Thesis for the Degree of Doctor of Philosophy

Lie Bialgebra Structures and their Quantization

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Abstract

This thesis consists of four papers in which we discuss various kinds of Lie bialgebra structures, their connection with solutions of the classical Yang-Baxter equation and explicit quantization.

In the first paper, we present the theory of rational solutions of the classical Yang-Baxter equation for a simple compact real Lie algebra $\mathbf{g}$. We prove that, up to gauge equivalence, any rational solution has the form $X(u,v) = \frac{\Omega}{u-v} + t_1 \wedge t_2 + ... + t_{2n-1} \wedge t_{2n}$, where Ω denotes the quadratic Casimir element of $\mathbf{g}$ and $\{t_i\}$ are linearly independent elements in a maximal torus $\mathbf{t}$ of $\mathbf{g}$. The quantization of these solutions is also emphasized.

In the second paper we investigate the rational solutions of the CYBE for $o(n)$ from the point of view of orders in the corresponding loop algebra. In the case of so-called singular vertices, we use the list of connected irreducible subgroups of $SO(n)$ locally transitive on the Grassmann manifolds IG_k^n of isotropic k-dimensional subspaces in $\mathbb{C}^n$, obtained by E. Vinberg and B. N. Kimel'fel'd . New arguments based on the analysis of the structure of the stationary subalgebra of a generic point allow us to find several rational solutions in $o(7)$, $o(8)$ and $o(12)$.

The third article is focused on some Lie bialgebra structures on parabolic subalgebras. Given a complex simple finite-dimensional Lie algebra $\mathbf{g}$ with fixed root system, there exists a so-called classical Drinfeld-Jimbo r-matrix, r. Consider any parabolic subalgebra $P_S \subseteq \mathbf{g}$ defined by a subset S of the set of simple roots. We prove that the Lie bialgebra structure on $\mathbf{g}$ defined by r can be restricted to P_S. Moreover, it turns out that the corresponding classical double $D(P_S)$ is isomorphic to $\mathbf{g} \oplus \mathbf{Red}(P_S)$, where $\mathbf{Red}(P_S)$ denotes the reductive part of P_S.

Finally, in the fourth article, we study classical twists of Lie bialgebra structures on the polynomial current algebra. We focus on the structures induced by so-called quasi-trigonometric solutions of the classical Yang-Baxter equation. We give complete classification for sl_2 and sl_3. For the sl_2 case we also emphasize quantization. We obtain a two-parameter twist of the quantum affine algebra and of the Yangian. Consequently, we determine the deformed quantum R matrices which correspond to quasi-trigonometric and rational solutions in sl_2.

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Keywords: classical Yang-Baxter equation, Lie bialgebra, rational solution, trigonometric solution, locally transitive group, twisting, quantum affine algebra, Yangian, quantization, parabolic subalgebra.

This doctoral thesis consists of an introduction, the following four papers and a discussion.

I. Iulia Pop and Alexander Stolin, *Rational solutions of CYBE for simple compact real Lie algebras.* Preprint 2005:2, Department of Mathematical Sciences, Chalmers, 2005. Submitted also to Journal of Geometry and Physics.

II. Iulia Pop, *Rational solutions of the CYBE and locally transitive actions on isotropic Grassmannians.* Revised version of preprint no. 14, 2003/2004, Mittag-Leffler Institute, Sweden. To appear in Journal of Algebra and Its Applications.

III. Iulia Pop, *On the classical double of parabolic subalgebras.* Communications in Algebra, Vol. **32** No. 10 (2004) 3787-3796.

IV. S. M. Khoroshkin, I. I. Pop, A. A. Stolin, V. N. Tolstoy, *On some Lie bialgebra structures on polynomial algebras and their quantization.* Revised version of preprint no. 21, 2003/2004, Mittag-Leffler Institute, Sweden. Submitted also to Communications in Mathematical Physics.

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I dedicate this thesis to my parents who were the first ones to show me the beauty of mathematics.

LIE BIALGEBRA STRUCTURES AND THEIR QUANTIZATION

IULIA POP

1. General introduction

1.1. **Quantum Yang-Baxter equation.** The Yang-Baxter equation, also known as the *star triangle relation*, the *factorization equation* or the *quantum Yang-Baxter equation*, appeared in several different contexts in literature and sometimes its solutions even preceded the equation. In the end of the 19th century, this equation was taught in the standard high school electricity courses as a device for solving problems in electrical network theory.

One can trace three streams of ideas from which the Yang-Baxter equation has emerged: the Bethe Ansatz, commuting transfer matrices in statistical mechanics, and factorizable S-matrices in field theory.

In the beginning of 1960's, the Yang-Baxter equation occured in the study of a one-dimensional quantum mechanical many-body problem with δ function interaction. By building the Bethe-type wavefunctions, J. M. McGuire [34], F. A. Berezin, V. N. Sushko [8] and others discovered that that the N-particle S-matrix factorized into the product of two-particles ones. In the late 1960's, C. N. Yang treated the case of arbitrary statistics of particles by introducing the so-called *nested Bethe Ansatz*. The Yang-Baxter equation appeared here as the consistency condition for the factorization.

In statistical mechanics, the importance of the Yang-Baxter equation was illustrated by L. Onsager who used it for proving commutation properties of certain *transfer matrices* [36, 37]. Later on, in the work of C. N. Yang [51, 52] and R. J. Baxter [1, 2] from 1970's, the method of producing such commuting matrices advanced, leading to breakthroughs in the study of two-dimensional lattice statistical mechanics and the quantum mechanics of many particle systems on the line. The method is based on the use of a parametrized family of matrices satisfying certain cubic relations. One of these equations became known as the (quantum) Yang-Baxter equation.

Nearly a decade after the pioneering works of C. N. Yang and R. J. Baxter, the theory of factorized S-matrices was resumed in the relativistic setting by A. A. Zamolodchikov and A. Al. Zamolodchikov [53], and other authors, showing how the S-matrix is determined by requiring the factorization problem along with unitarity and crossing symmetry.

Taking into consideration these works and also the development in soliton theory, L. D. Faddeev, L. A. Takhtajan, E. K. Sklyanin and their collaborators in Leningrad proposed the quantum inverse scattering method as a synthesis of classical and quantum integrable systems [42]. The traditional Bethe Ansatz found an algebraisation

in the framework of commutation relations of operators that are derived from the Yang-Baxter equation. E. K. Sklyanin reviewed [44] the quantum inverse scattering method on the examples treated earlier by J. M. McGuire, C. N. Yang and others. It was emphasized that the transition between the quantum and classical systems can be done by preserving integrability.

In [29], P. P. Kulish and E. K. Sklyanin proposed a system of terms and definitions for the theory of the Yang-Baxter equation and formulated the problem of classifying its solutions. They listed the known methods and also various applications of this equation to the theory of integrable quantum and classical systems. One year later, the same authors together with N. Yu. Reshetikhin realized that certain special cases of solutions of the YBE could be found via representation theory [30]. They systematically studied the rational R-matrix associated with $gl(n)$ and showed that the products of R-matrices yield new ones corresponding to symmetric and skew-symmetric tensor representations.

Since the quantum Yang-Baxter equation (QYBE) arises in a variety of contexts, it has different forms. Three fundamental forms are: the constant, the one-parameter and the two-parameter (see for instance [33]).

Let V denote a vector space over a field k and suppose that $R \in End(V \otimes V)$. Define $R^{12} = R \otimes 1_V$, $R^{23} = 1_V \otimes R$, $R^{13} = (1_V \otimes \tau)(R \otimes 1_V)(1_V \otimes \tau)$, where $\tau : V \otimes V \to V \otimes V$ is the twist map. The *constant form* of the QYBE is

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}.$$

Given an arbitrary set X, a subset Z of $X \times X$ and $\varphi : Z \to X$, the *one-parameter form* of the QYBE is defined as

$$R^{12}(x_1)R^{13}(\varphi(x_1,x_2))R^{23}(x_2) = R^{23}(x_2)R^{13}(\varphi(x_1,x_2))R^{12}(x_1),$$

for all $(x_1, x_2) \in Z$, where V is a vector space over k and $R : X \to End(V \otimes V)$. In the full generality of the definition, φ is not assumed to have any special properties. However, classically, X is the set of complex numbers and φ is addition or multiplication.

Finally, the *two-parameter form* is the following

$$R^{12}(u,v)R^{13}(u,w)R^{23}(v,w) = R^{23}(v,w)R^{13}(u,w)R^{12}(u,v),$$

for all u, v, w in X, where X is again a fixed set, and $R : X \times X \to End(V \otimes V)$. An important case is when one takes the complex numbers as parameter set X and an operator $R(u,v)$ depending only on $u - v$. In this case one can write $x = u - v$ and think of R as a one-parameter solution in x. Therefore the equation becomes

$$R^{12}(x)R^{13}(x+y)R^{23}(y) = R^{23}(y)R^{13}(x+y)R^{12}(x).$$

The quantum Yang-Baxter equation has connections not only with quantum integrable systems and statistical mechanics, but also with knot theory and invariants of 3-manifolds (see [38, 39]) The constant form is very closely related to the so-called braid equation:

$$R^{12}R^{23}R^{12} = R^{23}R^{12}R^{23},$$

which is of significant importance in the construction of knot and link invariants.

As an application in pure algebra, let us mention an interesting connection established between the QYBE and Frobenius algebras.

In [3] it was shown by K. I. Beidar, Y. Fong and A. Stolin that every Frobenius algebra over a commutative ring determines a class of solutions of the QYBE, which forms a subbimodule in the tensor square. It was also proved that a generator for this class is invertible in the tensor square if and only if the algebra is Azumaya.

In a second paper [4], these results were applied to Hopf algebras over commutative rings which are finitely generated and projective as modules. An explicit formula of the solution of QYBE was obtained in terms of the integral and antipode. Moreover, this solution was used to give characterizations of separable Hopf algebras over rings. The interaction between Frobenius extensions on the one hand and Hopf subalgebras, solutions of the QYBE, the Jones polynomials and 2-dimensional quantum field theories on the other hand was discussed further by L. Kadison in [26] and [27].

1.2. **Classical Yang-Baxter equation.** The classical Yang-Baxter equation has emerged from the QYBE through the correspondence principle. It was first introduced by E. K. Sklyanin in [43]. Compared to the quantum Yang-Baxter equation, an important and simplifying feature of the classical version is that it can be formulated in the realm of Lie algebras, independently of the way it is represented by matrices.

Like the quantum Yang-Baxter equation, the CYBE has several forms: without spectral parameter or with one (two) spectral parameters.

Let $\mathbf{g}$ denote a finite-dimensional complex Lie algebra and $U(\mathbf{g})$ be its universal enveloping algebra. We recall the following standard notation [5]: if $r = \sum_i y_i \otimes z_i \in \mathbf{g} \otimes \mathbf{g}$, we set $r^{12} = \sum_i y_i \otimes z_i \otimes 1 \in U(\mathbf{g})^{\otimes 3}$, $r^{13} = \sum_i y_i \otimes 1 \otimes z_i \in U(\mathbf{g})^{\otimes 3}$ and $r^{23} = \sum_i 1 \otimes y_i \otimes z_i \in U(\mathbf{g})^{\otimes 3}$.

Definition 1.1. The *classical Yang-Baxter equation without spectral parameter* is the following:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

Any solution of the CYBE is called a *(constant) r-matrix.*

For a general Lie algebra $\mathbf{g}$, there is no known classification or explicit construction of solutions. However, very important results have been obtained when $\mathbf{g}$ is a simple complex finite-dimensional Lie algebra. We will present this case later in detail.

Definition 1.2. The *classical Yang-Baxter equation with one spectral parameter* is

$$[r^{12}(z_1 - z_2), r^{13}(z_1 - z_3)] + [r^{12}(z_1 - z_2), r^{23}(z_2 - z_3)]$$

$$+[r^{13}(z_1 - z_3), r^{23}(z_2 - z_3)] = 0,$$

where $r(z)$ is function of one complex variable z with values in $\mathbf{g} \otimes \mathbf{g}$. Solutions of this equation are called *r-matrices with one spectral parameter.*

One also encounters a third form of the CYBE:

Definition 1.3. The *classical Yang-Baxter equation with two spectral parameters* is

$$[r^{12}(z_1, z_2), r^{13}(z_1, z_3)] + [r^{12}(z_1, z_2), r^{23}(z_2, z_3)]$$
$$+[r^{13}(z_1, z_3), r^{23}(z_2, z_3)] = 0,$$

where $r(z_1, z_2)$ is a function of two complex variables, assuming values in $\mathbf{g} \otimes \mathbf{g}$.

1.3. **The Belavin-Drinfeld classification of nondegenerate solutions with one spectral parameter.** In their famous paper [5] from 1982, A. A. Belavin and V. G. Drinfeld investigated solutions of the CYBE with one spectral parameter, under the assumption that $\mathbf{g}$ is a complex finite-dimensional simple Lie algebra. Moreover, they looked for solutions $r(z)$ in the class of meromorphic functions defined in a neighbourhood of 0 and satisfying one of the equivalent conditions:

(1) The determinant of the matrix formed by the coordinates of the tensor $r(z)$ is not identically equal to 0;
(2) The function $r(z)$ has at least one pole and there does not exist a Lie subalgebra $\mathbf{g}'$ of $\mathbf{g}$ such that $r(z) \in \mathbf{g}' \otimes \mathbf{g}'$;
(3) The function $r(z)$ has for $z = 0$ a pole of first order with residue of the form $c\Omega$, where c is a scalar and $\Omega = \sum I_\mu \otimes I_\mu$. Here $\{I_\mu\}$ denotes an orthonormal basis in $\mathbf{g}$ with respect to the Killing form.

A solution $r(z)$ satisfying one of the three equivalent conditions is called *nondegenerate.* A. A. Belavin and V. G. Drinfeld obtained the classification of nondegenerate solutions, up to the so-called "methods of propagation of solutions". The first method of propagation is the following:

Definition 1.4. Two solutions $r_1(z)$ and $r_2(z)$ defined in a neighbourhood U of the origin are called *equivalent* if there exists a meromorphic function Φ: $U \to Aut(\mathbf{g})$ such that

$$r_1(z - t) = (\Phi(z) \otimes \Phi(t)) r_2(z - t).$$

In order to describe the second method, let us recall the definition of invariant solution:

Definition 1.5. A solution $r(z)$ of the CYBE is called *invariant* with respect to a subalgebra $\mathbf{h} \subseteq \mathbf{g}$ if $[h \otimes 1 + 1 \otimes h, r(z)] = 0$ for any $h \in \mathbf{h}$.

Suppose $r(z)$ is an invariant solution with respect to a subalgebra $\mathbf{h} \subseteq \mathbf{g}$. Let $r_0 \in \mathbf{h} \wedge \mathbf{h}$ be a constant solution of the CYBE. Then $\widetilde{r}(z) := r(z) + r_0$ satisfies the CYBE. If $\mathbf{h}$ is a commutative subalgebra of $\mathbf{g}$, then any $r_0 \in \mathbf{h} \wedge \mathbf{h}$ provides a new solution $r(z) + r_0$ of the CYBE with spectral parameter.

Theorem 1.6. (Belavin-Drinfeld) *Any nondegenerate solution also satisfies the unitarity property, i.e.* $r^{12}(z) = -r^{21}(-z)$*, and extends meromorphically to the entire complex plane. All poles of* $r(z)$ *are simple and they form a discrete subgroup* Γ *of* $\mathbb{C}$.

(a) If Γ *has rank 2, then* $r(z)$ *is an elliptic function.*

(b) If Γ *has rank 1, then* $r(z)$ *is equivalent to a solution of the form* $f(e^{kz})$*, where* f *is a rational function. Such solutions are called* trigonometric.

(c) If $\Gamma = 0$*, then* $r(z)$ *is equivalent to a rational solution.*

Remark 1.7. Moreover, it was proved that elliptic solutions exist only for $\mathbf{g} = sl_n$.

Concerning trigonometric solutions, the complete classification was obtained. It turned out that, up to the methods of propagation of solutions and such trivial transformations like multiplication of a solution by a number and replacement of z by cz, the number of trigonometric solutions in finite.

Let us recall the main result concerning trigonometric solutions. Let σ be an automorphism of the Dynkin diagram Δ of $\mathbf{g}$, C be a corresponding Coxeter automorphism and h the Coxeter number of $(\mathbf{g}, \sigma)$.

We set $\mathbf{h} := \{x \in \mathbf{g} : Cx = x\}$, $\mathbf{g}_j := \{x \in \mathbf{g} : Cx = \omega^j x\}$, where $\omega = e^{2\pi i/h}$. For any $\alpha \in \mathbf{h}^*$, one considers $\mathbf{g}_j^\alpha := \{x \in \mathbf{g} : [a, x] = \alpha(a)x$, for all $a \in \mathbf{h}\}$. Let $\Gamma := \{\alpha \in \mathbf{h}^* : \mathbf{g}_1^\alpha \neq 0\}$ denote the set of simple weights. The projection of Ω on $\mathbf{g}_j \otimes \mathbf{g}_{-j}$ will be denoted by Ω_j.

Definition 1.8. An *admissible* triple is a triple $(\Gamma_1, \Gamma_2, \tau)$, where Γ_1 and Γ_2 are subsets of Γ, τ is a one-to-one map of Γ_1 onto Γ_2 such that

(i) For any $\alpha, \beta \in \Gamma_1$, $(\tau(\alpha), \tau(\beta)) = (\alpha, \beta)$;

(ii) For any $\alpha \in \Gamma_1$ there exists a natural number k such that $\tau^k(\alpha) \notin \Gamma_1$.

Let $(\Gamma_1, \Gamma_2, \tau)$ be an admissible triple. Denote by $\mathbf{a}_i$ the subalgebra of $\mathbf{g}$ which is spanned by the subspaces $\mathbf{g}_1^\alpha$ for all $\alpha \in \Gamma_i$. There exists a unique projector P of $\mathbf{g}$ onto $\mathbf{a}_1$ such that $P(\mathbf{g}_j^\alpha) = 0$ if $\mathbf{g}_j^\alpha \nsubseteq \mathbf{a}_1$. For any $\alpha \in \Gamma_1$ we fix an isomorphism of vector spaces $\mathbf{g}_1^\alpha \cong \mathbf{g}_1^{\tau(\alpha)}$ which can be extended to an isomorphism of Lie algebras $\theta : \mathbf{a}_1 \to \mathbf{a}_2$. This induces a nilpotent operator $\widetilde{\theta}$ by the formula $\widetilde{\theta} := \theta P$ and finally one defines $\psi := \widetilde{\theta}/(1 - \widetilde{\theta})$.

Theorem 1.9. (Belavin-Drinfeld) *Let σ be an automorphism of the Dynkin diagram Δ of $\mathbf{g}$ and C be a corresponding Coxeter automorphism. Let $(\Gamma_1, \Gamma_2, \tau)$ be an admissible triple and $r \in \mathbf{h} \otimes \mathbf{h}$ be a tensor satisfying the system of equations*

$$r^{12} + r^{21} = \Omega_0$$

$$(\tau\alpha \otimes 1)(r) + (1 \otimes \alpha)(r) = 0,$$

for any $\alpha \in \Gamma_1$. Then the function

$$r(z) = r + \frac{1}{e^z - 1}\sum_{j=0}^{h-1} e^{jz/h} - \sum_{j=1}^{h-1} e^{jz/h}(\Psi \otimes 1)\Omega_j + \sum_{j=1}^{h-1} e^{-jz/h}(1 \otimes \Psi)\Omega_{-j}$$

is a solution of the CYBE with set of poles $2\pi i\mathbb{Z}$ and residue Ω at zero. In addition $r(z + 2\pi i) = (C \otimes 1)r(z)$.

Any trigonometric solution with set of poles $2\pi i\mathbb{Z}$ and residue Ω at zero, corresponding to an automorphism σ of the Dynkin diagram, is equivalent to a solution of the above form.

Remark 1.10. [5] The solution constructed above is $\mathbf{h}_0$-invariant, where $\mathbf{h}_0$ denotes the set of all $a \in \mathbf{h}$ such that $\alpha(a) = \tau\alpha(a)$ for any $a \in \Gamma_1$. Thus, adding to this solution any skew-symmetric tensor from $\mathbf{h}_0 \otimes \mathbf{h}_0$ one obtains a new solution. Starting from one solution, one can get in this way all solutions corresponding to an admissible triple. Moreover, the solution $r(z)$ depends on the choice of the isomorphisms $\mathbf{g}_1^\alpha \cong \mathbf{g}_1^{\tau(\alpha)}$, $\alpha \in \Gamma_1$, and change of them leads to the replacement of solution by $(e^{\mathbf{ad}(a)} \otimes e^{\mathbf{ad}(a)})r(z)$, where $a \in \mathbf{h}$.

Regarding rational solutions of the CYBE, A. A. Belavin and V. G. Drinfeld succeeded in finding only some methods of constructing such solutions and left the classification problem open. One decade after, A. A. Stolin developed the theory of rational solutions based on so-called *orders* and obtained some classification [47, 48, 49].

1.4. **Lie bialgebras and classical double.** In 1983 a new meaning was given to the classical Yang-Baxter equation. In [10] V. G. Drinfeld showed that a classical r-matrix induces a Poisson-Lie group structure on the corresponding Lie group. We recall that a *Poisson-Lie group* is a Lie group G together with a grouped Poisson structure (i.e. G is endowed with a Poisson bracket and the group multiplication law is a Poisson map).

Theorem 1.11. (Drinfeld) *Let G be a Lie group and $\mathbf{g}$ its Lie algebra. Choose a basis $\{I_\mu\}$ in $\mathbf{g}$ and let ∂_μ be the right-invariant vector field on G defined by I_μ. Let $r = a^{\mu\nu} I_\mu \otimes I_\nu$ be such that $a^{\mu\nu} = -a^{\nu\mu}$. If $\varphi, \psi \in C^\infty(G)$ set $\{\varphi, \psi\} = a^{\mu\nu}\partial_\mu\varphi\partial_\nu\psi$. Then the following conditions are equivalent:*

(1) The operation $(\varphi, \psi) \mapsto \{\varphi, \psi\}$ is a Poisson bracket;

(2) r satisfies the CYBE.

It turns out that Poisson-Lie groups have a very strong connection with some new algebraic structures called *Lie bialgebras*. In fact, the category of connected and simply-connected Poisson-Lie groups is equivalent to the category of finite-dimensional Lie bialgebras.

Definition 1.12. (Drinfeld) Let $\mathbf{g}$ be a finite-dimensional vector space and suppose that both $\mathbf{g}$ and $\mathbf{g}^*$ have Lie algebra structures. These structures are called *compatible* if

$$c_{rs}^k f_k^{ij} = c_{\alpha r}^i f_s^{j\alpha} - c_{\alpha r}^j f_s^{i\alpha} - c_{\alpha s}^i f_r^{j\alpha} + c_{\alpha s}^j f_r^{i\alpha},$$

where c_{rs}^k and f_k^{ij} are the structure constants of $\mathbf{g}$ and $\mathbf{g}^*$ with respect to dual bases. We will say that $\mathbf{g}$ is given a *Lie bialgebra structure* if $\mathbf{g}$ and $\mathbf{g}^*$ have compatible Lie algebra structures.

Theorem 1.13. (Drinfeld) *Suppose $\mathbf{g}$ and $\mathbf{g}^*$ have fixed Lie algebra structures. Define the linear map $\varphi : \mathbf{g}^* \otimes \mathbf{g}^* \to \mathbf{g}^*$ by setting $\varphi(l_1 \otimes l_2) = [l_1, l_2]$. Then the following conditions are equivalent:*

(1) The Lie algebra structures on $\mathbf{g}$ and $\mathbf{g}^*$ are compatible.

