Thesis for the Degree of Doctor of Philosophy

On space-time means for solutions of nonlinear Klein-Gordon equations

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Göteborg, December 2003

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ISBN 91-7291-350-9

Doktorsavhandlingar vid Chalmers tekniska högskola
Ny serie nr 2032
ISSN 0346-718X

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Göteborg, Sweden 2003

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Abstract

The thesis consists of one paper about space-time means for solutions of the nonlinear Klein-Gordon equation, which also was my licentiate thesis (1986), and one paper about the rate of decay of long-time mean-values for solutions of the nonlinear Klein-Gordon equation.

In the first paper is studied if certain properties of finite energy solutions u_0 of the linear Klein-Gordon equation will be inherited of finite energy solutions u of the corresponding, (same data), nonlinear Klein-Gordon equation. An important step in our investigation is to establish that $\|u\|_{X_p^s}$, where X_p^s denotes L_p^s or $B_p^{s,q}$, satisfies a certain nonlinear Volterra integral inequality with singular kernel. It is proven that if u_0 has any one of the properties

$$\|u_0\|_{X_p^s} \leq w(t) \text{ as } t \geq t^*, \|u_0\|_{X_p^s} \in L_q \text{ or } (\frac{2}{t}\int_{\frac{t}{2}}^{t} \|u_0\|_{X_p^s}^q d\tau)^{\frac{1}{q}} \leq w(t) \text{ as } t \geq t^*,$$

then the same property is inherited by u under certain extra conditions. Results of this kind are important tools in proving the existence of everywhere defined scattering operators and uniqueness of weak solutions.

The second paper is devoted to the study of the rate of decay for space-time means of finite energy solutions to the nonlinear Klein-Gordon equation. From the main theorem follows that if certain long time mean-values of u_0 have an upper bound $O(t^{-\alpha})$ as $t \to \infty$, then the same upper bound also holds for the corresponding long time mean-values of u. For $n = 3$ the maximal rate of decay is obtained under certain extra conditions. Results of this kind are of interest since the decay properties of u, (in suitable L_p-spaces), can be related to the decay and the rate of the decay of local energy.

Key words and phrases: Nonlinear Klein-Gordon equation, linear Klein-Gordon equation, asymptotic properties, rate of decay, boundedness of space-time means, properties inherited from the corresponding linear case, nonlinear Volterra integral inequality with singular kernel.

This thesis consists of an introduction and the following papers:

- **Paper I:** *On space-time means of solutions to nonlinear Klein-Gordon equations and a nonlinear Volterra integral inequality with singular kernel.*
- **Paper II:** *On space-time means and rate of decay for solutions of nonlinear Klein-Gordon equations.*

Acknowledgements

My advisor in both areas of the thesis has been Professor Philip Brenner. I would like to thank him for his very encouraging support, for sharing his profound knowledge, and for showing a great deal of patience with my progress. Thanks also to Ulla for being there and for encouraging me. Finally, I thank Yumi Karlsson for typing the manuscript and for typing my licentiate thesis for the second time.

Paper I

Paper II

On space-time means for solutions of nonlinear Klein-Gordon equations - Introduction

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june 2003

1 Basic concepts and applications

1.1 Relativistic wave equations

In classical physics the fundamental law of energy is

$$E = \frac{|\mathbf{p}|^2}{2m} + V$$

In quantum mechanics physical quantities are represented by operators. The (Schrödinger) substitutions

$$E \to i\hbar\frac{\partial}{\partial t}, \mathbf{p} \to -i\hbar\nabla, V \to \hat{V}$$

lead to

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \hat{V}\psi$$

which is the Schrödinger equation. But the classical laws are not valid for very fast motion where relativity becomes important. The corresponding relativistic law for energy is

$$E^2 = |\mathbf{p}|^2c^2 + m^2c^4$$

The Schrödinger substitutions yield

$$-\hbar^2\frac{\partial^2\phi}{\partial t^2} = -\hbar^2c^2\Delta\phi + m^2c^4\phi$$

which, if we for simplicity take $\hbar = 1$ and $c = 1$, can be rewritten as

$$\phi_{tt} - \Delta\phi + m^2\phi = 0$$

This is the Klein-Gordon equation which holds for bosons. Bosons are a class of particles that have integer spin. Examples of bosons are fotons and mesons.

As the field ϕ is a scalar field there is no spin involved in the solutions but the Klein-Gordon equation is pertinent e.g. for the description of π^0 mesons and for the scalar components of the Higgs fields. For fermions, that have half integer spin, you have to use the "square-root version" of the Klein-Gordon equation

$$\frac{\partial\psi}{\partial t} + c\hat{\alpha}\cdot\nabla\psi + i\frac{\beta mc^2}{\hbar}\psi = 0$$

which is the Dirac equation.

1.2 The Klein-Gordon equation

In the following papers are studied the asymptotic behaviour of finite energy solutions to the nonlinear Klein-Gordon equation

$$\text{(NLKG)} \qquad \begin{cases} u_{tt} - \Delta u + m^2 u + f(u) = 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases}$$

where $x \in \mathbb{R}^n, t \in \mathbb{R}^+, m > 0, \Delta = \Delta_x$. Equations of this type appear as scalar versions of the field equations which describes weak nonlinear interaction between elementary particles. A typical form of the nonlinear term is

$$f(u) = \lambda u|u|^{\rho-1},\ \lambda > 0, \rho \geq 1$$

For $f(u) = 0$ we obtain the corresponding free equation, the Klein-Gordon equation (KG). (For $f(u) = 0$ and $m = 0$ we obtain the simpler wave equation.) We have investigated if some, mathematically and physically, interesting types of decay of the solution u_0 of the (KG) will be inherited by the solution u of the (NLKG). This research is presented in this introduction, and carried out in detail in the two attached papers below referred to as Paper I (Blomqvist [1]) and Paper II (Blomqvist [2]).

1.3 Energy

The Klein-Gordon lagrangian $L = \frac{1}{2}(\phi_t^2 - |\nabla_x\phi|^2 - m^2\phi^2)$ gives rise to the energy density $\frac{\partial L}{\partial\phi_t}\phi_t - L = \frac{1}{2}(\phi_t^2 + |\nabla_x\phi|^2 + m^2\phi^2)$ and the energy $E_o(t) = \frac{1}{2}\int(|\phi_t|^2 + |\nabla_x\phi|^2 + m^2|\phi|^2)dx$. For the system (NLKG) we define the energy $E(t)$ by

$$E(t) = \frac{1}{2}\int(|u_t|^2 + |\nabla_x u|^2 + m^2|u|^2)dx + \int F(u)dx$$

where $F(u) = \int_0^u f(v)dv$ is the potential energy density.

If we multiply the nonlinear Klein-Gordon equation by u_t and integrate over space it follows that the energy is a conserved quantity. We will assume that $F(u) \geq 0$ in order to ensure the existence of nonnegative energy, which is essential for the existence of global solutions when no restrictions are assumed on the size of the data. The space $X_e = H^1 \times L_2$ with norm

$$\|u(t)\|_e = \{\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{L_2}^2\}^{\frac{1}{2}}$$

is called the energy space. By a *finite energy solution* we mean a solution for which $(\varphi, \psi) \in X_e = H^1 \times L_2$.

1.4 Strichartz estimates for solutions of the (KG)

Let $\|g\|_{L_p(L_q^r)}$ denote the $L_p(\mathbb{R}^+)$-norm in t of $\|g(t)\|_{L_q^r(\mathbb{R}^n)}$. For a finite energy solution u_0 of the (KG) we then have that (Strichartz [26] ,[27] and Segal [22])

$$\|u_0\|_{L_p(H_p^{\frac{1}{2}})} \leq C_1(\|\varphi\|_{H^1} + \|\psi\|_{L_2}) \leq C$$

where $p = \frac{2(n+1)}{n-1}$, i.e. $\delta_p = \frac{1}{2} - \frac{1}{p} = \frac{1}{n+1}$.

More complex estimates bound u_o in $L_q(R, H_p^s(R^n))$ (see Strichartz [26], Marshall-Strauss-Wainger [17], Ginibre and Velo [11], and also Brenner [8]). For more detailed statements, see also below and Paper II.

1.5 Scattering theory

A large amount of work has been devoted to the theory of Scattering for nonlinear wave equations. The main purpose of Scattering theory is to study the asymptotic behaviour in time of solutions to the nonlinear equation by comparing them to solutions of the (simpler) linear equation. For a background and a definition of the scattering operator, see paper I. Under certain conditions, *there exists an everywhere defined scattering operator on X_e for the nonlinear Klein-Gordon equation.* (Brenner [5], [6], [7].) In particular it follows that there is a finite energy solution u_+ of (KG), such that for the corresponding soluton u of the (NLKG)

$$\|u(t) - u_+(t)\|_e \to 0, \quad \text{as } t \to \infty$$

1.6 Energy decay

For $\Omega \subset \mathbb{R}^n$ we define local energy $E_\Omega(t)$ by

$$E_\Omega(t) = \frac{1}{2}\int_\Omega (|u_t|^2 + |\nabla_x u|^2 + m^2|u|^2)dx + \int_\Omega F(u)dx$$

and the corresponding energy space by $X_e(\Omega) = H^1(\Omega) \times L_2(\Omega)$ with norm

$$\|u(t)\|_{e,\Omega} = \{\|u(t)\|^2_{H^1(\Omega)} + \|u_t(t)\|^2_{L_2(\Omega)}\}^{\frac{1}{2}}$$

For locally classical solutions of the (NLKG) Morawetz [18] in 1968 proved the following result of local energy decay on compact subsets of $\mathbb{R}^3$.

If u is locally a classical solution of the (NLKG) and Ω is a compact subset of $\mathbb{R}^3$, then $E_\Omega(t) \in L_1$ and $E_\Omega(t) \to 0$ as $t \to \infty$.

In particular it follows that

$$\|u(t)\|_{L_2(\Omega)} \in L_2 \textit{ and } \|u(t)\|_{L_2(\Omega)} \to 0 \textit{ as } t \to \infty$$

For finite energy solutions of the (NLKG) we can conclude, using the scattering theorem and the fact that u_+ is uniformly continuous in H^1_2, that u is uniformly continuous in $L_2(\Omega)$ and by the integrability it follows that

$$\|u(t)\|_{L_2(\Omega)} \to 0 \text{ as } t \to \infty$$

Some of these results can also be recaptured in other ways: By Hölder's inequality one can also directly show that for $p \geq 2$

$$\|u(t)\|_{L_2(\Omega)} = (\int_\Omega 1 \cdot |u(x,t)|^2 dx)^{\frac{1}{2}} \leq (m(\Omega))^{\frac{1}{2}-\frac{1}{p}} \|u(t)\|_{L_p(\Omega)}$$

If $\frac{1}{n} < \delta_p = \frac{1}{2} - \frac{1}{p} < \frac{1}{n-1}$ we have by the nonlinear counterpart of the Strichartz estimates (see Brenner [3],[4] and [8]) that $\|u(t)\|_{L_p} \in L_2$ and it follows again that

$$\|u(t)\|_{L_2(\Omega)} \in L_2$$

This method is of interest in the context of this exposition, since it relates the decay properties of u in suitable L_p-spaces and the decay, and rate of decay, of local energy. A third way to get hold of the limiting behavior is due to Strichartz [28] :

Let $Y_t = H^1_2(\mathbb{R}^n \setminus \{x : \varepsilon(t)t \leq |x| \leq (1-\varepsilon(t)t\})$ where $0 < \varepsilon(t) < 1, \varepsilon(t) \to 0$ as $t \to \infty$. Then, if u_0 is a finite energy solution of (KG),

$$\|u_0(t)\|_{Y_t} \to 0, \text{ as } t \to \infty.$$

With the aid of the scattering theorem this result can be extended to finite energy solutions u of (NLKG).

2 Some previous results on asymptotic behaviour of solutions to the Klein-Gordon equation

2.1 Global solutions of the Cauchy problem for the Klein-Gordon equation

A prerequisite to the study of asymptotic behaviour was earlier the existence and uniqueness of global solutions of the Cauchy problem for a reasonable large class of initial data. Jörgens [15] in 1961 proved that (NLKG) in the case $n = 3, f(u) = u|u|^{\rho-1}, 2 < \rho < 5$ has a classical solution for all $t \geq 0$ if data $\varphi, \psi \in C_0^\infty$. Pecher [21] in 1976 proved that (NLKG) has classical global solutions if

$$f(u) = u|u|^{\rho-1}, 2 < \rho < 1 + \frac{4}{n-2}, 3 \leq n \leq 5.$$

All these results can be extended to more common $f(u)$ such that $\int_0^u f(v)dv \geq 0$. The result of Pecher was carried over by Brenner [3] in 1979 to more general equations where Δ was replaced by a second order positive elliptic operator and the growth conditions on the derivatives of f were somewhat relaxed. This result was further improved by Brenner and von Wahl [9] in 1981. Here f is supposed to satisfy the conditions

$$f(0) = 0, f \in C^2, F(v) = \int_0^v f(u)du \geq 0 \text{ for } v \geq 0 \tag{1}$$

and

$$|f'(u)| = C(1 + |u|)^{\rho-1}, \rho < \frac{n+2}{n-2} \tag{2}$$

and data are assumed to belong to H_1^k for k large enough. The condition (2) is a "natural" growth condition on f in the sense that it together with (1) implies that the energy is equibounded for (KG) and (NLKG).

Uniqueness for finite energy solutions in all dimensions under the conditions (1),(2) was proved by Ginibre and Velo [10]. A short proof of the uniqueness was given by Brenner [8]

2.2 Asymptotic behaviour in time of solutions of the Klein-Gordon equation

The asymptotic behaviour in time of solutions of the Klein-Gordon equation has been more extensively studied during the last thirty years. A basic result was given in [19] which covers the case of dimension $n = 3$ and sufficiently regular solutions. This result was generalised in [20], [21] as regards the assumptions on the nonlinear term and extended in [5] to higher dimensions. The next progress was to relax the regularity assumptions on the solutions in order to cover the case of arbitrary finite energy solutions. In [6] it was proved that arbitrary large finite energy solutions of (NLKG) for space dimension $n \geq 3$ exhibit some of the time decay properties of the solutions of (KG) under the growth condition $1+\frac{4}{n} < \rho < 1+\frac{4}{n-1}$. In [8] these results were obtained for $1+\frac{4}{n} < \rho < 1+\frac{4}{n-2}$.

3 Integral inequalities

An important tool in our investigations of the (NLKG) will be the nonlinear Volterra integral equaiton with singular kernel derived below. The base for the equation is the solution formula for (KG)

$$u_0 = E_0(t)\varphi + E_1(t)\psi$$

where $E_0(t) = \cos(tB)$ and $E_1(t) = B^{-1}\sin(tB)$ are the solution operators of (KG). The solution of (NLKG) then can be written

$$u = u_0 - \int_0^t E_1(t-\tau)f(u)d\tau$$

by which we obtain that

$$\|u\|_{X^{s'}_{p'}} \leq \|u_0\|_{X^{s'}_{p'}} + \int_0^t \|E_1(t-\tau)f(u)\|_{X^{s'}_{p'}} d\tau$$

where here and in the following, X^s_p denotes H^s_p or $B^{s,q}_p$. A result of Brenner [5] now give us that

$$\|u\|_{X^{s'}_{p'}} \leq \|u_0\|_{X^{s'}_{p'}} + \int_0^t K(t-\tau)\|f(u)\|_{X^s_p} d\tau$$

where $0 \leq K \in L_1 \cap L_{1+\varepsilon}$, for some $\varepsilon > 0$. In Paper I is established that

$$\|f(u)\|_{X^s_p} \leq C\|u\|^{1-\eta}_{X^{s'}_{p'}},\ 0 \leq \eta < \eta_0 \quad \text{(small)}$$

by which we obtain the nonlinear Volterra integral inequality

$$\|u\|_{X^{s'}_{p'}} \leq \|u_0\|_{X^{s'}_{p'}} + C\int_0^1 K(t-\tau)\|u\|^{1-\eta}_{X^{s'}_{p'}} d\tau$$

which is an essential tool in our investigations. In Paper II the kernel $K(t-\tau)$ is replaced by $K(t-\tau)h(\tau)$ where $0 \leq K \in L_{q_0} \cap L_{q_0+\varepsilon}$, for some $\varepsilon > 0$, and $0 \leq h \in L_{q_0'}$, $\frac{1}{q_0} + \frac{1}{q_0'} = 1$.

4 Results of the following papers

4.1 Pointwise L_p-convergence in time

Over the years there have been a number of results on pointwise decay in $L_p(\mathbb{R}^n)$ for solutions of (NLKG). (Strauss 1968 [24], von Wahl 1970 [29], Morawetz and Strauss 1972 [19], Pecher 1974 [20], Brenner 1981-1985 [5], [7], [8]). See e.g. Brenner [5], where it is shown that if

$$\|u_0(t)\|_{H^1_{p'}} \leq C(1+t)^{-n\delta_{p'}}$$

the above property is inherited by u provided that data are sufficiently nice (i.e. have sufficiently many derivatives in L_1), $\delta_{p'} = \frac{1}{2} = \frac{1}{p'} < \min(\frac{1}{n-1}, \frac{\rho-1}{4}), p' \geq 2$ and that ρ satisfies certain conditions.

Results of this kind are important when showing existence and asymptotic completeness of the scattering operator associated with the (NLKG). What is the maximal rate of decay? Investigations of classical solutions of the nonlinear wave equation for critical exponents $\rho = 1 + \frac{4}{n-2}$ shows that we may use $\delta_{p'} = \frac{1}{2}$ ($p' = \infty$) for $n = 3$ (Grillakis [13], [14]). In general, how does the rate of decay of the (NLKG) relate to that of the (KG)? A result from Paper I is the following (See Theorem 5.1 in [1]):

Assume that

$$\|u_0(t)\|_{X^{s'}_{p'}} \leq w(t), t \geq 0, \text{ where } w(t) \to 0 \text{ as } t \to \infty$$

If $\|u_0\|_{X^{s'}_{p'}}$ is bounded for $t \geq 0$, then the above property is inherited by u provided that $K(t)$ and $w(t)$ satisfies certain conditions (again, see [1]).

4.2 Strichartz-type estimates for solutions of (NLKG)

In Paper I is considered "L_p-decay" in time. By Strichartz estimates we know that, for finite energy data and certain combination of q, p' and s',

$$\|u_0(t)\|_{X^{s'}_{p'}} \in L_q$$

It is shown that if $K \in L_1 \cap L_{1+\varepsilon}$ for some $\varepsilon > 0$, where $K(t)$ is the kernel of the nonlinear Volterra integral inequality derived above, then the above property is inherited by u. (See Theorem 9.1) Results of this kind are of importance e.g. to establish that the scattering operator of the (NLKG) is defined on all states of finite energy.

4.3 Long range mean-values

If we, e.g. by some Strichartz estimate, know that

$$\|u_0(t)\|_{X^{s'}_{p'}} \in L_q$$

it follows that

$$\int_{\frac{t}{2}}^{t} \|u_0\|^q_{X^{s'}_{p'}} d\tau \to 0 \text{ as } t \to \infty$$

so that this property is also inherited by u under certain conditions. This leads us to study the long range mean-value of u_0

$$M_{q,X}u_0(t) = \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t} \|u_0\|^q_X d\tau\Big)^{\frac{1}{q}},\ t > 0$$

What is known about the rate of decay of $M_{q,X}u_0(t)$ for nontrivial solutions u_0 of (KG)? If data are sufficiently smooth,i.e. φ and ψ have sufficiently many derivatives in L_1, it is known that [29], [20], [19], [16],

$$c_1 t^{-n\delta_{p'}} \leq M_{q,L_{p'}}u_0(t) \leq c_2 t^{-n\delta_{p'}},\ t \to\ \infty\ , p' \geq 2$$

For finite energy data, not necessarily smooth, it is only known that (Glassey [12])

$$ct^{-n\delta_{p'}} \leq M_{q,L_{p'}}u_0(t)$$

The rate of decay is in general not known. In Paper I it is shown that if, for finite energy data, the long range mean-value of u_0

$$M_{q',X^{s'}_{p'}}u_0(t) \leq w(t)$$

then this property is inherited by u under certain conditions. In Paper II is studied the rate of decay of solutions of the (NLKG). A bound for the rate of decay is given by Brenner in [8]:

If $u \in L^{loc}_q(\mathbb{R}, L^{loc}_p(\mathbb{R}^n))$ is a finite energy solution of (NLKG), then there is a constant $c > 0$ such that

$$M^T_{q,X}u(t) \geq \Big(\frac{1}{T}\int_t^{t+T} \|u(\tau)\|^q_{X_\tau} d\tau\Big)^{\frac{1}{q}} \geq ct^{-n\delta_p}$$

where $p, q \geq 2$ and $X_\tau = L_p\{X : |x| \leq t\} \subseteq X = L_p(R^n)$ provided that $1 + \frac{4}{n} < \rho < 1 + \frac{4}{n-2}$.

The question now arises: If the solutions of (KG) have maximal decay, is this property inherited by the solutions of (NLKG) ? A recent result of Brenner [8] is the following:

If $u_0 \in L_1(X^{s'}_{p'}) \cap L_q(X^{s'}_{p'})$ *is a finite energy solution of (KG) and if* $\|u_0(t)\|_{X^{s'}_{p'}}$ *has uniform decay in* L_q*, then*

$$\int_{\frac{t}{2}}^{t} \|u\|^q_{X^{s'}_{p'}} d\tau \approx \int_{\frac{t}{2}}^{t} \|u_0\|^q_{X^{s'}_{p'}} d\tau, \text{ as } t \to \infty.$$

provided that $1 + \frac{4}{n} < \rho < 1 + \frac{4}{n-2}$ *and that* $K(t)$ *fulfils the "standard" assumptions.*

(We say that $g : \mathbb{R}^+ \to \mathbb{R}^+$ has uniform decay in L_q if, for some $c, t^* \geq 1$, independent of t, $\int_{\frac{t}{4}}^{\frac{t}{2}} g(\tau)^q d\tau \leq c \int_{\frac{t}{2}}^{t} g(\tau)^q d\tau$, for $t > t^*$.) One corollary is that

if u_0 *has maximal decay in* $L_q((\frac{1}{2}, t), X^s_{p'})$*, then also* u *has maximal decay in* $L_q((\frac{t}{2}, t), X^s_{p'})$*.*

The main result in Paper II is the following (for detailed conditions on w and X we refer to that paper):

Main Theorem. *Assume that*

$$M_{q_1, L_r} u(t) = C_1(1+t)^{-\frac{1}{q_1}-\gamma}, \gamma > 0$$

for some $q_1, r \geq 2$ *and that*

$$M_{q,X} u_0(t) \leq w(t)$$

where $w(t)$ *satisfies certain growth conditions (given in Paper II) . Then*

$$M_{q,X} u(t) \leq C w(t)$$

One corollary is the following that gives an example of a case when the maximal rate of the long range mean-value is attained.

If $u \in L_{q'}(L^{\sigma}_{q'})$ *where* $\delta_{q'} = \frac{1}{n+1}$ *and* $M_{q,X} u_0(t)$ *has maximal decay, then* $M_{q,X} u(t)$ *also has maximal decay in the case* $n = 3$ *under certain conditions on* δ_q *and* ρ*.*

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Abstract

The aim of this paper is to prove, that certain properties of space-time means for solutions u to the *nonlinear* Klein-Gordon equation

(NLKG) $$\begin{cases} u_{tt} - \Delta u + m^2 u + f(u) = 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases}$$

are in fact inherited from the solutions u_0 to the corresponding *linear* ($f(u) = 0$) Klein-Gordon equation.

An important first step in our investigation is to establish the nonlinear Volterra integral inequality

(NLVI) $$\|u\|_{X_p^s} \leq \|u_0\|_{X_p^s} + \int_0^t K(t-\tau)\|u\|_{X_p^s}^{1-\eta} d\tau, |\eta| < \eta_0,$$

where X_p^s denotes L_p^s or $B_p^{s,q}$ and where $K(t) \leq Ct^{-1-\alpha}, t \geq 1$, for some $\alpha > 0$.

It is then proved, that if u_0 satisfies any one of the following three conditions

A) $\|u_0\|_{X_p^s} \leq w(t), w(t) \to 0$ as $t \to \infty$.

B) $\|u_0\|_{X_p^s} \in L_q, (|\eta| < \alpha q)$

C) $(\frac{2}{t}\int_{t/2}^t \|u_0\|_{X_p^s}^q d\tau)^{1/q} \leq w(t), w(t) \to 0$ as $t \to \infty$

then the same property is inherited by u under certain extra conditions.

Results of this kind are important tools in proving the existence of everywhere defined scattering operators and uniqueness of weak solutions.

0 Introduction

This chapter will present, and give some background to, the problems studied later.

Let us start by recalling some results for the Klein-Gordon equation

$$\text{(KG)} \qquad \begin{cases} v_{tt} - \Delta v + m^2 v = 0, \quad x \in R^n, \quad t \geq 0 \\ v(x,0) = \varphi(x); v_t(x,0) = \psi(x); \rho, \psi \in L_2^{\frac{1}{2}} \times L_2. \end{cases}$$

where $n \geq 3, m > 0$ and $\Delta = \Delta_x$.

The energy norm, $\|\cdot\|_e$, is defined by

$$\text{(0.1)} \qquad \|v\|_e = \left\{\frac{1}{2}\int(|\nabla_x v|^2 + |v_t|^2 + m^2|v|^2)dx\right\}^{\frac{1}{2}}.$$

In the following, $E_0(t)$ and $E_1(t)$ will denote the solution operators of $v_{tt} - \Delta v + m^2 v = 0$, and the solution of (KG) will be denoted by u_0. We then have that

$$u_0 = E_0(t)\varphi + E_1(t)\psi.$$

The following theorem says that L_2-conditions on the data imply a weak decay estimate for the solution of the (KG), and will later inspire our discussion of the corresponding nonlinear problem.

Theorem 0.1 (Brenner [6]). *Let* $2 \leq q < \infty, 2 \leq r \leq q, s > -\frac{1}{2}, 0 \leq \sigma \leq \frac{1}{2} + s, \delta_q = \frac{1}{2} - \frac{1}{q}$ *and* $\delta_r = \frac{1}{2} - \frac{1}{r}$.

Then $\varphi, \psi \in L_2^{\frac{1}{2}} \times L_2$ *implies that*

$$\|u_0\|_{L_q^{1/2+s-\sigma}} \in L_r^{\sigma}$$

provided that $s = \frac{1}{2}(1 - (n+1+\theta)\delta_q)$*, for some* $\theta \in [0,1]$*, and that* $(n-1+\theta)\delta_q + 2\delta_r > 1 >$
$> (n-1-\theta)\delta_q + 2\delta_r$*, or more generally,*

$$s \geq \frac{1}{2}(1-(n+2)\delta_q) \text{ and } 1 - 2s - 2\delta_q + 2\delta_r \geq 1 \geq -1 + 2s + 2n\delta_q + 2\delta_r$$

where equality signs may be used in the last inequality if $\delta_r \neq 0$.

Remark 0.1. Let X_e denote the set of solutions to (KG) having finite energy. By Theorem 0.1 it follows that for any $u_0 \in X_e$,

$$\|u_0\|_{L_q^{1/2}} \in L_q.$$

The norm on L_p^s is defined by

$$\|v\|_{L_p^s} = \|v\|_{p,s} = \|\mathcal{F}^{-1}(\hat{v}(\xi)(1+|\xi|^2)^{\frac{s}{2}})\|_p$$

where $\mathcal{F}v = \hat{v}$ denotes the Fourier transform of v.

Now consider the nonlinear Klein-Gordon equation

(NLKG) $$\begin{cases} u_{tt} - \Delta u + m^2 u + f(u) = 0, & x \in R^n, \quad t \geq 0 \\ u(x,0) = \varphi(x); u_t(x,0) = \psi(x); \varphi, \psi \in L_2^1 \times L_2 \end{cases}$$

where $n \geq 3, m > 0$ and $\Delta = \Delta_x$.

The nonlinearity $f(u)$ is supposed to satisfy the following conditions, which are motivated by the physics involved.

(i) $$F(u) = \int_0^u f(v)dv \geq 0 \text{ for all } u \in R, \text{ and } f(R) \subseteq R.$$

Condition (i) implies the existence of nonnegative energy for the solutions of the (NLKG), which is essential for the existence of global solutions when no restrictions are assumed on the size of the data.

(ii) $$|f^{(j)}(u)| \leq C|u|^{\rho - j}, j = 0, 1 \text{ and } \rho > 1$$
$$|f''(u)| \leq C|u|^{\rho-2}, \rho > 2$$
$$|f'(u) - f'(v)| \leq C|u - v|^{\rho - 1}, \rho \leq 2.$$

This condition gives growth restrictions at 0 and ∞. ρ merely provides a lower bound for the corresponding $\rho = \rho_0$ valid for $|u| \leq 1, |u - v| \leq 2$, and an upper bound for the corresponding $\rho = \rho_\infty$ valid for $|u| > 1, |u - v| > 2$.

(iii) $$uf(u) - 2F(u) \geq \alpha F(u) \text{ for some } \alpha > 0.$$
$$F(u) \geq \beta |u|^{\tilde{\rho}+1}(1 + |u|)^{-N} \text{ for some } \beta, \tilde{\rho}, N$$
$$\text{with } \beta > 0, \rho \leq \tilde{\rho} < \infty \text{ and } N \geq 0.$$

The condition $uf(u) - 2F(u) \geq 0$ ensures that no standing waves will appear as solutions. The appearance of standing waves would make decay and a scattering result impossible. The condition also implies the decay of local energy ([8]).

A consequence of (i) is that the energy norm of u is bounded. To realize this, we introduce the energy, $E(u)$, of the solution of the (NLKG) defined by

(0.2) $$E(u) = \|u\|_e^2 + \int F(u)dx.$$

If we multiply the nonlinear Klein-Gordon equation by u_t and integrate over space, it follows that

(0.3) $$E(u) = \text{constant}, t \geq 0,$$

and by (i), (0.2) and (0.3), we obtain that

(0.4) $$\|u\|_e \leq C.$$

On an interval where u exists as a solution to the (NLKG), we have that

(0.5) $$u = u_0 - \int_0^t E_1(t - \tau) f(u) d\tau.$$

Inspired by Theorem 0.1 and the above equation we now want to investigate what kind of conditions on u_0 will be inherited by u. The following $L_p - L_{p'}$-estimate for the (KG), will be an essential tool in our investigation. The proof is a consequence of the $L_p - L_q$-estimates proved by Marshall, Strauss and Wainger [7].

Lemma 0.1 (Brenner [5]).

$$(*) \qquad \begin{cases} \textit{Let } \frac{1}{p}+\frac{1}{p'}=1, \delta=\frac{1}{2}-\frac{1}{p'}, 2\le p'<\infty \textit{ and assume that} \\ \delta(n+1+\theta)\le 1+s-s', \textit{ for some } \theta\in(0,1] \textit{ and } s,s'\ge 0. \end{cases}$$

Then, if X_p^s *denotes* L_p^s *or* $B_p^{s,q}$, $\quad 1\le q\le\infty$,

$$\|E_1(t)g\|_{X_{p'}^{s'}} \le K(t)\|g\|_{X_p^s}, t\ge 0, \tag{0.6}$$

where

$$K(t)\le C\begin{cases} t^{-(n-1+\theta)\delta}, & t\ge 1 \\ t^{-(n-1-\theta)\delta}, & 0<t<1 \end{cases} \tag{0.7}$$

Remark 0.2. By the condition $\delta(n+1+\theta)\le 1+s-s'$, for $0<t<1$, we may choose $K(t)\le Ct^{1+s-s'-2n\delta}$.

The norm on $B_p^{s,q}$ is defined by

$$\|g\|_{B_p^{s,q}} = \|g\|_p + \Big(\int_0^1 (t^{-\sigma}\sum_{|\alpha|=S} w_p(t,D^\alpha g))^q \frac{dt}{t}\Big)^{\frac{1}{q}}$$

where $w_p(t,z)=\sup_{|h|\le t}\|z_h-z\|_p, z_h(x)=z(x+h), s=\sigma+S, 0<\sigma<1$, and S is an integer.

