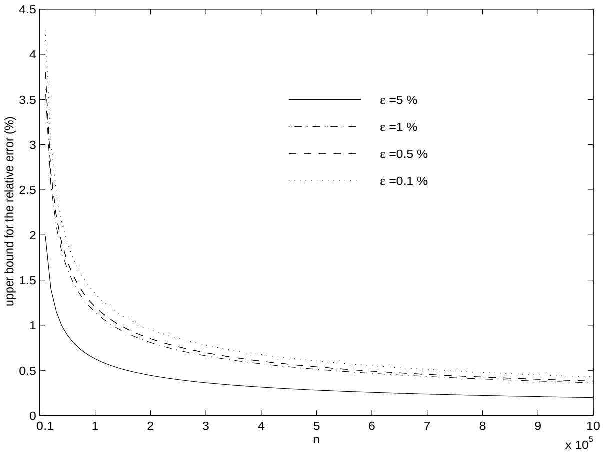
for $\epsilon > 0$. The reason that the minimum in the definition of γ_ϵ is taken over the set $\{4, 5, \ldots, 10\}$ is simply that we have computed the value of N_p for these numbers, see Table 6.1 in Section 6.1. Of course, one could take the minimum over a set larger than $\{4, 5, \ldots, 10\}$ and thereby obtain a smaller value of γ_ϵ. This will however not radically change the value of γ_ϵ, at least not for interesting values of ϵ, say between 0.05% and 5%.

Figure 6.1 shows the values of β_ϵ and γ_ϵ for $0.05\% \leq \epsilon \leq 5\%$. As the figure shows, if ϵ is greater than 2.5% then $\beta_\epsilon < \gamma_\epsilon$, but if ϵ is smaller than 2% then $\beta_\epsilon > \gamma_\epsilon$. Moreover, if ϵ is close to 0 then γ_ϵ is considerably much smaller than β_ϵ.

The next example, presented in Figure 6.2, describes the number of simulations that is required to obtain a certain accuracy of the Monte Carlo estimation of the quantity $E^Q[Y]$. According to the previous discussion, we may bound the upper endpoint of the confidence interval of degree $100(1 - \epsilon)\%$ for the relative error by

$$\frac{\min(\beta_\epsilon, \gamma_\epsilon)}{\sqrt{n}} \left(e^{\sigma_m^2 T} - 1\right)^{\frac{1}{2}}.$$

Figure 6.2: An upper bound for the relative error in the Monte Carlo estimation of the quantity $E^Q[Y]$ as a function of the number of simulations n. The option parameters are $\sigma_m = 0.3$ and $T = 1$. The value of n varies between 10^4 and 10^6.



In the example in Figure 6.2 the option parameters are $\sigma_m = 0.3$ and $T = 1$, the same as in the previous example. Figure 6.2 shows that the relative error is smaller than 1% if $n > 2 * 10^5$ for any $\epsilon \geq 0.1\%$.

6.3.2 Put Options

Much of the discussion in Subsection 6.3.1 is also of interest for put options. In particular, every $\Psi_K \in \mathcal{P}_K$ can be written

$$\Psi_K = K - \min(\Psi, K), \quad \Psi \in \mathcal{L}.$$

Thus, to price a derivative with the payoff $\Psi_K(S)$ it suffices to estimate the expectation of

$$Y = \min(\Psi(S), K).$$

Since $\min(\Psi, K) \in \mathcal{L}$ we obtain the same estimate of $D_p(Y)$ as in equation (6.12). Thus, the example given in section 6.3.1 is relevant for put options as well.

6.3.3 Options with a Floating Strike Price

There is a large class of options that are not included in neither $\mathcal{C}_K$ nor $\mathcal{P}_K$, namely options with a floating strike price. That is, the fixed strike price

K is replaced by $\Psi^\mu(S)$, where μ is a positive and bounded Borel measure on $[0,T]$ and

$$\Psi^\mu(xe^\omega) = \int_0^T \sum_{i=1}^m x_i e^{\omega_i(t)} \mu(dt).$$

This subsection discusses a method to estimate the relative error in the Monte Carlo pricing of these options.

Firstly, consider the payoff functions

$$X = \max\big(\Psi(S) - \Psi^\mu(S), 0\big) \quad \text{or} \quad X = \max\big(\Psi^\mu(S) - \Psi(S), 0\big),$$

where $\Psi \in \mathcal{L}$. For instance, if the measure μ is a positive linear combination of Dirac measures or the Lebesgue measure, then $E^Q[\Psi^\mu(S)]$ can easily be evaluated analytically and therefore it suffices to estimate the expectation of

$$Y = \max\big(\Psi(S), \Psi^\mu(S)\big) \quad \text{or} \quad Y = \min\big(\Psi(S), \Psi^\mu(S)\big).$$

As previously, the value of $D_p(Y)$ can be bounded as in equation (6.12).

6.3.4 A Remark on Barrier Options

Another large class of options which is not included in $\mathcal{C}_K$ or $\mathcal{P}_K$ are contracts which have a discontinuous payoff, that is, the payoff $X = \Psi(S)$ where $\Psi \notin C([0,T];[0,\infty)^m)$. Most barrier options are examples of such contracts. This subsection describes a method which gives an estimate of the relative error for some barrier options, namely those barrier options which have a bounded payoff.

Consider for instance an up-and-out call option that pays at time T the amount

$$X = \max(S_T^m - K, 0)1_{\{\tau > T\}},$$

where $K \geq 0$ and

$$\tau = \inf\{t \in M \,:\, S_t^m \geq H(t)\},$$

with $M \subseteq [0,T]$ closed and $H : M \to (0,\infty)$ continuous. Moreover, suppose $\inf_{t\in M} H(t) > S_0^m$ and $T \in M$. It is evident that

$$l\big(X\big) = H(T) - K.$$

One can easily find a lower bound α_{min} for $\alpha = E^Q[X]$. In fact,

$$\alpha \geq E^Q\big[\max(S_T^m - K, 0)1_{\{\max_{0\leq t\leq T} S_t^m < H_{min}\}}\big] = \alpha_{min},$$

where $H_{min} = \inf_{t\in M} H(t)$. Note that α_{min} can be evaluated analytically by using well known formulas, see e.g. Rich [102]. Thus, Theorem 6.3 implies that the relative error of the Monte Carlo pricing is, with probability $1-\epsilon$, bounded by

$$\frac{1}{\sqrt{n}}\sqrt{\frac{\ln(2/\epsilon)}{2}}\,\frac{H(T)-K}{\alpha_{min}}.$$

Chapter 7

Geometric Bounds on Certain Sublinear Functionals of Geometric Brownian Motion

Suppose $\{S_t\}_{0 \leq t \leq T}$ is an m-dimensional geometric Brownian motion with drift, μ is a bounded positive Borel measure in $[0, T]$, and $\phi : \mathbb{R}^m \mapsto [0, \infty)$ is a weighted $l^q(\mathbb{R}^m)$-norm, $1 \leq q \leq \infty$. The purpose of this chapter is to study the distribution and the moments of the random variable $X_\mu^{p,q}$ given by the $L^p(\mu)$-norm, $1 \leq p \leq \infty$, of the function $t \mapsto \phi(S_t)$, $0 \leq t \leq T$.

There are various sources of interest of the random variable $X_\mu^{p,q}$. In particular, in mathematical finance it is relevant in the pricing of Asian basket options ($p = q = 1$), options on the maximum of a basket ($p = \infty$, $q = 1$), and options on the maximum of several assets (μ equal to the Dirac measure at T and $q = \infty$). If μ is the Lebesgue measure on $[0, T]$ and the dimension m equals 1, the random variable $X_\mu^{p,q}$ is also of interest in the study of disorded systems as well as in the study of hyperbolic Brownian motion, see Yor [122]. The sum of lognormal random variables, which corresponds to the special case μ equal to the Dirac measure at T and $q = 1$, is of interest in geology, see Barouch et al. [9], and in radar theory, see Janos [67], to name a few areas. Note also that if μ is the Dirac measure at T and $q = \infty$ then the (generalised) moments of $X_\mu^{p,q}$ corresponds to the Laplace transform of the maximum of a discrete Gaussian process.