(2) The map $\varphi^* : \mathbf{g} \to \mathbf{g} \otimes \mathbf{g}$ is a 1-cocycle. It is understood that $\mathbf{g}$ acts on $\mathbf{g} \otimes \mathbf{g}$ by means of the adjoint representation.

(3) There is a Lie algebra structure on $\mathbf{g} \oplus \mathbf{g}^*$ inducing the Lie algebra structures on $\mathbf{g}$ and $\mathbf{g}^*$ which is such that the bilinear form Q given by the formula $Q((x_1, l_1), (x_2, l_2)) = l_1(x_2) + l_2(x_1)$ is invariant with respect to the adjoint representation of $\mathbf{g} \oplus \mathbf{g}^*$.

Moreover, the Lie algebra structure on $\mathbf{g} \oplus \mathbf{g}^*$ *referred to in (3) is unique if it exists. This Lie algebra is called the* classical double of $\mathbf{g}$.

In 1986, in his famous talk from Berkeley [11], V. G. Drinfeld defined the notion of Lie bialgebra structure for any Lie algebra $\mathbf{g}$, finite or infinite-dimensional, over a field of characteristic 0:

A *Lie bialgebra structure* on a Lie algebra $\mathbf{g}$ is a 1-cocycle $\delta : \mathbf{g} \to \mathbf{g} \wedge \mathbf{g}$ which satisfies the co-Jacobi identity:

$$Alt(\delta \otimes id)\delta(x) = 0,$$

for any $x \in \mathbf{g}$, where $Alt : \mathbf{g}^{\otimes 3} \to \mathbf{g}^{\otimes 3}$ is the alternation map:

$$Alt(a \otimes b \otimes c) = a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b.$$

The notion of *classical double* is defined as in the finite-dimensional case.

Remark 1.14. There is a one-to-one correspondence between Lie bialgebras and so-called *Manin triples* (see [11], [21]).

A *finite-dimensional Manin triple* is a triple of finite-dimensional Lie algebras $(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2)$, where $\mathbf{p}$ is equipped with a nondegenerate invariant bilinear form such that

(1) $\mathbf{p}_1$, $\mathbf{p}_2$ are Lie subalgebras of $\mathbf{p}$ and $\mathbf{p}_1 \oplus \mathbf{p}_2 = \mathbf{p}$ as vector spaces;

(2) $\mathbf{p}_1$, $\mathbf{p}_2$ are isotropic with respect to the fixed invariant form on $\mathbf{p}$.

The correspondence between Lie bialgebras and Manin triples is constructed in the following way: if $(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2)$ is a Manin triple, then we consider $\mathbf{g} = \mathbf{p}_1$ and define the cocommutator to be the dual to the commutator mapping $\mathbf{p}_2 \otimes \mathbf{p}_2 \to \mathbf{p}_2$. Conversely, if a Lie bialgebra $\mathbf{g}$ is given, then $(\mathbf{g} \oplus \mathbf{g}^*, \mathbf{g}, \mathbf{g}^*)$ is a Manin triple, where $\mathbf{g} \oplus \mathbf{g}^*$ is the classical double of $\mathbf{g}$.

The generalization to the *infinite-dimensional* setting requires an additional condition (see [21]):

(3) The invariant form on $\mathbf{p}$ induces an isomorphism $\mathbf{p}_2 \cong \mathbf{p}_1^*$.

With this definition, the notions of Manin triple and Lie bialgebra are again equivalent. Let us mention that in some papers (see [17], [15]) infinite-dimensional Manin triples are equipped with certain topologies. If $(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2)$ satisfies conditions (1)-(3), then one considers the discrete topology on $\mathbf{p}_1$, the weak topology on $\mathbf{p}_2$ and the product topology on $\mathbf{p} = \mathbf{p}_1 \oplus \mathbf{p}_2$. In a similar manner, if $\mathbf{g}$ is a Lie bialgebra, then $\mathbf{g}$ and $\mathbf{g}^*$ are regarded as topological Lie algebras with respect to the discrete and weak topologies. The classical double $\mathbf{g} \oplus \mathbf{g}^*$ is also equipped with the product of these two topologies.

Let us finally note that in the infinite-dimensional setting, the notion of Manin triple is not "symmetric": if $(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2)$ is a Manin triple and $\dim \mathbf{p}_1 = \infty$, then $(\mathbf{p}, \mathbf{p}_2, \mathbf{p}_1)$ is not necessarily a Manin triple.

1.5. **The Belavin-Drinfeld classification of non-skewsymmetric constant r-matrices.** Given a finite-dimensional Lie algebra $\mathbf{g}$, it is natural to try to find all Lie algebra structures on $\mathbf{g}^*$ compatible with it. This question is discussed in [7] for the case when $\mathbf{g}$ is a simple complex Lie algebra. Any 1-cocycle $\delta : \mathbf{g} \to \mathbf{g} \wedge \mathbf{g}$ is therefore a coboundary, i.e. there exists $s \in \mathbf{g} \wedge \mathbf{g}$ such that

$$\delta(a) = ds(a) := [a \otimes 1 + 1 \otimes a, s],$$

for any $a \in \mathbf{g}$.

On the other hand, δ must satisfy the co-Jacobi identity. This is equivalent to $CYB(s) := [s^{12}, s^{13}] + [s^{13}, s^{23}] + [s^{12}, s^{23}]$ is an invariant element of $\wedge^3 \mathbf{g}$. In literature a skewsymmetric tensor s satisfying this condition is sometimes called an r-matrix as well.

Let Ω be the quadratic Casimir element associated to a nondegenerate invariant form on $\mathbf{g}$. The subspace of $\mathbf{ad}$-invariant elements in $\wedge^3 \mathbf{g}$ is one-dimensional. Then $CYB(s) = \lambda[\Omega^{12}, \Omega^{23}]$, for some complex number λ.

In other terms, the classification of Lie bialgebra structures on $\mathbf{g}$ reduces to the classification of those $r \in \mathbf{g} \otimes \mathbf{g}$ satisfying $CYB(r) = 0$ and $r^{12} + r^{21} = \varepsilon\Omega$, where ε is a complex number (consider $s = r - \frac{\varepsilon}{2}\Omega$). This problem ramifies in two principal cases:

(1) $r^{12} + r^{21} \neq 0$ (strict quasi-triangular case);
(2) $r^{12} + r^{21} = 0$ (triangular case).

The first case reduces in fact to solving the following system

$$r^{12} + r^{21} = \Omega, \tag{1.1}$$

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0. \tag{1.2}$$

All solutions of this system have been found by A. A. Belavin and V. G. Drinfeld in [7]. Let $\mathbf{g}$ be a simple Lie algebra with a fixed nondegenerate invariant form. Let $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ be a triangular decomposition. Let Γ denote the system of simple roots and $(\Gamma_1, \Gamma_2, \tau)$ be an admissible triple (Γ_1 and Γ_2 are subsets of Γ, $\tau : \Gamma_1 \to \Gamma_2$ is an isometry and for any $\alpha \in \Gamma_1$ there exists a natural number k such that $\tau^k(\alpha) \notin \Gamma_1$). Choose a root vector X_α such that $(X_\alpha, X_{-\alpha}) = 1$ and $\tau(X_\alpha) = X_{\tau(\alpha)}$ for all $\alpha \in \mathbb{Z}\Gamma_1$. Define a partial order on the set of positive roots by $\alpha \prec \beta$ if there exists $n > 0$ such that $\tau^n(\alpha) = \beta$. Denote by Ω_0 the "Cartan part" of the Casimir element Ω.

Theorem 1.15. (Belavin-Drinfeld) *In the above setting, let* $r_0 \in \mathbf{h} \otimes \mathbf{h}$ *satisfy the system*

$$r_0^{12} + r_0^{21} = \Omega_0,$$

$$(\tau(\alpha) \otimes 1 + 1 \otimes \alpha)(r_0) = 0,$$

for any $\alpha \in \Gamma_1$. *Then the tensor*

$$r = r_0 + \sum_{\alpha > 0} X_{-\alpha} \otimes X_\alpha + \sum_{\alpha, \beta > 0, \alpha \prec \beta} X_{-\alpha} \wedge X_\beta$$

satisfies (1.1), (1.2). Moreover, any solution of (1.1), (1.2) is of the above form, for a suitable triangular decomposition of $\mathbf{g}$ *and a suitable choice of a basis* $\{X_\alpha\}$.

Remark 1.16. Among these solutions there is the so-called *standard Drinfeld-Jimbo r-matrix* which is obtained for $\Gamma_1 = \Gamma_2 = \emptyset$:

$$r_{DJ} = \sum_{\alpha>0} X_{-\alpha} \otimes X_\alpha + \frac{\Omega_0}{2}.$$

The second (triangular) case has not been completely solved. Its difficulty lies, for instance, in the fact that it contains a subproblem of classification of Frobenius Lie subalgebras. The following result is known from [5]:

Theorem 1.17. (Belavin-Drinfeld) *There is a one-to-one correspondence between triangular structures on* $\mathbf{g}$ *and pairs* (L, B), *where* L *is a Lie subalgebra of* $\mathbf{g}$ *and* B *is a nondegenerate 2-cocycle on* L *with values in the trivial representation* $\mathbb{C}$.

1.6. **Quantization of Lie bialgebras.** The first occurence of a quantum group is in the work [31] of P. P. Kulish and N. Yu. Reshetikhin. In the course of constructing trigonometric solutions of Yang-Baxter equation, they introduced a deformation of the universal enveloping algebra of sl_2. In [45] E. K. Sklyanin defined an elliptic version, in connection with the so-called "eight-vertex model".

The example of Kulish-Reshetikhin was later generalized to arbitrary simple or affine Lie algebras. The quantum group $U_\hbar(\mathbf{g})$ was introduced in 1985 independently by M. Jimbo [23] and V. G. Drinfeld [12]. These works were motivated by the quantum inverse scattering method.

In [12] V. G. Drinfeld observed that the natural language for the description of quantum groups as well as for the quantum inverse scattering method are Hopf algebras. In his report from Berkeley [11], one year later, Drinfeld presented the new theory of quantum groups. Let us also note that Faddeev, Reshetikhin and Takhtajan proposed in [22] a dual approach which is closer to the quantum inverse scattering method. Following the spirit of noncommutative geometry, these authors quantized, instead of a Lie group G, the algebra of functions $Fun(G)$ on it. They introduced the so-called *FRT construction* which connects the constant and one-parameter forms of the QYBE to bialgebras.

A Lie bialgebra is the classical analogue of a Hopf algebra, in the following meaning:

Definition 1.18. By a *(Hopf) quantized universal enveloping algebra* we mean a topological Hopf algebra A over $\mathbb{C}[[\hbar]]$ such that:

(i) the Hopf algebra $A/\hbar A$ is a universal enveloping algebra;

(ii) as a topological $\mathbb{C}[[\hbar]]$-module, A is isomorphic to $V[[\hbar]]$ for some vector space V over $\mathbb{C}$ (a base of neighbourhoods of zero in $V[[\hbar]]$ is given by $\hbar^n V[[\hbar]]$, $n \in \mathbb{N}$).

Remark 1.19. 1) The term "quantized universal enveloping algebra" will be abbreviated QUE-algebra.

2) The term "topological Hopf algebra" means in particular that the comultiplication Δ maps A into the completion $A\hat{\otimes}A$ of the tensor product.

3) The Lie algebra $\mathbf{g}$ such that $A/\hbar A = U(\mathbf{g})$ is unique:

$$\mathbf{g} = \{a \in A/\hbar A : \Delta(a) = a \otimes 1 + 1 \otimes a\}.$$

4) If A is a (Hopf) QUE-algebra with $A/\hbar A = U(\mathbf{g})$, then $\mathbf{g}$ has a Lie bialgebra structure with the cocommutator given by

$$\delta(x) = \hbar^{-1}(\Delta(a) - \Delta^{op}(a)) \bmod \hbar,$$

where a is the inverse image of x in A. The Lie bialgebra $(\mathbf{g}, \delta)$ is called the *classical limit* of A and A is the *quantization* of $(\mathbf{g}, \delta)$.

Recall [11] that a *coboundary Hopf algebra* is a pair (A, R) consisting of a Hopf algebra A and an invertible element $R \in A \otimes A$ such that $\Delta^{op}(a) = R\Delta(a)R^{-1}$, $R^{12}R^{21} = 1$, $R^{12}(\Delta \otimes id)(R) = R^{23}(id \otimes \Delta)(R)$, $(\varepsilon \otimes \varepsilon)(R) = 1$.

A *quasi-triangular Hopf algebra* is a pair (A, R) where R satisfies $(\Delta \otimes id)(R) = R^{13}R^{23}$, $(id \otimes \Delta)(R) = R^{13}R^{12}$, $\Delta^{op}(a) = R\Delta(a)R^{-1}$.

A quasi-triangular Hopf algebra is called *triangular* if $R^{12}R^{21} = 1$.

Definition 1.20. A *coboundary* QUE-algebra is a coboundary Hopf algebra (A, R) such that A is a QUE-algebra and $R \equiv 1 \bmod \hbar$.

We say that a coboundary QUE-algebra A is a *quantization* of a coboundary Lie bialgebra $(\mathbf{g}, dr)$ if

(1) A is a quantization of $\mathbf{g}$

(2) $r = \hbar^{-1}(R - 1) \bmod \hbar$.

Remark 1.21. Similar definitions are given for quasi-triangular and triangular QUE-algebras.

In [14] V. G. Drinfeld formulated a number of questions in quantum group theory, among which were the following:

Question 1. Can every Lie bialgebra be quantized?

Question 2. Does there exist a universal quantization for Lie bialgebras?

Question 3. Given an associative algebra A and a solution $r \in A \otimes A$ to the classical Yang-Baxter equation, does there always exist a formal series

$$R = R(\hbar) = 1 + \hbar r + \sum_{n=2}^{\infty} R_{(n)} \hbar^n$$

satisfying the quantum Yang-Baxter equation?

Question 4. Does there exist a universal solution to the above quantization problem?

Question 5. Does there exist a universal solution to the unitary quantization problem? (if $r^{21} = -r$, then it is natural to look for an R such that $R^{21} = R^{-1}$. This is called the *unitary quantization problem*).

In finite-dimensional case, question 5 was answered by Drinfeld in [13], where the quantization of constant solutions of the CYBE was obtained.

The answer to questions 1-4 is given by the work of P. Etingof and D. Kazhdan [17, 18, 19]. In [17] the existence of a quantization for Lie bialgebras is proved. In the first part the finite-dimensional case is presented. Then the construction is slightly modified to fit the infinite-dimensional setting. Additionally, they proved that any classical r-matrix over an associative algebra A can be quantized and also showed that R is unitary if r is unitary.

In [18] Drinfeld's question of the existence of a universal quantization is discussed. Etingof and Kazhdan showed that there exists a functor from the category of Lie bialgebras to the category of QUE-algebras. This functor defines an equivalence between these two categories.

The third article [19] of this series deals with $\mathbf{g}$-valued functions on a punctured rational or elliptic curve, where $\mathbf{g}$ is a finite-dimensional simple Lie algebra. The result of [17] applies to this case as well but is not sufficiently effective. The paper [19] gives a more manageable quantization procedure for this specific case.

Although any Lie bialgebra admits a quantization, to find explicit quantizations of classical r-matrices is difficult. Let us consider, for instance, the non-skewsymmetric classical r-matrices without spectral parameter from the Belavin-Drinfeld list [7] . In 1985 V. G. Drinfeld introduced the quantum group $U_h(\mathbf{g})$ which was a quantization of the Lie bialgebra structure induced by the standard Drinfeld-Jimbo r-matrix. However, the quantization of the entire list [7] has only recently been obtained by P. Etingof, T. Schedler and O. Schiffmann in [20].

The same question of finding an explicit quantization has been formulated for classical r-matrices with spectral parameter.

The simplest rational solution of the CYBE for a simple complex Lie algebra $\mathbf{g}$ is $r(z) = \frac{\Omega}{z}$, where $\Omega = \sum I_\lambda \otimes I_\lambda$, for a basis $\{I_\lambda\}$ orthonormal with respect to the Killing form on $\mathbf{g}$. This solution induces a Lie bialgebra structure on $\mathbf{g}[z] := \mathbf{g}\otimes_{\mathbb{C}}\mathbb{C}[z]$ by

$$\delta(a(z)) = [a(z) \otimes 1 + 1 \otimes a(t), r(z-t)].$$

In [12] V. G. Drinfeld constructed a quantization of this Lie bialgebra in the following way:

Theorem 1.22. *The Lie bialgebra* $(\mathbf{g}[u], \delta)$ *admits a unique homogeneous quantization* (A, Δ)*. The algebra A regarded as an associative topological algebra with unity is generated by elements* I_λ *and* J_λ *with defining relations:*

$$[I_\lambda, I_\mu] = c^\nu_{\lambda\mu} I_\nu, [I_\lambda, J_\nu] = c^\nu_{\lambda\mu} J_\nu,$$

$$[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = \hbar^2 a^{\alpha\beta\gamma}_{\lambda\mu\nu}\{I_\alpha, I_\beta, I_\gamma\},$$

$$[[J_\lambda, J_\mu], [I_r, J_s]] + [[J_r, J_s], [I_\lambda, J_\mu]] =$$
$$= \hbar^2(a^{\alpha\beta\gamma}_{\lambda\mu\nu} c^\nu_{rs} + a^{\alpha\beta\gamma}_{rs\nu} c^\nu_{\lambda\mu})\{I_\alpha, I_\beta, I_\gamma\},$$

where $c^\nu_{\lambda\mu}$ *are structure constants of* $\mathbf{g}$ *and*

$$a^{\alpha\beta\gamma}_{\lambda\mu\nu} = \frac{1}{24} c^i_{\lambda\alpha} c^j_{\mu\beta} c^k_{\nu\gamma} c^k_{ij},$$

$$\{x_1, x_2, x_3\} = \sum_{i\neq j\neq k} x_i x_j x_k.$$

Here $\deg I_\lambda = 0$ *and* $\deg J_\lambda = 1$*. Moreover,*

$$\Delta(I_\lambda) = I_\lambda \otimes 1 + 1 \otimes I_\lambda$$

$$\Delta(J_\lambda) = J_\lambda \otimes 1 + 1 \otimes J_\lambda + \frac{\hbar}{2} c^\nu_{\lambda\mu} I_\nu \otimes I_\mu.$$

The *Yangian*, denoted by $Y(\mathbf{g})$, is the Hopf algebra over $\mathbb{C}$ obtained by setting $\hbar = 1$ in the above relations. The Yangian is a so-called *pseudotriangular* Hopf algebra. More precisely, for any $a \in \mathbb{C}$ define an automorphism T_a of $Y(\mathbf{g})$ by $T_a(I_\lambda) = I_\lambda$, $T_a(J_\lambda) = J_\lambda + aI_\lambda$. There exists a unique formal series

$$R(z) = 1 + \sum_{k=1}^{\infty} R_k z^{-k},$$

where $R_k \in Y(\mathbf{g}) \otimes Y(\mathbf{g})$, such that

$$(T_z \otimes 1)\Delta^{op}(a) = R(z)((T_z \otimes 1)\Delta(a))R(z)^{-1}$$

for all $a \in Y(\mathbf{g})$. Moreover, R satisfies the QYBE, R is unitary and $R_1 = \Omega$.

Concerning the quantization of other rational solutions, some answer was given in [28, 32]. An explicit quantization of the simplest "nonstandard" rational r-matrix for sl_2, namely $r(z) = \frac{\Omega}{z} + h \wedge f$, was presented.

As we know, another type of nondegenerate solutions of the CYBE with spectral parameter are the trigonometric solutions. An immediate question would be whether it is possible to quantize them all. The most typical ones are the classical solutions associated to the generalized Toda system. In [24] M. Jimbo reported the explicit form of the quantum R-matrix in the fundamental representation for the generalized Toda system associated to non-exceptional affine Lie algebras. Consequently, one obtained a quantization of the corresponding classical solutions.

However, from the Belavin-Drinfeld classification of trigonometric solutions and the theory of rational solutions developed by A. Stolin [47, 48, 49], we see that there exist more solutions than the ones we mentioned. Their explicit quantization represents an interesting challenge.

2. Overview of the thesis

This thesis consists of four papers in which we discuss various kinds of Lie bialgebra structures, their connection with solutions of the classical Yang-Baxter equation and explicit quantization.

I. In the first article, we discuss the theory of rational solutions of the CYBE for a simple compact Lie algebra $\mathbf{g}$ over $\mathbb{R}$. Our method is similar to that developed by A. Stolin for the study of rational solutions for a simple complex Lie algebra in [47, 48, 49], based on the theory of so-called *orders*. We look for functions $X : \mathbb{R}^2 \to \mathbf{g} \otimes \mathbf{g}$ such that the following conditions are satisfied:

$$(2.1)\qquad [X^{12}(u_1,u_2), X^{13}(u_1,u_3)] + [X^{12}(u_1,u_2), X^{23}(u_2,u_3)]+$$
$$+[X^{13}(u_1,u_3), X^{23}(u_2,u_3)] = 0,$$

$$(2.2)\qquad X^{12}(u,v) = -X^{21}(v,u).$$

Let us consider the Killing form K on $\mathbf{g}$. Let $\{I_\lambda\}$ be an orthonormal basis in $\mathbf{g}$ with respect to $(-K)$ and $\Omega := -\sum I_\lambda \otimes I_\lambda$.

Definition 2.1. 1) A solution of (2.1) and (2.2) is called *rational* if it is of the form

$$X(u,v) = \frac{\Omega}{u-v} + r(u,v),$$

where $r(u,v)$ is a polynomial with coefficients in $\mathbf{g}\otimes\mathbf{g}$.

2) Two rational solutions X_1 and X_2 are said to be *gauge equivalent* if there exists $\sigma(u) \in Aut(\mathbf{g}[u])$ such that

$$X_1(u,v) = (\sigma(u)\otimes\sigma(v))X_2(u,v).$$

Here $Aut(\mathbf{g}[u])$ denotes the group of automorphisms of $\mathbf{g}[u]$ considered as an algebra over $\mathbb{R}[u]$.

The main result of the article is the following:

Theorem 2.2. *Up to gauge equivalence, any rational solution of the CYBE for a simple compact Lie algebra* $\mathbf{g}$ *over* $\mathbb{R}$ *has the form*

$$X(u,v) = \frac{\Omega}{u-v} + t_1\wedge t_2 + ... + t_{2n-1}\wedge t_{2n},$$

where $t_1,..., t_{2n}$ *are linearly independent elements in a maximal torus* $\mathbf{t}$ *of* $\mathbf{g}$.

The proof is done in several steps. First, we construct a one-to-one correspondence between rational solutions and certain Lagrangian subalgebras which turn out to be *orders* over $\mathbb{R}((u^{-1}))$.

Theorem 2.3. *Let* $\mathbf{g}$ *be a simple compact Lie algebra over* $\mathbb{R}$ *and* $\mathbf{g}[u] := \mathbf{g}\otimes_{\mathbb{R}}\mathbb{R}[u]$, $\mathbf{g}[[u^{-1}]] := \mathbf{g}\otimes_{\mathbb{R}}\mathbb{R}[[u^{-1}]]$, $\mathbf{g}((u^{-1})) := \mathbf{g}\otimes_{\mathbb{R}}\mathbb{R}((u^{-1}))$.

