THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Permutation patterns, continued fractions, and a group determined by an ordered set

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Göteborg, Sweden 2004

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ISBN 91-7291-404-1

Doktorsavhandlingar vid Chalmers tekniska högskola
Ny serie nr 2086
ISSN 0346-718x

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Matematiskt Centrum
Göteborg, Sweden January 2004

Abstract

We present six articles:

In the first and second article we give the first few results on generalized pattern avoidance, focusing on patterns of type $(1, 2)$ or $(2, 1)$. There are twelve such patterns, and they fall into three classes with respect to being equidistributed. We use 1-23, 1-32, and 2-13 as representatives for these classes. We prove that

$$|\mathcal{S}_n(1\text{-}23)| = |\mathcal{S}_n(1\text{-}32)| = B_n \quad \text{and} \quad |\mathcal{S}_n(2\text{-}13)| = C_n,$$

where B_n is the nth Bell number and C_n is the nth Catalan number. A complete solution for the number of permutations avoiding any pair of patterns of type $(1, 2)$ or $(2, 1)$ is also given.

In the third article we present an ordinary generating function for the number of permutations containing one occurrence of 1-23 (or 1-32). We also give the distribution of 2-13 in the form of a continued fraction, and explicit formulas for the number of permutations containing r occurrences of 2-13 when $r = 1$, 2, or 3.

In the fourth article the notion of a σ-segmented permutation is introduced: A permutation π is *σ-segmented* if every occurrence of σ in π is a contiguous subword in π. A bicoloured Dyck path is a Dyck path in which each up-step is assigned one of two colours, say, red and green. We show that 132-segmented permutations of length n with k occurrences of 132 are in one-to-one correspondence with bicoloured Dyck paths of length $2n - 4k$ with k red up-steps. Similarly, we show that 123-segmented permutations of length n with k occurrences of 123 are in one-to-one correspondence with bicoloured Dyck paths of length $2n - 4k$ with k red up-steps, each of height less than 2. We enumerate the permutations above by enumerating the corresponding bicoloured Dyck paths.

Continued fractions and patterns are the two main topics of the fifth article. Let $e_k(\pi)$ be the number of increasing subsequences of length $k+1$ in π. We prove that any Stieltjes continued fractions with monic monomial numerators is the generating function of a family of statistics on the 132-avoiding permutations, each consisting of a (possibly infinite) linear combination of the e_ks. Moreover, there is an invertible linear transformation that translates between linear combinations of e_ks and the corresponding continued fractions.

In the sixth article we study a permutation group determined by an ordered set. Let P be a finite ordered set, and let $J(P)$ be the distributive lattice of order ideals of P. The covering relations of $J(P)$ are naturally associated with elements of P; in this way, each element of P defines an involution on the set $J(P)$. Let $\Gamma(P)$ be the permutation group generated by these involutions. We show that if P is connected then $\Gamma(P)$ is either the alternating or the symmetric group.

Keywords: permutation pattern, generalized pattern, pattern avoidance, occurrence of a pattern, segmented permutation, continued fraction, ordered set, permutation group

AMS 2000 Subject classification: 05A05, 05A10, 05A15, 05A18, 06A07, 06A11, 20B99.

This thesis consists of an introduction and the following six articles:

[**C1**] A. Claesson,
Generalized Pattern Avoidance, *European Journal of Combinatorics* 22 (2001), 961–971.

[**CM1**] A. Claesson and T. Mansour,
Enumerating Permutations Avoiding a Pair of Babson-Steingrímsson Patterns, Accepted for publication in *Ars Combinatoria*.

[**CM2**] A. Claesson and T. Mansour,
Counting Occurrences of a Pattern of Type (1,2) or (2,1) in Permutations, *Advances in Applied Mathematics* 29 (2002), 293–310.

[**C2**] A. Claesson,
Bicoloured Dyck paths, segmented permutations, and Chebyshev polynomials

[**BCS**] P. Brändén, A. Claesson, and E. Steingrímsson,
Catalan Continued Fractions and Increasing Subsequences in Permutations, *Discrete Mathematics* 258 (2002), 275–287.

[**CGW**] A. Claesson, C. D. Godsil, and D. G. Wagner,
A Permutation Group Determined by an Ordered Set, *Discrete Mathematics* 269 (2003), 273–279.

The articles in this thesis are, however, not identical to the published articles above: I have made minor changes, mainly concerning notation, in order to make the thesis more homogeneous.

Acknowledgments

First of all, I thank my advisor, Einar Steingrímsson, for his inviting attitude towards mathematics, and for his friendship, generosity, and support; without him this thesis would not have seen the light of day.

There are many more who deserve my credit. I will mention only some of those directly related to my mathematical education and the making of this thesis:

Petter Brändén, Chris Godsil, Toufik Mansour, Einar Steingrímsson, and David Wagner for working together with me on one or more articles, and for sharing with me some of their vast knowledge of mathematics.

Mireille Bousquet-Mélou and David Wagner kindly invited me to Bordeaux and Waterloo, respectively. They were both very hospitable and taught me lots of mathematics. In this context, I would also like to thank Chalmersska forskningsfonden who gave me the financial support that made these visits possible.

Leif Tobiasson for teaching me some rigorous mathematics in upper secondary school.

My friends and fellow graduate students, Roger Bengtsson, Niklas Broberg, Martin Brundin, Petter Brändén, Fredrik Engström, Ola Helenius, Per Hörfelt, Sergey Kitaev, Håkan Samuelsson, Henrik Seppänen, and Marcus Sundhäll. With them I have had lots of fun and countless cups of coffee.

PERMUTATION PATTERNS, CONTINUED FRACTIONS, AND A GROUP DETERMINED BY AN ORDERED SET

ANDERS CLAESSON

Introduction

Let $V = \{v_1, v_2, \dots, v_n\}$ with $v_1 < v_2 < \dots < v_n$ be any finite subset of $\mathbb{N}$. A *permutation* π of V is a bijective map from V onto itself. We shall often view a permutation as a word over V. That is, if $\pi(v_i) = a_i$, we write

$$\pi = a_1 a_2 \cdots a_n.$$

Since π is injective, this word has no repeated letters. Conversely, any word with no repeated letters may be viewed as a permutation of its letters. As an example, the word 273 defines the permutation

$$\{\, 2 \mapsto 2,\ 3 \mapsto 7,\ 7 \mapsto 3 \,\}.$$

The set of all permutations of V is denoted $\mathcal{S}_V$; this is a group under composition of maps and is called *the symmetric group* on V. A commonly used n element set is $[n] = \{1, 2, \dots, n\}$ and for the set of permutations of $[n]$ we use the abbreviated notation $\mathcal{S}_n$; in addition, we define $\mathcal{S}$ as the disjoint union of the $\mathcal{S}_n$ for $n \geq 0$.

The *reduction* of a permutation π of V is the permutation $\operatorname{red}(\pi)$ of $[n]$ obtained from π by replacing the letter v_i with the letter i, for each i. As an example, $\operatorname{red}(19452) = 15342$. From a functional perspective, if ω is the unique order-preserving bijection $v_i \mapsto i$ from V to $[n]$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[d, "\omega"'] & V \arrow[d, "\omega"] \\ {[n]} \arrow[r, "\operatorname{red}(\pi)"] & {[n]} \end{tikzcd}$$

Let k be a nonnegative integer not larger than n. Given π in $\mathcal{S}_n$ and σ in $\mathcal{S}_k$, an *occurrence* of σ in π is a subword

$$o = \pi(i_1)\pi(i_2)\cdots\pi(i_k)$$

of π such that $\operatorname{red}(o) = \sigma$; in this context σ is called a *pattern*. If there is no occurrence of σ in π then we say that π *avoids* σ, or that π is σ-avoiding. The set of all σ-avoiding permutations of V is denoted $\mathcal{S}_V(\sigma)$.

An interesting and much studied problem is to enumerate the permutations of $[n]$ that avoid a fixed pattern σ; that is, to determine the sequence

$$n \mapsto |\mathcal{S}_n(\sigma)|.$$

The starting-point of this research seems to be Knuth's [20] enumeration of $\mathcal{S}_n(213)$ in 1969. The next step was the enumeration of $\mathcal{S}_n(123)$ by Hammersley [18] in 1972; this result was rediscovered by Knuth [21] in 1973, by Rotem [32] in 1975, and by

Rogers [31] in 1978. Somewhat surprisingly, the cardinality of $\mathcal{S}_n(213)$ and the cardinality of $\mathcal{S}_n(123)$ turned out to be the same:

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(213)| = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

This number is called the nth *Catalan number* and is usually denoted C_n. By a symmetry argument, explained below, this settles the problem for single patterns of length 3.

Let π be a permutation of $[n]$. Define the *reverse* of π, denoted π^r, by

$$\pi^r(i) = \pi(n+1-i), \qquad i \in [n],$$

and define the *complement* of π, denoted π^c, by

$$\pi^c(i) = n+1-\pi(i), \qquad i \in [n].$$

As a rule, we define operations on permutations as operations on $\mathcal{S}_n$. This will, however, not prevent us from using these operations on permutations of an arbitrary finite subset V of $\mathbb{N}$. What is then understood is that the definition of the operation, say, f, has been extended by requiring that the diagram

$$\begin{tikzcd} \mathcal{S}_V \arrow[r, "f"] \arrow[d, "\mathrm{red}"'] & \mathcal{S}_V \arrow[d, "\mathrm{red}"] \\ \mathcal{S}_n \arrow[r, "f"] & \mathcal{S}_n \end{tikzcd}$$

commute. For example, $(24918)^c = 84192$ and $(24918)^r = 81942$ (simply read the word backwards).

The operations reverse and complement together with $(\,\cdot\,)^{-1}$, the group theoretical inverse in $\mathcal{S}_V$, are all involutions on $\mathcal{S}_n$. These three operations generate the dihedral group D_4, the symmetry group of a square, acting on permutations; the orbits under this action are called *symmetry classes*. It is easy to see that if Φ is a member of D_4 then there are as many occurrences of σ in π as there are occurrences of $\Phi(\sigma)$ in $\Phi(\pi)$. In particular, $|\mathcal{S}_n(\sigma_1)| = |\mathcal{S}_n(\sigma_2)|$ whenever σ_1 and σ_2 are members of the same symmetry class. In general, if $|\mathcal{S}_n(\sigma_1)| = |\mathcal{S}_n(\sigma_2)|$ for all $n \geq 0$ then σ_1 and σ_2 are said to be *Wilf equivalent*; the equivalence classes under this relation are called *Wilf classes*. As we have seen, permutations belonging to the same symmetry class are Wilf equivalent. The converse is false in general. As an example, $\mathcal{S}_3$ is divided into two symmetry classes, namely

$$\{123, 321\} \quad \text{and} \quad \{132, 213, 231, 312\}.$$

But, by (1), they are subsets of the same Wilf class, $\mathcal{S}_3$. One basic question in the theory of pattern avoiding permutations is to classify all permutations up to Wilf equivalence. In the year 2002, Stankova and West [37] succeeded in this classification for all permutations of length up to 7; their results are shown in the table below.

n	1	2	3	4	5	6	7
symmetry classes in $\mathcal{S}_n$	1	1	2	7	23	115	694
Wilf classes in $\mathcal{S}_n$	1	1	1	3	17	91	595

As representatives for the three Wilf classes in $\mathcal{S}_4$ we take 1234, 1342, and 1324. Using the theory of symmetric functions Gessel [17] proved that

$$\sum_{n\geq 0} |\mathcal{S}_n(1\,2\cdots k)|\frac{t^{2n}}{(n!)^2} = \det\Big(I_{|i-j|}(2t)\Big)_{1\leq i,j\leq k},$$

where

$$I_i(2t) = \sum_{n\geq 0} \frac{t^{2n+i}}{n!(n+i)!}$$

is a modified Bessel function. Using that as a starting point he derived that

$$|\mathcal{S}_n(1234)| = 2\sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}^2 \frac{3k^2+2k+1-n-2kn}{(k+1)^2(k+2)(n-k+1)}.$$

Bóna [5] showed that the generating function for 1342-avoiding permutations is given by

$$\sum_{n\geq 0} |\mathcal{S}_n(1342)|t^n = \frac{32t}{1+12t-8t^2-(1-8t)^{3/2}}.$$

Enumerating the permutations avoiding 1324 is an open problem.

The *Stanley-Wilf conjecture* states that, for any pattern $\sigma \in \mathcal{S}_k$, the limit

$$\lim_{n\to\infty} |\mathcal{S}_n(\sigma)|^{1/n}$$

exists and is finite. This conjecture appears to have been formulated around 1990 by Richard Stanley and Herbert Wilf, but we are unable to provide an exact reference. Adam Marcus and Gábor Tardos [26] have very recently announced that they have a proof of the Stanley-Wilf conjecture. Their proof is not yet published. However, a manuscript is available on Tardos's homepage.

A function $f : \mathbb{N} \to \mathbb{C}$ is *P-recursive* (short for polynomially recursive) if it satisfies a homogeneous linear recurrence of finite degree with coefficients in $\mathbb{C}[n]$. We say that a power series F in $\mathbb{C}[[x]]$ is *D-finite* (short for differentialbly finite) if the vectorspace over $\mathbb{C}(x)$ spanned by F and all its derivatives F', F'', ... is finite-dimensional. It can be shown that f is P-recursive if and only if its (ordinary or exponential) generating function is D-finite.

The *Noonan-Zeilberger conjecture* states that, for any patterns $\sigma_1, \sigma_2, \ldots, \sigma_k$ and any nonnegative integers $r_1, r_2, \ldots, r_k$, the seqence

$$n \mapsto \operatorname{card}\big\{\ \pi \in \mathcal{S}_n \,:\, \pi \text{ has exactly } m_i \text{ occurences of } \sigma_i, \text{ for } i = 1, 2, \ldots, k.\ \big\}$$

is P-recursive. To resolve this conjecture is an open problem.

We also wish to consider permutations which avoid several patterns simultaneously. To this end, we define that, for any set of patterns Σ,

$$\mathcal{S}_n(\Sigma) = \bigcap_{\sigma\in\Sigma} \mathcal{S}_n(\sigma).$$

Already in 1935 Erdös and Szekeres [27] showed that

$$|\mathcal{S}_n(1\,2\cdots k,\ \ell\cdots 2\,1)| = 0,$$

for all $n \geq (k-1)(\ell-1)+1$. In 1981 Rotem [33] showed that

$$|\mathcal{S}_n(231, 312)| = 2^{n-1}.$$

The first systematic study of pattern avoiding permutations was undertaken by Simion and Schmidt; in a celebrated paper [35], published in 1985, they gave a complete solution for permutations avoiding any set of patterns of length 3.

The initial motivation for studying pattern avoiding permutations came from its connections with container data types in computer science. Knuth [20] pioneered this work by showing that the stack sortable permutations are exactly the 231-avoiding permutations. Here are some more results related to sorting: The set of permutations sortable by a restricted input deque is $\mathcal{S}(4231, 3241)$; see [20, 28]. The set of permutations sortable by an unlimited number of pop-stacks in series is $\mathcal{S}(2413, 3142)$; see [3, 2]. The set of permutations sortable by two pop-stacks in parallel is

$$\mathcal{S}(3214, 2143, 24135, 41352, 14352, 13542, 13524);$$

see [2]. The set of permutations expressible as the interleaving of two increasing subsequences is $\mathcal{S}(321)$ (see [20]). The set of permutations expressible as the interleaving of an increasing subsequence and a decreasing subsequence is $\mathcal{S}(3412, 2143)$; see [36, 19]. The set of permutations obtainable by a riffle shuffle of a deck of cards is $\mathcal{S}(321, 2143, 2413)$; see [1].

Problems involving pattern avoiding permutations have also appeared in other areas of mathematics. The permutations whose Stanley symmetric function is a Schur function are called *vexillary*; it is known that these are exactly the 2143-avoiding permutations (see [24] for an exposition). In [34] the permutations avoiding both 3142 and 2413 are considered in the context of bootstrap percolation.

By a *statistic* on $\mathcal{S}_n$ we simply mean a function $f : \mathcal{S}_n \to \mathbb{N}$. Two statistics f and g are said to *equidistributed* over a set of permutations $A \subseteq \mathcal{S}_n$ if

$$\sum_{\pi\in A} x^{f(\pi)} = \sum_{\pi\in A} x^{g(\pi)}.$$

With MacMahon's extensive study [25], in 1915, permutation statistics became an established field of mathematics. MacMahon considered four different statistics for a permutation π: the number of descents, $\operatorname{des}\pi$; the number of excedances, $\operatorname{exc}\pi$; the number of inversions, $\textsc{inv}\,\pi$; and the major index, $\textsc{maj}\,\pi$. With $\pi = a_1a_2\cdots a_n$ these statistics are defined as follows: A *descent* is an i such that $a_i > a_{i+1}$, an *excedance* is an i such that $a_i > i$, an *inversion* is a pair (i, j) such that $i < j$ and $a_i > a_j$, and the *major index* of π is the sum of descents in π.

MacMahon showed, algebraically, that exc is equidistributed with des and that $\textsc{inv}$ is equidistributed with $\textsc{maj}$ over $\mathcal{S}_n$, for any $n \geq 0$. The polynomials $\{A_n(x)\}$ defined by

$$A_n(x) = \sum_{\pi\in\mathcal{S}_n} x^{1+\operatorname{des}\pi}$$

are called *Eulerian polynomials*, and they satisfy the identity

$$\sum_{k\geq 0} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}.$$

These polynomials appear in Euler's work [13]. Any statistic that is equidistributed with des is called *Eulerian*. On the other hand, any statistic that is equidistributed with $\textsc{inv}$ is called *Mahonian*. Some 160 years ago, Rodriguez [30] showed that

$$\sum_{\pi\in\mathcal{S}_n} q^{\textsc{inv}\,\pi} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Babson and Steingrímsson [4] showed that all "descent based" Mahonian permutation statistics in the literature can be written as finite linear combinations of *generalized patterns*; there where 8 such statistics prior to their paper. They also found 6 new Mahonian statistics; for 3 of them they gave proofs, and for the remaining 3 Foata and Zeilberger [16] gave proofs. What are these generalized patterns? The essential property they possess is that they allow the requirement that two adjacent letters in the pattern must be adjacent in the permutation. We shall now define what a generalized pattern is.

Denote by $[a,b]$ the interval $\{\, i \in \mathbb{Z} : a \leq i \leq b \,\}$ between a and b in $\mathbb{Z}$. A *generalized pattern*, or a *Babson-Steingrímsson pattern*, or (briefly) a *pattern*, is a pair (σ, S) where $\sigma \in \mathcal{S}_k$ and $S \subseteq [0,k]$. Given π in $\mathcal{S}_n$, an *occurrence* of (σ, S) in π is a subword

$$o = \pi(i_1)\pi(i_2)\cdots\pi(i_k)$$

of π such that $\operatorname{red}(o) = \sigma$, and $i_{r+1} = i_r + 1$ if $r \in [0,k] \setminus S$, where $i_0 = 0$ and $i_{k+1} = n+1$. The "classical" patterns defined in the beginning of this introduction thus correspond to generalized patterns of the form $(\sigma, [0,k])$ where $\sigma \in \mathcal{S}_k$. To each pattern (σ, S) we also associate a function from $\mathcal{S}$ to $\mathbb{N}$ which, by abuse of notation, we also denote by (σ, S). To be precise, we let $(\sigma, S)(\pi)$ be the number of occurrences of (σ, S) in π. For instance, the permutation statistic des is identical to $(12, \{0,2\})$, as a function, and INV is identical to $(12, \{0,1,2\})$. We say that π avoids (σ, S) if $(\sigma, S)(\pi) = 0$.

In practice our notation for generalized patterns can be a bit awkward. Therefore we opt for a different notation, which also is the notation used by Babson and Steingrímsson. The pattern (σ, S) is denoted by the word obtained from $\sigma = \sigma(1)\sigma(2)\cdots\sigma(k)$ by inserting a dash "-" between $\sigma(i)$ and $\sigma(i+1)$ whenever $i \in S$, and subsequently bracket this word according to the following rules: The opening bracket is "(" if $0 \in S$ and "[" otherwise. The closing bracket is ")" if $k \in S$ and "]" otherwise. Here are some examples:

– The pattern $(123, \{0,1,2,3\})$ is denoted (1-2-3), and

$$(1\text{-}2\text{-}3)\,\pi = \operatorname{card}\{\, a_i a_j a_k : i < j < k \text{ and } a_i < a_j < a_k \,\},$$

in which $\pi = a_1 a_2 \cdots a_n$, as usual. In other words, $(1\text{-}2\text{-}3)\,\pi$ counts the number of increasing subsequence of length 3 in π; this is a classical pattern. For instance, we have $(1\text{-}2\text{-}3)\,15234 = \operatorname{card}\{123, 124, 134, 234\} = 4$.

– The pattern $(123, \{0,1,3\})$ is denoted (1-23), and

$$(1\text{-}23)\,\pi = \operatorname{card}\{\, a_i a_j a_{j+1} : i < j \text{ and } a_i < a_j < a_{j+1} \,\}.$$

Thus $(1\text{-}23)\,15234 = \operatorname{card}\{123, 134, 234\} = 3$.

– The pattern $(123, \{0,3\})$ is denoted (123), and

$$(123)\,\pi = \operatorname{card}\{\, a_i a_{i+1} a_{i+2} : a_i < a_{i+1} < a_{i+2} \,\}.$$

Thus $(123)\,\pi$ counts the number of increasing segments of length 3 in π, and $(123)\,15234 = \operatorname{card}\{234\} = 1$.

– The pattern $(123, \emptyset)$ is denoted [123], and

$$[123]\,\pi = \begin{cases} 1 & \text{if } \pi = 123, \\ 0 & \text{if } \pi \neq 123. \end{cases}$$

Returning to the paper [4] by Babson and Steingrímsson, we find a slightly more complex example:

$$\textsc{maj} = (1\text{-}32) + (2\text{-}31) + (3\text{-}21) + (21).$$

This identity is easy to see. Recall that MAJ sums the positions of descents. The position of a descent is the number of letters preceding the descent, and that is exactly what the right hand side counts. For instance, (2-31) counts the letters a_i preceding the descent $a_j a_{j+1}$ such that $a_j < a_i < a_{j+1}$.

The definition of reverse and the definition of complement easily extend to generalized patterns: if $p = (\sigma, S)$ with $|\sigma| = k$ then

$$p^r = (\sigma^r, k - S) \quad \text{and} \quad p^c = (\sigma^c, S),$$

where $k - S = \{k - i : i \in S\}$. For instance, $(1\text{-}32)^c = 3\text{-}12$ and $(1\text{-}32)^r = 23\text{-}1$. The definition of inverse does not seem to extend to generalized patterns. Reverse and complement alone generate the dihedral group D_2, the symmetry group of a rectangle, acting on generalized patterns. The orbits under this action are, again, called *symmetry classes*. It is plain that if Φ is a member of D_2 then there are as many occurrences of σ in π as there are occurrences of $\Phi(\sigma)$ in $\Phi(\pi)$. The definition of *Wilf class* extends trivially to generalized patterns.

A pattern $\sigma = (\sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k)$ containing exactly $k-1$ dashes is said to be of type $(|\sigma_1|, |\sigma_2|, \dots, |\sigma_k|)$. As it stands, this definition only applies to patterns enclosed in parentheses; this is merely to simplify the presentation. The corresponding definitions for the variations involving square brackets are almost identical. For example, the pattern [142-5-367) is of type $[3, 1, 3)$, the pattern [1-2] is of type $[1, 1]$, and any classical pattern of length k is of type $(\underbrace{1, 1, \dots, 1}_{k})$.

In what follows, every generalized pattern that we consider will be of the kind that is enclosed in parentheses. Therefore we take the liberty of omitting the parentheses when it is convenient. So we may write 1-23 instead of (1-23).

Before we proceed with a discussion of the articles in this thesis we would like to say a few words about continued fractions and generating functions. Recall that the formal power series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

is said to be the (ordinary) generating function for the sequence $\{a_n\}_{n\geq 0}$. We have already met generating functions in this introduction, and throughout the thesis we will meet many more. Sometimes a generating function will be presented in the form of a continued fraction, and this might call for some explanation. Let us look at an example:

$$\sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{\ddots}}}} \tag{2}$$

Here the right hand side should be understood as the limit of the approximants of the continued fraction, that is, the limit of the following sequence of rational functions:

$$1\,, \quad \frac{1}{1-x}, \quad \frac{1}{1-\dfrac{x}{1-x}}, \quad \frac{1}{1-\dfrac{x}{1-\dfrac{x}{1-x}}}, \;\ldots$$

The limit is taken with respect to the norm $\|F\| = 2^{-\lambda(F)}$, where $\lambda(F)$ is the the degree of the first non-zero term of the power series F. With this norm the ring of formal power series $\mathbb{C}[[x]]$ is complete. For the validity of (2) we may then reason as follows: The left hand side, let us call it $C(x)$, is known to be the generating function that counts Dyck paths by semilength. (A Dyck path of length $2n$ (semilength n) is a lattice path in $\mathbb{N}^2$ with steps $(1,1)$ and $(1,-1)$ starting at $(0,0)$ and ending at $(2n,0)$.) On the other hand, the hth approximant of the right hand side, let us call it $C^{[h]}(x)$, is the generating function that counts Dyck paths, by semilength, which stay below the line $y = h$. To reach the line $y = h$ we need h steps of the $(1,1)$ kind. Therefore $\lambda(C - C^{[h]}) = h$, and hence $\|C - C^{[h]}\| \to 0$ as $h \to \infty$.

For a general and thorough introduction to continued fractions we refer the reader to [23]. All the continued fractions that appear in this thesis belong to a class of continued fractions studied by Flajolet [14].

Let us now discuss the content of each of the articles comprising this thesis.

C1. We give the first few results on generalized pattern avoidance, focusing on patterns of type $(1,2)$ or $(2,1)$. With respect to being equidistributed, these twelve patterns fall into the three classes:

$$\begin{gathered}\{\text{ 1-23, 3-21, 12-3, 32-1 }\},\\ \{\text{ 1-32, 3-12, 21-3, 23-1 }\},\\ \{\text{ 2-13, 2-31, 13-2, 31-2 }\}.\end{gathered}$$

Let us agree on using 1-23, 1-32, and 2-13 as representatives for these classes. Using two very similar bijections we prove that

$$|\mathcal{S}_n(\text{1-23})| = |\mathcal{S}_n(\text{1-32})| = B_n,$$

where B_n is the nth *Bell number* (the number of partitions of $[n]$). For the third class we find that $\mathcal{S}_n(\text{2-13}) = \mathcal{S}_n(\text{2-1-3})$, and thus

$$|\mathcal{S}_n(\text{2-13})| = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In particular, this shows that even though there are three symmetry classes, there are only two Wilf classes.

We refine these results by looking at the distribution of left-to-right minima and the distribution of descents. A *left-to-right minimum* of $\pi = a_1 a_2 \cdots a_n$ is an element a_i such that $a_i < a_j$ for every $j < i$. Let $L(\pi)$ denote the number of left-to-right minima in π. We show that

$$\sum_{\pi \in \mathcal{S}_n(\text{1-23})} x^{L(\pi)} = \sum_{\pi \in \mathcal{S}_n(\text{1-32})} x^{L(\pi)} = \sum_{\pi \in \mathcal{S}_n(\text{1-32})} x^{1+\operatorname{des}\pi} = \sum_{k \ge 0} S(n,k) x^k.$$

where $S(n,k)$ is the number of partitions of $[n]$ into k blocks; these numbers are called the *Stirling numbers of the second kind*. We also show that

$$\sum_{\pi\in\mathcal{S}_n(2\text{-}13)} x^{L(\pi)} = \sum_{k\geq 0} \frac{k}{2n-k}\binom{2n-k}{n} x^k.$$

The numbers appearing as coefficients in this polynomial are the well known *Ballot numbers*.

Some results on avoiding several patterns are also presented:

$$|\mathcal{S}_n(1\text{-}23,\ 12\text{-}3)| = B_n^*;$$
$$|\mathcal{S}_n(1\text{-}23,\ 1\text{-}32)| = I_n;$$
$$|\mathcal{S}_n(1\text{-}23,\ 13\text{-}2)| = M_n.$$

where B_n^* is the nth Bessel number (the number of non-overlapping partitions of $[n]$ (see [15])), I_n is the number of involutions in $\mathcal{S}_n$, and M_n is the nth Motzkin number (the number of ways of drawing any number of non-intersecting chords among n points on a circle).

In the course of proving that $|\mathcal{S}_n(1\text{-}23,\ 12\text{-}3)| = B_n^*$, we first define a new class of set partitions—the monotone partitions. A partition is *monotone* if its non-singleton blocks can be written in increasing order of their least element and increasing order of their greatest element, simultaneously. We prove that there is a one-to-one correspondence between $\{1\text{-}23, 12\text{-}3\}$-avoiding permutations and monotone partitions and these are subsequently shown to be in one-to-one correspondence with non-overlapping partitions.

CM1. In this paper we follow up on the work in Article C1. A complete solution for the number of permutations avoiding a pair of patterns of type $(1,2)$ or $(2,1)$ is given. We also conjecture the number of permutations avoiding the patterns in any set of three or more such patterns. For the 66 pairs of patterns we find that there are 21 symmetry classes and 10 Wilf-classes. More details are given in Table 1.

CM2. In Article C1 and CM1 we were concerned with permutations avoiding one or more patterns. In this paper we address the more general problem of enumerating permutations with a prescribed number of occurrences of a given pattern.