With the exception of the maximum of a Gaussian process, previous studies of the random variable $X_\mu^{p,q}$ have been concentrated on the one-dimensional case (i.e. $m = 1$) with μ equal to the Lebesgue measure and $p = 1$, that is, on the random variable

$$\int_0^T S_t^1 dt, \tag{7.1}$$

where $\{S_t^1\}_{0\leq t\leq T}$ is an one-dimensional geometric Brownian motion. Yor, and co-authors, have written a large number of articles focusing on this random variable, articles which have been collected in the monograph Yor [122]. Here Yor, among other things, describes the density of the random variable in equation (7.1) in terms of series of one-dimensional integrals, see Yor [122] p. 43. Other results in the same direction can be found in Alili [3], Comtet and Monthus [36], and Dufresne [42],[43]. Moreover, Bhattacharya, Thomann, and Waymire [15] derives a partial differential equation for the density function. Explicit expressions for some of the generalised moments of the random variable in equation (7.1) are given in Yor [122] p. 31, Dufresne [42],[43], and Donati-Martin, Matsumoto, and Yor [41]. Recently, Nikeghbali [96] has proven that the law of the random variable in equation (7.1) is indetermined by its moments.

It should be mentioned that there is a large number of articles dealing with the problem of computing the expectation

$$E\big[\max(\int_0^T S_t^1\mu(dt) - K, 0)\big], \quad K > 0.$$

This problem is, as we can see, closely related to the problem of finding the law of $X_\mu^{1,1}$ in the one-dimensional case. For a further discussion about the pricing of Asian options the reader may consult Linetsky [88], Rogers and Shi [105], and the references therein.

For some results about the distribution and the moments of sums of lognormal random variables, see Barouch, Kaufman, and Glasser [9], Ben Slimane [12], Bondesson [18] p. 66, Janos [67], and Leipnik [83].

This chapter will prove that the distribution function of $X_\mu^{p,q}$ is log-concave and discuss conditions on p and μ that ensures that the distribution function is absolutely continuous. Moreover, the chapter will derive upper and lower bounds for the distribution function. The chapter will also present the asymptotic behaviour of the distribution function, discuss stochastic ordering, and give sharp inequalities for the moments. Moreover, it will proven that the distribution of $X_\mu^{p,q}$ is indetermined by its moment. We conclude this chapter by showing some financial applications of the results obtained.

7.1 Notation

This section will introduce some notation that will be used throughout this chapter. The sample space Ω is defined as in the previous chapter, i.e. $\Omega = C_0([0,T];\mathbb{R}^m)$ consists of all functions $\omega = (\omega_1, \omega_2, \dots, \omega_m)$ such that, for each $i = 1, \dots, m$, the function $\omega_i : [0,T] \to \mathbb{R}$ is continuous and $\omega_i(0) = 0$. The space Ω is equipped with the norm $\|\cdot\|_{C_0}$, defined by

$$\|\omega\|_{C_0} = \max_{i=1,\dots,m} \max_{0\le s\le T} |\omega_i(s)|, \quad \omega \in \Omega.$$

The measure P will henceforth denote Wiener measure on Ω. Setting $W_t(\omega) = \omega(t)$, $0 \le t \le T$, $\omega \in \Omega$, the process $\{W_t\}_{0\le t\le T}$ is a standard m-dimensional Brownian motion on $[0,T]$ with respect to P.

From now on the class $\mathcal{M}$ denotes all bounded positive measures μ on the Borel σ-algebra of $[0,T]$, where $0 < T < \infty$. The class $\mathcal{M}(0,T)$ consists of all $\mu \in \mathcal{M}$ such that $\sup\{t \le T : \mu((t,T]) > 0\} = T$ and if $0 < \tau \le T$ then $\mathcal{M}(\tau, T)$ consists of all $\mu \in \mathcal{M}(0,T)$ such that $\mu([0,\tau)) = 0$. The norm in $L^p([0,T],\mu)$ will be denoted $\|\cdot\|_{L^p(\mu)}$.

If $x \in \mathbb{R}^m$ then x_i will denote the i:th coordinate of x. The scalar product in $\mathbb{R}^m$ will be written $\langle \cdot, * \rangle$ and $|\cdot|_2 = \sqrt{\langle \cdot, \cdot \rangle}$ is the Euclidean norm in $\mathbb{R}^m$. Vectors in $\mathbb{R}^m$ are regarded as m by 1 matrices. Moreover, multiplication of two vectors in $\mathbb{R}^m$ should be understood as coordinate-wise multiplication.

The function $\phi_{\rho,q}$ is defined, for each $x \in (0,\infty)^m$, as

$$\phi_{\rho,q}(x) = \begin{cases} \left(\sum_{i=1}^m \rho_i x_i^q\right)^{\frac{1}{q}} & \text{if } 1 \le q < \infty, \\ \max_{\{i\,:\,\rho_i > 0\}} x_i, & \text{if } q = \infty, \end{cases}$$

where $\rho \in [0,\infty)^{m-1} \times (0,\infty)$. Henceforth we put $\Lambda = [0,\infty)^{m-1} \times (0,\infty)$.

Let $\{S_t\}_{0\le t\le T}$ be an m-dimensional geometric Brownian motion with drift, that is

$$dS_t = S_t(\,\eta dt + \sigma C dW_t\,), \quad 0 \le t \le T, \tag{7.2}$$

where $\eta \in \mathbb{R}^m$, S_0 is a constant in $(0,\infty)^m$, $\sigma \in (0,\infty)^m$, and C is non-singular m by m matrix with rows $c_1, \dots, c_m$ satisfying $|c_i|_2 = 1$, $i = 1, \dots, m$. Finally, suppose

$$\sigma_m = \max_{i=1,\dots,m} \sigma_i.$$

In what follows, the functional $\Psi^{p,q}_{\mu,\rho}$ will be defined as

$$\Psi^{p,q}_{\mu,\rho}(S) = \|\phi_{\rho,q}(S_{(\cdot)})\|_{L^p(\mu)} \tag{7.3}$$

where $\mu \in \mathcal{M}(0,T)$, $\rho \in \Lambda$, and $1 \le p, q \le \infty$, that is

$$\Psi^{p,q}_{\mu,\rho}(S) = \begin{cases} \Big(\int_0^T \phi_{\rho,q}(S_t)^p \mu(dt) \Big)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \max_{t \in \mathrm{supp}\, \mu} \phi_{\rho,q}(S_t), & \text{if } p = \infty, \end{cases}$$

where supp μ denotes the topological support of μ. Note in particular that $\Psi^{p,q}_{\mu,\rho} \in \mathcal{L}$, where $\mathcal{L}$ is defined as in Chapter 6. For simplicity, the functional $\Psi^{p,q}_{\mu,\rho}$ will mostly be abbreviated $\Psi_{\mu,\rho}$.

The distribution function of $\Psi^{p,q}_{\mu,\rho}(S)$ will be denoted $F^{p,q}_{\mu,\rho}$, that is

$$F^{p,q}_{\mu,\rho}(s) = P\big(\Psi^{p,q}_{\mu,\rho}(S) \le s\big), \quad s \ge 0.$$

Similarly, $F^{p,q}_{\mu,\rho}$ will mostly be written $F_{\mu,\rho}$. The distribution $F_{\mu,\rho}$ will often be compared to the lognormal distribution G_ς, defined by

$$G_\varsigma(s) = \Phi\left(\frac{\ln s}{\varsigma}\right), \quad s \ge 0,\ \varsigma > 0,$$

where Φ is the standard normal distribution function. Thus, G_ς is the distribution function of the random variable Y, where $\ln Y$ is a normal distributed random variable with mean 0 and variance ς^2.

7.2 Convexity Properties and Absolute Continuity

This section will prove that the distribution function $F_{\mu,\rho}$ is log-concave and discuss conditions that imply that $F_{\mu,\rho}$ is absolutely continuous with respect to Lebesgue measure. The results will be based on the celebrated Ehrhard inequality, see Theorem 7.1 below. In what follows, let the function Φ^{-1} denote the inverse of Φ.

Theorem 7.1. *Suppose A and B are closed convex sets in Ω. Then*

$$\Phi^{-1}\big(P\big(\lambda A + (1-\lambda)B\big)\big) \ge \lambda \Phi^{-1}\big(P(A)\big) + (1-\lambda)\Phi^{-1}\big(P(B)\big)$$

for every $0 \le \lambda \le 1$.

A finite dimensional version of Theorem 7.1, that is, the Wiener measure is replaced by the standard Gaussian measure in $\mathbb{R}^n$ and A and B are convex sets in $\mathbb{R}^n$, was first shown by Ehrhard [45]. Latala [80] has in the finite dimensional setting proved that the conclusion in Theorem 7.1 holds when only one of the two sets is convex. However, it is still an open question if the Ehrhard inequality remains true for arbitrary Borel sets A and B. A proof of the infinite dimensional version of the Ehrhard inequality, e.g. the version given in Theorem 7.1, can be found in Lifshits [86] p. 133.