There is a natural one-to-one correspondence between rational solutions of the CYBE and subalgebras $W \subseteq \mathbf{g}((u^{-1}))$ *such that*

(1) $W \supseteq u^{-N}\mathbf{g}[[u^{-1}]]$ *for some* $N > 0$*;*

(2) $W \oplus \mathbf{g}[u] = \mathbf{g}((u^{-1}))$*;*

(3) W *is a Lagrangian subspace with respect to the invariant nondegenerate bilinear form on* $\mathbf{g}((u^{-1}))$ *given by*

$$(x(u),y(u)) = Tr(\mathbf{ad}x(u)\cdot\mathbf{ad}y(u))_{-1},$$

meaning that we take the coefficient of u^{-1} *in the series expansion of* $Tr(\mathbf{ad}x(u)\cdot \mathbf{ad}y(u))$.

Definition 2.4. An $\mathbb{R}$-subalgebra $W \subseteq \mathbf{g}((u^{-1}))$ is called an *order* in $\mathbf{g}((u^{-1}))$ if there exist two non-negative integers N_1, N_2 such that

$$u^{-N_1}\mathbf{g}[[u^{-1}]] \subseteq W \subseteq u^{N_2}\mathbf{g}[[u^{-1}]].$$

Remark 2.5. Any subalgebra W which satisfies conditions (1) and (3) of Theorem 2.3 is an order. It also turns out that two rational solutions X_1 and X_2 are gauge equivalent if and only if the corresponding orders W_1 and W_2 (via the correspondence from Theorem 2.3) are gauge equivalent, i.e. there exists $\sigma(u) \in Aut(\mathbf{g}[u])$ such that $W_1 = \sigma(u)W_2$.

Then, using some results from the theory of Bruhat-Tits buildings [9], we prove

Theorem 2.6. *Let* $\mathbf{g}$ *be a simple compact Lie algebra over* $\mathbb{R}$*. Then any order* W *in* $\mathbf{g}((u^{-1}))$ *is gauge equivalent to an order contained in* $\mathbf{g}[[u^{-1}]]$*. Consequently, any rational solution of the CYBE for* $\mathbf{g}$ *is gauge equivalent to a solution of the form* $X(u,v) = \frac{\Omega}{u-v} + r$*, where* $r \in \mathbf{g}\wedge\mathbf{g}$ *is a constant* r*-matrix .*

Finally, using the correspondence between skew-symmetric constant r-matrices and pairs (L,B), where L is a subalgebra of $\mathbf{g}$ together with a nondegenerate 2-cocycle $B \in Z^2(L,\mathbb{R})$, we deduce Theorem 2.2.

We also discuss the quantization of the Lie bialgebra structures corresponding to solutions of the above form. The quantization is obtained by *twisting* the real Yangian $Y_\hbar(\mathbf{g})$. Explicit formulae are given.

II. In the second article, we investigate the rational solutions of the CYBE for the simple complex Lie algebra $o(n)$, from the point of view of orders in the corresponding loop algebra. The starting point for our study is the following result obtained in [48]. We refer to [48] for notation.

Theorem 2.7. (Stolin) *Let* α *be a singular vertex of the extended Dynkin diagram of a simple complex Lie algebra* $\mathbf{g}$ *and let* P_α^- *be the corresponding parabolic subalgebra. The set of subalgebras* $W \subseteq \mathbb{O}_\alpha$ *, which correspond to rational solutions, is in one-to-one correspondence with the set of pairs* (L,B) *such that:*

(1) L *is a subalgebra of* $\mathbf{g}$ *and* $L + P_\alpha^- = \mathbf{g}$*;*

(2) B *is a 2-cocycle on* L *which is nondegenerate on* $L\cap P_\alpha^-$*.*

We will say that a subalgebra L of $\mathbf{g}$ *provides a rational solution corresponding to* $\mathbb{O}_\alpha$ if there exists a 2-cocycle B on L such that (L,B) verifies conditions (1) and (2).

Remark 2.8. Condition (1) is equivalent to the fact that $G(L)$ acts locally transitively on $G(\mathbf{g})/G(P_\alpha^-)$ and $1\cdot G(P_\alpha^-)$ is a generic point of this action. For any Lie algebra Λ, $G(\Lambda)$ denotes the Lie group generated by $e^{\mathbf{ad}(x)}$.

Definition 2.9. [16, 47] A Lie algebra F is called *Frobenius* if there exists $f \in F^*$ such that the skew-symmetric bilinear form B_f defined by the formula $B_f(x,y) = f([x,y])$ for any x, $y \in F$, is nondegenerate.

A Lie algebra F is called *quasi-Frobenius* if there is a nondegenerate 2-cocycle B on F with values in $\mathbb{C}$.

Remark 2.10. Condition (2) from Theorem 2.7 implies that $L\cap P_\alpha^-$ is a quasi-Frobenius Lie algebra. On the other hand, if $L\cap P_\alpha^-$ is a Frobenius Lie algebra, then there exists a 2-cocycle B on L such that condition (2) is satisfied.

Our goal is to find examples of subalgebras L of $o(n)$ which provide rational solutions, using Theorem 2.7, particularly the previous remarks. E. Vinberg and B. N. Kimel'fel'd classified in [50] all the connected irreducible subgroups of $SO(n)$ locally transitive on the Grassmann manifolds IG_k^n of isotropic k-dimensional subspaces in

$\mathbb{C}^n$. We give a method which allows us to use the structure of the stationary subalgebra S of a generic point in order to find rational solutions. The idea is to decompose the Lie subalgebra $\widetilde{L}$ of a locally transitive irreducible subgroup of $SO(n)$ into a sum $\widetilde{L} = L+S$ such that $L\cap S$ is a Frobenius Lie algebra. In this way L induces a rational solution. In the cases that are considered in our paper, the stationary subalgebra S turns out to be either parabolic or Frobenius, fact which enables us to construct L which provides us with solutions in o(7), o(8) and o(12). We also show how the solutions in o(5) found in [48] induce solutions in o(8). Finally, in constructing a nonconstant solution in o(12), we encounter a stationary subalgebra S which is a 14-dimensional Frobenius Lie algebra whose unipotent part is noncommutative (see for comparison the classification for the commutative case [16]).

III. In the third article, we deal with Lie bialgebra structures on parabolic subalgebras. Let $\mathbf{g}$ be a simple complex finite-dimensional Lie algebra. We consider the root system R with respect to a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$, Δ the set of simple roots and $\{\mathbf{g}^\alpha\}_{\alpha\in R}$ the root spaces. For any $S \subseteq \Delta$, let us denote by $\Pi(S)$ the set of all $x \in R$ such that if $x = \sum k_\alpha \alpha$, then $k_\alpha \geq 0$ for any $\alpha \in S$. It is known that any parabolic subalgebra can be transformed by an inner automorphism to one of the following subalgebras:

$$P_S = \mathbf{h} \oplus \sum_{\alpha\in\Pi(S)} \mathbf{g}^\alpha.$$

Given a parabolic subalgebra P_S of $\mathbf{g}$, we prove that the Drinfeld-Jimbo r-matrix for $\mathbf{g}$ induces a Lie bialgebra structure δ_r on P_S and we describe the classical double associated to it. The main result of the paper is the following:

Theorem 2.11. *Let $\mathbf{g}$ be a complex simple finite-dimensional Lie algebra. Consider a parabolic subalgebra $P_S \subseteq \mathbf{g}$ defined by a subset S of the set of simple roots. Then the classical double $D(P_S)$, corresponding to the Lie bialgebra structure δ_r, is isomorphic to the Lie algebra $\mathbf{g} \oplus \mathbf{Red}(P_S)$.*

In particular, for $S = \emptyset$, the well-known result [40] $D(\mathbf{g}) \cong \mathbf{g} \oplus \mathbf{g}$ follows.

Remark 2.12. Our result has an infinite-dimensional analogue. Given $\mathbf{g}$ simple, we set $\mathbf{g}[u] := \mathbf{g} \otimes_{\mathbb{C}} \mathbb{C}[u]$, $\mathbf{g}[[u^{-1}]] := \mathbf{g} \otimes_{\mathbb{C}} \mathbb{C}[[u^{-1}]]$ and $\mathbf{g}((u^{-1})) := \mathbf{g} \otimes_{\mathbb{C}} \mathbb{C}((u^{-1}))$. Let us take $\{I_\mu\}$ an orthonormal basis in $\mathbf{g}$ with respect to the Killing form K and set $\Omega := \sum I_\mu \otimes I_\mu$. Let $r_0 = r + \frac{\Omega}{2}$. We consider the map δ: $\mathbf{g}[u] \to \mathbf{g}[u] \wedge \mathbf{g}[v]$ defined by

$$\delta(a(u)) = [\frac{u\Omega}{v-u} + r_0, a(u) \otimes 1 + 1 \otimes a(v)],$$

which gives a Lie bialgebra structure on $\mathbf{g}[u]$. According to the results of F. Montaner and E. Zelmanov [35], the classical double $D(\mathbf{g}[u])$ induced by δ is isomorphic to the direct sum of Lie algebras $\mathbf{g}((u^{-1})) \oplus \mathbf{g}$, which is equipped with the following invariant bilinear form:

$$Q((f(u), a), (g(u), b)) = K(f(u), g(u))_0 - K(a, b),$$

where the index zero means that we have taken the free term in the series expansion.

We notice that $\mathbf{g}[u]$ is a parabolic subalgebra of $\mathbf{g}((u^{-1}))$ and the reductive part of it, $\mathbf{Red}(\mathbf{g}[u]) = \frac{\mathbf{g}[u]}{u\mathbf{g}[u]}$, is isomorphic to $\mathbf{g}$. Therefore $D(\mathbf{g}[u]) \cong \mathbf{g}((u^{-1}))\oplus\mathbf{Red}(\mathbf{g}[u])$, exactly as in the finite-dimensional case.

IV. The goal of the fourth article is to reconsider Belavin-Drinfeld classification of trigonometric solutions [5] from the point of view of Lie bialgebra structures on $\mathbf{g}[z]$. As we have seen, the CYBE is strongly related to the fundamental concepts of Lie bialgebra and classical double. We are interested in the description of the classical double corresponding to Lie bialgebra structures on $\mathbf{g}[z]$. In the work of F. Montaner and E. Zelmanov [35] it was proved that there exist only four types of classical doubles. We will consider two of them: $\mathbf{g}((u^{-1}))$ and $\mathbf{g}((u^{-1})) \oplus \mathbf{g}$.

A natural question arises: which nondegenerate solutions, after a suitable change of variables, may be used to induce Lie bialgebra structures on $\mathbf{g}[z]$?

Let us consider rational solutions. It was proved in [47] that this type of solutions provide Lie bialgebra structures on $\mathbf{g}[z, z^{-1}]$ which can be reduced to $\mathbf{g}[z]$. The corresponding classical double was shown to be $\mathbf{g}((z^{-1}))$.

For trigonometric solutions, the situation is different. Any trigonometric solution has the form $f(e^{ku})$, where f is a rational function. After setting $e^{ku} = \frac{z}{t}$, this solution does not induce, generally speaking, a Lie bialgebra structure on $\mathbf{g}[z]$.

Therefore we are motivated to introduce a new class of solutions of *trigonometric type* that will induce Lie bialgebra structures on $\mathbf{g}[z]$. Let Ω denote the quadratic Casimir element of $\mathbf{g}$. We say that a solution X of the CYBE is *quasi-trigonometric* if it is of the form:

$$X(z, t) = \frac{t\Omega}{z-t} + p(z, t),$$

where $p(z, t)$ is a polynomial with coefficients in $\mathbf{g} \otimes \mathbf{g}$. We prove that by applying a certain holomorphic transformation and a change of variables, any quasi-trigonometric solution becomes trigonometric, in the sense of Belavin-Drinfeld classification.

We focus on the study of *quasi-trigonometric* r-matrices for a simple complex finite-dimensional Lie algebra $\mathbf{g}$. Any quasi-trigonometric solution X of the CYBE induces a Lie bialgebra structure on $\mathbf{g}[z]$ by considering the 1-cocycle δ_X defined by

$$\delta_X(a(z)) = [X(z, t), a(z) \otimes 1 + 1 \otimes a(t)],$$

for any $a(z) \in \mathbf{g}[z]$. The Lie bialgebra structures associated to different quasi-trigonometric solutions are *twisted* to each other, in the sense of [15]. Therefore one expects that the corresponding classical doubles are isomorphic. Indeed, we prove

Theorem 2.13. *Let X be a quasi-trigonometric solution and δ_X be the Lie bialgebra structure on $\mathbf{g}[z]$ induced by it. The corresponding classical double $D_X(\mathbf{g}[z])$ is isomorphic to the direct sum of Lie algebras $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$, where we consider the following invariant bilinear form:*

$$Q((f(z), a), (g(z), b)) = K(f(z), g(z))_0 - K(a, b).$$

Here the index zero means that we have taken the free term in the series expansion.

Remark 2.14. The Lie algebra $\mathbf{g}[z]$ is naturally identified with

$$V_0 := \{(a(z), a(0)); a(z) \in \mathbf{g}[z]\}.$$

Moreover we construct a one-to-one correspondence between this type of solutions and a special class of Lagrangian subalgebras of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$:

Theorem 2.15. *There exists a natural one-to-one correspondence between quasi-trigonometric solutions of the CYBE and linear subspaces W of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ such that*

1) W is a Lie subalgebra in $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ such that $W \supseteq z^{-N}\mathbf{g}[[z^{-1}]]$ for some $N > 0$;

2) $W \oplus V_0 = \mathbf{g}((z^{-1})) \oplus \mathbf{g}$;

3) W is a Lagrangian subspace with respect to the inner product of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$.

We consider quasi-trigonometric solutions up to gauge equivalence. As expected, two quasi-trigonometric solutions are gauge equivalent if and only if the corresponding Lagrangian subalgebras of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ are gauge equivalent.

In the case $\mathbf{g} = sl_n$, the correspondence given by Theorem 2.15 can be exploited further:

Theorem 2.16. *Let W be a Lie subalgebra corresponding to a quasi-trigonometric solution in sl_n. Then, up to a gauge equivalence, W is embedded into $L_k := d_k^{-1} sl_n[[z^{-1}]] d_k \oplus sl_n$, where $d_k = diag(1, ..., 1, z, ...z)$ (k-many 1's) and $0 \le k \le [\frac{n}{2}]$.*

Moreover, the problem of finding all such W which are contained in L_k can be replaced by a finite-dimensional problem.

Theorem 2.17. *There is a bijection between the set of subalgebras W of L_k, corresponding to quasi-trigonometric solutions, and the set of Lagrangian subalgebras $\overline{W}$ of $sl_n \oplus sl_n$ such that $\overline{W} \oplus \Delta_k = sl_n \oplus sl_n$, where*

$$\Delta_k = \{(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \begin{pmatrix} A & \tilde{B} \\ 0 & D \end{pmatrix})\}.$$

(here A and D are matrix blocks of order k and $n-k$ respectively).

In the cases sl_2 and sl_3, we describe the Lagrangian subalgebras $\overline{W}$ which are complementary to Δ_k, and we obtain the complete classification of *nontrivial* quasi-trigonometric solutions. Here *nontrivial* means that the polynomial part is nonconstant.

We are also interested in obtaining an explicit quantization of the unique (up to gauge equivalence) nontrivial quasi-trigonometric solution for sl_2:

$$X_{a,b}(z_1, z_2) = \frac{z_2\Omega}{z_1 - z_2} + \sigma^- \otimes \sigma^+ + \frac{1}{4}\sigma^z \otimes \sigma^z$$
$$+a(z_1\sigma^- \otimes \sigma^z - z_2\sigma^z \otimes \sigma^-) + b(\sigma^- \otimes \sigma^z - \sigma^z \otimes \sigma^-).$$

for any nonzero constants a and b (here σ^z, σ^-, σ^+ is the standard basis of sl_2). Our method is based on the following conjecture:

Conjecture 2.18. *Any classical twist can be extended to a quantum twist.*

We support this conjecture by constructing a two-parameter twist of the quantum affine algebra $U_q(\widehat{sl_2})$. Surprisingly, it has the simple form of a q-power function, but with q-commuting arguments (we refer to the article for notation). Using generalizations of Faddeev-Volkov identities, we prove:

Theorem 2.19. *The element*

$$\widetilde{F} = (1 - (2)_{q^2}(a \cdot 1 \otimes e_{\delta-\alpha} + bq^{-h} \otimes q^{-h}e_{-\alpha}))_{q^2}^{(-\frac{h\otimes 1}{2})}$$

is a quantum twist of $U_q(\widehat{sl_2})$ for any constants a and b.

Consequently, we determine the quantum R matrix which is a quantization of $X_{a,b}(z_1, z_2)$. Moreover, the Yangian degeneration of the quantum twist $\widetilde{F}$ becomes the usual power function whose arguments belong to an additive variant of the Manin q-plane. We obtain an explicit quantization of the unique nontrivial rational solution in sl_2:

$$r(u_1, u_2) = \frac{\Omega}{u_1 - u_2} + \xi(u_1\sigma^- \otimes \sigma^z - u_2\sigma^z \otimes \sigma^-).$$

Let us stress the fact that the quantization of this solution has not been obtained by other methods. Thus we answer the question of quantization of all the rational solutions of the CYBE for sl_2 (see also [28, 32]).

References

[1] Baxter R.J., *Partition function of the eight-vertex lattice model*, Ann.Phys. **70** (1972) 193-228.
[2] Baxter R.J., *Solvable eight-vertex model on an arbitrary planar lattice*, Phil. Trans. Royal Soc. London **289** (1978) 315-346.
[3] Beidar K.I., Fong Y., Stolin A., *On Frobenius algebras and the quantum Yang-Baxter equation*, Trans. AMS **349**, no. 9 (1997) 3823-3836.
[4] Beidar K.I., Fong Y., Stolin A., *On antipodes and integrals in Hopf algebras over rings and the quantum Yang-Baxter equation*, J. Algebra **194** (1997) 36-52.
[5] Belavin A.A., Drinfeld V.G., *On classical Yang-Baxter equation for simple Lie algebras*, Funct. Anal. and Appl. **16** no. 3 (1982) 1-29.
[6] Belavin A.A., Drinfeld V.G., *On classical Yang-Baxter equations for simple Lie algebras*, Funct. Anal. and Appl. **17** no. 3 (1983) 69-70.
[7] Belavin A.A., Drinfeld V.G., *Triangle equation and simple Lie algebras*, Soviet Sci. reviews, Section C **4** (1984) 93-165.
[8] Berezin F.A., Sushko V.N., *Relativistic two-dimensional model of a self-interacting fermion field with nonvanishing rest mass*, Sov. Phys. JETP **21** (1965) 865-873.
[9] Bruhat F., Tits J., *Groupes reductifs sur un corp local, I.*, Publ. Math. IHES, **41** (1972).
[10] Drinfeld V.G., *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations*, Soviet Math. Dokl. **27** (1983) 68-71.
[11] Drinfeld V.G., *Quantum groups*, Proceedings ICM (Berkeley 1986) **1** (1987), AMS 798-820.
[12] Drinfeld V.G., *Hopf algebras and the quantum Yang-Baxter equation*, Sov. Math. Doklady **32** (1985) 254-258.
[13] Drinfeld V.G., *On constant, quasiclassical solutions of the Yang-Baxter quantum equation*, Soviet Math. Dokl. **28** no. 3 (1983) 667-671.

[14] Drinfeld V.G., *On some unsolved problems in quantum groups theory*, Lect. Notes. Math. **1510** (1992) 1-8.
[15] Drinfeld V.G., *Quasi-Hopf algebras*, Leningrad Math. Journal **1** (1990) 1419-1457.
[16] Elashvili A., *Frobenius Lie algebras.* Funct. Anal. and Appl. **16**, no. 4, (1982) 94-95.
[17] Etingof P., Kazhdan D., *Quantization of Lie bialgebras I.* Selecta Math. **2**, no. 1 (1996) 1-41.
[18] Etingof P., Kazhdan D., *Quantization of Lie bialgebras II.* Selecta Math. **4**, no. 2 (1998) 213-231.
[19] Etingof P., Kazhdan D., *Quantization of Lie bialgebras III.* Selecta Math. **4**, no. 2 (1998) 233-269.
[20] Etingof P., Schedler T., Schiffmann O., *Explicit quantization of dynamical R-matrices for finite-dimensional semisimple Lie algebras*, J. AMS **13** (2000) 595-609.
[21] Etingof P., Schiffmann O., *Lectures on Quantum Groups*, International Press, Cambridge, MA, 1998.
[22] Faddeev L.D., Rheshetikhin N.Yu., Takhtajan L.A., *Quantization of Lie groups and Lie algebras*, in *Algebraic Analysis* vol. **1** Eds. M. Kashiwara and T. Kawai, Academic, Boston (1989) 129-139.
[23] Jimbo M., *A q-difference analogue of $U(\mathbf{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 63-69.
[24] Jimbo M., *Quantum R-matrix for the generalized Toda system*, Commun. Math. Phys. **102** no.1 (1986) 537-547.
[25] Jimbo M. (editor), *Yang-Baxter equation in integrable systems*, Advances Series in Mathematical Physics **10** (1989).
[26] Kadison L., *New examples of Frobenius extensions*, University Lecture Series **14**, AMS, 1999.
[27] Kadison L., Stolin A., *A Frobenius systems approach to Hopf algebras over rings*, Chalmers preprint, 1999.
[28] Khoroshkin S.M., Stolin A.A., Tolstoy V.N., *Deformation of the Yangian $Y(sl_2)$*, Commun. Alg. **26** no.3 (1998) 1041-1055.
[29] Kulish P.P., Sklyanin E.K., *Solutions of the Yang-Baxter equation*, Zapiski Nauch. Sem. LOMI **95** (1980) 129-160.
[30] Kulish P.P., Reshetikhin N.Yu., Sklyanin E.K., *Yang-Baxter equation and representation theory I*, Lett. Math. Phys. **5** (1981) 393-403.
[31] Kulish P.P., Reshetikhin N.Yu., *Quantum linear problem for the sine-Gordon equation and higher representations*, J. Soviet Math. **23** (1983) 2435-2441.
[32] Kulish P.P., Stolin A.A., *New rational solutions of Yang-Baxter equation and deformed Yangians*, Czech. J. Phys. **47** (1997) 123-129.
[33] Lambe L.A., Radford D.E., *Introduction to the Quantum Yang-Baxter Equation and Quantum Groups: An Algebraic Approach*, Kluwer Academic Publishers, Dordrecht, 1997.
[34] McGuire J.B., *Study of exactly solvable one-dimensional N-body problem*, J. Math.Phys. **5** (1964) 622-636.
[35] Montaner F., Zelmanov E., *Bialgebra structures on current Lie algebras*, preprint, University of Wisconsin, Madison, 1993.
[36] Onsager L., *Cristal statistics I. A two dimensional model with an order-disorder transition*, Physics Reports **65** (1944) 117-149.
[37] Onsager L., in *Critical Phenomena in Alloys, Magnets and Superconductors*, Eds. R.E. Mills, E. Ascher and R.J. Jaffe, McGraw-Hill, New York, 1971.
[38] Reshetikhin N., *Invariants of links and 3-manifolds related to quantum groups.* In *Proc. Int. Congr. Math, Kyoto, August 21-29, 1990*, vol. II, 1373-1375. Tokyo, Springer-Verlag.
[39] Reshetikhin N.Yu., Turaev V.G., *Invariants of 3-manifolds via link polynomials and quantum groups.* Invent. Math. **103** (3) 547-597.
[40] Reshetikhin N., Semenov-Tian-Shansky M., *Quantum R-matrix and factorization problems*, J. Geom. Phys. **5** (1988) 533-550.