As we have seen in Article C1 there are three different classes of patterns of type $(1,2)$ or $(2,1)$ with respect to being equidistributed. Let $u_r(n)$, $v_r(n)$, and $w_r(n)$ be the number of permutations of $[n]$ containing exactly r occurrences of patterns 1-23, 1-32, and 2-13, respectively. Moreover, let $U_r(x)$, $V_r(x)$, and $W_r(x)$ be the ordinary generating functions for the numbers $u_r(n)$, $v_r(n)$, and $w_r(n)$, respectively. We show that

$$u_1(n+2) = 2u_1(n+1) + \sum_{k=0}^{n-1}\binom{n}{k}\big[u_1(k+1) + B_{k+1}\big], \qquad u_1(0) = 0,$$

and

$$v_1(n+1) = v_1(n) + \sum_{k=1}^{n-1}\left[\binom{n}{k}v_1(k) + \binom{n-1}{k-1}B_k\right], \qquad v_1(0) = 0,$$

	$\{p, q\}$	$\lvert\mathcal{S}_n(p, q)\rvert$
1	1-23, 32-1 3-21, 12-3	0
2	1-23, 3-21 32-1, 12-3	$2(n-1)$
3	1-23, 2-31 3-21, 2-13 12-3, 31-2 32-1, 13-2	$\binom{n}{2}+1$
4a	1-23, 2-13 3-21, 2-31 12-3, 13-2 32-1, 31-2	2^{n-1}
4b	1-23, 23-1 3-21, 21-3 12-3, 3-12 32-1, 1-32	2^{n-1}
4c	1-23, 31-2 3-21, 13-2 12-3, 2-31 32-1, 2-13	2^{n-1}
4d	1-32, 2-13 3-12, 2-31 13-2, 21-3 23-1, 31-2	2^{n-1}
4e	1-32, 2-31 3-12, 2-13 31-2, 21-3 23-1, 13-2	2^{n-1}
4f	1-32, 3-12 23-1, 21-3	2^{n-1}
4g	1-32, 23-1 3-12, 21-3	2^{n-1}
4h	1-32, 31-2 3-12, 13-2 21-3, 2-31 23-1, 2-13	2^{n-1}
4i	2-13, 2-31 31-2, 13-2	2^{n-1}
4j	2-13, 13-2 2-31, 31-2	2^{n-1}
4k	2-13, 31-2 2-31, 13-2	2^{n-1}
5a	1-23, 13-2 3-21, 31-2 12-3, 2-13 32-1, 2-31	M_n (Motzkin no.)
5b	1-23, 21-3 3-21, 23-1 12-3, 1-32 32-1, 3-12	M_n (Motzkin no.)
6	1-32, 21-3 3-12, 23-1	a_n
7	1-23, 3-12 3-21, 1-32 23-1, 12-3 32-1, 21-3	b_n
8	1-23, 1-32 3-21, 3-12 21-3, 12-3 32-1, 23-1	I_n (# involutions)
9	1-32, 13-2 3-12, 31-2 21-3, 2-13 23-1, 2-31	C_n (Catalan no.)
10	1-23, 12-3 3-21, 32-1	B_n^* (Bessel no.)

TABLE 1. The enumeration of pairs of patterns of type $(1, 2)$ or $(2, 1)$

where B_n is the nth Bell number. On a slightly higher level of abstraction—the level on which the generating functions reside—these recursions amount to functional equations. From these equations we derive that

$$U_1(x) = \sum_{n\geq 1} \frac{x}{1-nx} \sum_{k\geq 0} \frac{kx^{k+n}}{(1-x)(1-2x)\cdots(1-(k+n)x)}$$

and

$$V_1(x) = \sum_{n\geq 1} \frac{x}{1-(n-1)x} \sum_{k\geq 0} \frac{kx^{k+n}}{(1-x)(1-2x)\cdots(1-(k+n)x)}.$$

Using a result due to Clarke et al. [11] we obtain a continued fraction expansion of the generating function for the distribution of occurrences the pattern 2-13. To be precise, we find that

$$\sum_{\pi\in\mathcal{S}} p^{(2\text{-}13)\pi} t^{|\pi|} = \cfrac{1}{1-\cfrac{[\,1\,]_p t}{1-\cfrac{[\,1\,]_p t}{1-\cfrac{[\,2\,]_p t}{1-\cfrac{[\,2\,]_p t}{\ddots}}}}},$$

where $[\,n\,]_p = 1+p+\cdots+p^{n-1}$. From this continued fraction we are then able to derive three closed formulas:

$$w_1(n) = \binom{2n}{n-3}; \quad w_2(n) = \frac{n(n-3)}{2(n+4)}\binom{2n}{n-3}; \quad w_3(n) = \frac{1}{3}\binom{n+2}{2}\binom{2n}{n-5}.$$

C2. Elizalde and Noy [12] presented exponential generating functions for the distribution of the number of segment-occurrences of any pattern of type (3): Let $h(x) = \sqrt{(x-1)(x+3)}$. Then

$$\sum_{\pi\in\mathcal{S}} x^{(123)\pi} \frac{t^{|\pi|}}{|\pi|!} = \frac{2h(x)e^{\frac{1}{2}(h(x)-x+1)t}}{h(x)+x+1+(h(x)-x-1)e^{h(x)t}},$$

$$\sum_{\pi\in\mathcal{S}} x^{(213)\pi} \frac{t^{|\pi|}}{|\pi|!} = \frac{1}{1-\int_0^t e^{(x-1)z^2/2}dz}.$$

We say that a permutation π is 132-*segmented* if $(1\text{-}3\text{-}2)\pi = (132)\pi$. In other words, π is 132-segmented if every occurrence of (1-3-2) in π also is an occurrence (132) in π. For instance, 4365172 contains 3 occurrences of (1-3-2), namely 465, 365, and 172. Of these occurrences, only 365 and 172 are occurrence of (132). Thus 4365172 is not 132-segmented.

In general, π is σ-*segmented* if $(\sigma,[0,k])(\pi) = (\sigma,\{0,k\})(\pi)$, where $k = |\sigma|$. Note that if π is σ-avoiding the π is also σ-segmented. In this article we try to enumerate the σ-segmented permutations by length and by the the number of occurrences of σ.

A *bicoloured Dyck path* is a Dyck path in which each up-step is assigned one of two colours, say, red and green. In [22] Krattenthaler gave two bijections: one between 132-avoiding permutations and Dyck paths, and one between 123-avoiding permutations and Dyck paths. We obtain two new results by extending these bijections:

– The 132-segmented permutations of length n with exactly k occurrences of 132 are in one-to-one correspondence with bicoloured Dyck paths of length $2n-4k$ with exactly k red up-steps.

– The 123-segmented permutations of length n with exactly k occurrences of 123 are in one-to-one correspondence with bicoloured Dyck paths of length $2n-4k$ with exactly k red up-steps, each of height less than 2.

We enumerate the permutations above by enumerating the corresponding bicoloured Dyck paths. To be more precise, let $\mathcal{B}_{n,k}$ be the set of bicoloured Dyck path of length $2n$ with k red up-steps, and let $\mathcal{B}_{n,k}^{[h]}$ be the subset of $\mathcal{B}_{n,k}$ consisting of those paths where the height of each red up-step is less than h. It is plain that $|\mathcal{B}_{n,k}| = \binom{n}{k}C_n$. We show that

$$\sum_{n,k\geq 0} |\mathcal{B}_{n,k}^{[h]}| q^k t^n = \frac{C(t) - 2xqU_h(x)U_{h-1}(x)}{1+q-qU_h^2(x)}, \qquad x = \frac{1}{2\sqrt{(1+q)t}},$$

where $C(t) = (1-\sqrt{1-4t})/2t$ is the generating function for the Catalan numbers, and U_n is the nth Chebyshev polynomial of the second kind. We also find formulas for $|\mathcal{B}_{n,k}^{[1]}|$ and $|\mathcal{B}_{n,k}^{[2]}|$:

$$|\mathcal{B}_{n,k}^{[1]}| = \frac{2k+1}{n+k+1}\binom{2n}{n-k}; \qquad \text{(ballot number)}$$

$$|\mathcal{B}_{n,k}^{[2]}| = \sum_{i\geq 0} \frac{2k+i+1}{n+k+i+1}\binom{k-1}{k-i}\binom{2n+i}{n-k}.$$

BCS. In this paper we only study classical patterns and the notation for generalized patterns is not used.

For $k \geq 1$, we denote by e_{k-1} the pattern/statistic $1\,2\cdots k$. Thus $e_0(\pi)$ is the length $|\pi|$ of π, and $e_1(\pi)$ counts non-inversions in π. We also define $e_{-1}(\pi) = 1$ for all permutations π.

A theorem of Robertson et al. [29] gives a simple continued fraction that records the joint distribution of the patterns $e_1 = 12$ and $e_2 = 123$ on permutations avoiding the pattern 132. We give the following generalization their result:

$$\sum_{\pi\in\mathcal{S}(132)} \prod_{k\geq 0} x_k^{e_k(\pi)} = \cfrac{1}{1 - \cfrac{x_0^{\binom{0}{0}}}{1 - \cfrac{x_0^{\binom{1}{0}} x_1^{\binom{1}{1}}}{1 - \cfrac{x_0^{\binom{2}{0}} x_1^{\binom{2}{1}} x_2^{\binom{2}{2}}}{1 - \cfrac{x_0^{\binom{3}{0}} x_1^{\binom{3}{1}} x_2^{\binom{3}{2}} x_3^{\binom{3}{3}}}{\ddots}}}}}$$

in which the $(n+1)$st numerator is $\displaystyle\prod_{k=0}^{n} x_k^{\binom{n}{k}}$.

To state the main theorem of this article we need some definitions: A *Stieltjes continued fraction* is a continued fraction of the form

$$C = \cfrac{1}{1 - \cfrac{m_1}{1 - \cfrac{m_2}{\ddots}}},$$

where each m_i is a monomial in some set of variables. We define a *Catalan continued fraction* to be a Stieltjes continued fraction with monic monomial numerators.

Let $\mathcal{A}$ be the ring of all infinite matrices with a finite number of non zero entries in each row, with multiplication defined by $(AB)_{nk} = \sum_{i=0}^{\infty} A_{ni}B_{ik}$. With each A in $\mathcal{A}$ we now associate a family of statistics $\{\langle \mathbf{e}, A_k\rangle\}_{k\geq 0}$, defined on $\mathcal{S}(132)$, where $\mathbf{e} = (e_0, e_1, \ldots)$, A_k is the kth column of A, and

$$\langle \mathbf{e}, A_k\rangle = \sum_i A_{ik}e_i.$$

Let $\mathbf{q} = (q_0, q_1, \ldots)$ and, for each A in $\mathcal{A}$, let

$$F_A(\mathbf{q}) = \sum_{\pi\in\mathcal{S}(132)} \prod_{k\geq 0} q_k^{\langle \mathbf{e}, A_k\rangle(\pi)}$$

and

$$C_A(\mathbf{q}) = \cfrac{1}{1 - \cfrac{\prod q_k^{A_{0k}}}{1 - \cfrac{\prod q_k^{A_{1k}}}{\ddots}}}.$$

Our main theorem states that if $A \in \mathcal{A}$ then

$$F_A(\mathbf{q}) = C_{BA}(\mathbf{q}),$$

where $B = [\binom{i}{j}]$, and conversely

$$C_A(\mathbf{q}) = F_{B^{-1}A}(\mathbf{q}).$$

In particular, all Catalan continued fractions are generating functions of statistics on $\mathcal{S}(132)$ consisting of (possibly infinite) linear combinations of e_ks.

We give several applications of this theorem. As an example, the Catalan continued fraction

$$R(q,t) = \cfrac{1}{1 - \cfrac{qt}{1 - \cfrac{q^3t}{1 - \cfrac{q^5t}{\ddots}}}}$$

was studied by Ramanujan. Applying our main theorem we find that $R(q,t)$ is the generating function for the distribution of the statistic $e_0 + 2e_1$ on 132-avoiding permutations.

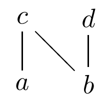
CGW. This last article is not at all concerned with patterns in permutations and is in that respect the odd one out in this thesis.

Let P be a finite ordered set, and let $J(P)$ be the distributive lattice of order ideals (also called down–sets) of P. For each $p \in P$, define a permutation σ_p on $J(P)$ as follows: for every $S \in J(P)$,

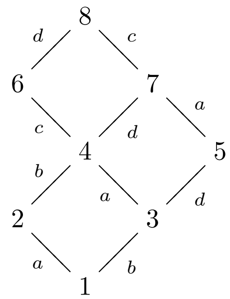
$$\sigma_p(S) := \begin{cases} S \cup \{p\} & \text{if } p \text{ is minimal in } P \setminus S, \\ S \setminus \{p\} & \text{if } p \text{ is maximal in } S, \\ S & \text{otherwise.} \end{cases}$$

Each of these permutations is an involution. We let $\Gamma(P)$ denote the subgroup of the symmetric group $\mathrm{Sym}(J(P))$ generated by all these involutions. The purpose of this paper is to determine $\Gamma(P)$.

As an example, for

$P =$ 

we may number the down–sets $\{\emptyset, a, b, ab, bd, abc, abd, abcd\}$ of P by 1 through 8, and then

$J(P) =$ 

and

$$\begin{cases} \sigma_a = (1\ 2)(3\ 4)(5\ 7) \\ \sigma_b = (1\ 3)(2\ 4) \\ \sigma_c = (4\ 6)(7\ 8) \\ \sigma_d = (3\ 5)(4\ 7)(6\ 8) \end{cases}$$

in which we have labeled the edges of the Hasse diagram of $J(P)$ to indicate the action of each σ_p on $J(P)$.

Let P and Q be disjoint finite ordered sets. Then it is plain that $\Gamma(P \cup Q) = \Gamma(P) \times \Gamma(Q)$. The problem is thus reduced to determining $\Gamma(P)$ for connected ordered sets P. Our main theorem is the following result: *If P is a finite connected ordered set then $\Gamma(P)$ is either the alternating group $\mathrm{Alt}(J(P))$ or the symmetric group $\mathrm{Sym}(J(P))$.* We also address the computational complexity of determining which case occurs. In the example above, $\Gamma(P)$ is the symmetric group $\mathrm{Sym}(J(P))$.

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[**C1**]

GENERALIZED PATTERN AVOIDANCE

ANDERS CLAESSON

Abstract. Recently, Babson and Steingrímsson have introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. We will consider pattern avoidance for such patterns, and give a complete solution for the number of permutations avoiding any single pattern of length three with exactly one adjacent pair of letters. For eight of these twelve patterns the answer is given by the Bell numbers. For the remaining four the answer is given by the Catalan numbers. We also give some results for the number of permutations avoiding two different patterns. These results relate the permutations in question to Motzkin paths, involutions and non-overlapping partitions. Furthermore, we define a new class of set partitions, called monotone partitions, and show that these partitions are in one-to-one correspondence with non-overlapping partitions.

1. Introduction

In the last decade a wealth of articles has been written on the subject of pattern avoidance, also known as the study of "restricted permutations" and "permutations with forbidden subsequences". Classically, a pattern is a permutation $\sigma \in \mathcal{S}_k$, and a permutation $\pi \in \mathcal{S}_n$ avoids σ if there is no subsequence in π whose letters are in the same relative order as the letters of σ. For example, $\pi \in \mathcal{S}_n$ avoids 132 if there is no $1 \leq i < j < k \leq n$ such that $\pi(i) < \pi(k) < \pi(j)$. In [4] Knuth established that for all $\sigma \in \mathcal{S}_3$, the number of permutations in $\mathcal{S}_n$ avoiding σ equals the nth Catalan number, $C_n = \frac{1}{1+n}\binom{2n}{n}$. One may also consider permutations that are required to avoid several patterns. In [5] Simion and Schmidt gave a complete solution for permutations avoiding any set of patterns of length three. Even patterns of length greater than three have been considered. For instance, West showed in [8] that permutations avoiding both 3142 and 2413 are enumerated by the Schröder numbers, $S_n = \sum_{i=0}^{n}\binom{2n-i}{i}C_{n-i}$.

In [1] Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. The motivation for Babson and Steingrímsson in introducing these patterns was the study of Mahonian statistics, and they showed that essentially all Mahonian permutation statistics in the literature can be written as linear combinations of such patterns. An example of a generalized pattern is (1-32). An occurrence of (1-32) in a permutation $\pi = a_1a_2\cdots a_n$ is a subword $a_ia_ja_{j+1}$, $(i < j)$, such that $a_i < a_{j+1} < a_j$. More generally, a pattern p is a word over the alphabet $\{1, 2, 3, \ldots\}$ where two adjacent letters may or may not be separated by a dash. The absence of a dash between two adjacent letters in a p indicates that the corresponding letters in an occurrence of p must be adjacent. Also, the ordering

Date: January 21, 2004.

of the letters in the occurrence of p must match the ordering of the letters in the pattern. This definition, as well as any other definition in the introduction, will be stated rigorously in Section 2. All classical patterns are generalized patterns where each pair of adjacent letters is separated by a dash. For example, the generalized pattern equivalent to 132 is (1-3-2).

We extend the notion of pattern avoidance by defining that a permutation avoids a (generalized) pattern p if it does not contain any occurrences of p. We show that this is a fruitful extension, by establishing connections to other well known combinatorial structures, not previously shown to be related to pattern avoidance. The main results are given below.

P	$\lvert\mathcal{S}_n(P)\rvert$	*Description*
1-23	B_n	Partitions of $[n]$
1-32	B_n	Partitions of $[n]$
2-13	C_n	Dyck paths of length $2n$
1-23, 12-3	B^*_n	Non-overlapping partitions of $[n]$
1-23, 1-32	I_n	Involutions in $\mathcal{S}_n$
1-23, 13-2	M_n	Motzkin paths of length n

Here $\mathcal{S}_n(P) = \{\pi \in \mathcal{S}_n : \pi \text{ avoids } p \text{ for all } p \in P\}$, and $[n] = \{1, 2, \ldots, n\}$. When proving that $\lvert\mathcal{S}_n(\text{1-23, 12-3})\rvert = B^*_n$ (the nth Bessel number), we first prove that there is a one-to-one correspondence between $\{\text{1-23}, \text{12-3}\}$-avoiding permutations and *monotone partitions*. A partition is monotone if its non-singleton blocks can be written in increasing order of their least element and increasing order of their greatest element, simultaneously. This new class of partitions is then shown to be in one-to-one correspondence with non-overlapping partitions.

2. Preliminaries

By an *alphabet* X we mean a non-empty set. An element of X is called a *letter*. A *word* over X is a finite sequence of letters from X. We consider also the *empty word*, that is, the word with no letters; it is denoted by ϵ. Let $x = x_1x_2\cdots x_n$ be a word over X. We call $|x| := n$ the *length* of x. A *subword* of x is a word $v = x_{i_1}x_{i_2}\cdots x_{i_k}$, where $1 \le i_1 < i_2 < \cdots < i_k \le n$. A *segment* of x is a word $v = x_ix_{i+1}\cdots x_{i+k}$. If X and Y are two linearly ordered alphabets, then two words $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ over X and Y, respectively, are said to be *order equivalent* if $x_i < x_j$ precisely when $y_i < y_j$.

Let $[n] := \{1, 2, \ldots, n\}$ (so $[0] = \emptyset$). A *permutation* of $[n]$ is bijection from $[n]$ to $[n]$. Let $\mathcal{S}_n$ be the set of permutations of $[n]$. We shall usually think of a permutation π as the word $\pi(1)\pi(2)\cdots\pi(n)$ over the alphabet $[n]$. In particular, $\mathcal{S}_0 = \{\epsilon\}$, since there is only one bijection from $\emptyset$ to $\emptyset$, the empty map.

Let $X = [k] \cup \{\text{-}\}$. For each word x in X let $\bar{x}$ be the word obtained from x by deleting all dashes in x. A word p over X is called a *pattern* if it contains no two consecutive dashes and $\bar{p}$ is a permutation of $[k]$. By slight abuse of terminology we refer to the *length of a pattern* p as the length of $\bar{p}$.

We say that a subword o of π is a p-*subword* if by replacing (possibly empty) segments of π with dashes we can obtain a word q such that $\bar{q} = o$, and the ith letter in q is a dash precisely when the ith letter in p is a dash.

All patterns that we consider will have a dash at the beginning and one at the end. For convenience, we therefore leave them out. For example, (1-23) is a pattern, and the permutation 491273865 contains three occurrences of (1-23), namely 127, 138, and 238. A permutation is said to be *p-avoiding* if it does not contain any occurrences of p. Define $\mathcal{S}_n(p)$ to be the set of p-avoiding permutations in $\mathcal{S}_n$ and, more generally, $\mathcal{S}_n(A) = \bigcap_{p\in A} \mathcal{S}_n(p)$.

We may think of a pattern p as a permutation statistic, that is, define $p\,\pi$ as the number of occurrences of p in π, thus regarding p as a function from $\mathcal{S}_n$ to $\mathbb{N}$. For example, $(1\text{-}23)\,491273865 = 3$. In particular, π is p-avoiding if and only if $p\,\pi = 0$. We say that two permutation statistics stat and stat$'$ are *equidistributed* over $A \subseteq \mathcal{S}_n$, if

$$\sum_{\pi\in A} x^{\operatorname{stat}\pi} = \sum_{\pi\in A} x^{\operatorname{stat}'\pi}.$$

In particular, this definition applies to patterns.

Let $\pi = a_1a_2\cdots a_n \in \mathcal{S}_n$. An i such that $a_i > a_{i+1}$ is called a *descent* in π. We denote by $\operatorname{des}\pi$ the number of descents in π. Observe that des can be defined as the pattern (ba), that is, $\operatorname{des}\pi = (ba)\pi$. A *left-to-right minimum* of π is an element a_i such that $a_i < a_j$ for every $j < i$. The number of left-to-right minima is a permutation statistic. Analogously we also define *left-to-right maximum*, *right-to-left minimum*, and *right-to-left maximum*.

In this paper we will relate permutations avoiding a given set of patterns to other better known combinatorial structures. Here follows a brief description of these structures. Two excellent references on combinatorial structures are [7] and [6].

Set partitions. A *partition* of a set S is a family, $\pi = \{A_1, A_2, \dots, A_k\}$, of pairwise disjoint non-empty subsets of S such that $S = \cup_i A_i$. We call A_i a *block* of π. The total number of partitions of $[n]$ is called a *Bell number* and is denoted B_n. For reference, the first few Bell numbers are

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597.$$

Let $S(n,k)$ be the number of partitions of $[n]$ into k blocks; these numbers are called the *Stirling numbers of the second kind*.

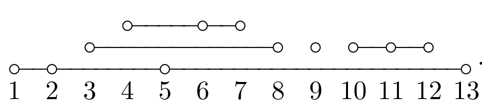
Non-overlapping partitions. Two blocks A and B of a partition π *overlap* if

$$\min A < \min B < \max A < \max B.$$

A partition is *non-overlapping* if no pairs of blocks overlap. Thus

$$\pi = \{\{1,2,5,13\},\{3,8\},\{4,6,7\},\{9\},\{10,11,12\}\}$$

is non-overlapping. A pictorial representation of π is

$\pi =$ 

.

Let B_n^* be the number of non-overlapping partitions of $[n]$; this number is called the nth *Bessel number* [3, p. 423]. The first few Bessel numbers are

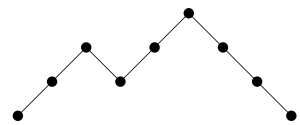
$$1, 1, 2, 5, 14, 43, 143, 509, 1922, 7651, 31965, 139685, 636712.$$

We denote by $S^*(n,k)$ the number of non-overlapping partitions of $[n]$ into k blocks.

Involutions. An *involution* is a permutation which is its own inverse. We denote by I_n the number of involutions in $\mathcal{S}_n$. The sequence $\{I_n\}_0^\infty$ starts with

$$1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152.$$

Dyck paths. A *Dyck path* of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ that never goes below the x-axis. Letting u and d represent the steps $(1,1)$ and $(1,-1)$ respectively, we code such a path with a word over $\{u,d\}$. For example, the path

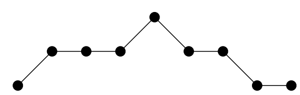


is coded by $uuduuddd$. A *return step* in a Dyck path δ is a d such that $\delta = \alpha u \beta d \gamma$, for some Dyck paths α, β, and γ. A useful observation is that every non-empty Dyck path δ can be uniquely decomposed as $\delta = u\alpha d\beta$, where α and β are Dyck paths. This is the so-called *first return decomposition* of δ.

The nth *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the number of Dyck paths of length $2n$. The sequence of Catalan numbers starts with

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012.$$

Motzkin paths. A *Motzkin path* of length n is a lattice path from $(0,0)$ to $(n,0)$ with steps $(1,0)$, $(1,1)$, and $(1,-1)$ that never goes below the x-axis. Letting ℓ, u, and d represent the steps $(1,0)$, $(1,1)$, and $(1,-1)$ respectively, we code such a path with a word over $\{\ell,u,d\}$. For example, the path



is coded by $u\ell\ell ud\ell d\ell$. If δ is a non-empty Motzkin path, then δ can be decomposed as $\delta = \ell\gamma$ or $\delta = u\alpha d\beta$, where α, β and γ are Motzkin paths.

The nth *Motzkin number* M_n is the number of Motzkin paths of length n. The first few of the Motzkin numbers are

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511.$$

3. Three classes of patterns

Let $\pi = a_1a_2\cdots a_n \in \mathcal{S}_n$. Define the *reverse* of π as $\pi^r := a_n\cdots a_2a_1$, and define the *complement* of π by $\pi^c(i) = n+1-\pi(i)$, where $i \in [n]$.

Proposition 1. *With respect to being equidistributed, the twelve pattern statistics of length three with one dash fall into the following three classes.*

(i) 1-23, 3-21, 12-3, 32-1.
(ii) 1-32, 3-12, 21-3, 23-1.
(iii) 2-13, 2-31, 13-2, 31-2.

Proof. The bijections $\pi \mapsto \pi^r$, $\pi \mapsto \pi^c$, and $\pi \mapsto (\pi^r)^c$ give the equidistribution part of the result. Calculations show that these three distributions differ pairwise on $\mathcal{S}_4$. $\square$

4. Permutations avoiding a pattern of class one or two

Proposition 2. *Partitions of* $[n]$ *are in one-to-one correspondence with* (1-23)*-avoiding permutations in* $\mathcal{S}_n$*. Hence* $|\mathcal{S}_n(\text{1-23})| = B_n$*.*

First proof. Recall that the Bell numbers satisfy $B_0 = 1$, and

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

We show that $|\mathcal{S}_n(\text{1-23})|$ satisfy the same recursion. Clearly, $\mathcal{S}_0(\text{1-23}) = \{\epsilon\}$. For $n > 0$, let $M = \{2, 3, \ldots, n+1\}$, and let S be a k element subset of M. For each (1-23)-avoiding permutation σ of S we construct a unique (1-23)-avoiding permutation π of $[n+1]$. Let τ be the word obtained by writing the elements of $M \setminus S$ in decreasing order. Define $\pi := \sigma 1 \tau$.

Conversely, if $\pi = \sigma 1 \tau$ is a given (1-23)-avoiding permutation of $[n+1]$, where $|\sigma| = k$, then the letters of τ are in decreasing order, and σ is an (1-23)-avoiding permutation of the k element set $\{2, 3, \ldots, n+1\} \setminus \{i : i \text{ is a letter in } \tau\}$. □

Second proof. Given a partition π of $[n]$, we introduce a standard representation of π by requiring that:

(a) Each block is written with its least element first, and the rest of the elements of that block are written in decreasing order.
(b) The blocks are written in decreasing order of their least element, and with dashes separating the blocks.

Define $\widehat{\pi}$ to be the permutation we obtain from π by writing it in standard form and erasing the dashes. We now argue that $\widehat{\pi} := a_1 a_2 \cdots a_n$ avoids (1-23). If $a_i < a_{i+1}$, then a_i and a_{i+1} are the first and the second element of some block. By the construction of $\widehat{\pi}$, a_i is a left-to-right minimum, hence there is no $j \in [i-1]$ such that $a_j < a_i$.

Conversely, π can be recovered uniquely from $\widehat{\pi}$ by inserting a dash in $\widehat{\pi}$ preceding each left-to-right minimum, apart from the first letter in $\widehat{\pi}$. Indeed, it easy to see that the partition, π, in this way obtained is written in standard form. Thus $\pi \mapsto \widehat{\pi}$ gives the desired bijection. □

Example. As an illustration of the map defined in the above proof, let

$$\pi = \{\{1,3,5\}, \{2,6,9\}, \{4,7\}, \{8\}\}.$$

Its standard form is 8-47-296-153. Thus $\widehat{\pi} = 847296153$.

Proposition 3. *Let* $L(\pi)$ *be the number of left-to-right minima of* π*. Then*

$$\sum_{\pi \in \mathcal{S}_n(\text{1-23})} x^{L(\pi)} = \sum_{k \geq 0} S(n,k) x^k.$$

Proof. This result follows readily from the second proof of Proposition 2. We here give a different proof, which is based on the fact that the Stirling numbers of the second kind satisfy

$$S(n,k) = S(n-1, k-1) + kS(n-1, k).$$

Let $T(n,k)$ be the number of permutations in $\mathcal{S}_n(\text{1-23})$ with k left-to-right minima. We show that the $T(n,k)$ satisfy the same recursion as the $S(n,k)$.

Let π be an (1-23)-avoiding permutation of $[n-1]$. To insert n in π, preserving (1-23)-avoidance, we can put n in front of π or we can insert n immediately after each left-to-right minimum. Putting n in front of π creates a new left-to-right minimum, while inserting n immediately after a left-to-right minimum does not. □

Proposition 4. *Partitions of* $[n]$ *are in one-to-one correspondence with* (1-32)*-avoiding permutations in* $\mathcal{S}_n$*. Hence* $|\mathcal{S}_n(1\text{-}32)| = B_n$*.*

Proof. Let π be a partition of $[n]$. We introduce a standard representation of π by requiring that:

(a) The elements of a block are written in increasing order.
(b) The blocks are written in decreasing order of their least element, and with dashes separating the blocks.

(Note that this standard representation is different from the one given in the second proof of Proposition 2.) Define $\widehat{\pi}$ to be the permutation we obtain from π by writing it in standard form and erasing the dashes. It easy to see that $\widehat{\pi}$ avoids (1-32). Conversely, π can be recovered uniquely from $\widehat{\pi}$ by inserting a dash in between each descent in $\widehat{\pi}$. □

Example. As an illustration of the map defined in the above proof, let

$$\pi = \{\{1,3,5\},\{2,6,9\},\{4,7\},\{8\}\}.$$

Its standard form is 8-47-269-135. Thus $\widehat{\pi} = 847269135$.

Proposition 5.

$$\sum_{\pi\in\mathcal{S}_n(1\text{-}32)} x^{L(\pi)} = \sum_{\pi\in\mathcal{S}_n(1\text{-}32)} x^{1+\operatorname{des}\pi} = \sum_{k\geq 0} S(n,k)x^k.$$

Proof. From the proof of Proposition 4 we see that a left-to-right minimum in π corresponds to a least element in a block of $\widehat{\pi}$. Moreover, π has $k+1$ blocks precisely when $\widehat{\pi}$ has k descents. □

Proposition 6. *Involutions in* $\mathcal{S}_n$ *are in one-to-one correspondence with permutations in* $\mathcal{S}_n$ *that avoid* (1-23) *and* (1-32)*. Hence*

$$|\mathcal{S}_n(1\text{-}23, 1\text{-}32)| = I_n.$$

Proof. We give a combinatorial proof using a bijection that is essentially identical to the one given in the second proof of Proposition 2.

Let $\pi \in \mathcal{S}_n$ be an involution. Recall that π is an involution if and only if each cycle of π is of length one or two. We now introduce a standard form for writing π in cycle notation by requiring that:

(a) Each cycle is written with its least element first.
(b) The cycles are written in decreasing order of their least element.