Recall that a function $f : A \to [0, \infty)$, where A is an open and convex set, is log-convex (log-concave) if

$$f\big(\lambda x + (1-\lambda)y\big) \le (\ge)\ f(x)^{\lambda} f(y)^{1-\lambda}$$

for all $0 < \lambda < 1$ and all $x, y \in A$. In particular, if $f > 0$ then f is log-convex (log-concave) if and only if $\ln f$ is convex (concave). The proof of Corollary 7.1 below uses that Φ is log-concave. This statement will be proven in the discussion that follows after the proof of Corollary 7.1.

Corollary 7.1. *Suppose* $\mu \in \mathcal{M}(0,T)$ *and* $\rho \in \Lambda$. *The function*

$$\mathbb{R} \ni s \mapsto \Phi^{-1} \circ F_{\mu,\rho} \circ \exp(s)$$

is concave. In particular, $F_{\mu,\rho}$ *is log-concave.*

Proof. If $\Upsilon : \Omega \mapsto \mathbb{R}$ is continuous and convex then the set $\{\Upsilon \le s\}$ is closed, convex, and

$$\{\Upsilon \le \lambda s + (1-\lambda)u\} \supseteq \lambda\{\Upsilon \le s\} + (1-\lambda)\{\Upsilon \le u\}$$

for all $s, u \in \mathbb{R}$ and all $0 \le \lambda \le 1$. Thus, by the Ehrhard inequality,

$$\begin{aligned} &\Phi^{-1}\big(P\big(\,\Upsilon \le \lambda s + (1-\lambda)u\,\big)\big) \\ &\qquad\qquad \ge \lambda \Phi^{-1}\big(P(\,\Upsilon \le s\,)\big) + (1-\lambda)\Phi^{-1}\big(P(\,\Upsilon \le u\,)\big) \end{aligned} \tag{7.4}$$

for all $s, u \in \mathbb{R}$ and all $0 \le \lambda \le 1$.

Thus, to prove that $\Phi^{-1} \circ F_{\mu,\rho} \circ \exp$ is concave it suffices to show that $\omega \mapsto \Psi_{\mu,\rho}(S(\omega))$ is log-convex. To prove this, suppose $0 < \lambda < 1$ and note

$$S_t(\,\lambda\omega + (1-\lambda)\tilde{\omega}\,) = S_t(\omega)^{\lambda}\, S_t(\tilde{\omega})^{1-\lambda}$$

for each $\omega, \tilde{\omega} \in \Omega$, $0 \le t \le T$, and any $i = 1, \dots, m$. Here x^{θ}, $x \in (0,\infty)^m$, $\theta > 0$, means $(x_1^{\theta}, \dots, x_m^{\theta})$. Hence

$$\phi_{\rho,q}\big(S_t(\,\lambda\omega + (1-\lambda)\tilde{\omega}\,)\big) \le \phi_{\rho,q}\big(S_t(\omega)\big)^{\lambda}\ \phi_{\rho,q}\big(S_t(\tilde{\omega})\big)^{1-\lambda},$$

according to the Hölder inequality in the case $q < \infty$. If $q = \infty$ the inequality is obvious. By applying the Hölder inequality once more we obtain

$$\Psi_{\mu,\rho}\big(S(\,\lambda\omega + (1-\lambda)\tilde{\omega}\,)\big) \le \Psi_{\mu,\rho}\big(S(\omega)\big)^{\lambda}\ \Psi_{\mu,\rho}\big(S(\tilde{\omega})\big)^{1-\lambda}.$$

Thus, $\omega \mapsto \Psi_{\mu,\rho}\big(S(\omega)\big)$ is log-convex.

It remains to show that $F_{\mu,\rho}$ is log-concave. Since $\Phi^{-1} \circ F_{\mu,\rho} \circ \exp$ is concave it follows

$$F_{\mu,\rho}\big(e^{\lambda s+(1-\lambda)u}\big) \geq \Phi\Big(\lambda\Phi^{-1}\big(F_{\mu,\rho}(e^s)\big) + (1-\lambda)\Phi^{-1}\big(F_{\mu,\rho}(e^u)\big)\Big)$$

for every $s, u \in \mathbb{R}$ and each $0 \leq \lambda \leq 1$. Hence, $F_{\mu,\rho}$ is log-concave because Φ is log-concave and the exponential function is convex. □

A positive Borel measure ν in a subinterval of $\mathbb{R}$ is log-concave if

$$\nu\big(\lambda A + (1-\lambda)B\big) \geq \nu(A)^\lambda \nu(B)^{1-\lambda},$$

for all $0 < \lambda < 1$ and all Borel sets A and B. Thus, if the measure ν is log-concave then the corresponding distribution function is log-concave. The reverse statement is not necessarily true. For instance, the measure with distribution function $F_{\mu,\rho}$ is not log-concave for all choices of μ and ρ. Indeed, an absolutely continuous, bounded, and positive Borel measure in some open subinterval of $\mathbb{R}$ is log-concave if and only if the density function is log-concave. The if part is the same as the Prékopa inequality, see Prékopa [100], and the only if part is shown in Borell [19]. Now, the lognormal distribution function G_ς has a density g_ς given by

$$g_\varsigma(s) = e^{-\frac{\ln^2 s}{2\varsigma^2}} \frac{1}{s\varsigma\sqrt{2\pi}}, \quad s > 0.$$

It is readily seen that the function g_ς is not log-concave and thus, the measure with distribution function G_ς is not log-concave. Note also that the Prékopa inequality implies that Φ is log-concave.

For a discussion about log-concavity in option pricing, see Borell [22]

The proof of Corollary 7.2 below exploits an idea in Hoffman-Jørgensen, Shepp, and Dudley [61].

Corollary 7.2. *Suppose $\mu \in \mathcal{M}(0,T)$, $1 \leq p, q \leq \infty$, $\rho \in \Lambda$, and put*

$$s_* = \inf\{s \geq 0 \,:\, F^{p,q}_{\mu,\rho}(s) > 0\}.$$

The distribution $F^{p,q}_{\mu,\rho}$ is absolutely continuous on (s_*, ∞). Moreover, if $\mu \in \mathcal{M}(\tau,T)$, $\tau > 0$, or if $p < \infty$ then $F^{p,q}_{\mu,\rho}$ is absolutely continuous on $[0,\infty)$.

Proof. Corollary 7.1 gives a concave function $\psi : (s_*, \infty) \to \mathbb{R}$ such that $F^{p,q}_{\mu,\rho}(s) = \exp(\psi(s))$ for all $s > s_*$. A concave function is absolutely continuous and hence, $F^{p,q}_{\mu,\rho}$ is absolutely continuous on (s_*, ∞). It remains to establish that $F^{p,q}_{\mu,\rho}(s)$ is continuous at s_* if $\mu \in \mathcal{M}(\tau, T)$, $\tau > 0$, or if $p < \infty$. Since a distribution function is right continuous this amounts to proving that $P(\Psi^{p,q}_{\mu,\rho}(S) = s_*) = 0$ if $\mu \in \mathcal{M}(\tau, T)$, $\tau > 0$, or $p < \infty$.

To begin with we will study the two special cases

$$\mu \in \mathcal{M}(\tau, T),\ \tau > 0, \quad \text{or} \quad p < \infty \text{ and } \mu(\{0\}) = 0. \tag{7.5}$$

It is obvious that there is a sequence of functions $\omega^{(n)} \in \Omega$, $n = 1, 2, \ldots$, such that $\phi_{p,q}(S_t(\omega^{(n)})) \to 0$, $n \to \infty$, point-wise for any $t \in (0, T]$ or uniformly for all $t \in [\tau, T]$. Thus, under any of the assumptions in equation (7.5), $\Psi^{p,q}_{\mu,\rho}\big(S(\omega^{(n)})\big) \to 0$ as $n \to \infty$. In particular, there is, for every $\epsilon > 0$, an $\tilde{\omega} \in \Omega$ such that $\Psi^{p,q}_{\mu,\rho}\big(S(\tilde{\omega})\big) < \epsilon$. Define $\Upsilon(\omega) = \Psi^{p,q}_{\mu,\rho}\big(S(\omega)\big)$ for each $\omega \in \Omega$. The map Υ is continuous which yields that the set $\Upsilon^{-1}\big((-\infty, \epsilon)\big)$ is an open non-empty subset of Ω. The topological support of P equals Ω and hence

$$P\Big(\Upsilon^{-1}\big((-\infty, \epsilon)\big)\Big) > 0$$

and therefore $s_* = 0$. But $\Upsilon(\omega) > 0$ for all $\omega \in \Omega$, which yields

$$P(\Psi^{p,q}_{\mu,\rho}(S) = s_*) = P(\Upsilon = 0) = 0.$$

Next, assume that

$$p < \infty \text{ and } \mu(\{0\}) > 0.$$

It is readily seen that

$$s_* \geq k = \phi_{\rho,q}(S_0)\, \mu(\{0\})^{\frac{1}{p}}.$$

Define $\nu(A) = \mu(A \cap (0, T])$ for every Borel set A of $[0, T]$. For all $s \geq k$ it holds

$$P\big(\ \Psi^{p,q}_{\mu,\rho}(S) = s\ \big) = P\Big(\ \Psi^{p,q}_{\nu,\rho}(S) = (s^p - k^p)^{\frac{1}{p}}\ \Big).$$