Now, if $(*)$ is satisfied, by the equation (0.5) and (0.6), we obtain that

$$\|u\|_{X_{p'}^{s'}} \le \|u_0\|_{X_{p'}^{s'}} + \int_0^t K(t-\tau)\|f(u)\|_{X_p^s}d\tau. \tag{0.8}$$

Remark 0.3. If $(*)$ is somewhat extended (see chapter 2), the nonlinearity $f(u)$ satisfies

$$\|f(u)\|_{X_p^s} \le C\|u\|_{X_{p'}^{s'}}^{1-\gamma}, |\gamma|=\eta.$$

In this paper we will consider three different conditions on u_0. These conditions, denoted by A, B and C below, are interesting from a purely mathematical point of view, but may also be justified by physical considerations. That is done in chapter 1.

A. Pointwise convergence in time:

$$\|u_0(t)\|_{X_{p'}^{s'}} \le w(t), t\ge 0, \text{ where } w(t)\to 0 \text{ as } t\to\infty.$$

If $\|u_0(t)\|_{X_{p'}^{s'}}$ is bounded for $t\ge 0$, the above property is inherited by $\|u(t)\|_{X_{p'}^{s'}}$ provided that $w(t)$ satisfies certain conditions. (See Theorem 5.1).

B. "L_q-decay" in time:

$$\|u_0(t)\|_{X_{p'}^{s'}} \in L_q.$$

If, in (0.8), $K\in L_1\cap L_{1+\epsilon}$ for some $\epsilon>0$, the above property is inherited by $\|u(t)\|_{X_{p'}^{s'}}$. (See Theorem 9.1.).

C. "Interpolation" between A and B:

$$\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t} \|u_0\|^q_{X^{s'}_{p'}} d\tau\Big)^{\frac{1}{q}} \le w(t), t > 0, \text{ where } w(t) \to 0 \text{ as } t \to \infty.$$

If, in (0.8), $K \in L_1 \cap L_{1+\epsilon}$, for some $\epsilon > 0$, and $\int_{\frac{t}{2}}^{t} K_N(t-\tau)\|u(\tau)\|_{X^{s'}_{p'}} d\tau$ decays to zero "roughly" as $w(t)$ for large t's, the above property is inherited by $\|u(t)\|_{X^{s'}_{p'}}$, provided that $w(t)$ satisfies certain conditions. (See Theorem 10.1).

It is a question for future research, whether or not we actually have convergence in the linear case with small data.

1 Applications on scattering theory

To better see the meaning of the physics involved, let us rewrite the (NLKG) as a system (NLKG)$_s$ by setting $\phi = \binom{u}{u_t}$.

(NLKG)$_s$
$$\frac{d\phi}{dt} = iH_0\phi + P\phi, \quad \phi(0) = \phi_0$$
,

where $iH_0 = \begin{pmatrix} 0 & I \\ \Delta - m^2 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & 0 \\ -G & 0 \end{pmatrix}$ with $Gu = f(u)$.

If $P = 0$ we have the corresponding linear system $(KG)_s$. The solutions of $(NLKG)_s$ describe a one parameter group $U(t)$ acting on a Hilbert space $X : \phi(t) = U(t)\phi_0, \phi_0 \in X$.

In this type of problems it is natural to take as a Hilbert space, the space defined by the energy norm

$$|\phi|_X = \Big(\frac{1}{2}\int (|\nabla_x\phi_1|^2 + |\phi_2|^2 + m^2|\phi_1|^2)dx\Big)^{\frac{1}{2}}.$$

Since this norm is an invariant for the solutions of $(KG)_s$, the solution operator $U_0(t)$ of $(KG)_s$ is unitary on X.

The following theorem says, that the unperturbed linear system "describes" the nonlinear system asymptotically as $t \to \pm\infty$, if the nonlinearity satisfies a certain conditon.

Theorem 1.1 (Strauss[13]). *Assume that $\phi \in X$ is a solution of $(NLKG)_s$ such that*

(1.1)
$$\int_{-\infty}^{\infty} |P\phi|_X dt < \infty.$$

Then there exist unique $\psi_-, \psi_+ \in X$ such that

$$|\phi(t) - U_0(t)\psi_\pm|_X \to 0 \text{ as } t \to \pm\infty.$$

Remark 1.1. The operator

$$S : U_0(t)\psi_- \to U_0(t)\psi_+$$

is called the scattering operator.

Now, let us examine condition (1.1). As

$$|P\phi|_X = \left|\begin{pmatrix} 0 & 0 \\ -G & 0 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix}\right|_X = \left|\begin{pmatrix} 0 \\ -f(u) \end{pmatrix}\right|_X =$$

$$= \left(\frac{1}{2}\int |f(u)|^2 dx\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\|f(u(t))\|_2$$

An alternative formulation of (1.1) is

$$(1.1)_2 \qquad \int_{-\infty}^{\infty} \|f(u)\|_2 dt < \infty.$$

If the nonlinearity satisfies $(1.1)_2$, the existence and asymptotic completeness of the scattering operator associated with the (NLKG) follows. (See the excellent book [12, Ch X 1.13] by Reed and Simon.)

For certain restrictions on ρ, one can prove that $(1.1)_2$ holds in the case of small data. The proof is carried out via L_p-decay estimates of the kind

$$(1.2) \qquad \|u(t)\|_{p',1} \leq C(1+t)^{-n\delta}, t \geq 0,$$

where $2 \leq p' < \infty$ and $\delta = \frac{1}{2} - \frac{1}{p'} < \frac{1}{n-1}$.

Thus, the following question is crucial:
If u_0 satisfies and L_p^s-estimate of type (1.2), is this property inherited by u?

Remark 1.2. From L_p-estimates of this kind one may also obtain maximum-norm decay results [1] using the jacking-up process suggested by Pecher [10].

The next theorem will give us another important condition.

Theorem 1.2 (Pecher [11]). *Let* $1+\frac{4}{n-1} < \rho < 1+\frac{4}{n-2}, n \geq 3, \delta_{q'} = \frac{1}{2}-\frac{1}{q'} = \frac{1}{n+1}$ *and* $\gamma+\sigma > \frac{n}{2(n+1)}$.

Then, if $u_- \in X_e$*, there exists a solution* u *of the (NLKG) such that*

$$\|u - u_-\|_e \to 0 \text{ as } t \to \infty.$$

If, in addition, $\|u\|_{L_{q'}^{\gamma+\sigma}} \in L_{q'}$*, then there also exists a solution* $u_+ \in X_e$ *such that*

$$\|u - u_+\|_e \to 0 \text{ as } t \to \infty.$$

Now recall the result of Remark 0.1 that for any $u_0 \in X_e$ we have that

$$\|u_0\|_{L_{q'}^{1/2}} \in L_{q'}.$$

If we could prove that this property is inherited by u, then by Theorem 1.2 it follows that the scattering operator for the (NLKG) is defined on all states of finite energy.

2 The integral inequality

In the present chapter we will derive an integral inequality, namely (2.7), which will be of crucial importance for the rest of the paper.

To begin with, we must sharpen conditon $(*)$ on page 4 to $(*)_s$.

$$(*)_s \quad \begin{cases} \textit{Let } \frac{1}{p}+\frac{1}{p'}=1, \delta=\frac{1}{2}-\frac{1}{p'}, 2\le p'<\infty \textit{ and assume that} \\ \textit{for some } \theta\in(0,1]: (n-1+\theta)\delta>1>(n-1-\theta)\delta, \\ \textit{and that for some } s,s'\in[0,1]: (n+1+\theta)\delta=1+s-s' \end{cases}$$

Remark 2.1. Under assumption $(*)_s$ Lemma 0.1 holds with $K\in L_1(R^+)\cap L_{1+\epsilon}(R^+)$ if $\epsilon>0$ is sufficiently small.

Remark 2.2. Let $r=r(p',s')=\frac{p'n}{n-p's'}$ (i.e. $\frac{1}{r}=\frac{1}{p'}-\frac{s'}{n}$) where p',s' satisfy $(*)_s$. Then

$$r=\frac{2n}{n-2(1+s)+2(1+\theta)\delta}$$

and so, for each $\bar{r}$ sufficiently close to r, there exist $\bar{p'},\bar{s'}$ such that $\bar{r}=r(\bar{p'},\bar{s'})$ and $(*)_s$ holds for $\bar{p'},\bar{s'}$.

If $(*)_s$ holds, we are able to estimate the nonlinearity $f(u)$ in the following way:
Let $\rho_\eta=1+2\delta(2-\eta)$ *and* $\rho_n=\frac{n(1+4\delta)+2s'-2(1+s)}{n-2}$. *For* $s=1$ *we set* $\rho_n=\bar{\rho}_n$.

Lemma 2.1. *Assume that* $(*)_1$ *and* (ii) *holds, and that for some* $\eta\in(0,1]$*:*

$$(2.1)_\rho \qquad \rho_\eta\le\rho\le\bar{\rho}_n-\eta\frac{2(n\delta+s'-1)}{n-2}, \quad \rho\ge 2-\eta.$$

Then

$$(2.1) \qquad \|f(u)\|_{p,1}\le C\|u\|_{2,1}^{\rho-1+\eta}\|u\|_{p',s'}^{1-\eta}$$

Corollary 2.1. *Assume that* $(*)_1$ *and* (ii) *holds, and that for some* $\eta\ge 0$*:*

$$(2.2)_\rho \qquad \rho_{-\eta}\le\rho\le\bar{\rho}_n, \rho\ge 2+\eta.$$

Then

$$(2.2) \qquad \|f(u)\|_{p,1}\le C\|u\|_{2,1}^{\rho-1-\eta}\|u\|_{p',s'}^{1+\eta}.$$

For $n\ge 6$ similar Besov space inequalities are used. For fractional derivatives Lemma 2.1 is replaced by

Lemma 2.2. *Assume that* $(*)_s$ *and* (ii) *holds, and that for some* $\eta\in(0,1]$*:*

$$(2.3)_\rho \qquad \rho_\eta\le\rho\le\rho_n-\eta\frac{2(n+\delta+s')}{n-2}, 1+\frac{s-s'}{1-s'}-\eta<\rho\le 2-\eta.$$

Then

$$(2.3) \qquad \|f(u)\|_{B_p^{s,2}}\le C\|u\|_{2,1}^{\rho-1+\eta}\|u\|_{p',s'}^{1-\eta}$$

Corollary 2.2. *Assume that* $(*)_s$ *and (ii) holds, and that for some* $\eta \geq 0$*:*

$$\rho_{-\eta} \leq \rho \leq \rho_n, \quad 1 + \frac{s - s'}{s - s'} + \eta < \rho \leq 2 + \eta. \tag{2.4$_\rho$}$$

Then

$$\|f(u)\|_{B_p^{s,2}} \leq C\|u\|_{2,1}^{\rho-1-\eta}\|u\|_{p',s'}^{1+\eta}. \tag{2.4}$$

For a proof of the above lemmas, see Brenner [3].

Remark 2.3. In the inequalities (2.1) to (2.4) above, $\|u\|_{p',s'}$ may be replaced by $\|u\|_r$ with $\frac{1}{r} = \frac{1}{p'} - \frac{s'}{n}$, i.e. $r = r(p', s')$ which is defined in Remark 2.2.

Next we will estimate $\|u\|_{2,1}$ in terms of $\|u\|_e$. As

$$\|u\|_{p,s} \sim \|u\|_p + \sum_{|\alpha|=s} \|D^\alpha u\|_p$$

if s is an integer and $1 < p < \infty$, we obtain that

$$\|u\|_{2,1} \leq C_1 \sum_{|\alpha|\leq 1} \|D^\alpha u\|_2 \leq$$

$$\leq C_1(\|u\|_2 + \sum_{j=1}^{n} \|\nabla_x u\|_2) \leq$$

$$\leq C_1\Big(\frac{1}{m} + n\Big)\sqrt{2}\|u\|_e$$

and so, since $\|u\|_e \leq C$, by the above estimate

$$\|u\|_{2,1} \leq C_2 \tag{2.5}$$

which together with the results of LemmaS 2.1-2.2 and the inclusions

$$B_p^{s,q} \supseteq L_p^s, 1 < p \leq 2 \text{ and } B_{p'}^{s',q'} \subseteq L_{p'}^{s'}, 2 \leq p' < \infty$$

gives us

$$\|f(u)\|_{X_p^s} \leq C\|u\|_{X_{p'}^{s'}}^{1-\gamma}, \gamma = \eta \text{ or } \gamma = -\eta. \tag{2.6}$$

Finally, by (0.8) and (2.6), we have that

$$\|u\|_{X_{p'}^{s'}} \leq \|u_0\|_{X_{p'}^{s'}} + C\int_0^t K(t-\tau)\|u\|_{X_{p'}^{s'}}^{1-\gamma} d\tau. \tag{2.7}$$

3 Local weak decay

The two lemmas presented in this chapter, will later make it possible to derive a kind of local weak decay of $\|u\|_{X^{s'}_{p'}}$.

The importance of Lemma 3.2 is better understood, if one recalls that, by Remark 2.3,

$$\|u(t)\|_{X^{s'}_{p'}} \leq \|u_0(t)\|_{X^{s'}_{p'}} + C\int_0^t K(t-\tau)\|u(\tau)\|_r d\tau$$

Lemma 3.1 (Morawetz and Strauss [9]). *Assume that for the* (NLKG)

(3.1) *data* φ, ψ *have compact supports contained in* $\{x : |x| \leq R_0 < \infty\}$

and let $\epsilon_0, T, a > 0$.
Then there exists a b depending boundedly on ϵ_0, T, a *and the energy* $E(u)$ *but not on* R_0, *and there exists an interval* $I = [t^* - 2T, t^*] \subseteq [a, b]$ *such that*

$$\iint_I F(u(x,t))dxdt < \epsilon_0. \tag{3.2}$$

Remark 3.1. The restriction (3.1) on the data for the (NLKG) can be removed in the applications that we have in mind.

From Lemma 3.1, (iii) and Remark 2.1-2.2 follows:

Lemma 3.2. *Assume that* $(*)_s$ *holds and that* $K_N \in L_1 \cap L_{1+\epsilon}$.
Then, for $t \in I^* = [t^* - T, t^*]$ *and* γ *small,*

$$\int_{t-T}^t K_N(t-\tau)\|u(\tau)\|_r^{1-\gamma} d\tau \leq A_0 \epsilon_0^{\kappa(1-\gamma)} \Big(\int_{t-T}^t K_N(t-\tau)\|u(\tau)\|_{\bar{r}} d\tau\Big)^{\nu(1-\gamma)} \tag{3.3}$$

where $0 < \nu, \kappa, A_0 < \infty$ *are independent of* $t, r = r(p', s')$ *where* $(*)_s$ *holds for* p', s' *and* $\bar{r} = r(\bar{p}, \bar{s'})$ *is so close to* r *that* $(*)_s$ *holds for* $\bar{p'}, \bar{s'}$.

Proof: Let $r = \frac{p'n}{n - p's'}$, where p', s' satisfy $(*)_s$, and let $\delta > 0$.

Then, for $t \in I^*$, by (iii) and Lemma 3.1

$$\begin{aligned}
&\int_{t-T}^t K_N(t-\tau)\|u(\tau)\|_r^{1-\gamma} d\tau \leq \\
&\leq \int_{t-T}^t K_N(t-\tau)\Big(\int (1+|u|)^{N\delta}|u|^{r-\delta}|u|^{\delta}(1+|u|)^{-N\delta}dx\Big)^{\frac{1-\gamma}{r}} d\tau \leq \\
&\leq \int_{t-T}^t K_N(t-\tau)\Big(\int (1+|u|)^{N\delta q}|u|^{(r-\delta)q}dx\Big)^{\frac{1-\gamma}{rq}} \cdot \Big(\int |u|^{\tilde{\rho}+1}(1+|u|)^{-N(\tilde{\rho}+1)}dx\Big)^{\frac{\delta(1-\gamma)}{r(\tilde{\rho}+1)}} d\tau \leq
\end{aligned}$$

$$
\begin{aligned}
&\le C_1\int_{t-T}^{t}K_N(t-\tau)\Big(\int|u|^{\bar r}dx\Big)^{\frac{1}{\bar r}\cdot\frac{\bar r(1-\gamma)}{rq}}\cdot\Big(\int\beta|u|^{\tilde\rho+1}(1+|u|)^{-N}dx\Big)^{\frac{\delta(1-\gamma)}{r(\tilde\rho+1)}}d\tau\le\\
&\le C_1\int_{t-T}^{t}K_N(t-\tau)^{1-\frac{\bar r(1-\gamma)}{rq}}\cdot K_N(t-\tau)^{\frac{\bar r(1-\gamma)}{rq}}\|u\|_{\bar r}^{\frac{\bar r(1-\gamma)}{rq}}\cdot\Big(\int F(u(x,\tau))dx\Big)^{\frac{\delta(1-\gamma)}{r(\tilde\rho+1)}}d\tau\le\\
&\le C_1\Big(\int_0^T K_N(\tau)^{\zeta}d\tau\Big)^{\frac{1}{\zeta}\cdot\frac{rq-\bar r(1-\gamma)}{rq}}\cdot\Big(\int_{t-T}^{t}K_N(t-\tau)\|u\|_{\bar r}d\tau\Big)^{\frac{\bar r(1-\gamma)}{rq}}\cdot\\
&\cdot\Big(\int_{t-T}^{t}\int F(u(x,\tau))dxd\tau\Big)^{\frac{\delta(1-\gamma)}{r(\tilde\rho+1)}}\le\\
&\le C_1\|K_N\|_{\zeta}^{\frac{rq-\bar r(1-\gamma)}{rq}}\Big(\int_{t-T}^{t}K_N(t-\tau)\|u\|_{\bar r}d\tau\Big)^{\frac{\bar r(1-\gamma)}{rq}}\cdot\Big(\int_{t^*-2T}^{t^*}\int F(u(x,\tau))dxd\tau\Big)^{\frac{\delta(1-\gamma)}{r(\tilde\rho+1)}}\le\\
&\le A_0\epsilon_0^{\kappa(1-\gamma)}\Big(\int_{t-T}^{t}K_N(t-\tau)\|u\|_{\bar r}d\tau\Big)^{\nu(1-\gamma)}
\end{aligned}
$$

by which (3.3) follows with $A_0=C_1\|K_N\|_{\zeta}^{\frac{rq-\bar r(1-\gamma)}{rq}}$, $\kappa=\frac{\delta}{r(\tilde\rho+1)}$ and $\nu=\frac{\bar r}{rq}$, as, by choosing δ small enough, it is possible to accomplish that

$$q=\frac{\tilde\rho+1}{\tilde\rho+1-\delta}$$

is close to 1, $\bar r$ is so close to r that $\bar r=r(\bar{p'},\bar{s'})$ with $\bar{p'},\bar{s'}$, satisfying $(*)_s$, as the conditions on r reads $N\delta q+(r-\delta)q\le\bar r$, $(r-\delta)q\ge 2$, and finally that

$$\zeta=\frac{\tilde\rho+1}{\tilde\rho+1-\frac{\delta q(1-\gamma)}{rq-\bar r(1-\gamma)}}$$

is so close to 1 that K_N^{ζ} is integrable. ///

4 Recapitulation

For simplicity we now introduce the following notation

$\|u_0(t)\|_{X_{p'}^{s'}}=U_0(t)$ and $\|u(t)\|_{X_{p'}^{s'}}=U(t)$.

$U(t)$ satisfies the nonlinear Volterra integral inequality

$$
(\#)\qquad\left\{\begin{array}{l}
U(t)\le U_0(t)+\int_0^t K(t-\tau)U(\tau)^{1-\eta}d\tau,\ |\eta|<\eta_0\quad\text{(small)},\\
\textit{where } 0\le K\in L_1\cap L_{1+\epsilon}, \textit{for some } \epsilon>0, \textit{and}\\
K(t)\le C\left\{\begin{array}{ll} t^{-1-\alpha}, & t\ge 1\\ t^{-1+\alpha}, & 0<t\le 1\end{array}\right., \textit{ for some } \alpha>0.
\end{array}\right.
$$

The above conditons on U and K will be referred to as (#). Functions that satisfy (#) will be studied in this paper, mainly with respect to the following questions:

1. $U_0(t)\le w(t), w(t)\to 0$ as $t\to\infty\Rightarrow U(t)\le Cw(t)$?

2. Weak decay properties.

3. $U_0 \in L_q \Rightarrow U \in L_q$?

4. $\left(\frac{2}{t}\int_{\frac{t}{2}}^{t} U_0(\tau)^q (d\tau)\right)^{\frac{1}{q}} \leq w(t), w(t) \to 0$ as $t \to \infty \Rightarrow \left(\frac{2}{t}\int_{\frac{t}{2}}^{t} U(\tau)^q d\tau\right)^{\frac{1}{q}} \leq Cw(t)$?

5 Pointwise convergence to zero

In this chapter we will investigate when the following property of $U_0(t)$:

$$U_0(t) \leq w(t), w(t) \to 0 \text{ as } t \to \infty,$$

is inherited by $U(t)$.

Lemma 5.1. *Assume that*

$$U(t) \leq U_0(t) + \int_0^t K(t-\tau)U(\tau)^{1-\eta} d\tau, 0 < \eta < 1,$$

where $K \in L_1$.

Then $U_0(t) \leq C_0, t \geq 0$, *implies that* $U(t) \leq C_1, t \geq 0$.

Proof: By the integral inequality

$$\begin{aligned} U(t) &\leq U_0(t) + \Big(\sup_{0\leq\tau\leq t} U(\tau)\Big)^{1-\eta} \int_0^t K(\tau)d\tau \leq \\ &\leq C_0 + \Big(\sup_{\tau\geq 0} U(\tau)\Big)^{1-\eta} \|K\|_1 \end{aligned} \tag{5.1}$$

It follows that $\sup_{t\geq 0} U(t) < \infty$. Let $S = \sup_{t\geq 0} U(t)$. Then, by (5.1),

$$S \leq C_0 + \|K\|_1 S^{1-\eta} \tag{5.2}$$

Asume that $S \geq (2\|K\|_1)^{1/\eta}$. Then, by (5.2),

$$S \leq C_0 + \frac{1}{2}S^\eta \cdot S^{1-\eta} = C_0 + \frac{1}{2}S.$$

Thus $S \geq (2\|K\|_1)^{1/\eta}$ implies that $S \leq 2C_0$.

Since $K \in L_1$, it follows that

$$S \leq \max((2\|K\|_1)^{1/\eta}, 2C_0) < \infty. \quad ///$$

Lemma 5.2. *Assume that U and K satisfy (#) and $U_0(t) \leq C_0, t \geq 0$. Then, if also*

$$U_0(t) < \frac{\epsilon}{4}, t \in I^*,$$

it follows that

$$\sup_{t \in I^*} U(t) < \epsilon.$$

Proof: By Lemma 5.1 we have that $\sup_{t \geq 0} \|u(t)\|_r \leq C_1, 2 \leq r \leq \bar{r}$. The integral inequality and Lemma 3.2 gives us that, for $t \in I^*$,

$$U(t) \leq U_0(t) + \int_0^{t-T} K(t-\tau)U(\tau)^{1-\eta} d\tau + \int_{t-T}^{t} K(t-\tau)U(\tau)^{1-\eta} d\tau <$$

$$< \frac{\epsilon}{4} + \left(\sup_{\tau \geq 0} U(\tau)\right)^{1-\eta} \int_T^t K(\tau) d\tau + A_0 \epsilon_0^{\kappa(1-\eta)} \left(\int_{t-T}^{t} K(t-\tau)\|u(\tau)\|_{\bar{r}} d\tau\right)^{\nu(1-\eta)} \leq$$

$$\leq \frac{\epsilon}{4} + C_1^{1-\eta} C \int_T^{\infty} \tau^{-1-\alpha} d\tau + A_0 \epsilon_0^{\kappa(1-\eta)} \left(\sup_{\tau \geq 0} \|u(\tau)\|_{\bar{r}}\right)^{\nu(1-\eta)} \left(\int_0^T K(\tau) d\tau\right)^{\nu(1-\eta)} \leq$$

$$\leq \frac{\epsilon}{4} + \frac{C_1^{1-\eta} C}{\alpha} T^{-\alpha} + A_0 (C_1 \|K\|_1)^{\nu(1-\eta)} \epsilon_0^{\kappa(1-\eta)} \leq \frac{3\epsilon}{4}$$

if $T \geq \Omega_1 = (\frac{4CC_1^{1-\eta}}{\alpha\epsilon})^{\frac{1}{\alpha}}$ and $\epsilon_0 \leq (\frac{\epsilon}{4A_0})^{\frac{1}{\kappa(1-\eta)}} (C_1\|K\|_1)^{-\frac{\nu}{\kappa}}$ and it follows that $\sup_{t \in I^*} U(t) < \epsilon$. ///

Lemma 5.3. *Assume that U and K satisfy (#) and $U_0(t) \leq C_0, t \geq 0$. Let $\epsilon < (4\|K\|_1)^{-1/\eta}$ so that $\|K\|_1 \epsilon^{1+\eta} < \frac{\epsilon}{4}$ and $T > \max[\Omega, \frac{1}{\epsilon_1}]$. ($\epsilon_1 > 0$ will be specified later.)*

Then $U_0(t) < \frac{\epsilon}{4}, t \geq t^* - T$ implies that $U(t) < \epsilon, t \geq t^* - T$.

Proof: By Lemma 5.2 we know that $U(t) < \epsilon, t^* - T \leq t \leq t^*$. Let

$$t^{**} = \sup_{t \geq t^* - T} \{t : U(t) < \epsilon\}$$

If $t^{**} = \infty$ the proposition of Lemma 5.3 is obvious, so assume that $t^{**} < \infty$ in order to arrive at a contradiction. By Lemma 5.1

$$\sup_{t \geq 0} U(t) = C_1 < \infty.$$

Let $t \in (t^{**}, t^{**} + \epsilon_1]$ where $0 < \epsilon_1 \leq (\frac{\alpha\epsilon}{4C_1^{1+\eta}})^{1/\alpha}$ so that $\frac{C_1^{1+\eta}}{\alpha} \cdot \epsilon_1^{\alpha} \leq \frac{\epsilon}{4}$ and that $\frac{C_1^{1+\eta}}{\alpha} \cdot T^{-\alpha} < \frac{\epsilon}{4}$.

By (#)

$$U(t) \leq U_0(t) + \Big(\int_0^{t-T} + \int_{t-T}^{t^{**}} + \int_{t^{**}}^{t}\Big) K(t-\tau)U(\tau)^{1+\eta} d\tau \leq$$

$$\leq U_0(t) + C_1^{1+\eta}\int_T^t K(\tau)d\tau + \Big(\sup_{t\in I^{**}} U(\tau)\Big)^{1+\eta} \int_{t-t^{**}}^{T} K(\tau)d\tau + C_1^{1+\eta}\int_0^{t-t^{**}} K(\tau)d\tau <$$

$$< \frac{\epsilon}{4} + C_1^{1+\eta}\int_T^{\infty} K(\tau)d\tau + \epsilon^{1+\eta}\|K\|_1 + C_1^{1+\eta}\int_0^{\epsilon_1} K(\tau)d\tau \leq$$

$$\leq \frac{\epsilon}{4} + \frac{C_1^{1+\eta}}{\alpha}T^{-\alpha} + \|K\|_1\epsilon^{1+\eta} + \frac{C_1^{1+\eta}}{\alpha}\epsilon_1^{\alpha} < \epsilon$$

for $t^{**} < t \leq t^{**} + \epsilon_1$ which contradicts the defintion of t^{**}. ///

Theorem 5.1. *Assume that in (#)*

$$U(t) \leq U_0(t) + \int_0^t K(t-\tau)U(\tau)^{1+\eta} d\tau, |\eta| < \eta_0 < 1,$$

where $K \in L_1, U_0(t) \leq C_0, t \geq 0$ *and* $U_0(t) < \frac{\epsilon_1}{4}, t \in [\frac{w}{2}, t^*]$.

Assume also that $w(t)$ is a function such that

1_w. $w(t) \to 0$ as $t \to \infty$

2_w. $\sup\limits_{0\leq t\leq\omega} w(t)^{-1} \leq E$

3_w. $\int_0^{t/2} K(t-\tau)w(\tau)d\tau \leq Bw(t), t > \omega, B \leq (4C_1^{\eta})^{-1}$

4_w. $\sup\limits_{\frac{t}{2}\leq\tau\leq t} w(\tau) \leq Aw(t), t > \omega$

where A, B and E are constants independent of t.

Then $U_0(t) \leq w(t), t \geq 0$ implies that $U(t) \leq Cw(t), t \geq 0$.

Proof: Let $S(t) = \sup\limits_{\tau \geq \frac{t}{2}} \Big(w(\tau)^{-1}U(\tau)\Big)$ and $R(t) = \sup\limits_{0\leq\tau\leq\frac{t}{2}} \Big(w(\tau)^{-1}U(\tau)\Big)$.

Then, for $t > \omega$, by $1_w, 4_w$ and Lemmas 5.2-5.3

$$(5.3)\qquad \int_{t/2}^{t} K(t-\tau)U(\tau)^{1+\eta}d\tau = \int_{t/2}^{t} U(\tau)^{\eta}w(\tau)w(\tau)^{-1}U(\tau)K(t-\tau)d\tau \le$$

$$\le \Big(\sup_{\frac{t}{2}\le\tau\le t} U(\tau)\Big)^{\eta}\Big(\sup_{\frac{t}{2}\le\tau\le t} w(\tau)\Big)\Big(\sup_{\frac{t}{2}\le\tau\le t} w(\tau)^{-1}U(\tau)\Big)\int_0^{t/2} K(\tau)d\tau \le$$

$$\le \Big(\sup_{\tau\ge\frac{\omega}{2}} U(\tau)\Big)^{\eta}\cdot Aw(t)\cdot S(t)\int_0^{\infty} K(\tau)d\tau \le$$

$$\le A\|K\|_1\epsilon_1^{\eta}w(t)S(t) \le \frac{1}{4}w(t)S(t)$$

if $\epsilon_1 \le (4A\|K\|_1)^{-1/\eta}$ and, by Lemma 5.1 and 3_w,

$$(5.4)\qquad \int_0^{t/2} K(t-\tau)U(\tau)^{1+\eta}d\tau = \int_0^{t/2} U(\tau)^{\eta}K(t-\tau)w(\tau)^{-1}U(\tau)w(\tau)d\tau \le$$

$$\le \Big(\sup_{0\le\tau\le\frac{t}{2}} U(\tau)\Big)^{\eta}\Big(\sup_{0\le\tau\le\frac{t}{2}} w(\tau)^{-1}U(\tau)\Big)\int_0^{t/2} K(t-\tau)w(\tau)d\tau \le$$

$$\le C_1^{\eta}R(t)Bw(t) = C_1^{\eta}Bw(t)R(t) \le \frac{1}{4}w(t)R(t)$$

if $B \le (4C_1^{\eta})^{-1}$.

By the integral inequality and 2_w, (5.3) and (5.4), for $t > \omega$,

$$(5.5)\qquad w(t)^{-1}U(t) \le w(t)^{-1}U_0(t) + \frac{1}{4}R(t) + \frac{1}{4}S(t) \le$$

$$\le w(t)^{-1}U_0(t) + \frac{1}{4}\sup_{0\le\tau\le\frac{t}{2}}\Big(w(\tau)^{-1}U(\tau)\Big) + \frac{1}{4}\sup_{\tau\ge\frac{t}{2}}\Big(w(\tau)^{-1}U(\tau)\Big) \le$$

$$\le 1 + \frac{2}{4}\sup_{0\le\tau\le\omega}\Big(w(\tau)^{-1}U(\tau)\Big) + \frac{2}{4}\sup_{\tau\ge\omega}\Big(w(\tau)^{-1}U(\tau)\Big) \le$$

$$\le 1 + \frac{1}{2}\Big(\sup_{0\le\tau\le\omega} w(\tau)^{-1}\Big)\Big(\sup_{0\le\tau\le\omega} U(\tau)\Big) + \frac{1}{2}\sup_{\tau\ge\omega}\Big(w(\tau)^{-1}U(\tau)\Big) \le$$

$$\le 1 + \frac{1}{2}EC_1 + \frac{1}{2}\sup_{\tau\ge\omega}\Big(w(\tau)^{-1}U(\tau)\Big).$$

Let $S = \sup_{t\ge\omega}\Big(w(t)^{-1}U(t)\Big)$. Then, by (5.5)

$$S \le 1 + \frac{1}{2}EC_1 + \frac{1}{2}S$$

so that

$$S \le EC_1 + 2$$

and, as $\sup_{0\le t\le\omega}\Big(w(t)^{-1}U(t)\Big) \le EC_1$, it follows that

$$U(t) \le (EC_1 + 2)w(t), t \ge 0. \quad ///$$

Corollary 5.1. If $K(t) \le C(1+t)^{-1-\epsilon}, t \ge 1, \epsilon > 0$, it is possible to take

$$w(t) = \tilde{C}(1+t)^{-\kappa}, 0 < \kappa \le 1+\delta < 1+\epsilon, 0 < \tilde{C} < \infty.$$

Proof: 1_w and 2_w are trivially true. Since

$$\sup_{\frac{t}{2} \le \tau \le t} \tilde{C}(1+t)^{-\kappa} = \tilde{C}(1+\frac{t}{2})^{-\kappa} = 2^\kappa \tilde{C}(2+t)^{-\kappa} \le 2^\kappa \tilde{C}(1+t)^{-\kappa}$$

4_w is true for all $\kappa > 0$.