Define $\widehat{\pi}$ to be the permutation obtained from π by writing it in standard form and erasing the parentheses separating the cycles.

Observe that $\widehat{\pi}$ avoids (1-23): Assume that $a_i < a_{i+1}$, that is $(a_i\, a_{i+1})$ is a cycle in π, then a_i is a left-to-right minimum in π. This is guaranteed by the construction of $\widehat{\pi}$. Thus there is no $j < i$ such that $a_j < a_i$.

The permutation $\widehat{\pi}$ also avoids (1-32): Assume that $a_i > a_{i+1}$, then a_{i+1} must be the smallest element of some cycle. Whence a_{i+1} is a left-to-right minimum in $\widehat{\pi}$.

Conversely, if $\widehat{\pi} := a_1 \ldots a_n$ is an $\{1\text{-}23, 1\text{-}32\}$-avoiding permutation then the involution π is given by: $(a_i\, a_{i+1})$ is a cycle in π if and only if $a_i < a_{i+1}$. $\square$

Example. The involution $\pi = 826543719$ written in standard form is

$$(9)(7)(4\,5)(3\,6)(2)(1\,8),$$

and hence $\widehat{\pi} = 974536218$.

Proposition 7. *The number of permutations in* $\mathcal{S}_n(1\text{-}23, 1\text{-}32)$ *with* $n-k-1$ *descents equals the number of involutions in* $\mathcal{S}_n$ *with* $n-2k$ *fixed points.*

Proof. Under the bijection $\pi \mapsto \widehat{\pi}$ in the proof of Proposition 6, a cycle of length two in π corresponds to an occurrence of (12) in $\widehat{\pi}$. Hence, if π has $n-2k$ fixed points, then $\widehat{\pi}$ has $n-k-1$ descents. $\square$

Corollary 8.

$$\sum_{\pi \in \mathcal{S}_n(1\text{-}23,1\text{-}32)} x^{1+\operatorname{des}\pi} = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k}\frac{k!}{2^k}x^{n-k}.$$

Proof. Let I_n^k denote the number of involutions in $\mathcal{S}_n$ with k fixed points. Then Proposition 7 is equivalently stated as

$$\sum_{\pi \in \mathcal{S}_n(1\text{-}23,1\text{-}32)} x^{1+\operatorname{des}\pi} = \sum_{k\geq 0} I_n^{n-2k} x^{n-k}. \tag{1}$$

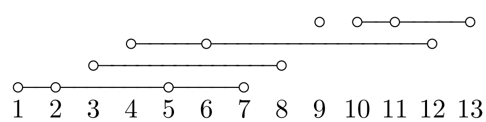
The result now follows from the well-known and easily to derived formula

$$I_n^k = \binom{n}{k}\binom{n-k}{r}\frac{r!}{2^r}, \text{ where } r = \frac{n-k}{2},$$

for $n-k$ even, with $I_n^k = 0$ for $n-k$ odd. $\square$

Definition 9. Let π be an arbitrary partition whose non-singleton blocks A_1, A_2, $\ldots$, A_k are ordered so that for all $i \in [k-1]$, $\min A_i > \min A_{i+1}$. If $\max A_i > \max A_{i+1}$ for all $i \in [k-1]$, then we call π a *monotone partition.* The set of monotone partitions of $[n]$ is denoted by $\mathcal{M}_n$.

Example. The partition

$\pi =$ 

is monotone.

Proposition 10. *Monotone partitions of* $[n]$ *are in one-to-one correspondence with permutations in* $\mathcal{S}_n$ *that avoid* (1-23) *and* (12-3)*. Hence*

$$|\mathcal{S}_n(1\text{-}23, 12\text{-}3)| = |\mathcal{M}_n|.$$

Proof. Given π in $\mathcal{M}_n$, let $A_1\text{-}A_2\text{-}\cdots\text{-}A_k$ be the result of writing π in the standard form given in the second proof of Proposition 2, and let $\widehat{\pi} = A_1 A_2 \cdots A_k$. By the construction of $\widehat{\pi}$ the first letter in each A_i is a left-to-right minimum. Furthermore, since π is monotone the second letter in each non-singleton A_i is a right-to-left maximum. Therefore, if xy is an occurrence of (ab) in $\widehat{\pi}$, then x is left-to-right minimum and y is a right-to-left maximum. Thus $\widehat{\pi}$ avoids both (1-23) and (12-3).

Conversely, given $\widehat{\pi}$ in $\mathcal{S}_n(1\text{-}23, 12\text{-}3)$, let $A_1\text{-}A_2\text{-}\cdots\text{-}A_k$ be the result of inserting a dash in $\widehat{\pi}$ preceding each left-to-right minimum, apart from the first letter in $\widehat{\pi}$. Since $\widehat{\pi}$ is (12-3)-avoiding, the second letter in each non-singleton A_i is a right-to-left maximum. The second letter in A_i is the maximal element of A_i when A_i is viewed as a set. Thus $\pi = \{A_1, A_2, \ldots, A_k\}$ is monotone. □

We now show that there is a one-to-one correspondence between monotone partitions and non-overlapping partitions. The proof we give is strongly influenced by the paper [3], in which Flajolet and Schot showed that the ordinary generating function of the Bessel numbers admits a nice continued fraction expansion

$$\sum_{n\geq 0} B_n^* x^n = \cfrac{1}{1 - 1\cdot x - \cfrac{x^2}{1 - 2\cdot x - \cfrac{x^2}{1 - 3\cdot x - \cfrac{x^2}{\ddots}}}},$$

and using that as a starting point they derived the asymptotic formula

$$B_n^* \sim \sum_{k\geq 0} \frac{k^{n+2}}{(k!)^2}.$$

Proposition 11. *Monotone partitions of $[n]$ are in one-to-one correspondence with non-overlapping partitions of $[n]$. Hence $|\mathcal{M}_n| = B_n^*$.*

Proof. Let π be a non-overlapping partition of $[n]$. From π we will create a new partition by successively inserting $1, 2, \ldots, n$, in this order, into this new partition. During this process a block is labelled as either *open* or *closed*. More formally, in each step $k = 1, 2, \ldots, n$ in this process we will have a partition σ of $[k]$ together with a function from σ to the set of labels $\{\text{open}, \text{closed}\}$. Before we start we also need a labelling of the blocks of π. Actually we need n such labellings, one for each $k \in [n]$: At step k a block B of π is labelled open if $\max B > k$ and closed otherwise. For ease of language, we say that a block is open if it is labelled open, and closed if it is labelled closed.

(a) If k is the minimal element of a non-singleton block of π, then create a new block $\{k\}$ and label it open.
(b) If k is the maximal element of a non-singleton block of π, then insert k into the open block with the smallest minimal element, and label it closed.
(c) If k belongs to a non-singleton block B of π and is not the minimal or the maximal element of B, and B has the ith largest minimal element of the open blocks of π, then insert k into the open block with the ith largest minimal element.
(d) If $\{k\}$ is a block of π then create a new block $\{k\}$ and label it closed.

Define $\Phi(\pi)$ as the partition obtained from π by applying the above process. Observe that $\Phi(\pi)$ is monotone. Indeed, the two crucial observations are (i) in (b) we label the open block with the smallest minimum closed, and (ii) a block labelled closed has received all its elements.

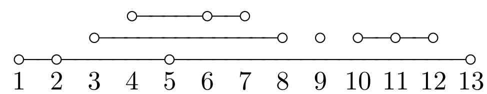
Conversely, we give a map Ψ that to each monotone partition π of $[n]$ gives a unique non-overlapping partition $\Psi(\pi)$ of $[n]$. Define Ψ the same way as Φ is defined, except for case (c), where we instead of inserting k into the block labelled open with the *smallest* minimal element, insert k into the block labelled open with the *largest* minimal element. It is easy to see that Φ and Ψ are each others inverses and hence they are bijections. □

Corollary 12. *The non-overlapping partitions of* $[n]$ *are in one-to-one correspondence with permutations in* $\mathcal{S}_n$ *that avoid* (1-23) *and* (12-3). *Hence*

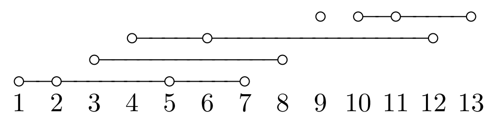
$$|\mathcal{S}_n(1\text{-}23, 12\text{-}3)| = B_n^*.$$

Proof. Follows immediately from Proposition 10 together with Proposition 11. □

Example. By the proof of Proposition 11, the non-overlapping partition

$\pi =$ 

corresponds to the monotone partition

$\Phi(\pi) =$ 

that according to the proof of Proposition 10 corresponds to the $\{1\text{-}23, 12\text{-}3\}$-avoiding permutation

$$\widehat{\Phi(\pi)} = 10\ 13\ 11\ 9\ 4\ 12\ 6\ 3\ 8\ 1\ 7\ 5\ 2.$$

Proposition 13. *Let* $L(\pi)$ *be the number of left-to-right minima of* π. *Then*

$$\sum_{\pi\in\mathcal{S}_n(1\text{-}23,12\text{-}3)} x^{L(\pi)} = \sum_{k\geq 0} S^*(n,k)x^k.$$

Proof. Under the bijection $\pi \mapsto \widehat{\pi}$ in the proof of Proposition 10, the number of blocks in π determines the number of left-to-right minima of $\widehat{\pi}$, and vice versa. The number of blocks is not changed by the bijection Ψ in the proof of Proposition 11. □

5. Permutations avoiding a pattern of class three

In [4] Knuth observed that there is a one-to-one correspondence between (2-1-3)-avoiding permutations and Dyck paths. For completeness and future reference we give this result as a lemma, and prove it using a bijection which rests on the first return decomposition of Dyck paths. First we need a definition. For each word $x = x_1x_2\cdots x_n$ without repeated letters, we define $\operatorname{red}(x)$—the *reduction* of x—as the permutation in $\mathcal{S}_n$ which is order equivalent to x. For example, $\operatorname{red}(265) = 132$.

Lemma 1. $|\mathcal{S}_n(2\text{-}1\text{-}3)| = C_n$.

Proof. Let $\pi = a_1a_2\cdots a_n$ be a permutation of $[n]$ such that $a_k = 1$. Then π is (2-1-3)-avoiding if and only if $\pi = \sigma 1 \tau$, where $\sigma := a_1\cdots a_{k-1}$ is a (2-1-3)-avoiding permutation of $\{n, n-1, \ldots, n-k+1\}$, and $\tau := a_{k+1}\cdots a_n$ is a (2-1-3)-avoiding permutation of $\{2, 3, \ldots, k\}$.

We define recursively a mapping Φ from $\mathcal{S}_n(2\text{-}1\text{-}3)$ onto the set of Dyck paths of length $2n$. If π is the empty word, then so is the Dyck path determined by π, that is, $\Phi(\epsilon) = \epsilon$. If $\pi \neq \epsilon$, then we can use the factorisation $\pi = \sigma 1 \tau$ from above, and define $\Phi(\pi) = u\,(\Phi\circ\mathrm{red})(\sigma)\,d\,(\Phi\circ\mathrm{red})(\tau)$. It is easy to see that Φ may be inverted, and hence is a bijection. $\square$

Lemma 2. *A permutation avoids* (2-13) *if and only if it avoids* (2-1-3).

Proof. The sufficiency part of the proposition is trivial. The necessity part is not difficult either. Assume that π contains an occurrence of (2-1-3). Then there is a segment $bm_1\cdots m_r$ of π, where, for some $j < r$, $m_j < b$ and $m_r > b$. Now choose the largest i such that $m_i < b$, then $m_{i+1} > b$. $\square$

Proposition 14. *Dyck paths of length $2n$ are in one-to-one correspondence with* (2-13)*-avoiding permutations in $\mathcal{S}_n$. Hence*

$$|\mathcal{S}_n(2\text{-}13)| = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Follows immediately from Lemmas 1 and 2. $\square$

Proposition 15. *Let $L(\pi)$ be the number of left-to-right minima of π. Then*

$$\sum_{\pi\in\mathcal{S}_n(2\text{-}13)} x^{L(\pi)} = \sum_{k\geq 0}\frac{k}{2n-k}\binom{2n-k}{n}x^k.$$

Proof. Let $R(\delta)$ denote the number of return steps in the Dyck path δ. It is well known (see [2]) that the distribution of R over all Dyck paths of length $2n$ is the distribution we claim that L has over $\mathcal{S}_n(2\text{-}13)$. .

Let γ be a Dyck path of length $2n$, and let $\gamma = u\alpha d\beta$ be its first return decomposition. Then $R(\gamma) = 1 + R(\beta)$. Let $\pi \in \mathcal{S}_n(2\text{-}13)$, and let $\pi = \sigma 1\tau$ be the decomposition given in the proof of Lemma 1. Then $L(\pi) = 1 + L(\sigma)$. The result now follows by induction. $\square$

In addition, it is easy to deduce that left-to-right minima, left-to-right maxima, right-to-left minima, and right-to-left maxima all share the same distribution over $\mathcal{S}_n(2\text{-}13)$.

Proposition 16. *Motzkin paths of length n are in one-to-one correspondence with permutations in $\mathcal{S}_n$ that avoid* (1-23) *and* (13-2). *Hence*

$$|\mathcal{S}_n(1\text{-}23, 13\text{-}2)| = M_n.$$

Proof. We mimic the proof of Lemma 1. Let $\pi \in \mathcal{S}_n(1\text{-}23, 13\text{-}2)$. Since π avoids (13-2) it also avoids (1-3-2) by Lemma 2 via $\pi \mapsto (\pi^c)^r$. Thus we may write $\pi = \sigma n\tau$, where $\pi(k) = n$, σ is an $\{1\text{-}23, 13\text{-}2\}$-avoiding permutation of $\{n-1, n-2, \ldots, n-k+1\}$, and τ is an $\{1\text{-}23, 13\text{-}2\}$-avoiding permutation of $[n-k]$. If $\sigma \neq \epsilon$ then $\sigma = \sigma' r$ where $r = n-k+1$, or else an occurrence of (1-23) would be formed with n as the '3' in (1-23). Define a map Φ from $\mathcal{S}_n(1\text{-}23, 13\text{-}2)$ to the set of Motzkin paths by $\Phi(\epsilon) = \epsilon$ and

$$\Phi(\pi) = \begin{cases} \ell\,(\Phi\circ\mathrm{red})(\sigma) & \text{if } \pi = n\sigma,\\ u\,(\Phi\circ\mathrm{red})(\sigma)\,d\,\Phi(\tau) & \text{if } \pi = \sigma r n\tau \text{ and } r = n-k+1.\end{cases}$$

It is routine to find the inverse of Φ. $\square$

Example. Let us find the Motzkin path associated with the $\{1\text{-}23, 13\text{-}2\}$-avoiding permutation 76453281.

$$\begin{aligned}
\Phi(76453281) &= u\Phi(54231)d\Phi(1)\\
&= u\ell\Phi(4231)d\ell\\
&= u\ell\ell\Phi(231)d\ell\\
&= u\ell\ell ud\Phi(1)d\ell\\
&= u\ell\ell ud\ell d\ell
\end{aligned}$$

Acknowledgement

I am greatly indebted to my advisor Einar Steingrímsson, who put his trust in me and gave me the opportunity to study mathematics on a postgraduate level. This work has benefited from his knowledge, enthusiasm and generosity.

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[**CM1**]

ENUMERATING PERMUTATIONS AVOIDING A PAIR OF BABSON-STEINGRíMSSON PATTERNS

ANDERS CLAESSON AND TOUFIK MANSOUR

Abstract. Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. Subsequently, Claesson presented a complete solution for the number of permutations avoiding any single pattern of type $(1,2)$ or $(2,1)$. For eight of these twelve patterns the answer is given by the Bell numbers. For the remaining four the answer is given by the Catalan numbers.

In the present paper we give a complete solution for the number of permutations avoiding a pair of patterns of type $(1,2)$ or $(2,1)$. We also conjecture the number of permutations avoiding the patterns in any set of three or more such patterns.

1. Introduction

Classically, a pattern is a permutation $\sigma \in \mathcal{S}_k$, and a permutation $\pi \in \mathcal{S}_n$ avoids σ if there is no subword of π that is order equivalent to σ. For example, $\pi \in \mathcal{S}_n$ avoids 132 if there is no $1 \le i < j < k \le n$ such that $\pi(i) < \pi(k) < \pi(j)$. We denote by $\mathcal{S}_n(\sigma)$ the set permutations in $\mathcal{S}_n$ that avoids σ.

In [6, Ch. 2.2.1] and [7, Ch. 5.1.4] Knuth shows that for any $\sigma \in S_3$, we have $|S_n(\sigma)| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. Later Simion and Schmidt [8] found the cardinality of $\mathcal{S}_n(P)$ for all $P \subseteq \mathcal{S}_3$.

In [1] Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. The motivation for Babson and Steingrímsson in introducing these patterns was the study of Mahonian statistics. Two examples of such patterns are 1-32 and 13-2 (1-32 and 13-2 are of type $(1,2)$ and $(2,1)$ respectively). A permutation $\pi = a_1a_2\cdots a_n$ avoids 1-32 if there are no subwords $a_ia_ja_{j+1}$ of π such that $a_i < a_{j+1} < a_j$. Similarly π avoids 13-2 if there are no subwords $a_ia_{i+1}a_j$ of π such that $a_i < a_j < a_{i+1}$.

Claesson [2] presented a complete solution for the number of permutations avoiding any single pattern of type $(1,2)$ or $(2,1)$ as follows.

Proposition 1 (Claesson [2])**.** *Let $n \in \mathbb{N}$. We have*

$$|\mathcal{S}_n(p)| = \begin{cases} B_n & \text{if } p \in \{1\text{-}23, 3\text{-}21, 12\text{-}3, 32\text{-}1, 1\text{-}32, 3\text{-}12, 21\text{-}3, 23\text{-}1\}, \\ C_n & \text{if } p \in \{2\text{-}13, 2\text{-}31, 13\text{-}2, 31\text{-}2\}, \end{cases}$$

where B_n and C_n are the nth Bell (# ways of placing n labelled balls into n indistinguishable boxes, see [9, A000110]*) and Catalan numbers, respectively.*

Date: January 21, 2004.

Key words and phrases. permutation, pattern avoidance.

In addition, Claesson gave some results for the number of permutations avoiding a pair of patterns.

Proposition 2 (Claesson [2]). *Let $n \in \mathbb{N}$. We have*

$$|\mathcal{S}_n(1\text{-}23,\ 12\text{-}3)| = B_n^*,\ |\mathcal{S}_n(1\text{-}23,\ 1\text{-}32)| = I_n,\ \text{and}\ |\mathcal{S}_n(1\text{-}23,\ 13\text{-}2)| = M_n,$$

where B_n^* is the nth Bessel number (# non-overlapping partitions of $[n]$ (see [4])), I_n is the number of involutions in $\mathcal{S}_n$, and M_n is the nth Motzkin number (# ways of drawing any number of nonintersecting chords among n points on a circle, see [9, A001006]).

This paper is organized as follows. In Section 2 we define the notion of a pattern and some other useful concepts. For a proof of Proposition 1 we could refer the reader to [2]. We will however prove Proposition 1 in Section 3 in the context of binary trees. The idea being that this will be a useful aid to understanding of the proofs of Section 4. In Section 4 we give a solution for the number of permutations avoiding any given pair of patterns of type $(1,2)$ or $(2,1)$. These results are summarized in the following table.

# pairs	2	2	4	34	8	2	4	4	4	2
$\lvert\mathcal{S}_n(p,q)\rvert$	0	$2(n-1)$	$\binom{n}{2}+1$	2^{n-1}	M_n	a_n	b_n	I_n	C_n	B_n^*

Here

$$\sum_{n\geq 0} a_n x^n = \frac{1}{1 - x - x^2 \sum_{n\geq 0} B_n^* x^n}$$

and

$$b_{n+2} = b_{n+1} + \sum_{k=0}^{n} \binom{n}{k} b_k.$$

Finally, in Section 5 we conjecture the sequences $\{\#\mathcal{S}_n(P)\}_n$ for sets P of three or more patterns of type $(1,2)$ or $(2,1)$.

2. Preliminaries

By an *alphabet* X we mean a non-empty set. An element of X is called a *letter*. A *word* over X is a finite sequence of letters from X. We consider also the *empty word*, that is, the word with no letters; it is denoted by ϵ. Let $w = x_1x_2\cdots x_n$ be a word over X. We call $|w| := n$ the *length* of w. A *subword* of w is a word $v = x_{i_1}x_{i_2}\cdots x_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Let $[n] := \{1,2,\dots,n\}$ (so $[0] = \emptyset$). A *permutation* of $[n]$ is bijection from $[n]$ to $[n]$. Let $\mathcal{S}_n$ be the set of permutations of $[n]$, and $\mathcal{S} = \cup_{n\geq 0}\mathcal{S}_n$. We shall usually think of a permutation π as the word $\pi(1)\pi(2)\cdots\pi(n)$ over the alphabet $[n]$.

Define the *reverse* of π by $\pi^r(i) = \pi(n+1-i)$, and define the *complement* of π by $\pi^c(i) = n+1-\pi(i)$, where $i \in [n]$.

For each word $w = x_1x_2\cdots x_n$ over the alphabet $\{1,2,3,4,\dots\}$ without repeated letters, we define the *reduction* of w, which we denote $\mathrm{red}(w)$, by

$$\mathrm{red}(w) = a_1a_2\cdots a_n\,, \quad \text{where} \quad a_i = |\{j \in [n] : x_j \leq x_i\}|.$$

Equivalently, red(w) is the permutation in $\mathcal{S}_n$ which is order equivalent to w. For example, red$(2659) = 1324$.

We may regard a *pattern* as a function from $\mathcal{S}_n$ to the set $\mathbb{N}$ of natural numbers. The patterns of main interest to us are defined as follows. Let $xyz \in \mathcal{S}_3$ and $\pi = a_1a_2\cdots a_n \in \mathcal{S}_n$, then

$$(x\text{-}yz)\,\pi \;=\; |\{a_ia_ja_{j+1} : \mathrm{red}(a_ia_ja_{j+1}) = xyz, 1 \le i < j < n\}|$$

and similarly $(xy\text{-}z)\,\pi = (z\text{-}yx)\,\pi^r$. For instance

$$(1\text{-}23)\,491273865 = |\{127, 138, 238\}| = 3.$$

A pattern $p = p_1\text{-}p_2\text{-}\cdots\text{-}p_k$ containing exactly $k-1$ dashes is said to be of type $(|p_1|, |p_2|, \ldots, |p_k|)$. For example, the pattern 142-5-367 is of type $(3, 1, 3)$, and any classical pattern of length k is of type $(\underbrace{1, 1, \ldots, 1}_{k})$.

We say that a permutation π *avoids* a pattern p if $p\,\pi = 0$. The set of all permutations in $\mathcal{S}_n$ that avoids p is denoted $\mathcal{S}_n(p)$ and, more generally, $\mathcal{S}_n(P) = \bigcap_{p\in P}\mathcal{S}_n(p)$ and $\mathcal{S}(P) = \bigcup_{n\ge 0}\mathcal{S}_n(P)$.

We extend the definition of reverse and complement to patterns the following way. Let us call π the *underlying permutation* of the pattern p if π is obtained from p by deleting all the dashes in p. If p is a pattern with underlying permutation π, then p^c is the pattern with underlying permutation π^c and with dashes at precisely the same positions as there are dashes in p. We define p^r as the pattern we get from regarding p as a word and reading it backwards. For example, $(1\text{-}23)^c = 3\text{-}21$ and $(1\text{-}23)^r = 32\text{-}1$. Observe that

$$\begin{aligned}\sigma \in \mathcal{S}_n(p) &\iff \sigma^r \in \mathcal{S}_n(p^r)\\ \sigma \in \mathcal{S}_n(p) &\iff \sigma^c \in \mathcal{S}_n(p^c).\end{aligned}$$

These observations of course generalize to $\mathcal{S}_n(P)$ for any set of patterns P.

The operations reverse and complement generates the dihedral group D_2 (the symmetry group of a rectangle). The orbits of D_2 in the set of patterns of type $(1, 2)$ or $(2, 1)$ will be called *symmetry classes*. For instance, the symmetry class of 1-23 is

$$\{1\text{-}23, 3\text{-}21, 12\text{-}3, 32\text{-}1\}.$$

We also talk about symmetry classes of sets of patterns (defined in the obvious way). For example, the symmetry class of $\{1\text{-}23, 3\text{-}21\}$ is

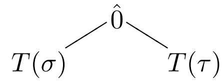
$$\{\{1\text{-}23, 3\text{-}21\}, \{32\text{-}1, 12\text{-}3\}\}.$$

A set of patterns P such that if $p, p' \in P$ then, for each n, $|\mathcal{S}_n(p)| = |\mathcal{S}_n(p')|$ is called a *Wilf-class*. For instance, by Proposition 1, the Wilf-class of 1-23 is

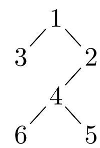
$$\{1\text{-}23, 3\text{-}21, 12\text{-}3, 32\text{-}1, 1\text{-}32, 3\text{-}12, 21\text{-}3, 23\text{-}1\}.$$

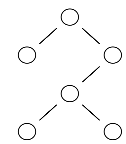
We also talk about Wilf-classes of sets of patterns (defined in the obvious way). It is clear that symmetry classes are Wilf-classes, but as we have seen the converse does not hold in general.

In what follows we will frequently use the well known bijection between increasing binary trees and permutations (e.g. see [10, p. 24]). Let π be any word on the alphabet $\{1,2,3,4,\ldots\}$ with no repeated letters. If $\pi \neq \epsilon$ then we can factor π as $\pi = \sigma\,\hat{0}\,\tau$, where $\hat{0}$ is the minimal element of π. Define $T(\epsilon) = \bullet$ (a leaf) and

$T(\pi) =$ 

In addition, we define $U(t)$ as the unlabelled counterpart of the labelled tree t. For instance

$T(316452) =$ 

$U \circ T(316452) =$ 

Note that, for sake of simplicity, the leafs are not displayed.

3. Single patterns

There are 3 symmetry classes and 2 Wilf-classes of single patterns. The details are as follows.

Proposition 3 (Claesson [2])**.** *Let $n \in \mathbb{N}$. We have*

$$|\mathcal{S}(p)| = \begin{cases} B_n & \text{if } p \in \{1\text{-}23, 3\text{-}21, 12\text{-}3, 32\text{-}1\}, \\ B_n & \text{if } p \in \{1\text{-}32, 3\text{-}12, 21\text{-}3, 23\text{-}1\}, \\ C_n & \text{if } p \in \{2\text{-}13, 2\text{-}31, 13\text{-}2, 31\text{-}2\}, \end{cases}$$

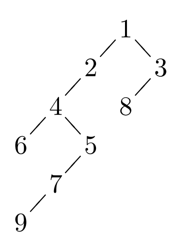
where B_n and C_n are the nth Bell and Catalan numbers, respectively.

Proof of the first case. Note that

$$\sigma 1 \tau \in \mathcal{S}(1\text{-}23) \iff \begin{cases} \mathrm{red}(\sigma) \in \mathcal{S}(1\text{-}23) \\ \mathrm{red}(\tau) \in \mathcal{S}(12) \\ \sigma 1 \tau \in \mathcal{S} \end{cases}$$

where of course $\mathcal{S}(12) = \{\epsilon, 1, 21, 321, 4321, \ldots\}$. This enables us to give a bijection Φ between $\mathcal{S}_n(1\text{-}23)$ and the set of partitions of $[n]$, by induction. Let $\Phi(\epsilon)$ be the empty partition. Let the first block of $\Phi(\sigma 1\tau)$ be the set of letters of 1τ, and let the rest of the blocks of $\Phi(\sigma 1 \tau)$ be as in $\Phi(\sigma)$. $\square$

The most transparent way to see the above correspondence is perhaps to view the permutation as an increasing binary tree. For instance, the tree

$T(649752183) =$ 

corresponds to the partition $\{\{1,3,8\},\{2\},\{4,5,7,9\},\{6\}\}$.

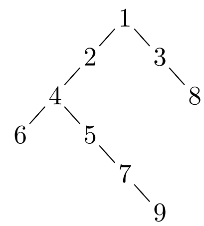
Proof of the second case. This case is analogous to the previous one. We have

$$\sigma 1 \tau \in \mathcal{S}(1\text{-}32) \iff \begin{cases} \mathrm{red}(\sigma) \in \mathcal{S}(1\text{-}32) \\ \mathrm{red}(\tau) \in \mathcal{S}(21) \\ \sigma 1 \tau \in \mathcal{S} \end{cases}$$

We give a bijection Φ between $\mathcal{S}_n(1\text{-}23)$ and the set of partitions of $[n]$, by induction. Let $\Phi(\epsilon)$ be the empty partition. Let the first block of $\Phi(\sigma 1 \tau)$ be the set of letters of 1τ, and let the rest of the blocks of $\Phi(\sigma 1 \tau)$ be as in $\Phi(\sigma)$. $\square$

As an example, the tree

$T(645792138) =$



corresponds to the partition $\{\{1,3,8\},\{2\},\{4,5,7,9\},\{6\}\}$.

Now that we have seen the structure of $\mathcal{S}(1\text{-}23)$ and $\mathcal{S}(1\text{-}32)$, it is trivial to give a bijection between the two sets. Indeed, if $\Theta : \mathcal{S}(1\text{-}23) \to \mathcal{S}(1\text{-}32)$ is given by $\Theta(\epsilon) = \epsilon$ and $\Theta(\sigma 1 \tau) = \Theta(\sigma)\, 1\, \tau^r$ then Θ is such a bijection. Actually Θ is its own inverse.

Proof of the third case. It is plain that a permutation avoids 2-13 if and only if it avoids 2-1-3 (see [2]). Note that

$$\sigma 1 \tau \in \mathcal{S}(2\text{-}1\text{-}3) \iff \begin{cases} \mathrm{red}(\sigma), \mathrm{red}(\tau) \in \mathcal{S}(2\text{-}1\text{-}3) \\ \tau > \sigma \\ \sigma 1 \tau \in \mathcal{S} \end{cases}$$

where $\tau > \sigma$ means that any letter of τ is greater than any letter of σ. Hence we get a unique labelling of the binary tree corresponding to $\sigma 1 \tau$, that is, if $\pi_1, \pi_2 \in \mathcal{S}(2\text{-}1\text{-}3)$ and $U \circ T(\pi_1) = U \circ T(\pi_2)$ then $\pi_1 = \pi_2$. It is well known that there are exactly C_n (unlabelled) binary trees with n (internal) nodes. The validity of the last statement can be easily deduced from the following simple bijection between Dyck words and binary trees. Fixing notation, we let the set of Dyck words be the smallest set of words over $\{u, d\}$ that contains the empty word and is closed under $(\alpha, \beta) \mapsto u\alpha d\beta$. Now the promised bijection is given by $\Psi(\bullet) = \epsilon$ and

$$\Psi\left(\begin{smallmatrix} & \circ & \\ \swarrow & & \searrow \\ L & & R \end{smallmatrix}\right) = u\Psi(L)d\Psi(R).$$

$\square$

4. Pairs of patterns

There are $\binom{12}{2} = 66$ pairs of patterns altogether. It turns out that there are 21 symmetry classes and 10 Wilf-classes. The details are given in Table 1, and the numbering of the symmetry classes in the titles of the subsections below is taken from that table.