Since $\nu(\{0\}) = 0$, the previous results implies that $P\big(\Psi^{p,q}_{\nu,\rho}(S) = s\big) = 0$ for all $s \geq 0$, and therefore $P\big(\Psi^{p,q}_{\mu,\rho}(S) = s\big) = 0$ for each $s \geq k$. In particular, $P\big(\Psi^{p,q}_{\mu,\rho}(S) = s_*\big) = 0$ and the proof is done. □

If $\mu \in \mathcal{M}(0, T)$ then $F^{\infty,q}_{\mu,\rho}$ is not necessarily continuous at s_*, where s_* is defined as in Corollary 7.2. For instance, if $\mu = \delta_0 + \delta_T$, where δ_s is the Dirac measure at s, then it is readily seen that $F^{\infty,q}_{\mu,\rho}$ is discontinuous at s_*.

7.3 Bounds on the Distribution Function and Tail Probabilities

The first theorem in this section gives bounds on the distribution function.

Theorem 7.2. *Suppose* $0 \le \tau \le T$, $\mu \in \mathcal{M}(\tau, T)$, $\rho \in \Lambda$, *and* $\varsigma = \sigma_m \sqrt{T}$. *Assume* $\theta \ge 1$ *and choose* $a, b > 0$ *such that*

$$F_{\mu,\rho}(a) = G_\varsigma(b),$$

then

$$F_{\mu,\rho}(\theta a) \ge G_\varsigma(\theta b). \tag{7.6}$$

If, in addition, $\tau > 0$ *then*

$$F_{\mu,\rho}(\theta a) \le G_\varsigma(\theta^\gamma b), \tag{7.7}$$

where

$$\gamma = \frac{\sigma_m \sqrt{T}}{\alpha \sqrt{\tau}} \quad \textit{with} \quad \alpha = \max_{|x|_2=1} \min_{\{i\,:\,\rho_i>0\}} \sigma_i \langle c_i, x\rangle > 0.$$

Moreover, if $0 < \theta < 1$ *then the inequalities in equations* (7.6) *and* (7.7) *are reversed.*

Proof. Equation (7.6) follows at once from Corollary 6.2 in the previous chapter. To prove equation (7.7) we will use Theorem 6.7. Let the Cameron-Martin space $\mathcal{H}$ be defined as in Section 6.2. Now, suppose $\omega \in \Omega$, $h \in \mathcal{H}$, $\lambda > 0$, and note that

$$S_t(\omega + \lambda h) = e^{\lambda \sigma C h(t)} S_t(\omega)$$

for each $0 \le t \le T$. Here $e^x = (e^{x_1}, \ldots, e^{x_m})$, $x \in \mathbb{R}^m$. Thus, if $\mu \in \mathcal{M}(\tau, T)$, $\tau > 0$, and $I = \{i : \rho_i > 0\}$ then

$$\Psi_{\mu,\rho}\big(S(\omega + \lambda h)\big) \le \big(\max_{\tau \le t \le T} \max_{i \in I} e^{\lambda \sigma_i \langle c_i, h(t)\rangle}\big)\, \Psi_{\mu,\rho}\big(S(\omega)\big). \tag{7.8}$$

Let $\chi_{[0,\tau]}$ be the characteristic function of the interval $[0, \tau]$ and fix $h \in \mathcal{H}$ such that

$$h'(t) = \frac{x}{\sqrt{\tau}} \chi_{[0,\tau]}(t), \quad 0 \le t \le T,$$

where $x \in \mathbb{R}^m$ satisfies $|x|_2 = 1$ and

$$\max_{i \in I} \sigma_i \langle c_i, x \rangle = \min_{|y|_2=1} \max_{i \in I} \sigma_i \langle c_i, y \rangle.$$

Observe that $\|h\|_{\mathcal{H}} = 1$ and $\max_{i \in I} \sigma_i \langle c_i, x \rangle = -\alpha$, where α is defined as in Proposition 7.2. Since $h(t) = \sqrt{\tau} x$ for all $t \in [\tau, T]$ it follows

$$\max_{\tau \le t \le T} \max_{i \in I} e^{\lambda \sigma_i \langle c_i, h(t) \rangle} = \max_{i \in I} e^{\lambda \sigma_i \sqrt{\tau} \langle c_i, x \rangle} = e^{-\lambda \alpha \sqrt{\tau}}.$$

Consequently,

$$\Psi_{\mu,\rho}\big(S(\omega + \lambda h)\big) \le e^{-\lambda \alpha \sqrt{\tau}} \Psi_{\mu,\rho}(S(\omega)).$$

Recall that $\|h\|_{\mathcal{H}} = 1$ and

$$P\Big(\Psi_{\mu,\rho}(S) \le a\Big) = \Phi\Big(\frac{\ln b}{\varsigma}\Big).$$

Theorem 6.7 implies

$$P\Big(\Psi_{\mu,\rho}\big(S(\cdot + \lambda h)\big) \le a\Big) \le \Phi\Big(\frac{\ln b}{\varsigma} + \lambda\Big),$$

for each $\lambda \ge 0$, and therefore

$$P\Big(\Psi_{\mu,\rho}(S) \le e^{\lambda \alpha \sqrt{\tau}} a\Big) \le \Phi\Big(\frac{\ln b}{\varsigma} + \lambda\Big). \tag{7.9}$$

The constant α is strictly greater than 0. In fact, if A denotes the convex hull of the vectors $\{c_i\}_{i \in I}$ then, since $0 \notin A$, $\langle c_i, x \rangle > 0$ for all $i \in I$ if x denotes the point in A closest to the origin. Hence, equation (7.7) follows by setting $\lambda = \frac{\ln \theta}{\alpha \sqrt{\tau}}$ in equation (7.9).

The last part of Theorem 7.2 is obvious. □

Next we will consider the tail probabilities for the law of $\Psi_{\mu,\rho}(S)$. To begin with we will study the upper tail probability.

In what follows we write $f(s) \sim_s g(s)$ if $f(s)/g(s) \to 1$ as $s \to \infty$. The lognormal distribution satisfies

$$\ln\big(1 - G_\varsigma(\lambda s)\big) \sim_s -\frac{1}{2} \frac{\ln^2 s}{\varsigma^2}, \tag{7.10}$$

for any $\lambda > 0$ and any $\varsigma > 0$. This follows at once from the well-known estimates

$$\frac{1}{\sqrt{2\pi}} \frac{s}{1+s^2} e^{-\frac{s^2}{2}} \le 1 - \Phi(s) \le \frac{1}{\sqrt{2\pi}} \frac{1}{s} e^{-\frac{s^2}{2}}, \quad s > 0,$$

and the definition of G_ς. The next result, Theorem 7.3 below, generalises this observation as well as a previous result by Janos in [67]. Janos obtains by other methods the same result in the special case $p, q < \infty$, and μ equal to a positive linear combination of Dirac measures.