The proof of 3_w will be split into three cases.

1. $0 < \kappa < 1$

$$\int_0^{t/2} K(t-\tau)w(\tau)d\tau \le C\tilde{C}(1+\frac{t}{2})^{-1-\epsilon} \int_0^{t/2} (1+\tau)^{-\kappa} d\tau \le$$

$$\le \frac{C}{1-\kappa}\tilde{C}(1+\frac{t}{2})^{-1-\epsilon+1-\kappa} \le \frac{2^\kappa C}{1-\kappa}(1+\frac{t}{2})^{-\epsilon}\tilde{C}(1+t)^{-\kappa} \le$$

$$\le (4C_1^\eta)^{-1} w(t)$$

if $t \ge 2\Big(\dfrac{2^{\kappa+2}CC_1^\eta}{1-\kappa}\Big)^{\frac{1}{\epsilon}} - 2.$

2. $\kappa = 1$

$$\int_0^{t/2} K(t-\tau)w(\tau)d\tau \le C\tilde{C}(1+\frac{t}{2})^{-1-\epsilon} \int_0^{t/2} (1+\tau)^{-1} d\tau =$$

$$= C\frac{\ln(1+\frac{t}{2})}{(1+\frac{t}{2})^\epsilon}\tilde{C}(1+\frac{t}{2})^{-1} \le 2C\frac{\ln(1+\frac{t}{2})}{(1+\frac{t}{2})^\epsilon}\tilde{C}(1+t)^{-1} \le$$

$$\le (4C_1^\eta)^{-1} w(t)$$

if t is large enough, since $\dfrac{\ln(1+\frac{t}{2})}{(1+\frac{t}{2})^\epsilon} \to 0$ as $t \to \infty$.

3. $1 < \kappa \le 1+\delta$

$$\int_0^{t/2} K(t-\tau)w(\tau)d\tau \le C\tilde{C}\Big(1+\frac{t}{2}\Big)^{-1-\epsilon} \int_0^{t/2} (1+\tau)^{-\kappa} d\tau \le$$

$$\le 2^{1+\epsilon}C\tilde{C}(1+t)^{-1-\epsilon} \cdot \frac{1}{\kappa-1} = \frac{2^{1+\epsilon}C}{\kappa-1}(1+t)^{\delta-\epsilon}\tilde{C}(1+t)^{-1-\delta} \le$$

$$\le (4C_1^\eta)^{-1} w(t)$$

if $t \ge \Big(\dfrac{2^{3+\epsilon}CC_1^\eta}{\kappa-1}\Big)^{\frac{1}{\epsilon-\delta}} - 1.$ ///

6 Local L_q-decay

In this chapter we will prove that if $U_0 \in L_q^{\text{loc}}$ this property is inherited by U provided that $K \in L_1 \cap L_{1+\epsilon}$.

Let Q denote the convolution operator $K*$, defined by

$$QU(t) = \int_0^t K(t-\tau)U(\tau)d\tau$$

and let K_N be the kernel of $Q^N = \underbrace{K * K * \ldots * K*}_{N-\text{times}}$, i.e.

$$Q^N U(t) = \int_0^t K_N(t-\tau)U(\tau)d\tau$$

K_N has the following useful properties.

Lemma 6.1. *For $N \in Z^+$ we have that*

(6.1) $$\|K_N\|_1 \leq \|K\|_1^N$$

(6.2) $$\|K_N\|_{r(N)} \leq (\|K\|_{1+\epsilon})^N, r(N) = \frac{1+\epsilon}{1-(N-1)\epsilon}, 0 < \epsilon < \frac{1}{N}$$

Proof: The proof is carried out by induction. Both propositions are trivially true for $N = 1$. Assume they are true for $N = p$.

The last step of the proof is a consequence of Young's inequality:

(6.1)′ $$\|K_{p+1}\|_1 = \|K * K_p\|_1 \leq \|K\|_1 \|K_p\|_1 \leq \|K\|_1^{p+1}$$

(6.2)′ $$\|K_{p+1}\|_{r(p+1)} = \|K * K_p\|_{r(p+1)} \leq \|K\|_{1+\epsilon}\|K_p\|_{r(p)} \leq (\|K\|_{1+\epsilon})^{p+1}$$

as

$$\frac{1}{r(p+1)} = \frac{1-p\epsilon}{1+\epsilon} = \frac{1}{1+\epsilon} + \frac{1-(p-1)\epsilon}{1+\epsilon} - 1 = \frac{1}{1+\epsilon} + \frac{1}{r(p)} - 1. \quad ///$$

Next we will show that if N is large enough and $U_0 \in L_q$, then $K_N * U_0 \in L_\infty$ and $K_N * U \in L_\infty$.

Lemma 6.2. *Assume that $K \in L_1 \cap L_{1+\epsilon}$, some $\epsilon > 0$.*
Then $U_0 \in L_q$ implies that $K_N * U_0 \in L_\infty, N \geq N_0 = [\frac{1+\epsilon}{\epsilon q}] + 1$.

Proof: Let $q' = \frac{1+\tilde{\epsilon}}{1-(N-1)\tilde{\epsilon}}$ where $0 < \tilde{\epsilon} = \frac{1}{Nq-1}$.

Then $\tilde{\epsilon} \leq \epsilon$ for $N \geq N_0$.

By Young's inequality and (6.2)

$$\|K_N * U_0\|_\infty \leq \|K_N\|_{q'}\|U_0\|_q \leq \|K\|_{1+\tilde{\epsilon}}^N \|U_0\|_q < \infty, \quad N \geq N_0. \quad ///$$

Lemma 6.3. *Assume that $U(t) \geq 0$ satisfies the nonlinear Volterra integral inequality*

$$U(t) \leq U_0(t) + \int_0^t K(t-\tau)U(\tau)^{1-\eta}d\tau, 0 < \eta < \eta_0 \leq 1,$$

where $0 \leq K \in L_1 \cap L_{1+\epsilon}$.

Then $U_0 \in L_q$ implies that $K_N * U \in L_\infty, N \geq N_0$.

Proof: Multiply the integral inequality by $K_N(s-t)$ and integrate over $(0, s)$.

$$\int_0^s K_N(s-t)U(t)dt \leq \tag{6.3}$$
$$\leq \int_0^s K_N(s-t)U_0(t)dt + \int_0^s K_N(s-t)\int_0^t K(t-\tau)U(\tau)^{1-\eta}d\tau dt.$$

By Lemma 6.2 $K_N * U_0 \in L_\infty, N \geq N_0$, and for the last term we obtain the estimate

$$Q^N Q U^{1-\eta} = Q Q^N U^{1-\eta} = \int_0^s K(s-t)\int_0^t K_N(t-\tau)U(\tau)^{1-\eta}d\tau dt = \tag{6.4}$$
$$= \int_0^s K(s-t)\int_0^t K_N(t-\tau)^\eta K_N(t-\tau)^{1-\eta}U(\tau)^{1-\eta}d\tau dt \leq$$
$$\leq \int_0^s K(s-t)\Big(\int_0^t K_N(t-\tau)d\tau\Big)^\eta\Big(\int_0^t K_N(t-\tau)U(\tau)d\tau\Big)^{1-\eta}dt \leq$$
$$\leq \Big(\sup_{0\leq t\leq s}\int_0^t K_N(t-\tau)U(\tau)d\tau\Big)^{1-\eta}\int_0^s K(t)\Big(\int_0^{s-t} K_N(\tau)d\tau\Big)^\eta dt \leq$$
$$\leq \|K\|_1^{1+\eta N}\Big(\sup_{t\geq 0}\int_0^t K_N(t-\tau)U(\tau)d\tau\Big)^{1-\eta}.$$

Altogether, as $K \in L_1$, by (6.3) together with Lemma 6.2 and (6.4) we have that

$$\sup_{s\geq 0}\int_0^s K_N(s-t)U(t)dt \leq C + C\Big(\sup_{t\geq 0}\int_0^t K_N(t-\tau)U(\tau)d\tau\Big)^{1-\eta} \tag{6.5}$$

where $C < \infty$.

Let

$$\sup_{t\geq 0}\int_0^t K_N(t-\tau)U(\tau)d\tau = M.$$

Assume that $M \geq (2C)^{1/\eta}$. Then by (6.5)

$$M \leq C + C\cdot M^{1-\eta} \leq C + \frac{1}{2}M^{1-\eta}\cdot M^\eta = C + \frac{M}{2}.$$

Thus $M \geq (2C)^{1/\eta}$ implies that $M \leq 2C$ and it follows that

$$M \leq \max((2C)^{1/\eta}, 2C). \quad ///$$

Corollary 6.3. $K_N * U^{1-\eta} \in L_\infty, N \geq N_0$.

Proof:

$$\int_0^t K_N(t-\tau)U(\tau)^{1-\eta}d\tau \leq \Big(\int_0^t K_N(\tau)d\tau\Big)^\eta \Big(\int_0^t K_N(t-\tau)U(\tau)d\tau\Big)^{1-\eta} \leq$$

$$\leq \|K_N\|_1^\eta \Big(\sup_{t\geq 0}\int_0^t K_N(t-\tau)U(\tau)d\tau\Big)^{1-\eta} \leq \|K\|_1^{\eta N} M^{1-\eta}$$

by which $\|K_N * U^{1-\eta}\|_\infty \leq \|K\|_1^{\eta N} M^{1-\eta} < \infty$. ///

Remark 6.3. By Lemma 3.2 and Lemma 6.3 it follows that

$$\int_{t-T}^t K_N(t-\tau)U(\tau)^{1-\eta}d\tau \leq C_0\epsilon_0^{\kappa(1-\eta)}, t \in I^* = [t^*-T, t^*].$$

where $C_0 < \infty$ is independent of t.

By the technique used in Lemma 6.3 it is possible to prove the local version of result 4.3.

Theorem 6.1. *Assume that $U(t) \geq 0$ satisfies*

$$U(t) \leq U_0(t) + \int_0^t K(t-\tau)U(\tau)^{1-\eta}d\tau, 0 \leq \eta < 1,$$

where $0 \leq K \in L_1 \cap L_{1+\epsilon}$.

Then $U_0 \in L_q^{\mathrm{loc}}$ implies that $U \in L_q^{\mathrm{loc}}$.

Proof: In the same way as in the proof of Lemma 6.3, we obtain that $K_N * U \in L_\infty^{\mathrm{loc}}$, and as $L_\infty^{\mathrm{loc}} \subseteq L_q^{\mathrm{loc}}$, it follows that $K_N * U \in L_q^{\mathrm{loc}}$ for $N \geq N_0$.

Now assume that $K_p * U \in L_q^{\mathrm{loc}}$. By the integral inequality for $\eta = 0$ we have that

$$K_{p-1} * U \leq K_{p-1} * U_0 + K_p * U$$

and, since $K_{p-1} * U_0 \in L_q^{\mathrm{loc}}$ by Young's inequality, it follows that $K_{p-1} * U \in L_q^{\mathrm{loc}}$.
By induction we obtain that $K * U \in L_q^{\mathrm{loc}}$, and as

$$U \leq U_0 + K * U$$

it follows that $U \in L_q^{\mathrm{loc}}$. ///

7 Weak decay to zero

Let us first recall some properties of the kernel K. By Lemma 0.1 and $(*)_s$ we can assume that

$$K(t) \leq \begin{cases} Ct^{-1-\alpha}, & t \geq 1 \\ Ct^{-1+\alpha}, & 0 < t \leq 1 \end{cases}, \text{ for some } \alpha > 0.$$

The above property is, in fact, inherited by K_N.

Lemma 7.1. *Assume that, for some $\alpha > 0$,*

$$K(t) \leq \begin{cases} C_1 t^{-1-\alpha}, & t \geq 1 \\ C_1 t^{-1+\alpha}, & 0 < t \leq 1 \end{cases} . \tag{7.1}$$

Then, for $N \in Z^+$,

$$K_N(t) \leq \begin{cases} C_N t^{-1-\alpha}, & t \geq 1 \\ C_N t^{-1+\alpha}, & 0 < t \leq 1 \end{cases} \tag{7.2}$$

where $C_N < \infty$ is independent of t.

Proof: By assumption, (7.2) is true for $N = 1$. Assume that (7.2) is true for $N = p$.

We will use that

$$K_{p+1}(t) = K * K_p(t) = \int_0^t K(t-\tau) K_p(\tau) d\tau$$

and split the proof into three cases.

1. $t \geq 2$

$$\begin{aligned} K_{p+1}(t) &\leq C_1 C_p \Big(2 \int_0^1 (t-\tau)^{-1-\alpha} \tau^{-1+\alpha} d\tau + \int_1^{t-1} (t-\tau)^{-1-\alpha} \tau^{-1-\alpha} d\tau \Big) \leq \\ &\leq C_1 C_p (2(t-1)^{-1-\alpha} \int_0^1 \tau^{-1+\alpha} d\tau + t^{-1-\alpha} \int_1^{t-1} (\tau^{-1} + (t-\tau)^{-1})^{1+\alpha} d\tau) \leq \\ &\leq C_1 C_p (\frac{2}{\alpha}(t-1)^{-1-\alpha} + 2^\alpha t^{-1-\alpha} \int_1^{t-1} (\tau^{-1-\alpha} + (t-\tau)^{-1-\alpha}) d\tau \Big) \leq \\ &\leq C_1 C_p \Big(\frac{2}{\alpha}(t-1)^{-1-\alpha} + \frac{2^{1+\alpha}}{\alpha} t^{-1-\alpha} \Big) \leq \\ &\leq \frac{2 C_1 C_p}{\alpha} \Big(\Big(\frac{t}{t-1} \Big)^{1+\alpha} t^{-1-\alpha} + 2^\alpha t^{-1-\alpha} \Big) \leq \frac{2^\alpha C_1 C_p}{\alpha} \cdot 6 t^{-1-\alpha}. \end{aligned}$$

2. $1 \leq t < 2$

$$\begin{aligned} K_{p+1}(t) &\leq C_1 C_p \Big(2 \int_0^{t-1} (t-\tau)^{-1-\alpha} \tau^{-1+\alpha} d\tau + \int_{t-1}^1 (t-\tau)^{-1+\alpha} \tau^{-1+\alpha} d\tau \Big) \leq \\ &\leq C_1 C_p \Big(2 \int_0^{t-1} \tau^{-1+\alpha} d\tau + t^{-1+2\alpha} \int_{1-\frac{1}{t}}^{\frac{1}{t}} (1-x)^{\alpha-1} x^{\alpha-1} dx \Big) \leq \\ &\leq C_1 C_p \Big(\frac{2}{\alpha}(t-1)^\alpha + t^{-1+2\alpha} \int_0^1 (1-x)^{\alpha-1} x^{\alpha-1} dx \Big) \leq \\ &\leq C_1 C_p \Big(\frac{2}{\alpha} + t^{-1+2\alpha} \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)} \Big) \leq \\ &\leq C_1 C_p \Big(\frac{2}{\alpha} \cdot 2^{1+\alpha} t^{-1-\alpha} + \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)} \cdot 2^{3\alpha} t^{-1-\alpha} \Big) = \frac{2^\alpha C_1 C_p}{\alpha} C'_\alpha t^{-1-\alpha}. \end{aligned}$$

3. $0 < t \le 1$

$$\begin{aligned} K_{p+1}(t) &\le C_1 C_p \int_0^t (t-\tau)^{-1+\alpha}\tau^{-1+\alpha} d\tau = \\ &= C_1 C_p \cdot t^{-1+\alpha} \cdot t^{\alpha} \int_0^1 (1-x)^{\alpha-1} x^{\alpha-1} dx \le \\ &\le C_1 C_p t^{-1+\alpha} \cdot 1 \cdot \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)} = \frac{2^{\alpha} C_1 C_p}{\alpha} C''_{\alpha} t^{-1+\alpha} \end{aligned}$$

and the Lemma is proven with $C_{p+1} = \frac{2^{\alpha} C_1 C_p}{\alpha} \max[6, C'_{\alpha'}, C''_{\alpha}]$. ///

To proceed we will need the following result of weak decay of U.

Lemma 7.2. *Assume that the conditions (#) are satisfied by K and U. By Remark 6.3 we can also assume that*

$$\int_{t-T}^{t} K_N(t-\tau) U(\tau)^{1-\eta} d\tau \le C_0 \epsilon_0^{\kappa(1-\eta)}, \quad t \in I^* = [t^* - T, t^*].$$

Then $U_0 \in L_q$ implies that for $t^* - \frac{T}{2} \le t \le t^*$

(7.3)
$$\int_{t^*-T}^{t} K_N(t-\tau) U(\tau) d\tau < \epsilon_1, N \ge N_0, T \ge \Omega_0, t^* > \omega_0.$$

(ω_0 and Ω_0 are specified in the proof.)

Proof: By the integral inequality for $\eta > 0$

$$U(\tau) \le U_0(\tau) + \int_0^{\tau} K(\tau - s) U(s)^{1-\eta} ds$$

and it follows that for $t^* - \frac{T}{2} \le t \le t^*$

$$\begin{aligned} &\int_{t^*-T}^{t} K_N(t-\tau) U(\tau) d\tau \le \\ &\le \int_{t^*-T}^{t} K_N(t-\tau) U_0(\tau) d\tau + \int_{t^*-T}^{t} K_N(t-\tau) \int_0^{\tau} K(\tau-s) U(s)^{1-\eta} ds d\tau \le \\ &\le \int_{t^*-T}^{t} K_N(t-\tau) U_0(\tau) d\tau + \int_0^{t^*-T} K(t-\tau) \int_0^{\tau} K_N(\tau-s) U(s)^{1-\eta} ds d\tau + \\ &+ \int_{t^*-T}^{t} K(t-\tau) \int_0^{\tau-T} K_N(\tau-s) U(s)^{1-\eta} ds d\tau + \\ &+ \int_{t^*-T}^{t} K(t-\tau) \int_{\tau-T}^{\tau} K_N(\tau-s) U(s)^{1-\eta} ds d\tau = \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now

$$I_1 \le \|K\|_{1+\tilde{\epsilon}}^N \Big(\int_{t^*-T}^{t} U_0(\tau)^q d\tau \Big)^{\frac{1}{q}}$$

and as $U_0 \in L_q$ there is a ω_0 such that

$$\int_{t^*-T}^{t} U_0(\tau)^q d\tau < 4^{-q} \|K\|_{1+\tilde{\epsilon}}^{-Nq} \epsilon_1^q, \quad t^* > \omega_0,$$

so that $I_1 < \frac{\epsilon_1}{4}$ for $t^* > \omega_0$.

By Corollary 6.3 for $T \geq 2$

$$I_2 \leq \Big(\sup_{\tau \geq 0} \int_0^{\tau} K_N(\tau - s) U(s)^{1-\eta} ds\Big) \int_{t-t^*+T}^{t} K(\tau) d\tau \leq$$

$$\leq \|K\|_1^{\eta N} M^{1-\eta} C \int_{\frac{T}{2}}^{\infty} \tau^{-1-\alpha} d\tau = \frac{\|K\|_1^{\eta N} M^{1-\eta} C}{\alpha} \Big(\frac{T}{2}\Big)^{-\alpha} \leq \frac{\epsilon_1}{4}$$

$$\text{if } T \geq 2 \Big(\frac{4 \|K\|_1^{\eta N} M^{1-\eta} C}{\alpha \epsilon_1}\Big)^{\frac{1}{\alpha}} = \Omega_0'.$$

By Lemma 6.3 and Lemma 7.1

$$I_3 = \int_{t^*-T}^{t} K(t-\tau) \int_0^{\tau - T} K_N(\tau - s)^{\eta} K_N(\tau - s)^{1-\eta} U(s)^{1-\eta} ds d\tau \leq$$

$$\leq \Big(\sup_{\tau \geq 0} \int_0^{\tau} K_N(\tau - s) U(s) ds\Big)^{1-\eta} \int_{t^*-T}^{t} K(t-\tau) \Big(\int_T^{\tau} K_N(s) ds\Big)^{\eta} d\tau \leq$$

$$\leq M^{1-\eta} \Big(\int_T^{\infty} K_N(s) ds\Big)^{\eta} \int_0^{t-t^*+T} K(\tau) d\tau \leq$$

$$\leq M^{1-\eta} \Big(\frac{C_N}{\alpha} T^{-\alpha}\Big)^{\eta} \|K\|_1 \leq \frac{\epsilon_1}{4}$$

$$\text{if } T \geq \Big(\frac{4 \|K\|_1 M^{1-\eta} C_N^{\eta}}{\alpha^{\eta} \epsilon_1}\Big)^{\frac{1}{\alpha \eta}} = \Omega_0''.$$

By (6.1) and Remark 6.3

$$I_4 = \int_{t^*-T}^{t} K(t-\tau) \Big(\int_{\tau - T}^{\tau} K_N(\tau - s) U(s)^{1-\eta} ds\Big) d\tau \leq$$
$$\leq C_0 \epsilon_0^{\kappa(1-\eta)} \int_0^{t-t^*+T} K(\tau) d\tau \leq$$
$$\leq \|K\|_1 C_0 \epsilon_0^{\kappa(1-\eta)} \leq \frac{\epsilon_1}{4}$$

if $\epsilon_0 \le \Big(\frac{\epsilon_1}{4\|K\|_1 C_0}\Big)^{\frac{1}{\kappa(1-\eta)}}$.

Let $\Omega_0 = \max[2, \Omega_0', \Omega_0'']$. Altogether we obtain that

$$\int_{t^*-T}^{t} K_N(t-\tau)U(\tau)d\tau \le I_1 + I_2 + I_3 + I_4 < \epsilon_1$$

if $N \ge N_0, T \ge \Omega_0$ and $t^* > \omega_0$. ///

In the end of this chapter we will prove that U and ϵ_1 in (7.3) for $\eta > 0$ can be replaced by $U^{1+\eta}$ and $C\epsilon_1^{1+\eta}$ respectively, but to be able to do so, we must introduce a more complicated kernel $K_{M,N}^{\eta}$, which, for $M \ge 1, N \ge 0$ and $\eta > 0$, is defined by

$$K_{M,N}^{\eta}(U)(t,\tau) = K(t-\tau)(Q^{M-1}(Q^N U)^{1+\eta})(\tau).$$

$K_{j,N-j}$ has the following property. (In the rest of the chapter K and U are assumed to satisfy (#) with $|\eta| < \eta_0 < 1$. η_0 will be specified in Chapter 9.)

Lemma 7.3. *Assume that $I \subseteq [0,t]$ and that $N \ge N_1$, where $N_1 = [(1+|\eta|)\frac{1+\epsilon}{\epsilon q}] + 1$.*

Then, if $U_0 \in L_q$,

$$\sup_{t \ge 0} \int_I K_{j,N-j}^{\eta}(U)(\tau)d\tau < \infty, j = 1, 2, \ldots, N. \tag{7.4}$$

Proof: For $|\gamma| \le |\eta|$,

$$\begin{aligned}
&\int_I K_{j,N-j}^{\eta}(U)(\tau)d\tau = \\
&= \int_I K(t-\tau)Q^{j-1}\Big(\int_0^s K_{N-j}(s-\sigma)U(\sigma)d\sigma\Big)^{1+\eta}(\tau)d\tau \le \\
&\le 2^{\eta} \int_I K(t-\tau)Q^{j-1}\Big(\int_0^s K_{N-j}(s-\sigma)U_0(\sigma)d\sigma\Big)^{1+\eta}(\tau)d\tau + \\
&+2^{\eta} \int_I K(t-\tau)Q^{j-1}\Big(\int_0^s K_{N-j}(s-\sigma)\int_0^{\sigma} K(\sigma-x)U(x)^{1-\gamma}dx\Big)^{1+\eta}(\tau)d\tau = \\
&= 2^{\eta} \int_I K(t-\tau)Q^{j-1}(Q^{N-j}U_0)^{1+\eta}(\tau)d\tau + \\
&+2^{\eta} \int_I K(t-\tau)Q^{j-1}(Q^{N-j+1}U^{1-\gamma})^{1+\eta}(\tau)d\tau = \\
&= 2^{\eta} \int_I K_{j,N-j}^{\eta}(U_0)(\tau)d\tau + 2^{\eta} \int_I K_{j,N-j+1}^{\eta}(U^{1-\gamma})(\tau)d\tau.
\end{aligned} \tag{7.5}$$

As $0 \le \frac{\eta}{1+\eta} \le \eta$ if $\eta \ge 0$ we are allowed to take $\gamma = \frac{\eta}{1+\eta}$ in (2.7) by which $1 - \gamma = \frac{1}{1+\eta}$. Now (7.4) follows if we can prove that the last two integrals are bounded.

As $\tilde{\epsilon}_1 = \frac{1+\eta}{Nq-1-\eta} \le \epsilon$ for $N \ge N_1$,

$$\begin{aligned}
\int_I K^\eta_{j,N-j}(U_0)d\tau &= \int_I K(t-\tau)Q^{j-1}\Big(\int_0^s K_{N-j}(s-\sigma)U_0(\sigma)d\sigma\Big)^{1+\eta}d\tau \le \\
&\le \int_I K(t-\tau)Q^{j-1}\Big(\int_0^s K_{N-j}(s-\sigma)^{\frac{\eta}{1+\eta}}K_{N-j}(s-\sigma)^{\frac{1}{1+\eta}}U_0(\sigma)d\sigma\Big)^{1+\eta}d\tau \le \\
&\le \int_I K(t-\tau)Q^{j-1}\Big\{\Big(\int_0^s K_{N-j}(\sigma)d\sigma\Big)^\eta \int_0^s K_{N-j}(s-\sigma)U_0(\sigma)^{1+\eta}d\sigma\Big\}d\tau \le \\
&\le \|K_{N-j}\|_1^\eta \int_I K(t-\tau)\int_0^\tau K_{j-1}(\tau-s)\int_0^s K_{N-j}(s-\sigma)U_0(\sigma)^{1+\eta}d\sigma ds d\tau \le \\
&\le \|K\|_1^{\eta(N-j)}\int_I K(t-\tau)\int_0^\tau K_{N-1}(\tau-s)U_0(s)^{1+\eta}dsd\tau \le \\
&\le \|K\|_1^{\eta(N-j)}\int_I K(t-\tau)\Big(\int_0^\tau K_{N-1}(s)^r ds\Big)^{\frac{1}{r}}\Big(\int_0^\tau U_0(s)^q ds\Big)^{\frac{1+\eta}{q}}d\tau \le \\
&\le \|K\|_1^{\eta(N-j)}\|K_{N-1}\|_r\|U_0\|_q^{1+\eta}\int_I K(t-\tau)d\tau \le \\
&\le \|K\|_1^{1+\eta(N-j)}\|K\|_{1+\tilde{\epsilon}_1}^{N-1}\|U_0\|_q^{1+\eta} < \infty, \quad N \ge N_1.
\end{aligned} \tag{7.6}$$

By Lemma 6.3,

$$\begin{aligned}
&\int_I K^\eta_{j,N-j+1}\Big(U^{\frac{1}{1+\eta}}\Big)d\tau = \\
&= \int_I K(t-\tau)Q^{j-1}\Big(\int_0^s K_{N-j+1}(s-\sigma)^{\frac{\eta}{1+\eta}}K_{N-j+1}(s-\sigma)^{\frac{1}{1+\eta}}U(\sigma)^{\frac{1}{1+\eta}}d\sigma\Big)^{1+\eta}d\tau \le \\
&\le \int_I K(t-\tau)Q^{j-1}\Big(\Big(\int_0^s K_{N-j+1}(\sigma)d\sigma\Big)^\eta\Big(\int_0^s K_{N-j+1}(s-\sigma)U(\sigma)d\sigma\Big)\Big)d\tau \le \\
&\le \|K\|_1^{(N-j+1)\eta}\int_I K(t-\tau)Q^{j-1}\Big(\int_0^s K_{N-j+1}(s-\sigma)U(\sigma)d\sigma\Big)d\tau = \\
&= \|K\|_1^{(N-j+1)\eta}\int_I K(t-\tau)\int_0^\tau K_N(\tau-s)U(s)dsd\tau \le \\
&\le \|K\|_1^{(N-j+1)\eta}\Big(\sup_{\tau\ge 0}\int_0^\tau K_N(\tau-s)U(s)ds\Big)\int_I K(t-\tau)d\tau \le \\
&\le \|K\|_1^{1+\eta(N-j+1)}\cdot M < \infty, \quad N \ge N_0.
\end{aligned} \tag{7.7}$$

Now, by (7.5) to (7.7), for $N \ge N_1$

$$\begin{aligned}
&\sup_{t\ge 0}\int_I K^\eta_{j,N-j}(U)d\tau \le \\
&\le \|K\|_1^{1+\eta(N-j)}(\|K\|_{1+\tilde{\epsilon}_1}^{N-1}\|U_0\|_q^{1+\eta} + \|K\|_1^\eta M) < \infty. \quad ///
\end{aligned}$$

Remark 7.1. For $I = [0,t]$ and $j = N$ it follows that

$$\sup_{t\ge 0}\int_0^t K_N(t-\tau)U(\tau)^{1+\eta}d\tau < \infty.$$

Remark 7.2.

$$N_1 = \Big[(1+|\eta|)\frac{1+\epsilon}{\epsilon q}\Big] + 1 \geq \Big[\frac{1+\epsilon}{\epsilon q}\Big] + 1 = N_0.$$

Lemma 7.4. *Let $t \geq \sigma \geq \sigma_0 \geq t^* - \frac{T}{2} \geq T \geq 2$ and $\beta \geq 1$.*

Then there are constants $A, B < \infty$ such that for $N \geq N_1$

$$\int_{t^*-T}^{\sigma-\frac{T}{2}} K(t-\tau)\Big(\int_0^{\tau} K_N(\tau-s)U(s)ds\Big)^{\beta} d\tau \leq AT^{-\alpha} \tag{7.8}$$

and

$$\int_{\sigma-\sigma_0}^{\sigma} K(t-\tau)\Big(\int_0^{\tau-\frac{T}{2}} K_N(\tau-s)U(s)ds\Big)^{\beta} d\tau \leq BT^{-\frac{\eta\beta\alpha}{1+\eta}}. \tag{7.9}$$

Proof: By Lemma 6.3 and Remark 7.1 for $N \geq N_1$ we have that

$$\sup_{\tau\geq 0}\int_0^{\tau} K_N(\tau-s)U(s)ds \leq M < \infty$$

and that

$$\sup_{\tau\geq 0}\int_0^{\tau} K_N(\tau-s)U(\sigma)^{1+\eta}ds \leq M_1 < \infty,$$

and so

$$\begin{aligned}
&\int_{t^*-T}^{\sigma-\frac{T}{2}} K(t-\tau)\Big(\int_0^{\tau} K_N(\tau-s)U(s)ds\Big)^{\beta} d\tau \leq \\
&\leq \Big(\sup_{\tau\geq 0}\int_0^{\tau} K_N(\tau-s)U(s)ds\Big)^{\beta}\int_{t-\sigma+\frac{T}{2}}^{t-t^*+T} K(\tau)d\tau \leq \\
&\leq M^{\beta}\int_{\frac{T}{2}}^{\infty} K(\tau)d\tau = M^{\beta}C_1\int_{\frac{T}{2}}^{\infty}\tau^{-1-\alpha}d\alpha = \frac{2^{\alpha}M^{\beta}C_1}{\alpha}T^{-\alpha},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\sigma-\sigma_0}^{\sigma} K(t-\tau)\Big(\int_0^{\tau-\frac{T}{2}} K_N(\tau-s)U(s)ds\Big)^{\beta} d\tau = \\
&= \int_{\sigma-\sigma_0}^{\sigma} K(t-\tau)\Big(\int_0^{\tau-\frac{T}{2}} K_N(\tau-s)^{\frac{\eta}{1+\eta}}K_N(\tau-s)^{\frac{1}{1+\eta}}U(s)ds\Big)^{\beta} d\tau \leq \\
&\leq \int_{\sigma-\sigma_0}^{\sigma} K(t-\tau)\Big(\int_0^{\tau-\frac{T}{2}} K_N(\tau-s)ds\Big)^{\frac{\eta\beta}{1+\eta}}\Big(\int_0^{\tau-\frac{T}{2}} K_N(\tau-s)U(s)^{1+\eta}ds\Big)^{\frac{\beta}{1+\eta}} d\tau \leq \\
&\leq \Big(\sup_{\tau\geq\frac{T}{2}}\int_{T/2}^{\tau} K_N(s)ds\Big)^{\frac{\eta\beta}{1+\eta}} M_1^{\frac{\beta}{1+\eta}}\cdot\int_{t-\sigma}^{t-\sigma+\sigma_0} K(\tau)d\tau \leq \\
&\leq M_1^{\frac{\beta}{1+\eta}}\|K\|_1\Big(\int_{\frac{T}{2}}^{\infty} K_N(s)ds\Big)^{\frac{\eta\beta}{1+\eta}} \leq \Big(\frac{2^{\alpha}M_1^{\frac{1}{\eta}}C_N}{\alpha}\Big)^{\frac{\eta\beta}{1+\eta}}\|K\|_1 T^{-\frac{\eta\beta\alpha}{1+\eta}}
\end{aligned}$$

so that we can take $A = \dfrac{2^{\alpha}M^{\beta}C_1}{\alpha}$ and $B = \Big(\dfrac{2^{\alpha}M_1^{\frac{1}{\eta}}C_N}{\alpha}\Big)^{\frac{\eta\beta}{1+\eta}}\|K\|_1$. ///

Corollary 7.4. If $T > \{\max[A\epsilon_1^{-1-\eta}, (B\epsilon_1^{-1-\eta})^{\frac{1+\eta}{\eta\beta}}]\}^{\frac{1}{\alpha}} = \Omega_1$, both integrals are less than $\epsilon_1^{1+\eta}$.