	$\{p,q\}$	$\lvert\mathcal{S}_n(p,q)\rvert$
1	1-23, 32-1 3-21, 12-3	0
2	1-23, 3-21 32-1, 12-3	$2(n-1)$
3	1-23, 2-31 3-21, 2-13 12-3, 31-2 32-1, 13-2	$\binom{n}{2}+1$
4a	1-23, 2-13 3-21, 2-31 12-3, 13-2 32-1, 31-2	2^{n-1}
4b	1-23, 23-1 3-21, 21-3 12-3, 3-12 32-1, 1-32	2^{n-1}
4c	1-23, 31-2 3-21, 13-2 12-3, 2-31 32-1, 2-13	2^{n-1}
4d	1-32, 2-13 3-12, 2-31 13-2, 21-3 23-1, 31-2	2^{n-1}
4e	1-32, 2-31 3-12, 2-13 31-2, 21-3 23-1, 13-2	2^{n-1}
4f	1-32, 3-12 23-1, 21-3	2^{n-1}
4g	1-32, 23-1 3-12, 21-3	2^{n-1}
4h	1-32, 31-2 3-12, 13-2 21-3, 2-31 23-1, 2-13	2^{n-1}
4i	2-13, 2-31 31-2, 13-2	2^{n-1}
4j	2-13, 13-2 2-31, 31-2	2^{n-1}
4k	2-13, 31-2 2-31, 13-2	2^{n-1}
5a	1-23, 13-2 3-21, 31-2 12-3, 2-13 32-1, 2-31	M_n (Motzkin no.)
5b	1-23, 21-3 3-21, 23-1 12-3, 1-32 32-1, 3-12	M_n (Motzkin no.)
6	1-32, 21-3 3-12, 23-1	a_n
7	1-23, 3-12 3-21, 1-32 23-1, 12-3 32-1, 21-3	b_n
8	1-23, 1-32 3-21, 3-12 21-3, 12-3 32-1, 23-1	I_n (# involutions)
9	1-32, 13-2 3-12, 31-2 21-3, 2-13 23-1, 2-31	C_n (Catalan no.)
10	1-23, 12-3 3-21, 32-1	B_n^* (Bessel no.)

TABLE 1. The cardinality of $\mathcal{S}_n(P)$ for $|P|=2$.

Symmetry class 1. We have

$$\sigma 1 \tau \in \mathcal{S}(1\text{-}23, 32\text{-}1) \iff \begin{cases} \mathrm{red}(\sigma) \in \mathcal{S}(21, 1\text{-}23) \\ \mathrm{red}(\tau) \in \mathcal{S}(12, 32\text{-}1) \\ \sigma 1 \tau \in \mathcal{S} \end{cases}$$

The result now follows from $\mathcal{S}(21, 1\text{-}23) = \{\epsilon, 1, 12\}$ and $\mathcal{S}(12, 32\text{-}1) = \{\epsilon, 1, 21\}$.

Symmetry class 2. Since 3-21 is the complement of 1-23, the cardinality of $\mathcal{S}_n(1\text{-}23, 3\text{-}21)$ is twice the number of permutations in $\mathcal{S}_n(1\text{-}23, 3\text{-}21)$ in which 1 precedes n. In addition, 1 and n must be adjacent letters in a permutation avoiding 1-23 and 3-21. Let $\sigma 1 n \tau$ be such a permutation. Note that τ must be both increasing and decreasing, that is, $\tau \in \{\epsilon, 2, 3, 4, \ldots, n-1\}$, so there are $n-1$ choices for τ. Furthermore, there is exactly one permutation in $\mathcal{S}_n(1\text{-}23, 3\text{-}21)$ of the form $\sigma 1 n$, namely $(\lceil \frac{n+1}{2} \rceil, \ldots, n-2, 3, n-1, 2, n, 1)$, and similarly there is exactly one of the form $\sigma 1 n k$ for each $k \in \{2, 3, \ldots, n-1\}$. This completes our argument.

Symmetry class 3. Note that

$$\sigma 1 \tau \in \mathcal{S}(1\text{-}23, 2\text{-}31) \iff \begin{cases} \mathrm{red}(\sigma), \mathrm{red}(\tau) \in \mathcal{S}(12) \\ \sigma 1 \tau \in \mathcal{S}(2\text{-}31) \end{cases}$$

It is now rather easy to see that $\pi \in \mathcal{S}_n(1\text{-}23, 2\text{-}31)$ if and only if $\pi = n \cdots 21$ or π is constructed in the following way. Choose i and j such that $1 \le j < i \le n$. Let $\pi(i-1) = 1$, $\pi(i) = n+1-j$ and arrange the rest of the elements so that $\pi(1) > \pi(2) > \cdots > \pi(i-1)$ and $\pi(i) > \pi(i+1) > \cdots > \pi(n)$ (this arrangement is unique). Since there are $\binom{n}{2}$ ways of choosing i and j we get the desired result.

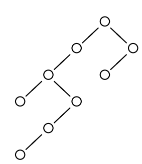
Symmetry class 4a. We have

$$\sigma 1 \tau \in \mathcal{S}(1\text{-}23, 2\text{-}13) \iff \begin{cases} \mathrm{red}(\sigma) \in \mathcal{S}(1\text{-}23, 2\text{-}13) \\ \mathrm{red}(\tau) \in \mathcal{S}(12) \\ \sigma > \tau \\ \sigma 1 \tau \in \mathcal{S}, \end{cases}$$

where $\sigma > \tau$ means that any letter of τ is greater than any letter of σ. This enables us to give a bijection between $\mathcal{S}_n(1\text{-}23, 2\text{-}13)$ and the set of compositions (ordered formal sums) of n. Indeed, such a bijection Ψ is given by $\Psi(\epsilon) = \epsilon$ and $\Psi(\sigma 1 \tau) = \Psi(\sigma) + |1\tau|$.

As an example, the tree

$$U \circ T(958764132) \;=$$



corresponds to the composition $1 + 4 + 1 + 3$ of 9.

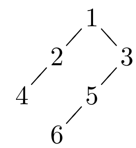
Symmetry class 4b. We have

$$\sigma 1 \tau \in \mathcal{S}(1\text{-}23, 23\text{-}1) \iff \begin{cases} \mathrm{red}(\sigma), \mathrm{red}(\tau) \in \mathcal{S}(12) \\ \sigma 1 \tau \in \mathcal{S} \end{cases}$$

Hence a permutation in $\mathcal{S}(1\text{-}23, 23\text{-}1)$ is given by the following procedure. Choose a subset $S \subseteq \{2, 3, 4, \ldots, n\}$, let σ be the word obtained by writing the elements of S in decreasing order, and let τ be the word obtained by writing the elements of $\{2, 3, 4, \ldots, n\} \setminus S$ in decreasing order.

For instance, the tree

$$T(421653) \;=$$



corresponds to the subset $\{2,4\}$ of $\{2,3,4,5,6\}$.

Symmetry class 4c. This case is essentially identical to the case dealt with in (4a).

Symmetry class 4d. The bijection Θ between $\mathcal{S}(1\text{-}23)$ and $\mathcal{S}(1\text{-}32)$ (see page [CM1]-5) provides a bijection between $\mathcal{S}_n(1\text{-}32, 2\text{-}13)$ and $\mathcal{S}_n(1\text{-}23, 2\text{-}13)$. Consequently the result follows from (4a).

Symmetry class 4e. We have

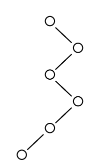
$$\sigma 1 \tau \in \mathcal{S}(3\text{-}12, 2\text{-}13) \iff \begin{cases} \mathrm{red}(\sigma), \mathrm{red}(\tau) \in \mathcal{S}(3\text{-}12, 2\text{-}13) \\ \sigma = \epsilon \;\text{ or }\; \tau = \epsilon \\ \sigma 1 \tau \in \mathcal{S} \end{cases}$$

Thus a bijection between $\mathcal{S}_n(3\text{-}12, 2\text{-}13)$ and $\{0,1\}^{n-1}$ is given by $\Psi(\epsilon) = \epsilon$ and

$$\Psi(\sigma 1 \tau) = x \Psi(\sigma\tau) \text{ where } x = \begin{cases} 1 & \text{if } \sigma \neq \epsilon, \\ 0 & \text{if } \tau \neq \epsilon, \\ \epsilon & \text{otherwise.} \end{cases}$$

As an example, the tree

$$U \circ T(136542) \;=$$



corresponds to $01011 \in \{0,1\}^5$.

Symmetry class 4f. Since 3-12 is the complement of 1-32, the cardinality of $\mathcal{S}_n(1\text{-}32, 3\text{-}12)$ is twice the number of permutations in $\mathcal{S}_n(1\text{-}32, 3\text{-}12)$ in which 1 precedes n. In addition, n must be the last letter in such a permutation or else a hit of 1-32 would be formed. We have

$$\begin{aligned} \sigma 1 \tau n \in \mathcal{S}(1\text{-}32, 3\text{-}12) \quad &\iff \quad \begin{cases} \mathrm{red}(\sigma 1 \tau) \in \mathcal{S}(1\text{-}32, 3\text{-}12) \\ \mathrm{red}(\tau) \in \mathcal{S}(21) \\ \sigma 1 \tau \in \mathcal{S} \end{cases} \\ &\iff \quad \begin{cases} \mathrm{red}(\sigma) \in \mathcal{S}(1\text{-}32, 3\text{-}12) \\ \mathrm{red}(\tau) \in \mathcal{S}(21) \\ \sigma < \tau \\ \sigma 1 \tau \in \mathcal{S} \end{cases} \end{aligned}$$

The rest of the proof follows the same lines as the proof of (4a).

Symmetry class 4g. We can copy almost verbatim the proof of (4e); indeed, it is easy to see that $\mathcal{S}_n(1\text{-}32, 23\text{-}1) = \mathcal{S}_n(1\text{-}32, 2\text{-}31)$.

Symmetry class 4h. We can copy almost verbatim the proof of (4f); indeed, it is easy to see that $\mathcal{S}_n(1\text{-}32, 31\text{-}2) = \mathcal{S}_n(1\text{-}32, 3\text{-}12)$.

Symmetry class 4i. $|\mathcal{S}_n(2\text{-}13, 2\text{-}31)| = |\mathcal{S}_n(2\text{-}1\text{-}3, 2\text{-}3\text{-}1)| = 2^{n-1}$ by [8, Lemma 5(d)].

Symmetry class 4j. $|\mathcal{S}_n(2\text{-}13, 13\text{-}2)| = |\mathcal{S}_n(1\text{-}3\text{-}2, 2\text{-}1\text{-}3)| = 2^{n-1}$ by [8, Lemma 5(b)].

Symmetry class 4k. $|\mathcal{S}_n(2\text{-}13, 31\text{-}2)| = |\mathcal{S}_n(2\text{-}1\text{-}3, 3\text{-}1\text{-}2)| = 2^{n-1}$ by [8, Lemma 5(c)].

Symmetry class 5a. See Proposition 2.

Symmetry class 5b. We give a bijection

$$\Lambda : \mathcal{S}_n(1\text{-}23, 21\text{-}3) \to \mathcal{S}_n(1\text{-}23, 13\text{-}2)$$

by means of induction. Let $\pi \in \mathcal{S}_n(1\text{-}23, 21\text{-}3)$. Define $\Lambda(\pi) = \pi$ for $n \leq 1$. Assume $n \geq 2$ and $\pi = a_1 a_2 \cdots a_n$. It is plain that either $a_1 = n$ or $a_2 = n$, so we can define $\Lambda(\pi)$ by

$$\begin{cases} (a'_1+1, \ldots, a'_{n-1}+1, a'_{n-2}+1, 1) & \text{if } \begin{cases} a_1 = n \quad \text{and} \\ a'_1 \cdots a'_{n-1} = \Lambda(a_2 a_3 a_4 \cdots a_n), \end{cases} \\ (a'_1+1, \ldots, a'_{n-1}+1, 1, a'_{n-2}+1) & \text{if } \begin{cases} a_2 = n \quad \text{and} \\ a'_1 \cdots a'_{n-1} = \Lambda(a_1 a_3 a_4 \cdots a_n). \end{cases} \end{cases}$$

Observing that if $\sigma \in \mathcal{S}_n(1\text{-}23, 13\text{-}2)$ then $\sigma(n-1) = 1$ or $\sigma(n) = 1$, it easy to find the inverse of Λ.

Symmetry class 6. In [2] Claesson introduced the notion of a monotone partition. A partition is *monotone* if its non-singleton blocks can be written in increasing order of their least element and increasing order of their greatest element, simultaneously. He then proved that monotone partitions and non-overlapping partitions are in one-to-one correspondence. Non-overlapping partitions were first studied by Flajolet and Schot in [4]. A partition π is *non-overlapping* if for no two blocks A and B of π we have $\min A < \min B < \max A < \max B$. Let B^*_n be the number of non-overlapping partitions of $[n]$; this number is called the nth *Bessel number*. Proposition 2 tells us that there is a bijection between non-overlapping partitions and permutations avoiding 1-23 and 12-3. Below we define a new class of partitions called strongly monotone partitions and then show that there is a bijection between strongly monotone partitions and permutations avoiding 1-32 and 21-3.

Definition 4. Let π be an arbitrary partition whose blocks $\{A_1, \ldots, A_k\}$ are ordered so that for all $i \in [k-1]$, $\min A_i > \min A_{i+1}$. If $\max A_i > \max A_{i+1}$ for all $i \in [k-1]$, then we call π a *strongly monotone partition*.

In other words a partition is strongly monotone if its blocks can be written in increasing order of their least element and increasing order of their greatest element, simultaneously. Let us denote by a_n the number of strongly monotone partitions of $[n]$. The sequence $\{a_n\}_0^\infty$ starts with

$$1, 1, 2, 4, 9, 22, 58, 164, 496, 1601, 5502, 20075, 77531, 315947, 1354279.$$

It is routine to derive the continued fraction expansion

$$\sum_{n\geq 0} a_n x^n = \cfrac{1}{1 - 1\cdot x - \cfrac{x^2}{1 - 1\cdot x - \cfrac{x^2}{1 - 2\cdot x - \cfrac{x^2}{1 - 3\cdot x - \cfrac{x^2}{1 - 4\cdot x - \cfrac{x^2}{\ddots}}}}}}$$

using the standard machinery of Flajolet [3] and Françon and Viennot [5]. One can also note that there is a one-to-one correspondence between strongly monotone partitions and non-overlapping partition, π, such that if $\{x\}$ and B are blocks of π then either $x < \min B$ or $\max B < x$. In addition, we observe that

$$\sum_{n\geq 0} a_n x^n = \frac{1}{1 - x - x^2 B^*(x)},$$

where $B^*(x) = \sum_{n\geq 0} B_n^* x^n$ is the ordinary generating function for the Bessel numbers.

Suppose $\pi \in \mathcal{S}_n$ has $k+1$ left-to-right minima $1, 1', 1'', \ldots, 1^{(k)}$ such that

$$1 < 1' < 1'' < \cdots < 1^{(k)}, \text{ and } \pi = 1^{(k)}\tau^{(k)} \cdots 1'\tau' 1\tau.$$

Then π avoids 1-32 if and only if, for each i, $\tau^{(i)} \in \mathcal{S}(21)$. If π avoids 1-32 and $x_i = \max 1^{(i)}\tau^{(i)}$ then π avoids 21-3 precisely when $x_0 < x_1 < \cdots < x_k$. This follows from observing that the only potential (21-3)-subwords of π are $x_{i+1}1^{(k)}x_j$ with $j \leq i$.

Mapping π to the partition $\{1\sigma, 1'\sigma', \ldots, 1^{(k)}\tau^{(k)}\}$ we thus get a one-to-one correspondence between permutations in $\mathcal{S}_n(1\text{-}32, 21\text{-}3)$ and strongly monotone partitions of $[n]$.

Symmetry class 7. Let the sequence $\{b_n\}$ be defined by $b_0 = 1$ and, for $n \geq -2$,

$$b_{n+2} = b_{n+1} + \sum_{k=0}^{n} \binom{n}{k} b_k.$$

The first few of the numbers b_n are

$$1, 1, 2, 4, 9, 23, 65, 199, 654, 2296, \ldots$$

Suppose $\pi \in \mathcal{S}_n$ has $k+1$ left-to-right minima $1, 1', 1'', \ldots, 1^{(k)}$ such that

$$1 < 1' < 1'' < \cdots < 1^{(k)}, \text{ and } \pi = 1^{(k)}\tau^{(k)} \cdots 1'\tau' 1\tau.$$

Then π avoids 1-23 if and only if, for each i, $\tau^{(i)} \in \mathcal{S}(12)$. If π avoids 1-23 and $x_i = \max 1^{(i)}\tau^{(i)}$ then π avoids 3-12 precisely when

$$j > i \ \text{ and } \ x_i \neq 1^{(i)} \implies x_j < x_i.$$

This follows from observing that the only potential (3-12)-subwords of π are $x_j 1^{(k)} x_i$ with $j \leq i$. Thus we have established

$$\sigma 1 \tau \in \mathcal{S}_n(1\text{-}23, 3\text{-}12) \iff \begin{cases} \mathrm{red}(\sigma) \in \mathcal{S}(1\text{-}23, 3\text{-}12) \\ \tau \neq \epsilon \quad \Rightarrow \quad \tau = \tau' n \text{ and } \mathrm{red}(\tau') \in \mathcal{S}(12) \\ \sigma 1 \tau \in \mathcal{S}_n \end{cases}$$

If we know that $\sigma 1 \tau' n \in \mathcal{S}_n(1\text{-}23, 3\text{-}12)$ and $\mathrm{red}(\tau') \in \mathcal{S}_k(12)$ then there are $\binom{n-2}{k}$ candidates for τ'. In this way the recursion follows.

Symmetry class 8. See Proposition 2.

Symmetry class 9. $\mathcal{S}_n(1\text{-}32, 13\text{-}2) = \mathcal{S}_n(1\text{-}3\text{-}2)$.

Symmetry class 10. See Proposition 2.

5. More than two patterns

Let P be a set of patterns of type $(1, 2)$ or $(2, 1)$. With the aid of a computer we have calculated the cardinality of $\mathcal{S}_n(P)$ for sets P of three or more patterns. From these results we arrived at the plausible conjectures of table 2 (some of which are trivially true). We use the notation $m \times n$ to express that there are m symmetric classes each of which contains n sets. Moreover, we denote by F_n the nth *Fibonacci number* $(F_0 = F_1 = 1, F_{n+1} = F_n + F_{n-1})$.

Acknowledgements

The first author wishes to express his gratitude towards Einar Steingrímsson, Kimmo Eriksson, and Mireille Bousquet-Mélou; Einar for his guidance and infectious enthusiasm; Kimmo for useful suggestions and a very constructive discussion on the results of this paper; Mireille for her great hospitality during a stay at LaBRI, where some of the work on this paper was done.
We would like to thank N. J. A. Sloane for his excellent web site "The On-Line Encyclopedia of Integer Sequences"

`http://www.research.att.com/~njas/sequences/`.

It is simply an indispensable tool for all studies concerned with integer sequences.

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For $|P| = 3$ there are 220 sets, 55 symmetry classes and 9 Wilf-classes.

cardinality	# sets
0	7×4
3	1×4
n	24×4
$1 + \binom{n}{2}$	2×4
F_n	7×4
$\binom{n}{[n/2]}$	1×4
$2^{n-2} + 1$	1×4
2^{n-1}	10×4
M_n	2×4

For $|P| = 4$ there are 495 sets, 135 symmetry classes, and 9 Wilf-classes.

cardinality	# sets
0	$1 \times 1 + 6 \times 2 + 30 \times 4$
2	$2 \times 1 + 5 \times 2 + 35 \times 4$
3	1×4
n	$37 \times 4 + 1 \times 2$
$1 + \binom{n}{2}$	1×4
F_n	$9 \times 4 + 1 \times 2$
$\binom{n}{[n/2]}$	1×2
$2^{n-2} + 1$	1×2
2^{n-1}	$1 \times 4 + 3 \times 2$

For $|P| = 5$ there are 792 sets, 198 symmetry classes, and 5 Wilf-classes.

cardinality	# sets
0	84×4
1	16×4
2	74×4
n	20×4
F_n	4×4

For $|P| = 6$ there are 924 sets, 246 symmetry classes, and 4 Wilf-classes.

cardinality	# sets
0	$17 \times 2 + 124 \times 4$
1	$4 \times 2 + 38 \times 4$
2	$7 \times 2 + 51 \times 4$
n	$1 \times 2 + 3 \times 4$
F_n	1×2

For $|P| = 7$ there are 792 sets, 198 symmetry classes, and 3 Wilf-classes.

cardinality	# sets
0	140×4
1	40×4
2	18×4

For $|P| = 8$ there are 495 sets, 135 symmetry classes, and 3 Wilf-classes.

cardinality	# sets
0	$2 \times 1 + 14 \times 2 + 94 \times 4$
1	$4 \times 2 + 18 \times 4$
2	$1 \times 1 + 2 \times 4$

For $|P| = 9$ there are 220 sets, 55 symmetry classes, and 2 Wilf-classes.

cardinality	# sets
0	50×4
1	5×4

For $|P| = 10$ there are 66 sets, 21 symmetry classes, and 2 Wilf-classes.

cardinality	# sets
0	$8 \times 2 + 12 \times 4$
1	1×2

For $|P| = 11$ there are 12 sets, 3 symmetry classes, and 1 Wilf-class.

cardinality	# sets
0	3×4

For $|P| = 12$ there is 1 set, 1 symmetry class, and 1 Wilf-class.

cardinality	# sets
0	1×1

TABLE 2. The cardinality of $\mathcal{S}_n(P)$ for $|P| > 2$.

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[CM2]

COUNTING OCCURRENCES OF A PATTERN OF TYPE $(1,2)$ OR $(2,1)$ IN PERMUTATIONS

ANDERS CLAESSON AND TOUFIK MANSOUR

Abstract. Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. Claesson presented a complete solution for the number of permutations avoiding any single pattern of type $(1,2)$ or $(2,1)$. For eight of these twelve patterns the answer is given by the Bell numbers. For the remaining four the answer is given by the Catalan numbers.

With respect to being equidistributed there are three different classes of patterns of type $(1,2)$ or $(2,1)$. We present a recursion for the number of permutations containing exactly one occurrence of a pattern of the first or the second of the aforementioned classes, and we also find an ordinary generating function for these numbers. We prove these results both combinatorially and analytically. Finally, we give the distribution of any pattern of the third class in the form of a continued fraction, and we also give explicit formulas for the number of permutations containing exactly r occurrences of a pattern of the third class when $r \in \{1,2,3\}$.

1. Introduction and preliminaries

Let $[n] = \{1,2,\dots,n\}$ and denote by $\mathcal{S}_n$ the set of permutations of $[n]$. We shall view permutations in $\mathcal{S}_n$ as words with n distinct letters in $[n]$.

Classically, a pattern is a permutation $\sigma \in \mathcal{S}_k$, and an occurrence of σ in a permutation $\pi = a_1a_2\cdots a_n \in \mathcal{S}_n$ is a subword of π that is order equivalent to σ. For example, an occurrence of 132 is a subword $a_ia_ja_k$ $(1 \le i < j < k \le n)$ of π such that $a_i < a_k < a_j$. We denote by $s_\sigma^r(n)$ the number of permutations in $\mathcal{S}_n$ that contain exactly r occurrences of the pattern σ.

In the last decade much attention has been paid to the problem of finding the numbers $s_\sigma^r(n)$ for a fixed $r \ge 0$ and a given pattern σ (see [1, 2, 4, 6, 7, 8, 11, 13, 14, 16, 17, 18, 19, 20, 21]). Most of the authors consider only the case $r = 0$, thus studying permutations *avoiding* a given pattern. Only a few papers consider the case $r > 0$, usually restricting themselves to patterns of length 3. Using two simple involutions (*reverse* and *complement*) on $\mathcal{S}_n$ it is immediate that with respect to being equidistributed, the six patterns of length three fall into the two classes $\{123, 321\}$ and $\{132, 213, 231, 312\}$. Noonan [15] proved that $s_{123}^1(n) = \frac{3}{n}\binom{2n}{n-3}$. A general approach to the problem was suggested by Noonan and Zeilberger [16]; they gave another proof of Noonan's result, and conjectured that

$$s_{123}^2(n) = \frac{59n^2 + 117n + 100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$$

Date: January 21, 2004.

and $s^1_{132}(n) = \binom{2n-3}{n-3}$. The latter conjecture was proved by Bóna in [7]. A conjecture of Noonan and Zeilberger states that $s^r_\sigma(n)$ is P-recursive in n for any r and σ. It was proved by Bóna [5] for $\sigma = 132$.

Mansour and Vainshtein [14] suggested a new approach to this problem in the case $\sigma = 132$, which allows one to get an explicit expression for $s^r_{132}(n)$ for any given r. More precisely, they presented an algorithm that computes the generating function $\sum_{n\geq 0} s^r_{132}(n)x^n$ for any $r \geq 0$. To get the result for a given r, the algorithm performs certain routine checks for each element of the symmetric group S_{2r}. The algorithm has been implemented in C, and yields explicit results for $1 \leq r \leq 6$.

In [3] Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. The motivation for Babson and Steingrímsson in introducing these patterns was the study of Mahonian permutation statistics. Two examples of (generalized) patterns are 1-32 and 13-2. An occurrence of 1-32 in a permutation $\pi = a_1a_2\cdots a_n$ is a subword $a_ia_ja_{j+1}$ of π such that $a_i < a_{j+1} < a_j$. Similarly, an occurrence of 13-2 is a subword $a_ia_{i+1}a_j$ of π such that $a_i < a_j < a_{i+1}$.

For each word $w = x_1x_2\cdots x_n$ over the alphabet $\{1, 2, 3, 4, \ldots\}$ without repeated letters, we define the *reduction* of w, which we denote $\operatorname{red}(w)$, by

$$\operatorname{red}(w) = a_1a_2\cdots a_n\,, \quad \text{where} \quad a_i = |\{j \in [n] : x_j \leq x_i\}|.$$

Equivalently, $\operatorname{red}(w)$ is the permutation in $\mathcal{S}_n$ which is order equivalent to w. For example, $\operatorname{red}(2659) = 1324$. For $xyz \in \mathcal{S}_3$ and $\pi = a_1a_2\cdots a_n \in \mathcal{S}_n$ we define

$$(x\text{-}yz)\,\pi = \operatorname{card}\{\, a_ia_ja_{j+1} : \operatorname{red}(a_ia_ja_{j+1}) = xyz,\ 1 \leq i < j < n \,\}$$

and, similarly,

$$(xy\text{-}z)\,\pi = \operatorname{card}\{\, a_ia_{i+1}a_j : \operatorname{red}(a_ia_{i+1}a_j) = xyz,\ 2 < i+1 < j \leq n \,\}.$$

For any word (finite sequence of letters), w, we denote by $|w|$ the length of w, that is, the number of letters in w. A pattern $\sigma = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ containing exactly $k-1$ dashes is said to be of type $(|\sigma_1|, |\sigma_2|, \ldots, |\sigma_k|)$. For example, the pattern 142-5-367 is of type $(3, 1, 3)$, and any classical pattern of length k is of type $(\underbrace{1, 1, \ldots, 1}_{k})$.

In [11] Elizalde and Noy presented the following theorem regarding the distribution of the number of occurrences of any pattern of type (3).

Theorem 1 (Elizalde and Noy [11])**.** *Let $h(x) = \sqrt{(x-1)(x+3)}$. Then*

$$\sum_{\pi\in\mathcal{S}} x^{(123)\pi}\frac{t^{|\pi|}}{|\pi|!} = \frac{2h(x)e^{\frac{1}{2}(h(x)-x+1)t}}{h(x)+x+1+(h(x)-x-1)e^{h(x)t}},$$

$$\sum_{\pi\in\mathcal{S}} x^{(213)\pi}\frac{t^{|\pi|}}{|\pi|!} = \frac{1}{1-\int_0^t e^{(x-1)z^2/2}dz}.$$

The easy proof of the following proposition can be found in [9].

Proposition 2 (Claesson [9])**.** *With respect to being equidistributed, the twelve patterns of type* $(1, 2)$ *or* $(2, 1)$ *fall into the three classes*

$$\{\ 1\text{-}23,\ 3\text{-}21,\ 12\text{-}3,\ 32\text{-}1\ \},$$
$$\{\ 1\text{-}32,\ 3\text{-}12,\ 21\text{-}3,\ 23\text{-}1\ \},$$
$$\{\ 2\text{-}13,\ 2\text{-}31,\ 13\text{-}2,\ 31\text{-}2\ \}.$$

In the subsequent discussion we refer to the classes of the proposition above (in the order that they appear) as Class 1, 2 and 3 respectively.

Claesson [9] also gave a solution for the number of permutations avoiding any pattern of the type $(1,2)$ or $(2,1)$ as follows.

Proposition 3 (Claesson [9])**.** *Let $n \in \mathbb{N}$. We have*

$$|\mathcal{S}_n(\sigma)| = \begin{cases} B_n & \text{if } \sigma \in \{1\text{-}23, 3\text{-}21, 12\text{-}3, 32\text{-}1, 1\text{-}32, 3\text{-}12, 21\text{-}3, 23\text{-}1\}, \\ C_n & \text{if } \sigma \in \{2\text{-}13, 2\text{-}31, 13\text{-}2, 31\text{-}2\}, \end{cases}$$

where B_n and C_n are the nth Bell and Catalan numbers, respectively.

In particular, since B_n is not P-recursive in n, this result implies that for generalized patterns the conjecture that $s_\sigma^r(n)$ is P-recursive in n is false for $r = 0$ and, for example, $\sigma = 1\text{-}23$.

This paper is organized as follows. In Section 2 we find a recursion for the number of permutations containing exactly one occurrence of a pattern of Class 1, and we also find an ordinary generating function for these numbers. We prove these results both combinatorially and analytically. Similar results are also obtained for patterns of Class 2. In Section 3 we give the distribution of any pattern of Class 3 in the form of a continued fraction, and we also give explicit formulas for the number of permutations containing exactly r occurrences of a pattern of Class 3 when $r \in \{1, 2, 3\}$.