Theorem 7.3. *If $\mu \in \mathcal{M}(0,T)$ and $\rho \in \Lambda$ then*

$$\ln\big(1 - F_{\mu,\rho}(s)\big) \sim_s -\frac{1}{2}\frac{\ln^2 s}{\sigma_m^2 T}.$$

Proof. Suppose $\varsigma = \sigma_m\sqrt{T}$ and fix $a, b > 0$ so that

$$F_{\mu,\rho}(a) = G_\varsigma(b).$$

By equation (7.6) with $\theta = s/a$ it follows

$$F_{\mu,\rho}(s) \geq G_\varsigma(\frac{b}{a}s), \quad s \geq a,$$

which gives $\ln\big(1 - F_{\mu,\rho}(s)\big) \leq \ln\big(1 - G_\varsigma(\frac{b}{a}s)\big)$ and therefore

$$\big|\ln\big(1 - F_{\mu,\rho}(s)\big)\big| \geq \big|\ln\big(1 - G_\varsigma(\frac{b}{a}s)\big)\big|$$

for each $s \geq a$. Hence, if

$$l^- = \liminf_{s\to\infty} \frac{\ln\big(1 - F_{\mu,\rho}(s)\big)}{\ln\big(1 - G_\varsigma(s)\big)}$$

then

$$l^- \geq \liminf_{s\to\infty} \frac{\ln\big(1 - G_\varsigma(\frac{b}{a}s)\big)}{\ln\big(1 - G_\varsigma(s)\big)} = 1,$$

according to equation (7.10).

The next aim is to find an upper bound. Fix ϵ such that $0 < \epsilon < T$ and define $\nu(A) = \mu\big(A \cap (T-\epsilon, T]\big)$ for each Borel set A of $[0,T]$. Note that $\nu \in \mathcal{M}(T-\epsilon, T)$. Moreover, set $\tilde{\rho} = (0, 0, \ldots, 0, \rho_m)$. It is evident that

$$F_{\mu,\rho}(s) \leq F_{\nu,\tilde{\rho}}(s) \tag{7.11}$$

for each $s > 0$.

Next, because $\nu([0,T]) > 0$ there are $a, b > 0$ such that

$$F_{\nu,\tilde{\rho}}(a) = G_\varsigma(b).$$

Equation (7.7) with $\theta = s/a$ implies

$$F_{\nu,\tilde{\rho}}(s) \leq G_\varsigma\big(\frac{b}{a^\gamma}s^\gamma\big), \quad s \geq a,$$

where

$$\gamma = \sqrt{\frac{T}{T-\epsilon}}.$$

In view of equation (7.11) we find

$$\Big|\ln\Big(1 - F_{\mu,\rho}(s)\Big)\Big| \leq \Big|\ln\Big(1 - G_\varsigma\big(\frac{b}{a^\gamma}s^\gamma\big)\Big)\Big|$$

for each $s \geq a$. Thus, if

$$l^+ = \limsup_{s\to\infty} \frac{\ln\big(1 - F_{\mu,\rho}(s)\big)}{\ln\big(1 - G_\varsigma(s)\big)}$$

then

$$l^+ \leq \limsup_{s\to\infty} \frac{\ln\Big(1 - G_\varsigma\big(\frac{b}{a^\gamma}s^\gamma\big)\Big)}{\ln\big(1 - G_\varsigma(s)\big)} = \gamma^2,$$

according to equation (7.10).

To sum up, for every $\epsilon > 0$,

$$1 \leq l^- \leq l^+ \leq \frac{T}{T-\epsilon},$$

and the proof is done. □

Some further details about the upper tail probability of $F^{p,\infty}_{\delta_T,\rho}$, with δ_T the Dirac measure at T, can be found in Lifshits [85].

It is far more difficult to state any general results about the lower tail probability. For instance, if $\mu(\{0\}) > 0$ or if $p = \infty$ and $\mu \in \mathcal{M}(0,T) \setminus \bigcup_{t>0} \mathcal{M}(t,T)$ then $\inf\{s\,;\, F^{p,q}_{\mu,\rho}(s) > 0\} > 0$. However, the next proposition shows that it is possible to find upper bounds for $F_{\mu,\rho}(s)$ as $s \to 0^+$.

Proposition 7.1. *Suppose $\mu \in \mathcal{M}(0,T)$ and $\rho \in \Lambda$. There exist $\kappa > 0$ and $\epsilon > 0$ such that*

$$F_{\mu,\rho}(s) \leq e^{-\kappa \ln^2 s}$$

for all $s \leq \epsilon$.

Proof. Theorem 7.2 implies that $F_{\mu,\rho}(s) \leq G_\varsigma(\lambda s)$ for some $\lambda > 0$ and all sufficiently small $s > 0$. In particular, since $G_\varsigma(\lambda s) = 1 - G_\varsigma(1/\lambda s)$ equation (7.10) gives that there is a $\kappa > 0$ such that

$$F_{\mu,\rho}(s) \leq e^{-\kappa \ln^2 s}$$

for all sufficiently small $s > 0$. □

7.4 Stochastic Ordering

The first theorem in this section is a slight modification of Corollary 6.3 in Chapter 6. The proof is identical and will be omitted.

Theorem 7.4. *Suppose $\mu \in \mathcal{M}(0,T)$, $\rho \in \Lambda$, and let A be an open subinterval of $\mathbb{R}$. Assume that the random variable Y has distribution G_ς with $\varsigma = \sigma_m\sqrt{T}$. Moreover, suppose $\varphi : (0,\infty) \to A$ is a monotonic function such that $\varphi(\Psi_{\mu,\rho}(S)) \in L^1(P)$ and $\psi : A \to \mathbb{R}$ is a convex function. Finally, assume there is a $\kappa > 0$ such that $\varphi(\kappa Y) \in L^1(P)$ and*

$$E\big[\varphi\big(\Psi_{\mu,\rho}(S)\big)\big] = E\big[\varphi(\kappa Y)\big].$$

Then

$$E\big[\psi \circ \varphi\big(\Psi_{\mu,\rho}(S)\big)\big] \leq E\big[\psi \circ \varphi\big(\kappa Y\big)\big],$$

where the expectations are finite or $+\infty$.

The next theorem considers the special cases $p = q = 1$ and $p = q = \infty$.

Theorem 7.5. *Let $\mu \in \mathcal{M}(0,T)$ and $\rho \in \Lambda$. Suppose C and D are two m by m matrices with rows $c_1, \ldots, c_m$ and $d_1, \ldots, d_m$, respectively, such that $\langle c_i, c_j\rangle \leq \langle d_i, d_j\rangle$ and $|c_i|_2 = |d_i|_2 = 1$ for all $1 \leq i,j \leq m$. Moreover, assume $\{S_t^C\}_{0\leq t\leq T}$ and $\{S_t^D\}_{0\leq t\leq T}$ are defined by*

$$dS_t^C = S_t^C(\,\eta dt + \sigma C dW_t\,) \quad \text{and} \quad dS_t^D = S_t^D(\,\eta dt + \sigma D dW_t\,),$$

where $S_0^C = S_0^D = (s_1, s_2, \ldots, s_m) \in (0,\infty)^m$ and η and σ are defined as previous.
(i): If $\psi : (0,\infty) \to \mathbb{R}$ is a convex function then

$$E\big[\,\psi\big(\Psi_{\mu,\rho}^{1,1}(S^C)\big)\,\big] \leq E\big[\,\psi\big(\Psi_{\mu,\rho}^{1,1}(S^D)\big)\,\big], \tag{7.12}$$

where the expectations are finite or $+\infty$.
(ii): If $\varphi : (0,\infty) \to \mathbb{R}$ is a non-decreasing and continuous function then

$$E\big[\,\varphi\big(\Psi_{\mu,\rho}^{\infty,\infty}(S^C)\big)\,\big] \geq E\big[\,\varphi\big(\Psi_{\mu,\rho}^{\infty,\infty}(S^D)\big)\,\big], \tag{7.13}$$

provided both integrands are in $L^1(P)$.

The proof of Theorem 7.5 will be based on the so called comparison principle for Gaussian random variables. This principle can be formulated in different ways. Theorem 7.6 below gives two versions of the comparison principle.

Theorem 7.6. *Assume that* $X = (X_1, \dots, X_n)$ *and* $X' = (X'_1, \dots, X'_n)$ *are centred Gaussian random variables with values in* $\mathbb{R}^n$ *and* $E[\,X_i X_j\,] \leq E[\,X'_i X'_j\,]$, $1 \leq i, j \leq n$.
(i): If $f : \mathbb{R}^n \to \mathbb{R}$ *is two times continuously differentiable with*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0, \quad \text{and} \quad \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = O\big(e^{k\,|x|_2}\big), \quad \text{as } |x|_2 \to \infty,$$

for all $1 \leq i, j \leq n$ *and some constant* $k > 0$, *then*

$$E\big[f(X)\big] \leq E\big[f(X')\big].$$

(ii): If $Var(X_i) = Var(X'_i)$, $i = 1, \dots, n$, *then*

$$P\big(\,X_i \leq a_i,\ i = 1, \dots, n\big) \leq P\big(\,X'_i \leq a_i,\ i = 1, \dots, n\big)$$

for all $a_i \in \mathbb{R}$, $i = 1, \dots, n$.