Lemma 7.5. *Assume that $t^* - \frac{T}{2} \leq t$ where*

$$T \geq \Big(\frac{2^\alpha C_N M_1^{\frac{1}{\eta}}}{\alpha}\Big)^{\frac{1}{\alpha}} \epsilon_1^{-\frac{1+\eta}{\eta\alpha}} = \Omega_2.$$

Then, if

$$\int_{t^*-T}^{t} K_N(t-\tau)U(\tau)d\tau < \epsilon_1, \quad N \geq N_1,$$

we have that

$$\int_{t^*-\frac{3}{2}T}^{t} K_N(t-\tau)U(\tau)d\tau < 2\epsilon_1, \quad N \geq N_1 \tag{7.10}$$

Proof:

$$\int_{t^*-\frac{3}{2}T}^{t} K_N(t-\tau)U(\tau)d\tau \leq$$

$$\leq \int_{t^*-\frac{3}{2}T}^{t^*-T} K_N(t-\tau)^{\frac{\eta}{1+\eta}} K_N(t-\tau)^{\frac{1}{1+\eta}} U(\tau)d\tau + \int_{t^*-T}^{t} K_N(t-\tau)U(\tau)d\tau <$$

$$< \Big(\int_{t-t^*+T}^{t-t^*+\frac{3}{2}T} K_N(\tau)d\tau\Big)^{\frac{\eta}{1+\eta}} \Big(\int_{t^*-\frac{3T}{2}}^{t^*-T} K_N(t-\tau)U(\tau)^{1+\eta}d\tau\Big)^{\frac{1}{1+\eta}} + \epsilon_1 \leq$$

$$\leq \Big(\int_{T/2}^{\infty} K_N(\tau)d\tau\Big)^{\frac{\eta}{1+\eta}} \Big(\sup_{t\geq 0}\int_0^t K_N(t-\tau)U(\tau)^{1+\eta}d\tau\Big)^{\frac{1}{1+\eta}} + \epsilon_1 \leq$$

$$\leq \Big\{\frac{C_N}{\alpha}\Big(\frac{T}{2}\Big)^{-\alpha}\Big\}^{\frac{\eta}{1+\eta}} M_1^{\frac{1}{1+\eta}} + \epsilon_1 \leq 2\epsilon_1. \quad ///$$

Lemma 7.6. *Assume that $t \geq t^* - \frac{T}{2}$ where $t^* > \omega_1, T > \Omega_3$ and that $N_2 = [(1+|\eta|)^2\frac{1+\epsilon}{\epsilon q}] + 1$. ($\omega_1$ and Ω_3 are specified later.)*

Then $U_0 \in L_q$ implies that for $j = 1, 2, \ldots, N$

$$\int_{t^*-T}^{t} K_{j,N-j}^{\eta}(U_0)(\tau)d\tau < \epsilon_1^{1+\eta}, \quad N \geq N_2. \tag{7.11}$$

Proof:

$$\int_{t^*-T}^{t} K^{\eta}_{j,N-j}(U_0)(\tau)d\tau =$$

$$= \int_{t^*-T}^{t} K(t-\tau)\int_0^{\tau} K_{j-1}(\tau-s)\cdot$$

$$\cdot\Big(\int_0^s K_{N-j}(s-\sigma)^{\frac{\eta}{1+\eta}}K_{N-j}(s-\sigma)^{\frac{1}{1+\eta}}U_0(\sigma)d\sigma\Big)^{1+\eta}dsd\tau \le$$

$$\le \|K_{N-j}\|_1^{\eta}\int_{t^*-T}^{t} K(t-\tau)\int_0^{\tau} K_{j-1}(\tau-s)\int_0^s K_{N-j}(s-\sigma)U_0(\sigma)^{1+\eta}d\sigma dsd\tau \le$$

$$\le \|K\|_1^{\eta(N-j)}\int_{t^*-T}^{t} K(t-\tau)\int_0^{\tau} K_{N-1}(\tau-s)U_0(s)^{1+\eta}dsd\tau \le$$

$$\le C\int_0^{t^*-T} K_N(t-s)U_0(s)^{1+\eta}ds + C\int_{t^*-T}^{t} K_N(t-s)U_0(s)^{1+\eta}ds =$$

$$= CI_1 + CI_2.$$

Next we intend to prove that both I_1 and I_2 are less thatn $\frac{\epsilon_1^{1+\eta}}{2C}$.

$$I_1 = \int_0^{t^*-T} K_N(t-s)^{\frac{\eta}{1+\eta}}K_N(t-s)^{\frac{1}{1+\eta}}U_0(s)^{1+\eta}ds \le$$

$$\le \Big(\int_{t-t^*+T}^{t} K_N(s)ds\Big)^{\frac{\eta}{1+\eta}}\Big(\int_0^{t^*-T} K_N(t-s)U_0(s)^{(1+\eta)^2}ds\Big)^{\frac{1}{1+\eta}} \le$$

$$\le C_N^{\frac{\eta}{1+\eta}}\Big(\int_{T/2}^{\infty} s^{-1-\alpha}ds\Big)^{\frac{\eta}{1+\eta}}\Big(\int_{t-t^*+T}^{t} K_N(s)^r ds\Big)^{\frac{1}{r(1+\eta)}}\Big(\int_0^{t^*-T} U_0(s)^q ds\Big)^{\frac{1+\eta}{q}} \le$$

$$\le C_N^{\frac{\eta}{1+\eta}}\|K_N\|_r^{\frac{1}{1+\eta}}\|U_0\|_q^{1+\eta}\Big(\frac{1}{\alpha}\Big(\frac{T}{2}\Big)^{-\alpha}\Big)^{\frac{\eta}{1+\eta}} \le$$

$$\le \Big(\frac{2^{\alpha}C_N}{\alpha}\Big)^{\frac{\eta}{1+\eta}}\|K\|_{1+\tilde{\epsilon}_2}^{\frac{N}{1+\eta}}\|U_0\|_q^{1+\eta}T^{-\frac{\eta\alpha}{1+\eta}} < \frac{\epsilon_1^{1+\eta}}{2C}$$

if $T > (2^{\frac{1}{\eta}}\|K\|_1^{N-j})^{\frac{1+\eta}{\alpha}}\Big(\frac{2^\alpha C_N}{\alpha}\Big)^{\frac{1}{\alpha}}\|K\|_{1+\tilde\epsilon_2}^{\frac{N}{\eta\alpha}}\|U_0\|_q^{\frac{(1+\eta)^2}{\eta\alpha}}\epsilon_1^{-\frac{(1+\eta)^2}{\eta\alpha}} = \Omega_3$, and

$$I_2 = \int_{t^*-T}^{t} K_N(t-s)U_0(s)^{1+\eta}ds \le$$

$$\le \Big(\int_0^{t^*-t+T} K_N(s)^p ds\Big)^{\frac{1}{p}}\Big(\int_{t^*-T}^{t} U_0(s)^q ds\Big)^{\frac{1+\eta}{q}} \le$$

$$\le \|K_N\|_p\Big(\int_{t^*-T}^{t} U_0(s)^q ds\Big)^{\frac{1+\eta}{q}} \le \|K\|_{1+\tilde\epsilon_1}^{N}\Big(\int_{t^*-T}^{t} U_0(s)^q ds\Big)^{\frac{1+\eta}{q}} \le$$

$$< \frac{\epsilon_1^{1+\eta}}{2C}$$

since $U_0 \in L_q$ implies that there is a $\omega_1 \ge \omega_0$ such that

$$\int_{t^*-T}^{t} U_0(s)^q ds < (2C\|K\|_{1+\tilde\epsilon_1}^{N})^{-\frac{q}{1+\eta}} \cdot \epsilon_1^q, \quad t^* > \omega_1,$$

and $0 < \tilde\epsilon_1 \le \tilde\epsilon_2 \le \epsilon$ for $N \ge N_2$.

Altogether, we obtain that, for $j = 1, 2, \ldots, N$

$$\int_{t^*-T}^{t} K_{j,N-j}^{\eta}(U_0)(\tau)d\tau < \epsilon_1^{1+\eta}, T > \Omega_3, t^* > \omega_1, N \ge N_2. \quad ///$$

Lemma 7.7. *Let $t^* - \frac{T}{2} \le t \le t^*$ with $t^* > \omega_1, T > \max\limits_{i=1,2,3} \Omega_i$.*

Then, if

$$\int_{t^*-T}^{t} K_N(t-\tau)U(\tau)d\tau < \epsilon_1, \quad N \ge N_2,$$

it follows that

$$\int_{t^*-T}^{t} K_{1,N-1}^{\eta}(U)(\tau)d\tau < \tilde C_1\epsilon_1^{1+\eta}, \quad N \ge N_2.$$

Proof: By (7.5) for $\gamma = 0, j = 1, I = [t^* - T, t]$ and by Lemma 7.4 – Lemma 7.6 we obtain that for $\eta > 0$

$$\int_{t^*-T}^{t} K_{1,N-1}^{\eta}(U)(\tau)d\tau =$$

$$= 2^\eta\int_{t^*-T}^{t} K_{1,N-1}^{\eta}(U_0)(\tau)d\tau + 2^\eta\int_{t^*-T}^{t} K_{1,N}(U)(\tau)d\tau \le$$

$$\le 2^\eta\int_{t^*-T}^{t} K_{1,N-1}^{\eta}(U_0)(\tau)d\tau + 2^\eta\int_{t^*-T}^{t-\frac{T}{2}} K(t-\tau)\Big(\int_0^\tau K_N(\tau-s)U(s)ds\Big)^{1+\eta}d\tau +$$

$$+ 2^{2\eta} \int_{t-\frac{T}{2}}^{t} K(t-\tau)\Big(\int_0^{\tau-\frac{T}{2}} K_N(\tau-s)U(s)ds\Big)^{1+\eta} d\tau +$$

$$+ 2^{2\eta} \int_{t-\frac{T}{2}}^{t} K(t-\tau)\Big(\int_{\tau-\frac{T}{2}}^{\tau} K_N(\tau-s)U(s)ds\Big)^{1+\eta} d\tau <$$

$$< 2^{\eta}\epsilon_1^{1+\eta} + 2^{\eta}\epsilon_1^{1+\eta} + 2^{2\eta}\epsilon_1^{1+\eta} +$$

$$+ 2^{2\eta} \int_{t-\frac{T}{2}}^{t} K(t-\tau)\Big(\int_{t^*-3\frac{T}{2}}^{\tau} K_N(\tau-s)U(s)ds\Big)^{1+\eta} d\tau \leq$$

$$\leq 2^{\eta}(2+2^{\eta})\epsilon_1^{1+\eta} 2^{2_\eta}(2\epsilon_1)^{1+\eta} \int_0^{\frac{T}{2}} K(\tau)d\tau \leq$$

$$\leq 2^{\eta}(2+2^{\eta})\epsilon_1^{1+\eta} + 2^{2_\eta}(2\epsilon_1)^{1+\eta}\|K\|_1 =$$

$$= 2^{1+\eta}(2^{2_\eta}\|K\|_1 + 1 + 2^{\eta-1})\epsilon_1^{1+\eta} =$$

$$= \tilde{C}_1\epsilon_1^{1+\eta}. \quad ///$$

Lemma 7.8. *Let* $t^* - \frac{T}{2} \leq t \leq t^*$ *with* $t^* > \omega_1$ *and* $T > \max_{i=1,2,3} \Omega_i$.

If

$$\int_{t^*-T}^{t} K_N(t-\tau)U(\tau)d\tau < \epsilon_1, \quad N \geq N_2,$$

U *and* ϵ_1 *above can be replaced by* $U^{1+\eta}$ *and* $C\epsilon_1^{1+\eta}$ *respectively where* $\eta > 0$ *and* $C < \infty$.

Proof: The proof is carried out by induction.

By Lemma 7.7

$$\int_{t^*-T}^{t} K_{1,N-1}^{\eta}(U)(\tau)d\tau < \tilde{C}_1\epsilon_1^{1+\eta}, \quad N \geq N_2.$$

Assume that, for some $j \in Z^+$,

$$\int_{t^*-T}^{t} K_{j-1,N-j+1}^{\eta}(U)(\tau)d\tau < \tilde{C}_{j-1}\epsilon_1^{1+\eta}.$$

By (7.5) for $\gamma = 0, \eta > 0$ and $I = [t^* - T, t]$

$$\int_{t^*-T}^{t} K^{\eta}_{j,N-j}(U)(\tau)d\tau \leq$$

$$\leq 2^{\eta} \int_{t^*-T}^{t} K^{\eta}_{j,N-j}(U_0)(\tau)d\tau + 2^{\eta} \int_{t^*-T}^{t} K^{\eta}_{j,N-j+1}(U)(\tau)d\tau =$$

$$= 2^{\eta} \int_{t^*-T}^{t} K^{\eta}_{j,N-j}(U_0)(\tau)d\tau + 2^{\eta} \int_{t^*-T}^{t} K(t-\tau) \int_0^{\tau} K^{\eta}_{j-1,N-j+1}(U)(s,\tau)dsd\tau$$

so, by Lemma 7.6,

$$\int_{t^*-T}^{t} K^{\eta}_{j,N-j}(U)d\tau < 2^{\eta}\epsilon_1^{1+\eta} + 2^{\eta}\tilde{C}_{j-1}\epsilon_1^{1+\eta}\|K\|_1 =$$

$$= 2^{\eta}(\tilde{C}_{j-1}\|K\|_1 + 1)\epsilon_1^{1+\eta} =$$

$$= \tilde{C}_j \epsilon_1^{1+\eta}$$

and we obtain by induction that

$$\int_{t^*-T}^{t} K^{\eta}_{N,0}(U)d\tau < \tilde{C}_N \epsilon_1^{1+\eta}$$

i.e.

$$\tilde{C}_N \epsilon_1^{1+\eta} > \int_{t^*-T}^{t} K(t-\tau) \int_0^{\tau} K_{N-1}(\tau - s)U(s)^{1+\eta}dsd\tau =$$

$$= \int_0^{t^*-T} U(\tau)^{1+\eta} \int_{t^*-T-\tau}^{t-\tau} K_{N-1}(x)K(t-\tau-x)dxd\tau + \int_{t^*-T}^{t} K_N(t-\tau)U(\tau)^{1+\eta}d\tau$$

by which

$$\int_{t^*-T}^{t} K_N(t-\tau)U(\tau)^{1+\eta}d\tau < \tilde{C}_N \epsilon_1^{1+\eta}. \quad ///$$

8 Uniform convergence to zero

Lemma 8.1. *Assume that* $0 \leq K \in L_1 \cap L_{1+\epsilon}$.

Then, if $U_0 \in L_q$ *and* $N \geq N_0$, *we have that*

$$\sup_{t>s} \int_s^t K_N(t-\tau)U_0(\tau)d\tau \to 0 \text{ as } s \to \infty. \tag{8.1}$$

Proof:

$$\int_s^t K_N(t-\tau)U_0(\tau)d\tau \le$$

$$\le \|K_N\|_{q'}\Big(\int_s^t U_0(\tau)^q d\tau\Big)^{\frac{1}{q}} \le \|K\|_{1+\tilde{\epsilon}}^N\Big(\int_s^t U_0(\tau)^q d\tau\Big)^{\frac{1}{q}}.$$

As $U_0 \in L_q$ to each $\epsilon_0 > 0$ there is a ω^* such that

$$\Big(\int_s^\infty U_0(\tau)^q d\tau\Big)^{\frac{1}{q}} < \frac{\epsilon_0}{2\|K\|_{1+\tilde{\epsilon}}^N}, \quad s > \omega^*,$$

and so

$$\sup_{t>s}\int_s^t K_N(t-\tau)U_0(\tau)d\tau < \epsilon_0, s > \omega^*. \quad ///$$

The results achieved in chapter 7 enable us to show that the above property is inherited by U.

Theorem 8.1. *Assume that K and U satisfy (#).*

Then $U_0 \in L_q$ implies that for $t > s$

(8.2)
$$\lim_{s\to\infty}\int_s^t K_N(t-\tau)U(\tau)d\tau = 0, \quad N \ge N_2.$$

Proof: Define, for $N \ge N_2, t^{**}$ by

$$t^{**} = \sup\{t : \int_{t^*-T}^t K_N(t-\tau)U(\tau)d\tau < \epsilon_1\}.$$

By assumption and Lemma 7.2 $t^{**} \ge t^*$. If $t^{**} = \infty$ there is nothing to prove, so assume that $t^{**} < \infty$ in order to arrive at a contradiction.

Now, choose t such that $t^{**} < t \le t^{**} + \epsilon_2$, where $\epsilon_2 > 0$ is small (specified later).

As $U(\tau) \le U_0(\tau) + \int_0^\tau K(\tau-s)U(s)^{1-\gamma}ds$, we obtain that

$$\int_{t^*-T}^t K_N(t-\tau)U(\tau)d\tau \le$$

$$\le \int_{t^*-T}^t K_N(t-\tau)U_0(\tau)d\tau +$$

$$+ \int_0^{t^*-T} K(t-\tau)\int_0^\tau K_N(\tau-s)U(s)dsd\tau +$$

$$+\int_{t^*-T}^{t^*-\frac{T}{2}} K(t-\tau)\int_0^\tau K_N(\tau-s)U(s)dsd\tau+$$

$$+\int_{t^*-\frac{T}{2}}^{t^{**}} K(t-\tau)\int_0^{\tau-\frac{T}{2}} K_N(\tau-s)U(s)dsd\tau+$$

$$+\int_{t^*-\frac{T}{2}}^{t^{**}} K(t-\tau)\int_{\tau-\frac{T}{2}}^{\tau} K_N(\tau-s)U(s)^{1+\eta}dsd\tau+$$

$$+\int_{t^{**}}^{t} K(t-\tau)\int_0^\tau K_N(\tau-s)U(s)dsd\tau=$$

$$=I_1+I_2+I_3+I_4+I_5+I_6.$$

Here, for $N\geq N_0$ and $\eta>0$,

$$I_1\leq \|K\|_{1+\tilde{\epsilon}}^N\Big(\int_{t^*-T}^{t} U_0(\tau)^q d\tau\Big)^{\frac{1}{q}}$$

and as $U_0\in L_q$ there is a $\omega^*\geq\omega_1$ such that

$$I_1<\epsilon_1^{1+\eta},\quad t^*>\omega^*.$$

By Lemma 7.4

$$I_3+I_4<2\epsilon_1^{1+\eta}$$

By Lemma 6.3

$$I_2\leq \big(\sup_{\tau\geq 0}\int_0^\tau K_N(\tau-s)U(s)ds\big)\int_{t-t^*+T}^{t} K(\tau)d\tau\leq$$

$$\leq M\int_T^\infty K(\tau)d\tau\leq \frac{C_1M}{\alpha}T^{-\alpha}<\epsilon_1^{1+\eta},$$

$$\text{if } T>\Big(\frac{C_1M}{\alpha\epsilon_1^{1+\eta}}\Big)^{\frac{1}{\alpha}}=\Omega_4.$$

By Lemma 7.8 for $N\geq N_2$

$$I_5\leq\Big(\sup_{t^*-\frac{T}{2}\leq\tau\leq t^{**}}\int_{t^*-T}^{\tau} K_N(\tau-s)U(s)^{1+\eta}ds\Big)\int_{t-t^{**}}^{t-t^*+\frac{T}{2}} K(\tau)d\tau\leq$$

$$\leq \tilde{C}_N\epsilon_1^{1+\eta}\|K\|_1.$$

By Lemma 6.3, for $N \geq N_0$,

$$I_6 \leq \Big(\sup_{\tau \geq 0} \int_0^{\tau} K_N(\tau - s)U(s)ds \Big) \int_0^{t-t^{**}} K(\tau)d\tau \leq$$

$$\leq MC_1 \int_0^{\epsilon_2} \tau^{-1+\alpha} d\tau = \frac{MC_1}{\alpha} \epsilon_2^{\alpha} < \epsilon_1^{1+\eta}$$

if $\epsilon_2 < \Big(\frac{\alpha \epsilon_1^{1+\eta}}{MC_1} \Big)^{\frac{1}{\alpha}}$.

Altogether, for $t^* > \omega^*, T > \max_{i=0,1,2,3,4} \Omega_i = \Omega$, $\epsilon_1 < (\tilde{C}_N \|K\|_1 + 6)^{-\frac{1}{\eta}}$ and $\epsilon_2 < \Big(\frac{\alpha \epsilon_1^{1+\eta}}{MC_1} \Big)^{\frac{1}{\alpha}}$, we obtain that

$$\int_{t^*-T}^{t} K_N(t-\tau)U(\tau)d\tau < \epsilon_1, t^{**} < t \leq t^{**} + \epsilon_2, \quad N \geq N_2, \tag{8.3}$$

which contradicts the maximality of t^{**}, and so $t^{**} = \infty$. ///

Corollary 8.1. For $t > t^*, N \geq N_2$, we have that

$$\sup_{t > t^*} \int_{t^*}^{t} K_N(t-\tau)U(\tau)d\tau \to 0 \text{ as } t^* \to \infty. \tag{8.4}$$

9 L_q-decay

Theorem 9.1. *Assume that* $U(\tau) \geq 0$ *satisfies*

$$U(t) \leq U_0(t) + \int_0^t K(t-\tau)U(\tau)^{1-\eta}d\tau, |\eta| < \alpha q, \tag{9.1}$$

where $0 \leq K \in L_1 \cap L_{1+\epsilon}$*, some* $\epsilon > 0$*, and*

$$K(t) \leq \begin{cases} C_1 t^{-1-\alpha}, & t \geq 1 \\ C_1 t^{-1+\alpha}, & 0 < t \leq 1 \end{cases}, \text{ some } \alpha > 0.$$

Then $U_0 \in L_q$ *implies that* $U \in L_q$.

Proof: (9.1) for $\eta > 0$ implies that

$$K_{N-1} * U \leq K_{N-1} * U_0 + K_N * U^{1-\eta}. \tag{9.2}$$

Here $K_{N-j} * U_0 \in L_q, j = 1, 2, \ldots, N$, as, by Young's inequality and (6.1),

$$\|K_{N-j} * U_0\|_q \leq \|K_{N-j}\|_1 \|U_0\|_q \leq \|K\|_1^{N-j} \|U_0\|_q < \infty$$

and by Lemma 6.3, (6.1), Lemma 7.1 and Theorem 8.1, for $N \geq N_2$,

$$K_N * U^{1-\eta} = \Big(\int_0^{t/2} + \int_{t/2}^t\Big) K_N(t-\tau)^\eta K_N(t-\tau)^{1-\eta} U(\tau)^{1-\eta} d\tau \leq$$

$$\leq \Big(\frac{2^\alpha C_N}{\alpha}\Big)^\eta t^{-\eta\alpha} \Big(\int_0^t K_N(t-\tau)U(\tau)d\tau\Big)^{1-\eta} + \|K\|_1^{\eta N} \Big(\int_{\frac{t}{2}}^t K_N(t-\tau)U(\tau)d\tau\Big)^{1-\eta} =$$

$$= L_\infty - \text{function } \cdot t^{-\eta\alpha} + C_1 \cdot o(1) \quad \text{as } t \to \infty,$$

if N is large enough, and so, by (9.2), we obtain that

$$K_{N-1} * U \in L_q + o(1)L_\infty \text{ as } t \to \infty.$$

Now assume that $K_j * U \in L_q + o(1)L_\infty$.

Then, since (9.1) with $\eta = 0$ implies that

$$K_{j-1} * U \leq K_{j-1} * U_0 + K_j * U$$

we obtain that

$$K_{j-1} * U \in L_q + o(1)L_\infty,$$

and by induction it follows that

$$U \in L_q + o(1)L_\infty.$$

Next, we shall prove that, in fact, $U \in L_q + o(1)L_{q/\eta} \cap L_\infty$.
For $t \geq 2t^*$, (9.1) and Corollary 8.1 implies that

$$K_{N-1} * U = \int_0^t K_{N-1}(t-\tau)U(\tau)d\tau \leq$$

$$\leq \int_0^t K_{N-1}(t-\tau)U_0(\tau)d\tau + \int_0^{t^*} K_N(t-\tau)U(\tau)^{1+\eta}d\tau + \int_{t^*}^t K_N(t-\tau)U(\tau)^{1+\eta}d\tau =$$

$$= I_1 + I_2 + I_3$$

where $I_1 \in L_q$.

Now, since $U \in L_q + o(1)L_\infty$,

$$\begin{aligned} I_2 &= \int_0^{t^*} K_N(t-\tau)U(\tau)^{1+\eta}d\tau \leq \\ &\leq L_q - \text{function} + C_2 \int_{t-t^*}^t K_N(\tau)d\tau = \\ &\leq L_q - \text{function} + C_3(t-t^*)^{-(\alpha-\gamma)}(t-t^*)^{-\gamma} \end{aligned}$$

and it follows that, for $0 < \gamma < \alpha, 0 < \eta < \gamma q$,

$$I_2 \in L_q + o(1)L_{q/\eta} \cap L_\infty.$$

Finally, in order to estimate I_3, split U in two parts U_1 and U_2 such that $U = U_1 + U_2$ where $U_1 \in L_q$ and $U_2 \in o(1)L_\infty$. Then,

$$I_3 = \int_{t^*}^t K_N(t-\tau)U(\tau)(U_1(\tau)+U_2(\tau))^\eta d\tau \leq$$

$$\leq \int_{t^*}^t K_N(t-\tau)U(\tau)(U_1(\tau)^\eta + U_2(\tau)^\eta)d\tau =$$

$$= \int_{t^*}^t K_N(t-\tau)U_1(\tau)U_1(\tau)^\eta d\tau + \int_{t^*}^{\frac{t}{2}} K_N(t-\tau)U_1(\tau)U_2(\tau)^\eta d\tau +$$

$$+ \int_{t^*}^t K_N(t-\tau)U_2(\tau)U_1(\tau)^\eta d\tau + \int_{t^*}^{\frac{t}{2}} K_N(t-\tau)U_2(\tau)U_2(\tau)^\eta d\tau +$$

$$+ \int_{\frac{t}{2}}^t K_N(t-\tau)U(\tau)U_2(\tau)^\eta d\tau =$$

$$= I_{1,1} + I_{1,2} + I_{2,1} + I_{2,2} + I_5.$$

Next we will prove that $I_{1,i} \in L_q$ and that $I_{2,i} \in o(1)L_{q/\eta} \cap L_\infty$ for $i = 1, 2$.

$$\begin{aligned} \|I_{1,1}\|_q &\leq \|K_N * U_1^{1+\eta}\|_q \leq \|K_N\|_r \|U_1^{1+\eta}\|_{\frac{q}{1+\eta}} \leq \\ &\leq \|K\|_{1+\tilde{\epsilon}}^N \|U_1\|_q^{1+\eta} < \infty \end{aligned} \tag{9.3}$$

where $\tilde{\epsilon} = \frac{\eta}{Nq-\eta}$, so $I_{1,1} \in L_q$.

$$I_{1,2} = \int_{t^*}^{\frac{t}{2}} K_N(t-\tau)U_1(\tau)U_2(\tau)^\eta d\tau \leq C_4 \int_{t^*}^{\frac{t}{2}} K_N(t-\tau)U_1(\tau)d\tau \tag{9.4}$$

so $I_{1,2} \in L_q$.

$$\begin{aligned} I_{2,1} &= \int_{t^*}^{t} K_N(t-\tau)^\eta U_1(\tau)^\eta K_N(t-\tau)^{1-\eta} U_2(\tau) d\tau \leq \\ &\leq \Big(\int_{t^*}^{t} K_N(t-\tau)U_1(\tau)d\tau\Big)^\eta \Big(\int_{t^*}^{t} K_N(t-\tau)U_2(\tau)^{\frac{1}{1-\eta}} d\tau\Big)^{1-\eta} \leq \\ &\leq C_5 \Big(\int_{t^*}^{t} K_N(t-\tau)U_1(\tau)d\tau\Big)^\eta \Big(\int_{0}^{t-t^*} K_N(\tau)d\tau\Big)^{1-\eta} \leq \\ &\leq C_6 \Big(\int_{t^*}^{t} K_N(t-\tau)U_1(\tau)d\tau\Big)^\eta \end{aligned} \tag{9.5}$$

so $I_{2,1} \in o(1)L_{q/\eta} \cap L_\infty$.

$$I_{2,2} = \int_{t^*}^{\frac{t}{2}} K_N(t-\tau)U_2(\tau)^{1+\eta} d\tau \leq C_7 t^{-(\alpha-\gamma)} t^{-\gamma} \tag{9.6}$$

so $I_{2,2} \in o(1)L_{q/\eta} \cap L_\infty$ if $0 < \gamma < \alpha$, $\eta < \gamma q$.