2. Counting occurrences of a pattern of Class 1 or 2

Theorem 4. *Let $u_1(n)$ be the number of permutations of length n containing exactly one occurrence of the pattern 1-23 and let B_n be the nth Bell number. The numbers $u_1(n)$ satisfy the recurrence*

$$u_1(n+2) = 2u_1(n+1) + \sum_{k=0}^{n-1} \binom{n}{k} \big[u_1(k+1) + B_{k+1}\big],$$

whenever $n \geq -1$, with the initial condition $u_1(0) = 0$.

Proof. Each permutation $\pi \in \mathcal{S}_{n+2}^1(1\text{-}23)$ contains a unique subword abc such that $a < b < c$ and bc is a segment of π. Let x be the last letter of π and define the sets $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ by

$$\pi \in \begin{cases} \mathcal{T} & \text{if } x = 2, \\ \mathcal{T}' & \text{if } x \neq 2 \text{ and } a = 1, \\ \mathcal{T}'' & \text{if } x \neq 2 \text{ and } a \neq 1. \end{cases}$$

Then $\mathcal{S}_{n+2}^1(1\text{-}23)$ is the disjoint union of $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$, so

$$u_1(n+2) = |\mathcal{T}| + |\mathcal{T}'| + |\mathcal{T}''|.$$

Since removing/adding a trailing 2 from/to a permutation does not affect the number of hits of 1-23, we immediately get

$$|\mathcal{T}| = u_1(n+1).$$

For the cardinality of $\mathcal{T}'$ we observe that if $x \neq 2$ and $a = 1$ then $b = 2$: If the letter 2 precedes the letter 1 then every hit of 1-23 with $a = 1$ would cause an additional hit of 1-23 with $a = 2$ contradicting the uniqueness of the hit of 1-23; if 1 precedes 2 then $a = 1$ and $b = 2$. Thus we can factor any permutation $\pi \in \mathcal{T}'$

uniquely in the form $\pi = \sigma 2 \tau$, where σ is (1-23)-avoiding, the letter 1 is included in σ, and τ is nonempty and (12)-avoiding. Owing to Proposition 3 we have showed

$$|\mathcal{T}'| = \sum_{k=0}^{n-1} \binom{n}{k} B_{k+1}.$$

Suppose $\pi \in \mathcal{T}''$. Since $x \neq 2$ and $a \neq 1$ we can factor π uniquely in the form $\pi = \sigma 1 \tau$, where σ contains exactly one occurrence of 1-23, the letter 2 is included in σ, and τ is nonempty and (12)-avoiding. Consequently,

$$|\mathcal{T}''| = \sum_{k=0}^{n} \binom{n}{k} u_1(k+1),$$

which completes the proof. $\square$

Example 5. Let us consider all permutations of length 5 that contain exactly one occurrence of 1-23, and give a small illustration of the proof of Theorem 4. If $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ are defined as above then

$$\mathcal{T} = \underline{135}4|2 \quad \underline{1}4\underline{35}|2 \quad \underline{145}3|2 \quad \underline{1}5\underline{34}|2 \quad 4\underline{135}|2 \quad 5\underline{134}|2 \quad \underline{345}1|2$$

$$\mathcal{T}' = \begin{array}{llllll} \underline{1}|\underline{25}43 & \underline{1}3|\underline{25}4 & \underline{1}4|\underline{25}3 & \underline{1}43|\underline{25} & \underline{1}5|\underline{24}3 & \underline{1}53|\underline{24} \\ \underline{1}54|\underline{23} & 3\underline{1}|\underline{25}4 & 3\underline{1}4|\underline{25} & 3\underline{1}5|\underline{24} & 34\underline{1}|\underline{25} & 35\underline{1}|\underline{24} \\ 4\underline{1}|\underline{25}3 & 4\underline{1}3|\underline{25} & 4\underline{1}5|\underline{23} & 43\underline{1}|\underline{25} & 45\underline{1}|\underline{23} & 5\underline{1}|\underline{24}3 \\ 5\underline{1}3|\underline{24} & 5\underline{1}4|\underline{23} & 53\underline{1}|\underline{24} & 54\underline{1}|\underline{23} & & \end{array}$$

$$\mathcal{T}'' = \begin{array}{lllll} \underline{234}|15 & \underline{235}|14 & \underline{235}4|1 & \underline{2}4\underline{35}|1 & \underline{245}|13 \\ \underline{245}3|1 & \underline{2}5\underline{34}|1 & \underline{345}2|1 & 4\underline{235}|1 & 5\underline{234}|1 \end{array}$$

where the underlined subword is the unique hit of 1-23, and the bar indicates how the permutation is factored in the proof of Theorem 4.

Theorem 6. *Let $v_1(n)$ be the number of permutations of length n containing exactly one occurrence of the pattern* 1-32 *and let B_n be the nth Bell number. The numbers $v_1(n)$ satisfy the recurrence*

$$v_1(n+1) = v_1(n) + \sum_{k=1}^{n-1} \left[\binom{n}{k} v_1(k) + \binom{n-1}{k-1} B_k \right],$$

whenever $n \geq 0$, with the initial condition $v_1(0) = 0$.

Proof. Each permutation $\pi \in \mathcal{S}_{n+2}^1(\text{1-32})$ contains a unique subword acb such that $a < b < c$ and cb is a segment of π. Define the sets $\mathcal{T}$ and $\mathcal{T}'$ by

$$\pi \in \begin{cases} \mathcal{T} & \text{if } a = 1, \\ \mathcal{T}' & \text{if } a \neq 1. \end{cases}$$

Then $\mathcal{S}_{n+2}^1(\text{1-32})$ is the disjoint union of $\mathcal{T}$ and $\mathcal{T}'$, so

$$v_1(n+2) = |\mathcal{T}| + |\mathcal{T}'|.$$

For the cardinality of $\mathcal{T}$ we observe that if $a = 1$ then $b = 2$: If the letter 2 precedes the letter 1 or 12 is a segment of π then every hit of 1-23 with $a = 1$ would cause an additional hit of 1-32 with $a = 2$ contradicting the uniqueness of the hit of 1-23; if 1 precedes 2 then $a = 1$ and $b = 2$. Thus we can factor π uniquely in the form $\pi = \sigma x 2\tau$, where σx is (1-32)-avoiding, the letter 1 is included in σ, and τ is nonempty and (12)-avoiding. Let $\mathcal{R}_n$ be the set of (1-32)-avoiding permutations of $[n]$ that do not end with the letter 1. Since the letter 1 cannot be the last letter of a hit of 1-32, we have, by Proposition 3, that $|\mathcal{S}_n(1\text{-}32) \setminus \mathcal{R}_n| = B_{n-1}$. Consequently, $|\mathcal{R}_n| = B_n - B_{n-1}$ and

$$\begin{aligned} |\mathcal{T}| &= \sum_{k=1}^{n} \binom{n-1}{k-1} |\mathcal{R}_k| \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} (B_k - B_{k-1}) \\ &= \sum_{k=1}^{n-1} \binom{n-1}{k-1} B_k. \end{aligned}$$

For the last identity we have used the familiar recurrence $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$.

Suppose $\pi \in \mathcal{T}'$. Since $a \neq 1$ we can factor π uniquely in the form $\pi = \sigma 1 \tau$, where σ contains exactly one occurrence of 1-32, and τ is nonempty and (12)-avoiding. Accordingly,

$$|\mathcal{T}''| = \sum_{k=0}^{n} \binom{n}{k} v_1(k),$$

which completes the proof. $\square$

Let σ be a pattern of Class 1 or 2. Using combinatorial reasoning we have found a recursion for the number of permutations containing exactly one occurrence of the pattern σ (Theorem 4 and 6). More generally, given $r \geq 0$, we would like to find a recursion for the number of permutations containing exactly r occurrence of the pattern σ. Using a more general and analytic approach we will now demonstrate how this (at least in principle) can be achieved.

Let $S_\sigma^r(x)$ be the generating function $S_\sigma^r(x) = \sum_n s_\sigma^r(n) x^n$. To find functional relations for $S_\sigma^r(x)$ the following lemma will turn out to be useful.

Lemma 1. *If $\{a_n\}$ is a sequence of numbers and $A(x) = \sum_{n\geq 0} a_n x^n$ is its ordinary generating function, then, for any $d \geq 0$,*

$$\sum_{n\geq 0} \left[\sum_{j=0}^{n} \binom{n}{j} a_{j+d} \right] x^n = \frac{(1-x)^{d-1}}{x^d} \left[A\Big(\frac{x}{1-x}\Big) - \sum_{j=0}^{d-1} a_j \Big(\frac{x}{1-x}\Big)^j \right].$$

Proof. It is plain that

$$\sum_{n\geq 0} \Big[\sum_{j=0}^{n} \binom{n}{j} a_j \Big] x^n = \frac{1}{1-x} A\Big(\frac{x}{1-x}\Big).$$

See for example [12, p 192]. On the other hand,

$$\sum_{n\geq 0} a_{n+d} x^n = \frac{1}{x^d} \Big[A(x) - \sum_{j=0}^{d-1} a_j x^j \Big].$$

Combining these two identities we get the desired result. $\square$

Define $\mathcal{S}_n^r(\sigma)$ to be the set of permutations $\pi \in S_n$ such that $(\sigma)\pi = r$. Let $s_\sigma^r(n) = |\mathcal{S}_n^r(\sigma)|$ for $r \geq 0$ and $s_\sigma^r(n) = 0$ for $r < 0$. Given $b_1, b_2, \dots, b_k \in \mathbb{N}$, we also define

$$s_\sigma^r(n; b_1, b_2, \dots, b_k) = \#\{a_1 a_2 \cdots a_n \in \mathcal{S}_n^r(\sigma) \mid a_1 a_2 \cdots a_k = b_1 b_2 \cdots b_k\}.$$

As a direct consequence of the above definitions, we have

$$s_\sigma^r(n) = \sum_{j=1}^{n} s_\sigma^r(n; j). \tag{1}$$

We start by considering patterns that belong to Class 1 and we use 12-3 as a representative of this class. Let us define

$$\begin{aligned} u_r(n; b_1, \dots, b_k) &= s_{12\text{-}3}^r(n; b_1, \dots, b_k), \\ u_r(n) &= s_{12\text{-}3}^r(n), \\ U_r(x) &= S_{12\text{-}3}^r(x). \end{aligned}$$

Lemma 2. *Let $n \geq 1$. We have $u_r(n; n-1) = u_r(n; n) = u_r(n-1)$ and*

$$u_r(n; i) = \sum_{j=1}^{i-1} u_r(n-1; j) + \sum_{j=0}^{n-i-1} u_{r-j}(n-1; n-1-j),$$

whenever $1 \leq i \leq n-2$.

Proof. If $a_1 a_2 \cdots a_n$ is any permutation of $[n]$ then

$$(12\text{-}3)a_1 a_2 \cdots a_n = (12\text{-}3)a_2 a_3 \cdots a_n + \begin{cases} n - a_2 & \text{if } a_1 < a_2, \\ 0 & \text{if } a_1 > a_2. \end{cases}$$

Hence,

$$\begin{aligned} u_r(n; i) &= \sum_{j=1}^{i-1} u_r(n; i, j) + \sum_{j=i+1}^{n} u_r(n; i, j) \\ &= \sum_{j=1}^{i-1} u_r(n-1; j) + \sum_{j=i+1}^{n} u_{r-n+j}(n-1; j-1) \\ &= \sum_{j=1}^{i-1} u_r(n-1; j) + \sum_{j=0}^{n-i-1} u_{r-j}(n-1; n-1-j). \end{aligned}$$

For $i = n-1$ or $i = n$ it is easy to see that $u_r(n; i) = u_r(n-1)$. $\square$

Using Lemma 2 we quickly generate the numbers $u_r(n)$; the first few of these numbers are given in Table 1. Given $r \in \mathbb{N}$ we can also use Lemma 2 to find a functional relation determining $U_r(x)$. Here we present such functional relations for $r = 0, 1, 2$ and also explicit formulas for $r = 0, 1$.

Equation 1 tells us how to compute $u_r(n)$ if we are given the numbers $u_r(n; i)$. For the case $r = 0$ Lemma 3, below, tells us how to do the converse.

Lemma 3. *If $1 \leq i \leq n-2$ then*

$$u_0(n; i) = \sum_{j=0}^{i-1} \binom{i-1}{j} u_0(n-2-j).$$

$n\backslash r$	0	1	2	3	4	5	6
0	1						
1	1						
2	2						
3	5	1					
4	15	7	1	1			
5	52	39	13	12	2	1	1
6	203	211	112	103	41	24	17
7	877	1168	843	811	492	337	238
8	4140	6728	6089	6273	4851	3798	2956
9	21147	40561	43887	48806	44291	38795	33343
10	115975	256297	321357	386041	394154	379611	355182

TABLE 1. The number of permutations of length n containing exactly r occurrences of the pattern 12-3.

Proof. For $n = 1$ the identity is trivially true. Assume the identity is true for $n = m$. We have

$$\begin{aligned} u_0(m+1;i) &= \sum_{j=1}^{i-1} u_0(m;j) + u_0(m-1) && \text{by Lemma 2} \\ &= \sum_{j=1}^{i-1}\sum_{k=0}^{j-1} \binom{j-1}{k} u_0(m-2-k) + u_0(m-1) && \text{by the induction hypothesis} \\ &= \sum_{j=1}^{i-1}\sum_{k=j-1}^{i-2} \binom{k}{j-1} u_0(m-1-j). \end{aligned}$$

Using the familiar equality $\binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$ we then get

$$u_0(m+1;i) = \sum_{j=1}^{i-1} \binom{i-1}{j} u_0(m-1-j).$$

Thus the identity is true for $n = m+1$ and by the principle of induction the desired identity is true for all $n \geq 1$. □

The following proposition is a direct consequence of Proposition 3. However, we give a different proof. The proof is intended to illustrate the general approach. It is advisable to read this proof before reading the proof of Theorem 4′ below.

Proposition 7. *The generating function, $U_0(x)$, for the number of (12-3)-avoiding permutations of length n is*

$$U_0(x) = \sum_{k\geq 0} \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

Proof. We have

$$\begin{aligned}
u_0(n) &= \sum_{k=1}^{n} u_0(n;k) && \text{by Equation 1} \\
&= 2u_0(n-1) + \sum_{i=1}^{n-2}\sum_{j=0}^{i-1}\binom{i-1}{j}u_0(n-2-j) && \text{by Lemma 2 and 3} \\
&= u_0(n-1) + \sum_{i=0}^{n-2}\binom{n-2}{i}u_0(n-1-i) && \text{by } \textstyle\sum_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1} \\
&= u_0(n-1) + \sum_{i=0}^{n-2}\binom{n-2}{i}u_0(i+1).
\end{aligned}$$

Therefore, by Lemma 1, we have

$$U_0(x) = xU_0(x) + 1 - x + xU_0\left(\frac{x}{1-x}\right),$$

which is equivalent to

$$U_0(x) = 1 + \frac{x}{1-x}U_0\left(\frac{x}{1-x}\right).$$

An infinite number of applications of this identity concludes the proof. □

We now derive a formula for $U_1(x)$ that is somewhat similar to the one for $U_0(x)$. The following lemma is a first step in this direction.

Lemma 4. *If* $1 \le i \le n-2$ *then*

$$u_1(n;i) = \sum_{j=0}^{i-1}\binom{i-1}{j}u_1(n-2-j) + u_0(n;i).$$

Proof. For $n = 1$ the identity is trivially true. Assume the identity is true for $n = m$. Lemma 2 and the induction hypothesis imply

$$\begin{aligned}
u_1(m+1;i) &= \sum_{j=1}^{i-1}u_1(m;j) + u_1(m-1) + u_0(m-1) \\
&= \sum_{j=0}^{i-1}\binom{j-1}{k}u_1(m-1-j) + \sum_{j=1}^{i-1}u_0(m;j) + u_0(m-1).
\end{aligned}$$

In addition, Lemma 3 implies

$$\begin{aligned}
u_0(m+1;i) &= \sum_{j=1}^{i-1}\sum_{k=0}^{j-1}\binom{j-1}{k}u_0(n-2-k) + u_0(n-1) \\
&= \sum_{j=0}^{i-1}\binom{i-1}{j}u_0(n-1-j) \\
&= \sum_{j=1}^{i-1}u_0(m;j) + u_0(m-1).
\end{aligned}$$

Thus the identity is true for $n = m+1$ and by the principle of induction the desired identity is true for all $n \geq 1$. $\square$

Next, we rediscover Theorem 4.

Theorem 4′. Let $u_1(n)$ be the number of permutations of length n containing exactly one occurrence of the pattern 12-3 and let B_n be the nth Bell number. The numbers $u_1(n)$ satisfy the recurrence

$$u_1(n+2) = 2u_1(n+1) + \sum_{k=0}^{n-1} \binom{n}{k} \big[u_1(k+1) + B_{k+1}\big],$$

whenever $n \geq -1$, with the initial condition $u_1(0) = 0$.

Proof. Similarly to the proof of Proposition 7, we use Equation 1, Lemma 2, 3, and 4 to get

$$\begin{aligned}
u_1(n) &= 2u_1(n-1) + \sum_{i=1}^{n-2} \left[\sum_{j=0}^{i-1} \binom{i-1}{j} u_1(n-2-j) + u_0(n;i) \right] \\
&= 2u_1(n-1) + \sum_{i=1}^{n-2} \sum_{j=0}^{i-1} \binom{i-1}{j} \big(u_1(n-2-j) + u_0(n-2-j)\big) \\
&= u_1(n-1) - u_0(n-1) + \sum_{i=0}^{n-2} \binom{n-2}{i} \big(u_1(i+1) + u_0(i+1)\big) \\
&= 2u_1(n-1) + \sum_{i=0}^{n-3} \binom{n-2}{i} \big(u_1(i+1) + u_0(i+1)\big).
\end{aligned}$$

$\square$

Corollary 8. *The ordinary generating function, $U_1(x)$, for the number of permutations of length n containing exactly one occurrence of the pattern* 12-3 *satisfies the functional equation*

$$U_1(x) = \frac{x}{1-x} \left(U_1\Big(\frac{x}{1-x}\Big) + U_0\Big(\frac{x}{1-x}\Big) - U_0(x) \right).$$

Proof. The result follows from Theorem 4 together with Lemma 1. $\square$

Corollary 9. *The ordinary generating function for the number of permutations of length n containing exactly one occurrence of the pattern* 12-3 *is*

$$U_1(x) = \sum_{n \geq 1} \frac{x}{1-nx} \sum_{k \geq 0} \frac{kx^{k+n}}{(1-x)(1-2x)\cdots(1-(k+n)x)}.$$

Proof. We simply apply Corollary 8 an infinite number of times and in each step we perform some rather tedious algebraic manipulations. $\square$

Theorem 10. *The ordinary generating function, $U_2(x)$, for the number of permutations of length n containing exactly two occurrences of the pattern* 12-3 *satisfies* *the functional equation*

$$U_2(x) = \frac{x}{(1-x)^2(1-2x)}\left(\begin{array}{l} U_2\left(\frac{x}{1-x}\right) - (1-x)U_2(x) + \\ U_1\left(\frac{x}{1-x}\right) - (1-x)^2U_1(x) + \\ U_0\left(\frac{x}{1-x}\right) - (1-x)^2U_0(x) \end{array} \right).$$

Proof. The proof is similar to the proofs of Lemma 4, Theorem 4' and Corollary 8, and we only sketch it here.

Lemma 2 yields

$$\begin{aligned} u_2(n;n) &= u_2(n-1) \\ u_2(n;n-1) &= u_2(n-1) \\ u_2(n;n-2) &= u_2(n-1) - u_2(n-2) + u_1(n-2) \end{aligned}$$

and, by means of induction,

$$u_2(n;i) = u_1(n;i) + u_0(n;i) - u_0(n-1;i) + \sum_{j=0}^{i-1}\binom{i-1}{j}u_2(n-2-j),$$

whenever $1 \le i \le n-3$. Therefore, $u_2(0) = u_2(1) = u_2(2) = 0$ and

$$\begin{aligned} &u_2(n) = 3u_2(n-1) - u_2(n-2) + u_1(n-2) + \\ &\sum_{i=1}^{n-3}\binom{n-3}{i}(u_2(n-1-i) + u_1(n-1-i) + u_0(n-1-i) - u_0(n-2-i)). \end{aligned}$$

whenever $n \ge 3$. Thus, the result follows from Lemma 1. □

We now turn our attention to patterns that belong to Class 2 and we use 23-1 as a representative of this class. The results found below regarding the 23-1 pattern are very similar to the ones previously found for the 12-3 pattern, and so are the proofs; therefore we choose to omit most of the proofs. However, we give the necessary lemmas from which the reader may construct her/his own proofs.

Define

$$\begin{aligned} v_r(n;b_1,\ldots,b_k) &= s^r_{23\text{-}1}(n;b_1,\ldots,b_k), \\ v_r(n) &= s^r_{23\text{-}1}(n), \\ V_r(x) &= S^r_{23\text{-}1}(x). \end{aligned}$$

If $a_1a_2\cdots a_n$ is any permutation of $[n]$ then

$$(23\text{-}1)a_1a_2\cdots a_n = (23\text{-}1)a_2a_3\cdots a_n + \begin{cases} a_1 - 1 & \text{if } a_1 < a_2, \\ 0 & \text{if } a_1 > a_2. \end{cases}$$

Lemma 5. *Let $n \ge 1$. We have $v_r(n;1) = v_r(n;n) = v_r(n-1)$ and*

$$v_r(n;i) = \sum_{j=1}^{i-1} v_r(n-1;j) + \sum_{j=i}^{n-1} v_{r-i+1}(n-1;j),$$

whenever $2 \le i \le n-1$.

Using Lemma 5 we quickly generate the numbers $v_r(n)$; the first few of these numbers are given in Table 2.

$n\backslash r$	0	1	2	3	4	5	6
0	1						
1	1						
2	2						
3	5	1					
4	15	6	3				
5	52	32	23	10	3		
6	203	171	152	98	62	22	11
7	877	944	984	791	624	392	240
8	4140	5444	6460	6082	5513	4302	3328
9	21147	32919	43626	46508	46880	41979	36774
10	115975	208816	304939	360376	396545	393476	377610

TABLE 2. The number of permutations of length n containing exactly r occurrences of the pattern 23-1.

Lemma 6. *If $2 \leq i \leq n-1$ then*

$$v_0(n;i) = \sum_{j=0}^{i-2} \binom{i-2}{j} v_0(n-2-j).$$

Proposition 11. *The generating function, $V_0(x)$, for the number of* (23-1)*-avoiding permutations of length n is*

$$V_0(x) = \sum_{k\geq 0} \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

Lemma 7. *If $2 \leq i \leq n-1$ then*

$$v_1(n;i) = \sum_{j=0}^{i-2} \binom{i-2}{j} v_1(n-2-j) + v_0(n;i-1) - v_0(n-1,i-1).$$

Theorem 6′. Let $v_1(n)$ be the number of permutations of length n containing exactly one occurrence of the pattern 23-1 and let B_n be the nth Bell number. The numbers $v_1(n)$ satisfy the recurrence

$$v_1(n+1) = v_1(n) + \sum_{k=1}^{n-1} \left[\binom{n}{k} v_1(k) + \binom{n-1}{k-1} B_k \right],$$

whenever $n \geq 0$, with the initial condition $v_1(0) = 0$.

Corollary 12. *The ordinary generating function for the number of permutations of length n containing exactly one occurrence of the pattern* 23-1 *satisfies the functional equation*

$$V_1(x) = \frac{x}{1-x} V_1\Big(\frac{x}{1-x}\Big) + x\bigg(V_0\Big(\frac{x}{1-x}\Big) - V_0(x)\bigg).$$

Corollary 13. *The ordinary generating function for the number of permutations of length n containing exactly one occurrence of the pattern* 23-1 *is*

$$V_1(x) = \sum_{n\geq 1} \frac{x}{1-(n-1)x} \sum_{k\geq 0} \frac{kx^{k+n}}{(1-x)(1-2x)\cdots(1-(k+n)x)}.$$

Theorem 14. *The ordinary generating function, $V_2(x)$, for the number of permutations of length n containing exactly two occurrences of the pattern* 23-1 *satisfies the functional equation*

$$V_2(x) = \frac{x}{1-x}\left(V_2\Big(\frac{x}{1-x}\Big) + (1-2x)V_1\Big(\frac{x}{1-x}\Big) + (1-3x+x^2)V_0\Big(\frac{x}{1-x}\Big)\right) - x + x^2$$

Proof. By Lemma 5

$$\begin{aligned}
v_2(n;n) &= v_2(n-1)\\
v_2(n;1) &= v_2(n-1)\\
v_2(n;2) &= v_2(n-2) + v_1(n-1) - v_1(n-2)\\
v_2(n;3) &= v_2(n-2) + v_2(n-3) + v_1(n-2) - v_1(n-3) +\\
&\qquad + v_0(n-1) - v_0(n-2) - v_0(n-3)
\end{aligned}$$

and, by means of induction,

$$v_2(n;i) = \sum_{j=0}^{i-2} \binom{i-2}{j} v_2(n-2-j) + v_1(n;i-1) + v_1(n-1;i-1) - v_0(n-1;i-2)$$

for $n-1 \geq i \geq 4$. Thus $v_2(0) = v_2(1) = v_2(2) = 0$ and for all $n \geq 3$

$$\begin{aligned}
v_2(n) = v_2(n-1) &+ \sum_{j=0}^{n-2} \binom{n-2}{j} v_2(n-1-j) +\\
&+ \sum_{j=0}^{n-3} \binom{n-3}{j} \big(v_1(n-1-j) - v_1(n-2-j)\big) +\\
&+ \sum_{j=0}^{n-4} \binom{n-4}{j} \big(v_0(n-1-j) - v_0(n-2-j) - v_0(n-3-j)\big).
\end{aligned}$$

The result now follows from Lemma 1. □

3. Counting occurrences of a pattern of Class 3

We choose 2-13 as our representative for Class 3 and we define $w_r(n)$ as the number of permutations of length n containing exactly r occurrences of the pattern 2-13. We could apply the analytic approach from the previous section to the problem of determining $w_r(n)$. However, a result by Clarke, Steingrímsson and Zeng [10, Corollary 11] provides us with a better option.

Theorem 15. *The following Stieltjes continued fraction expansion holds*

$$\sum_{\pi\in\mathcal{S}} x^{1+(12)\pi} y^{(21)\pi} p^{(2\text{-}31)\pi} q^{(31\text{-}2)\pi} t^{|\pi|} = \cfrac{1}{1-\cfrac{x[\,1\,]_{p,q}t}{1-\cfrac{y[\,1\,]_{p,q}t}{1-\cfrac{x[\,2\,]_{p,q}t}{1-\cfrac{y[\,2\,]_{p,q}t}{\ddots}}}}}$$

where $[\,n\,]_{p,q} = q^{n-1} + pq^{n-2} + \cdots + p^{n-2}q + p^{n-1}$.

Proof. In [10, Corollary 11] Clarke, Steingrímsson and Zeng derived the following continued fraction expansion

$$\sum_{\pi\in\mathcal{S}} y^{\operatorname{des}\pi} p^{\operatorname{Res}\pi} q^{\operatorname{Ddif}\pi} t^{|\pi|} = \cfrac{1}{1-\cfrac{[\,1\,]_p t}{1-\cfrac{yq[\,1\,]_p t}{1-\cfrac{q[\,2\,]_p t}{1-\cfrac{yq^2[\,2\,]_p t}{\ddots}}}}}$$

where $[\,n\,]_p = 1+p+\cdots+p^{n-1}$. We refer the reader to [10] for the definitions of Ddif and Res. However, given these definitions, it is easy to see that Res = (2-31) and Ddif = (21) + (2-31) + (31-2). Moreover, des = (21) and $|\pi| = 1 + (12)\pi + (21)\pi$. Thus, substituting $y(xq)^{-1}$ for y, pq^{-1} for p, and xt for t, we get the desired result. □

The following corollary is an immediate consequence of Theorem 15.

Corollary 16. *The bivariate ordinary generating function for the distribution of occurrences of the pattern* 2-13 *admits the Stieltjes continued fraction expansion*

$$\sum_{\pi\in\mathcal{S}} p^{(2\text{-}13)\pi} t^{|\pi|} = \cfrac{1}{1-\cfrac{[\,1\,]_p t}{1-\cfrac{[\,1\,]_p t}{1-\cfrac{[\,2\,]_p t}{1-\cfrac{[\,2\,]_p t}{\ddots}}}}}$$

where $[\,n\,]_p = 1 + p + \cdots + p^{n-1}$

Using Corollary 16 we quickly generate the numbers $w_r(n)$; the first few of these numbers are given in Table 3.

Corollary 17. *The number of* (2-13)*-avoiding permutations of length* n *is*

$$w_0(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. This result is explicitly stated in Proposition 3, but it also follows from Corollary 16 by putting $p = 0$. □

$n\backslash r$	0	1	2	3	4	5	6
0	1						
1	1						
2	2						
3	5	1					
4	14	8	2				
5	42	45	25	7	1		
6	132	220	198	112	44	12	2
7	429	1001	1274	1092	700	352	140
8	1430	4368	7280	8400	7460	5392	3262
9	4862	18564	38556	56100	63648	59670	47802
10	16796	77520	193800	341088	470934	541044	535990

TABLE 3. The number of permutations of length n containing exactly r occurrences of the pattern 2-13.

Corollary 18. *The number of permutations of length n containing exactly one occurrence of the pattern* 2-13 *is*

$$w_1(n) = \binom{2n}{n-3}.$$

Proof. For $m > 0$ let

$$W(p,t;m) = \cfrac{1}{1 - \cfrac{[\,m\,]_p t}{1 - \cfrac{[\,m\,]_p t}{1 - \cfrac{[\,m+1\,]_p t}{1 - \cfrac{[\,m+1\,]_p t}{\ddots}}}}}$$

Note that

$$W(p,t;m) = \cfrac{1}{1 - \cfrac{[\,m\,]_p t}{1 - [\,m\,]_p t W(p,t;m+1).}}$$

Assume $m > 1$. Differentiating $W(p,t;m)$ with respect to p and evaluating the result at $p = 0$ we get

$$D_p W(p,t;m)\big|_{p=0} = tC(t)^3 + t^2C(t)^5 + t^2C(t)^4 D_p W(p,t;m+1)\big|_{p=0}$$

where $C(t) = W(0,t,1)$ is the generating function for the Catalan numbers. Applying this identity an infinite number of times we get

$$D_p W(p,t,m)\big|_{p=0} = tC(t)^3 + t^2C(t)^5 + t^3C(t)^7 + \cdots = \frac{tC(t)^3}{1 - tC(t)^2}.$$

On the other hand, $D_p W(p,t;1)\big|_{p=0} = t^2C(t)^4 D_p W(p,t;2)\big|_{p=0}$. Combining these two identities we get

$$D_p W(p,t;1)\big|_{p=0} = \frac{t^3C(t)^7}{1 - tC(t)^2}.$$

Since $\sum_{n\geq 0} w_1(n)t^n = D_pW(p,t;1)\big|_{p=0}$ the proof is completed on extracting coefficients in the last identity. □

The proofs of the following two corollaries are similar to the proof of Corollary 18 and are omitted.