Theorem 7.6 (i) is stated in Lifshits [86], p. 188, for a function f that, in addition, has bounded second derivatives. However, the proof in Lifshits [86] of Theorem 7.6 (i) also works given the assumptions in Theorem 7.6 (i). Statement (ii) is due to Slepian, see [114]. A geometrical proof of an important special case of the Slepian inequality can be found in Chartres [32].

We are now in the position to prove Theorem 7.5.

Proof of Theorem 7.5. First we prove (i). Suppose $i = 1, \dots, n$, $n \geq 1$, and $0 \leq t_1 < t_2 < \dots < t_n \leq T$. Set $N = nm$ and let $x \in \mathbb{R}^N$. Define $u_i : \mathbb{R}^N \to \mathbb{R}^m$ as

$$u_i(x) = \Big(s_1 e^{(\eta_1 - \sigma_1^2/2)t_i + \sigma_1 x_{(i-1)m+1}}, \dots, s_m e^{(\eta_m - \sigma_m^2/2)t_i + \sigma_m x_{(i-1)m+m}}\Big)$$

and $g : \mathbb{R}^N \to \mathbb{R}$ by

$$g(x) = \sum_{i=1}^{n} \tilde{\rho}_i \phi_{\rho,1}\big(u_i(x)\big)$$

with $\tilde{\rho} \in (0, \infty)^n$.

Since

$$E\left[\,\Psi^{1,1}_{\mu,\rho}(S^C)\,\right] = E\left[\,\Psi^{1,1}_{\mu,\rho}(S^D)\,\right]$$

it is, according to the discussion preceding Corollary 6.3, enough to prove equation (7.12) for all non-decreasing convex functions. In addition, an approximation argument yields that it is sufficient to prove the relation (7.12) for all ψ that are two times differentiable and $\psi(s) = s - k$ for all sufficiently large $s > 0$ and some $k \geq 0$. If $f(x) = \psi \circ g(x)$ then

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(x) = \psi'' \circ g(x)\,\frac{\partial g}{\partial x_j}(x)\frac{\partial g}{\partial x_i}(x) + \psi' \circ g(x)\,\frac{\partial^2 g}{\partial x_j \partial x_i}(x)$$

for all $1 \leq i, j \leq N$. It is readily seen that the first two derivatives of g are non-negative and thus

$$\frac{\partial^2 f}{\partial x_j \partial x_i} \geq 0.$$

Let X^C and X^D be two centred Gaussian random variables in $\mathbb{R}^N$ with

$$E\left[\,X^C_{im+j}X^C_{km+l}\,\right] = \min(t_i, t_k)\,\langle c_j, c_l\rangle, \quad 0 \leq i, k \leq n-1,\ 1 \leq j, l \leq m,$$

and

$$E\left[\,X^D_{im+j}X^D_{km+l}\,\right] = \min(t_i, t_k)\,\langle d_j, d_l\rangle, \quad 0 \leq i, k \leq n-1,\ 1 \leq j, l \leq m.$$

Theorem 7.6 (i) gives $E\left[\,f(X^C)\,\right] \leq E\left[\,f(X^D)\,\right]$ and hence,

$$E\left[\,\psi\big(\Psi^{1,1}_{\mu_n,\rho}(S^C)\big)\,\right] \leq E\left[\,\psi\big(\Psi^{1,1}_{\mu_n,\rho}(S^D)\big)\,\right] \tag{7.14}$$

for all measures $\mu_n \in \mathcal{M}(0,T)$ such that μ_n is a positive linear combination of Dirac measures. There is a sequence of such measures μ_n so that μ_n converges weakly to μ for any $\mu \in \mathcal{M}(0,T)$ and thus, bounded convergence yields that equation (7.14) is valid for all $\mu \in \mathcal{M}(0,T)$.

To prove the second statement (ii), suppose X^C, X^D, and u_i are defined as before and put

$$g(x) = \max_{i=1,\dots,n} \tilde\rho_i \phi_{\rho,\infty}\big(u_i(x)\big), \quad x \in \mathbb{R}^N,\ \tilde\rho \in (0,\infty)^n.$$

The set $\{x\,:\,\varphi \circ g(x) \leq t\}$, $t \in \mathbb{R}$, is of the form $\{x\,:\,x_i \leq a_i,\ 1 \leq i \leq N\}$ and thus, by the Slepian inequality, $E[\,\varphi \circ g(X^C)\,] \geq E[\,\varphi \circ g(X^D)\,]$. Hence,

$$E\left[\,\varphi\big(\Psi^{\infty,\infty}_{\mu_n,\rho}(S^C)\big)\,\right] \geq E\left[\,\varphi\big(\Psi^{\infty,\infty}_{\mu_n,\rho}(S^D)\big)\,\right]$$

for all measures $\mu_n \in \mathcal{M}(0,T)$ such that μ_n is a positive linear combination of Dirac measures. The general case with $\mu \in \mathcal{M}(0,T)$ now follows by bounded convergence and the fact that φ is continuous. $\square$

7.5 Moment Inequalities

The purpose of this section is to derive inequalities for the moments of $\Psi^{p,q}_{\mu,\rho}$. Define, for all $r \in \mathbb{R} \setminus \{0\}$,

$$\|X\|_r = E\big[|X|^r\big]^{\frac{1}{r}},$$

where X is a random variable in (Ω, P). By Theorem 7.3 and Proposition 7.1 we can draw the conclusion that $0 < \|\Psi_{\mu,\rho}(S)\|_r < \infty$ for all $r \neq 0$.

If $-\infty < r_0 < r_1 < \infty$ and $r_0 r_1 \neq 0$ then it is well-known that

$$\|\Psi_{\mu,\rho}(S)\|_{r_0} \leq \|\Psi_{\mu,\rho}(S)\|_{r_1}.$$

The main result in this section, Theorem 7.7 below, is a sharp reversed inequality.

Theorem 7.7. *Suppose $\mu \in \mathcal{M}(0,T)$ and $\rho \in \Lambda$. If $-\infty < r_0 < r_1 < \infty$ and $r_0 r_1 \neq 0$ then*

$$\|\Psi_{\mu,\rho}(S)\|_{r_1} \leq e^{\frac{1}{2}\sigma_m^2 T(r_1 - r_0)}\, \|\Psi_{\mu,\rho}(S)\|_{r_0}. \tag{7.15}$$

Moreover, there is equality in equation (7.15) *if $\mu = k\delta_T$, $k > 0$, and $\rho_i = 0$ for $i = 1, \dots, m-1$.*

Proof. Suppose $\varsigma = \sigma_m \sqrt{T}$ and assume that Y is a random variable with distribution function G_ς. Set $f(s) = s^{r_0}$, $\psi(s) = \operatorname{sign}(r_1)\, s^{r_1/r_0}$, and

$$\kappa = \frac{\|\Psi_{\mu,\rho}(S)\|_{r_0}}{\|Y\|_{r_0}}.$$

The function $\psi : (0,\infty) \to \mathbb{R}$ is convex, so Theorem 7.5 shows

$$E\big[\, \psi \circ f\big(\Psi_{\mu,\rho}(S)\big)\,\big] \leq E\big[\, \psi \circ f\big(\kappa Y\big)\,\big].$$

Thus,

$$\|\Psi_{\mu,\rho}(S)\|_{r_1} \leq \frac{\|Y\|_{r_1}}{\|Y\|_{r_0}}\, \|\Psi_{\mu,\rho}(S)\|_{r_0}.$$

Since $\|Y\|_r = e^{\frac{1}{2}\varsigma^2 r}$, we have established the desired inequality. The last part of the theorem is obvious. $\square$

By Theorem 7.7 one can derive an estimate for the Laplace transform of the maximum of a Gaussian process. Indeed, assume $X = (X_1, \dots, X_m)$ is a Gaussian random variable with values in $\mathbb{R}^m$ and $\sigma_m^2 = \max_{i=1,\dots,m} \operatorname{Var}(X_i)$. By setting $\mu = \delta_T$, $q = \infty$, and $r_1 = -r_0 = r$ in Theorem 7.7 we obtain

$$E\big[e^{r \max_{i=1,\dots,m} X_i}\big] E\big[e^{-r \max_{i=1,\dots,m} X_i}\big] \leq e^{\sigma_m^2 r^2}, \quad r \in \mathbb{R},$$

with equality if $m = 1$.