Finally, since

$$I_5 \leq C_8 \int_{t^*}^{t} K_N(t-\tau)U(\tau)d\tau$$

(9.3) to (9.6) and Corollary 8.1 gives us that

$$K_{N-1} * U \in L_q + o(1)L_{q/\eta} \cap L_\infty,$$

and by induction,

$$U \in L_q + o(1)L_{q/\eta} \cap L_\infty.$$

Repetition of the above arguments, but now with $U_2 \in o(1)L_{q/\eta} \cap L_\infty$, yields

$$K_{N-1} * U \in L_q + o(1)L_{q/(2\eta)} \cap L_\infty$$

and, by induction

$$U \in L_q + o(1)L_{q/(2\eta)} \cap L_\infty.$$

After a finite number of similar steps, we obtain that

$$U \in L_q + o(1)L_q = L_q = L_q([2t^*, \infty))$$

by which $U \in L_q(R^+)$, as we have earlier proven that $U \in L_q([0, 2t^*])$. ///

10 Decay of a long-range mean value

In this chapter we will study the decay of the long-range mean value

$$m_U(t) = \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t} U(\tau)^q d\tau\Big)^{\frac{1}{q}}, t > 0.$$

Theorem 10.1. *Assume that $U(t) \geq 0$ satisfies*

$$(1^\circ) \qquad U(t) \leq U_0(t) + \int_0^t K(t-\tau)U(\tau)^{1-\eta}d\tau, 0 \leq \eta < 1,$$

where $U_0 \in L_q, K(\tau) \leq \tilde{C}_1(1+\tau)^{-1-\alpha}$ for $\tau \geq 1$ and $0 \leq K \in L_1 \cap L_{1+\epsilon}$ for some $\epsilon > 0$.

$$(2^\circ) \qquad \int_{\frac{\tau}{2}}^{\tau} K_N(\tau - s)U(s)ds \leq Ew(\tau), N \geq N_0,$$

where $w(t)$ is a function such that, for $t \geq t^*$,

$$(1_w) \qquad \int_{\frac{t}{2}}^{t} w(\tau)^q d\tau \leq \frac{A}{2} t w(t)^q$$

$$(2_w) \qquad w(t)^{-1} \leq B(1+t)^{\frac{1}{q}+\alpha}$$

$$(3_w) \qquad \int_0^{\frac{t}{2}} w(\tau)^q d\tau \leq C$$

$$(4_w) \qquad w\Big(\frac{t}{2}\Big) \leq Dw(t)$$

and A, B, C, D and E are constants independent of t.

Then, if $m_{U_0}(t) \leq w(t)$, we also have that

$$m_U(t) \leq C_0 w(t), \quad t \geq t^*.$$

Proof: Repeated use of the integral inequality 1° for $\eta = 0$ yields

$$U(\tau) \leq \sum_{j=0}^{\ell-1} K_j * U_0(\tau) + K_\ell * U(\tau), \quad \ell \in Z^+, K_0 * U_0(\tau) = U_0(\tau),$$

and by convexity, for $\ell = N \geq N_0$, it follows that

$$(10.0) \qquad U(\tau)^q \leq \sum_{j=0}^{N-1} 2^{(j+2)(q-1)}(K_j * U_0(\tau))^q + 2^{q-1}(K_N * U(\tau))^q$$

Here, by Lemma 6.1, for $j \in Z^+$

$$\begin{aligned}
K_j * U_0(\tau) &= \int_0^\tau K_j(\tau-s)U_0(s)ds = \\
&= \int_0^\tau K_j(\tau-s)^{\frac{1}{q'}} K_j(\tau-s)^{\frac{1}{q}} U_0(s)ds \le \\
&\le \Big(\int_0^\tau K_j(s)ds\Big)^{\frac{1}{q'}} \Big(\int_0^\tau K_j(\tau-s)U_0(s)^q ds\Big)^{\frac{1}{q}} \le \\
&\le \|K_j\|_1^{\frac{1}{q'}} \Big(\int_0^\tau K_j(\tau-s)U_0(s)^q ds\Big)^{\frac{1}{q}} \le \\
&\le \|K\|_1^{\frac{j}{q'}} \Big(\int_0^\tau K_j(\tau-s)U_0(s)^q ds\Big)^{\frac{1}{q}}
\end{aligned}$$

and, as a consequence of Fubini's theorem, it follows that

$$\begin{aligned}
&\int_0^{\frac{t}{2}} (K_j * U_0(\tau))^q d\tau \le \\
&\le \|K\|_1^{\frac{jq}{q'}} \int_0^{\frac{t}{2}} \int_0^\tau K_j(\tau-s)U_0(s)^q ds d\tau = \\
&= \|K\|_1^{j(q-1)} \int_0^{\frac{t}{2}} U_0(s)^q \int_0^{\frac{t}{2}-s} K_j(\tau) d\tau ds \le \\
&\le \|K\|_1^{jq} \int_0^{\frac{t}{2}} U_0(s)^q ds
\end{aligned} \tag{10.1}$$

Again by Lemma 6.1, for $j \in Z^+, \tau \ge 2$

$$\begin{aligned}
&K_j * U_0(\tau) = \\
&= \int_0^{\frac{\tau}{2}} K_j(\tau-s)U_0(s)ds + \int_{\frac{\tau}{2}}^\tau K_j(\tau-s)^{\frac{1}{q'}} K_j(\tau-s)^{\frac{1}{q}} U_0(s)ds \le \\
&\le \Big(\int_{\frac{\tau}{2}}^\tau K_j(s)^{q'} ds\Big)^{\frac{1}{q'}} \Big(\int_0^{\frac{\tau}{2}} U_0(s)^q ds\Big)^{\frac{1}{q}} + \\
&+ \Big(\int_0^{\frac{\tau}{2}} K_j(s)ds\Big)^{\frac{1}{q'}} \Big(\int_{\frac{\tau}{2}}^\tau K_j(\tau-s)U_0(s)^q ds\Big)^{\frac{1}{q}} \le \\
&\le 2^{\frac{1}{q}+\alpha} \tilde{C}_j (1+\tau)^{-\frac{1}{q}-\alpha} \Big(\int_0^{\frac{\tau}{2}} U_0(s)^q ds\Big)^{\frac{1}{q}} + \\
&+ \|K\|_1^{\frac{j}{q'}} \Big(\int_{\frac{\tau}{2}}^\tau K_j(\tau-s)U_0(s)^q ds\Big)^{\frac{1}{q}}
\end{aligned}$$

and, as a consequence of Fubini's theorem, it follows that

$$\begin{aligned}
&\int_{\frac{t}{2}}^t (K_j * U_0(\tau))^q d\tau \le \\
&\le 2^{(1+\alpha)q} \tilde{C}_j^q \int_{\frac{t}{2}}^t (1+\tau)^{-1-\alpha q} \int_0^{\frac{\tau}{2}} U_0(s)^q ds d\tau + \\
&+ 2^{q-1} \|K\|_1^{j(q-1)} \int_{\frac{t}{2}}^t \int_{\frac{\tau}{2}}^\tau K_j(\tau-s)U_0(s)^q ds d\tau \le
\end{aligned} \tag{10.2}$$

$$\leq 2^{(1+2\alpha)q}\tilde{C}_j^q(1+t)^{-\alpha q}\int_0^{\frac{t}{2}} U_0(s)^q ds +$$

$$+2^{q-1}\|K\|_1^{j(q-1)}\Big(\int_{\frac{t}{4}}^{\frac{t}{2}} U_0(s)^q \int_{\frac{t}{2}-s}^{s} K_j(\tau)d\tau ds + \int_{\frac{t}{2}}^{t} U_0(s)^q \int_0^{t-s} K_j(\tau)d\tau ds\Big) \leq$$

$$\leq 2^{(1+2\alpha)q}\tilde{C}_j^q(1+t)^{-\alpha q}\int_0^{\frac{t}{2}} U_0(s)^q ds +$$

$$+2^{q-1}\|K\|_1^{jq}\int_{\frac{t}{4}}^{\frac{t}{2}} U_0(s)^q ds +$$

$$+2^{q-1}\|K\|_1^{jq}\int_{\frac{t}{2}}^{t} U_0(s)^q ds$$

When estimating the convolution $K_N * U(\tau)$, we will need that, for $\tau \geq 2$,

$$\begin{aligned}
&\int_0^{\frac{\tau}{2}} K_N(\tau-s)U(s)ds \leq \\
&\leq \Big(\int_{\frac{\tau}{2}}^{\tau} K_N(s)^{q'}ds\Big)^{\frac{1}{q'}}\Big(\int_0^{\frac{\tau}{2}} U(s)^q ds\Big)^{\frac{1}{q}} \leq \\
&\leq \tilde{C}_N\Big(1+\frac{\tau}{2}\Big)^{-\frac{1}{q}-\alpha}\Big(\int_0^{\frac{\tau}{2}} U(s)^q ds\Big)^{\frac{1}{q}} \leq \\
&\leq 2^{\frac{1}{q}+\alpha}\tilde{C}_N(1+\tau)^{-\frac{1}{q}-\alpha}\Big(\int_0^{\frac{\tau}{2}} U(s)^q ds\Big)^{\frac{1}{q}}
\end{aligned} \tag{10.3}$$

Now, by (10.1), (10.2), (10.4) and $2^\circ, 3_w$, for $t \geq t^*$ it follows that

$$\begin{aligned}
&\int_0^{\frac{t}{2}} U(\tau)^q d\tau \leq \\
&\leq \sum_{j=0}^{N-1} 2^{(j+2)(q-1)}\int_0^{\frac{t}{2}} (K_j * U_0(\tau))^q d\tau + \\
&+2^{2q-2}\int_0^{\frac{t}{2}}\Big(\int_0^{\frac{\tau}{2}} K_N(\tau-s)U(s)ds\Big)^q d\tau + \\
&+2^{2q-2}\int_0^{\frac{t}{2}}\Big(\int_{\frac{\tau}{2}}^{\tau} K_N(\tau-s)U(s)ds\Big)^q d\tau \leq \\
&\leq \sum_{j=0}^{N-1} 2^{(j+2)(q-1)}\|K\|_1^{jq}\int_0^{\frac{t}{2}} U_0(s)^q ds + \\
&+2^{2q-2}\int_0^{2}\Big(\int_{\frac{\tau}{2}}^{\tau} K_N(s)^{q'}ds\Big)^{\frac{q}{q'}}\Big(\int_0^{\frac{\tau}{2}} U(s)^q ds\Big)d\tau + \\
&+2^{2q-1+\alpha q}\tilde{C}_N^q\int_2^{\frac{t^*}{2}}(1+\tau)^{-1-\alpha q}\Big(\int_0^{\frac{\tau}{2}} U(s)^q ds\Big)d\tau + \\
&+2^{2q-1+\alpha q}\tilde{C}_N^q\int_{\frac{t^*}{2}}^{\frac{t}{2}}(1+\tau)^{-1-\alpha q}\Big(\int_0^{\frac{\tau}{2}} U(s)^q ds\Big)d\tau +
\end{aligned} \tag{10.4}$$

$$+2^{2q-2}E^q\int_0^{\frac{t}{2}} w(\tau)^q d\tau \le$$

$$\le \Big(\sum_{j=0}^{N-1} 2^{(j+2)(q-1)}\|K\|_1^{jq}\Big)\|U_0\|^q_{L_q(0,\frac{t}{2})} + 2^{2q-1}\|K_N\|^q_{q'}\int_0^1 U(s)^q ds +$$

$$+2^{2q-1+\alpha q}3^{-\alpha q}\frac{\tilde{C}_N^q}{\alpha q}\int_0^{\frac{t^*}{4}} U(s)^q ds + 2^{2q-1+2\alpha q}\frac{\tilde{C}_N^q}{\alpha q}(1+t^*)^{-\alpha q}\int_0^{\frac{t}{4}} U(s)^q ds +$$

$$+2^{2q-2}CE^q$$

For $t^* \ge 2^{2(1+\frac{1}{\alpha})}\tilde{C}_N^{\frac{1}{\alpha}}(\alpha q)^{-\frac{1}{\alpha q}} - 1$, it follows by (10.5) that

$$\int_0^{\frac{t}{2}} U(s)^q ds \le$$

$$\le C_1\|U_0\|_q^q + C_2\|U\|^q_{L_q(0,\frac{t^*}{4})} + C_3 + \frac{1}{2}\int_0^{\frac{t}{2}} U(s)^q ds$$

and, as $U_0 \in L_q^{\text{loc}} \Rightarrow U \in L_q^{\text{loc}}$ by Theorem 6.1, we have that

$$\int_0^{\frac{t}{2}} U(\tau)^q d\tau \le C_4 \tag{10.5}$$

Finally, for $t \ge t^*$, by (10.1)-(10.6), $1^\circ, 2^\circ$ and 1_w - 4_w, we obtain that

$$w(t)^{-q} m_U(t)^q = w(t)^{-q}\cdot\frac{2}{t}\int_{\frac{t}{2}}^t U(\tau)^q d\tau \le$$

$$\le \sum_{j=0}^{N-1} 2^{(j+2)(q-1)} w(t)^{-q}\cdot\frac{2}{t}\int_{\frac{t}{2}}^t (K_j * U_0(\tau))^q d\tau +$$

$$+ 2^{q-1} w(t)^{-q}\cdot\frac{2}{t}\int_{\frac{t}{2}}^t \Big(\Big(\int_0^{\frac{\tau}{2}} + \int_{\frac{\tau}{2}}^{\tau}\Big) K_N(\tau-s)U(s)ds\Big)^q d\tau \le$$

$$\le 2^{2q-2} w(t)^{-q}\cdot\frac{2}{t}\int_{\frac{t}{2}}^t U_0(\tau)^q d\tau +$$

$$+ \sum_{j=1}^{N-1} 2^{(j+2)(q-1)} w(t)^{-q}\cdot\frac{2}{t}\int_{\frac{t}{2}}^t (K_j * U_0(\tau))^q d\tau +$$

$$+ 2^{2q-2} w(t)^{-q}\cdot\frac{2}{t}\int_{\frac{t}{2}}^t \Big(\int_0^{\frac{\tau}{2}} K_N(\tau-s)U(s)ds\Big)^q d\tau +$$

$$+ 2^{2q-2} w(t)^{-q}\cdot\frac{2}{t}\int_{\frac{t}{2}}^t \Big(\int_{\frac{\tau}{2}}^{\tau} K_N(\tau-s)U(s)ds\Big)^q d\tau \le$$

$$\le 2^{2q-2} + \sum_{j=1}^{N-1} 2^{(j+2)(q-1)}\Big(2^{(1+2\alpha)q}\tilde{C}_j^q w(t)^{-q}\cdot\frac{2}{t}(1+t)^{-\alpha q}\int_0^{\frac{t}{2}} U_0(s)^q ds +$$

$$+ 2^{q-1}\|K\|_1^{jq} w(t)^{-q}\cdot\frac{2}{t}\int_{\frac{t}{4}}^{\frac{t}{2}} U_0(s)^q ds +$$

$$+ 2^{q-1}\|K\|_1^{jq} w(t)^{-q} \cdot \frac{2}{t}\int_{\frac{t}{2}}^{t} U_0(s)^q ds\Big)+$$

$$+ 2^{(2+\alpha)q-1}\tilde{C}_N^q w(t)^{-q} \cdot \frac{2}{t}\int_{\frac{t}{2}}^{t}(1+\tau)^{-1-\alpha q}\int_0^{\frac{\tau}{2}} U(s)^q ds d\tau+$$

$$+ 2^{2q-2}E^q w(t)^{-q} \cdot \frac{2}{t}\int_{\frac{t}{2}}^{t} w(\tau)^q d\tau \leq$$

$$\leq 2^{2q-2} + \sum_{j=1}^{N-1} 2^{(j+2)(q-1)}\Big(2^{(1+2\alpha)q+2} w(t)^{-q}(1+t)^{-1-\alpha q}\|U_0\|^q_{L_q(0,\frac{t}{2})}+$$

$$+ 2^{q-2}D^q\|K\|_1^{jq} + 2^{q-1}\|K\|_1^{jq}\Big)+$$

$$+ 2^{2(1+\alpha)q}\tilde{C}_N^q w(t)^{-q}(1+t)^{-1-\alpha q}\int_0^{\frac{t}{2}} U(s)^q ds + 2^{2q-2}AE^q \leq$$

$$\leq 2^{2q-2}+$$

$$+ \sum_{j=1}^{N-1} 2^{(j+2)(q-1)}(2^{(1+2\alpha)q+2}B^q\|U_0\|_q^q + 2^{q-1}(2^{-1}D^q+1)\|K\|_1^{jq}+$$

$$+ 2^{2(1+\alpha)q}\tilde{C}_N^q B^q C_4 + 2^{2q-2}AE^q$$

by which

$$\sup_{t\geq t^*}(w(t)^{-1}w_U(t)) \leq$$

$$\leq 2^{2-\frac{2}{q}}+$$

$$+ \sum_{j=1}^{N-1} 2^{(j+2)(1-\frac{1}{q})}(2^{1+2\alpha+\frac{2}{q}}B\|U_0\|_q + 2^{1-\frac{1}{q}}(2^{-\frac{1}{q}}D+1)\|K\|_1^j)+$$

$$+ 2^{2+2\alpha}\tilde{C}_N B C_4^{\frac{1}{q}} + 2^{2-\frac{2}{q}}A^{\frac{1}{q}}E =$$

$$= C_0. \quad ///$$

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Popular version

The wave equation is an equation that describes wave propagation. For relativistic wave propagation (very high speed) you have to use more complicated equations e.g. the Klein-Gordon equation. The Klein-Gordon equation was first considered by Schrödinger, but was abandoned as it could not explain the fine structure of the hydrogen atom, and Schrödinger instead published his famous nonrelativistic equation. The first one to publish the Klein-Gordon equation (1926) was the Swedish theoretical physicist Oskar Klein (1894–1977) in a work where he attempted to extend Schrödinger's theory to an even more ambitious, unified theory framed in five dimensions. The Klein-Gordon equation was independently discovered (also in 1926) by the German physicist Walter Gordon (1893–1939). Equations of this type appears as scalar versions of the field equations which describes weak nonlinear interaction between elementary particles. The Klein-Gordon equation holds for bosons. Bosons are a class of particles that have integer spin. Examples of bosons are fotons and mesons. There is no spin involved in the solutions but the Klein-Gordon equation gives e.g. a good description of π^0-mesons and for the scalar components of the Higgs fields.

In this thesis are studied the asymptotic behaviour (the behaviour after a long period of time) of finite energy solutions to the nonlinear Klein-Gordon equation. The nonlinear Klein-Gordon equation is a more complicated version of the linear Klein-Gordon equation. The assumptions on the extra, more complicated, term that is added to the linear Klein-Gordon equation are such that they ensure the existence of nonnegative energy, which is essential for the existence of global solutions when no restrictions are assumed on the size of the data. It is shown that some, mathematically and physically, interesting types of decay of the solution of the linear Klein-Gordon equation will be inherited by the solutions of the corresponding (same data) linear Klein-Gordon equation. This is of interest in the theory of scattering. Scattering is irregular reflection or dispersal of waves or particles. That the sky is blue is, for instance, a scattering phenomenon. These kind of results are also of interest since you can relate the decay properties of the solutions (in suitable spaces) and the decay and the rate of the decay of local energy. If data are sufficiently smooth both a lower and an upper bound is known for the decay. For finite energy data, not necessarily smooth, only a lower bound is known. The rate of decay is in general not known. In this thesis is also shown that if the solution of the linear Klein-Gordon eqution in three space-dimensions has maximal rate of decay, so has the solution of the nonlinear Klein-Gordon equation under certain conditions on the nonlinear term.

1 Introduction

1.1 The linear Klein-Gordon equation

The Klein-Gordon equation

$$(\mathrm{KG}) \begin{cases} v_{tt} - \Delta v + m^2 v = 0 \ , \ x \in R^n, \ t \geq 0, \\ v(x,0) = \varphi(x), v_t(x,0) = \psi(x) \end{cases}$$

where $n \geq 3, \ m > 0$ and $\Delta = \Delta_x = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, is the equation for relativistic wave-propagation.

1.1.1 Energy

The energy $E(t)$ is defined by

$$E(t) = \frac{1}{2} \int (|\nabla_x v|^2 + |v_t|^2 + m^2 |v|^2) dx = \|v\|_e^2$$

where $\|\cdot\|_e$ denotes the energy norm. If we multiply the Klein-Gordon equation by v_t and integrate over space, it follows that $E(t)$ is a conserved quantity

$$E(t) = \text{constant} = E(0), t \geq 0.$$

1.1.2 Sobolev and Besov spaces

We now introduce the Sobolev spaces H_p^s (or L_p^s). The norm on H_p^s is defined by

$$\|g\|_{H_p^s} = \|g\|_{p,s} = \|\mathcal{F}^{-1}(w_s \mathcal{F} g)\|_p$$

where $w_s(\xi) = (1 + |\xi|^2)^{\frac{s}{2}}, \mathcal{F}g = \hat{g}$ denotes the Fourier transform of g and $\|\cdot\|_p$ denotes the norm on L_p. For $p = 2$ we usually drop the reference to L_2 and write $H_2^s = H^s$. Let $Y^s = H^{s+1} \times H^s$. For $s = 0$ we set $Y^0 = H^1 \times L_2 = X_e$, the energy space. A finite energy solution of the (KG), i.e. a solution of the (KG) with data $(\varphi, \psi) \in X_e$, will in the following be denoted by u_0.

Let $B_p^{s,q}$ denote the L_p-based Besov space of order $s \geq 0$. The norm on $B_p^{s,q}$ is defined by

$$\|g\|_{B_p^{s,q}} = \|g\|_p + \Big(\int_0^1 \Big(t^{-\sigma} \sum_{|\alpha|=S} w_p(t, D^\alpha g) \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$

where $w_p(t,z) = \sup_{|h| \leq t} \|z_h - z\|_p, z_h(x) = z(x+h)$ and $s = \sigma + S, 0 < \sigma < 1$, S an integer. Between Besov spaces, $B_p^{s,q}$, and Sobolev spaces, H_p^s, we have the following inclusions:

$$B_p^{s,p} \subseteq H_p^s \subseteq B_p^{s,2}, \ 1 < p \leq 2, \quad B_{p'}^{s',2} \subseteq H_{p'}^{s'} \subseteq B_{p'}^{s',p'}, \ 2 \leq p' < \infty.$$

For a proof, see e.g. Bergh and Löfström [1].

1.1.3 Solution formula

Define the operator B by

$$Bg = \mathcal{F}^{-1}(\hat{B}\hat{g}), \hat{B}(\xi) = (|\xi|^2 + m^2)^{\frac{1}{2}}$$

Then, if $(\varphi, \psi) \in X_e$ and $u_0(t) = u_0(\cdot, t)$ for each t

$$\widehat{u_0(t)}(\xi) = \cos(t\hat{B}(\xi))\hat{\varphi}(\xi) + \frac{\sin(t\hat{B}(\xi))}{\hat{B}(\xi)}\hat{\psi}(\xi)$$

and thus u_0 has the solution formula

$$u_0(t) = E_0(t)\varphi + E_1(t)\psi = \frac{\partial R}{\partial t} * \varphi + R * \psi$$

where, for each t

$$R(x,t) = \mathcal{F}_\xi^{-1}\left\{\frac{\sin(t\hat{B}(\xi))}{\hat{B}(\xi)}\right\}$$

in the sense of distributions.

$E_0(t)$ and $E_1(t)$ are called the solution operators of the (KG) and are often denoted by $\cos(tB)$ and $B^{-1}\sin(tB)$. For $E_1(t)$ we will need the following result by Brenner [3], [4].

Lemma 1.1. *Let*

$$(*) \begin{cases} 2 \le p' < \infty, \frac{1}{p} + \frac{1}{p'} = 1, 1 \le q \le \infty \\ \delta_{p'}(n+1+\theta) \le 1 + s - s' \textit{ where } \delta_{p'} = \frac{1}{2} - \frac{1}{p'}, 0 \le \theta \le 1 \textit{ and } s, s' \ge 0. \end{cases}$$

Then, if X_p^s denotes H_p^s or $B_p^{s,q}$,

$$\|E_1(t)g\|_{X_{p'}^{s'}} \le K(t)\|g\|_{X_p^s}, t \ge 0$$

where

$$K(t) \le C \begin{cases} t^{-(n-1+\theta)\delta_{p'}}, & 1 \le t \\ t^{-(n-1-\theta)\delta_{p'}}, & 0 < t < 1 \end{cases}$$

1.1.4 Strichartz estimates

Let $\|g\|_{L_p^s(L_q^r)}$ denote the $L_p^s(R^+)$-norm in t of $\|g(t)\|_{L_q^r(R^n)}$. For a finite energy solution u_0 of the (KG) we have that (Strichartz [13], [15], Segal [12])

$$\|u_0\|_{L_p(L_p^{\frac{1}{2}})} \le C_1\|\varphi\|_{H^1} + \|\psi\|_{L_2}) \le C$$

where $p = \frac{2(n+1)}{n-1}$, i.e. $\delta_p = \frac{1}{2} - \frac{1}{p} = \frac{1}{n+1}$. More complex estimates bound u_0 in $L_q(R, H_p^s(R^n))$. (Strichartz [15], Marshall-Strauss-Wainger [10], Brenner [5].) The following Strichartz estimate (Brenner [5]) will later inspire our discussion of the corresponding nonlinear problem.

Theorem 1.1. *Let $2 \le r \le \mu < \infty, s > -\frac{1}{2}$ and $0 \le \sigma \le s + \frac{1}{2}$. Then*

$$u_0 \in L_r^\sigma(L_\mu^{\frac{1}{2}+s-\sigma})$$

if data $(\varphi, \psi) \in X_e$, provided that

$$s = \frac{1}{2}(1 - (n + 1 + \theta)\delta_\mu), \text{ some } \theta \in [0, 1], \text{ and}$$
$$(n - 1 + \theta)\delta_\mu > 1 - 2\delta_r > (n - 1 - \theta)\delta_\mu$$

or, more generally,

$$s \ge \frac{1}{2}(1 - (n + 2)\delta_\mu) \text{ and}$$
$$\delta_\mu + s < \delta_r < 1 - n\delta_\mu - s.$$

If $\delta_r \neq 0$ we may use equality signs in the last inequality.

The following special cases will be useful later.

Example 1: Let $2 < q \le p' < \infty$ and $s' > 0$. Then $u_0 \in L_q(L_{p'}^{s'})$, provided that

$$s' \ge 1 - \frac{n+2}{2}\delta_{p'} \text{ and } \delta_{p'} + s' - \frac{1}{2} \le \delta_q \le \frac{3}{2} - n\delta_{p'} - s'.$$

Example 2: Let $2 = q \le p' < \infty$ and $\gamma > 0$. Then $u_0 \in L_2(L_{p'}^\gamma)$ provided that

$$1 - \frac{n+2}{2}\delta_{p'} \le \gamma < \min(\frac{1}{2} - \delta_{p'}, \frac{3}{2} - n\delta_{p'}).$$

Remark: It follows that $\frac{1}{n} < \delta_{p'} < \frac{1}{n-2}$. If $\delta_{p'} = \dfrac{1-\epsilon}{n-1}$, $0 < \epsilon < \frac{1}{n}$, we must have that

$$\frac{1}{2} - \frac{3}{2(n-1)} + \epsilon\frac{n+2}{2(n-1)} \le \gamma < \frac{1}{2} - \frac{1}{n-1} + \epsilon\frac{1}{n-1}$$

Example 3: Let $2 < r \le q' = \frac{2(n+1)}{n-1}$ (i.e. $\delta_{q'} = \frac{1}{n+1}$), $\frac{n}{2(n+1)} \le \sigma \le \frac{1}{2}$. Then $u_0 \in L_r(L_{q'}^\sigma)$, provided that

$$\sigma - \frac{n-1}{2(n+1)} \le \delta_r \le \frac{n+3}{2(n+1)} - \sigma.$$

1.1.5 Decay of solutions of the linear Klein-Gordon equation

Theorem 1.1 implies that for $(\varphi, \psi) \in X_e$

$$\int_{\frac{t}{2}}^t \|u_0(\tau)\|_{L_{p'}^{s'}}^q d\tau \to 0 \text{ as } t \to \infty$$

for certain combinations of q, p' and s'. This leads us to introduce the "long range mean-value" $M_{q,X}$ defined by

$$M_{q,X}v(t) = \Big(\frac{2}{t}\int_{\frac{t}{2}}^t \|v(\tau)\|_X^q d\tau\Big)^{\frac{1}{q}}, t > 0.$$

By Theorem 1.1 $M_{q,X}u_0(t) \to 0$ as $t \to \infty$ for certain q and X if $(\varphi, \psi) \in X_e$. What is known about the rate of decay of $M_{q,X}u_0(t)$ for non-trivial solutions u_0? If data are sufficiently smooth, i.e. φ and ψ have sufficiently many derivatives in L_1, it is known that (Strichartz [13], Kumlin [9])

$$c_1 t^{-n\delta_{p'}} \leq M_{q,L_{p'}} u_0(t) \leq c_2 t^{-n\delta_{p'}}, t \to \infty, p' \geq 2$$

For finite energy data, not necessarily smooth, it is only known that (Glassey [8])

$$ct^{-n\delta_{p'}} \leq M_{q,L_{p'}} u_0(t)$$

The rate of decay is, in general, not known.

1.2 The nonlinear Klein-Gordon equation

The aim of this paper is to study the decay of solutions of the nonlinear Klein-Gordon equation

$$\text{(NLKG)} \begin{cases} u_{tt} - \Delta u + m^2 u + f(u) = 0, x \in R^n, t \geq 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x), (\varphi, \psi) \in X_e \end{cases}$$

where $n \geq 3, m > 0$ and $\Delta = \Delta_x = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. A solution of the (NLKG) will in the following be denoted by u.

1.2.1 Conditions on the nonlinearity

The nonlinearity $f(u)$ is supposed to satisfy the following conditions which are motivated by the physics involved. We will assume that $f \in C^1, f(0) = 0$ with $f(\mathbb{R}) \subseteq \mathbb{R}$ and that

(i) $F(u) = \int_0^u f(v)dv \geq 0$ for all $u \in \mathbb{R}$

(ii) $|f'(u)| \leq C|u|^{\rho-1}$ where $1 + \frac{4}{n} = \rho_* < \rho < \rho^* = 1 + \frac{4}{n-2}$.

(iii) $uf(u) - 2F(u) \geq \alpha F(u)$ for some $\alpha > 0$ and F is not flat at 0 or ∞.

1.2.2 Energy

To better understand condition (i) we introduce the energy, $E(t)$, of a solution to the (NLKG)

$$E(t) = \frac{1}{2} \int (|\nabla_x u|^2 + |u_t|^2 + m^2|u|^2)dx + \int F(u)dx$$

Condition (i) implies the existence of nonnegative energy for the solutions of the (NLKG), which is essential for the existence of global solutions when no restrictions are assumed on the size of the data.

Another consequence of (i) is that the energy norm of u is bounded. To realize this, we multiply the nonlinear Klein-Gordon equation by u_t and integrate over space. It follows that

$$E(t) = \text{constant} \ , t \geq 0$$

which together with (i) implies that

$$\|u\|_e \leq C.$$

Condition (ii) gives growth restrictions at 0 and ∞. Typically $f(u) \sim u|u|^{\rho-1}$ at 0 and ∞ with $\rho > \rho_*$ for $|u| \leq 1$ and $\rho < \rho^*$ for $|u| \geq 1$. The condition $uf(u) - 2F(u) \geq \alpha F(u)$ ensures that no standing waves will appear as solutions. The appearance of standing waves would make decay impossible. The condition also implies the decay of local energy (Morawetz [11]).

1.2.3 The rate of decay of solutions of the nonlinear Klein-Gordon equation

A bound for the rate of decay of solutions to the (NLKG) is given by the following therorem.

Theorem 1.2 (Brenner [7]). *Let $1 + \frac{4}{n} < \rho < 1 + \frac{4}{n-2}$, and $X_t = L_p(\{x : |x| \leq t\}) \subseteq X = L_p(R^n)$. Then, if $u \in L_q^{\text{loc}}(R, L_p^{\text{loc}}(R^n))$, where $p, q \geq 2$, there is a constant $c > 0$ such that*

$$M_{q,X}^T u(t) \geq \Big(\frac{1}{T}\int_{t+T}^{T} \|u(\tau)\|_{X_\tau}^q d\tau\Big)^{\frac{1}{q}} \geq ct^{-n\delta_p}$$

for $T > 0$ and $t \geq \max\{1, T\}$.