Corollary 19. *The number of permutations of length n containing exactly two occurrences of the pattern* 2-13 *is*

$$w_2(n) = \frac{n(n-3)}{2(n+4)}\binom{2n}{n-3}.$$

Corollary 20. *The number of permutations of length n containing exactly three occurrences of the pattern* 2-13 *is*

$$w_3(n) = \frac{1}{3}\binom{n+2}{2}\binom{2n}{n-5}.$$

As a concluding remark we note that there are many questions left to answer. What is, for example, the formula for $w_k(n)$ in general? What are the combinatorial explanations of $ns^1_{1\text{-}2\text{-}3}(n) = 3s^1_{2\text{-}13}(n)$ and

$$(n+3)(n+2)(n+1)s^1_{2\text{-}13}(n) = 2n(2n-1)(2n-2)s^1_{2\text{-}1\text{-}3}(n)?$$

In addition, Corollary 18 obviously is in need of a combinatorial proof.

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[**C2**]

BICOLOURED DYCK PATHS, SEGMENTED PERMUTATIONS, AND CHEBYSHEV POLYNOMIALS

ANDERS CLAESSON

ABSTRACT. A bicoloured Dyck path is a Dyck path in which each up-step is assigned one of two colours, say, red and green. We say that a permutation π is *σ-segmented* if every occurrence o of σ in π is a segment-occurrence (i.e., o is a contiguous subword in π).

We show combinatorially the following results: The 132-segmented permutations of length n with k occurrences of 132 are in one-to-one correspondence with bicoloured Dyck paths of length $2n-4k$ with k red up-steps. Similarly, the 123-segmented permutations of length n with k occurrences of 123 are in one-to-one correspondence with bicoloured Dyck paths of length $2n-4k$ with k red up-steps, each of height less than 2.

We enumerate the permutations above by enumerating the corresponding bicoloured Dyck paths. More generally, we present a bivariate generating function for the number of bicoloured Dyck path of length $2n$ with k red up-steps, each of height less than h. This generating function is expressed in terms of Chebyshev polynomials of the second kind.

1. INTRODUCTION

It is relatively straightforward to show that number of permutations of $[n] = \{1, 2, \ldots, n\}$ avoiding a pattern of length 3 is the Catalan number, $C_n = \binom{2n}{n}/(n+1)$ (e.g., see [8] or [5]). In contrast, to count the permutations containing r occurrences of a fixed pattern of length 3, for a general r, is a very hard problem. The best result on this latter problem has been achieved by Mansour and Vainshtein [6]. They presented an algorithm that computes the generating function for the number of permutations with r occurrences of 132 for any $r \geq 0$. The algorithm has been implemented in C. It yields explicit results for $1 \leq r \leq 6$.

We say that an occurrence o of σ in π is a *segment-occurrence* if o is a segment of π, in other words, if o is a contiguous subword in π. Elizalde and Noy [2] presented exponential generating functions for the distribution of the number of segment-occurrences of any pattern of length 3. Related problems have also been studied by Kitaev [3] and by Kitaev and Mansour [4].

We say that π is *σ-segmented* if every occurrence of σ in π is a segment-occurrence. For instance, 4365172 contains 3 occurrences of 132, namely 465, 365, and 172. Of these occurrences, only 365 and 172 are segment-occurrence. Thus 4365172 is not 132-segmented. Note that if π is σ-avoiding then π is also σ-segmented. In this article we try to enumerate the σ-segmented permutations by length and by the the number of occurrences of σ. In [5] Krattenthaler gave two bijections: one between 132-avoiding permutations and Dyck paths, and one

Date: January 21, 2004.

between 123-avoiding permutations and Dyck paths. We obtain two new results by extending these bijections:

– The 132-segmented permutations of length n with k occurrences of 132 are in one-to-one correspondence with bicoloured Dyck paths of length $2n-4k$ with k red up-steps.

– The 123-segmented permutations of length n with k occurrences of 123 are in one-to-one correspondence with bicoloured Dyck paths of length $2n-4k$ with k red up-steps, each of height less than 2.

Here a bicoloured Dyck path is a Dyck path in which each up-step is assigned one of two colours, say, red and green. We enumerate the permutations above by enumerating the corresponding bicoloured Dyck paths. To be more precise, let $\mathcal{B}_{n,k}$ be the set of bicoloured Dyck path of length $2n$ with k red up-steps. Let $\mathcal{B}^{[h]}_{n,k}$ be the subset of $\mathcal{B}_{n,k}$ consisting of those paths where the height of each red up-step is less than h. It is plain that $|\mathcal{B}_{n,k}| = \binom{n}{k} C_n$. We show that

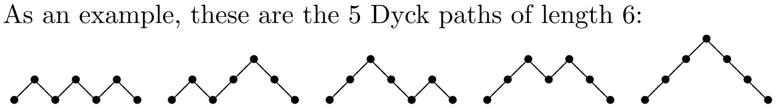
$$\sum_{n,k\geq 0} |\mathcal{B}^{[h]}_{n,k}| q^k t^n = \frac{C(t) - 2xqU_h(x)U_{h-1}(x)}{1+q-qU_h^2(x)}, \qquad x = \frac{1}{2\sqrt{(1+q)t}},$$

where $C(t) = (1-\sqrt{1-4t})/2t$ is the generating function for the Catalan numbers, and U_n is the nth Chebyshev polynomial of the second kind. We also find formulas for $|\mathcal{B}^{[1]}_{n,k}|$ and $|\mathcal{B}^{[2]}_{n,k}|$.

2. Bicoloured Dyck paths

By a *lattice path* we shall mean a path in $\mathbb{Z}^2$ with steps $(1,1)$ and $(1,-1)$; the steps $(1,1)$ and $(1,-1)$ will be called *up-* and *down-steps*, respectively. Furthermore, a lattice path that never falls below the x-axis will be called *nonnegative*.

Recall that a *Dyck path* of length $2n$ is a nonnegative lattice path from $(0,0)$ to $(2n,0)$. As an example, these are the 5 Dyck paths of length 6:



Letting u and d represent the steps $(1,1)$ and $(1,-1)$, we code a Dyck path with a word over $\{u,d\}$. For example, the paths above are coded by

$$udud ud \quad uduudd \quad uuddud \quad uududd \quad uuuddd$$

Let $\mathcal{D}_n$ be the language over $\{u,d\}$ obtained from Dyck paths of length $2n$ via this coding, and let $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$. In general, if $\mathcal{A}$ is a language over some alphabet X, then the *characteristic series* of $\mathcal{A}$, also (by slight abuse of notation) denoted $\mathcal{A}$, is the element of $\mathbb{C}\langle\langle X\rangle\rangle$ defined by

$$\mathcal{A} = \sum_{w\in\mathcal{A}} w.$$

A nonempty Dyck path β can be written uniquely as $u\beta_1 d\beta_2$ where β_1 and β_2 are Dyck paths. This decomposition is called the *first return decomposition* of β, because the d in $u\beta_1 d\beta_2$ corresponds to the first place, after $(0,0)$, where the path touches the x-axis. By this decomposition, the characteristic series of $\mathcal{D}$ is uniquely determined by the functional equation

$$\mathcal{D} = 1 + u\mathcal{D}d\mathcal{D}, \tag{1}$$

where 1 denotes the empty word.

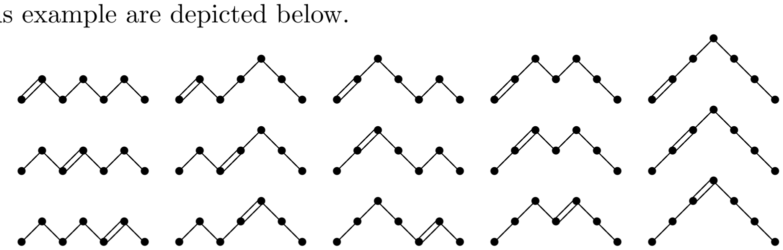
In a similar vein, we now consider the language $\mathcal{B}$ over $\{u, \bar{u}, d\}$ whose characteristic series is uniquely determined by the functional equation

$$\mathcal{B} = 1 + (u + \bar{u})\mathcal{B}d\mathcal{B}. \tag{2}$$

Let $\mathcal{B}_n$ be the set of words in $\mathcal{B}$ that are of length $2n$, and let $\mathcal{B}_{n,k}$ be the set of words in $\mathcal{B}_n$ with k occurrences of $\bar{u}$. As an example, when $n = 3$ and $k = 1$ there are 15 such words, namely

$$\begin{array}{ccccc} \bar{u}dudud & \bar{u}duudd & \bar{u}uddud & \bar{u}ududd & \bar{u}uuddd \\ ud\bar{u}dud & ud\bar{u}udd & u\bar{u}ddud & u\bar{u}dudd & u\bar{u}uddd \\ udud\bar{u}d & udu\bar{u}dd & uudd\bar{u}d & uud\bar{u}dd & uu\bar{u}ddd \end{array}$$

We may view the elements of $\mathcal{B}$ as *bicoloured Dyck paths.* The words from the previous example are depicted below.



Here steps represented by double edges are, say, red, and steps represented by simple edges are, say, green.

Proposition 1. *With $C_n = \operatorname{card} \mathcal{D}_n$, we have*

$$\operatorname{card} \mathcal{B}_{n,k} = \binom{n}{k} C_n \quad \textit{and} \quad \operatorname{card} \mathcal{B}_n = 2^n C_n.$$

Proof. A bicoloured Dyck paths β of length $2n$ naturally breaks up into two parts: (a) The Dyck path obtained from β by replacing each red up-step with a green ditto. (b) The subset of $[n] = \{1, 2, \ldots, n\}$ consisting of those integers i for which the ith up-step is red. □

For $h \geq 1$, let $\mathcal{B}^{[h]}$ be the subset of $\mathcal{B}$ whose characteristic series is the solution to

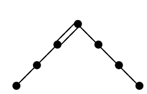
$$\mathcal{B}^{[h]} = 1 + (u + \bar{u})\mathcal{B}^{[h-1]}d\mathcal{B}^{[h]}, \tag{3}$$

with the initial condition $\mathcal{B}^{[0]} = \mathcal{D}$, where $\mathcal{D}$ is defined as above. Let

$\mathcal{B}_n^{[h]}$ be the set of words in $\mathcal{B}^{[h]}$ that are of length $2n$, and let

$\mathcal{B}_{n,k}^{[h]}$ be the set of words in $\mathcal{B}_n^{[h]}$ with k occurrences of $\bar{u}$.

To translate these definitions in terms of lattice paths we define the *height* of a step in a (bicoloured) lattice path as the height above the x-axis of its left point. Then $\mathcal{B}^{[h]}$ is the set of bicoloured Dyck paths whose red up-steps all are of height less than h. As an example, there is exactly one element in $\mathcal{B}_{3,1}$ that is not in $\mathcal{B}^{[2]}$, namely



To count words of given length in $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{B}^{[h]}$, we will study the commutative counterparts of the functional equations (1), (2) and (3). Formally, we define the substitution $\mu : \mathbb{C}\langle\langle u, \bar{u}, d \rangle\rangle \to \mathbb{C}[[q, t]]$ by

$$\mu = \{ u \mapsto t, \bar{u} \mapsto qt, d \mapsto 1 \}.$$

Let $C = \mu(\mathcal{D})$, $B = \mu(\mathcal{B})$, and $B^{[h]} = \mu(\mathcal{B}^{[h]})$. We then get

$$C = 1 + tC^2; \tag{4}$$

$$B = 1 + (1+q)tB^2; \tag{5}$$

$$B^{[h]} = 1 + (1+q)tB^{[h-1]}B^{[h]}; \qquad B^{[0]} = C. \tag{6}$$

By an easy application of the Lagrange inversion formula it follows from (4) that

$$[t^n]C(t)^i = \frac{i}{i+n}\binom{2n+i-1}{n}. \tag{7}$$

In particular, we obtain that C is the familiar generating function of the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$. Thus we have derived the well known fact that the number of Dyck paths of length $2n$ is the nth Catalan number. Furthermore, it follows from (5) that

$$B(q,t) = C((1+q)t), \tag{8}$$

and it follows from (6) that

$$B^{[h]}(q,t) = \frac{1}{1-(1+q)tB^{[h-1]}C(t)}; \qquad B^{[0]} = C. \tag{9}$$

From these series we generate the first few values of $|\mathcal{B}_{n,k}|$, $|\mathcal{B}^{[1]}_{n,k}|$ and $|\mathcal{B}^{[2]}_{n,k}|$; tables with these values are given in Section 4.

Recall that the *Chebyshev polynomials of the second kind*, denoted $U_n(x)$, are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where n is an integer, $x = \cos\theta$, and $0 \le \theta \le \pi$. Equivalently, these polynomials can be defined as the solution to the difference equation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),$$

with $U_{-1}(x) = 0$ and $U_0(x) = 1$.

Via a bijection between Dyck paths and 132-avoiding permutations due to Krattenthaler [5, Lemma Φ and Theorem 2] it follows by a result of Chow and West [1, Theorem 3.1] that

$$C^{[h]}(t) = \frac{U_{h-1}\left(\frac{1}{2\sqrt{t}}\right)}{\sqrt{t}\cdot U_h\left(\frac{1}{2\sqrt{t}}\right)} \tag{10}$$

is the generating function for Dyck paths which stay below height h. Note that, since $C^{[0]} = 1$ and $C^{[h]} = (1 - tC^{[h-1]})^{-1}$, this result is also easy to prove by induction on h.

Theorem 2. *With $B^{[h]}$ being the generating function for the number of Dyck paths whose red up-steps all are of height less than h, and U_n being the nth Chebyshev polynomial of the second kind we have*

$$B^{[h]}(q,t) = \frac{4x^2U_{h-1}(x) - 2xU_{h-2}(x)C(t)}{2xU_h(x) - U_{h-1}(x)C(t)} = \frac{C(t) - 2xqU_h(x)U_{h-1}(x)}{1+q-qU_h^2(x)},$$

where $x = 1/2\sqrt{(1+q)t}$, and $C(t) = (1-\sqrt{1-4t})/2t$ is the generating function for the Catalan numbers.

Proof. We shall prove the first equality by induction. To this end, we let

$$F^{[h]}(q,t) = \frac{4x^2U_{h-1}(x) - 2xU_{h-2}(x)C(t)}{2xU_h(x) - U_{h-1}(x)C(t)}.$$

From $U_{-2}(x) = -1$, $U_{-1}(x) = 0$, and $U_0(x) = 1$ it readily follows that $F^{[0]}(q,t) = C(t) = B^{[0]}(q,t)$. If $B^{[h]} = F^{[h]}$, for some fixed $h \geq 0$, then

$$\begin{aligned}
B^{[h+1]} &= \frac{1}{1-(1+q)tB^{[h]}}\\
&= \frac{1}{1-(1+q)tF^{[h]}}\\
&= \frac{2xU_h - U_{h-1}C}{2xU_h - U_{h-1}C - (1+q)t\big(4x^2U_{h-1} - 2xU_{h-2}C\big)}\\
&= \frac{2xU_h - U_{h-1}C}{2xU_h - (1+q)t4x^2U_{h-1} - \big(U_{h-1} - (1+q)t2xU_{h-2}\big)C}\\
&= \frac{4x^2U_h - 2xU_{h-1}C}{2x\big(2xU_h - (1+q)t4x^2U_{h-1}\big) - \big(2xU_{h-1} - (1+q)t4x^2U_{h-2}\big)C}\\
&= \frac{4x^2U_h - 2xU_{h-1}C}{2x\big(2xU_h - U_{h-1}\big) - \big(2xU_{h-1} - U_{h-2}\big)C}\\
&= \frac{4x^2U_h - 2xU_{h-1}C}{2xU_{h+1} - U_hC}\\
&= F^{[h+1]}.
\end{aligned}$$

This completes the induction step, and thus the first equality holds for all $h \geq 0$. The second equality is plain algebra/trigonometry. □

Proposition 3. *Let $|\beta|_{\bar{u}}^{h}$ denote the number of occurrences of $\bar{u}$ in β at height h. Then*

$$\sum_{\beta\in\mathcal{B}} q^{|\beta|_{\bar{u}}^{h}} t^{|\beta|} = \frac{8x^2U_h(x) - 2x(1+q)U_{h-1}(x)C(2t)}{4xU_{h+1}(x) - (1+q)U_h(x)C(2t)}, \qquad x = \frac{1}{2\sqrt{2t}}.$$

Proof. Let $G = G(q,t)$ denote the left hand side of the above identity. It is plain that $G^{[1]} = (1-qtC)^{-1}$ and $G^{[h]} = (1-tG^{[h-1]})^{-1}$. Similarly to the proof of Theorem 2, the result now follows by induction on h. □

Proposition 4. *For $n, k \geq 0$ we have*

$$\operatorname{card} \mathcal{B}^{[1]}_{n,k} = b(n+k, n-k) = \frac{2k+1}{n+k+1}\binom{2n}{n-k};$$

$$\operatorname{card} \mathcal{B}^{[1]}_{n} = \binom{2n}{n},$$

where $b(n,k) = \frac{n-k+1}{n+1}\binom{n+k}{n}$ is a ballot number.

Proof. The ballot number $b(n,k)$ is the number of nonnegative lattice paths from $(0,0)$ to $(n+k, n-k)$. Thus, the first claim of the lemma is that $|\mathcal{B}^{[1]}_{n,k}|$ equals the number of nonnegative lattice paths from $(0,0)$ to $(2n, 2k)$; let $\mathcal{A}_{n,k}$ denote the language over $\{u, d\}$ obtained from these paths via the usual coding. In addition, let $\mathcal{A}_n = \cup_{k\geq 0}\mathcal{A}_{n,k}$ and $\mathcal{A} = \cup_{n\geq 0}\mathcal{A}_n$. The characteristic series of $\mathcal{A}$ satisfies

$$\mathcal{A} = 1 + u\mathcal{D}(u+d)\mathcal{A}.$$

From (3) we also know that

$$\mathcal{B}^{[1]} = 1 + (u+\bar{u})\mathcal{D}d\mathcal{B}^{[1]}.$$

We exploit the obvious similarity between these two functional equations to define, by recursion, a length preserving bijection f from $\mathcal{B}^{[1]}$ onto $\mathcal{A}$ such that $\beta \in \mathcal{B}^{[1]}$ has exactly k occurrences of $\bar{u}$ precisely when $f(\beta) \in \mathcal{A}$ ends at height $2k$:

$$f(\beta) = \begin{cases} 1 & \text{if } \beta = 1, \\ u\beta_1 d f(\beta_2) & \text{if } \beta = u\beta_1 d\beta_2,\ \beta_1 \in \mathcal{D},\ \beta_2 \in \mathcal{B}^{[1]}, \\ u\beta_1 u f(\beta_2) & \text{if } \beta = \bar{u}\beta_1 d\beta_2,\ \beta_1 \in \mathcal{D},\ \beta_2 \in \mathcal{B}^{[1]}. \end{cases}$$

For $\beta \in \mathcal{B}$, let $|\beta|_{\bar{u}}$ denote the number of occurrences of $\bar{u}$ in β, and for $\alpha \in \mathcal{A}$ let $h(\alpha)$ denote the height at which α ends. To prove that f is length preserving, bijective, and that $2|\cdot|_{\bar{u}} = h \circ f$, we use induction on path-length: f trivially has these properties as a function from $\mathcal{B}^{[1]}_0$ to $\mathcal{A}_0$. Let n be a positive integer and assume that f has the desired properties as a function from $\cup_{k=0}^{n-1}\mathcal{B}^{[1]}_k$ to $\cup_{k=0}^{n-1}\mathcal{A}_k$. Any β in $\mathcal{B}^{[1]}_n$ can be written as $\beta = x\beta_1 d\beta_2$ for some $x \in \{u, \bar{u}\}$, $\beta_1 \in \mathcal{D}$ and $\beta_2 \in \mathcal{B}^{[1]}$. Therefore,

$$|f(\beta)| = 2 + |\beta_1| + |f(\beta_2)| = 2 + |\beta_1| + |\beta_2| = |\beta|$$

and

$$(h \circ f)(\beta) = 2|x|_{\bar{u}} + (h \circ f)(\beta_2) = 2|x|_{\bar{u}} + 2|\beta_2|_{\bar{u}} = 2|\beta|_{\bar{u}}$$

To prove that f is injective, assume that $f(\beta) = f(\beta')$, where $\beta' = x'\beta'_1 d\beta'_2$ for some $x' \in \{u, \bar{u}\}$, $\beta'_1 \in \mathcal{D}$, and $\beta'_2 \in \mathcal{B}^{[1]}$. Then

$$f(\beta) = u\beta_1 y f(\beta_2) = u\beta'_1 y' f(\beta'_2) = f(\beta'),$$

in which $y, y' \in \{u, d\}$. Thus $\beta_1 = \beta'_1$, $y = y'$, and $f(\beta_2) = f(\beta'_2)$. By the induction hypothesis, $f(\beta_2) = f(\beta'_2)$ implies that $\beta_2 = \beta'_2$, and hence $\beta = \beta'$.

To prove hat f is surjective, take any $\alpha = u\alpha' y\alpha''$ in $\mathcal{A}_n$, where $y \in \{u, d\}$, $\alpha' \in \mathcal{D}$, and $\alpha'' \in \mathcal{A}$. By the induction hypothesis, there exists β'' in $\mathcal{B}^{[1]}$ such that $f(\alpha'') = \beta''$; so $f(u\alpha' y\beta'') = \alpha$. This concludes the proof of the first part of the lemma.

Given the first result, the second result may be formulated as saying that the central binomial coefficient $\binom{2n}{n}$ is the sum of the ballot numbers $b(n+k, n-k)$ for $k = 0, 1, \dots, n$. This is a known fact (see [7, p. 79]); indeed

$$\frac{2k+1}{n+k+1}\binom{2n}{n-k} = \binom{2n}{n-k} - \binom{2n}{n-k-1},$$

and hence the sum these numbers is alternating.

For a bijective proof of the second part we consider the set of all lattice paths from $(0,0)$ to $(2n, 0)$; let $\mathcal{P}_n$ be the language over $\{u, d\}$ obtained from these $\binom{2n}{n}$ paths via the usual coding, and let $\mathcal{P} = \cup_{n\geq 0}\mathcal{P}_n$. The characteristic series of $\mathcal{P}$ satisfies

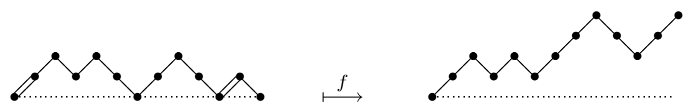
$$\mathcal{P} = 1 + u\mathcal{D}d\mathcal{P} + d\widehat{\mathcal{D}}u\mathcal{P},$$

where $\widehat{\mathcal{D}}$ is the image of $\mathcal{D}$ under the involution on $\mathbb{C}\langle\langle u, d\rangle\rangle$ defined by $u \mapsto d$ and $d \mapsto u$; this involution has the effect of reflecting a Dyck path in the x-axis. A length preserving bijection g from $\mathcal{B}^{[1]}$ onto $\mathcal{P}$ is then recursively defined by

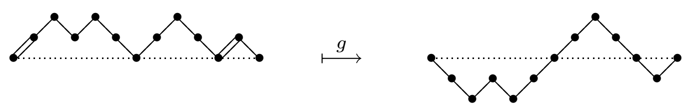
$$g(\beta) = \begin{cases} 1 & \text{if } \beta = 1, \\ u\beta_1 d g(\beta_2) & \text{if } \beta = u\beta_1 d\beta_2,\ \beta_1 \in \mathcal{D},\ \beta_2 \in \mathcal{B}^{[1]}, \\ d\hat{\beta}_1 u g(\beta_2) & \text{if } \beta = \bar{u}\beta_1 d\beta_2,\ \beta_1 \in \mathcal{D},\ \beta_2 \in \mathcal{B}^{[1]}. \end{cases}$$

Again, by induction on path-length it follows that g is a bijection. □

Example. As an illustration of the bijections in the proof of Proposition 4, we have



and



Proposition 5. *For $n, k \geq 0$ we have*

$$\operatorname{card} \mathcal{B}^{[2]}_{n,k} = \sum_{i\geq 0} \frac{2k+i+1}{n+k+i+1}\binom{k-1}{k-i}\binom{2n+i}{n-k}.$$

Proof. From (9) it follows that

$$B^{[2]}(q,t) = \frac{1 - t(1+q)C(t)}{1 - t(1+q)(1+C(t))}.$$

Using (4) we rewrite this as

$$B^{[2]}(q,t) = \frac{(1 - qtC(t)^2)C(t)}{1 - (1+C(t))qtC(t)^2}, \tag{11}$$

and on expanding the right hand side as a geometric series we get

$$[q^k]B^{[2]}(q,t) = t^k C(t)^{2k+1}(1+C(t))^{k-1}(\delta_{k,0} + C(t)), \tag{12}$$

where $\delta_{k,0}$ is 1 if $k = 0$, and 0 otherwise. The result is easy to check for $k = 0$, so let us assume that $k \geq 1$. Then

$$[q^k]B^{[2]}(q,t) = t^k \sum_{i\geq 0}\binom{k-1}{i} C(t)^{3k-i+1} = t^k \sum_{i\geq 0}\binom{k-1}{3k-i} C(t)^{i+1}.$$

From (7) we get

$$
\begin{aligned}
[t^n q^k] B^{[2]}(q,t) &= \sum_{i\geq 0} \frac{i+1}{n-k+i+1} \binom{k-1}{3k-i} \binom{2n-2k+i}{n-k} \\
&= \sum_{i\geq 0} \frac{i+1}{n+k+i+1} \binom{k-1}{i-1} \binom{2n-i}{n-k},
\end{aligned}
$$

which concludes the proof. □

3. Segmented permutations

Let $v = v_1 v_2 \cdots v_n$ be a word over $\mathbb{N}$ without repeated letters. We define the *reduction* of v, denoted $\mathrm{red}(v)$, by

$$\mathrm{red}(v)(i) = \mathrm{card}\{\, j : v_j \leq v_i \,\}.$$

In other words, $\mathrm{red}(v)$ is the permutation of $[n]$ obtained from v by replacing the smallest letter in v with 1, the second smallest with 2, etc. For instance, $\mathrm{red}(19453) = 15342$. We will also need a map that is a kind of inverse to red. For a finite subset V of $\mathbb{N}$, with $n = |V|$, and a permutation π of $[n]$, we denote by $\mathrm{red}_V^{-1}(\pi)$ the word over V obtained from π by replacing i in π with the ith smallest element in V, for all i. Here is an example: If $V = \{1,3,4,5,9\}$ then $\mathrm{red}_V^{-1}(15342) = 19453$.

Given π in $\mathcal{S}_n$ and σ in $\mathcal{S}_k$, an *occurrence* of σ in π is a subword

$$o = \pi(i_1)\pi(i_2)\cdots\pi(i_k)$$

of π such that $\mathrm{red}(o) = \sigma$. If, in addition, $i_r + 1 = i_{r+1}$ for each $r = 1, 2, \ldots, k-1$, then o is a *segment-occurrence* of σ in π. We say that π is $(\sigma)^k$*-segmented* if there are exactly k occurrences of σ in π, each of which is a segment-occurrence of σ in π. A $(\sigma)^0$-segmented permutation is usually called σ*-avoiding*, and the set of σ-avoiding permutations of $[n]$ is denoted $\mathcal{S}_n(\sigma)$.

If π is $(\sigma)^k$-segmented for some k, then we say that π is σ*-segmented.* We also define

$$\mathcal{R}_n^k(\sigma) = \{\, \pi \in \mathcal{S}_n : \pi \text{ is } (\sigma)^k\text{-segmented} \,\}$$

and $\mathcal{R}_n(\sigma) = \cup_{k\geq 0} \mathcal{R}_n^k(\sigma)$. In other words, $\mathcal{R}_n(\sigma)$ is the set of σ-segmented permutations of length n. Let

$$R(\sigma; q, t) = \sum_{k,n\geq 0} \mathrm{card}\, \mathcal{R}_n^k(\sigma)\, q^k t^n.$$

The first nontrivial case is $\sigma = 12$. A permutation is 12-segmented if all its non-inversions are rises. For instance, the permutation 7653412 is 12-segmented while 7643512 is not (45 is a non-inversion, but not a rise).

Let $\pi \in \mathcal{R}_n(12)$ with $n \geq 1$. If the letter 1 precedes the letter b in π, then $1b$ is an occurrence of 12 in π. Thus, either 1 is the last letter in π, or 1 is the next last letter in π and 2 is the last letter in π. In terms of the generating function $R = R(12; q, t)$ this amounts to

$$R = 1 + tR + qt^2 R.$$

So R is a rational function in t and q. Extracting coefficients we get

$$\mathrm{card}\, \mathcal{R}_n^k(12) = \tbinom{n-k}{k} \quad \text{and} \quad \mathrm{card}\, \mathcal{R}_n(12) = F_n,$$

where F_n is the nth Fibonacci number (i.e., $F_{n+1} = F_n + F_{n-1}$ with $F_0 = F_1 = 1$).

For the cases $\sigma = 123$ and $\sigma = 132$, we have the following result.

Theorem 6. *Let $k \geq 0$ and $n \geq 3k$.*

The 132-segmented permutations of length n with k occurrences of 132 are in one-to-one correspondence with bicoloured Dyck paths of length $2n-4k$ with k red up-steps. Thus

$$\operatorname{card} \mathcal{R}_n^k(132) = \operatorname{card} \mathcal{B}_{n-2k,k} = \binom{n-2k}{k} C_{n-2k},$$

where the last equality is a consequence of Proposition 1.

The 123-segmented permutations of length n with k occurrences of 123 are in one-to-one correspondence with bicoloured Dyck paths of length $2n-4k$ with k red up-steps, each of height less than 2. Thus

$$\operatorname{card} \mathcal{R}_n^k(123) = \operatorname{card} \mathcal{B}^{[2]}_{n-2k,k} = \sum_{i \geq 0} \frac{2k+i+1}{n-k+i+1} \binom{k-1}{k-i} \binom{2n-4k+i}{n-3k},$$

where the last equality is a consequence of Proposition 5.