[41] Sklyanin E.K., Faddeev L.D., *Quantum-mechanical approach to completely integrable field theory models*, Dokl. Akad. Nauk. CCCP **243** (1978) 1430-1433.
[42] Sklyanin E.K., Takhtajan L.A., Faddeev L.D., *Quantum inverse problem method **I**,* Teoret. Mat. Fiz. **40** (1979) 194-220.
[43] Sklyanin E.K., *On complete integrability of the Landau-Lifschitz equation*, preprint E-3-1979 (1979).
[44] Sklyanin E.K., *Quantum version of the method of inverse scattering problem*, J. Soviet Math. **19** (1982) 1546-1596.
[45] Sklyanin E.K., *Some algebraic structures connected with the Yang-Baxter equation*, Funct. Anal. Appl. **16** (1983) 263-270; **17** (1984) 273-284.
[46] Schnider S., Sternberg S., *Quantum groups. From coalgebras to Drinfeld algebras. A guided tour*, volume II of *Graduate Texts in Mathematical Physics.* International Press, Cambridge, MA, 1993.
[47] Stolin A.A., *On rational solutions of Yang-Baxter equation for $sl(n)$*, Math. Scand. **69** (1991) 57-80.
[48] Stolin A. A., *On rational solutions of Yang-Baxter equation. Maximal orders in loop algebra.* Comm. Math. Phys., **141** (1991) 533-548.
[49] Stolin A., *A geometrical approach to rational solutions of the classical Yang-Baxter equation. Part I.* Symposia Gaussiana, Conf. Alg., ed. by Walter de Gryter & Co., Berlin, New York, 1995, 347-357.
[50] Vinberg E., Kimel'fel'd B. N., *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups.* Sov. J. Funct. Anal. **12** no. 3 (1978) 12-19.
[51] Yang C.N., *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19** (1967) 1312-1314.
[52] Yang C.N., *S matrix for the one-dimensional N-body problem with repulsive or attractive δ-function interaction*, Phys. Rev. **168** (1968) 1920-1923.
[53] Zamolodchikov A.B., Zamolodchikov Al.B., *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models*, Ann. Phys. **120** (1979) 253-291.

RATIONAL SOLUTIONS OF CYBE FOR SIMPLE COMPACT REAL LIE ALGEBRAS

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Abstract. In [8,9,10] a theory of rational solutions of the classical Yang-Baxter equation for a simple complex Lie algebra $\mathbf{g}$ was presented. We discuss this theory for simple compact real Lie algebras $\mathbf{g}$. We prove that up to gauge equivalence all rational solutions have the form $X(u,v) = \frac{\Omega}{u-v} + t_1 \wedge t_2 + ... + t_{2n-1} \wedge t_{2n}$, where Ω denotes the quadratic Casimir element of $\mathbf{g}$ and $\{t_i\}$ are linearly independent elements in a maximal torus $\mathbf{t}$ of $\mathbf{g}$. The quantization of these solutions is also emphasized.

Keywords: classical Yang-Baxter equation, rational solution, Bruhat-Tits building, quasi-Frobenius Lie algebra, Yangian.
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1. Introduction

In their outstanding paper from 1982, A. A. Belavin and V. G. Drinfeld obtained an almost complete classification of solutions of the classical Yang-Baxter equation with spectral parameter for a simple complex Lie algebra $\mathbf{g}$. These solutions are functions $X(u,v)$ which depend only on the difference $u-v$ and satisfy the CYBE and some additional nondegeneracy condition. It was proved in [1] that nondegenerate solutions are of three types: rational, trigonometric and elliptic. The last two kinds were fully classified in [1]. However, the similar question for rational solution remained open. This problem was solved in [8,9] by classifying instead solutions of the form

$$X(u,v) = \frac{\Omega}{u-v} + r(u,v), \tag{1.1}$$

where $r(u,v)$ is a polynomial with coefficients in $\mathbf{g} \otimes \mathbf{g}$ and Ω denotes the quadratic Casimir element of $\mathbf{g}$. This new type of solutions, which will also be called *rational*, look somehow different from those in the Belavin-Drinfeld approach. However, as it turned out in [2], any solution of this type can be transformed into one which depends only on $u-v$, by means of a change of variables and a holomorphic transformation.

In [8,9,10] a correspondence was established between rational solutions of the form (1.1) and so-called *orders* in $\mathbf{g}((u^{-1}))$, i.e. subalgebras W of $\mathbf{g}((u^{-1}))$ which satisfy the condition

$$u^{-N_1}\mathbf{g}[[u^{-1}]] \subseteq W \subseteq u^{N_2}\mathbf{g}[[u^{-1}]] \tag{1.2}$$

for some non-negative integers N_1 and N_2. The study of rational solutions is essentially based on this correspondence and the description of the maximal orders.

In the present paper, we follow the method developed in [8,9,10] to study rational solutions of the CYBE for a simple compact Lie algebra $\mathbf{g}$ over $\mathbb{R}$. We establish a similar correspondence between solutions and orders and we are interested in the description of the maximal orders. We obtain that there is only one maximal order, the trivial one. Therefore all rational solutions will have the form

$$X(u,v) = \frac{\Omega}{u-v} + r, \tag{1.3}$$

where $r \in \mathbf{g} \wedge \mathbf{g}$ is a constant r-matrix. Here we would like to note that this theorem was communicated to the second author by V. Drinfeld without proof.

On the other hand, there exists a 1-1 correspondence between skew-symmetric constant r-matrices and pairs (L, B), where L is a subalgebra of $\mathbf{g}$ together with a non-degenerate 2-cocycle $B \in Z^2(L,\mathbb{R})$. A subalgebra L for which there exists a non-degenerate B is called *quasi-Frobenius*. We prove that any quasi-Frobenius subalgebra of a compact simple Lie algebra is commutative. Consequently, up to gauge equivalence, any rational solution has the form

$$X(u,v) = \frac{\Omega}{u-v} + t_1 \wedge t_2 + ... + t_{2n-1} \wedge t_{2n}, \tag{1.4}$$

where t_1,..., t_{2n} are linearly independent elements in a maximal torus $\mathbf{t}$ of $\mathbf{g}$.

Finally we discuss the quantization of the Lie bialgebra structures corresponding to solutions of the form (1.4). The quantization is obtained by twisting the real Yangian $Y_\hbar(\mathbf{g})$.

2. Rational solutions and orders

Let $\mathbf{g}$ denote a simple compact Lie algebra over $\mathbb{R}$ and $U(\mathbf{g})$ its universal enveloping algebra. Let $[\ ,\ ]$ be the usual Lie bracket on the associative algebra $U(\mathbf{g})^{\otimes 3}$.

We recall the following notation [1]: φ_{12}, φ_{13}, φ_{23}, φ_{21}: $\mathbf{g}\otimes\mathbf{g} \to U(\mathbf{g})^{\otimes 3}$ are the linear maps respectively defined by $\varphi_{12}(a\otimes b) = a\otimes b\otimes 1$, $\varphi_{13}(a\otimes b) = a\otimes 1\otimes b$, $\varphi_{23}(a\otimes b) = 1\otimes a\otimes b$ and $\varphi_{21}(a\otimes b) = b\otimes a\otimes 1$, for any a, $b \in \mathbf{g}$.

For a function $X : \mathbb{R}^2 \to \mathbf{g}\otimes\mathbf{g}$, we consider X^{ij}: $\mathbb{R}^2 \to U(\mathbf{g})^{\otimes 3}$ defined by $X^{ij}(u_i,u_j) = \varphi_{ij}(X(u_i,u_j))$.

Definition 2.1. [1] A solution of the classical Yang-Baxter equation (CYBE) is a function $X : \mathbb{R}^2 \to \mathbf{g}\otimes\mathbf{g}$ such that the following conditions are satisfied:

$$[X^{12}(u_1,u_2), X^{13}(u_1,u_3)] + [X^{12}(u_1,u_2), X^{23}(u_2,u_3)] + \tag{2.1}$$

$$+[X^{13}(u_1,u_3), X^{23}(u_2,u_3)] = 0$$

$$X^{12}(u,v) = -X^{21}(v,u). \tag{2.2}$$

Let us consider the Killing form K on $\mathbf{g}$. Then $(-K)$ is a positive definite invariant bilinear form on $\mathbf{g}$. Let $\{I_\lambda\}$ be an orthonormal basis in $\mathbf{g}$ with respect to $(-K)$.

We denote by Ω the quadratic Casimir element of $\mathbf{g}$, i.e. $\Omega = -\sum I_\lambda \otimes I_\lambda$. Now we define rational solutions as in the complex case [8,9,10]:

Definition 2.2. A solution of the CYBE is called *rational* if it is of the form

$$X(u,v) = \frac{\Omega}{u-v} + r(u,v), \tag{2.3}$$

where $r(u,v)$ is a polynomial with coefficients in $\mathbf{g}\otimes\mathbf{g}$.

Remark 2.3. The simplest example of a rational solution is $X_0(u,v) = \frac{\Omega}{u-v}$. By adding to $X_0(u,v)$ any skew-symmetric constant r-matrix, we also obtain a rational solution.

We will consider rational solutions up to a certain equivalence relation:

Definition 2.4. Two rational solutions X_1 and X_2 are said to be *gauge equivalent* if there exists $\sigma(u) \in Aut(\mathbf{g}[u])$ such that

$$X_1(u,v) = (\sigma(u)\otimes\sigma(v))X_2(u,v). \tag{2.4}$$

Here $Aut(\mathbf{g}[u])$ denotes the group of automorphisms of $\mathbf{g}[u]$ considered as an algebra over $\mathbb{R}[u]$.

Remark 2.5. One can check that gauge transformations applied to rational solutions also give rational solutions.

Let $\mathbb{R}[[u^{-1}]$ be the ring of formal power series in u^{-1} and $\mathbb{R}((u^{-1}))$ its field of quotients. Set $\mathbf{g}[u] := \mathbf{g}\otimes_{\mathbb{R}}\mathbb{R}[u]$, $\mathbf{g}[[u^{-1}]] := \mathbf{g}\otimes_{\mathbb{R}}\mathbb{R}[[u^{-1}]]$ and $\mathbf{g}((u^{-1})) := \mathbf{g}\otimes_{\mathbb{R}}\mathbb{R}((u^{-1}))$. There exists a nondegenerate $\mathbf{ad}$-invariant bilinear form on $\mathbf{g}((u^{-1}))$ given by

$$(x(u), y(u)) = Tr(\mathbf{ad}x(u)\cdot\mathbf{ad}y(u))_{-1}, \tag{2.5}$$

meaning that we take the coefficient of u^{-1} in the series expansion of $Tr(\mathbf{ad}x(u)\cdot\mathbf{ad}y(u))$.

In [8, Th.1] a correspondence between rational solutions and a special class of subalgebras of $\mathbf{g}((u^{-1}))$ was presented. The same result holds when $\mathbf{g}$ is real compact:

Theorem 2.6. *Let $\mathbf{g}$ be a simple compact Lie algebra over $\mathbb{R}$. There is a natural one-to-one correspondence between rational solutions of the CYBE and subalgebras $W \subseteq \mathbf{g}((u^{-1}))$ such that*

(1) $W \supseteq u^{-N}\mathbf{g}[[u^{-1}]]$ for some $N > 0$;

(2) $W \oplus \mathbf{g}[u] = \mathbf{g}((u^{-1}))$;

(3) W is a Lagrangian subspace with respect to the bilinear form on $\mathbf{g}((u^{-1}))$ given by (2.5), i.e. $W = W^\perp$.

Proof. We briefly sketch the proof which is similar to that in the complex case. Let $V := \mathbf{g}[u]$. Then $V^* = u^{-1}\mathbf{g}[[u^{-1}]]$. If $f \in V^*$ and $x \in V$ then $f(x) := (f,x)$, where $(\ ,\ )$ is the bilinear form given by (2.5).

Denote by $Hom_{cont}(V^*,V)$ the space of those linear maps $F: V^* \to V$ such that $Ker(F) \supseteq u^{-N}V^*$ for some $N \geq 0$. To motivate the notation, we make the remark that this space consists of all linear maps F which are continuous with respect to the

"u^{-1} - adic" topology. $\mathbb{R}[[u^{-1}]]$ is a topological valuation ring and V^* is a topological free $\mathbb{R}[[u^{-1}]]$ - module. We also put the discrete topology on V.

There exists an isomorphism $\Phi: V\otimes V \to Hom_{cont}(V^*,V)$ defined by

$$\Phi(x\otimes y)(f) = f(y)x, \tag{2.6}$$

for any x, $y \in V$ and $f \in V^*$. The inverse map is given by

$$\Phi^{-1}(F) = -\sum_{i=1}^{n}\sum_{k=0}^{\infty} F(I_i u^{-k-1})\otimes I_i u^k, \tag{2.7}$$

for any $F \in Hom_{cont}(V^*,V)$. We make the remark that $F(I_i u^{-k-1}) = 0$ for $k \geq N$ so that the sum which appears in (2.7) is finite.

There is a natural bijection between $Hom_{cont}(V^*,V)$ and the set of all subspaces W of $\mathbf{g}((u^{-1}))$ which are complementary to V and such that $W \supseteq u^{-N}V^* = u^{-N-1}\mathbf{g}[[u^{-1}]]$ for some $N \geq 0$. Indeed, for any $F \in Hom_{cont}(V^*,V)$, we consider the following subspace of $\mathbf{g}((u^{-1}))$

$$W(F) := \{f + F(f) : f \in V^*\} \tag{2.8}$$

which satisfies the required properties.

The inverse mapping associates to any W the linear function F_W such that for any $f \in V^*$, $F_W(f) = -x$, uniquely defined by the decomposition $f = w + x$ with $w \in W$ and $x \in V$.

One can easily see that $W(\Phi(r))$ is Lagrangian with respect to the bilinear form (2.5) if and only if $r^{12}(u,v) = -r^{21}(v,u)$. Consequently, $\Omega/(u-v) + r(u,v)$ satisfies the unitarity condition (2.2) if and only if $W(\Phi(r))$ is Lagrangian subspace.

Finally, if $\Omega/(u-v) + r(u,v)$ is a solution of (2.1)-(2.2), then

$$([f + \Phi(r)(f), g + \Phi(r)(g)], h + \Phi(r)(h)) = 0 \tag{2.9}$$

for any elements f, g, h in V^*. Because $W(\Phi(r))$ is Lagrangian, (2.9) implies that $W(\Phi(r))$ is a subalgebra of $\mathbf{g}((u^{-1}))$. □

Remark 2.7. One can easily see that if W is contained in $\mathbf{g}[[u^{-1}]]$ and satisfies the above properties, then the corresponding rational solution has the form $X(u,v) = \Omega/(u-v) + r$, where r is a constant polynomial.

Definition 2.8. An $\mathbb{R}$-subalgebra $W \subseteq \mathbf{g}((u^{-1}))$ is called an *order* in $\mathbf{g}((u^{-1}))$ if there exist two non-negative integers N_1, N_2 such that

$$u^{-N_1}\mathbf{g}[[u^{-1}]] \subseteq W \subseteq u^{N_2}\mathbf{g}[[u^{-1}]]. \tag{2.10}$$

Obviously $\mathbf{g}[[u^{-1}]]$ is an order.

Remark 2.9. Let W satisfy conditions (1) and (3) of Theorem 2.6. Then W is an order.

Concerning gauge equivalence, the result of Theorem 2 in [8] remains true:

Theorem 2.10. *Let $\mathbf{g}$ be simple compact Lie algebra over $\mathbb{R}$. Let X_1 and X_2 be rational solutions of the CYBE and W_1, W_2 the corresponding orders in $\mathbf{g}((u^{-1}))$. Let $\sigma(u) \in Aut(\mathbf{g}[u])$. Then the following conditions are equivalent:*

(1) $X_1(u,v) = (\sigma(u)\otimes\sigma(v))X_2(u,v)$*;*
(2) $W_1 = \sigma(u)W_2$*.*

Definition 2.11. Let V_1 and V_2 be subalgebras of $\mathbf{g}((u^{-1}))$. We say that V_1 and V_2 are *gauge equivalent* if there exists $\sigma(u) \in Aut(\mathbf{g}[u])$ such that $V_1 = \sigma(u)V_2$.

3. Maximal orders for compact Lie algebras

We will prove the following result:

Theorem 3.1. *Let* $\mathbf{g}$ *be a simple compact Lie algebra over* $\mathbb{R}$*. Then any order* W *in* $\mathbf{g}((u^{-1}))$ *is gauge equivalent to an order contained in* $\mathbf{g}[[u^{-1}]]$*.*

Proof. Let G be a connected compact Lie group whose Lie algebra is $\mathbf{g}$. Then G is embedded into $SL(n,\mathbb{C})$ via any irreducible complex representation. Without any loss of generality, we may suppose that the image of a maximal torus T of G is included into the diagonal torus H of $SL(n,\mathbb{C})$.

Let W denote an order of $\mathbf{g}((u^{-1}))$. Since we have the following sequence of embeddings

$$W \hookrightarrow W\otimes_{\mathbb{R}}\mathbb{C} \hookrightarrow (\mathbf{g}\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}\mathbb{C}((u^{-1})) \hookrightarrow sl(n,\mathbb{C}((u^{-1}))), \tag{3.1}$$

we may view any $w\in W$ as a matrix in $sl(n,\mathbb{C}((u^{-1})))$.

Let us prove that for each $w\in W$, the exponential $\exp(w)$ defined formally by

$$\exp(w) := \sum_{k\geq 0}\frac{w^k}{k!}. \tag{3.2}$$

makes sense as an element of $SL(n,\mathbb{C}((u^{-1})))$.

Without any loss of generality, we may suppose that W is an $\mathbb{R}[[u^{-1}]]$-module of finite rank. We set $\mathbb{O} := \mathbb{C}[[u^{-1}]]$ and consider the $\mathbb{O}$-module

$$M := \mathbb{O}^n + W\mathbb{O}^n + \ldots + WW\ldots W\mathbb{O}^n + \ldots. \tag{3.3}$$

Let us show that there exists some integer l such that

$$M \subseteq u^l\mathbb{O}^n. \tag{3.4}$$

If $x_1,\ldots, x_r$ is a basis of the $\mathbb{R}[[u^{-1}]]$-module W, then obviously

$$M \subseteq \sum_{k_i\geq 0} x_1^{k_1}\ldots x_r^{k_r}\mathbb{O}^n. \tag{3.5}$$

It is well-known that the field $\mathbb{K} := \mathbb{C}((u^{-1}))$ may be endowed with the discrete valuation $v(\sum_{k\geq N} a_k u^{-k}) = N$. For any $f \in\mathbb{K}$, we consider its norm:

$$|f| = 2^{-v(f)}. \tag{3.6}$$

Note that $\mathbb{O}$ is the set of all f such that $|f|\leq 1$.

On the other hand, one can define a norm on $gl(n,\mathbb{K})$ which is compatible with the norm of $\mathbb{K}$. Given a matrix A of $gl(n,\mathbb{K})$, one sets

$$|A| = 2^g, \tag{3.7}$$

where $g := \inf k$ such that $A\mathbb{O}^n \subseteq u^k\mathbb{O}^n$.

This norm satisfies the properties: $|A_1A_2| \leq |A_1||A_2|$, $|f(u)\cdot A| = |f(u)||A|$, $|A_1+A_2| \leq \sup\{|A_1|, |A_2|\}$.

We make the remark that, since W is an order, there exists $N\geq 0$ such that $|w|\leq 2^N$ for all $w\in W$.

In order to prove (3.4), it is enough to show that

$$\sup_{(k_1,\ldots,k_r)} \left|x_1^{k_1}\ldots x_m^{k_r}\right| < \infty. \tag{3.8}$$

This means that for each $1\leq i\leq r$ there exists a positive integer M_i such that

$$\sup_k \left|x_i^k\right| \leq M_i. \tag{3.9}$$

It suffices to prove that the norms of the eigenvalues of x_i for the action of x_i on $\mathbb{K}^n$ are less or equal to 1. Indeed, let us suppose that this requirement is fulfilled. Then the coefficients of the characteristic polynomial of x_i have norm less or equal to 1, so they belong to $\mathbb{O}$. In follows that x_i is integral over $\mathbb{O}$, i.e. there exists a_{ip_i}, ..., a_{i1} in $\mathbb{O}$ such that $x_i^{p_i} + a_{ip_i}x^{p_i-1} + \ldots + a_{i1} = 0$. One can check by induction that x_i^k, for any $k\geq 0$, is a linear combination of 1, x_i, ..., $x_i^{p_i-1}$ with coefficients in $\mathbb{O}$. Since the elements of $\mathbb{O}$ have norm less or equal to 1, we get that

$$\left|x_i^k\right| \leq \sup\{1, |x_i|, \ldots, \left|x_i^{p_i-1}\right|\} \tag{3.10}$$

for any $k\geq 0$ and thus (3.9) will be fulfilled.

Let w be an arbitrary element of W. Let $\varepsilon_1(w)$,..., $\varepsilon_n(w)$ be the eigenvalues of w for the action of w on $\mathbb{K}^n$. We will show that $|\varepsilon_i(w)|\leq 1$ for all i. Without any loss of generality, we may suppose that w is a diagonalizable element. Consider the eigenvalues $\alpha_1(w)$,..., $\alpha_m(w)$ for the action of w on $(\mathbf{g}\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}\mathbb{C}((u^{-1}))$. Some of them are zero and some behave as roots. For any $\alpha_j(w)$ there exists a corresponding eigenvector which belongs to W. Since $W\otimes_{\mathbb{R}}\mathbb{C}$ is an $\mathbb{O}$-module of finite type, it follows that $|\alpha_j(w)|\leq 1$ for all j. On the other hand, because the weights of a representation are linear combinations of simple roots, we have that $\varepsilon_1(w)$,..., $\varepsilon_n(w)$ are linear combinations of some $\alpha_j(w)$ with rational coefficients. This implies that $|\varepsilon_i(w)|\leq 1$ for all i.

Thus (3.9) holds and this implies (3.4). Since (3.4) holds for some integer l, $\exp(w)$ belongs to $SL(n,\mathbb{C}((u^{-1})))$, for any $w\in W$. We denote by S the connected subgroup generated by $\exp(w)$ for all $w\in W$. Its Lie algebra is W.

Recall that G is embedded into $SL(n,\mathbb{C})$ such that the image of a maximal torus T of G is contained in a maximal torus H of $SL(n,\mathbb{C})$. Let $\mathcal{T}$ be the affine Bruhat-Tits building associated to $G(\mathbb{R}((u^{-1})))$ and the valuation v. Let $\mathcal{T}'$ be the affine Bruhat-Tits building associated to $SL(n,\mathbb{C}((u^{-1})))$ and the valuation v. According to [4, p. 202-204] there exists an embedding

$$\mathcal{T} \hookrightarrow \mathcal{T}' \tag{3.11}$$

which is compatible with the preceding embedding $G\hookrightarrow SL(n,\mathbb{C})$.

Since W is contained in $\mathbf{g}((u^{-1}))$, one has that

$$S \subseteq G(\mathbb{R}((u^{-1}))) \hookrightarrow SL(n, \mathbb{C}((u^{-1}))). \tag{3.12}$$

The module M given by (3.3) satisfies the property $SM \subseteq M$. Since $\mathbb{O}^n \subseteq M \subseteq u^l\mathbb{O}^n$, it follows that $S\mathbb{O}^n \subseteq u^l\mathbb{O}^n$. Therefore S must be a bounded subgroup of $SL(n, \mathbb{C}((u^{-1})))$, i.e. there is an upper bound on the absolute values of the matrix entries of the elements of S.