The question now arises: If u_0 has maximal decay, is this property inherited by u? The following theorem of Brenner is a recent result announced in [7], but before we state the theorem, we must define the concept of uniform decay. We say that $g : R_+ \to R_+$ has uniform decay in L_q if, for some $c, t^* \geq 1$ independent of t,

$$\int_{\frac{t}{4}}^{\frac{t}{2}} g(\tau)^q d\tau \leq c \int_{\frac{t}{2}}^{t} g(\tau)^q d\tau, t \geq t^*$$

Theorem 1.3. *Assume that u is a solution of the (NLKG) with finite energy data, and that u_0 is the corresponding solution of the (KG) with the same data. Assume that $(*)$, $(*)'$ hold, $1 + \frac{4}{n} < \rho < \frac{n+2}{n-2}$ and that $u_0 \in L_q(X_{p'}^{s'}) \cap L_1(X_{p'}^{s'})$. Then, if $\|u_0\|_{X_{p'}^{s'}}$ has uniform decay in L_q,*

$$\int_{\frac{t}{2}}^{t} \|u(\tau)\|_{X_{p'}^{s'}}^q d\tau \sim \int_{\frac{t}{2}}^{t} \|u_0(\tau)\|_{X_{p'}^{s'}}^q d\tau, \ \textit{as } t \to \infty$$

Corollary 1.1. *If u_0 has maximal rate of decay in $L_q((\frac{t}{2}, t), X_{p'}^{s'})$, then also u has maximal rate of decay in $L_q((\frac{t}{2}, t), X_{p'}^{s'})$.*

In an earlier work of the author [2] it is shown that if the "long range mean-value" of u_0

$$M_{q,X_{p'}^{s'}} u_0(t) = \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t} \|u_0\|_{X_{p'}^{s'}}^q d\tau\Big)^{\frac{1}{q}} \leq w(t)$$

where $w(t) \to 0$ as $t \to \infty$, then this property is inherited by u provided that $K \in L_1 \cap L_{1+\epsilon}$, some $\epsilon > 0$, $\int_{\frac{t}{2}}^{t} K_N(t-\tau)\|u(\tau)\|_{X_{p'}^{s'}} d\tau$ decays to zero "roughly" as $w(t)$ for large values of t and that $w(t)$ satisfies certain conditions. The main result in this paper is the following. (See Theorem 2.2.)

Let u be a finite energy solution of the (NLKG), and let u_0 be the corresponding solution of the (KG) with the same data. Assume that

$$M_{q_1,L_r}u(t) \leq C_1(1+t)^{-\frac{1}{q_1}-\gamma}, \gamma > 0$$

for some $q_1, r \geq 2$, and that

$$M_{q,X}u_0(t) \leq w(t)$$

where $w(t)$ satisfies certain growth conditions. Then

$$M_{q,X}u(t) \leq Cw(t)$$

One corollary is the following.

If $u \in L_r(L_{q'}^{\sigma})$ where $\delta_{q'} = \frac{1}{n+1}$ and u_0 has maximal rate of decay, then u also has maximal rate of decay in the case $n = 3$ for certain r, σ and ρ.

1.2.4 A nonlinear Volterra integral inequality with singular kernel

On an interval where u exists as a solution of the (NLKG), we have that

$$u = u_0 - \int_0^t E_1(t-\tau)f(u)d\tau$$

where u_0 is the corresponding solution of the (KG) with the same data as u. If condition $(*)$ in Lemma 1.1 is satisfied, we obtain by the solution formula above, Minkowski's inequality, Minkowski's inequality for integrals and Lemma 1.1 that

$$\|u\|_{X_{p'}^{s'}} \leq \|u_0\|_{X_{p'}^{s'}} + \int_0^t K(t-\tau)\|f(u)\|_{X_p^s} d\tau$$

Inspired by Theorem 1.1 and the above inequality, we have investigated if some (mathematically and physically) interesting types of decay of u_0 will be inherited by u. In an earlier work of the author [2] the integral inequality

$$\|u(t)\|_X \leq \|u_0(t)\|_X + C\int_0^t K(t-\tau)\|u(\tau)\|_X^{1-\eta} d\tau, |\eta| < \eta_0 < 1$$

where $X = X_{p'}^{s'}$ denotes $L_{p'}^{s'}$ or $B_{p'}^{s',2}$ was an essential tool. The kernel K was supposed to satisfy the conditions

$$0 \leq K \in L_1 \cap L_{1+\epsilon}, \text{ some } \epsilon > 0,$$

and

$$K(t) \leq C \begin{cases} t^{-1-\alpha}, 1 \leq t \\ t^{-1+\alpha}, 0 < t \leq 1 \end{cases}, \text{ some } \alpha > 0.$$

These conditions have their origin i Lemma 1.1 which stated that

$$\|E_1(t)g\|_{X^{s'}_{p'}} \leq K(t)\|g\|_{X^s_p}, t \geq 0, 2 \leq p' < \infty, \frac{1}{p}+\frac{1}{p'}=1$$

where

$$K(t) \leq C \begin{cases} t^{-(n-1+\theta)\delta_{p'}} & 1 \leq t \\ t^{-(n-1-\theta)\delta_{p'}} & 0 < t \leq 1 \end{cases}$$

provided that

$$(*) \qquad (n+1+\theta)\delta_{p'} \leq 1+s-s', 0 \leq \theta \leq 1 \text{ and } s, s' \geq 0.$$

In paper 1 also the stronger assumption was used that p', s' should be such that $K \in L_1$, i.e.

$$(*)_s \begin{cases} (n+1+\theta)\delta_{p'} \leq 1+s-s' \\ (n-1-\theta)\delta_{p'} < 1 < (n-1+\theta)\delta_{p'} \end{cases} \quad \text{some } \theta \in (0,1], s, s' \in [0,1]$$

In this paper the kernel $K(t-\tau)$ is replaced by $K(t-\tau)h(\tau)$ where $K(t)$ and $h(t)$ are supposed to satisfy the conditions

$$\begin{cases} 0 \leq K \in L_{q_0}(1,\infty) \cap L_{q_0+\epsilon}(1,\infty) \text{ some } \epsilon > 0, 0 \leq h \in L_{q_0'}(0,\infty), \ \frac{1}{q_0}+\frac{1}{q_0'}=1 \\ K \in L_{r_0}(0,1) \cap L_{r_0+\varepsilon}(0,1) \text{ some } \varepsilon > 0, h \in L_{r_0'}(0,\infty), \frac{1}{r_0}+\frac{1}{r_0'}=1 \end{cases}$$

and

$$\begin{cases} K(t) \leq Ct^{-\beta}, t > 0, \beta q_0 > 1 > \beta r_0 \\ \left(\frac{2}{t}\int_{\frac{t}{2}}^{t} h(\tau)^{r_0'} d\tau\right)^{\frac{1}{r_0'}} \leq C(1+t)^{-\frac{1}{r_0'}-\gamma}, t > 0, \gamma > 0, r_0' \geq q_0' \end{cases}$$

To motivate the above conditions, we will study the important special case $\delta_{p'} = \frac{1}{n+1}$. If we use Lemma 1.1, for $\delta_{q'} = \frac{1}{n+1}$ it follows that

$$\|E_1(t)f(u)\|_{q',\sigma} \leq K(t)\|f(u)\|_{q,\sigma}$$

where $\frac{1}{q}+\frac{1}{q'}=1$ and $\sigma \geq 0$. From Lemma 1.1 also follows in this case that $\theta = 0$ by which

$$K(t) \leq Ct^{-\frac{n-1}{n+1}}, t > 0$$

so that $\beta = \frac{n-1}{n+1}$. We also have that $K \in L^{\text{loc}}_{r_0} \cap L^{\text{loc}}_{r_0+\varepsilon}$, some $\varepsilon > 0$, if $r_0 < \frac{n+1}{n-1}$ and that $K \in L_{q_0}([1,\infty)) \cap L_{q_0+\epsilon}([1,\infty))$, some $\epsilon > 0$, if $q_0 > \frac{n+1}{n-1}$. From $q_0 > \frac{n+1}{n-1}$ also follows that $q_0' < \frac{n+1}{2}$.

To derive the new integral inequality we must estimate $\|f(u)\|_{q,\sigma}$. We have, by (ii), and Sobolev's embedding theorem that

$$\|f(u)\|_{q,\sigma} \leq C\|u\|^{\rho-1+\eta}_{q',\gamma+\sigma}\|u\|^{1-\eta}_{q',\sigma}, 0 \leq |\eta| < \eta_0 \quad \text{(small)}$$

provided that $\rho \geq 1-\eta$ and that

$$\frac{1}{q}-\frac{\sigma}{n} \geq (\rho-1+\eta)\Big(\frac{1}{q'}-\frac{\gamma+\sigma}{n}\Big)+(1-\eta)\Big(\frac{1}{q'}-\frac{\sigma}{n}\Big)$$

that is, for

$$\rho \leq 1 + \frac{2\delta_{q'} + \eta\frac{\gamma}{n}}{\frac{1}{2} - \delta_{q'} - \frac{\gamma+\sigma}{n}} = 1 + \frac{4 + \eta\frac{2(n+1)\gamma}{n}}{n - 1 - \frac{2(n+1)(\gamma+\sigma)}{n}} = \rho_u.$$

Notice that $\rho_u \geq 1 + \frac{4}{n-2}$ if $\gamma + \sigma \geq \frac{n}{2(n+1)} - \eta\frac{n-2}{4}\gamma$ The lower bound for ρ is obtained from

$$\frac{1}{q} \leq \frac{\rho}{q'}$$

by which

$$\rho \geq \frac{\frac{1}{2} + \delta_{q'}}{\frac{1}{2} - \delta_{q'}} = 1 + \frac{4}{n-1}.$$

We have now derived the new integral inequality

$$\|u(t)\|_{q',\sigma} \leq \|u_0(t)\|_{q',\sigma} + C\int_0^t K(t-\tau)h(\tau)\|u(\tau)\|_{q',\sigma}^{1-\eta}d\tau$$

where $h(\tau) = \|u(\tau)\|_{q',\gamma+\sigma}^{\rho-1+\eta}$. It remains to investigate for which q_0' we have that $h \in L_{q_0'}$. From Brenner [6] we know that $u \in L_{q'}(L_{q'}^{\gamma+\sigma})$ where $q' = \frac{2(n+1)}{n-1}$, i.e, $\delta_{q'} = \frac{1}{n+1}$ and $\frac{n}{2(n+1)} \leq \gamma + \sigma \leq \frac{1}{2}$. The proof is based on interpolation between $L_\infty(L_2^1)$ and $L_2(L_{p'}^\gamma)$ where $\delta_{p'} = \frac{1-\varepsilon}{n-1}$ and $\gamma = \frac{1}{2} - \frac{1}{n-1}$. That $u \in L_\infty(L_2^1)$ follows from

$$\|u\|_{2,1} \leq C\|u\|_e$$

and that $u \in L_2(L_{p'}^\gamma)$ follows from the Remark of example 2, page 3 in this paper, and the following Strichartz-type estimate.

THEOREM 1.4 (Brenner [6]): Let $1 + 4\delta_{p'} < \rho < \frac{n+4n\delta_{p'}-2-2(s-s')}{n-2}, n \geq 3$. Assume that $(*)_s$ holds (see page 7 in this paper) with $\frac{s-s'}{1-s'} < \rho - 1$. Then

$$u_0 \in L_q(L_{p'}^{s'}) \Rightarrow u \in L_q(L_{p'}^{s'})$$

Finally $u \in L_{q'}(L_{q'}^{\gamma+\sigma})$ implies that $h \in L_{q_0'}$ where

$$\frac{1}{q_0'} = \frac{\rho - 1 + \eta}{q'} = (\rho - 1 + \eta)\Big(\frac{1}{2} - \delta_{q'}\Big)$$

1.2.5 Decay function

The conditions on the decay function $w(t)$ in the main theorem are

(1_w) $$\Big(\frac{2}{t}\int_{\frac{t}{2}}^t w(\tau)^q d\tau\Big)^{\frac{1}{q}} \leq Aw(t), t > 0$$

(2_w) $$\frac{1}{E} \leq w(t)^{-1} \leq Bt^{\beta - \frac{1}{q_0} + \frac{1}{q}}, t \geq \frac{t^*}{4}$$

(3_w) $$w(\frac{t}{2}) \leq Fw(t)$$

where A, B, E and F are positive constants.

Example: The function $w(t) = C(1+t)^{-\alpha}, C > 0$, satisfies the conditions $1_w - 3_w$ with $A = F = 2^\alpha$, $E = \frac{C}{(1+\frac{t^*}{4})^\alpha}$ and $B = \frac{(1+\frac{4}{t^*})^\alpha}{C}$, provided that $0 < \alpha \le \beta - \frac{1}{q_0} + \frac{1}{q}$, as

$$(1_w) \qquad \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t} w(\tau)^q d\tau\Big)^{\frac{1}{q}} \le C\Big(1+\frac{t}{2}\Big)^{-\alpha} \le 2^\alpha C(1+t)^{-\alpha}$$

$$(2_w) \qquad \frac{(1+\frac{t^*}{4})^\alpha}{C} \le \frac{(1+t)^\alpha}{C} \le \frac{(1+\frac{4}{t^*})^\alpha}{C} t^\alpha$$

$$(3_w) \qquad C\Big(1+\frac{t}{2}\Big)^{-\alpha} \le 2^\alpha C(1+t)^{-\alpha}$$

1.2.6 Rate of decay of solutions to the (NLKG)

Let $0 < \delta_r \le \delta_{q'} = \frac{1}{n+1}$. From example 3, page 3 in this paper, we have that

$$u_0 \in L_r(L^\sigma_{q'}) \text{ provided that}$$

$$\sigma \ge \frac{n}{2(n+1)} \text{ and } \sigma - \frac{n-1}{2(n+1)} \le \delta_r \le \frac{n+3}{2(n+1)} - \sigma.$$

From Brenner [6] we also know that

$$u_0 \in L_r(L^\sigma_{q'}) \Rightarrow u \in L_r(L^\sigma_{q'})$$

provided that $\delta_{q'} = \dfrac{1}{n+1}, 0 \le \sigma \le 1$, and

$$1 + \frac{4}{n-1} \le \rho < 1 + \frac{4}{n-2}$$

By the main theorem, it follows that if

$$M_{r,L^\sigma_{q'}} u_0(t) \le C(1+t)^{-\alpha}$$

then also

$$M_{r,L^\sigma_{q'}} u(t) \le C(1+t)^{-\alpha}.$$

for t large and $0 < \alpha \le \beta - \frac{1}{q_0} + \frac{1}{r}$ where

$$\begin{aligned}
\beta - \frac{1}{q_0} &= \beta + \frac{1}{q_0'} - 1 = \\
&= \frac{n-1}{n+1} + (\rho - 1 + \eta)\Big(\frac{1}{2} - \delta_{q'}\Big) - 1 = \\
&= (\rho - 1 + \eta)\frac{n-1}{2(n+1)} - \frac{2}{n+1}
\end{aligned}$$

by which

$$\alpha_{max} = \beta - \frac{1}{q_0} + \frac{1}{r} = (\rho - 1 + \eta)\frac{n-1}{2(n+1)} + \frac{n-3}{2(n+1)} - \delta_r$$

1.2.7 Estimates of the rate of decay α

For non-vanishing finite energy data, one can prove that (Glassey [8])

$$M_{q',L^{\sigma}_{q'}} u_0(t) \geq ct^{-n\delta_{q'}}$$

where $\delta_{q'} = \dfrac{1}{n+1}$ and $\sigma = \dfrac{1}{2}$. From the main theorem it follows that if this mean-value, for the solution u_0 of the (KG), has an upper bound $O(t^{-\alpha})$ as $t \to \infty$, then the same upper bound also holds for the same mean-value of the solution u of the (NLKG). In the special case $n = 3$, $\delta_{q'} = \frac{1}{n+1} = \frac{1}{4}$, we have found that

$$\alpha_{max} = (\rho - 1 + \eta)\frac{n-1}{2(n+1)} + \frac{n-3}{2(n+1)} - \delta_r = \frac{\rho - 1 + \eta}{4} - \delta_r$$

where

$$1 + \frac{4}{n-1} = 3 < \rho < 1 + \frac{4}{n-2} = 5$$

and we obtain maximal rate of decay if

$$\alpha_{max} = \frac{\rho - 1 + \eta}{4} - \delta_r \geq n\delta_{q'} = \frac{3}{4}$$

i.e. if

$$\rho \geq 4 + 4\delta_r - \eta$$

For $\delta_r = \frac{1}{4} - \frac{\xi}{2}$, $\sigma = \frac{1-\xi}{2}, 0 < \xi \leq \frac{1}{4}$ maximal rate of decay is obtained for $5 - 2\xi \leq \rho < 5$. More generally we have that

$$\alpha_{max} \leq (\rho - 1 + \eta)\frac{n-1}{2(n+1)} + \frac{2n - 4 + \theta}{2(n+1)} - \frac{1}{n-1} = \alpha^*$$

for $\sigma = \frac{1}{n-1} - \frac{\theta}{2(n+1)}$ some $\theta \in (0, 1]$. If $\rho = 1 + \frac{4}{n-1}$ and $\theta = \varepsilon$ we have that

$$\alpha^*_{\min} = n\delta_{q'} - \frac{1}{n-1} + \frac{\varepsilon + \eta(n-1)}{2(n+1)}$$

and, if $\rho = 1 + \frac{4}{n-2} - \varepsilon$ and $\theta = 1$,

$$\alpha^*_{\max} = n\delta_{q'} + \frac{2}{3(n-2)} - \frac{1}{6(n+1)} - \frac{1}{n-1} + (\eta - \varepsilon)\frac{n-1}{2(n+1)} =$$
$$= n\delta_{q'} - \frac{1}{n-1} + \frac{n+2}{2(n+1)(n-2)} + (\eta - \varepsilon)\frac{n-1}{2(n+1)}$$

by which we get the following table

n	$n\delta_{q'} - \frac{1}{n-1}$	$n\delta_{q'} - \frac{1}{n-1} + \frac{n+2}{2(n+1)(n-2)}$	$n\delta_{q'}$
3	$\frac{3}{4} - \frac{1}{2} = \frac{1}{4} = \frac{2}{8}$	$\frac{3}{4} + \frac{1}{8} = \frac{7}{8}$	$\frac{6}{8}$
4	$\frac{4}{5} - \frac{1}{3} = \frac{7}{15} = \frac{14}{30}$	$\frac{4}{5} - \frac{1}{30} = \frac{23}{30}$	$\frac{24}{30}$
5	$\frac{5}{6} - \frac{1}{4} = \frac{7}{12} = \frac{21}{36}$	$\frac{5}{6} - \frac{1}{18} = \frac{7}{9} = \frac{28}{36}$	$\frac{30}{36}$
6	$\frac{6}{7} - \frac{1}{5} = \frac{23}{35}$	$\frac{6}{7} - \frac{2}{35} = \frac{4}{5} = \frac{28}{35}$	$\frac{30}{35}$
10	$\frac{10}{11} - \frac{1}{9} = \frac{79}{99} = \frac{316}{396}$	$\frac{10}{11} - \frac{17}{396} = \frac{343}{396}$	$\frac{360}{396}$

2 Decay of long time mean-values of solutions to the nonlinear Klein-Gordon equation

2.1 Basic definitions

Assume that

$$\begin{cases} 0 \leq K \in L_{q_0}(1,\infty) \cap L_{q_0+\epsilon}(1,\infty) \text{ some } \epsilon > 0, h \in L_{q_0'}(0,\infty) \text{ with } \frac{1}{q_0} + \frac{1}{q_0'} = 1 \\ K \in L_{r_0}(0,1) \cap L_{r_0+\varepsilon}(0,1) \text{ some } \varepsilon > 0, h \in L_{r_0'}(0,\infty) \text{ with } \frac{1}{r_0} + \frac{1}{r_0'} = 1 \end{cases}$$

and let $\mathcal{K}^j$ be the integral operators defined by

$$\mathcal{K}^0 g(t) = g(t)$$
$$\mathcal{K}^1 g(t) = \mathcal{K} g(t) = \int_0^t K(t-\tau)h(\tau)g(\tau)d\tau = K * (hg)(t)$$
$$\mathcal{K}^j g(t) = \mathcal{K}(\mathcal{K}^{j-1}g)(t), j \in Z_+$$

From the definitions above follows by Fubini's theorem that

$$\mathcal{K}^j g(t) = \int_0^t K_j(t,\tau)g(\tau)d\tau$$

where

$$K_1(t,\tau) = K(t-\tau)h(\tau)$$

and

$$K_j(t,\tau) = \int_\tau^t K_1(t,\sigma)K_{j-1}(\sigma,\tau)d\sigma, j \in Z_+ \setminus \{1\}$$

2.2 Some useful properties of the kernel

To make the proof of the main theorem more accessible, we will in the following give some of the details as lemmas.

Lemma 2.1. *Let* $\|\cdot\|_r$ *denote* $\|\cdot\|_{L_r(0,t)}$. *Then*

$$\|\mathcal{K}g\|_q \leq 2^{1-\frac{1}{q}}(2\|K\|_{L_{r_0}(0,1)}\|h\|_{r_0'} + \|K\|_{L_{q_0}(1,\infty)}\|h\|_{q_0'})\|g\|_q$$

Proof. By Hölder's inequality and Fubini's theorem, for $t \geq 1$,

$$
\begin{aligned}
&\|\mathcal{K}g\|_q = ((\int_0^1 + \int_1^t)(\mathcal{K}g(\tau))^q d\tau)^{\frac{1}{q}} \leq \\
&\leq (\int_0^1 (\int_0^\tau K_1(\tau,s)g(s)ds)^q d\tau)^{\frac{1}{q}} + (\int_1^t (\int_0^{\tau-1} + \int_{\tau-1}^\tau)K_1(\tau,s)g(s)ds)^q d\tau)^{\frac{1}{q}} \leq \\
&\leq (\int_0^1 (\int_0^\tau h(s)^{r_0'}ds)^{\frac{q}{r_0'}} (\int_0^\tau K(\tau-s)^{\frac{qr_0-r_0^2}{q}} K(\tau-s)^{\frac{r_0^2}{q}} g(s)^{r_0}ds)^{\frac{q}{r_0}} d\tau)^{\frac{1}{q}} + \\
&+ 2^{1-\frac{1}{q}}(\int_1^t (\int_0^{\tau-1} h(s)^{q_0'}ds)^{\frac{q}{q_0'}} (\int_0^{\tau-1} K(\tau-s)^{\frac{qq_0-q_0^2}{q}} K(\tau-s)^{\frac{q_0^2}{q}} g(s)^{q_0}ds)^{\frac{q}{q_0}} d\tau)^{\frac{1}{q}} + \\
&+ 2^{1-\frac{1}{q}}(\int_1^t (\int_{\tau-1}^\tau h(s)^{r_0'}ds)^{\frac{q}{r_0'}} (\int_{\tau-1}^\tau K(\tau-s)^{\frac{qr_0-r_0^2}{q}} K(\tau-s)^{\frac{r_0^2}{q}} g(s)^{r_0}ds)^{\frac{q}{r_0}} d\tau)^{\frac{1}{q}} \leq \\
&\leq \|h\|_{L_{r_0'}(0,1)}(\int_0^1 (\int_0^\tau K(s)^{r_0}ds)^{\frac{q-r_0}{r_0}} (\int_0^\tau K(\tau-s)^{r_0}g(s)^q ds)d\tau)^{\frac{1}{q}} + \\
&+ 2^{1-\frac{1}{q}}\|h\|_{L_{q_0'}(0,t-1)}(\int_1^t (\int_1^\tau K(s)^{q_0}ds)^{\frac{q-q_0}{q_0}} (\int_0^{\tau-1} K(\tau-s)^{q_0}g(s)^q ds)d\tau)^{\frac{1}{q}} + \\
&+ 2^{1-\frac{1}{q}}\|h\|_{L_{r_0'}(0,t)}(\int_1^t (\int_0^1 K(s)^{r_0}ds)^{\frac{q-r_0}{r_0}} (\int_{\tau-1}^\tau K(\tau-s)^{r_0}g(s)^q ds)d\tau)^{\frac{1}{q}} \leq \\
&\leq \|h\|_{L_{r_0'}(0,1)}\|K\|_{L_{r_0}(0,1)}^{\frac{q-r_0}{q}}(\int_0^1 g(s)^q \int_s^1 K(\tau-s)^{r_0}d\tau ds)^{\frac{1}{q}} + \\
&+ 2^{1-\frac{1}{q}}\|h\|_{L_{q_0'}(0,t-1)}\|K\|_{L_{q_0}(1,t)}^{\frac{q-q_0}{q}}(\int_0^{t-1} g(s)^q \int_{s+1}^t K(\tau-s)^{q_0}d\tau ds)^{\frac{1}{q}} + \\
&+ 2^{1-\frac{1}{q}}\|h\|_{L_{r_0'}(0,t)}\|K\|_{L_{r_0}(0,1)}^{\frac{q-r_0}{q}}(\int_0^1 g(s)^q \int_1^{s+1} K(\tau-s)^{r_0}d\tau ds + \\
&+ \int_1^{t-1} g(s)^q \int_s^{s+1} K(\tau-s)^{r_0}d\tau ds + \int_{t-1}^t g(s)^q \int_s^t K(\tau-s)^{r_0}d\tau ds)^{\frac{1}{q}} \leq
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|h\|_{L_{r_0'}(0,1)}\|K\|_{L_{r_0}(0,1)}\|g\|_{L_q(0,1)} + \\
&+ 2^{1-\frac{1}{q}}\|h\|_{L_{q_0'}(0,t-1)}\|K\|_{L_{q_0}(1,t)}\|g\|_{L_q(0,t-1)} + \\
&+ 2^{1-\frac{1}{q}}\|h\|_{L_{r_0'}(0,t)}\|K\|_{L_{r_0}(0,1)}\|g\|_{L_q(0,t)} \leq \\
&\leq 2^{1-\frac{1}{q}}(2\|K\|_{L_{r_0}(0,1)}\|h\|_{r_0'} + \|K\|_{L_{q_0}(1,t)}\|h\|_{q_0'})\|g\|_q \qquad ///
\end{aligned}
$$

In the same way it follows that, for $0 < t \leq 1$,

$$\|\mathcal{K}g\|_q \leq \|h\|_{L_{r_0'}(0,1)}\|\mathcal{K}\|_{L_{r_0}(0,1)}\|g\|_{L_q(0,t)} \qquad ///$$

Corollary 2.1. *Repeated use of lemma 2.1 yields*

$$
\begin{aligned}
\|\mathcal{K}^j g\|_q &\leq 2^{(1-\frac{1}{q})j}(2\|K\|_{L_{r_0}(0,1)}\|h\|_{r_0'} + \|K\|_{L_{q_0}(1,\infty)}\|h\|_{q_0'})^j\|g\|_q \leq \\
&\leq 2^{(2-\frac{1}{q})j}(\|K\|_{L_{r_0}(0,1)}\|h\|_{r_0'} + \|K\|_{L_{q_0}(1,\infty)}\|h\|_{q_0'})^j\|g\|_q
\end{aligned}
$$

for $j \in Z_+$.

Lemma 2.2. *Let $0 < \tau - 1 \le s \le \tau$. Then*

$$K_N(\tau, s) = G_{N-1}(\tau, s)h(s) \le \|h\|^{N-1}_{L_{r_0'}(s,\tau)} \tilde{G}_{N-1}(\tau - s)h(s) \text{ where}$$

$$\tilde{G}_{N-1}(x) = \underbrace{(K^{r_0} * K^{r_0} * \ldots * K^{r_0})}_{N}(x)^{\frac{1}{r_0}} \text{ for } 2 \le N \in \mathbb{Z}^+.$$

Proof.

$$K_2(\tau, s) = \int_s^\tau K(\tau - \sigma)h(\sigma)K(\sigma - s)h(s)d\sigma =$$
$$= h(s)\int_s^\tau K(\tau - \sigma)h(\sigma)K(\sigma - s)d\sigma = G_1(\tau, s)h(s)$$

so the first part of the proposition holds for $N = 2$. Now assume that the first part of the proposition holds for $N = p$. Then

$$K_{p+1}(\tau, s) = \int_s^\tau K_1(\tau, \sigma)K_p(\sigma, s)d\sigma =$$
$$= h(s)\int_s^\tau K(\tau - \sigma)h(\sigma)G_{p-1}(\sigma, s)d\sigma = G_p(\tau, s)h(s)$$

where $G_p((\tau, s) = \int_s^\tau K_1(\tau, \sigma)G_{p-1}(\sigma, s)d\sigma$ and so the first part of the proposition holds by induction for $\tau > s \ge 0$. By Hölder's inequality it also follows that, for $0 < \tau - 1 \le s \le \tau$,

$$G_1(\tau, s) = \int_s^\tau K(\tau - \sigma)h(\sigma)K(\sigma - s)d\sigma \le$$
$$\le \Big(\int_s^\tau h(\sigma)^{r_0'}d\sigma\Big)^{\frac{1}{r_0'}}\Big(\int_s^\tau K(\tau - \sigma)^{r_0}K(\sigma - s)^{r_0}d\sigma\Big)^{\frac{1}{r_0}} =$$
$$= \|h\|_{L_{r_0'}(s,\tau)}\Big(\int_0^{\tau - s} K(\tau - s - x)^{r_0}K(x)^{r_0}dx\Big)^{\frac{1}{r_0}} = \|h\|_{L_{r_0'}(s,\tau)}\tilde{G}_1(\tau - s)$$

where $\tilde{G}_1(\tau - s) = (K^{r_0} * K^{r_0})(\tau - s)^{\frac{1}{r_0}}$ so the second part of the proposition holds for $N = 2$. Now assume that the second part of the proposition holds for $N = p$. Then

$$G_{p+1}(\tau, s) = \int_s^\tau K_1(\tau, \sigma)G_p(\sigma, s)d\sigma \le$$
$$\le \int_s^\tau K_1(\tau, \sigma)\|h\|^p_{L_{r_0'}(s,\sigma)}\tilde{G}_p(\sigma - s)d\sigma \le$$
$$\le \|h\|^p_{L_{r_0'}(s,\tau)}\Big(\int_s^\tau h(\sigma)^{r_0'}d\sigma\Big)^{\frac{1}{r_0'}}\Big(\int_s^\tau K(\tau - \sigma)^{r_0}\tilde{G}_p(\sigma - s)^{r_0}d\sigma\Big)^{\frac{1}{r_0}} =$$
$$= \|h\|^{p+1}_{L_{r_0'}(s,\tau)}\Big(\int_0^{\tau - s} K(\tau - s - x)^{r_0}\tilde{G}_p(x)^{r_0}dx\Big)^{\frac{1}{r_0}} =$$
$$= \|h\|^{p+1}_{L_{r_0'}(s,\tau)}(K^{r_0} * \tilde{G}_p)(\tau - s)^{\frac{1}{r_0}} = \|h\|^{p+1}_{L_{r_0'}(s,\tau)}\tilde{G}_{p+1}(\tau - s)$$

where $\tilde{G}_{p+1}(x) = \underbrace{(K^{r_0} * K^{r_0} * \ldots * K^{r_0}}_{p+2}(x)^{\frac{1}{r_0}}$ and so the second part of the proposition also holds by induction. ///

Remark. If we define $\tilde{G}_0(x) = K(x)$, Lemma 2.2 also holds for $N = 1$.

Lemma 2.3. *Let $\tau \geq 1$. Then*

$$\Big(\int_{\tau-1}^{\tau} K_N(\tau,s)^{r(N)} ds\Big)^{\frac{1}{r(N)}} \leq \|K\|^N_{L_{r_0+\varepsilon}(0,1)} \|h\|^N_{L_{r_0'}(\tau-1,\tau)}$$

where $\varepsilon \geq 0$, $r(N) = \dfrac{1}{1 - \frac{N\epsilon}{r_0(r_0+\epsilon)}}$ and ϵ is so small that $r(N) > 0$, i.e. $0 \leq \epsilon \leq \frac{r_0^2}{N-r_0}$ for $N \geq r_0$.