First proof. Let n be a positive integer, and let π be a 132-segmented permutation of length n. If the letter n is not part of any occurrence of 132, then we can factor π as $\pi = \pi_1 n \pi_2$, where π_1 and π_2 are 132-segmented permutations, and $\pi_2 < \pi_1$ (i.e., every letter in π_2 is smaller than every letter in π_1). On the other hand, if n is part of an occurrence of 132, then we can factor π as

$$\pi = \pi_1 a n b \, \pi_2, \quad \text{where} \quad \pi_2 < a < b < \pi_1,$$

and π_1 and π_2 are 132-segmented permutations. In particular, $a = |\pi_2| + 1$ and $b = a + 1$. Thus the generating function $R = R(132; q, t)$ satisfies the functional equation

$$R = 1 + (t + qt^3)R^2.$$

It follows that $R = C(t+qt^3)$, where $C(t)$ is the generating function for the Catalan numbers, and hence $[t^n q^k]R = \operatorname{card} \mathcal{B}_{n-2k,k}$, as claimed.

Let $\pi \in \mathcal{R}_n^k(123)$ with $n \geq 1$. Then, either $k = 0$ and π is 123-avoiding, or $k \geq 1$ and π contains at least one occurrence of 123. Let us focus on the latter case, and let

$$\pi = \pi_1 abc \, \pi_2,$$

where abc is the leftmost occurrence of 123 in π. Then $a\pi_2$ is $(123)^{k-1}$-segmented and $\pi_1 c$ is 123-avoiding, with the additional restriction that $a\pi_2$ may not begin with an occurrence of 123. Moreover,

$$a\pi_2 < b < \pi_1 c,$$

or else a non segment-occurrence of 123 would be present. With regard to the generating function $R = R(123; q, t)$ this decomposition of 123-segmented permutations amounts to the functional equation

$$R = C + qt(\tilde{R} - 1)(C - 1), \tag{13}$$

where $C = C(t)$ is the generating function of the Catalan numbers, and the coefficient of $q^k t^n$ in $\tilde{R} = \tilde{R}(q, t)$ is the number of $(123)^k$-segmented permutations of length n that do not begin with an occurrence of 123. Considering the decomposition above in the special case when π_1 is the empty word, we see that $t^2 q(\tilde{R} - 1)$ is the generating function of the number of 123-segmented permutations that begin with an occurrence of 123; so

$$R = \tilde{R} + qt^2(\tilde{R} - 1). \tag{14}$$

Solving equations (13) and (14) for R, eliminating $\tilde{R}$, we get

$$R = \frac{(1 - qt^3C^2)C}{1 - (1 + C)qt^3C^2}. \tag{15}$$

It follows from (11) that $R = B^{[2]}(qt^2, t)$, as claimed. □

Second proof. We shall define a bijection

$$f : \mathcal{R}(132) \to \mathcal{B}^{[2]},$$

such that $|f(\pi)| = 2(n-2k)$ and $|f(\pi)|_{\bar{u}} = k$ whenever $\pi \in \mathcal{R}_n^k(132)$. Our definition of f will be recursive and we start by defining that $f(\epsilon) = \epsilon$, where ϵ denotes the empty word. Now, assume that n is a positive integer, and let π be a 132-segmented permutation of length n. As in the first proof, if the letter n is not part of any occurrence of 132, then we can factor π as $\pi = \pi_1 n \pi_2$, where π_1 and π_2 are 132-segmented permutations, and $\pi_2 < \pi_1$; in this case we define

$$f(\pi) = u(f \circ \mathrm{red})(\pi_1)d(f \circ \mathrm{red})(\pi_2).$$

If n is part of an occurrence of 132, then we can factor π as $\pi = \pi_1 anb\, \pi_2$ where $\pi_2 < a < b < \pi_1$ and π_1 and π_2 are 132-segmented permutations; in this case we define

$$f(\pi) = \bar{u}(f \circ \mathrm{red})(\pi_1)d(f \circ \mathrm{red})(\pi_2).$$

For any β in $\mathcal{B}$, let

$$\lambda(\beta) = \tfrac{1}{2}|\beta| + 2|\beta|_{\bar{u}} = |\beta|_u + 3|\beta|_{\bar{u}}.$$

Using induction, it is plain to show that the inverse of f is given by

$$\begin{aligned} f^{-1}(\epsilon) &= \epsilon; \\ f^{-1}(u\beta_1 d\beta_2) &= (\mathrm{red}_{V_1}^{-1} \circ f^{-1})(\beta_1)\, n\, (\mathrm{red}_{V_2}^{-1} \circ f^{-1})(\beta_2), \end{aligned}$$

where $n = \lambda(\beta_1) + \lambda(\beta_2) + 1$, $V_1 = [\lambda(\beta_2) + 1, n - 1]$, and $V_2 = [1, \lambda(\beta_2)]$;

$$f^{-1}(\bar{u}\beta_1 d\beta_2) = (\mathrm{red}_{V_1}^{-1} \circ f^{-1})(\beta_1)\, anb\, (\mathrm{red}_{V_2}^{-1} \circ f^{-1})(\beta_2),$$

where $a = \lambda(\beta_2) + 1$, $b = a + 1$, $n = \lambda(\beta_1) + b + 1$, $V_1 = [b + 1, n - 1]$, and $V_2 = [1, a - 1]$.

To find a bijective proof of the second part of Theorem 6 we will first discuss a decomposition of paths in $\mathcal{B}^{[2]}$ which is similar to the decomposition of permutations in $\mathcal{R}(123)$ underlying (13). Let $\beta \in \mathcal{B}^{[2]}$. If there is a leftmost occurrence of $\bar{u}$ in β then the height of that $\bar{u}$ must be either 0 or 1. Thus we have

$$\mathcal{B}^{[2]} = \mathcal{D} + \mathcal{D}\bar{u}\mathcal{B}^{[1]}d\mathcal{B}^{[2]} + \mathcal{D}u\mathcal{D}\bar{u}\mathcal{D}d\mathcal{B}^{[1]}d\mathcal{B}^{[2]} \tag{16}$$

whose commutative counterpart is

$$\begin{aligned} B^{[2]} &= C + qtCB^{[1]}B^{[2]} + qt^2C^3B^{[1]}B^{[2]} \\ &= C + qt^{-1}(tC + t^2C^3)tB^{[1]}B^{[2]}. \end{aligned} \tag{17}$$

Since $C = 1 + tC^2$, the factor $tC + t^2C^3$ simplifies to $C - 1$. Moreover, if we let $\tilde{\mathcal{B}}^{[2]}$ denote the set of paths in $\mathcal{B}^{[2]}$ whose first step is u (i.e., not $\bar{u}$), then

$$\tilde{\mathcal{B}}^{[2]} = 1 + u\mathcal{B}^{[1]}d\mathcal{B}^{[2]},$$

and, as a consequence, $tB^{[1]}B^{[2]} = \tilde{B}^{[2]} - 1$. Thus (17) can be rewritten as

$$B^{[2]} = C + qt^{-1}(C-1)(\tilde{B}^{[2]} - 1),$$

which should be compared to (13). This suggests that we should be able to uniquely decompose any path β in $\mathcal{B}^{[2]} \setminus \mathcal{D}$ into two nonempty paths $\beta' \in \mathcal{D}$ and $\beta'' \in \tilde{B}^{[2]}$ such that $|\beta| = |\beta'| + |\beta''| - 1$ and $|\beta|_{\bar{u}} = |\beta''|_{\bar{u}} + 1$. Indeed, using (16), such a decomposition is defined by the map

$$\beta_1\bar{u}\beta_2 d\beta_3 \mapsto \langle\, \beta_1 ud,\, u\beta_2 d\beta_3 \,\rangle,$$
$$\beta_1 u\beta_1'\bar{u}\beta_2'' d\beta_2 d\beta_3 \mapsto \langle\, \beta_1 u\beta_1' u\beta_1'' dd,\, u\beta_2 d\beta_3 \,\rangle,$$

where $\beta_1, \beta_1', \beta_1'' \in \mathcal{D}$, $\beta_2 \in \mathcal{B}^{[1]}$, and $\beta_3 \in \mathcal{B}^{[2]}$. We denote by Φ the inverse of this map; it is obtained by simply reversing the arrows.

Let h be any bijection from $\mathcal{S}_n(123)$ to $\mathcal{D}_n$. For definiteness, we can take h to be the bijection Ψ given by Krattenthaler in [5, p. 522]. (A description of Ψ can be found in the example following this proof.) We shall define a bijection

$$g : \mathcal{R}(123) \to \mathcal{B}^{[2]}$$

such that $|g(\pi)| = 2(n-2k)$ and $|g(\pi)|_{\bar{u}} = k$, whenever $\pi \in \mathcal{R}_n^k(123)$. If π avoids 123 then let $g(\pi) = h(\pi)$. If π does not avoid 123 then, as in the first proof, we can write $\pi = \pi_1 abc\pi_2$, where abc is the leftmost occurrence of 123 in π; in this case, we let

$$g(\pi) = \Phi\big\langle\, (g \circ \mathrm{red})(\pi_1 c),\, (g \circ \mathrm{red})(a\pi_2) \,\big\rangle.$$

Proving that g is invertible is similar to proving that f is invertible. □

We remark that the bijection f from the first part of the preceding proof maps 132-avoiding permutations onto Dyck paths. In fact, the restriction of f to $\mathcal{S}(132)$ is a bijection due to Krattenthaler [5, p. 512].

Example 7. The permutation 846572931 is 132-segmented. It has two occurrences of 132, namely 465 and 293. We illustrate the bijection f, from the first part of the preceding proof, by finding the image of 846572931 under f:

$$f(846572931) = \bar{u}f(84657)df(1) = \bar{u}udf(4657)dud =$$
$$= \bar{u}uduf(465)ddud = \bar{u}udu\bar{u}dddud.$$

For convenience we have not reduced the permutations in the intermediate steps.

To give an example of how g, from the second part of the preceding proof, is applied, we first need to describe Krattenthaler's [5, p. 522] bijection Ψ from $\mathcal{S}_n(123)$ to $\mathcal{D}_n$. Let $\pi = a_1a_2\cdots a_n$ be a 123-avoiding permutation. Determine all the right-to-left maxima in π. A *right-to-left maximum* is an element a_i such that $a_i > a_j$ for all $j > i$. Let the right-to-left maxima in π be $m_1, m_2, \ldots, m_s$, from right to left, so that

$$\pi = \pi_s m_s \cdots \pi_2 m_2 \pi_1 m_1,$$

where π_i is the subword of π between m_{i+1} and m_i. If there is an occurrence ab of 12 in π then abm_i is an occurrence of 123 in π. Therefore, the elements in π_i are in decreasing order. Moreover, we have $\pi_i < \pi_{i+1}$.

The Dyck path $\Psi(\pi)$ is generated from right to left: Read π from right to left. Any right-to-left maximum m_i is translated into $m_i - m_{i-1}$ up-steps (with the convention $m_0 = 0$). Any subword π_i is translated into $|\pi| + 1$ down-steps.

We are now ready for an illustration of g. The permutation 957841362 is 123-segmented. It has two occurrences of 123, namely 578 and 136. To find the image of 957841362 under g we proceed as follows:

$$\begin{aligned}
g(957841362) &= \Phi\langle (g\circ \mathrm{red})(98), (g\circ \mathrm{red})(541362)\rangle;\\
(g\circ \mathrm{red})(98) &= \Psi(21) = udud;\\
(g\circ \mathrm{red})(541362) &= \Phi\langle (g\circ \mathrm{red})(546), (g\circ \mathrm{red})(12)\rangle;\\
(g\circ \mathrm{red})(546) &= \Psi(213) = uuuddd;\\
(g\circ \mathrm{red})(12) &= \Psi(12) = uudd;\\
\Phi\langle uuuddd, uudd\rangle &= u\bar{u}uddudd;\\
\Phi\langle udud, u\bar{u}uddudd\rangle &= ud\bar{u}\bar{u}uddudd.
\end{aligned}$$

Thus $g(957841362) = ud\bar{u}\bar{u}uddudd$.

Corollary 8. *For $k \geq 0$ and $n \geq 0$ we have*

$$\operatorname{card} R_n^k(123) \leq \operatorname{card} \mathcal{R}_n^k(132).$$

Proof. The result follows immediately from $\mathcal{B}_{n,k}^{[2]} \subseteq \mathcal{B}_{n,k}$ and Theorem 6. □

Corollary 9. *The generating functions $R(132; q, t)$ and $R(123; q, t)$ admit the following continued fraction expansions*:

$$R(132;q,t) = \cfrac{1}{1 - \cfrac{t+qt^3}{1 - \cfrac{t+qt^3}{1 - \cfrac{t+qt^3}{\ddots}}}}; \quad R(123;q,t) = \cfrac{1}{1 - \cfrac{t+qt^3}{1 - \cfrac{t+qt^3}{1 - \cfrac{t}{1 - \cfrac{t}{1 - \cfrac{t}{\ddots}}}}}}.$$

Proof. Using (9) and Theorem 6 the result follows from iterating the identity $C(t) = 1/(1 - tC(t))$. □

Proposition 10. *The generating function*

$$R(123, 132;\, p, q, t) = \sum_{\pi \in \mathcal{R}(123)\cap\mathcal{R}(132)} p^{(123)\,\pi} q^{(132)\,\pi} t^{|\pi|}$$

counting $\{123, 132\}$-segmented permutations by occurrences of 123 and 132 is the following rational function:

$$R(123, 132;\, p, q, t) = \frac{1-t}{1 - 2t - (p+q)t^3} = \cfrac{1}{1 - \cfrac{t + (p+q)t^3}{1-t}}.$$

Proof. Let n be a positive integer, and let π be a $\{123, 132\}$-segmented permutation of length n. We distinguish between three cases:

(a) If the letter n is not part of any occurrence of 123 or 132, then we can factor π as $\pi = \pi_1 n \pi_2$, where π_1 is 12-avoiding, π_2 is $\{123, 132\}$-segmented, and $\pi_2 < \pi_1$.

(b) If the letter n is part of an occurrence of 123, then we can factor π as $\pi = \pi_1 abn \pi_2$, where π_1 is 12-avoiding, π_2 is $\{123, 132\}$-segmented, and $\pi_2 < a < b < \pi_1$.

(c) If the letter n is part of an occurrence of 132, then we can factor π as $\pi = \pi_1 anb \pi_2$, where π_1 is 12-avoiding, π_2 is $\{123, 132\}$-segmented, and $\pi_2 < a < b < \pi_1$.

It is clear that an occurrence of 123 can not overlap with an occurrence of 132 without creating a non-segment occurrence of 123 or 132. Therefore, the cases (a) and (b) are mutually exclusive. Thus the generating function $R = R(123, 132; p, q, t)$ satisfies

$$R = 1 + R(12; 0, t)(t + pt^3 + qt^3)R, \tag{18}$$

where $R(12; 0, t) = 1/(1-t)$ is the generating function for 12-avoiding permutations. Solving (18) for R we obtain the desired result. □

4. Tables

card $B_{n,k}$:

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	4	2					
3	5	15	15	5				
4	14	56	84	56	14			
5	42	210	420	420	210	42		
6	132	792	1980	2640	1980	792	132	
7	429	3003	9009	15015	15015	9009	3003	429

card $B^{[1]}_{n,k}$:

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	3	1					
3	5	9	5	1				
4	14	28	20	7	1			
5	42	90	75	35	9	1		
6	132	297	275	154	54	11	1	
7	429	1001	1001	637	273	77	13	1

card $B^{[2]}_{n,k}$:

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	4	2					
3	5	14	13	4				
4	14	48	62	36	8			
5	42	165	264	217	92	16		
6	132	572	1066	1104	670	224	32	
7	429	2002	4186	5130	3965	1912	528	64

5. Acknowledgments

I thank Petter Brändén for several discussions on the subject of this paper. I also thank Einar Steingrímsson for proofreading a preliminary version of this article.

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[BCS]

CATALAN CONTINUED FRACTIONS AND INCREASING SUBSEQUENCES IN PERMUTATIONS

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ABSTRACT. We call a Stieltjes continued fraction with monic monomial numerators a Catalan continued fraction. Let $e_k(\pi)$ be the number of increasing subsequences of length $k+1$ in the permutation π. We prove that any Catalan continued fraction is the multivariate generating function of a family of statistics on the 132-avoiding permutations, each consisting of a (possibly infinite) linear combination of the e_ks. Moreover, there is an invertible linear transformation that translates between linear combinations of e_ks and the corresponding continued fractions.

Some applications are given, one of which relates fountains of coins to 132-avoiding permutations according to number of inversions. Another relates ballot numbers to such permutations according to number of right-to-left maxima.

1. INTRODUCTION AND MAIN RESULTS

We denote by $\mathcal{S}_n$ the set of permutation on $\{1, 2, \ldots, n\}$. Given $\pi = a_1 a_2 \cdots a_n$ in $\mathcal{S}_n$ and $\tau = b_1 b_2 \cdots b_k$ in $\mathcal{S}_k$, we say that π has j occurrences of the *pattern* τ if there are exactly j different sequences $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that the numbers $a_{i_1} a_{i_2} \cdots a_{i_k}$ are in the same relative order as $b_1 b_2 \cdots b_k$. We use the symbol τ also for the permutation statistics defined by $\tau(\pi) = j$ if π has j occurrences of the pattern τ. If $\tau(\pi) = 0$ we say that π is τ-*avoiding*.

Everywhere in this paper a permutation on $S \subset \mathbb{N}$, with $|S| = n$, will be identified with the permutation in $\mathcal{S}_n$ whose letters are in the same relative order as the letters of the given permutation on S. As an example, the permutation 17358 on $\{1, 3, 5, 7, 8\}$ is identified with 14235 in $\mathcal{S}_5$.

Let $\mathcal{S}_n(132)$ be the set of 132-avoiding permutations of length n, and let $\mathcal{S}(132) = \bigcup_{n \ge 0} \mathcal{S}_n(132)$. Suppose $\pi = \pi_1 n \pi_2 \in \mathcal{S}_n(132)$. Then each letter in π_1 must be greater than any letter in π_2, where both π_1 and π_2 must necessarily be 132-avoiding. Conversely, every permutation of this form is clearly 132-avoiding. This observation immediately yields a functional relation for the generating function, $C(x)$, for the number of 132-avoiding permutations according to length, namely

$$C(x) = 1 + xC(x)^2. \tag{1}$$

Readers unfamiliar with the symbolic method implicitly used in this derivation may consult, for example, [3]. Solving for $C(x)$ in (1) we obtain

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x},$$

Date: January 21, 2004.

Key words and phrases. Catalan, continued fraction, 132-avoiding, increasing subsequence, permutation.

which is the familiar generating function of the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$. Thus we have derived the well known fact [5, p. 239] that the cardinality of $\mathcal{S}_n(132)$ is the nth Catalan number. Rewriting (1) in the form

$$C(x) = \frac{1}{1 - xC(x)}$$

and iterating this identity we arrive at the formal continued fraction expansion

$$C(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{\ddots}}},$$

which is the simplest instance of the continued fractions studied in this paper.

A *Stieltjes continued fraction* is a continued fraction of the form

$$C = \cfrac{1}{1 - \cfrac{m_1}{1 - \cfrac{m_2}{\ddots}}},$$

where each m_i is a monomial in some set of variables. We define a *Catalan continued fraction* to be a Stieltjes continued fraction with monic monomial numerators.

For $k \geq 1$, we denote by e_{k-1} the pattern/statistic $1\,2 \cdots k$. Thus $e_0(\pi)$ is the length $|\pi|$ of π, and $e_1(\pi)$ counts the number of non-inversions in π. We also define $e_{-1}(\pi) = 1$ for all permutations π (that is, we declare all permutations to have exactly one increasing subsequence of length 0).

The main purpose of this paper is to show that any Catalan continued fraction is the multivariate generating function of a family of statistics, consisting of linear combinations of the e_ks. Moreover, there is an invertible linear transformation that translates between linear combinations of e_ks and the corresponding continued fractions.

A theorem of Robertson, Wilf and Zeilberger [12] gives a simple continued fraction that records the joint distribution of the patterns 12 and 123 on permutations avoiding the pattern 132.

Generalizations of this theorem have already been given, by Krattenthaler [6], by Mansour and Vainshtein [8] and by Jani and Rieper [4]. However, in none of these papers is there explicit mention of the *joint* distribution of the statistics under consideration. We now state this theorem; it is a generalization of [12, Theorem 1]. Moreover, this theorem is implicit in [8, Proposition 2.3] and it also follows, with minor changes, from the corresponding proofs in [4, Corollary 7] and [6, Theorem 1].

Theorem 1. *The following continued fraction expansion holds:*

$$\sum_{\pi\in\mathcal{S}(132)}\prod_{k\geq 0} x_k^{e_k(\pi)} = \cfrac{1}{1-\cfrac{x_0^{\binom{0}{0}}}{1-\cfrac{x_0^{\binom{1}{0}}x_1^{\binom{1}{1}}}{1-\cfrac{x_0^{\binom{2}{0}}x_1^{\binom{2}{1}}x_2^{\binom{2}{2}}}{1-\cfrac{x_0^{\binom{3}{0}}x_1^{\binom{3}{1}}x_2^{\binom{3}{2}}x_3^{\binom{3}{3}}}{\ddots}}}}}$$

in which the $(n+1)$*st numerator is* $\prod_{k=0}^{n} x_k^{\binom{n}{k}}$.

Proof. Let $\pi = \pi_1 n\pi_2 \in \mathcal{S}_n(132)$. Since every increasing subsequence of length $k+1$ is contained either in π_1, or in π_2, or may consist of a subsequence of length k in π_1 ending with the n in $\pi_1 n\pi_2$, we have

$$e_k(\pi) = e_k(\pi_1) + e_{k-1}(\pi_1) + e_k(\pi_2), \quad k \geq 0.$$

Let $\mathbf{x} = (x_0, x_1, \ldots)$, where the x_is are indeterminates, and let

$$w(\pi;\mathbf{x}) = \prod_{k\geq 0} x_k^{e_k(\pi)}.$$

Then $w(\pi;\mathbf{x}) = x_0 w(\pi_1;\mathbf{x}^*)w(\pi_2;\mathbf{x})$, where $\mathbf{x}^* = (x_0x_1, x_1x_2, \ldots)$. Consequently, the generating function

$$C(\mathbf{x}) = \sum_{\pi\in\mathcal{S}(132)} w(\pi,\mathbf{x})$$

satisfies

$$C(\mathbf{x}) = 1 + x_0 C(\mathbf{x}^*)C(\mathbf{x}),$$

or, equivalently,

$$C(\mathbf{x}) = \frac{1}{1 - x_0 C(\mathbf{x}^*)},$$

and the theorem follows by induction. □

To state and prove our main theorem we need some definitions: Let

$$\mathcal{A} = \{A : \mathbb{N}\times\mathbb{N} \to \mathbb{Z} \mid \forall n\,(A_{nk} = 0 \text{ for all but finitely many } k)\},$$

be the ring of all infinite matrices with a finite number of non zero entries in each row, with multiplication defined by $(AB)_{nk} = \sum_{i=0}^{\infty} A_{ni}B_{ik}$.

With each $A \in \mathcal{A}$ we now associate a family of statistics $\{\langle \mathbf{e}, A_k\rangle\}_{k\geq 0}$, defined on $\mathcal{S}(132)$, where $\mathbf{e} = (e_0, e_1, \ldots)$, A_k is the kth column of A, and

$$\langle \mathbf{e}, A_k\rangle = \sum_i A_{ik} e_i.$$

Let $\mathbf{q} = (q_0, q_1, \ldots)$, where the q_is are indeterminates. For each $A \in \mathcal{A}$ and $\pi \in \mathcal{S}(132)$ we define:

(1) the *weight* $\mu(\pi, A; \mathbf{q})$ of π with respect to A, by

$$\mu(\pi, A; \mathbf{q}) = \prod_{k\geq 0} q_k^{\langle \mathbf{e}, A_k\rangle(\pi)},$$

(2) the multivariate generating function, associated with A, of the statistics $\{\langle \mathbf{e}, A_k\rangle\}_{k\geq 0}$, by

$$F_A(\mathbf{q}) = \sum_{\pi\in\mathcal{S}(132)} \mu(\pi, A; \mathbf{q}),$$

(3) the Catalan continued fraction associated with A, by

$$C_A(\mathbf{q}) = \cfrac{1}{1-\cfrac{\prod q_k^{A_{0k}}}{1-\cfrac{\prod q_k^{A_{1k}}}{1-\cfrac{\prod q_k^{A_{2k}}}{1-\cfrac{\prod q_k^{A_{3k}}}{\ddots}}}}}.$$

Note that the product in part 1 above is finite by the definition of $\mathcal{A}$ together with the fact that $e_i(\pi) = 0$ whenever $i > |\pi|$.

In what follows we will use the convention that $\binom{n}{k} = 0$ whenever $n < k$ or $k < 0$.

Theorem 2. *Let $A \in \mathcal{A}$. Then*

$$F_A(\mathbf{q}) = C_{BA}(\mathbf{q}),$$

where $B = [\binom{i}{j}]$, and conversely

$$C_A(\mathbf{q}) = F_{B^{-1}A}(\mathbf{q}).$$

In particular, all Catalan continued fractions are generating functions of statistics on $\mathcal{S}(132)$ consisting of (possibly infinite) linear combinations of e_ks.

Proof. We have

$$\begin{aligned}
\mu(\pi, A; \mathbf{q}) &= \prod_{k\geq 0} q_k^{\langle \mathbf{e}, A_k\rangle(\pi)} \\
&= \prod_{k\geq 0}\prod_{j\geq 0} q_k^{A_{jk}e_j(\pi)} \\
&= \prod_{j\geq 0}\left(\prod_{k\geq 0} q_k^{A_{jk}}\right)^{e_j(\pi)}.
\end{aligned}$$

Let $x_j = \prod_{k\geq 0} q_k^{A_{jk}}$. Applying Theorem 1 we get a continued fraction in which the $(n+1)$st numerator is

$$\prod_{j\geq 0} x_j^{\binom{n}{j}} = \prod_{j\geq 0}\left(\prod_{k\geq 0} q_k^{A_{jk}}\right)^{\binom{n}{j}} = \prod_{k\geq 0} q_k^{\langle(\binom{n}{0},\binom{n}{1},\binom{n}{2},\dots), A_k\rangle},$$

which is the $(n+1)$st numerator in $C_{BA}(\mathbf{q})$. Hence

$$F_A(\mathbf{q}) = C_{BA}(\mathbf{q}).$$

Observing that $B^{-1} = [(-1)^{i-j}\binom{i}{j}] \in \mathcal{A}$ we also get

$$C_A(\mathbf{q}) = F_{B^{-1}A}(\mathbf{q}).$$

□

Corollary 3. *If $f = \sum_{k\geq 0} \lambda_k e_k$ with $\lambda_k \in \mathbb{Z}$, then the generating function for the statistic f over $\mathcal{S}(132)$ admits the Catalan continued fraction expansion*

$$\sum_{\pi\in\mathcal{S}(132)} x^{f(\pi)} t^{|\pi|} = \cfrac{1}{1 - \cfrac{x^{f(e_0)}t}{1 - \cfrac{x^{f(e_1)-f(e_0)}t}{1 - \cfrac{x^{f(e_2)-f(e_1)}t}{\ddots}}}}.$$

where in the continued fraction e_{k-1} is the permutation $12\cdots k$.

Proof. The result follows from Theorem 2 and the observation

$$\begin{aligned}
f(e_n) - f(e_{n-1}) &= \sum_k \lambda_k \Big(e_k(e_n) - e_k(e_{n-1}) \Big) \\
&= \sum_k \lambda_k \left(\binom{n+1}{k+1} - \binom{n}{k+1} \right) \\
&= \sum_k \lambda_k \binom{n}{k}.
\end{aligned}$$

□

2. Dyck paths

Before giving applications of Theorem 2 we review some theory on Dyck paths and their relation to 132-avoiding permutations.

A *Dyck path* of length $2n$ is a path in the integral plane from $(0,0)$ to $(2n,0)$, consisting of steps of type $u = (1,1)$ and $d = (1,-1)$ and never going below the x-axis. We call the steps of type u *up-steps* and those of type d we call *down-steps*. The *height* of a step in a Dyck path is the height above the x-axis of its left point.

A nonempty Dyck path w can be written uniquely as uw_1dw_2 where w_1 and w_2 are Dyck paths. This decomposition is called the *first return decomposition* of w, because the d in uw_1dw_2 corresponds to the first place, after $(0,0)$, where the path touches the x-axis.

In [6] a bijection Φ between $\mathcal{S}_n(132)$ and the set of Dyck paths of length $2n$ is studied. This bijection, as a function defined on $\mathcal{S}(132)$, can also be defined recursively by

$$\Phi(\varepsilon) = \varepsilon \text{ and } \Phi(\pi) = u\Phi(\pi_1)d\Phi(\pi_2),$$

where $\pi = \pi_1 n \pi_2 \in \mathcal{S}_n(132)$ and ε is the empty permutation/Dyck path. For example, letting Φ operate on the permutation 453612 we successively obtain

$$453612 \;\rightarrow\; u453d12 \;\rightarrow\; uu4d3du1d \;\rightarrow\; uuudduddduudd.$$

In what follows, when we talk about a correspondence between a Dyck path and a 132-avoiding permutation, we will always mean the correspondence defined by Φ.

Using Φ we can express $e_k(\pi)$ in terms of the Dyck path corresponding to π. Namely (see [6]),

$$e_k(\pi) = \sum_{d \text{ in } \Phi(\pi)} \binom{h(d)-1}{k}, \tag{2}$$

where the sum is over all down-steps d in $\Phi(\pi)$ and $h(d)$ is the height of the left point of d. This can also be shown by induction over the length of π. Indeed, for a nonempty 132-avoiding permutation $\pi = \pi_1 n \pi_2$, we have

$$e_k(\pi) = e_k(\pi_1) + e_{k-1}(\pi_1) + e_k(\pi_2).$$

On the other hand, defining $f_k(w) = \sum_{d \textit{ in } w} \binom{h(d)-1}{k}$ for $w = uw_1dw_2$ we have

$$\begin{aligned}
f_k(w) &= \sum_{d \textit{ in } w} \binom{h(d)-1}{k} \\
&= \sum_{d \textit{ in } w_1} \binom{h(d)}{k} + \sum_{d \textit{ in } w_2} \binom{h(d)-1}{k} \\
&= \sum_{d \textit{ in } w_1} \binom{h(d)-1}{k} + \sum_{d \textit{ in } w_1} \binom{h(d)-1}{k-1} + f_k(w_2) \\
&= f_k(w_1) + f_{k-1}(w_1) + f_k(w_2).
\end{aligned}$$

Since $e_k(\varepsilon) = f_k(\varepsilon)$, it follows by induction over the length of π that $f_k(\Phi(\pi)) = e_k(\pi)$, which is the same as (2).