According to [3, p. 161], S is bounded in the sense of Bruhat-Tits bornology for the building $\mathcal{T}'$ (see [3, p. 160]). Because the embedding $\mathcal{T} \hookrightarrow \mathcal{T}'$ is compatible with the building metric, it follows that S is a bounded subgroup of $G(\mathbb{R}((u^{-1})))$, in the sense of Bruhat-Tits bornology corresponding to the building $\mathcal{T}$.

Now the Bruhat-Tits fixed point theorem ([3, p. 157, 161]) implies that S fixes a point p of the building $\mathcal{T}$.

It was proved in [7] that the action of $G(\mathbb{R}[u])$ on the Bruhat-Tits building associated to $G(\mathbb{R}(u))$ and the valuation ω defined by $\omega(f/g) = \deg(g) - \deg(f)$, admits as simplicial fundamental domain a so-called “sector”. This result remains true when we pass to our building $\mathcal{T}$ since, by taking the completion $\mathbb{R}((u^{-1}))$, the building does not change, only the apartment system gets completed. Moreover, the action of $G(\mathbb{R}[u])$ is continuous. Let $\mathcal{H}$ denote the Cartan subalgebra of $sl(n, \mathbb{C})$ corresponding to H and $\mathcal{H}_{\mathbb{R}}$ its real part. The simplicial fundamental domain for the action of $G(\mathbb{R}[u])$ on $\mathcal{T}$ is contained in the standard apartment of the building $\mathcal{T}'$ which is identified with $\mathcal{H}_{\mathbb{R}}$.

Let h be the point of $\mathcal{H}_{\mathbb{R}}$ which is equivalent to p via the action of $G(\mathbb{R}[u])$. There exists $X \in G(\mathbb{R}[u])$ such that $Xp = h$, which implies that XSX^{-1} is contained in the stabilizer P_h of h under the action of $G(\mathbb{R}((u^{-1})))$ on $\mathcal{T}$.

On the other hand, $P_h = P'_h \cap G(\mathbb{R}((u^{-1})))$, where P'_h is the stabilizer of h under the action of $SL(n, \mathbb{C}((u^{-1})))$ on $\mathcal{T}'$. It follows that

$$\mathbf{Ad}(X)W \subseteq \mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap Lie(P'_h). \tag{3.13}$$

The stabilizer P'_h was computed in [4, p. 238] and its Lie algebra is

$$\mathcal{O}_h = \{(g_{ij}) \in sl(n, \mathbb{C}((u^{-1}))) : v(g_{ij}) \geq \alpha_{ij}(h)\}. \tag{3.14}$$

Let us prove that

$$\mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathcal{O}_h \subseteq \mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}[[u^{-1}]]. \tag{3.15}$$

We know that

$$\mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathcal{O}_h \subseteq su(n) \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathcal{O}_h. \tag{3.16}$$

It is enough to show the following:

$$su(n) \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathcal{O}_h \subseteq su(n) \otimes_{\mathbb{R}} \mathbb{R}[[u^{-1}]]. \tag{3.17}$$

If a matrix (g_{ij}) belongs to $su(n) \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathcal{O}_h$, then $v(g_{ij}) \geq \alpha_{ij}(h)$ for all i, j and $g_{ij} + \overline{g_{ji}} = 0$. We have $v(g_{ij}) = v(-\overline{g_{ij}}) = v(g_{ji})$. On the other hand, $v(g_{ji}) \geq -\alpha_{ij}(h)$. We conclude that $v(g_{ij}) \geq 0$ and therefore (g_{ij}) belongs to $su(n) \otimes_{\mathbb{R}} \mathbb{R}[[u^{-1}]]$.

In conclusion, for some $X \in G(\mathbb{R}[u])$, one has that

$$\mathbf{Ad}(X)W \subseteq \mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}[[u^{-1}]] \tag{3.18}$$

which completes the proof. □

4. Description of rational solutions

Theorem 3.1 has an important consequence:

Corollary 4.1. *Let $\mathbf{g}$ be a simple compact Lie algebra over $\mathbb{R}$. Any rational solution of the CYBE for $\mathbf{g}$ is gauge equivalent to a solution of the form*

$$X(u, v) = \frac{\Omega}{u - v} + r, \tag{4.1}$$

where $r \in \mathbf{g} \wedge \mathbf{g}$ is a constant r-matrix .

Proof. We know that any order W of $\mathbf{g}((u^{-1}))$ is gauge equivalent to an order contained in $\mathbf{g}[[u^{-1}]]$. On the other hand, if a rational solution $X(u, v)$ corresponds to an order $W \subseteq \mathbf{g}[[u^{-1}]]$ then, by Remark 2.7, $X(u, v) = \Omega/(u - v) + r$, where r is a constant polynomial. Because $X(u, v)$ is a solution of the CYBE, it results that r itself is a solution of the CYBE. □

Let us recall a result which describes constant solutions in a different way. This theorem was formulated for the complex case in [1], but the proof obviously works for any simple compact Lie algebra $\mathbf{g}$ over $\mathbb{R}$.

Theorem 4.2. *Any rational solution of the CYBE of the form (4.1) induces a pair (L, B), where L is a subalgebra of $\mathbf{g}$ and B is a non-degenerate 2-cocycle on L. The Lie subalgebra L is the smallest vector subspace in $\mathbf{g}$ such that $r \in L \wedge L$ and B is the bilinear form on L which is the inverse of r. Conversely, any pair (L, B) provides a rational solution of the form (4.1), where $r \in L \wedge L$ is the inverse of B.*

Remark 4.3. In particular, if L is a commutative subalgebra of $\mathbf{g}$ and B is a non-degenerate skew-symmetric form on L, then there exists the corresponding solution of the form (4.1).

Recall that a subalgebra L of $\mathbf{g}$ is called *quasi-Frobenius* if there exists a non-degenerate 2-cocycle $B \in Z^2(L, \mathbb{R})$.

Theorem 4.4. *Let $\mathbf{g}$ be a simple compact Lie algebra over $\mathbb{R}$. Any quasi-Frobenius Lie subalgebra L of $\mathbf{g}$ is commutative.*

Proof. Any subalgebra of a compact Lie algebra is compact. Therefore L must be compact as well. Moreover (see for example [6, p. 97]), the derived algebra L' of L is semisimple and if $\zeta(L)$ denotes the center of L, then

$$L = L' \oplus \zeta(L). \tag{4.2}$$

Let us assume that $L' \neq 0$ and there exists a non-degenerate 2-cocycle B on L. We have the following identity

$$B([x, y], z) + B([y, z], x) + B([z, x], y) = 0 \tag{4.3}$$

for any x, $y \in L'$ and $z \in \zeta(L)$. This implies $B([x,y],z) = 0$ for arbitrary x, $y \in L'$ and $z \in \zeta(L)$. Since L' is semisimple, its derived algebra coincides with L'. We obtain

$$B(w,z) = 0 \tag{4.4}$$

for any $w \in L'$ and $z \in \zeta(L)$.

On the other hand, since L' is semisimple, the restriction of B to L' is a coboundary, i.e. there exists a non-zero functional f on L' such that $B(w_1,w_2) = f([w_1,w_2])$, for all w_1, w_2 in L'. Let a_0 be the element of L' which corresponds to f via the isomorphism $L' \cong (L')^*$ defined by the Killing form. Then for all $w \in L'$ one has

$$B(a_0,w) = K(a_0,[a_0,w]) = K([a_0,a_0],w) = 0. \tag{4.5}$$

Together with (4.2) and (4.4) this implies that

$$B(a_0,l) = 0 \tag{4.6}$$

for all elements l of L. Thus B is degenerate on L, which is a contradiction. □

Corollary 4.5. *Up to gauge equivalence, any rational solution of the CYBE for a simple compact Lie algebra* $\mathbf{g}$ *over* $\mathbb{R}$ *has the form*

$$X(u,v) = \frac{\Omega}{u-v} + t_1 \wedge t_2 + ... + t_{2n-1} \wedge t_{2n}, \tag{4.7}$$

where t_1,..., t_{2n} *are linearly independent elements in a maximal torus* $\mathbf{t}$ *of* $\mathbf{g}$.

Proof. We have seen that rational solutions are determined by pairs (L,B), where L is a quasi-Frobenius Lie subalgebra and B a non-degenerate 2-cocycle on L. By the previous result, L is a commutative subalgebra and B is a non-degenerate skew-symmetric form on L. Then L is contained in a maximal commutative subalgebra $\mathbf{t}$ of $\mathbf{g}$ and the dimension of L is even, say $2n$.

Moreover, it is well-known that there exists a basis t_1,..., t_{2n} in L such that $B(t_{2i-1},t_{2i}) = -B(t_{2i},t_{2i-1}) = -1$ for $1 \le i \le n$ and $B(t_j,t_k) = 0$ otherwise. The rational solution induced by the pair (L,B) is precisely (4.7). □

5. Quantization

Let $\mathbf{g}$ be a simple compact Lie algebra over $\mathbb{R}$. Let us recall that the rational solution $X_0(u,v) = \frac{\Omega}{u-v}$ induces a Lie bialgebra structure on $\mathbf{g}[u]$ via the 1-cocycle δ_0 given by

$$\delta_0(a(u)) = [a(u) \otimes 1 + 1 \otimes a(v), X_0(u,v)], \tag{5.1}$$

for any $a(u) \in \mathbf{g}[u]$.

We have seen that, up to gauge equivalence, rational solutions have the form (4.7). To any such solution one can associate a Lie bialgebra structure on $\mathbf{g}[u]$ by defining the 1-cocycle

$$\delta_r(a(u)) = [a(u) \otimes 1 + 1 \otimes a(v), X(u,v)]. \tag{5.2}$$

Here $r = t_1 \wedge t_2 + ... + t_{2n-1} \wedge t_{2n}$. In other words, the Lie bialgebra $(\mathbf{g}[u],\delta_r)$ is obtained from the Lie bialgebra $(\mathbf{g}[u],\delta_0)$ by so-called *twisting* via r.

Remark 5.1. This notion was introduced by V. G. Drinfeld in a more general setting for Lie quasi-bialgebras.

The purpose of this section is to give a quantization of the Lie bialgebra $(\mathbf{g}[u],\delta_r)$.

Let us begin by pointing out that the Lie algebra $(\mathbf{g}[u],\delta_0)$ admits a unique quantization which we will denote by $Y_\hbar(\mathbf{g})$ (here $\hbar$ is Planck's constant). The construction is analogous to that of the Yangian introduced in [5]. We recall that if K denotes the Killing form of a simple compact $\mathbf{g}$, then $(-K)$ is a positive definite invariant bilinear form. Let $\{I_\lambda\}$ be an orthonormal basis in $\mathbf{g}$ with respect to $(-K)$. Then $Y_\hbar(\mathbf{g})$ is the topological Hopf algebra over $\mathbb{R}[[\hbar]]$ generated by elements I_λ and J_λ with defining relations

$$[I_\lambda, I_\mu] = c^{\nu}_{\lambda\mu} I_\nu \tag{5.3}$$

$$[I_\lambda, J_\mu] = c^{\nu}_{\lambda\mu} J_\nu \tag{5.4}$$

$$[J_\lambda,[J_\mu,I_\nu]] - [I_\lambda,[J_\mu,J_\nu]] = \hbar^2 a^{\alpha\beta\gamma}_{\lambda\mu\nu}\{I_\alpha,I_\beta,I_\gamma\} \tag{5.5}$$

$$\begin{aligned}&[[J_\lambda,J_\mu],[I_r,J_s]] + [[J_r,J_s],[I_\lambda,J_\mu]] = \\ &\hbar^2 (a^{\alpha\beta\gamma}_{\lambda\mu\nu} c^{\nu}_{rs} + a^{\alpha\beta\gamma}_{rs\nu} c^{\nu}_{\lambda\mu})\{I_\alpha,I_\beta,I_\gamma\},\end{aligned} \tag{5.6}$$

where $a^{\alpha\beta\gamma}_{\lambda\mu\nu} := \frac{1}{24} c^{i}_{\lambda\alpha} c^{j}_{\mu\beta} c^{k}_{\nu\gamma} c^{k}_{ij}$ and $\{x_1,x_2,x_3\} := \Sigma_{i\neq j\neq k} x_i x_j x_k$. The co-multiplication, the co-unit and the antipode are given by the following:

$$\Delta(I_\lambda) = I_\lambda \otimes 1 + 1 \otimes I_\lambda \tag{5.7}$$

$$\Delta(J_\lambda) = J_\lambda \otimes 1 + 1 \otimes J_\lambda - \frac{\hbar}{2} c^{\nu}_{\lambda\mu} I_\nu \otimes I_\mu \tag{5.8}$$

$$\varepsilon(I_\lambda) = \varepsilon(J_\lambda) = 0, \varepsilon(1) = 1 \tag{5.9}$$

$$S(I_\lambda) = -I_\lambda \tag{5.10}$$

$$S(J_\lambda) = -J_\lambda + \frac{\hbar}{4} I_\lambda. \tag{5.11}$$

Clearly $Y_\hbar(\mathbf{g})$ contains $U(\mathbf{g})[[\hbar]]$ as a Hopf subalgebra.

Since the generators of $Y_\hbar(\mathbf{g})$ are simultaneously generators for the complex Yangian and all the structure constants are real, it follows immediately from [5, Th.3] that $Y_\hbar(\mathbf{g})$ is a pseudotriangular Hopf algebra. More precisely, for any real number a, define an automorphism T_a of $Y_\hbar(\mathbf{g})$ by the formulae

$$T_a(I_\lambda) = I_\lambda \tag{5.12}$$

$$T_a(J_\lambda) = J_\lambda + a I_\lambda. \tag{5.13}$$

Then there exists an element $R(u) = 1 + \sum_{k=1}^{\infty} R_k u^{-k}$, where $R_1 = \Omega$ and $R_k \in Y_\hbar(\mathbf{g})^{\otimes 2}$, such that the following conditions are satisfied:

$$(T_a \otimes T_b)R(u) = R(u+a-b) \tag{5.14}$$

$$(T_u \otimes 1)\Delta^{op}(x) = R(u)((T_u \otimes 1)\Delta(x))R(u)^{-1} \tag{5.15}$$

$$(\Delta \otimes 1)R(u) = R^{13}(u)R^{23}(u) \tag{5.16}$$

$$R^{12}(u)R^{21}(-u) = 1 \otimes 1 \tag{5.17}$$

$$\begin{aligned} R^{12}(u_1-u_2)R^{13}(u_1-u_3)R^{23}(u_2-u_3) &= \\ = R^{23}(u_2-\lambda_3)R^{13}(u_1-u_3)R^{12}(u_1-u_2). \end{aligned} \tag{5.18}$$

Here Δ^{op} denotes the opposite comultiplication.

In order to give a quantization of $(\mathbf{g}[u], \delta_r)$, we introduce a deformation of the Yangian $Y_\hbar(\mathbf{g})$ by a so-called *quantum twist*. The approach is based on [11, Th.5] that we recall below:

Theorem 5.2. *Let $F \in (U(\mathbf{g})[[\hbar]])^{\otimes 2}$ such that*

$$F \equiv 1 (\mathrm{mod} \hbar) \tag{5.19}$$

$$(\varepsilon \otimes 1)F = (1 \otimes \varepsilon)F = 1 \otimes 1 \tag{5.20}$$

$$(\Delta \otimes 1)F \cdot F^{12} = (1 \otimes \Delta)F \cdot F^{23}. \tag{5.21}$$

Denote by $\widetilde{Y}_\hbar(\mathbf{g})$ the associative unital algebra which has the same multiplication m as $Y_\hbar(\mathbf{g})$ but the comultiplication is

$$\widetilde{\Delta} := F^{-1}\Delta F. \tag{5.22}$$

Then the following statements hold:

1) $\widetilde{Y}_\hbar(\mathbf{g})$ is a Hopf algebra with antipode

$$\widetilde{S} := Q^{-1}SQ, \tag{5.23}$$

where $Q = m(S \otimes 1)(F)$).

2) Let $\widetilde{R}(u) := (F^{21})^{-1}R(u)F$. Then the equations (5.14)-(5.18) hold for $\widetilde{R}(u)$ and $\widetilde{\Delta}(u)$.

Remark 5.3. In literature, an element F satisfying (5.19)-(5.21) is called a *quantum twist* of $Y_\hbar(\mathbf{g})$. The Hopf algebra $\widetilde{Y}_\hbar(\mathbf{g})$ is the *twisted* (or *deformed*) *Yangian* by the tensor F.

We can easily construct a quantum twist in the following way:

Proposition 5.4. *Suppose that t_1,..., t_{2n} are linearly independent elements in a maximal torus $\mathbf{t}$ of $\mathbf{g}$. Then the two-tensor*

$$F = \exp(\hbar(t_1 \otimes t_2 + ... + t_{2n-1} \otimes t_{2n})) \tag{5.24}$$

is a quantum twist of $Y_\hbar(\mathbf{g})$.

Proof. Conditions (5.19)-(5.21) can be checked by straightforward computations. □

Theorem 5.2 implies the following

Corollary 5.5. *The deformed Hopf algebra $\widetilde{Y}_\hbar(\mathbf{g})$, obtained by applying the quantum twist F given by (5.24), is a quantization of $(\mathbf{g}, \delta_r)$, where $r = t_1 \wedge t_2 + ... + t_{2n-1} \wedge t_{2n}$.*

Proof. For any $a \in \widetilde{Y}_\hbar(\mathbf{g})$, we have to check the following:

$$\hbar^{-1}(\widetilde{\Delta}(a) - \widetilde{\Delta}^{op}(a)) \bmod \hbar = \delta_r(a \bmod \hbar). \tag{5.25}$$

Since $\widetilde{\Delta} = F^{-1}\Delta F$, we obtain

$$\widetilde{\Delta}(a) - \widetilde{\Delta}^{op}(a) = F^{-1}\Delta(a)F - (F^{21})^{-1}\Delta^{op}(a)F^{21}. \tag{5.26}$$

On the other hand, since $Y_\hbar(\mathbf{g})$ is a quantization of $(\mathbf{g}, \delta_0)$, we have that

$$\Delta(a) - \Delta^{op}(a) = \hbar\delta_0(a \bmod \hbar) + O(\hbar^2). \tag{5.27}$$

Using (5.26), (5.27) and $(F^{21})^{-1}F = \exp(\hbar r)$, we obtain

$$\begin{aligned} \widetilde{\Delta}(a) - \widetilde{\Delta}^{op}(a) &= \hbar([\Delta(a), r] + \delta_0(a \bmod \hbar)) + O(\hbar^2) \\ &= \hbar\delta_r(a \bmod \hbar) + O(\hbar^2). \end{aligned} \tag{5.28}$$

□

Finally, we give the explicit formulae for the comultiplication and antipode of the twisted Yangian $\widetilde{Y}_\hbar(\mathbf{g})$. Let us recall the root system of $\mathbf{g}$ with respect to a torus, according to [6, p. 98-99]. We denote by $\mathbf{h}$ a Cartan subalgebra of $\mathbf{g} \otimes_{\mathbb{R}} \mathbb{C}$ and let Λ be the root system with respect to $\mathbf{h}$, together with a lexicographic ordering of Λ. We choose the root vectors e_α, corresponding to each root α, such that $K(e_\alpha, e_{-\alpha}) = -1$. Let $\mathbf{h}_{\mathbb{R}} = \{h \in \mathbf{h}\colon \alpha(h) \in \mathbb{R} \text{ for all } \alpha\}$. We put

$$C_\alpha := \frac{1}{\sqrt{2}}(e_\alpha + e_{-\alpha}) \tag{5.29}$$

$$S_\alpha := \frac{i}{\sqrt{2}}(e_\alpha - e_{-\alpha}) \tag{5.30}$$

It is well-known that

$$\mathbf{g} = i\mathbf{h}_{\mathbb{R}} \oplus \sum_{\alpha>0}(\mathbb{R}C_\alpha \oplus \mathbb{R}S_\alpha). \tag{5.31}$$

An orthonormal basis in $\mathbf{g}$, with respect to the bilinear form$(-K)$, is formed by the elements C_α, S_α and $p_j := ik_j$, where $\{k_j\}$ is an orthonormal basis in $\mathbf{h}_{\mathbb{R}}$. We choose this basis as our $\{I_\lambda\}$. The role of $\{J_\lambda\}$ is played correspondingly by some elements denoted by U_α, V_α and P_j. For any $h \in \mathbf{h}_{\mathbb{R}}$ we have the following:

$$[ih, C_\alpha] = \alpha(h)S_\alpha \tag{5.32}$$

$$[ih, S_\alpha] = -\alpha(h)C_\alpha \tag{5.33}$$

$$[ih, U_\alpha] = \alpha(h)V_\alpha \tag{5.34}$$

$$[ih, V_\alpha] = -\alpha(h)U_\alpha. \tag{5.35}$$

Let us consider now a quantum twist F as in (5.24). Since F is a a product of exponents, it is enough to perform computations for

$$F=\exp(\hbar(t_1\otimes t_2)), \tag{5.36}$$

where t_1 and t_2 are two linearly independent elements in the torus $\mathbf{t}=i\mathbf{h}_{\mathbb{R}}$. Let $t_1=ih_1$ and $t_2=ih_2$, where h_1 and h_2 are elements of $\mathbf{h}_{\mathbb{R}}$.

Lemma 5.6. *Let $T_{1\alpha}:=i\hbar\alpha(h_1)h_2$ and $T_{2\alpha}:=i\hbar\alpha(h_2)h_1$. The following identities hold:*

$$F^{-1}(C_\alpha\otimes 1)F=C_\alpha\otimes\cos(T_{1\alpha})-S_\alpha\otimes\sin(T_{1\alpha}) \tag{5.37}$$

$$F^{-1}(1\otimes C_\alpha)F=\cos(T_{2\alpha})\otimes C_\alpha-\sin(T_{2\alpha})\otimes S_\alpha \tag{5.38}$$

$$F^{-1}(S_\alpha\otimes 1)F=S_\alpha\otimes\cos(T_{1\alpha})+C_\alpha\otimes\sin(T_{1\alpha}) \tag{5.39}$$

$$F^{-1}(1\otimes S_\alpha)F=\cos(T_{2\alpha})\otimes S_\alpha+\sin(T_{2\alpha})\otimes C_\alpha. \tag{5.40}$$

$$F^{-1}(U_\alpha\otimes 1)F=U_\alpha\otimes\cos(T_{1\alpha})-V_\alpha\otimes\sin(T_{1\alpha}) \tag{5.41}$$

$$F^{-1}(1\otimes U_\alpha)F=\cos(T_{2\alpha})\otimes U_\alpha-\sin(T_{2\alpha})\otimes V_\alpha \tag{5.42}$$

$$F^{-1}(V_\alpha\otimes 1)F=V_\alpha\otimes\cos(T_{1\alpha})+U_\alpha\otimes\sin(T_{1\alpha}) \tag{5.43}$$

$$F^{-1}(1\otimes V_\alpha)F=\cos(T_{2\alpha})\otimes V_\alpha+\sin(T_{2\alpha})\otimes U_\alpha. \tag{5.44}$$

Proof. To prove the first identity, we use relations (5.32)-(5.33) and the formula

$$\exp(\lambda)\mu\exp(-\lambda)=\exp(\mathbf{ad}(\lambda))\mu=\mu+[\lambda,\mu]+\frac{1}{2!}[\lambda,[\lambda,\mu]]+... \tag{5.45}$$

for $\lambda:=-\hbar(ih_1\otimes ih_2)$ and $\mu:=C_\alpha\otimes 1$.