Proof. By Lemma 2.2, Hölder's and Young's inequalities

$$\begin{aligned}
&(\int_{\tau-1}^{\tau} K_N(\tau,s)^{r(N)} ds)^{\frac{1}{r(N)}} \leq \\
&\leq \Big(\int_{\tau-1}^{\tau} h(s)^{r(N)} \|h\|^{(N-1)r(N)}_{L_{r_0'}(s,\tau)} \tilde{G}_{N-1}(\tau-s)^{r(N)} ds\Big)^{\frac{1}{r(N)}} \leq \\
&\leq \|h\|^{N-1}_{L_{r_0'}(\tau-1,\tau)} \Big(\int_{\tau-1}^{\tau} h(s)^{r_0'} ds\Big)^{\frac{1}{r_0'}} \Big(\int_{\tau-1}^{\tau} \tilde{G}_{N-1}(\tau-s)^{\gamma_{N-1}} ds\Big)^{\frac{1}{\gamma_{N-1}}} = \\
&= \|h\|^{N-1}_{L_{r_0'}(\tau-1,\tau)} \Big(\int_0^1 \tilde{G}_{N-1}(x)^{\gamma_{N-1}} dx\Big)^{\frac{1}{\gamma_{N-1}}} = \\
&= \|h\|^{N-1}_{L_{r_0'}(\tau-1,\tau)} \|K^{r_0} * \tilde{G}^{r_0}_{N-2}\|^{\frac{1}{r_0}}_{L_{\frac{\gamma_{N-1}}{r_0}}(0,1)} \leq \\
&\leq \|h\|^{N-1}_{L_{r_0'}(\tau-1,\tau)} \|K^{r_0}\|^{\frac{1}{r_0}}_{L_{\frac{r_0+\epsilon}{r_0}}(0,1)} \|\tilde{G}^{r_0}_{N-2}\|^{\frac{1}{r_0}}_{L_{\frac{\gamma_{N-2}}{r_0}}(0,1)} = \\
&= \|h\|^{N-1}_{L_{r_0'}(\tau-1,\tau)} \|K\|_{L_{r_0+\epsilon}(0,1)} \|\tilde{G}_{N-2}\|_{L_{\gamma_{N-2}}(0,1)} \leq \dots \leq \\
&= \|h\|^{N}_{L_{r_0'}(\tau-1,\tau)} \|K\|^{N-1}_{L_{r_0+\epsilon}(0,1)} \|\tilde{G}_0\|_{L_{r_0}(0,1)}
\end{aligned}$$

Here, by repeated use of Young's inequality

$$\frac{1}{r(N)} - \frac{1}{r_0'} = \frac{1}{\gamma_{N-1}} = \frac{N-1}{r_0+\varepsilon} - \frac{N-1}{r_0} + \frac{1}{\gamma_0}$$

by which $\gamma_0 = r_0 + \varepsilon$, and the proposition follows. ///

Corollary 2.2. *$\epsilon = 0$ gives that*

$$\int_{\tau-1}^{\tau} K_N(\tau,s) ds \leq \|K\|^N_{L_{r_0}(0,1)} \|h\|^N_{L_{r_0'}(\tau-1,\tau)}$$

Corollary 2.3. *Let $\epsilon_0 = \frac{r_0^2}{Nq-r_0}$. For N so large that $\varepsilon_0 \leq \varepsilon$, i.e $N \geq N_0 = \Big[\frac{r_0(r_0+\epsilon)}{\epsilon q}\Big] + 1$, we have that*

$$\Big(\int_{\tau-1}^{\tau} K_N(\tau,s)^{q'} ds\Big)^{\frac{1}{q'}} \leq \|K\|^N_{L_{r_0+\varepsilon_0}(0,1)} \|h\|^N_{L_{r_0'}(\tau-1,\tau)}$$

Lemma 2.4. *Let $K(t) \leq \tilde{C}_1 t^{-\beta}, t > 0, \beta > 0$, and let q_0 and r_0 be such that $\beta q_0 > 1$ and $\beta r_0 < 1$. Then, for $\tau > s \geq 0$,*

$$G_{N-1}(\tau,s) \leq \tilde{C}_N (\tau-s)^{-\beta} (\|h\|_{L_{q_0'}(s,\tau)} + \|h\|_{L_{r_0'}(s,\tau)})^{N-1}, N \in Z_+.$$

Proof. The proposition is trivially true for $N=1$, as

$$G_0(\tau,s)=K(\tau-s)\le \tilde{C}_1(\tau-s)^{-\beta},\tau>s,$$

by assumption. Now assume that the proposition holds for $N=p$. Then

$$G_p(\psi,s)=(\int_s^{\frac{\tau+s}{2}}+\int_{\frac{\tau+s}{2}}^{\tau})K_1(\tau,\sigma)G_{p-1}(\sigma,s)d\sigma\le$$
$$\le \tilde{C}_1\tilde{C}_p\int_s^{\frac{\tau+s}{2}}(\tau-\sigma)^{-\beta}h(\sigma)(\sigma-s)^{-\beta}(\|h\|_{L_{q_0'}(s,\sigma)}+\|h\|_{L_{r_0'}(s,\sigma)})^{p-1}d\sigma+$$
$$+\tilde{C}_1\tilde{C}_p\int_{\frac{\tau+s}{2}}^{\tau}(\tau-\sigma)^{-\beta}h(\sigma)(\sigma-s)^{-\beta}(\|h\|_{L_{q_0'}(s,\sigma)}+\|h\|_{L_{r_0'}(s,\sigma)})^{p-1}d\sigma\le$$
$$\le \tilde{C}_1\tilde{C}_p\Big(\frac{\tau-s}{2}\Big)^{-\beta}(\|h\|_{L_{q_0'}(s,\tau)}+\|h\|_{L_{r_0'}(s,\tau)})^{p-1}\cdot$$
$$\cdot(\int_s^{\frac{\tau+s}{2}}h(\sigma)(\sigma-s)^{-\beta}d\sigma+\int_{\frac{\tau+s}{2}}^{\tau}h(\sigma)(\tau-\sigma)^{-\beta}d\sigma)$$

If $\tau-s\ge 2$, we have that $s+1\le\frac{\tau+s}{2}\le\tau-1$ and by Hölder's inequality it follows that

$$(\int_s^{s+1}+\int_{s+1}^{\frac{\tau+s}{2}})h(\sigma)(\sigma-s)^{-\beta}d\sigma+(\int_{\frac{\tau+s}{2}}^{\tau-1}+\int_{\tau-1}^{\tau})h(\sigma)(\tau-\sigma)^{-\beta}d\sigma\le$$
$$\le(\int_s^{s+1}h(\sigma)^{r_0'}d\sigma)^{\frac{1}{r_0'}}(\int_0^1\sigma^{-\beta r_0}d\sigma)^{\frac{1}{r_0}}+(\int_{s+1}^{\frac{\tau+s}{2}}h(\sigma)^{q_0'}d\sigma)^{\frac{1}{q_0'}}(\int_1^{\frac{\tau-s}{2}}\sigma^{-\beta q_0}d\sigma)^{\frac{1}{q_0}}+$$
$$+(\int_{\frac{\tau+s}{2}}^{\tau-1}h(\sigma)^{q_0'}d\sigma)^{\frac{1}{q_0'}}(\int_1^{\frac{\tau-s}{2}}\sigma^{-\beta q_0}d\sigma)^{\frac{1}{q_0}}+(\int_{\tau-1}^{\tau}h(\sigma)^{r_0'}d\sigma)^{\frac{1}{r_0'}}(\int_0^1\sigma^{-\beta r_0}d\sigma)^{\frac{1}{r_0}}\le$$
$$\le 2(\beta q_0-1)^{-\frac{1}{q_0}}\|h\|_{L_{q_0'}(s,\tau)}+2(1-\beta r_0)^{-\frac{1}{r_0}}\|h\|_{L_{r_0'}(s,\tau)}\le$$
$$\le 2\max((\beta q_0-1)^{-\frac{1}{q_0}},(1-\beta r_0)^{-\frac{1}{r_0}})(\|h\|_{L_{q_0'}(s,\tau)}+\|h\|_{L_{r_0'}(s,\tau)})$$

Now consider the case $0<\tau-s<2$. We then have that $\tau-1<\frac{\tau+s}{2}<s+1$ and by Hölder's inequality it follows that

$$\int_s^{\frac{\tau+s}{2}}h(\sigma)(\sigma-s)^{-\beta}d\sigma+\int_{\frac{\tau+s}{2}}^{\tau}h(\sigma)(\tau-\sigma)^{-\beta}d\sigma\le$$
$$\le(\int_s^{\frac{\tau+s}{2}}h(\sigma)^{r_0'}d\sigma)^{\frac{1}{r_0'}}(\int_0^{\frac{\tau-s}{2}}\sigma^{-\beta r_0}d\sigma)^{\frac{1}{r_0}}+(\int_{\frac{\tau+s}{2}}^{\tau}h(\sigma)^{r_0'}d\sigma)^{\frac{1}{r_0'}}(\int_0^{\frac{\tau-s}{2}}\sigma^{-\beta r_0}d\sigma)^{\frac{1}{r_0}}\le$$
$$\le 2(1-\beta r_0)^{-\frac{1}{r_0}}\|h\|_{L_{r_0'}(s,\tau)}$$

and the lemma follows by induction. ///

Corollary 2.4. *By Lemma 2.4 and Hölder's inequality follows that, for* $\tau > 0$,

$$(\int_0^{\frac{\tau}{2}} K_N(\tau,s)^{q'}ds)^{\frac{1}{q'}} = (\int_0^{\frac{\tau}{2}} h(s)^{q'}G_{N-1}(\tau,s)^{q'}ds)^{\frac{1}{q'}} \leq$$
$$\leq \tilde{C}_N(\int_0^{\frac{\tau}{2}} h(s)^{q'}(\tau-s)^{-\beta q'}(\|h\|_{L_{q'_0}(s,\tau)} + \|h\|_{L_{r'_0}(s,\tau)})^{(N-1)q'}ds)^{\frac{1}{q'}} \leq$$
$$\leq \tilde{C}_N(\|h\|_{L_{q'_0}(0,\tau)} + \|h\|_{L_{r'_0}(0,\tau)})^{N-1}(\int_0^{\frac{\tau}{2}} h(s)^{q'_0}ds)^{\frac{1}{q'_0}}(\int_{\frac{\tau}{2}}^{\tau} s^{-\frac{\beta qq_0}{q-q_0}}ds)^{\frac{q-q_0}{qq_0}} \leq$$
$$\leq 2^{\frac{\beta q_0-1}{q_0}+\frac{1}{q}}\tilde{C}_N(\|h\|_{L_{q'_0}(0,\tau)} + \|h\|_{L_{r'_0}(0,\tau)})^N\tau^{-\frac{\beta q_0-1}{q_0}-\frac{1}{q}}$$

Corollary 2.5. *By Lemma 2.4, Hölder's inequality and Corollary 2.2 follows that, for* $\tau > 1$,

$$\int_0^{\tau} K_N(\tau,s)ds = \int_0^{\tau-1} h(s)G_{N-1}(\tau,s)ds + \int_{\tau-1}^{\tau} K_N(\tau,s)ds \leq$$
$$\leq \tilde{C}_N(\|h\|_{L_{q'_0}(0,\tau)} + \|h\|_{L_{r'_0}(0,\tau)})^{N-1}(\int_0^{\tau-1} h(s)^{q'_0}ds)^{\frac{1}{q'_0}}(\int_1^{\tau} s^{-\beta q_0}ds)^{\frac{1}{q_0}} +$$
$$+ \|K\|^N_{L_{r_0}(0,1)}\|h\|^N_{L_{r'_0}(\tau-1,\tau)} \leq$$
$$\leq ((\beta q_0-1)^{-\frac{1}{q_0}}\tilde{C}_N + \|K\|^N_{L_{r_0}(0,1)})(\|h\|_{L_{q'_0}(0,\tau)} + \|h\|_{L_{r'_0}(0,\tau)})^N$$

and, for $0 < \tau \leq 1$, *that*

$$\int_0^{\tau} K_N(\tau,s)ds = \int_0^{\tau} h(s)G_{N-1}(\tau,s)ds \leq$$
$$\leq \tilde{C}_N(\|h\|_{L_{q'_0}(0,\tau)} + \|h\|_{L_{r'_0}(0,\tau)})^{N-1}(\int_0^{\tau} h(s)^{r'_0}ds)^{\frac{1}{r'_0}}(\int_0^{\tau} s^{-\beta r_0}ds)^{\frac{1}{r_0}} \leq$$
$$\leq (1-\beta r_0)^{-\frac{1}{r_0}}\tilde{C}_N(\|h\|_{L_{q'_0}(0,\tau)} + \|h\|_{L_{r'_0}(0,\tau)})^N$$

so that

$$\int_0^{\tau} K_N(\tau,s)ds \leq M_N(\|h\|_{L_{q'_0}(0,\tau)} + \|h\|_{L_{r'_0}(0,\tau)})^N$$

where

$$M_N = \|K\|^N_{L_{r_0}(0,1)} + \tilde{C}_N max((1-\beta r_0)^{-\frac{1}{r_0}}, (\beta q_0-1)^{-\frac{1}{q_0}})$$

Lemma 2.5. *If* $\tau \geq a+1$ *where* $a \geq 0$, *then*

$$(\int_a^{\tau-1} K_N(\tau,s)^{q'}ds)^{\frac{1}{q'}} \leq \Big(\frac{\beta qq_0}{q-q_0} - 1\Big)^{-\frac{q-q_0}{qq_0}}\tilde{C}_N(\|h\|_{L_{q'_0}(a,\tau)} + \|h\|_{L_{r'_0}(a,\tau)})^N$$

Proof. By Lemma 2.4 and Hölder's inequality

$$
\begin{aligned}
&(\int_a^{\tau-1} K_N(\tau,s)^{q'}ds)^{\frac{1}{q'}} = (\int_a^{\tau-1} h(s)^{q'}G_{N-1}(\tau,s)^{q'}ds)^{\frac{1}{q'}} \leq \\
&\leq \tilde{C}_N(\|h\|_{L_{q_0'}(a,\tau)} + \|h\|_{L_{r_0'}(a,\tau)})^{N-1}(\int_a^{\tau-1} h(s)^{q'}(\tau-s)^{-\beta q'}ds)^{\frac{1}{q'}} \leq \\
&\leq \tilde{C}_N(\|h\|_{L_{q_0'}(a,\tau)} + \|h\|_{L_{r_0'}(a,\tau)})^{N-1}(\int_a^{\tau-1} h(s)^{q_0'}ds)^{\frac{1}{q_0'}}(\int_1^{\tau-a} s^{-\frac{\beta q q_0}{q-q_0}}ds)^{\frac{q-q_0}{qq_0}} \leq \\
&\leq \Big(\frac{\beta q q_0}{q-q_0}-1\Big)^{-\frac{q-q_0}{qq_0}}\tilde{C}_N(\|h\|_{L_{q_0'}(a,\tau)} + \|h\|_{L_{r_0'}(a,\tau)})^{N}
\end{aligned}
$$

as $\frac{\beta q q_0}{q-q_0} = \beta q_0 \frac{q}{q-q_0} > \frac{q}{q-q_0} > 1$ ///

Corollary 2.6. *If we use Lemma 2.5 for $a = 0$ together with Corollary 2.3 it follows that, for $\tau \geq 1$, and $N \geq N_0$*

$$
\begin{aligned}
&(\int_0^{\tau} K_N^{q'}(\tau,s)ds)^{\frac{1}{q'}} \leq (\int_0^{\tau-1} K_N(\tau,s)^{q'}ds)^{\frac{1}{q'}} + (\int_{\tau-1}^{\tau} K_N(\tau,s)^{q'}ds)^{\frac{1}{q'}} \leq \\
&\leq \Big(\Big(\frac{\beta q q_0}{q-q_0}-1\Big)^{-\frac{q-q_0}{qq_0}}\tilde{C}_N + \|K\|^N_{L_{r_0+\varepsilon_0}(0,1)}\Big)(\|h\|_{L_{r_0'}(0,\tau)} + \|h\|_{L_{q_0'}(0,\tau)})^N \leq \\
&\leq L_N(\|h\|_{L_{r_0'}(0,\infty)} + \|h\|_{L_{q_0'}(0,\infty)})^N
\end{aligned}
$$

Corollary 2.7. *If we use Lemma 2.5 for $a = \frac{\tau}{2}$ together with Corollary 2.3, it follows that, for $\tau \geq 2$ and $N \geq N_0$,*

$$
\begin{aligned}
&(\int_{\frac{\tau}{2}}^{\tau} K_N(\tau,s)^{q'}ds)^{\frac{1}{q'}} \leq \\
&\leq \Big(\Big(\frac{\beta q q_0}{q-q_0}-1\Big)^{-\frac{q-q_0}{qq_0}}\tilde{C}_N + \|K\|^N_{L_{r_0+\varepsilon_0}(0,1)}\Big)(\|h\|_{L_{r_0'}(\frac{\tau}{2},\tau)} + \|h\|_{L_{q_0'}(\frac{\tau}{2},\tau)})^N = \\
&= L_N(\|h\|_{L_{r_0'}(\frac{\tau}{2},\tau)} + \|h\|_{L_{q_0'}(\frac{\tau}{2},\tau)})^N
\end{aligned}
$$

2.3 Local boundedness

In this section we will show that the property $\|u_0\|_X \in L_q^{\text{loc}}$ is inherited by u.

Lemma 2.6. *Assume that $U_0 \in L_q^{\text{loc}}$ and that*

(1) $$U(t) \leq U_0(t) + \mathcal{K}(U^{1-\eta})(t), 0 \leq |\eta| < \eta_0 < 1 (\eta_0 \textit{ small}),$$

Then

(2) $$\sup_{0\leq t\leq t^*} \mathcal{K}^N U(t) \leq M < \infty, N \geq N_0.$$

Proof. By the inequality (1), for $\eta > 0$,

(3) $$\mathcal{K}^N U(t) \le \mathcal{K}^N U_0(t) + \mathcal{K}^{N+1}(U^{1-\eta})(t) = \mathcal{K}^N U_0(t) + \mathcal{K}(\mathcal{K}^N(U^{1-\eta}))(t)$$

where, by Hölder's inequality and Corollary 2.5, for $0 \le t \le t^*$, and $0 < \eta < \eta_0 < 1$

(4) $$\begin{aligned}\mathcal{K}(\mathcal{K}^N(U^{1-\eta}))(t) &= \int_0^t K_1(t,\tau) \int_0^\tau K_N(\tau,s)U(s)^{1-\eta} ds d\tau = \\ &= \int_0^t K_1(t,\tau) \int_0^\tau K_N(\tau,s)^\eta K_N(\tau,s)^{1-\eta} U(s)^{1-\eta} ds d\tau \le \\ &\le \int_0^t K_1(t,\tau) (\int_0^\tau K_N(\tau,s) ds)^\eta (\mathcal{K}^N U(\tau))^{1-\eta} d\tau \le \\ &\le M_N^\eta (\|h\|_{q_0'} + \|h\|_{r_0'})^{\eta N} \Big(\sup_{0 \le \tau \le t} \mathcal{K}^N U(\tau) \Big)^{1-\eta} \int_0^t K_1(t,\tau) d\tau \le \\ &\le M_1 M_N^\eta (\|h\|_{q_0'} + \|h\|_{r_0'})^{1+\eta N} \Big(\sup_{0 \le t \le t^*} \mathcal{K}^N U(t) \Big)^{1-\eta}\end{aligned}$$

so that, by (3) and (4), for $0 \le t \le t^*$

(5) $$\sup_{0 \le t \le t^*} \mathcal{K}^N U(t) \le \sup_{0 \le t \le t^*} \mathcal{K}^N U_0(t) + M_1 M_N^\eta (\|h\|_{q_0'} + \|h\|_{r_0'})^{1+\eta N} \Big(\sup_{0 \le t \le t^*} \mathcal{K}^N U(t) \Big)^{1-\eta}$$

Assume that $N \ge N_0$. By Corollary 2.6 it follows that

(6) $$\sup_{0 \le t \le t^*} \mathcal{K}^N U_0(t) \le L_N (\|h\|_{q_0'} + \|h\|_{r_0'})^N \|U_0\|_{L_q[0,t^*]}$$

Let $S = \sup_{0 \le t \le t^*} \mathcal{K}^N U(t)$. As $U_0 \in L_q^{\text{loc}}$ we have by (5) and (6) that

(7) $$S \le C_1 + C_2 S^{1-\eta}$$

Now assume that $S \ge (2C_2)^{\frac{1}{\eta}}$. Then, by (7), $S \le 2C_1$, and it follows that

(8) $$S \le \max(2L_N(\|h\|_{q_0'} + \|h\|_{r_0'})^N \|U_0\|_{L_q[0,t^*]}, 2^{\frac{1}{\eta}} M_1^{\frac{1}{\eta}} M_N (\|h\|_{q_0'} + \|h\|_{r_0'})^{\frac{1}{\eta}+N}) = M < \infty. \quad ///$$

Corollary 2.8. *As $L_\infty^{\text{loc}} \subseteq L_q^{\text{loc}}$, it follows that $\mathcal{K}^N U \in L_q[0,t^*]$, for $N \ge N_0$.*

Theorem 2.1. *Assume that $\|u_0\|_X \in L_q[0,t^*]$ and that*

(1) $$\|u(t)\|_X \le \|u_0(t)\|_X + (\mathcal{K}\|u(t)\|_X^{1-\eta})(t), 0 \le |\eta| < \eta_0 < 1.$$

Then $\|u\|_X \in L_q[0,t^*]$.

Proof. By Corollary 2.8 we have that $\mathcal{K}^N \|u\|_X \in L_q[0,t^*]$, for $N \ge N_0$. Now assume that $\mathcal{K}^p \|u\|_X \in L_q[0,t^*]$, $p \in Z_+, p \le N$. Using the inequality (1), for $\eta = 0$, we obtain that

(2) $$\mathcal{K}^{p-1} \|u\|_X \le \mathcal{K}^{p-1} \|u_0\|_X + \mathcal{K}^p \|u\|_X.$$

Here $\mathcal{K}^{p-1} \|u_0\|_X \in L_q[0,t^*]$ by Corollary 2.1, and it follows that $\mathcal{K}^{p-1} \|u\|_X \in L_q[0,t^*]$. By induction it follows that $\mathcal{K}\|u\|_X \in L_q[0,t^*]$, and as, by (1),

$$\|u\|_X \le \|u_0\|_X + \mathcal{K}\|u\|_X$$

we have that $\|u\|_X \in L_q[0,t^*]$. ///

2.4 The main theorem

Let $M_{q,X,j}g(t) = \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}(\mathcal{K}^j\|g\|_X(\tau))^q d\tau\Big)^{\frac{1}{q}}$ for $t > 0, j \in N$. We will show that the property $M_{q,X}u_0(t) = M_{q,X,0}u_0(t) \leq w(t)$ is inherited by u under certain assumptions on $\mathcal{K}$ and w.

Theorem 2.2. *Assume that* $\|u_0(t)\|_X \in L_q$ *and that*

$$(A_1) \qquad \|u(t)\|_X \leq \|u_0(t)\|_X + (\mathcal{K}\|u\|_X^{1-\eta})(t), 0 \leq |\eta| < 1,$$

where $\mathcal{K}f(t) = \int_0^t K(t-\tau)h(\tau)f(\tau)d\tau$ *with*

$$0 \leq K \in L_{q_0}(1,\infty) \cap L_{q_0+\epsilon}(1,\infty) \text{ some } \epsilon > 0, 0 \leq h \in L_{q_0'}(0,\infty), \frac{1}{q_0} + \frac{1}{q_0'} = 1 \text{ and}$$

$$K \in L_{r_0}(0,1) \cap L_{r_0+\varepsilon}(0,1) \text{ some } \varepsilon > 0, h \in L_{r_0'}(0,\infty), \frac{1}{r_0} + \frac{1}{r_0'} = 1$$

Assume also that K *and* h *have the properties*

$$(A_2) \qquad K(t) \leq \tilde{C}_1 t^{-\beta}, t > 0, \beta r_0 < 1 < \beta q_0$$

$$(A_3) \qquad \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t} h(\tau)^{r_0'} d\tau\Big)^{\frac{1}{r_0'}} \leq C_0(1+t)^{-\frac{1}{r_0'}-\gamma}, t > 0, \gamma > 0, r_0' \geq q_0'$$

Then if

$$(M_0) \qquad M_{q,X}u_0(t) = M_{q,X,0}u_0(t) \leq w(t)$$

where $w(t)$ *is a function such that*

$$(1_w) \qquad \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t} w(\tau)^q d\tau\Big)^{\frac{1}{q}} \leq Aw(t)$$

$$(2_w) \qquad \frac{1}{E} \leq w(t)^{-1} \leq Bt^{\beta+\frac{1}{q}-\frac{1}{q_0}} \text{ for } t \geq \frac{t^*}{4}$$

$$(3_w) \qquad w\Big(\frac{t}{2}\Big) \leq Fw(t)$$

where A, B, E *and* F *are positive constants, and* t^* *will be specified in the theorem, it follows that*

$$M_{q,X}u(t) \leq Cw(t) \text{ for } t \geq t^*.$$

Proof. Repeated use of the integral inequality (A_1) for $\eta = 0$ yields

(1) $$\|u(\tau)\|_X \leq \sum_{j=0}^{N-1} \mathcal{K}^j \|u_0\|_X(\tau) + \mathcal{K}^N \|u\|_X(\tau)$$

From (1) follows, for $q \geq 1$, by convexity that

(2) $$\|u(\tau)\|_X^q \leq \sum_{j=0}^{N-1} 2^{(j+2)(q-1)} (\mathcal{K}^j \|u_0\|_X(\tau))^q + 2^{q-1} (\mathcal{K}^N \|u\|_X(\tau))^q$$

by which, for $I \subseteq R^+$,

(3) $$\Big(\int_I \|u(\tau)\|_X^q d\tau \Big)^{\frac{1}{q}} \leq$$
$$\leq \sum_{j=0}^{N-1} 2^{(j+2)(1-\frac{1}{q})} \Big(\int_I (\mathcal{K}^j \|u_0\|_X(\tau))^q d\tau \Big)^{\frac{1}{q}} + 2^{1-\frac{1}{q}} \Big(\int_I (\mathcal{K}^N \|u\|_X(\tau))^q d\tau \Big)^{\frac{1}{q}}$$

Taking $I = (\frac{t}{2}, t)$ in (3) we obtain that, for $t > 0$

(4) $$M_{q,X,0} u(t) \leq \sum_{j=0}^{N-1} 2^{(j+2)(1-\frac{1}{q})} M_{q,X,j} u_0(t) + 2^{1-\frac{1}{q}} M_{q,X,N} u(t)$$

Our first goal will be to estimate $M_{q,X,N}u(t)$. By convexity and Hölder's inequality

(5) $$M_{q,X,N}u(t) =$$
$$= \Big(\frac{2}{t} \int_{\frac{t}{2}}^t \Big(\Big(\int_0^{\frac{\tau}{2}} + \int_{\frac{\tau}{2}}^{\tau} \Big) K_N(\tau, s) \|u(s)\|_X ds \Big)^q d\tau \Big)^{\frac{1}{q}} \leq$$
$$\leq 2^{1-\frac{1}{q}} \Big(\frac{2}{t} \int_{\frac{t}{2}}^t \Big(\int_0^{\frac{\tau}{2}} K_N(\tau, s) \|u(s)\|_X ds \Big)^q d\tau \Big)^{\frac{1}{q}} +$$
$$+ 2^{1-\frac{1}{q}} \Big(\frac{2}{t} \int_{\frac{t}{2}}^t \Big(\int_{\frac{\tau}{2}}^{\tau} K_N(\tau, s) \|u(s)\|_X ds \Big)^q d\tau \Big)^{\frac{1}{q}} \leq$$
$$\leq 2^{1-\frac{1}{q}} \Big(\frac{2}{t} \int_{\frac{t}{2}}^t \Big(\int_0^{\frac{\tau}{2}} K_N(\tau, s)^{q'} ds \Big)^{\frac{q}{q'}} \Big(\int_0^{\frac{\tau}{2}} \|u(s)\|_X^q ds \Big) d\tau \Big)^{\frac{1}{q}} +$$
$$+ 2^{1-\frac{1}{q}} \Big(\frac{2}{t} \int_{\frac{t}{2}}^t \Big(\int_{\frac{\tau}{2}}^{\tau} K_N(\tau, s)^{q'} ds \Big)^{\frac{q}{q'}} \Big(\int_{\frac{\tau}{2}}^{\tau} \|u(s)\|_X^q ds \Big) d\tau \Big)^{\frac{1}{q}}$$

Here, by Corollary 2.4,

(6) $$\Big(\int_0^{\frac{\tau}{2}} K_N(\tau, s)^{q'} ds \Big)^{\frac{1}{q'}} \leq 2^{\frac{\beta q_0 - 1}{q_0} + \frac{1}{q}} \tilde{C}_N (\|h\|_{q_0'} + \|h\|_{r_0'})^N \tau^{-\frac{\beta q_0 - 1}{q_0} - \frac{1}{q}}$$

and, by Corollary 2.7 and A_3 for $\tau \geq 2$ and $N \geq N_0$,

(7) $$\Big(\int_{\frac{\tau}{2}}^{\tau} K_N(\tau, s)^{q'} ds \Big)^{\frac{1}{q'}} \leq$$
$$\leq L_N (\|h\|_{L_{q_0'}(\frac{\tau}{2}, \tau)} + \|h\|_{L_{r_0'}(\frac{\tau}{2}, \tau)})^N \leq$$
$$\leq (2^{-\frac{1}{q_0'}} + 2^{-\frac{1}{r_0'}})^N L_N C_0^N (1 + \tau)^{-N\gamma}$$

Now, let $m_{q,X,j}g(t) = \Big(\frac{2}{t}\int_0^{\frac{t}{2}} (\mathcal{K}^j \|g\|_X(\tau))^q d\tau\Big)^{\frac{1}{q}}$ for $t > 0, j \in N$. Then, by (5) - (7), 1_w and 2_w for $t \geq \frac{t^*}{4} \geq 4, N > max(N_0, \frac{1}{\gamma q})$

$$
\begin{aligned}
(8)\ \ & w(t)^{-1} M_{q,X,N}u(t) \leq \\
& \leq 2^{\beta-\frac{1}{q_0}+1-\frac{1}{q}} (\|h\|_{q_0'} + \|h\|_{r_0'})^N \tilde{C}_N \Big(\sup_{\frac{t}{2}\leq\tau\leq t} (\tau^{\frac{1}{q}} m_{q,X,0}u(\tau) \Big) w(t)^{-1} \Big(\frac{2}{t}\int_{\frac{t}{2}}^t \tau^{-\frac{\beta q_0 - 1}{q_0}q-1} d\tau\Big)^{\frac{1}{q}} + \\
& + 2^{1-\frac{2}{q}} (2^{-\frac{1}{q_0'}} + 2^{-\frac{1}{r_0'}})^N L_N C_0^N \Big(\sup_{\frac{t}{2}\leq\tau\leq t} w(\tau)^{-1} M_{q,X,0}u(\tau) \Big) w(t)^{-1} \Big(\frac{2}{t}\int_{\frac{t}{2}}^t (1+\tau)^{1-N\gamma q} w(\tau)^q d\tau\Big)^{\frac{1}{q}} \leq \\
& \leq 2^{2\beta-\frac{2}{q_0}+1} \Big(\frac{q_0}{(\beta q_0 - 1)q}\Big)^{\frac{1}{q}} (\|h\|_{q_0'} + \|h\|_{r_0'})^N \tilde{C}_N \Big(\sup_{\frac{t}{2}\leq\tau\leq t} (\tau^{\frac{1}{q}} m_{q,X,0}u(\tau) \Big) w(t)^{-1} t^{-\beta-\frac{1}{q}+\frac{1}{q_0}} + \\
& + 2^{1-\frac{2}{q}} (2^{-\frac{1}{q_0'}} + 2^{-\frac{1}{r_0'}})^N L_N C_0^N (1+\frac{t}{2})^{\frac{1}{q}-N\gamma} \Big(\sup_{\frac{t}{2}\leq\tau\leq t} (w(\tau)^{-1} M_{q,X,0}u(\tau) \Big) w(t)^{-1} \Big(\frac{2}{t}\int_{\frac{t}{2}}^t w(\tau)^q d\tau\Big)^{\frac{1}{q}} \leq \\
& \leq 2^{2\beta-\frac{2}{q_0}+1} \Big(\frac{q_0}{(\beta q_0 - 1)q}\Big)^{\frac{1}{q}} (\|h\|_{q_0'} + \|h\|_{r_0'})^N \tilde{C}_N B \Big(\sup_{\frac{t}{2}\leq\tau\leq t} (\tau^{\frac{1}{q}} m_{q,X,0}u(\tau)) \Big) + \\
& + 2^{1-\frac{2}{q}} (2^{-\frac{1}{q_0'}} + 2^{-\frac{1}{r_0'}})^N L_N C_0^N A (1+\frac{t}{2})^{\frac{1}{q}-N\gamma} \Big(\sup_{\frac{t}{2}\leq\tau\leq t} (w(\tau)^{-1} M_{q,X,0}u(\tau)) \Big) = \\
& = BC_1 \Big(\sup_{\frac{t}{2}\leq\tau\leq t} \Big(\tau^{\frac{1}{q}} m_{q,X,0}u(\tau)\Big)\Big) + AC_2 \Big(1+\frac{t}{2}\Big)^{\frac{1}{q}-N\gamma} \Big(\sup_{\frac{t}{2}\leq\tau\leq t} (w(\tau)^{-1} M_{q,X,0}u(\tau))\Big)
\end{aligned}
$$

where $C_1 = 2^{1+2\beta-\frac{2}{q_0}} \Big(\frac{q_0}{(\beta q_0 - 1)q}\Big)^{\frac{1}{q}} \tilde{C}_N (\|h\|_{q_0'} + \|h\|_{r_0'})^N$ and $C_2 = 2^{1-\frac{2}{q}} (2^{-\frac{1}{q_0'}} + 2^{-\frac{1}{r_0'}})^N L_N C_0^N$. As we now want to estimate $\tau^{\frac{1}{q}} m_{q,X,0}u(\tau)$ for $\frac{t}{2} \leq \tau \leq t$, we start by taking $I = (0, \frac{t}{2})$ in (3) and obtain for $t > 0$ that

$$
(9) \qquad m_{q,X,0}u(t) \leq \sum_{j=0}^{N-1} 2^{(j+2)(1-\frac{1}{q})} m_{q,X,j}u_0(t) + 2^{1-\frac{1}{q}} m_{q,X,N}u(t)
$$