3. Applications

We now give some applications of Theorem 2. Some of these relate known continued fractions to the statistics e_k, whereas others relate these statistics to various other combinatorial structures.

3.1. **A continued fraction of Ramanujan.** The continued fraction

$$R(q,t) = \cfrac{1}{1 - \cfrac{qt}{1 - \cfrac{q^3t}{1 - \cfrac{q^5t}{1 - \cfrac{q^7t}{\ddots}}}}}$$

was studied by Ramanujan (see [10, p. 126]). It was shown in [2] that the coefficient to $t^n q^k$ in the expansion of $R(q,t)$ is the number of Dyck paths of length $2n$ and area k. Using the converse part of Theorem 2, we would like to find the linear combinations of the statistics e_ks that have as bivariate generating function the continued fraction $R(q,t)$. Comparing $R(q,t)$ with the $C_A(\mathbf{q})$ defined just before Theorem 2, we have

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & \cdots \\ 5 & 1 & 0 & 0 & \cdots \\ 7 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since

$$\sum_{k\geq 0} (2k+1)(-1)^{n-k}\binom{n}{k} = \delta_{n0} + 2\delta_{n1},$$

where δ_{ij} is the Kronecker delta, we get

$$B^{-1}A = \begin{pmatrix} 1 & 1 & 0 & \cdots \\ 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and hence, recalling that the coefficient of the linear combinations of the statistics e_k are the columns of this matrix, we have

$$R(q,t) = \sum_{\pi \in \mathcal{S}(132)} q^{e_0(\pi)+2e_1(\pi)} t^{|\pi|},$$

where we prefer to use two different notations $e_0(\pi)$ and $|\pi|$ for the length of π. Thus $R(q,t)$ records the statistic $e_0 + 2e_1$ on 132-avoiding permutations. In fact, the bijection Φ translates the statistic $e_0 + 2e_1$ into the sum of the heights of the steps in the corresponding Dyck path, which in turn is easily seen to equal area.

3.2. **Fountains of coins.** A *fountain of coins* is an arrangement of coins in rows such that the bottom row is full (that is, there are no "holes"), and such that each coin in a higher row rests on two coins in the row below (see Figure 1). Let $F(x,t) = \sum_{n,k} f(n,k) x^k t^n$, where $f(n,k)$ counts the number of fountains with n coins in the bottom row and k coins in total. In [9] it is shown that

$$F(x,t) = \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{x^2 t}{1 - \cfrac{x^3 t}{1 - \cfrac{x^4 t}{\ddots}}}}}$$

A straightforward application of Theorem 2 gives the following result.

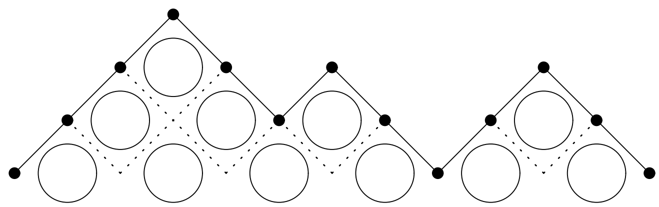
Proposition 4. *The number $f(n,k)$ equals the number of permutations $\pi \in \mathcal{S}_n(132)$ with $(e_0 + e_1)\pi = k$. Equivalently, $f(n,k)$ equals the number of permutations in $\mathcal{S}_n(132)$ with $k - n$ non-inversions.*

If we reverse each permutation in $\mathcal{S}_n(132)$ we see that $f(n,k)$ also equals the number of 231-avoiding permutations in $\mathcal{S}_n$ with exactly $k - n$ inversions.

We also give a combinatorial proof of Proposition 4, by constructing a bijection between the set of Dyck paths of length $2n$ and the set of fountains with n coins in the bottom row. Let Ψ be the bijection that maps a Dyck path to the fountain obtained by placing coins at the centre of all lattice squares inside the path, in the way that Figure 1 suggests.

The ith *slant line* in a fountain is the sequence of coins starting with the ith coin from the left in the bottom row and continuing in the northeast direction. The height of a down-step thus corresponds to the number of coins in the slant line ending at the left point of the down-step d. Now, e_0 counts the number of coins in the bottom row and $\binom{h(d)-1}{1}$ is one less than the number of coins in the

Figure 1. A fountain of coins and the corresponding Dyck path.



corresponding slant line (see the end of Section 2). Thus $e_0 + e_1$ counts the total number of coins in the fountain.

3.3. **Increasing subsequences.** The total number of increasing subsequences in a permutation is counted by $e_0 + e_1 + \cdots$. An application of Theorem 2 gives the following continued fraction for the distribution of $e_0 + e_1 + \cdots$:

$$\sum_{\pi \in \mathcal{S}(132)} x^{e_0\pi + e_1\pi + \cdots} t^{|\pi|} = \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{x^2t}{1 - \cfrac{x^4t}{1 - \cfrac{x^8t}{\ddots}}}}}$$

3.4. **Right-to-left maxima and ballot numbers.** We say that an increasing subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ of $\pi \in \mathcal{S}_n$ is *right maximal* if $\pi(i_k) < \pi(j)$ implies $j < i_k$ (so that the sequence can not be extended to the right).

Proposition 5. *Let $\pi \in \mathcal{S}_n(132)$ and let $m_k(\pi)$ be the number of right maximal increasing subsequences of π of length $k+1$. Then*

$$m_k(\pi) = e_k(\pi) - e_{k+1}(\pi) + e_{k+2}(\pi) - \cdots .$$

In particular, the number of right-to-left maxima in π equals

$$e_0(\pi) - e_1(\pi) + e_2(\pi) - e_3(\pi) + \cdots .$$

Proof. It suffices to prove that for all $\pi \in \mathcal{S}(132)$ and $k \geq 0$ we have $m_k(\pi) + m_{k+1}(\pi) = e_k(\pi)$. The statistic e_k counts all increasing sequences of length $k+1$ in π. If such a sequence is right maximal, it is counted by m_{k+1}. It therefore suffices to show that every increasing subsequence of length k that is not right maximal can be associated to a unique right maximal subsequence of length $k+1$, and conversely.

If an increasing subsequence of length k is not right maximal, it can be extended to a right maximal one of length $k+1$ and we show that this can only be done in one way. Suppose x is the last letter of the original sequence and that the sequence can be extended to a right maximal one by adjoining either y or z, where y comes before z in π. Then y must be greater than z, so x, y, z form a 132-sequence which is contrary to the assumption that π is 132-avoiding.

Conversely, deleting the last letter in a right maximal sequence of length $k+1$ clearly gives a non-right maximal sequence of length k. $\square$

Define
$$M_k(x,t) = \sum_{\pi\in\mathcal{S}(132)} x^{m_k(\pi)} t^{|\pi|}.$$
To apply Corollary 3 we note that
$$m_k(e_n) - m_k(e_{n-1}) = \binom{n}{k} - \binom{n}{k+1} + \binom{n}{k+2} - \cdots = \binom{n-1}{k-1},$$
so the $(n+1)$st numerator in the Catalan continued fraction expansion of $M_k(x,t)$ is $tx^{\binom{n-1}{k-1}}$. Define
$$E_k(x,t) = \sum_{\pi\in\mathcal{S}(132)} x^{e_k(\pi)} t^{|\pi|}.$$
Since $\binom{n-1}{-1}$ is naturally defined to be δ_{n0}, Theorem 2 yields, for all $k \geq -1$, that $E_k(x,t)$ is the continued fraction with $(n+1)$st numerator $tx^{\binom{n}{k}}$. This leads to the following observation.

Proposition 6. *For all $k \geq 0$ we have*
$$M_k(x,t) = \frac{1}{1 - tE_{k-1}(x,t)}.$$

The *ballot number* $b(n,k)$ is the number of paths from $(0,0)$ to $(n+k, n-k)$ that do not go below the x-axis. It is well known that the ballot number $b(n,k)$ is equal to $\frac{n+1-k}{n+1}\binom{n+k}{n}$. Define $B(x,t) = \sum_{n,k} b(n,k)x^k t^n$. Then (see [11, p 152])
$$B(x,t) = \frac{C(xt)}{1 - tC(xt)},$$
where $C(x)$ is the generating function for the Catalan numbers.

Proposition 7. *The number of permutations in $\mathcal{S}_n(132)$ with k right-to-left maxima equals the ballot number*
$$b(n-1, n-k) = \frac{k}{2n-k}\binom{2n-k}{n},$$
and
$$b(n-1,k) = \frac{n-k}{n+k}\binom{n+k}{k}$$
counts the number of permutations of length n with k right maximal increasing subsequences of length two.

Proof. By Proposition 6,
$$M_0(x,t) = \frac{1}{1 - xtC(t)}$$
records the distribution of right-to-left maxima. Since
$$B(x^{-1}, xt) = \frac{C(t)}{1 - xtC(t)}$$
we have
$$M_0(x,t) = 1 + xtB(x^{-1}, xt) = 1 + \sum_{n,k} b(n-1, n-k)x^k t^n,$$

and the first assertion follows. For the second assertion, observe that by Proposition 6,

$$M_1(x,t) = \frac{1}{1 - tC(xt)}.$$

Furthermore,

$$M_1(x,t) = M_0(x^{-1}, xt) = 1 + tB(x,t),$$

which concludes the proof. □

The first assertion of Proposition 7 can be proved bijectively using the map Φ in Section 2. In fact, the number of right-to-left maxima of π is equal to the *number of returns* in $\Phi(\pi)$, that is, the number of times the path $\Phi(\pi)$ intersects the x-axis. This number is known to have a distribution given by $b(n-1, n-k)$ (see [1]).

3.5. **Narayana numbers.** The generating function $N(x,t) = \sum_{n,k} N(n,k)x^k t^n$ for the Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ satisfies the functional equation (see for example [13])

$$N(x,t) = 1 + xtN^2(x,t) - xtN(x,t) + tN(x,t).$$

Equivalently,

$$N(x,t) = \cfrac{1}{1 - \cfrac{t}{1 - xtN(x,t)}}.$$

This allows us to express $N(x,t)$ as a continued fraction:

$$N(x,t) = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{tx}{1 - \cfrac{t}{1 - \cfrac{tx}{\ddots}}}}}.$$

Proposition 8. *The statistic $s = e_1 - 2e_2 + 4e_3 - \cdots$ has the Narayana distribution over $\mathcal{S}(132)$, that is,*

$$\sum_{\pi \in \mathcal{S}(132)} x^{s(\pi)} t^{|\pi|} = \sum_{n,k} N(n,k) x^k t^n.$$

Proof. This follows immediately from Theorem 2 and the identity

$$\sum_{k \text{ odd}} (-1)^{n-k} \binom{n}{k} = (-2)^{n-1}, \text{ for } n > 0.$$

□

Now

$$\sum_{k \geq 1} (-2)^{k-1} f_k(w) = \sum_{k \geq 1} \sum_{d \text{ in } w} (-2)^{k-1} \binom{h(d)-1}{k} = \sum_{d \text{ in } w} \frac{1 + (-1)^{h(d)}}{2}$$

so the interpretation of $e_1 - 2e_2 + 4e_3 - \cdots$ in terms of Dyck paths is the number of down-steps starting at even height, whose distribution is known [7] to be given by the Narayana numbers.

Acknowledgements

Thanks to the anonymous referees, who made several valuable comments on the presentation of this paper.

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A PERMUTATION GROUP DETERMINED BY AN ORDERED SET

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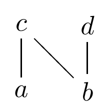
Abstract. Let P be a finite ordered set, and let $J(P)$ be the distributive lattice of order ideals of P. The covering relations of $J(P)$ are naturally associated with elements of P; in this way, each element of P defines an involution on the set $J(P)$. Let $\Gamma(P)$ be the permutation group generated by these involutions. We show that if P is connected then $\Gamma(P)$ is either the alternating or the symmetric group. We also address the computational complexity of determining which case occurs.

Let P be a finite ordered set, and let $J(P)$ be the distributive lattice of order ideals (also called down–sets) of P. For each $p \in P$, define a permutation σ_p on $J(P)$ as follows: for every $S \in J(P)$,

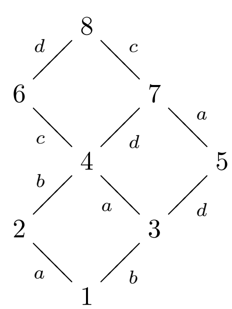
$$\sigma_p(S) := \begin{cases} S \cup \{p\} & \text{if } p \text{ is minimal in } P \smallsetminus S, \\ S \smallsetminus \{p\} & \text{if } p \text{ is maximal in } S, \\ S & \text{otherwise.} \end{cases}$$

Each of these permutations is an involution. We let $\Gamma(P)$ denote the subgroup of the symmetric group $\mathrm{Sym}(J(P))$ generated by all these involutions. Plain curiosity led us to wonder about the structure of these permutation groups. As we shall see, this can be determined quite precisely.

As an example, for

$P =$ 

we may number the down–sets $\{\emptyset, a, b, ab, bd, abc, abd, abcd\}$ of P by 1 through 8, and then

$J(P) =$ 

and

$$\begin{cases} \sigma_a &= (1\ 2)(3\ 4)(5\ 7) \\ \sigma_b &= (1\ 3)(2\ 4) \\ \sigma_c &= (4\ 6)(7\ 8) \\ \sigma_d &= (3\ 5)(4\ 7)(6\ 8) \end{cases}$$

Date: January 21, 2004.

Key words and phrases. ordered set, distributive lattice, permutation group.

Research supported by operating grants from the Natural Sciences and Engineering Research Council of Canada.

in which we have labeled the edges of the Hasse diagram of $J(P)$ to indicate the action of each σ_p on $J(P)$. By using GAP [1] (or otherwise) one finds that $\Gamma(P)$ is the symmetric group $\mathrm{Sym}(J(P))$ in this case.

We use the following notation for ordered sets. The set of minimal elements of P is $P_{\min}$ and the set of maximal elements of P is $P_{\max}$. A covering relation in P is denoted by $a \lessdot b$. For $S \subseteq P$ we let $\downarrow S = \{p \in P : \quad p \leq b \text{ for some } b \in S\}$ denote the down–set (order ideal) generated by S, we let $\uparrow S = \{p \in P : \quad b \leq p \text{ for some } b \in S\}$ denote the up–set (dual order ideal) generated by S, and we let $\updownarrow S = \downarrow S \cup \uparrow S$ be the set of elements comparable with S. The set P with the opposite order is denoted by P^{op}. For more background on finite ordered sets and distributive lattices, see Chapter 3 of Stanley [3], for instance.

The first observation is completely elementary.

Lemma 1. *Let P and Q be disjoint finite ordered sets. Then*

$$\Gamma(P \cup Q) = \Gamma(P) \times \Gamma(Q).$$

Proof. Since $P \cup Q$ is the disjoint union of P and Q we may regard $J(P \cup Q)$ as $J(P) \times J(Q)$ via the bijection $S \leftrightarrow (S \cap P, S \cap Q)$. For such a down–set S of $P \cup Q$ we have $\sigma_p(S) = (\sigma_p(S \cap P), S \cap Q)$ for all $p \in P$, and $\sigma_q(S) = (S \cap P, \sigma_q(S \cap Q))$ for all $q \in Q$. This proves the result. $\square$

The problem is thus reduced to determining $\Gamma(P)$ for connected ordered sets P.

Theorem 1. *Let P be a finite connected ordered set. Then $\Gamma(P)$ is either the alternating group $\mathrm{Alt}(J(P))$ or the symmetric group $\mathrm{Sym}(J(P))$.*

This is, of course, something of a disappointment – we had hoped that some ordered sets would exhibit groups with more interesting structure. Our proof of Theorem 2 is by induction on $|J(P)|$. We begin with a few simple observations.

Lemma 2. *For any finite ordered set P, the permutation group $\Gamma(P)$ acts transitively on $J(P)$.*

Proof. This follows immediately from connectedness of the Hasse diagram of $J(P)$. $\square$

Lemma 3. *For any finite ordered set P, $\Gamma(P^{\mathrm{op}}) \simeq \Gamma(P)$.*

Proof. One checks that the bijection $S \mapsto P \smallsetminus S$ from $J(P)$ to $J(P^{\mathrm{op}})$ commutes with the actions of $\Gamma(P)$ on $J(P)$ and $\Gamma(P^{\mathrm{op}})$ on $J(P^{\mathrm{op}})$. $\square$

An element of an ordered set is *extremal* if it is either minimal or maximal.

Lemma 4. *Every finite connected ordered set P with at least two elements has an extremal element $p \in P$ such that $P \smallsetminus \{p\}$ is also connected.*

Proof. Form the bipartite graph G with bipartition $(P_{\min}, P_{\max})$ and with edges $a \sim b$ whenever $a < b$ in P. Then G has at least two elements, and P is connected if and only if G is connected. Let T be a spanning tree of G, and let p be a leaf of T. Then $G \smallsetminus \{p\}$ is connected, so that $P \smallsetminus \{p\}$ is connected. $\square$

Lemma 5. *Let P be a finite ordered set, and let $p \in P_{\max}$. Then*

$$\frac{1}{2}|J(P)| \leq |J(P \smallsetminus \{p\})| < |J(P)|.$$

Further, if P is connected and $|P| \geq 2$ then the first inequality is strict.

Proof. The second inequality is trivial. Let L be the set of down–sets of P which contain p, so that $J(P) = J(P \smallsetminus \{p\}) \cup L$. The function from L to $J(P \smallsetminus \{p\})$ given by $S \mapsto S \smallsetminus \{p\}$ is injective, so that $|L| \leq |J(P \smallsetminus \{p\})|$ and the first inequality follows. If equality holds then the above function is a bijection, so that $p \in P_{\mathrm{min}} \cap P_{\mathrm{max}}$. When $|P| \geq 2$ this implies that P is not connected. □

Lemma 6. *Let P be a finite ordered set, and let $p \in P_{\mathrm{max}}$. Then $\Gamma(P \smallsetminus \{p\})$ is a quotient of a subgroup of $\Gamma(P)$.*

Proof. The subgroup $H = \langle \sigma_a : \ a \in P \smallsetminus \{p\} \rangle$ of $\Gamma(P)$ has two orbits on $J(P)$ – namely $J(P \smallsetminus \{p\})$ and L, with the notation of the proof of Lemma 6. The homomorphism $\gamma \mapsto \gamma|_{J(P \smallsetminus \{p\})}$ from H to $\Gamma(P \smallsetminus \{p\})$ is surjective, and the result follows. □

Proposition 2. *Let P be a finite connected ordered set. Then $\Gamma(P)$ is 2–transitive (and hence primitive).*

Proof. Since $\Gamma(P)$ is transitive, by Lemma 3, it suffices to show that the stabilizer $\Gamma(P)_\emptyset$ of $\emptyset$ in $\Gamma(P)$ is transitive on $J(P) \smallsetminus \{\emptyset\}$. We prove this by induction on $|P|$, the basis $|P| = 1$ being trivial.

For the induction step $|P| \geq 2$, so that by Lemma 5 there is an extremal element $p \in P$ such that $P \smallsetminus \{p\}$ is connected. By Lemma 4, (replacing P by P^{op} if necessary) we may assume that p is maximal in P.

For each $A \subseteq P_{\mathrm{min}}$, let $J_A(P)$ be the set of down–sets $S \in J(P)$ such that $S \cap P_{\mathrm{min}} = A$. Each of these is a distributive lattice – in fact $J_A(P) \simeq J(P_A)$ in which P_A is obtained by deleting the up–set $\uparrow (P_{\mathrm{min}} \smallsetminus A)$ from P, then deleting the set A of minimal elements of the result; see Figure 1 for an example. The covering relations of $J(P_A)$ correspond to elements of $P_A \subseteq P \smallsetminus P_{\mathrm{min}}$. By Lemma 3, $\Gamma(P_A)$ acts transitively on $J(P_A)$. Therefore, the subgroup $D = \langle \sigma_v : \ v \in P \smallsetminus P_{\mathrm{min}} \rangle$ of $\Gamma(P)$ acts transitively on each of the sets $J_A(P)$ separately, for all $A \subseteq P_{\mathrm{min}}$. In fact, these are the orbits of D acting on $J(P)$. The subgroup D is contained in the stabilizer $\Gamma(P)_\emptyset$.

Now, $P \smallsetminus \{p\}$ is connected, so that $\Gamma(P \smallsetminus \{p\})$ is 2–transitive on $J(P \smallsetminus \{p\})$, by induction. Since $\Gamma(P \smallsetminus \{p\})$ is a quotient of a subgroup of $\Gamma(P)$, it follows that $\Gamma(P)_\emptyset$ is transitive on $J(P \smallsetminus \{p\}) \smallsetminus \{\emptyset\}$ as well. Since $J(P_{\mathrm{min}}) \smallsetminus \{\emptyset\} \subseteq J(P \smallsetminus \{p\}) \smallsetminus \{\emptyset\}$, it follows that $J(P_{\mathrm{min}}) \smallsetminus \{\emptyset\}$ is contained in a single orbit of $\Gamma(P)_\emptyset$ acting on $J(P)$. Since $J(P) \smallsetminus \{\emptyset\}$ is the union of the $J_A(P)$ for all $\emptyset \neq A \subseteq P_{\mathrm{min}}$, it follows that $\Gamma(P)_\emptyset$ acts transitively on $J(P) \smallsetminus \{\emptyset\}$. This completes the induction step, and the proof. □

A well–known lemma ([4] Theorem 13.3) states that if a primitive permutation group of degree n contains a 3–cycle then it contains $\mathrm{Alt}(n)$. We can apply this in the following circumstance. A covering relation $a \lessdot b$ in P is *dominant* provided that every element of P is comparable with either a or b.

Proposition 3. *If a finite ordered set P has a dominant covering relation, then $\mathrm{Alt}(J(P)) \leq \Gamma(P)$.*

Proof. Notice that since P has a dominant covering relation $a \lessdot b$, it follows that P is connected. Proposition 8 thus implies that $\Gamma(P)$ is primitive. We claim that the element $\gamma = \sigma_b \sigma_a \sigma_b \sigma_a$ of $\Gamma(P)$ is a 3–cycle, which suffices to prove the result.

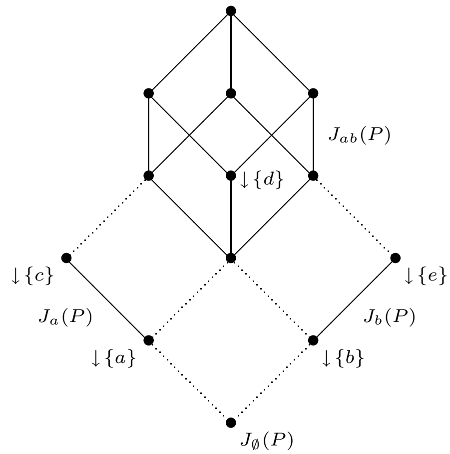
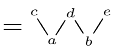


FIGURE 1. The partition of $J(P)$ for $P =$ 

Consider any down–set S of P on which both σ_a and σ_b act nontrivially. Then we have either $a \in S_{\max}$ or $a \in (P \smallsetminus S)_{\min}$, and either $b \in S_{\max}$ or $b \in (P \smallsetminus S)_{\min}$. Since $a < b$ and S is a down–set, the only consistent possibility is that $a \in S_{\max}$ and $b \in (P \smallsetminus S)_{\min}$. If $c \in S_{\max}$ and $c \neq a$, then a and c are incomparable – since $a \lessdot b$ is dominant it follows that $c < b$. Therefore, $S \subseteq \downarrow\{b\} \smallsetminus \{b\}$. Since $b \in (P \smallsetminus S)_{\min}$, it follows that $S = \downarrow\{b\} \smallsetminus \{b\}$. That is, this down–set $\downarrow\{b\} \smallsetminus \{b\}$ is the only element of $J(P)$ on which both σ_a and σ_b act nontrivially. From this and the fact that σ_a and σ_b are involutions, it follows that $\sigma_b\sigma_a$ consists of one 3–cycle and some 2–cycles and fixed points. Therefore $\gamma = (\sigma_b\sigma_a)^2$ is a 3–cycle, as claimed. □

The induction step for the proof of Theorem 2 is a consequence of the following lemma.

Lemma 7. *Let Γ be a primitive group of permutations on a set X with $|X| \geq 9$. Assume that Γ has a subgroup H which has exactly two orbits Y and $\overline{Y}$ on X, such that $|Y| > |\overline{Y}|$ and $\mathrm{Alt}(Y) \leq H|_Y$. Then $\mathrm{Alt}(X) \leq \Gamma$.*

Proof. Let K be the preimage of $\mathrm{Alt}(Y)$ under the quotient map $H \to H|_Y$. If the pointwise stabilizer $K_{\overline{Y}}$ is trivial then K acts faithfully on $\overline{Y}$, and therefore $\mathrm{Alt}(Y)$ acts faithfully on $\overline{Y}$. Since $|\overline{Y}| < |Y|$ this is not possible, so that $K_{\overline{Y}}$ is not trivial. Therefore, H contains a nontrivial element h fixing $\overline{Y}$ pointwise. The conjugates of h under H generate a normal subgroup G of H which has a nontrivial image in $H|_Y$. Since $\mathrm{Alt}(Y)$ is simple it follows that $\mathrm{Alt}(Y) \leq G|_Y$, and since G fixes $\overline{Y}$ pointwise this implies that G (and hence Γ) contains a three–cycle. Since Γ is primitive, it follows that $\mathrm{Alt}(X) \leq \Gamma$. □

Proof of Theorem 2. We prove Theorem 2 by induction on $|J(P)|$. If P is a connected ordered set of width at most two then P contains a dominant covering relation, so that $\mathrm{Alt}(J(P)) \leq \Gamma(P)$ by Proposition 9. If P is a connected ordered set of width at least three, then $|J(P)| \geq 9$. Thus, the basis of induction $|J(P)| \leq 8$ is established. For the induction step, let P be a connected ordered set with $|J(P)| \geq 9$. Replacing P by P^{op}, if necessary (by Lemma 4) we may assume that $p \in P_{\max}$ is such that $P \smallsetminus \{p\}$ is connected (by Lemma 5). Now Lemmas 6 and 7, Proposition 8, and the induction hypothesis imply that $\Gamma = \Gamma(P)$, $X = J(P)$, $H = \langle \sigma_a : \ a \in P \smallsetminus \{p\} \rangle$, and $Y = J(P \smallsetminus \{p\})$ satisfy the hypotheses of Lemma 10. It follows that $\mathrm{Alt}(J(P)) \le \Gamma(P)$, completing the induction step and the proof. $\square$

The only remaining issue is to determine, for each finite connected ordered set, which case of the conclusion of Theorem 2 holds. This seems to be difficult, but it is equivalent to a problem which appears superficially to be easier.

Proposition 4. *Let P be a finite connected ordered set. Then $\Gamma(P) = \mathrm{Alt}(J(P))$ if and only if for every $p \in P$, the cardinality of $J(P \smallsetminus \updownarrow \{p\})$ is even.*

Proof. The statement follows by observing that for each $p \in P$, the two–cycles of the involution σ_p correspond bijectively with the elements of $J(P \smallsetminus \updownarrow \{p\})$. Thus, the condition is equivalent to requiring that $\Gamma(P)$ is contained in $\mathrm{Alt}(J(P))$. $\square$

Proposition 11 suggests the following two decision problems.

The Group Problem:
Instance: A finite connected ordered set P.
Problem: Determine whether $\Gamma(P)$ equals $\mathrm{Alt}(J(P))$ or $\mathrm{Sym}(J(P))$.

The Parity Problem:
Instance: A finite ordered set P.
Problem: Determine whether $|J(P)|$ is even or odd.

A decision problem $\mathcal{A}$ is *polynomially reducible* to a decision problem $\mathcal{B}$ when the following holds: from any instance A of $\mathcal{A}$ of size n one can compute several instances $B_1, \ldots, B_m$ of $\mathcal{B}$ such that:
- the number of operations required to compute $\{B_i\}$ is bounded by a polynomial function of n; and
- given a solution to $\mathcal{B}$ for each B_i, a solution to $\mathcal{A}$ for A can be computed using a number of operations which is bounded by a polynomial function of n.

Two decision problems each of which is polynomially reducible to the other are said to be *polynomially equivalent*. [We are being rather informal with these issues of computational complexity. To be more precise, the size of an instance is the number of bits required to represent it, and the operations discussed above are bit operations. For more details, see Shmoys and Tardos [2].]

Theorem 5. *The Group Problem and the Parity Problem are polynomially equivalent.*

Proof. First, we reduce the Parity Problem to the Group Problem. Given a finite ordered set P as an instance of the Parity Problem, let x, y, z be distinct new elements, and construct the ordered set Q with elements $P \cup \{x, y, z\}$ and order relations given by those of P together with $\{x, y\} \times (P \cup \{z\})$. Then Q is a finite connected ordered set. Assume that we have a solution to the Group Problem for Q. By Proposition 11, we know whether or not all of the $|J(Q \smallsetminus \updownarrow \{b\})|$ for $b \in Q$ are even. Now, if $b \in P$ then $Q \smallsetminus \updownarrow \{b\} = (P \smallsetminus \updownarrow \{b\}) \cup \{z\}$, so that $J(Q \smallsetminus \updownarrow \{b\}) = J(P \smallsetminus \updownarrow \{b\}) \times J(\{z\})$ has even cardinality since $|J(\{z\})| = 2$. Also, if $b \in \{x, y\}$ then $|Q \smallsetminus \updownarrow \{b\}| = 1$ so that $|J(Q \smallsetminus \updownarrow \{b\})| = 2$. Thus, $\Gamma(Q) = \mathrm{Alt}(J(Q))$ if and only if $|J(Q \smallsetminus \updownarrow \{z\})|$ is even. Since $Q \smallsetminus \updownarrow \{z\} = P$, this reduces the Parity Problem to the Group Problem. One checks easily that the computations can be made with only polynomially many operations.

Conversely, we reduce the Group Problem to the Parity Problem. Given a connected finite ordered set P as an instance of the Group Problem, consider the set $\{P \setminus \updownarrow \{p\} :\ p \in P\}$ of instances of the Parity Problem. This set can be computed from P using only polynomially many operations. Given a solution to the Parity Problem for each instance in this set, we check whether all these parities are even – Proposition 11 implies that if so, then $\Gamma(P) = \mathrm{Alt}(J(P))$; otherwise $\Gamma(P) = \mathrm{Sym}(J(P))$. This reduces the Group Problem to the Parity Problem, and completes the proof. $\square$

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