Identities (5.38)-(5.44) can be proved in a similar way. □

Consequently we obtain the following result:

Proposition 5.7. *The comultiplication $\widetilde{\Delta}$ of the twisted Yangian $\widetilde{Y}_\hbar(\mathbf{g})$ is given on its generators by the following:*

$$\begin{aligned}
\widetilde{\Delta}(C_\alpha)&=C_\alpha\otimes\cos(T_{1\alpha})-S_\alpha\otimes\sin(T_{1\alpha})+\cos(T_{2\alpha})\otimes C_\alpha-\sin(T_{2\alpha})\otimes S_\alpha\\
\widetilde{\Delta}(S_\alpha)&=S_\alpha\otimes\cos(T_{1\alpha})+C_\alpha\otimes\sin(T_{1\alpha})+\cos(T_{2\alpha})\otimes S_\alpha+\sin(T_{2\alpha})\otimes C_\alpha\\
\widetilde{\Delta}(U_\alpha)&=U_\alpha\otimes\cos(T_{1\alpha})-V_\alpha\otimes\sin(T_{1\alpha})+\cos(T_{2\alpha})\otimes U_\alpha-\sin(T_{2\alpha})\otimes V_\alpha\\
&\quad-\frac{\hbar}{2}[C_\alpha\otimes\cos(T_{1\alpha})-S_\alpha\otimes\sin(T_{1\alpha}),\widetilde{\Omega}]\\
\widetilde{\Delta}(V_\alpha)&=V_\alpha\otimes\cos(T_{1\alpha})+U_\alpha\otimes\sin(T_{1\alpha})+\cos(T_{2\alpha})\otimes V_\alpha+\sin(T_{2\alpha})\otimes U_\alpha\\
&\quad-\frac{\hbar}{2}[S_\alpha\otimes\cos(T_{1\alpha})+C_\alpha\otimes\sin(T_{1\alpha}),\widetilde{\Omega}]\\
\widetilde{\Delta}(p_j)&=p_j\otimes 1+1\otimes p_j\\
\widetilde{\Delta}(P_j)&=P_j\otimes 1+1\otimes P_j-\frac{\hbar}{2}[p_j\otimes 1,\widetilde{\Omega}],
\end{aligned}$$

where

$$\widetilde{\Omega}=\sum_{\alpha>0}(C_\alpha\cos(T_{2\alpha})+S_\alpha\sin(T_{2\alpha}))\otimes(\cos(T_{1\alpha})C_\alpha+\sin(T_{1\alpha})S_\alpha)+$$

$$+(C_\alpha\sin(T_{2\alpha})-S_\alpha\cos(T_{2\alpha})\otimes(\sin(T_{1\alpha})C_\alpha-\cos(T_{1\alpha})S_\alpha)+\sum_j p_j\otimes p_j.$$

We conclude by expliciting the antipode $\widetilde{S}$ of the twisted Yangian $\widetilde{Y}_\hbar(\mathbf{g})$. It is given by $\widetilde{S}=Q^{-1}SQ$, where $Q=\exp(\hbar h_1h_2)$.

Similarly to Lemma 5.6, one can prove

Lemma 5.8. *Let $T_\alpha:=i\hbar(\alpha(h_2)h_1+\alpha(h_1)h_2)$. The following identities hold:*

$$Q^{-1}C_\alpha Q=\exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha)C_\alpha+\sin(T_\alpha)S_\alpha) \tag{5.46}$$

$$Q^{-1}S_\alpha Q=\exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha)S_\alpha-\sin(T_\alpha)C_\alpha) \tag{5.47}$$

$$Q^{-1}U_\alpha Q=\exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha)U_\alpha+\sin(T_\alpha)V_\alpha) \tag{5.48}$$

$$Q^{-1}V_\alpha Q=\exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha)V_\alpha-\sin(T_\alpha)U_\alpha). \tag{5.49}$$

Proposition 5.9. *The antipode $\widetilde{S}$ of the deformed Yangian $\widetilde{Y}_\hbar(\mathbf{g})$ is given on its generators by*

$$\widetilde{S}(C_\alpha)=-\exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha)C_\alpha+\sin(T_\alpha)S_\alpha) \tag{5.50}$$

$$\widetilde{S}(S_\alpha)=-\exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha)S_\alpha-\sin(T_\alpha)C_\alpha) \tag{5.51}$$

$$\widetilde{S}(U_\alpha)=\exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha)(-U_\alpha+\frac{\hbar}{4}C_\alpha)+ \tag{5.52}$$

$$+\sin(T_\alpha)(-V_\alpha+\frac{\hbar}{4}S_\alpha))$$

$$\widetilde{S}(V_\alpha)=\exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha)(-V_\alpha+\frac{\hbar}{4}S_\alpha)+ \tag{5.53}$$

$$+\sin(T_\alpha)(U_\alpha-\frac{\hbar}{4}C_\alpha)).$$

$$\widetilde{S}(p_j)=-p_j \tag{5.54}$$

$$\widetilde{S}(P_j)=-P_j+\frac{\hbar}{4}p_j. \tag{5.55}$$

References

[1] Belavin A. A., Drinfeld V. G., *On classical Yang-Baxter equation for simple Lie algebras*, Funct. An. Appl., **16**, no. 3 (1982) 1-29.
[2] Belavin A. A., Drinfeld V. G., *On classical Yang-Baxter equation for simple Lie algebras*, Funct. An. Appl., **17**, no. 3 (1983) 69-70.
[3] Brown K. S., *Buildings*, Springer-Verlag, New York, 1989.
[4] Bruhat F., Tits J., *Groupes reductifs sur un corp local, I.*, Publ. Math. IHES, **41** (1972).
[5] Drinfeld V. G., *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Doklady, **32** (1985) 254-258.
[6] Goto M., *An introduction to Lie algebra for Lie group*, Lecture Notes Series No. **36** (1973).
[7] Soule C., *Chevalley groups over polynomial rings*, Homological Group Theory. London MS, Lecture Notes, Cambridge, **36** (1979) 359-367.
[8] Stolin A. A., *On rational solutions of Yang-Baxter equation for* $sl(n)$, Math. Scand. **69** (1991) 57-80.
[9] Stolin A. A., *On rational solutions of Yang-Baxter equation. Maximal orders in loop algebra.* Comm. Math. Phys., **141** (1991) 533-548.
[10] Stolin A., *A geometrical approach to rational solutions of the classical Yang-Baxter equation. Part I.* Symposia Gaussiana, Conf. Alg., ed. by Walter de Gryter & Co., Berlin, New York, 1995, 347-357.
[11] Stolin A., Kulish P. P., *New rational solutions of Yang-Baxter equation and deformed Yangians*, Czechoslovak J. Phys. **47** (1997) 123-129.

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RATIONAL SOLUTIONS OF THE CYBE AND LOCALLY TRANSITIVE ACTIONS ON ISOTROPIC GRASSMANNIANS

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Abstract. This paper continues the investigation of the theory of rational solutions of the CYBE for $o(n)$ from the point of view of orders in the corresponding loop algebra, as it was developed in [8]. As suggested by [8], in the case of "singular vertices", we use the list of connected irreducible subgroups of $SO(n)$ locally transitive on the Grassmann manifolds IG_k^n of isotropic k-dimensional subspaces in $\mathbb{C}^n$ obtained in [11]. New arguments based on the analysis of the structure of the stationary subalgebra of a generic point allow us to construct several rational solutions in $o(7)$, $o(8)$ and $o(12)$.

Keywords: classical Yang-Baxter equation, rational solution, locally transitive group, Grassmannian.
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1. Introduction

In 1982 A. A. Belavin and V. G. Drinfeld listed all elliptic and trigonometric solutions $X(u, v)$ of the classical Yang-Baxter equation, where X takes values in a simple complex Lie algebra $\mathbf{g}$ and left the classification problem of the rational solutions open. In 1984 V. G. Drinfeld conjectured that if a rational solution is equivalent to a solution of the form $X(u, v) = \Omega/(u-v) + r(u, v)$, where Ω is the quadratic Casimir element and r is a polynomial in u, v, then $\deg_u r = \deg_v r \leq 1$. In [7], [8] A. Stolin proved Drinfeld's conjecture and developed a theory of rational solutions based on the classification of the so-called "orders" in simple Lie algebras over the field $\mathbb{C}((u^{-1}))$. The maximal orders in the loop algebra of $\mathbf{g}$ correspond in turn to the vertices of the extended Dynkin diagram $D^e(\mathbf{g})$. In the case of so-called "singular" vertices, A. Stolin established a relationship between orders and connected subgroups of a simple Lie group which are locally transitive on a quotient space by some maximal parabolic subgroup. On the other hand, E. Vinberg and B. N. Kimel'fel'd classified in [11] all the connected irreducible subgroups of $SO(n)$ locally transitive on the Grassmann manifolds IG_k^n of isotropic k-dimensional subspaces in $\mathbb{C}^n$. The list of irreducible subgroups corresponding to the case $k = 1$ was obtained earlier by B. N. Kimel'fel'd in [5]. Using [11], A. Stolin found in [8] all nonconstant solutions in $o(5)$ and examples for $o(7)$, $o(10)$ and $o(14)$.

In the present paper we continue the investigation of the same list. We give a method which allows us to use the actual structure of the stationary subalgebra S of a generic point in order to find rational solutions. The idea is to decompose the Lie subalgebra $\widetilde{L}$ of a locally transitive irreducible subgroup of $SO(n)$ into a sum

$\widetilde{L} = L + S$ such that $L \cap S$ is a Frobenius Lie algebra. In this way L induces a rational solution. In the cases that we will consider, the stationary subalgebra S turns out to be either parabolic or Frobenius, fact which enables us to construct L and find solutions in o(7), o(8) and o(12). We also show how the solutions in o(5) found in [8] provide solutions in o(8). Finally, in constructing a nonconstant solution in o(12), we encounter a stationary subalgebra S which is a 14-dimensional Frobenius Lie algebra whose unipotent part is noncommutative (see for comparison the classification in the commutative case given in [3]).

2. Introduction to the theory of rational solutions of the CYBE

Let us first recall some definitions, notations and main results from [8] that will be used in our paper.

2.1. The correspondence between rational solutions and orders in $\mathbf{g}((u^{-1}))$. Let $\mathbf{g}$ be a simple complex finite-dimensional Lie algebra. Consider the universal enveloping algebra $U(\mathbf{g})$ and let φ_{12}, φ_{13}, φ_{23}, φ_{21}: $\mathbf{g} \otimes \mathbf{g} \to U(\mathbf{g})^{\otimes 3}$ be the linear maps respectively defined by $\varphi_{12}(a \otimes b) = a \otimes b \otimes 1$, $\varphi_{13}(a \otimes b) = a \otimes 1 \otimes b$, $\varphi_{23}(a \otimes b) = 1 \otimes a \otimes b$ and $\varphi_{21}(a \otimes b) = b \otimes a \otimes 1$. For a function $X : \mathbb{C}^2 \to \mathbf{g} \otimes \mathbf{g}$, we consider X^{ij}: $\mathbb{C}^2 \to U(\mathbf{g})^{\otimes 3}$ defined by $X^{ij}(u_i, u_j) = \varphi_{ij}(X(u_i, u_j))$. Let $[,]$ be the usual Lie bracket on the associative algebra $U(\mathbf{g})^{\otimes 3}$.

A solution of the classical Yang-Baxter equation (CYBE) is a function $X : \mathbb{C}^2 \to \mathbf{g} \otimes \mathbf{g}$ such that the following conditions are satisfied:

$$[X^{12}(u_1, u_2), X^{13}(u_1, u_3)] + [X^{12}(u_1, u_2), X^{23}(u_2, u_3)] + \tag{2.1}$$

$$+[X^{13}(u_1, u_3), X^{23}(u_2, u_3)] = 0$$

$$X^{12}(u, v) = -X^{21}(v, u). \tag{2.2}$$

Definition 2.1. [8] A solution of the CYBE is called *rational* if it is of the form

$$X(u, v) = \frac{\Omega}{u - v} + r(u, v), \tag{2.3}$$

where $r(u, v)$ is a polynomial with coefficients in $\mathbf{g} \otimes \mathbf{g}$ and Ω is the quadratic Casimir element of $\mathbf{g}$.

Remark 2.2. 1) The rational solutions we are dealing with look somehow different from those in Belavin-Drinfeld approach (see [1], [2]), where the solutions $X(u, v)$ depend only on $u - v$. As it turns out in [8], the above definition is more suitable for a classification of rational solutions.

2) The simplest example of a rational solution is Yang's r-matrix: $X_0(u, v) = \frac{\Omega}{u-v}$. By adding to $X_0(u, v)$ any skew-symmetric constant r-matrix, we also obtain a rational solution. However there are many more solutions than these.

Definition 2.3. [8] Two rational solutions X_1 and X_2 are said to be *gauge equivalent* if there exists $\sigma(u) \in Aut(\mathbf{g}[u])$ such that

$$X_1(u,v) = (\sigma(u) \otimes \sigma(v))X_2(u,v). \tag{2.4}$$

Remark 2.4. Here $\mathbf{g}[u] := \mathbf{g} \otimes \mathbb{C}[u]$ and $Aut(\mathbf{g}[u])$ denotes the group of automorphisms of $\mathbf{g}[u]$ seen as an algebra over $\mathbb{C}[u]$. One can check that gauge transformations applied to rational solutions also give rational solutions.

Let $\mathbb{O} := \mathbb{C}[[u^{-1}]$ be the ring of formal power series in u^{-1} and $\mathbb{K} := \mathbb{C}((u^{-1}))$ be the field of quotients of $\mathbb{O}$. Set $\mathbf{g}[[u^{-1}]] := \mathbf{g}\otimes\mathbb{O}$ and $\mathbf{g}((u^{-1})) := \mathbf{g}\otimes\mathbb{K}$. There exists a nondegenerate **ad**-invariant inner product on $\mathbf{g}((u^{-1}))$ given by

$$(x(u), y(u)) = Res_{u=0} tr(\mathbf{ad}x(u) \cdot \mathbf{ad}y(u)). \tag{2.5}$$

Theorem 2.5. *[8] There is a natural one-to-one correspondence between rational solutions of the CYBE and subalgebras $W \subseteq \mathbf{g}((u^{-1}))$ such that*
(1) $W \supseteq u^{-N}\mathbf{g}[[u^{-1}]]$ for some $N > 0$;
(2) $W \oplus \mathbf{g}[u] = \mathbf{g}((u^{-1}))$;
(3) W is a Lagrangian subspace with respect to the inner product on $\mathbf{g}((u^{-1}))$ given in (2.5), i.e. $W = W^{\perp}$.

Definition 2.6. [8] A $\mathbb{C}$-subalgebra $W \subseteq \mathbf{g}((u^{-1}))$ is said to be an *order* in $\mathbf{g}((u^{-1}))$ if there exist two non-negative integers N_1, N_2 such that $u^{-N_1}\mathbf{g}[[u^{-1}]] \subseteq W \subseteq u^{N_2}\mathbf{g}[[u^{-1}]]$.

Remark 2.7. Let W satisfy (1) and (3) from Theorem 2.5. Then W is an order.

Theorem 2.8. *[8] Let X_1 and X_2 be rational solutions of the CYBE and W_1, W_2 the corresponding orders in $\mathbf{g}((u^{-1}))$. Let $\sigma(u) \in Aut(\mathbf{g}[u])$. Then the following conditions are equivalent:*
(1) $X_1(u,v) = (\sigma(u) \otimes \sigma(v))X_2(u,v)$;
(2) $W_1 = \sigma(u)W_2$.

Definition 2.9. [8] Let V_1 and V_2 be subalgebras of $\mathbf{g}((u^{-1}))$. We say that V_1 and V_2 are *gauge equivalent* if there exists $\sigma(u) \in Aut(\mathbf{g}[u])$ such that $V_1 = \sigma(u)V_2$.

2.2. **Maximal orders in $\mathbf{g}((u^{-1}))$.** Let $\mathbf{h}$ be a Cartan subalgebra in $\mathbf{g}$, Λ the set of roots of $\mathbf{g}$ and $\{\mathbf{g}^{\alpha}\}_{\alpha\in\Lambda}$ the root spaces. Let us denote by $\mathbf{h}(\mathbb{R})$ the set of all $h \in \mathbf{h}$ such that $\alpha(h) \in\mathbb{R}$ for all $\alpha \in \Lambda$. One can define a valuation on $\mathbb{K}$ by $v(\sum_{k\geq N} a_k u^{-k}) = N$. For any root α and any $h \in \mathbf{h}(\mathbb{R})$ one sets $M_{\alpha}(h) = \{f \in \mathbb{K} : v(f) \geq \alpha(h)\}$ and

$$\mathbb{O}_h = \mathbf{h}[[u^{-1}]] \oplus (\oplus_{\alpha\in\Lambda} M_{\alpha}(h) \otimes_{\mathbb{O}} \mathbf{g}^{\alpha}). \tag{2.6}$$

Obviously, $\mathbb{O}_h$ is an order for any h.

The following result was formulated in V. G. Drinfeld's correspondence with J. P. Serre (cf. [8]) and then proved by A. Stolin in [9]:

Theorem 2.10. *[8] Let $\Delta_{st} = \{h \in \mathbf{h}(\mathbb{R})$: $\alpha(h) \geq 0$ for all simple roots α and $\alpha_{max}(h) \leq 1\}$. Then the following statements hold:*

(1) Any order M in $\mathbf{g}((u^{-1}))$ such that $M + \mathbf{g}[u] = \mathbf{g}((u^{-1}))$ is gauge equivalent to an order contained in $\mathbb{O}_h$ for $h \in \Delta_{st}$.
(2) Any maximal order T such that $T + \mathbf{g}[u] = \mathbf{g}((u^{-1}))$ is gauge equivalent to a maximal order corresponding to a vertex of the standard simplex Δ_{st}.

Moreover, to any such maximal order corresponds a vertex of the extended Dynkin diagram. Explicitly, the vertices of Δ_{st} are 0 and h_1, ..., h_r, where r denotes the rank of $\mathbf{g}$. To every order $\mathbb{O}_h$ we assign a vertex of the extended Dynkin diagram $D^e(\mathbf{g})$ of $\mathbf{g}$ according to the following rule:

$$0 \leftrightarrow \alpha_{max}$$

$$h_i \leftrightarrow \alpha_i,$$

where $\alpha_i(h_j) = \delta_{ij}/k_j$ and k_j are given by the relation $\sum k_j\alpha_j = \alpha_{max}$. We will denote by $\mathbb{O}_{\alpha}$ the maximal order corresponding to the vertex α of the extended Dynkin diagram.

Let G be the simply connected Lie group with Lie algebra $\mathbf{g}$. Denote by $G_{\mathbf{ad}}$ the Lie group $G/center$ and by $G_{\mathbf{ad}}(\mathbb{K})$ the set of its $\mathbb{K}$-points.

Proposition 2.11. *[8] Let H be the Cartan subgroup of G, $H_{\mathbf{ad}}$ its image in $G_{\mathbf{ad}}$, α a vertex of $D(\mathbf{g})$ such that $\sigma(\alpha) = -\alpha_{max}$ for some $\sigma \in Aut(D^e(\mathbf{g}))$. Then $\mathbb{O}_{\alpha} = H_{\alpha}^{-1}\mathbb{O}_0 H_{\alpha}$ for some $H_{\alpha} \in H_{\mathbf{ad}}(\mathbb{K})$.*

Remark 2.12. 1) If there exists an automorphism of $\mathbf{g}$ sending α_1 to α_2 then the corresponding orders are gauge equivalent and there is a one-to-one correspondence between solutions corresponding to α_1 and to α_2.
2) An order W corresponding to a rational solution $X(u,v) = \frac{\Omega}{u-v} + r(u,v)$ is contained in $\mathbb{O}_0$ if and only if r is a skew-symmetric constant r-matrix.

2.3. **Solutions corresponding to singular vertices.**

Definition 2.13. A vertex of the extended Dynkin diagram $D^e(\mathbf{g})$ is called *singular* if there exists an automorphism of $D^e(\mathbf{g})$ sending the vertex to $-\alpha_{max}$ and *regular* otherwise.

We suppose that we have fixed the Cartan subalgebra $\mathbf{h}$ and the root system. For any simple root α, let us denote by $\Pi_{\alpha}^{\pm}$ the set of all positive (respectively, negative) roots which do not contain α in their decomposition into linear combination of simple roots. We consider the following parabolic subalgebras:

$$P_{\alpha}^{+} = \mathbf{h} \oplus (\oplus_{\beta>\mathbf{0}}\mathbf{g}^{\beta}) \oplus (\oplus_{\gamma\in\mathbf{\Pi}_{\alpha}^{-}}\mathbf{g}^{\gamma}) \tag{2.7}$$

$$P_{\alpha}^{-} = \mathbf{h} \oplus (\oplus_{\beta<\mathbf{0}}\mathbf{g}^{\beta}) \oplus (\oplus_{\gamma\in\mathbf{\Pi}_{\alpha}^{+}}\mathbf{g}^{\gamma}). \tag{2.8}$$

Let now α be a singular vertex of $D^e(\mathbf{g})$. The following classification theorem was given in [8]:

Theorem 2.14. *[8] The set of subalgebras $W \subseteq \mathbb{O}_\alpha$ satisfying the conditions from Theorem 2.5 is in one-to-one correspondence with the set of pairs (L, B) such that:*
(1) L is a subalgebra of $\mathbf{g}$ and $L + P_\alpha^- = \mathbf{g}$;
(2) B is a 2-cocycle on L which is nondegenerate on $L \cap P_\alpha^-$.

Remark 2.15. 1) A similar result is true if one takes P_α^+ instead of P_α^-.
2) By dimension considerations, condition (1) is equivalent to the fact that $G(L)$ acts locally transitively on $G(\mathbf{g})/G(P_\alpha^-)$ and $1 \cdot G(P_\alpha^-)$ is a generic point of this action. From now on, for any Lie algebra Λ, $G(\Lambda)$ denotes the Lie group generated by $exp(\mathbf{ad}x)$.

We will say that a subalgebra L of $\mathbf{g}$ *provides* a rational solution corresponding to $\mathbb{O}_\alpha$ if there exists a 2-cocycle B on L such that (L, B) verifies conditions (1) and (2).

Definition 2.16. [3], [6], [8] A Lie algebra F is called *Frobenius* if there exists $f \in F^*$ such that the skew-symmetric bilinear form B_f defined by the formula $B_f(x, y) = f([x, y])$ for any x, $y \in F$, is nondegenerate.
A Lie algebra F is called *quasi-Frobenius* if there is a nondegenerate 2-cocycle B on F with values in $\mathbb{C}$.

Obviously any Frobenius Lie algebra is quasi-Frobenius.

Remark 2.17. Condition (2) from Theorem 2.14 implies that $L \cap P_\alpha^-$ is a quasi-Frobenius Lie algebra. On the other hand, if $L \cap P_\alpha^-$ is a Frobenius Lie algebra, then there exists a 2-cocycle B on L such that condition (2) is satisfied. Indeed any linear functional on $L \cap P_\alpha^-$ can be extended to L and the corresponding 2-cocycle is nondegenerate on $L \cap P_\alpha^-$.