7.6 The Stieltjes Moment Problem

A distribution function F with support on the positive real axis is said to be *indetermined* (or *Stieltjes-indetermined*) by its moments if there exist a distribution function $\tilde{F}$ with support on the positive real axis such that $F \neq \tilde{F}$ and

$$\int_0^\infty s^k F(ds) = \int_0^\infty s^k \tilde{F}(ds) \quad \text{for all } k = 1, 2, \dots .$$

It is well-known that the lognormal distribution is indetermined by its moments. This result goes back to Heyde [59] who also construct a class of distribution functions with the same moments as the lognormal distribution.

Suppose $\tilde{\rho} = (0, \dots, 0, \tilde{\rho}_m)$ with $\tilde{\rho}_m > 0$. Nikeghbali [96] has recently proved that for any $\mu \in \mathcal{M}(0,T)$ and any $p, q < \infty$ the distribution $F^{p,q}_{\mu,\tilde{\rho}}$ is indetermined by its moments. Nikeghbali proved this result using a criterion by Pakes in [97]. Namely, Theorem 5 in Pakes [97] states that if F is a distribution function with support on the positive real axis such that

$$\int_u^\infty \frac{-\ln\big(1 - F(s)\big)}{s^{3/2}}\, ds < \infty,$$

for some $u > 0$, then F is indetermined by its moment. This result is a modification of the so called Krein criterion, see [97] for further details.

The result by Pakes shows that $F_{\mu,\rho}$ is indetermined by its moment. Indeed, by Theorem 7.3 it follows

$$\int_u^\infty \frac{-\ln\big(1 - F_{\mu,\rho}(s)\big)}{s^{3/2}}\, ds < \infty, \quad u > 0,$$

and thus, we have

Theorem 7.8. *Suppose $\mu \in \mathcal{M}(0,T)$ and $\rho \in \Lambda$. The distribution $F_{\mu,\rho}$ is indetermined by its moments.*

7.7 Applications in Option Pricing

This section discusses some applications of the results in the previous sections. The first example considers an upper bound for the price of an Asian call option. Assume for simplicity that the dimension $m = 1$. Suppose that $\{S^1_t\}_{0 \le t \le T}$ denotes an one-dimensional geometric Brownian motion and the measure Q is defined as in Theorem 1.2. It is well known that if $\mu \in \mathcal{M}(0,T)$

with $\mu([0,T]) = 1$ then the price (at time $t = 0$) of a European call option does not fall below the price of the corresponding Asian call option, that is

$$e^{-rT} E^Q\big[\max(\int_0^T S_t^1 \mu(dt) - K, 0)\big] \leq e^{-rT} E^Q\big[\max(S_T^1 - K, 0)\big].$$

This follows at once from the Jensen inequality and the fact that the price of an European call option is an increasing function of T. By Theorem 7.4 we can establish a better bound for the price an Asian option, namely

$$e^{-rT} E^Q\big[\max(\int_0^T S_t^1 \mu(dt) - K, 0)\big] \leq e^{-rT} E^Q\big[\max(\kappa S_T^1 - K, 0)\big]$$

with

$$\kappa = \frac{E^Q\big[\int_0^T S_t^1 \mu(dt)\big]}{E^Q\big[S_T^1\big]} = \frac{\int_0^T e^{rt} \mu(dt)}{e^{rT}} \leq 1.$$

For a further discussion about other upper bounds on the price of an Asian option, see Rogers et al. [105].

Next we will discuss the so-called moment-matching method for the pricing of Asian basket options. To value these contracts one must determine the expectation

$$E^Q\big[\max(\Psi_{\mu,\rho}^{1,1}(S) - K, 0)\big],$$

where $\mu \in \mathcal{M}(0,T)$, $\rho \in \Lambda$, and K is a constant. Note that for many choices of μ the quantities $E^Q[(\Psi_{\mu,\rho}^{1,1})^k]$, $k \in \mathbb{N}_+$, can be computed analytically. A common approach to estimate the price is the so-called moment-matching method, which means that one determines a random variable X such that

$$E^Q\big[X^k\big] = E^Q\big[(\Psi_{\mu,\rho}^{1,1})^k\big], \quad k = 1, \dots, n,$$

and then make the approximation

$$E^Q\big[\max(\Psi_{\mu,\rho}^{1,1}(S) - K, 0)\big] \approx E^Q\big[\max(X - K, 0)\big]. \tag{7.16}$$

If $K > 0$, Theorem 7.8 shows that even with $n = \infty$ it is not guaranteed that there is equality in equation (7.16). To be more specific, there is a random variable X such that $E^Q[X^k] = E^Q[(\Psi_{\mu,\rho}^{1,1})^k]$ for all $k \in \mathbb{N}_+$ and a constant $K > 0$ such that the left hand side is not equal to the right hand side in equation (7.16). Recall that if X and X' are two non-negative random variables with finite expectation such that $E^Q[\max(X - K, 0)] =$

$E^Q[\max(X' - K, 0)]$ for all $K \geq 0$, then X and X' are equal in law. For other aspects on the moment-matching method, see Brigo et al. [27].

We conclude this chapter by discussing how the price of an Asian call or put option on a basket or a call or put option on the maximum of several assets depends on the correlation matrix. Let, for some m by m matrix C and $\chi \in \{-1, 1\}$,

$$v_\chi^{p,q}(C) = e^{-rT} E^Q\big[\max\big(\chi\,(\Psi_{\mu,\rho}^{p,q}(S^C) - K),\, 0\big)\big],$$

where S^C is defined as in Theorem 7.5 and $K \geq 0$. If C and D are two m by m matrices with rows $c_1, \ldots, c_m$ and $d_1, \ldots, d_m$, respectively, such that $\langle c_i, c_j \rangle \leq \langle d_i, d_j \rangle$ and $|c_i|_2 = |d_i|_2 = 1$ for all $1 \leq i, j \leq m$, then, if $v_\chi^{p,q}(D)$ is defined in analogy with $v_\chi^{p,q}(C)$ but with S^C replaced by S^D, Theorem 7.5 shows that

$$v_\chi^{1,1}(C) \leq v_\chi^{1,1}(D) \quad \text{and} \quad \chi\, v_\chi^{\infty,\infty}(C) \geq \chi\, v_\chi^{\infty,\infty}(D),$$

for $\chi \in \{-1, 1\}$. For a further discussion about monotonicity in option prices, see Kijima [74], Janson and Tysk [68], and the references therein.

Chapter 8

Dividends and the Pricing of Path-Dependent Options

Although traders in most equity option markets regard the Black-Scholes model as the premier option valuation model, the Black-Scholes model has certain limitations. The model assumes, for instance, that the underlying asset does not pay dividends. Of course, in practice, this is not usually the case. In addition, dividends are known to have a significant influence on the price of American call options, see for instance Jarrow and Rudd [69].

The purpose of this chapter is to describe how the Black-Scholes model can be extended to also handle stocks that pay dividends and show how path-dependent options can be priced in this model. The chapter will moreover present numerical examples illustrating the influence dividends have on the price of some path-dependent options.

8.1 The Heath-Jarrow Model

The price of a dividend paying stock will drop just after the dividend date. The size of the drop is dependent on many factors. The size of the dividend amount will of course affect the price change, but other factors as taxes and transaction costs may also influence the price change at the ex-dividend date. In most cases, the stock price will go down by somewhat less than the amount of the dividend, see for instance Heath and Jarrow [57] for a further discussion.

In the Black-Scholes model the underlying asset price follows a geometric Brownian motion. In particular, the price process is continuous. Since the price change at the ex-dividend date is discontinuous, it is no longer realistic to assume that the underlying price process of a dividend paying stock equals a geometric Brownian motion. Heath and Jarrow [57] suggest a more realistic model for the price of a dividend paying stock during a bounded time interval, say $[0, T]$ where T can be thought of as the maturity date of some option. The purpose of the remaining part of this section is to discuss the model by Heath et al..

Assume that dividends are to be paid at the dates $t_1, t_2, \ldots, t_n$, where $0 \leq t_i \leq T$, $i = 1, \ldots, n$, and that the stock price will drop by the amount $d_1, d_2, \ldots, d_n$ at the corresponding dividends dates. Heath et al. make the assumption that the dividend dates are known in advance while the price changes can be random variables. For simplicity, we will assume that both the dividend dates $t_1, t_2, \ldots, t_n$ and the price changes $d_1, d_2, \ldots, d_n$ are fixed constants.