Here, by Corollary 2.1, for $t > 0$

$$
(10) \qquad m_{q,X,j}u_0(t) \leq 2^{(2-\frac{1}{q})j} (\|K\|_{L_{r_0}(0,1)} \|h\|_{r_0'} + \|K\|_{L_{q_0}(1,\infty)} \|h\|_{q_0'})^j m_{q,X,0}u_0(t)
$$

so that, by (9) and (10),

$$
(11) \qquad m_{q,X,0}u(t) \leq S_N m_{q,X,0}u_0(t) + 2^{1-\frac{1}{q}} m_{q,X,N}u(t)
$$

where $S_N = \sum_{j=0}^{N-1} 2^{(3-\frac{2}{q})j+2-\frac{2}{q}}(\|K\|_{L_{r_0}(0,1)}\|h\|_{r_0'} + \|K\|_{L_{q_0}(1,\infty)}\|h\|_{q_0'})^j$. Next step will be to estimate $m_{q,X,N}u(\tau)$ for $8 \le \frac{t^*}{2} \le \frac{t}{2} \le \tau \le t$. By (6), (7), Lemma 2.6 and 2_w for $N > N_1 = max(N_0 - 1, [\frac{2}{\gamma q}])$

$$\tau^{\frac{1}{q}} m_{q,X,N}u(\tau) \le 2^{\frac{1}{q}}\Big(\int_0^2 (\mathcal{K}^N\|u\|_X(s))^q ds\Big)^{\frac{1}{q}} + \tag{12}$$

$$+\Big(2\Big(\int_2^{\frac{t^*}{2}} + \int_{\frac{t^*}{4}}^{\frac{\tau}{2}}\Big)\Big(\Big(\int_0^{\frac{s}{2}} + \int_{\frac{s}{2}}^{s}\Big)K_N(s,\sigma)\|u(\sigma)\|_X d\sigma\Big)^q ds\Big)^{\frac{1}{q}} \le$$

$$\le 2^{\frac{2}{q}} \sup_{0\le s\le 2} (\mathcal{K}^N\|u(s)\|_X) +$$

$$+2^{1+\beta-\frac{1}{q_0}+\frac{1}{q}}\tilde{C}_N(\|h\|_{q_0'} + \|h\|_{r_0'})^N\Big(\int_2^{\frac{t^*}{4}} s^{-\frac{\beta q_0-1}{q_0}q-1}\Big(\int_0^{\frac{s}{2}} \|u(\sigma)\|_X^q d\sigma\Big)ds\Big)^{\frac{1}{q}} +$$

$$+2^{1+\beta-\frac{1}{q_0}}\tilde{C}_N(\|h\|_{q_0'} + \|h\|_{r_0'})^N\Big(\int_{\frac{t^*}{4}}^{\frac{\tau}{2}} s^{-\frac{\beta q_0-1}{q_0}q-1} s m_{q,X,0}u(s)^q ds\Big)^{\frac{1}{q}} +$$

$$+2C_0^N(2^{-\frac{1}{q_0'}} + 2^{-\frac{1}{r_0'}})^N L_N\Big(\int_2^{\frac{t^*}{4}} (1+s)^{-N\gamma q}\Big(\int_{\frac{s}{2}}^{s} \|u(\sigma)\|_X^q d\sigma\Big)ds\Big)^{\frac{1}{q}} +$$

$$+2^{1-\frac{1}{q}}C_0^N(2^{-\frac{1}{q_0'}} + 2^{-\frac{1}{r_0'}})^N L_N\Big(\int_{\frac{t^*}{4}}^{\frac{\tau}{2}} (1+s)^{1-N\gamma q} w(s)^q w(s)^{-q} M_{q,X,0}u(s)^q ds\Big)^{\frac{1}{q}} \le$$

$$\le 2^{\frac{2}{q}}M + 2^{1+\beta-\frac{1}{q_0}+\frac{1}{q}}\tilde{C}_N(\|h\|_{q_0'} + \|h\|_{r_0'})^N\Big(\frac{q_0}{(\beta q_0 - 1)q}\Big)^{\frac{1}{q}}\Big(\int_0^{\frac{t^*}{8}} \|u(\sigma)\|_X^q d\sigma\Big)^{\frac{1}{q}} +$$

$$+2C_0^N(2^{-\frac{1}{q_0'}} + 2^{-\frac{1}{r_0'}})^N L_N \frac{1}{(N\gamma q - 1)^{\frac{1}{q}}}\Big(\int_1^{\frac{t^*}{4}} \|u(\sigma)\|_X^q d\sigma\Big)^{\frac{1}{q}} +$$

$$+2^{1+\beta-\frac{1}{q_0}}\tilde{C}_N(\|h\|_{q_0'} + \|h\|_{r_0'})^N\Big(\frac{q_0}{(\beta q_0 - 1)q}\Big)^{\frac{1}{q}}\Big(\frac{t^*}{4}\Big)^{-\frac{\beta q_0-1}{q_0}}\Big(\sup_{\frac{t^*}{4}\le s\le\frac{\tau}{2}} (s^{\frac{1}{q}} m_{q,X,0}u(s))\Big) +$$

$$+2^{1-\frac{1}{q}}C_0^N(2^{-\frac{1}{q_0'}} + 2^{-\frac{1}{r_0'}})^N L_N \frac{E}{(N\gamma q - 2)^{\frac{1}{q}}}\Big(\frac{t^*}{4}\Big)^{\frac{2}{q}-N\gamma}\Big(\sup_{\frac{t^*}{4}\le s\le\frac{\tau}{2}} (w(s)^{-1}M_{q,X,0}u(s))\Big)$$

Thus, by (11) and (12), for $8 \le \frac{t^*}{2} \le \frac{t}{2} \le \tau \le t$ and $N > N_1$

$$\tau^{\frac{1}{q}} m_{q,X,0}u(\tau) \le S_N \tau^{\frac{1}{q}} m_{q,X,0}u_0(\tau) + 2^{1-\frac{1}{q}}\tau^{\frac{1}{q}} m_{q,X,N}u(\tau) \le \tag{13}$$

$$\le 2^{\frac{1}{q}} S_N\Big(\int_0^{\frac{\tau}{2}} \|u_0(s)\|_X^q ds\Big)^{\frac{1}{q}} + 2^{1+\frac{1}{q}}M +$$

$$+2\Big(2^{-\beta+\frac{1}{q_0}}C_1 + \Big(\frac{2}{N\gamma q - 1}\Big)^{\frac{1}{q}}C_2\Big)\Big(\int_0^{\frac{t^*}{4}} \|u(\sigma)\|_X^q d\sigma\Big)^{\frac{1}{q}} +$$

$$+2^{1-\frac{1}{q}}C_1\Big(\frac{t^*}{2}\Big)^{-\frac{\beta q_0-1}{q_0}}\Big(\sup_{\frac{t^*}{4}\le s\le\frac{t^*}{2}} (s^{\frac{1}{q}} m_{q,X,0}u(s)) + \sup_{\frac{t^*}{2}\le s\le\frac{t}{2}} (s^{\frac{1}{q}} m_{q,X,0}u(s))\Big) +$$

$$+2\Big(\frac{1}{N\gamma q - 2}\Big)^{\frac{1}{q}} EC_2\Big(\frac{t^*}{4}\Big)^{\frac{2}{q}-N\gamma}\Big(\sup_{\frac{t^*}{4}\le s\le\frac{t}{2}} (w(s)^{-1}M_{q,X,0}u(s))\Big)$$

and if we take $t^* \geq 2(2^{2-\frac{1}{q}}C_1)^{\frac{q_0}{\beta q_0 - 1}}$ in (13), it follows that, for $N > N_1$,

$$
\begin{aligned}
(14) \qquad & \tau^{\frac{1}{q}} m_{q,X,0} u(\tau) \leq \\
& \leq 2^{\frac{1}{q}} S_N \Big(\int_0^{\frac{\tau}{2}} \|u_0(s)\|_X^q ds \Big)^{\frac{1}{q}} + 2^{1+\frac{1}{q}} M + \\
& + 2\Big(2^{-\beta+\frac{1}{q_0}} C_1 + \Big(\frac{2}{N\gamma q - 1}\Big)^{\frac{1}{q}} C_2\Big)\Big(\int_0^{\frac{t^*}{4}} \|u(\sigma)\|_X^q d\sigma\Big)^{\frac{1}{q}} + \\
& + 2^{\frac{1}{q}-1}\Big(\int_0^{\frac{t^*}{4}} \|u(s)\|_X^q ds\Big)^{\frac{1}{q}} + 2^{-1} \sup_{\frac{t^*}{2} \leq s \leq t} (s^{\frac{1}{q}} m_{q,X,0} u(s)) + \\
& + 2\Big(\frac{1}{N\gamma q - 2}\Big)^{\frac{1}{q}} E C_2 \Big(\frac{t^*}{4}\Big)^{\frac{2}{q} - N\gamma} \Big(\sup_{\frac{t^*}{4} \leq s \leq \frac{t}{2}} (w(s)^{-1} M_{q,X,0} u(s)) \Big)
\end{aligned}
$$

by which

$$
\begin{aligned}
(15) \qquad & \sup_{\frac{t^*}{2} \leq \tau \leq t} \Big(\tau^{\frac{1}{q}} m_{q,X,0} u(\tau)\Big) \leq \\
& \leq 2^{1+\frac{1}{q}} S_N \Big(\int_0^{\frac{t}{2}} \|u_0(s)\|_X^q ds \Big)^{\frac{1}{q}} + 2^{2+\frac{1}{q}} M + \\
& + \Big(4\Big(2^{-\beta+\frac{1}{q_0}} C_1 + \Big(\frac{2}{N\gamma q - 1}\Big)^{\frac{1}{q}} C_2\Big) + 2^{\frac{1}{q}}\Big)\Big(\int_0^{\frac{t^*}{4}} \|u(s)\|_X^q ds\Big)^{\frac{1}{q}} + \\
& + 4\Big(\frac{1}{N\gamma q - 2}\Big)^{\frac{1}{q}} E C_2 \Big(\frac{t^*}{4}\Big)^{\frac{2}{q} - N\gamma} \Big(\sup_{\frac{t^*}{4} \leq s \leq \frac{t}{2}} (w(s)^{-1} M_{q,X,0} u(s)) \Big)
\end{aligned}
$$

so that, by (8) and (15), for $t \geq t^*$ and $N > N_1$

$$
\begin{aligned}
(16) \qquad & w(t)^{-1} M_{q,X,N} u(t) \leq \\
& \leq B C_1 \Big(\sup_{\frac{t^*}{2} \leq \tau \leq t} (\tau^{\frac{1}{q}} m_{q,X,0} u(\tau)) \Big) + \\
& + A C_2 \Big(1 + \frac{t^*}{2}\Big)^{\frac{1}{q} - N\gamma} \Big(\sup_{\frac{t}{2} \leq \tau \leq t} (w(\tau)^{-1} M_{q,X,0} u(\tau)) \Big) \leq \\
& \leq 2^{1+\frac{1}{q}} B C_1 S_N \Big(\int_0^{\frac{t}{2}} \|u_0(s)\|_X^q ds \Big)^{\frac{1}{q}} + 2^{2+\frac{1}{q}} B C_1 M + \\
& + B C_1 \Big(4\Big(2^{-\beta+\frac{1}{q_0}} C_1 + \Big(\frac{2}{N\gamma q - 1}\Big)^{\frac{1}{q}} C_2\Big) + 2^{\frac{1}{q}}\Big)\Big(\int_0^{\frac{t^*}{4}} \|u(s)\|_X^q ds\Big)^{\frac{1}{q}} + \\
& + 4\Big(\frac{1}{N\gamma q - 2}\Big)^{\frac{1}{q}} B E C_1 C_2 (\frac{t^*}{4})^{\frac{2}{q} - N\gamma} \Big(\sup_{\frac{t^*}{4} \leq s \leq \frac{t}{2}} (w(s)^{-1} M_{q,X,0} u(s)) \Big) + \\
& + A C_2 (\frac{t^*}{2})^{\frac{1}{q} - N\gamma} \Big(\sup_{\frac{t}{2} \leq \tau \leq t} (w(\tau)^{-1} M_{q,X,0} u(\tau)) \Big) =
\end{aligned}
$$

$$= 2^{2+\frac{1}{q}}BC_1M + C_3\Big(\int_0^{\frac{t}{2}}\|u_0(s)\|_X^q ds\Big)^{\frac{1}{q}} + C_4\Big(\int_0^{\frac{t^*}{4}}\|u(s)\|_X^q ds\Big)^{\frac{1}{q}} +$$
$$+C_5(\frac{t^*}{4})^{\frac{2}{q}-N\gamma}\Big(\sup_{\frac{t^*}{4}\le s\le\frac{t}{2}}(w(s)^{-1}M_{q,X,0}u(s))\Big) +$$
$$+C_6(\frac{t^*}{2})^{\frac{1}{q}-N\gamma}\Big(\sup_{\frac{t}{2}\le\tau\le t}(w(\tau)^{-1}M_{q,X,0}u(\tau))\Big)$$

We have now reached our first goal that was to estimate $M_{q,X,N}u(t)$ (see (4) page 20). It remains to estimate $M_{q,X,j}u_0(t)$. For $j = 1,2,\dots,N-1$, by convexity,Lemma 2.2, Lemma 2.4 and Hölder's inequality for $t \ge \frac{t^*}{4} \ge 4$

(17) $$w(t)^{-1}M_{q,X,j}u_0(t) =$$
$$= w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\Big(\int_0^{\frac{\tau}{2}} + \int_{\frac{\tau}{2}}^{\tau-1} + \int_{\tau-1}^{\tau}\Big)K_j(\tau,s)\|u_0(s)\|_X ds\Big)^q d\tau\Big)^{\frac{1}{q}} \le$$
$$\le 2^{1-\frac{1}{q}}\tilde{C}_j w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_0^{\frac{\tau}{2}}(\tau-s)^{-\beta}(\|h\|_{L_{q_0'}(s,\tau)} + \|h\|_{L_{r_0'}(s,\tau)})^{j-1}h(s)\|u_0(s)\|_X ds)^q d\tau\Big)^{\frac{1}{q}} +$$
$$+2^{2-\frac{2}{q}}\tilde{C}_j w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}(\int_{\frac{\tau}{2}}^{\tau-1}(\tau-s)^{-\beta}(\|h\|_{L_{q_0'}(s,\tau)} + \|h\|_{L_{r_0'}(s,\tau)})^{j-1}h(s)\|u_0(s)\|_X ds\Big)^q d\tau\Big)^{\frac{1}{q}} +$$
$$+2^{2-\frac{2}{q}}\tilde{C}_j w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{\tau-1}^{\tau}(\tau-s)^{-\beta}(\|h\|_{L_{q_0'}(s,\tau)} + \|h\|_{L_{r_0'}(s,\tau)})^{j-1}h(s)\|u_0(s)\|_X ds\Big)^q d\tau\Big)^{\frac{1}{q}} \le$$

$$\le 2^{1-\frac{1}{q}}\tilde{C}_j(\|h\|_{L_{q_0'}(0,t)} + \|h\|_{L_{r_0'}(0,t)})^{j-1}w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_0^{\frac{\tau}{2}}(\tau-s)^{-\beta}h(s)\|u_0(s)\|_X ds\Big)^q d\tau\Big)^{\frac{1}{q}} +$$
$$+2^{2-\frac{2}{q}}\tilde{C}_j(\|h\|_{L_{q_0'}(\frac{t}{4},t)} + \|h\|_{L_{r_0'}(\frac{t}{4},t)})^{j}w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{\frac{\tau}{2}}^{\tau-1}(\tau-s)^{-\beta q_0}\|u_0(s)\|_X^{q_0}ds\Big)^{\frac{q}{q_0}}d\tau\Big)^{\frac{1}{q}} +$$
$$+2^{2-\frac{2}{q}}\tilde{C}_j(\|h\|_{L_{r_0'}(\frac{t}{2}-1,t)} + \|h\|_{L_{r_0'}(\frac{t}{2}-1,t)})^{j}w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{\tau-1}^{\tau}(\tau-s)^{-\beta r_0}\|u_0(s)\|_X^{r_0}ds\Big)^{\frac{q}{r_0}}d\tau\Big)^{\frac{1}{q}}$$

Here, by Hölder's inequality and 2_w for $t \ge \frac{t^*}{4} \ge 4$

(18) $$w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_0^{\frac{\tau}{2}}(\tau-s)^{-\beta}h(s)\|u_0(s)\|_X ds\Big)^q d\tau\Big)^{\frac{1}{q}} \le$$
$$\le w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{\frac{\tau}{2}}^{\tau}s^{-\frac{\beta q q_0}{q-q_0}}ds\Big)^{\frac{q-q_0}{q_0}}\Big(\int_0^{\frac{\tau}{2}}h(s)^{q_0'}ds\Big)^{\frac{q}{q_0'}}\Big(\int_0^{\frac{\tau}{2}}\|u_0(s)\|_X^q ds\Big)d\tau\Big)^{\frac{1}{q}} \le$$
$$\le 2^{\beta-\frac{1}{q_0}+\frac{1}{q}}\|h\|_{L_{q_0'}(0,\frac{t}{2})}\Big(\int_0^{\frac{t}{2}}\|u_0(s)\|_X^q\Big)^{\frac{1}{q}}w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}(\tau^{-\beta-\frac{1}{q}+\frac{1}{q_0}})^q d\tau\Big)^{\frac{1}{q}} \le$$
$$\le 4^{\beta-\frac{1}{q_0}+\frac{1}{q}}B\|h\|_{L_{q_0'}(0,\frac{t}{2})}\Big(\int_0^{\frac{t}{2}}\|u_0(s)\|_X^q ds\Big)^{\frac{1}{q}}$$

and again by Hölder's inequality,

$$(19)\qquad \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{\frac{\tau}{2}}^{\tau-1}(\tau-s)^{-\beta q_0}\|u_0(s)\|_X^{q_0}ds\Big)^{\frac{q}{q_0}}d\tau\Big)^{\frac{1}{q}}\le$$

$$\le\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{\frac{\tau}{2}}^{\tau-1}(\tau-s)^{-\beta\frac{qq_0-q_0^2}{q}}(\tau-s)^{-\beta\frac{q_0^2}{q}}\|u_0(s)\|_X^{q_0}ds)^{\frac{q}{q_0}}d\tau\Big)^{\frac{1}{q}}\le$$

$$\le\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{1}^{\frac{\tau}{2}}s^{-\beta q_0}ds\Big)^{\frac{q-q_0}{q_0}}\Big(\int_{\frac{\tau}{2}}^{\tau-1}(\tau-s)^{-\beta q_0}\|u_0(s)\|_X^{q}ds\Big)d\tau\Big)^{\frac{1}{q}}\le$$

$$\le(\beta q_0-1)^{-\frac{q-q_0}{qq_0}}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\int_{\frac{\tau}{2}}^{\tau-1}(\tau-s)^{-\beta q_0}\|u_0(s)\|_X^{q}dsd\tau\Big)^{\frac{1}{q}}$$

and

$$(20)\qquad \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{\tau-1}^{\tau}(\tau-s)^{-\beta r_0}\|u_0(s)\|_X^{r_0}ds\Big)^{\frac{q}{r_0}}d\tau\Big)^{\frac{1}{q}}\le$$

$$\le\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{\tau-1}^{\tau}(\tau-s)^{-\beta\frac{qr_0-r_0^2}{q}}(\tau-s)^{-\beta\frac{r_0^2}{q}}\|u_0(s)\|_X^{r_0}ds\Big)^{\frac{q}{r_0}}d\tau\Big)^{\frac{1}{q}}\le$$

$$\le\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{0}^{1}s^{-\beta r_0}ds\Big)^{\frac{q-r_0}{r_0}}\Big(\int_{\tau-1}^{\tau}(\tau-s)^{-\beta r_0}\|u_0(s)\|_X^{q}ds\Big)d\tau\Big)^{\frac{1}{q}}\le$$

$$\le(1-\beta r_0)^{-\frac{q-r_0}{qr_0}}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\int_{\tau-1}^{\tau}(\tau-s)^{-\beta r_0}\|u_0(s)\|_X^{q}dsd\tau\Big)^{\frac{1}{q}}$$

Fubini's theorem now gives that, for $t\ge 4$

$$(21)\qquad \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\int_{\frac{\tau}{2}}^{\tau-1}(\tau-s)^{-\beta q_0}\|u_0(s)\|_X^{q}dsd\tau\Big)^{\frac{1}{q}}\le$$

$$\le\Big(\frac{2}{t}\Big(\int_{\frac{t}{4}}^{\frac{t}{2}-1}\|u_0(s)\|_X^q\int_{\frac{t}{2}-s}^{s}\sigma^{-\beta q_0}d\sigma ds+\int_{\frac{t}{2}-1}^{\frac{t}{2}}\|u_0(s)\|_X^q\int_{1}^{s}\sigma^{-\beta q_0}d\sigma ds+$$

$$+\int_{\frac{t}{2}}^{t-1}\|u_0(s)\|_X^q\int_{1}^{t-s}\sigma^{-\beta q_0}d\sigma ds\Big)\Big)^{\frac{1}{q}}\le$$

$$\le(\beta q_0-1)^{-\frac{1}{q}}\cdot\Big(2^{-\frac{1}{q}}\Big(\frac{4}{t}\int_{\frac{t}{4}}^{\frac{t}{2}}\|u_0(s)\|_X^q ds\Big)^{\frac{1}{q}}+\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\|u_0(s)\|_X^q ds\Big)^{\frac{1}{q}}\Big)$$

and that

$$(22)\qquad \Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\int_{\tau-1}^{\tau}(\tau-s)^{-\beta r_0}\|u_0(s)\|_X^{q}dsd\tau\Big)^{\frac{1}{q}}\le$$

$$\le(1-\beta r_0)^{-\frac{1}{q}}\cdot\Big(2^{-\frac{1}{q}}\Big(\frac{4}{t}\int_{\frac{t}{4}}^{\frac{t}{2}}\|u_0(s)\|_X^q ds\Big)^{\frac{1}{q}}+\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\|u_0(s)\|_X^q ds\Big)^{\frac{1}{q}}\Big)$$

and, by (19) - (22) and 3_w follows that

$$(23)\qquad w(t)^{-1}\Big(\frac{2}{t}\int_{\frac{t}{2}}^{t}\Big(\int_{\frac{\tau}{2}}^{\tau}K_j(\tau,s)\|u_0(s)\|_X ds\Big)^{q}d\tau\Big)^{\frac{1}{q}}\le$$

$$\le 2^{1-\frac{1}{q}}\tilde{C}_j((\beta q_0-1)^{-\frac{1}{q_0}}+(1-\beta r_0)^{-\frac{1}{r_0}})(\|h\|_{q_0'}+\|h\|_{r_0'})^j(2^{-\frac{1}{q}}F+1)$$

so that, by (17), (18) and (23), for $j = 1, 2, \ldots, N-1$

$$(24)\qquad \begin{aligned} & w(t)^{-1} M_{q,X,j} u_0(t) \leq \\ & \leq 2^{2+2\beta-\frac{2}{q_0}} B\tilde{C}_j (\|h\|_{q_0'} + \|h\|_{r_0'})^j \Big(\int_0^{\frac{t}{2}} \|u_0(s)\|_X^q ds \Big)^{\frac{1}{q}} + \\ & + 2^{2-\frac{2}{q}} \tilde{C}_j ((\beta q_0 - 1)^{-\frac{1}{q_0}} + (1-\beta r_0)^{-\frac{1}{r_0}})(\|h\|_{q_0'} + \|h\|_{r_0'})^j (2^{-\frac{1}{q}} F + 1) \end{aligned}$$

By (4), (16) and (24) now follows that, for $t \geq t^*$ and $N > N_1$

$$(25)\qquad \begin{aligned} & w(t)^{-1} M_{q,X,0} u(t) \leq \\ & \leq 2^{2-\frac{2}{q}} w(t)^{-1} M_{q,X,0} u_0(t) + \sum_{j=1}^{N-1} 2^{(j+2)(1-\frac{1}{q})} w(t)^{-1} M_{q,X,j} u_0(t) + \\ & + 2^{1-\frac{1}{q}} w(t)^{-1} M_{q,X,N} u(t) \leq \\ & \leq 2^{2-\frac{2}{q}} + \Big(\sum_{j=1}^{N-1} 2^{(j+2)(1-\frac{1}{q})+2+2\beta-\frac{2}{q_0}} (\|h\|_{q_0'} + \|h\|_{r_0'})^j B\tilde{C}_j \Big) \Big(\int_0^{\frac{t}{2}} \|u_0(s)\|_X^q ds \Big)^{\frac{1}{q}} + \\ & + \sum_{j=1}^{N-1} 2^{(j+4)(1-\frac{1}{q})} (2^{-\frac{1}{q}} F + 1)((\beta q_0 - 1)^{-\frac{1}{q_0}} + (1-\beta r_0)^{-\frac{1}{r_0}}) \tilde{C}_j (\|h\|_{q_0'} + \|h\|_{r_0'})^j + \\ & + 8BC_1 M + \\ & + 2^{1-\frac{1}{q}} C_3 \Big(\int_0^{\frac{t}{2}} \|u_0(s)\|_X^q ds \Big)^{\frac{1}{q}} + 2^{1-\frac{1}{q}} C_4 \Big(\int_0^{\frac{t^*}{4}} \|u(s)\|_X^q ds \Big)^{\frac{1}{q}} + \\ & + 2^{1-\frac{1}{q}} C_5 (\frac{t^*}{4})^{\frac{2}{q} - N\gamma} \Big(\sup_{\frac{t^*}{4} \leq s \leq t^*} (w(s)^{-1} M_{q,X,0} u(s)) + \sup_{s \geq t^*} (w(s)^{-1} M_{q,X,0} u(s)) \Big) + \\ & + 2^{1-\frac{1}{q}} C_6 (\frac{t^*}{2})^{\frac{1}{q} - N\gamma} \Big(\sup_{\frac{t^*}{2} \leq \tau \leq t^*} (w(\tau)^{-1} M_{q,X,0} u(\tau)) + \sup_{\tau \geq t^*} (w(\tau)^{-1} M_{q,X,0} u(\tau)) \Big) \end{aligned}$$

Now take

$$(26)\qquad t^* \geq \max\Big(16, 2(2^{2-\frac{1}{q}} C_1)^{\frac{q_0}{\beta q_0 - 1}}, 4(2^{3-\frac{1}{q}} C_5)^{\frac{q}{N\gamma q - 2}}, 2(2^{3-\frac{1}{q}} C_6)^{\frac{q}{N\gamma q - 1}}\Big)$$

By (25), for $t \geq t^*$, we obtain that

$$(27)\qquad \begin{aligned} & \sup_{t \geq t^*} (w(t)^{-1} M_{q,X,0} u(t)) \leq \\ & \leq C_7 + C_8 \Big(\int_0^{\infty} \|u_0(s)\|_X^q ds \Big)^{\frac{1}{q}} + \\ & + 2^{2-\frac{1}{q}} C_4 \Big(\int_0^{\frac{t^*}{4}} \|u(s)\|_X^q ds \Big)^{\frac{1}{q}} + \sup_{\frac{t^*}{4} \leq \tau \leq t^*} (w(\tau)^{-1} M_{q,X,0} u(\tau)) \end{aligned}$$

Here, for $\frac{t^*}{4} \le t \le t^*$ and $N > N_1$, by (4)-(7), (24) and 2_w

(28)
$$\begin{aligned}
&w(t)^{-1}M_{q,X,0}u(t) \le \\
&\le \sum_{j=0}^{N-1} 2^{(j+2)(1-\frac{1}{q})} w(t)^{-1}M_{q,X,j}u_0(t) + 2^{1-\frac{1}{q}} w(t)^{-1}M_{q,X,N}u(t) \le \\
&\le 2^{2-\frac{2}{q}} + \sum_{j=1}^{N-1} 2^{(j+4)(1-\frac{1}{q})}(2^{-\frac{1}{q}}F+1)((\beta q_0-1)^{-\frac{1}{q_0}} + (1-\beta r_0)^{-\frac{1}{r_0}})\tilde{C}_j(\|h\|_{q_0'} + \|h\|_{r_0'})^j + \\
&+\Big(\sum_{j=0}^{N-1} 2^{(j+2)(1-\frac{1}{q})+2+2\beta-\frac{2}{q_0}} B\tilde{C}_j(\|h\|_{q_0'} + \|h\|_{r_0'})^j\Big)\Big(\int_0^{\frac{t^*}{2}} \|u_0(s)\|_X^q ds\Big)^{\frac{1}{q}} + \\
&+2^{2^{\frac{\beta q_0-1}{q_0}}+2}\Big(\frac{q_0}{(\beta q_0-1)q}\Big)^{\frac{1}{q}} B\tilde{C}_N(\|h\|_{q_0'} + \|h\|_{r_0'})^N\Big(\int_0^{\frac{t^*}{2}} \|u(s)\|_X^q ds\Big)^{\frac{1}{q}} + \\
&+2^{3N\gamma+2-\frac{2}{q}}(2^{-\frac{1}{q_0'}} + 2^{-\frac{1}{r_0'}})^N L_N C_0^N B(t^*)^{\frac{\beta q_0-1}{q_0}+\frac{1}{q}-N\gamma}\Big(\int_{\frac{t^*}{16}}^{t^*} \|u(s)\|_X^q ds\Big)^{\frac{1}{q}} \le \\
&\le C_9 + C_{10}\Big(\int_0^{\frac{t^*}{2}} \|u_0(s)\|_X^q ds\Big)^{\frac{1}{q}} + C_{11}\Big(\int_0^{t^*} \|u(s)\|_X^q ds\Big)^{\frac{1}{q}}
\end{aligned}$$

where C_{11} is independent of t^* if $N > \frac{\beta q_0-1}{\gamma q_0} + \frac{1}{\gamma q}$. Finally, by (27) and (28), it follows that

(29)
$$\begin{aligned}
&\sup_{t\ge t^*}(w(t)^{-1}M_{q,X,0}u(t)) \le \\
&\le C_{12} + C_{13}\Big(\int_0^{\infty} \|u_0(s)\|_X^q ds\Big)^{\frac{1}{q}} + C_{14}\Big(\int_0^{t^*} \|u(s)\|_X^q ds\Big)^{\frac{1}{q}}
\end{aligned}$$

Now recall from Theorem 2.1 that $\|u_0\|_X \in L_q[0,t^*] \Rightarrow \|u\|_X \in L_q[0,t^*]$ so that, as $\|u_0\|_X \in L_q[0,\infty)$, it follows by (29) that

$$\sup_{t\ge t^*}(w(t)^{-1}M_{q,X,0}u(t)) \le C.$$

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