2.4. **Solutions corresponding to singular vertices for the case $\mathbf{g} = o(n)$.** Suppose α is a singular vertex of $D^e(o(n))$. In order to find subalgebras L of $o(n)$ which satisfy condition (1) of Theorem 2.14, we can use the list of connected irreducible subgroups of $SO(n)$ locally transitive on the Grassmann manifolds IG_k^n of isotropic k-dimensional subspaces in $\mathbb{C}^n$ given in [11]. Moreover, k is not arbitrary. The next result shows how one has to choose k.

Lemma 2.18. *[8] The following statements hold:*
1) Let α_1 be the singular vertex of $D(o(2n+1))$. To find solutions corresponding to $\mathbb{O}_{\alpha_1}$ we have to take $k = 1$ in the list from [11].
2) Let α_1, α_{n-1} and α_n be the singular vertices of $D(o(2n))$. To find solutions corresponding to $\mathbb{O}_{\alpha_1}$ or $\mathbb{O}_{\alpha_n}$ (gauge equivalent $\mathbb{O}_{\alpha_{n-1}}$), we have to take $k = 1$ or $k = n$.

Let us finally recall the following result about nonconstant solutions in $o(5)$. We will use it later for constructing solutions in $o(8)$. We denote by $4\Lambda_1$ the representation of $sl(2)$ in the vector space $S^4(\mathbb{C}^2)$ induced by the standard representation Λ_1.

Lemma 2.19. *[8] 1) There is a unique nonconstant solution corresponding to the singular vertex $\alpha_1 \in D(o(5))$ and an irreducible $L \subset o(5)$. In this case $L = sl(2)$, via the embedding $4\Lambda_1$.*
2) Besides this there are 3 nonconstant solutions. They correspond to α_1 and to the following subalgebras $L \subset o(5)$:
a)

$$L = \begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$$

b)

$$L = g^{-1} \begin{bmatrix} * & * & 0 & * & 0 \\ * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix} g$$

c)

$$L = g^{-1} \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & -b \\ 0 & 0 & 0 & -c & -a \end{bmatrix} g$$

where $g \in SO(5)$ is such that $g(0, 0, 0, 0, 1)^t = (1, 0, \sqrt{2}, 1, -1)^t$.

3. Locally transitive actions and their applications to rational solutions

We intend to give concrete applications of Theorem 2.14 and find solutions corresponding to $\mathbb{O}_{\alpha_1}$ (as in Lemma 2.18). From now on we consider $k = 1$. We will use the list given in [5] (case $k = 1$ in [11]), to find subalgebras $L \subset o(n)$ which verify the conditions of Theorem 2.14. For this purpose, we will prove two lemmata which allow us to simplify the problem by using the structure of the stationary subalgebra of a generic point.

Let $\mathbf{g}$ be any simple complex finite-dimensional Lie algebra and α a singular vertex of $D^e(\mathbf{g})$.

Lemma 3.1. *Suppose that L is a Lie subalgebra of $\mathbf{g}$ such that $G(L)$ acts locally transitively on $G(\mathbf{g})/G(P_\alpha^-)$. Suppose that $x \cdot G(P_\alpha^-)$ is a generic point of this action, $x \in G(\mathbf{g})$. Then $L + \mathbf{Ad}(x)(P_\alpha^-) = \mathbf{g}$.*

Proof. Let S be the stationary subalgebra of the generic point $x \cdot G(P_\alpha^-)$. It results that $S = L \cap \mathbf{Ad}(x)(P_\alpha^-)$ and since we also have $\dim L - \dim S = \dim \mathbf{g} - \dim P_\alpha^-$, it follows that $L + \mathbf{Ad}(x)(P_\alpha^-) = \mathbf{g}$. □

Lemma 3.2. *Let $\widetilde{L}$ be a subalgebra of $\mathbf{g}$ such that $G(\widetilde{L})$ acts locally transitively on $G(\mathbf{g})/G(P_\alpha^-)$. Let S be the stationary subalgebra of a generic point $x \cdot G(P_\alpha^-)$, $x \in G(\mathbf{g})$. Suppose that L is a subalgebra of $\widetilde{L}$ such that*
(1)' $\widetilde{L} = L + S$;
(2)' $L \cap S$ is a Frobenius Lie algebra.
Then $\mathbf{Ad}(x^{-1})(L)$ provides a rational solution corresponding to $\mathbb{O}_\alpha$.

Proof. We know that $S = \widetilde{L} \cap \mathbf{Ad}(x)(P_\alpha^-)$. By Lemma 3.1, $\widetilde{L} + \mathbf{Ad}(x)(P_\alpha^-) = \mathbf{g}$. Since $\widetilde{L} = L + S$, it follows that $L + \mathbf{Ad}(x)(P_\alpha^-) = \mathbf{g}$ and thus $\mathbf{Ad}(x^{-1})(L) + P_\alpha^- = \mathbf{g}$. We also have $L \cap S = L \cap \mathbf{Ad}(x)(P_\alpha^-)$ or, equivalently, $\mathbf{Ad}(x^{-1})(L \cap S) = \mathbf{Ad}(x^{-1})(L) \cap P_\alpha^-$. It results that $\mathbf{Ad}(x^{-1})(L) \cap P_\alpha^-$ is a Frobenius Lie algebra. The conclusion follows immediately by Theorem 2.14 and Remark 2.17. □

Remark 3.3. The result remains true if condition (2)' is replaced by: there exists a 2-cocycle B on L which is nondegenerate on $L \cap S$.

Lemma 3.2 allows us to find examples of rational solutions in the following way:

We start with a subalgebra $\widetilde{L}$ which is the Lie algebra of a connected irreducible Lie subgroup of $SO(n)$ locally transitive on IG_1^n. We choose a generic point $p = [v]$ and we find its stationary subalgebra S. In [5] B.N. Kimel'fel'd gave a classification of the stationary subalgebras of generic points. However for our purpose we will need specific presentations. The stationary subalgebra may be Frobenius or not. If S is Frobenius then $\widetilde{L}$ (or a conjugate of it) provides a solution for our problem. If the answer is negative, we will look instead for a subalgebra L of $\widetilde{L}$ such that we have the decomposition $\widetilde{L} = L + S$ and $L \cap S$ is a Frobenius Lie algebra.

3.1. **Rational solutions provided by trivial stationary subalgebra of a generic point.** If the stationary subalgebra S of a generic point is trivial, then we obviously obtain a solution. By looking at the classification list, we find two such situations:

I. $L = sl(2)$, $\mathbf{g} = o(5)$, via the embedding $4\Lambda_1$. This has already been noticed in [8].

II. $L = sl(2) \times sl(2)$, $\mathbf{g} = o(8)$, via the embedding $\Lambda_1 \times 3\Lambda_1$. This example is presented in [5] but not in [11], probably a misprint in [11].

3.2. **Rational solutions provided by nontrivial stationary subalgebra of a generic point.** In this section we will treat the following cases from the list of connected irreducible subgroups locally transitive on isotropic Grassmannians:

I. $SL(3) \subset SO(8)$ (the embedding is via $\mathbf{Ad}$-representation). According to [5], $SL(3)$ acts locally transitively on the 6-dimensional manifold IG_1^8 via $\mathbf{Ad}$-representation. Here IG_1^8 is considered with respect to the Killing form of $sl(3)$. Let $\mathbf{h}$ be the Cartan subalgebra of $sl(3)$. One can check that for any isotropic $h \in \mathbf{h}$ the corresponding line is a generic point and its stationary subalgebra is $\mathbf{h}$.

On the other hand, there exists a subalgebra $L \subset sl(3)$ such that $L \oplus \mathbf{h} = sl(3)$. Indeed, it is clear that L must be 6-dimensional and therefore a parabolic subalgebra.

Let us consider the "standard" parabolic subalgebra of $sl(3)$:

$$P = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}. \tag{3.1}$$

The following conjugate of P is a complement to $\mathbf{h}$:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1}. \tag{3.2}$$

If $P_{\alpha_1}^-$ denotes as usual the negative parabolic subalgebra of $o(8)$ corresponding to $\alpha_1 \in D(o(8))$, then Lemma 3.2 (the second condition simply disregarded) implies that $\mathbf{Ad}(x^{-1})(L) \oplus P_{\alpha_1}^- = o(8)$, for some $x \in SO(8)$. Thus $\mathbf{Ad}(x^{-1})(L)$ provides a solution corresponding to $\mathbb{O}_{\alpha_1}$.

II. $G_2 \subset SO(7)$ (the embedding is via Λ_1). Let us find the stationary subalgebra of a generic point.

Let $\mathbf{h}$ be the Cartan subalgebra of $\mathbf{g}_2$, β_1 and β_2 be the simple roots of $\mathbf{g}_2$ (β_1 the short one). For any root β denote by $\mathbf{g}_2^\beta$ the corresponding root space in $\mathbf{g}_2$ and let us choose $e_\beta \in \mathbf{g}_2^\beta$ such that $K(e_\beta, e_{-\beta}) = 1$. We consider the parabolic subalgebra

$$P_{\beta_1}^+ = \mathbf{h} \oplus \sum_{\beta > 0} \mathbf{g}_2^\beta \oplus \mathbf{g}_2^{-\beta_2}. \tag{3.3}$$

The standard 7-dimensional representation of G_2, Λ_1, has the highest weight $\lambda = 2\beta_1 + \beta_2$. We consider the action of G_2 on the projective space $\mathbb{P}^6$ induced by Λ_1. Let $p \in \mathbb{P}^6$ be the point corresponding to the eigenspace with eigenvalue λ. According to [4, p. 388], the stabilizer of the point p is the parabolic subgroup corresponding to the subset of simple roots that are orthogonal on λ. Since only β_2 is orthogonal on λ, it follows that the stabilizer of the point p is the parabolic subgroup whose Lie algebra is $P_{\beta_1}^+$.

On the other hand, the orbit of the point p is a 5-dimensional projective algebraic variety. According to [4, p. 391], it is a quadric hypersurface. Therefore there exists a nondegenerate symmetric bilinear form Q on $\mathbb{C}^7$ which is preserved by the action of G_2. We consider the Grassmannian IG_1^7 with respect to this Q. It follows that the orbit of p coincides with IG_1^7 (the action is transitive) and its stationary subalgebra is $P_{\beta_1}^+$.

This fact enables us to construct a subalgebra of $\mathbf{g}_2$ which produces a solution corresponding to the singular vertex $\alpha_1 \in D(o(7))$. Let α_2 and α_3 denote the other two simple roots of $o(7)$.

Proposition 3.4. *Consider the following parabolic subalgebra of* $\mathbf{g}_2$

$$P^-_{\beta_2} = \mathbf{h} \oplus \sum_{\beta<0} \mathbf{g}_2^{\beta} \oplus \mathbf{g}_2^{\beta_1}. \tag{3.4}$$

Then the subalgebra $\mathbf{Ad}(x^{-1})(P^-_{\beta_2})$ *provides a rational solution corresponding to* $\mathbb{O}_{\alpha_1}$*, where* $x = \exp(e_{-\alpha_2-2\alpha_3})$*. Here* $e_{-\alpha_2-2\alpha_3}$ *is a root vector corresponding to the root* $-\alpha_2 - 2\alpha_3$.

Proof. G_2 acts transitively on IG^7_1. Let $p \in \mathbb{P}^6$ be the point corresponding to the eigenspace with eigenvalue $\lambda = 2\beta_1 + \beta_2 = -\alpha_2 - 2\alpha_3$. Its stationary subalgebra is $P^+_{\beta_1}$. Let us check if conditions (1)' and (2)' of Lemma 3.2 are verified. Here $\widetilde{L} = \mathbf{g}_2 \subset o(7)$, $S = P^+_{\beta_1}$ and $L = P^-_{\beta_2}$. The condition $L+S = \mathbf{g}_2$ is obviously satisfied. On the other hand, $L \cap S = \mathbf{h} \oplus \mathbf{g}_2^{\beta_1} \oplus \mathbf{g}_2^{-\beta_2}$. This 4-dimensional Lie subalgebra is Frobenius. Indeed, let us define the linear functional f by $f(h+ae_{\beta_1}+be_{-\beta_2}) = a+b$ for any $h \in \mathbf{h}$ and complex numbers a, b. Let B_f be the induced 2-cocycle. Let $h_{\beta_i} = [e_{\beta_i}, e_{-\beta_i}]$, $i = 1,2$. One can check that the matrix of B_f in the basis h_{β_1}, h_{β_2}, e_{β_1}, $e_{-\beta_2}$ is nondegenerate and therefore $L \cap S$ is Frobenius. Lemma 3.2 implies that $\mathbf{Ad}(x^{-1})(P^-_{\beta_2})$ provides a solution corresponding to $\mathbb{O}_{\alpha_1}$. $\square$

III. $Sp(2s) \times SL(2) \subset SO(4s)$, $s \geq 2$. The embedding is via the representation $\Lambda_1 \times \Lambda_1$. According to [5], $Sp(2s) \times SL(2)$ acts locally transitively on IG^{4s}_1. In order to find the stationary subalgebra corresponding to a generic point, we need the explicit form of the embedding of $sp(2s) \times sl(2)$ in $o(4s)$.

Let $\{e_i\}_{i=1,\dots,2s}$ and $\{f_j\}_{j=1,2}$ be the canonical bases in $\mathbb{C}^{2s}$ and $\mathbb{C}^2$ respectively. We consider the following skew-symmetric bilinear forms: Q_1 on $\mathbb{C}^{2s}$ given by $Q_1(e_i, e_{2s+1-i}) = 1$ for $i \leq s$ and $Q_1(e_i, e_j) = 0$ if $i + j \neq 2s + 1$ and Q_2 on $\mathbb{C}^2$ defined by $Q_2(f_1, f_2) = 1$. It is clear that Q_1 and Q_2 induce a symmetric bilinear form Q on $\mathbb{C}^{2s} \otimes \mathbb{C}^2$ by $Q(v_1 \otimes w_1, v_2 \otimes w_2) = Q_1(v_1, v_2)Q_2(w_1, w_2)$.

We consider $sp(2s)$ defined as the Lie algebra of invariant matrices with respect to Q_1 and $sl(2)$ as the Lie algebra of invariant matrices with respect to Q_2. We will write any matrix from $sp(2s)$ in the form $\begin{bmatrix} A & B \\ C & \widetilde{A} \end{bmatrix}$, where B, C are symmetric with respect to the second diagonal and $\widetilde{A}$ is obtained from A by skew-symmetry. It follows immediately that the symmetric form Q defined previously is invariant with respect to the representation $\Lambda_1 \times \Lambda_1$ and therefore the image of $sp(2s) \times sl(2)$ is included in $o(4s)$. Let us choose the following basis in $\mathbb{C}^{2s} \otimes \mathbb{C}^2$: $e_1 \otimes f_1$, $e_1 \otimes f_2$,..., $e_s \otimes f_1$, $e_s \otimes f_2$,$-e_{s+1} \otimes f_1$, $e_{s+1} \otimes f_2$,...,$-e_{2s} \otimes f_1$, $e_{2s} \otimes f_2$. Obviously the matrix of Q with respect to this basis has 1 on the second diagonal and 0 elsewhere.

Any pair $(X_1, X_2) \in sp(2s) \times sl(2)$ acts on $\mathbb{C}^{2s} \otimes \mathbb{C}^2$ in the canonical way: $(X_1, X_2) \cdot (v \otimes w) = X_1 v \otimes w + v \otimes X_2 w$. With respect to the chosen basis in $\mathbb{C}^{2s} \otimes \mathbb{C}^2$, the operator induced by (X_1, X_2) has a matrix which can be computed explicitly and is skew-symmetric with respect to the second diagonal.

Let us consider the Grassmannian IG^{4s}_1 of isotropic lines in $\mathbb{C}^{4s}$ (identified with $\mathbb{C}^{2s} \otimes \mathbb{C}^2$) with respect to the symmetric bilinear form Q. By direct calculations using the embedding of $sp(2s) \times sl(2)$ in $o(4s)$, one can prove the following

Proposition 3.5. *The line spanned by* $e_1 \otimes f_1 + e_2 \otimes f_2$ *is a generic point for the locally transitive action of* $Sp(2s) \times SL(2)$ *on* IG^{4s}_1 *and its stationary subalgebra* S *is the set of pairs*

$$\begin{bmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & \widetilde{A} \end{bmatrix} + \varphi(A) \in sp(2s) \times sl(2),$$

for all $A \in gl(2)$, $B \in sp(2s-4)$, *where* $\varphi : gl(2) \to sl(2)$ *is defined by*

$$\varphi \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{d-a}{2} & -c \\ -b & \frac{a-d}{2} \end{bmatrix}.$$

We make the remark that the dimension of S is $2s^2 - 3s + 5$, as expected. It is clear that S cannot be a Frobenius algebra if s is even because the dimension of S would be odd. In this case, in order to find a solution to our problem, we shall look instead for a subalgebra L of $sp(2s) \times sl(2)$ such that $L + S = sp(2s) \times sl(2)$ and $L \cap S$ is Frobenius. In the case when s is odd, S has even dimension and it might happen that it is Frobenius.

We will apply these general ideas to construct several solutions in the cases $s = 2$ and $s = 3$.

Let us consider $s = 2$. In this situation, the stationary subalgebra S is a 7-dimensional subalgebra:

$$S = \{ \begin{pmatrix} A & * \\ 0 & \widetilde{A} \end{pmatrix} + \varphi(A); A \in gl(2)\}. \tag{3.5}$$

We can easily construct a suitable subalgebra L:

Proposition 3.6. *Let* X *be any element of* $SO(8)$ *sending the isotropic vector* $(1, 0, 0, 1, 0, 0, 0, 0)^t$ *to* $(1, 0, 0, 0, 0, 0, 0, 0)^t$ *in* $\mathbb{C}^8$*. Consider the following subalgebra* L *of* $sp(4) \times sl(2)$:

$$L = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix} \times sl(2).$$

Then the subalgebra $\mathbf{Ad}(X)(L)$ *provides a rational solution corresponding to* $\mathbb{O}_{\alpha_1}$*, where* α_1 *is the first singular vertex of* $D(o(8))$.

Proof. One can easily check that $L + S = sp(4) \times sl(2)$ and that

$$L \cap S = \{ \begin{bmatrix} a & 0 & 0 & 0 \\ b & c & d & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & -b & -a \end{bmatrix} + \varphi \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \} \tag{3.6}$$

is a 4-dimensional Frobenius algebra. The conclusion follows by applying Lemma 3.2. □

Let us recall that all nonconstant solutions corresponding to the singular vertex in $o(5)$ were given in Lemma 2.19. The following result shows how the solutions in $o(5)$ provide solutions in $o(8)$:

Proposition 3.7. *Suppose that L is a subalgebra of $o(5)$ which provides a rational solution corresponding to $\alpha_1 \in D(o(5))$. Then $\mathbf{Ad}(X)(L \times sl(2)) \subset o(5) \times sl(2)$ provides a rational solution in $o(8)$, where X is chosen as in Prop. 3.6.*

Proof. Let us fix un isomorphism $i : o(5) \to sp(4)$. Let $\widetilde{L} = i(L) \times sl(2)$. Let $S \subset sp(4) \times sl(2)$ be the stationary subalgebra from (3.5). In other words, $S = (id \oplus \pi)(P)$, where

$$P = \begin{bmatrix} a & b & e & f \\ c & d & g & e \\ 0 & 0 & -d & -b \\ 0 & 0 & -c & -a \end{bmatrix} \tag{3.7}$$

and π is the projection of $sp(4)$ onto $sl(2)$ given by $\pi \begin{bmatrix} A & B \\ C & \widetilde{A} \end{bmatrix} = \varphi(A)$. It is clear that P is a 7-dimensional parabolic subalgebra of $sp(4)$. Via the isomorphism i, its image in $o(5)$ will be also a 7-dimensional parabolic subalgebra. One may choose the isomorphism i such that this parabolic subalgebra is the one corresponding to the singular vertex $\alpha_1 \in D(o(5))$:

$$P^+_{\alpha_1} = \begin{bmatrix} * & * & * & * & 0 \\ 0 & * & * & 0 & * \\ 0 & * & 0 & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}. \tag{3.8}$$

It is clear that S is isomorphic to $(id \oplus \pi i)(P^+_{\alpha_1}) \subset o(5) \times sl(2)$. Since L provides a solution corresponding to α_1 in $o(5)$, $L + P^+_{\alpha_1} = o(5)$ and there is a 2-cocycle on L which is nondegenerate on $L \cap P^+_{\alpha_1}$. We have that $\widetilde{L} + S \cong L \times sl(2) + (id \oplus \pi i)(P^+_{\alpha_1}) = o(5) \times sl(2)$ and $\widetilde{L} \cap S \cong (L \times sl(2)) \cap (id \oplus \pi i)(P^+_{\alpha_1}) \cong L \cap P^+_{\alpha_1}$. Remark 3.3 implies that $\mathbf{Ad}(X)(\widetilde{L})$ provides a solution in $o(8)$. □

This result combined with Lemma 2.19 gives the following:

Corollary 3.8. *Let L be one of the four subalgebras from Lemma 2.19. Then $\mathbf{Ad}(X)(L \times sl(2))$ provides a rational solution in $o(8)$, where X is chosen as in Prop. 3.6.*

Remark 3.9. It follows that the solution obtained in Proposition 3.6 is equivalent to one solution from the corollary.

Let us consider now $s = 3$. In this case, the stationary subalgebra S found in Proposition 3.5 is 14-dimensional:

$$S = \{ \begin{bmatrix} a & b & * & * & * & * \\ c & d & * & * & * & * \\ 0 & 0 & e & f & * & * \\ 0 & 0 & g & -e & * & * \\ 0 & 0 & 0 & 0 & -d & -b \\ 0 & 0 & 0 & 0 & -c & -a \end{bmatrix} + \begin{bmatrix} \frac{d-a}{2} & -c \\ -b & \frac{a-d}{2} \end{bmatrix} \}. \tag{3.9}$$

We intend to prove that S is a Frobenius Lie algebra. It is useful to decompose S into the unipotent part U and the reductive part R, both of them 7-dimensional subalgebras. More precisely,

$$U = \begin{bmatrix} 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \subset sp(6) \tag{3.10}$$

and

$$R = \{ \begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & e & f & 0 & 0 \\ 0 & 0 & g & -e & 0 & 0 \\ 0 & 0 & 0 & 0 & -d & -b \\ 0 & 0 & 0 & 0 & -c & -a \end{bmatrix} + \begin{bmatrix} \frac{d-a}{2} & -c \\ -b & \frac{a-d}{2} \end{bmatrix} \}. \tag{3.11}$$

Obviously, R acts on U by means of the Lie bracket. We make the important remark that U is not commutative and a problem arises here. Several criteria have been given in literature but for Lie algebras with commutative unipotent part. For instance, A. Elashvili obtained in [3] a classification theorem. A more general result about nondegenerate 2-cocycles on a semidirect sum of an arbitrary Lie algebra and a commutative ideal was also obtained in [10]. Since our Lie algebra does not fit into this picture, we will give a constructive proof.