Next, let D_t, $0 \leq t \leq T$, be defined by

$$D_t = \sum_{i=1}^{n} d_i e^{-r(t_i - t)} \chi_{[0,t_i]}(t),$$

where χ_A is the indicator function of the interval A and r is the continuously compounded risk-free interest rate. Now, Heath et al. suggest that under the risk-neutral measure Q, the price process $\{S_t\}_{0 \leq t \leq T}$ of a dividend paying stock evolves according to

$$S_t = Y_t + D_t, \quad 0 \leq t \leq T, \tag{8.1}$$

where Y_t is a geometric Brownian motion with drift starting at $Y_0 = S_0 - D_0$, to be more specific, Y_t solves

$$\begin{cases} dY_t = Y_t(rdt + \sigma dW_t^Q), & 0 \leq t \leq T, \\ Y_0 = S_0 - D_0, \end{cases}$$

where $\{W_t^Q\}_{t \geq 0}$ is a standard one-dimensional Brownian motion with respect to Q and σ is a positive constant.

In this model, the stock price will fall by the amount d_i at the dividend date t_i, while between the dividend dates the price process will evolve almost (note that D_t is not constant between the dividend dates) as in the Black-Scholes model. It can be observed that if $n = 0$ then the Heath-Jarrow model will coincide with the Black-Scholes model.

We conclude this section by making some comments about the parameter T. In practice, when some option shall be priced, T is sat to the maturity date of the option. By setting T equal to the maturity date implies that the model will be dependent of the option, which is unfortunate. For instance, the value of an American call option will not necessary be an increasing function of the maturity date.

8.2 Pricing Path-Dependent Options in the Heath-Jarrow Model

This section describes how the theoretical value of a European barrier option and a European styled Asian option can be determined in the Heath-Jarrow model. To begin with we will focus on barrier options.

Consider, for instance, a continuous down-and-out call option with maturity date T, constant barrier H, and strike price K. Recall that the payoff function of this contract is given by

$$\max(S_T - K, 0)1_{\{\, S_t > H,\ \text{for all}\ t \in [0,T]\,\}}.$$

Assume that the underlying asset price is described as in equation (8.1). The theoretical price of the barrier option can be written

$$\begin{aligned} & e^{-rT} E^Q \big[\max(\, S_T - K, 0\,) 1_{\{\, S_t > H,\ \text{for all}\ t \in [0,T]\,\}} \big] \\ & \qquad = e^{-rT} E^Q \big[\max \big(Y_T - (K - D_T), 0 \big) 1_{\{\, Y_t > H - D_t,\ \text{for all}\ t \in [0,T]\,\}} \big]. \end{aligned}$$

The right hand side can be recognised as the theoretical price *in the Black-Scholes model* of a continuous down-and-out call with strike price $K - D_T$, initial asset price $S_0 - D_0$, and a (moving) barrier at the level $H - D_t$ at time t. Thus, the problem of pricing down-and-out call options in the Heath-Jarrow model is equivalent to valuing down-and-out call options with a *moving* barrier in the Black-Scholes model. This problem has been addressed earlier in the literature, see for instance Derman et al. [39].

Next we will consider Asian contracts. Recall that the payoff function of a discrete Asian call option with strike price K and maturity date T is given by

$$\max\Big(\sum_{j=1}^{m} \rho_j S_{\tau_j} - K,\ 0\,\Big),$$

where τ_j, $0 \le \tau_j \le T$, $j = 1, \dots, m$, represent the time points when the stock price is sampled and the weights ρ_j, $j = 1, \dots, m$, are strictly positive numbers. The theoretical value of an Asian call option in the Heath-Jarrow model equals

$$e^{-rT}E^Q\left[\max(\sum_{j=1}^{m}\rho_j S_{\tau_j} - K,\, 0)\right]$$
$$= e^{-rT}E^Q\left[\max\left(\sum_{j=1}^{m}\rho_j Y_{\tau_j} - (K - \sum_{j=1}^{m}\rho_j D_{\tau_j}),\, 0\right)\right].$$

In other words, the options value equals the price in the Black-Scholes model of an Asian call with strike price $K - \sum_{j=1}^{m}\rho_j D_{\tau_j}$ and initial price $S_0 - D_0$.

8.3 Numerical Examples and Conclusions

This section presents numerical examples that show how the price of path-dependent options varies with the dividend amount as well as with the dividend date. Figure 8.1 displays the price of a European down-and-out call option and a European styled Asian call option as a function of the size on the price change of the underlying asset price at the dividend date. Figure 8.2 shows the price of the same options as a function of the dividend date. The theoretical prices of the barrier option has been calculated using a numerical procedure described in Derman et al. [39] and the Asian contract has been valued with a moment matching method (cf. Section 7.6).

Observe that the option prices differs substantially in both cases. In particular, the price of the barrier and the Asian option is considerably more sensitive to the dividend date than the price of a similar American call option, see the caption in Figure 8.2.

The conclusion is that dividends may have a great influence on the price of a path-dependent option. Not only does the amount of the dividend influence the price of a path-dependent option, but also the dividend day may affect the price.

Thus, to properly price a path-dependent option dividends must be included in the pricing model. For this reason it would be of great interest to investigate what the consequences would be if the price changes at the ex-dividend dates are random variables instead of constants as in our example. This would make the model more realistic if the option has a long lifetime, which is the case for many barrier and Asian options. Hopefully future research will investigate this question.

Figure 8.1: The price at time $t = 0$ of a down-and-out call and an Asian call as a function of d_1, the amount by which the stock will fall at the dividend date. The stock will only pay one dividend and the dividend date is $t_1 = 1/2$. The option parameters are in both cases $S_0 = 110$, $K = 110$, $\sigma = 0.3$, $r = 0.047$, and $T = 1$ year. The barrier is $H = 100$ and the Asian option has weekly sampling with weights $\rho_j = 1/52$, $j = 1, \ldots, 52$. The price of a similar American call option will vary between 18.77 and 15.13 as d_1 varies between 0 and 6 unit of currency.

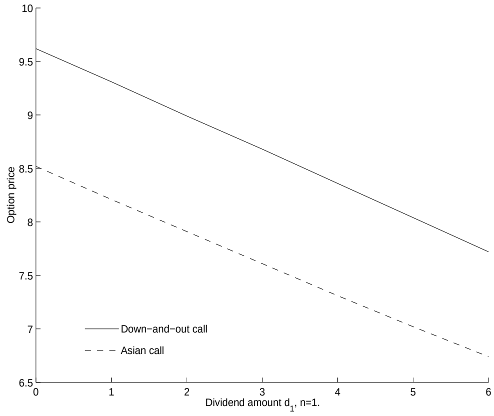
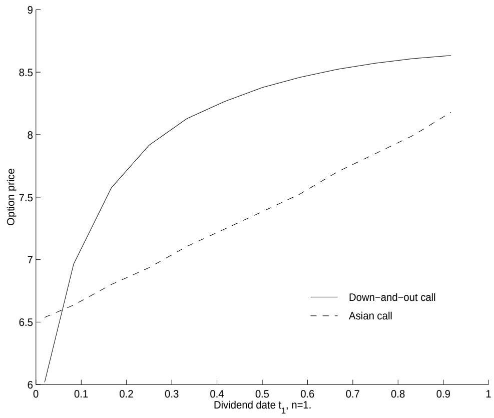


Figure 8.2: The price at time $t = 0$ of a down-and-out call and an Asian call as a function of the dividend date t_1. We assume that the stock will drop by the amount $d_1 = 4$ at the dividend date. The option parameters are as in Figure 8.2. The price of a similar American call option will vary between 16.35 and 16.44 as the dividend date t_1 varies between 2 weeks and 11 months.



Notation and Conventions

This section explains some notation and conventions employed in this text.

Sets

$\mathbb{N}_+$	natural numbers, 1,2, ...
$\mathbb{N}$	non-negative integers, 0,1,2, ...
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers

If A and B are subsets of a real vector space V then

$$\lambda A = \{\lambda x \,:\, x \in A\}, \quad \lambda \in \mathbb{R},$$
$$A + B = \{x + y \,:\, x \in A,\, y \in B\},$$

and

$$A + x = \{y + x \,:\, y \in A\}, \quad x \in V.$$

Moreover, we assume that $\inf \emptyset = \infty$.

Functions

$x \wedge y$	minimum of x and y, $x, y \in \mathbb{R}$
$\text{sign}(x)$	the sign of x, $x \in \mathbb{R} \setminus \{0\}$
$\Phi(x)$	standard normal distribution function, i.e.

$$\Phi(x) = \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}, \quad -\infty \leq x \leq \infty.$$

If X and Y are two real valued random variables defined on the same probability space, then $X = Y$ $(X > Y)$ means that X are equal to (larger than) Y almost surely.

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