Proposition 3.10. *S is a Frobenius Lie algebra.*

Proof. We will construct $f \in S^*$ such that the skew-symmetric bilinear form defined by $B_f(x, y) = f([x, y])$, for any $x, y \in S$, is nondegenerate. Let us decompose the subalgebra S into the unipotent part U and the reductive part R as above. Let us choose the canonical basis for U: $E_1 = e_{13} - e_{46}$, $E_2 = e_{14} + e_{36}$, $E_3 = e_{15} + e_{26}$, $E_4 = e_{24} + e_{35}$, $E_5 = e_{23} - e_{45}$, $E_6 = e_{16}$, $E_7 = e_{25}$ and for R: $F_1 = e_{11} - e_{66}$, $F_2 = e_{22} - e_{55}$, $F_3 = e_{33} - e_{44}$, $F_4 = e_{34}$, $F_5 = e_{43}$, $F_6 = e_{12} - e_{56}$, $F_7 = e_{21} - e_{65}$. We put $f(E_i) = \alpha_i$ and $f(F_i) = \beta_i$ for all $i = 1, ..., 7$. We will prove that we can choose α_i, β_i such that B_f is nondegenerate. Let us make the remark that the matrix of B_f with respect to the basis E_1,...,E_7, F_1,..., F_7 consists of four blocks of dimension 7:

$$M_{B_f} = \begin{bmatrix} A & B \\ -B^t & C \end{bmatrix}, \tag{3.12}$$

where $A = (a_{ij}) = (B_f(E_i, E_j))$, $B = (b_{ij}) = (B_f(E_i, F_j))$ and $C = (c_{ij}) = (B_f(F_i, F_j))$. These matrices can be computed explicitly, A and B depend on α_i's and C on β_i's. We make the remark that U is not commutative and therefore A is not zero. Simple computations show that B is degenerate for any choice of the numbers α_i, but there exist α_i's such that the rank equals 6. Moreover one can choose α_i's such that the first 7 rows are linearly independent and then find β_i such that the whole matrix is nondegenerate. Such a choice is $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = 1$, $\alpha_2 = 0$, $\alpha_6 = \frac{1}{2}$, $\alpha_7 = -\frac{1}{2}$ and $\beta_i = 0$ except $\beta_3 = 1$. Thus S is Frobenius. □

Corollary 3.11. *Let Y be any element of $SO(12)$ sending the isotropic vector $(1,0,0,1,0,..,0)^t$ to $(1,0,...,0)^t$ in $\mathbb{C}^{12}$. Then $\mathbf{Ad}(Y)(sp(6) \times sl(2))$ provides a non-constant rational solution in $o(12)$.*

Proof. The result follows by Lemma 3.2. □

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References

[1] Belavin A. A., Drinfeld V. G., *On classical Yang-Baxter equations for simple Lie algebras*, Funct. Anal. and Appl. **16** no. 3 (1982) 1-29.

[2] Belavin A. A., Drinfeld V. G., *On classical Yang-Baxter equations for simple Lie algebras*, Funct. Anal. and Appl. **17** no. 3 (1983) 69-70.

[3] Elashvili A., *Frobenius Lie algebras*, Funct. Anal. and Appl. **16** no. 4 (1982) 94-95.

[4] Fulton W., Harris J., *Representation Theory*, Springer-Verlag (1991).

[5] Kimel'fel'd B. N., *Open orbits of reductive groups on complex quadrics*, Soobshch. Akad. Nauk. Gruz. SSR **81** no. 2 (1976) 305-308.

[6] Ooms A. I., *On Frobenius Lie algebras*, Commun. Alg. **8** no. 1 (1980) 13-52.

[7] Stolin A., *On rational solutions of Yang-Baxter equation for $sl(n)$*, Math. Scand. **69** (1991) 57-80.

[8] Stolin A., *On rational solutions of Yang-Baxter equation. Maximal orders in loop algebra*, Commun. Math. Phys. **141** (1991) 533-548.

[9] Stolin A., *A geometrical approach to rational solutions of the classical Yang-Baxter equation. Part I*, Symposia Gaussiana, Conf. Alg., Walter de Gryter&Co., Berlin, New York (1995) 347-357.

[10] Stolin A., *Rational solutions of the classical Yang-Baxter equation and quasi Frobenius Lie algebras*, J. Pure. Appl. Alg. **137** (1999) 285-293.

[11] Vinberg E., Kimel'fel'd B. N., *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups*, Sov. J. Funct. Anal. **12** no. 3 (1978) 12-19.

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ON THE CLASSICAL DOUBLE OF PARABOLIC SUBALGEBRAS

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Abstract. Given a complex simple finite-dimensional Lie algebra $\mathbf{g}$ with fixed root system, there exists a so-called classical Drinfeld-Jimbo r-matrix, r. Consider any parabolic subalgebra $P_S \subseteq \mathbf{g}$ defined by a subset S of the set of simple roots. We prove that the Lie bialgebra structure on $\mathbf{g}$ defined by r can be restricted to P_S. Moreover, it turns out that the corresponding classical double $D(P_S)$ is isomorphic to $\mathbf{g} \oplus \mathbf{Red}(P_S)$, where $\mathbf{Red}(P_S)$ denotes the reductive part of P_S.

Keywords: Lie bialgebra, parabolic subalgebra, Drinfeld-Jimbo r-matrix, classical double.
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1. Introduction

The notion of *Lie bialgebra* was introduced by V. G. Drinfeld in [3]. Let us recall the definition:

Definition 1.1. Let $\mathbf{g}$ be a finite-dimensional vector space over $\mathbb{C}$ and suppose that both $\mathbf{g}$ and $\mathbf{g}^*$ have Lie algebra structures. The Lie algebra structures are called *compatible* if

$$c^k_{rs} d^{ij}_k = c^i_{\alpha r} d^{j\alpha}_s - c^j_{\alpha r} d^{i\alpha}_s - c^i_{\alpha s} d^{j\alpha}_r + c^j_{\alpha s} d^{i\alpha}_r, \tag{1.1}$$

where c^k_{rs} and d^{ij}_k are the structure constants of $\mathbf{g}$ and $\mathbf{g}^*$ with respect to bases of $\mathbf{g}$ and $\mathbf{g}^*$ which are dual to each other. We will say that $\mathbf{g}$ is equipped with a *Lie bialgebra structure* if $\mathbf{g}$ and $\mathbf{g}^*$ have compatible Lie algebra structures.

In [3] it was proved the following result:

Theorem 1.2. *Suppose that $\mathbf{g}$ and $\mathbf{g}^*$ have fixed Lie algebra structures. Define the linear map $\varphi : \mathbf{g}^* \otimes \mathbf{g}^* \to \mathbf{g}^*$ by setting $\varphi(l_1 \otimes l_2) = [l_1, l_2]$. Then the following conditions are equivalent:*
1) The Lie algebra structures on $\mathbf{g}$ and $\mathbf{g}^*$ are compatible.
2) The map $\varphi^* : \mathbf{g} \to \mathbf{g} \otimes \mathbf{g}$ is a 1-cocycle, where it is understood that $\mathbf{g}$ acts on $\mathbf{g} \otimes \mathbf{g}$ by means of the adjoint representation, i.e. for any a, $b \in \mathbf{g}$,

$$\varphi^*([a,b]) = a.\varphi^*(b) - b.\varphi^*(a), \tag{1.2}$$

where $a.(b \otimes c) = [a \otimes 1 + 1 \otimes a, b \otimes c] = [a,b] \otimes c + b \otimes [a,c]$.
3) There is a Lie algebra structure on the vector space $\mathbf{g} \oplus \mathbf{g}^*$ inducing the given Lie algebra structures on $\mathbf{g}$ and $\mathbf{g}^*$, which is such that the bilinear form Q on $\mathbf{g} \oplus \mathbf{g}^*$ defined by the formula

$$Q((x_1, l_1), (x_2, l_2)) = l_1(x_2) + l_2(x_1), \tag{1.3}$$

is invariant with respect to the adjoint representation of $\mathbf{g} \oplus \mathbf{g}^*$. Moreover, such a Lie algebra structure on $\mathbf{g} \oplus \mathbf{g}^*$ is unique if it exists.

The linear space $\mathbf{g} \oplus \mathbf{g}^*$ equipped with this Lie algebra structure is called the *classical double* of the Lie bialgebra $\mathbf{g}$ and is denoted by $D(\mathbf{g})$.

Let us remark that in order to equip $\mathbf{g}$ with a Lie bialgebra structure, we need a 1-cocycle $\delta : \mathbf{g} \to \mathbf{g} \wedge \mathbf{g}$ which satisfies the co-Jacobi identity:

$$Per(\delta \otimes 1) \circ \delta(a) = 0, \tag{1.4}$$

for any $a \in \mathbf{g}$, where $Per(a \otimes b \otimes c) = a \otimes b \otimes c + c \otimes a \otimes b + b \otimes c \otimes a$.

Let us suppose now that $\mathbf{g}$ is a simple finite-dimensional Lie algebra over $\mathbb{C}$. In this case, any 1-cocycle δ is a co-boundary and therefore it is given by the formula $\delta(a) = [a \otimes 1 + 1 \otimes a, r]$ for some $r \in \mathbf{g} \otimes \mathbf{g}$. It was shown in [3] that r can be chosen skew-symmetric and satisfying the following condition: $< r, r >= [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$ is an $\mathbf{ad}$-invariant element of $\mathbf{g} \otimes \mathbf{g} \otimes \mathbf{g}$. Since $\mathbf{g}$ is simple, there is only one $\mathbf{ad}$-invariant tensor in $\mathbf{g} \otimes \mathbf{g} \otimes \mathbf{g}$ up to a scalar factor, namely $a \cdot c_{ijk} I_i \otimes I_j \otimes I_k$, where $\{I_i\}$ is an orthonormal basis of $\mathbf{g}$ with respect to the Killing form, c_{ijk} are the corresponding structure constants and $a \in \mathbb{C}$.

In the case $a \neq 0$, all these tensors r were found by Belavin and Drinfeld in [1]. A. Stolin has proved in [9] that any classical double $D(\mathbf{g})$ is graded by its root system. As a consequence, $D(\mathbf{g}) \cong \mathbf{g} \otimes A$, where A is a unital commutative associative algebra of dimension 2. There are two possibilities for A: nilpotent or semisimple. The nilpotent case corresponds to the case $a = 0$ and the semisimple to the case $a \neq 0$.

In our paper, we replace $\mathbf{g}$ by a parabolic subalgebra and we give a description of the classical double induced by the classical Drinfeld-Jimbo r-matrix.

We recall that a *parabolic* subalgebra of a Lie algebra is a subalgebra that contains a Borel subalgebra (i.e. a maximal solvable subalgebra).

Let $\mathbf{g}$ be a complex simple finite-dimensional Lie algebra. We consider the root system R with respect to a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$, Δ the set of simple roots and $\{\mathbf{g}^\alpha\}_{\alpha \in R}$ the root spaces. For any $S \subseteq \Delta$, let us denote by $\Pi(S)$ the set of all $x \in R$ such that if $x = \sum k_\alpha \alpha$, then $k_\alpha \geq 0$ for any $\alpha \in S$. It is known that any parabolic subalgebra can be transformed by an inner automorphism to one of the following subalgebras:

$$P_S = \mathbf{h} \oplus \sum_{\alpha \in \Pi(S)} \mathbf{g}^\alpha. \tag{1.5}$$

Let us consider the Killing form K on $\mathbf{g}$. For any nonzero element $e_\alpha \in \mathbf{g}^\alpha$, there exists an element $e_{-\alpha} \in \mathbf{g}^{-\alpha}$ such that $K(e_\alpha, e_{-\alpha}) = 1$. With these notations, the classical Drinfeld-Jimbo r-matrix is the following:

$$r = \frac{1}{2} \sum_{\alpha > 0} e_\alpha \wedge e_{-\alpha}. \tag{1.6}$$

It is well-known that r has the following properties:

$$r^{12} + r^{21} = 0 \tag{1.7}$$

and the tensor $< r, r >$ defined by

$$< r, r >= [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \tag{1.8}$$

is **ad** - invariant.

Given a parabolic subalgebra P_S of $\mathbf{g}$, we prove that r defines a Lie bialgebra structure on P_S and we describe the classical double associated to it. We will show that the classical double of a parabolic subalgebra P_S of $\mathbf{g}$, induced by the tensor r, is isomorphic to $\mathbf{g} \oplus \mathbf{Red}(P_S)$.

In particular, for $S = \emptyset$, the well-known result $D(\mathbf{g}) \cong \mathbf{g} \oplus \mathbf{g}$ (see [7]) will follow.

2. Basic Lie bialgebra structure for a parabolic subalgebra

Let r be the classical Drinfeld-Jimbo r-matrix defined in (1.6) and let us consider the 1-cocycle δ_r: $\mathbf{g} \to \mathbf{g} \wedge \mathbf{g}$ defined by

$$\delta_r(a) = [a \otimes 1 + 1 \otimes a, r] \tag{2.1}$$

for all $a \in \mathbf{g}$.

Let S be a subset of Δ and P_S the corresponding parabolic subalgebra of $\mathbf{g}$.

Proposition 2.1. *The map δ_r provides a Lie bialgebra structure for any parabolic subalgebra P_S.*

Proof. Because δ_r is a 1-cocycle that satisfies the co-Jacoby identity, it is enough to show that $\delta_r(P_S) \subseteq P_S \wedge P_S$. For all $\gamma \in R$, we have chosen $e_\gamma \in \mathbf{g}^\gamma$ such that $K(e_\gamma, e_{-\gamma}) = 1$. It is known that $\{e_\gamma\}_{\gamma>0}$ is a basis for $\mathbf{n}_+ = \sum_{\gamma>0} \mathbf{g}^\gamma$ and $\{e_{-\gamma}\}_{\gamma>0}$ is a basis for $\mathbf{n}_- = \sum_{\gamma>0} \mathbf{g}^{-\gamma}$. Therefore it is enough to check that $\delta_r(e_\gamma) \in P_S \wedge P_S$ for any $\gamma > 0$ and $\gamma < 0$ which does not contain roots belonging to S. Define $N_{\alpha,\beta}$ by the formula $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ for α, β, $\alpha + \beta \in R$. If α, $\beta \in R$, $\alpha + \beta \notin R$ and $\alpha \neq -\beta$, we set $N_{\alpha,\beta} = 0$.

Consider $\gamma > 0$. We have:

$$\delta_r(e_\gamma) = [e_\gamma \otimes 1 + 1 \otimes e_\gamma, \frac{1}{2} \sum_{\beta>0} e_\beta \wedge e_{-\beta}] = \frac{1}{2} \sum_{\beta>0} ([e_\gamma, e_\beta] \wedge e_{-\beta} + e_\beta \wedge [e_\gamma, e_{-\beta}]).$$

We make the following remarks: If $-\beta \in \Pi(S)$, then $[e_\gamma, e_\beta] \wedge e_{-\beta} = N_{\gamma,\beta} e_{\gamma+\beta} \wedge e_{-\beta}$ or it is zero. In any case it is an element from $P_S \wedge P_S$. If $\gamma - \beta$ is not a root or $\gamma - \beta \in \Pi(S)$, then $e_\beta \wedge [e_\gamma, e_{-\beta}] = N_{\gamma,-\beta} e_\beta \wedge e_{\gamma-\beta} \in P_S \wedge P_S$ or it is contained in $P_S \wedge \mathbf{h}$. Thus the terms that remain to be considered are $N_{\gamma,\beta} e_{\gamma+\beta} \wedge e_{-\beta}$ when β contains at least one root from S, and $N_{\gamma,-\beta} e_\beta \wedge e_{\gamma-\beta}$ when $\delta = \beta - \gamma > 0$ contains roots from S. We consider $B_{\gamma,S} = \{\delta > 0 : \delta$ contains simple roots from S in its decomposition and $\gamma + \delta \in R\}$. Consequently, from the previous remarks, it follows that

$$\delta_r(e_\gamma) - \sum_{\delta \in B_{\gamma,S}} (N_{\gamma,\delta} + N_{\gamma,-\gamma-\delta}) e_{\gamma+\delta} \wedge e_{-\delta} \in P_S \wedge P_S. \tag{2.2}$$

Similarly, for $\gamma > 0$ which does not contain roots from S,

$$\delta_r(e_{-\gamma}) = \frac{1}{2} \sum_{\beta>0} ([e_{-\gamma}, e_\beta] \wedge e_{-\beta} + e_\beta \wedge [e_{-\gamma}, e_{-\beta}])$$

and the only "problematic" terms are $N_{-\gamma,\beta} e_{-\gamma+\beta} \wedge e_{-\beta}$ for $\beta > 0$ containing roots from S and $N_{-\gamma,-\beta} e_\beta \wedge e_{-\gamma-\beta}$ when $\beta + \gamma$ contains roots from S. Therefore we have proved that

$$\delta_r(e_{-\gamma}) - \sum_{\delta \in B_{-\gamma,S}} (N_{-\gamma,\delta} + N_{-\gamma,\gamma-\delta}) e_{-\gamma+\delta} \wedge e_{-\delta} \in P_S \wedge P_S. \tag{2.3}$$

The following lemma will prove that the two sums that appear in (2.2) and (2.3) vanish and thus $\delta_r(e_\gamma) \in P_S \wedge P_S$ for all $\gamma > 0$ and $\gamma < 0$ which does not contain roots belonging to S. □

Lemma 2.2. *Suppose that α, β, $\gamma \in R$ satisfy the relation $\alpha + \beta + \gamma = 0$. Then $N_{\alpha,\gamma} + N_{\beta,\gamma} = 0$.*

Proof. One has the following:

$$N_{\alpha,\gamma} = N_{\alpha,\gamma} K(e_{-\beta}, e_\beta) = K(N_{\alpha,\gamma} e_{\alpha+\gamma}, e_\beta) =$$
$$= K([e_\alpha, e_\gamma], e_\beta) = K(e_\alpha, [e_\gamma, e_\beta]) = K(e_\alpha, N_{\gamma,\beta} e_{-\alpha}) = N_{\gamma,\beta}.$$

□

3. The classical double

In the previous section we proved that the 1-cocycle δ_r provides a Lie bialgebra structure for a parabolic subalgebra P_S of a complex simple finite-dimensional Lie algebra $\mathbf{g}$. Therefore there exists a classical double associated to it, which we denote by $D(P_S)$. We will give a description for $D(P_S)$. We will prove that the classical double is isomorphic to $\mathbf{g} \oplus \mathbf{Red}(P_S)$. We recall that $\mathbf{Red}(P_S)$ denotes the reductive part of P_S and is isomorphic to $\frac{P_S}{P_S^\perp}$, where the orthogonal is considered with respect to the Killing form of $\mathbf{g}$.

Let $A_S = \{\mu \in R : \mu$ does not contain simple roots from S in its decomposition$\}$ and $B_S = \{\delta > 0 : \delta$ contains simple roots from S in its decomposition$\}$. Then it is well-known that

$$P_S^\perp = \sum_{\delta \in B_S} \mathbf{g}^\delta \tag{3.1}$$

and therefore

$$\mathbf{Red}(P_S) = \mathbf{h} \oplus \sum_{\mu \in A_S} \mathbf{g}^\mu. \tag{3.2}$$

This allows us to compute the dimension of $\mathbf{Red}(P_S)$ and we get that the linear spaces $D(P_S) = P_S \oplus P_S^*$ and $\mathbf{g} \oplus \mathbf{Red}(P_S)$ have the same dimension. Thus they are isomorphic as vector spaces.

Consider now the map $d : P_S \to \mathbf{g} \oplus \mathbf{Red}(P_S)$ defined by $d(x) = (x, \pi(x))$, where π: $P_S \to \frac{P_S}{P_S^\perp}$ is the canonical projection. $\mathbf{Red}(P_S)$ is equipped with the Killing form of $\mathbf{g}$, which is nondegenerate on $\mathbf{Red}(P_S)$. Therefore we can define a nondegenerate symmetric bilinear form Q on $\mathbf{g}\oplus\mathbf{Red}(P_S)$ by

$$Q((a,b),(c,d)) = K(a,c) - K(b,d). \tag{3.3}$$

Q is also invariant with respect to the adjoint representation.

We are interested in finding a Lagrangian subalgebra W of $\mathbf{g}\oplus\mathbf{Red}(P_S)$, with respect to the bilinear form Q, which satisfies the additional condition $W\cap d(P_S)=0$. We give its construction in the next lemma:

Lemma 3.1. *There exists a Lagrangian subalgebra W in $\mathbf{g}\oplus\mathbf{Red}(P_S)$, with respect to the bilinear form Q, such that $W\cap d(P_S)=0$.*

Proof. Suppose that for all $\gamma\in R$ we have chosen $e_\gamma\in\mathbf{g}^\gamma$ such that $K(e_\gamma,e_{-\gamma})=1$. Put $[e_\gamma,e_{-\gamma}]=h_\gamma\in\mathbf{h}$. The canonical basis of P_S is formed by h_β for all $\beta\in\Delta$ (set of simple roots) and e_γ for $\gamma\in\Pi(S)$. We make the following remarks: i) if $\gamma>0$ contains simple roots from S, then $\pi(e_\gamma)=0$; ii) if $\gamma\in R$ does not contain roots from S, then $\pi(e_\gamma)=e_\gamma$; iii) for any $\beta\in\Delta$, $\pi(h_\beta)=h_\beta$.

Let $C_S=\{\delta>0$: δ does not contain simple roots from S in its decomposition$\}$. Put $\mathbf{b}_-=\mathbf{h}\oplus\mathbf{n}_-$ and $\mathbf{b}_S=\mathbf{h}\oplus\sum_{\delta\in C_S}\mathbf{g}^\delta$. We consider the following Lie subalgebra of $\mathbf{g}\oplus\mathbf{Red}(P_S)$:

$$W=\{(x,y)\in\mathbf{b}_-\oplus\mathbf{b}_S : x_{\mathbf{h}}+y_{\mathbf{h}}=0\}, \tag{3.4}$$

where $x_{\mathbf{h}}$, $y_{\mathbf{h}}$ denote the "Cartan parts" of x and y respectively. Let us choose the following basis in W: F^γ denoting $(e_{-\gamma},0)$ if $\gamma>0$ and $(0,-e_{-\gamma})$ for $\gamma<0$ which does not contain simple roots from S in its decomposition, and $H^\beta=(h_\beta,-h_\beta)$ for all $\beta\in\Delta$. We will prove that W is a Lagrangian subalgebra. Firstly, $\dim W=\dim P=\frac{1}{2}\dim(\mathbf{g}\oplus\mathbf{Red}(P_S))$. So it suffices to show that $W\subseteq W^\perp$. We have the following: $Q((e_{-\gamma},0),(e_{-\delta},0))=K(e_{-\gamma},e_{-\delta})=0$ for $\gamma,\delta>0$; $Q((0,-e_{-\gamma}),(0,-e_{-\delta}))=-K(e_{-\gamma},e_{-\delta})=0$ for $\gamma,\delta<0$ which do not contain simple roots from S; $Q((e_{-\gamma},0),(h_\beta,-h_\beta))=K(e_{-\gamma},h_\beta)=0$ for $\gamma>0$, $\beta\in\Delta$; $Q((0,-e_{-\gamma}),(h_\beta,-h_\beta))=-K(e_{-\gamma},h_\beta)=0$ and $Q((h_\beta,-h_\beta),(h_\theta,-h_\theta))=0$ when $\beta,\theta\in\Delta$. In conclusion, W is a Lagrangian subalgebra.

On the other hand, the canonical basis in $d(P_S)$ is formed by the following elements: E_γ representing $(e_\gamma,0)$ for $\gamma>0$ which contains simple roots from S; (e_γ,e_γ) for $\gamma\in R$ which does not contain simple roots from S; $G_\beta=(h_\beta,h_\beta)$ for all $\beta\in\Delta$. Now it follows immediately that $W\cap d(P_S)=0$. This completes the proof. □

In order to get to the main result of this paper, we will build a basis in W which is dual to the canonical basis of $d(P_S)$ described in the previous lemma. Firstly, we notice that $Q((e_\gamma,0),(e_{-\delta},0))=K(e_\gamma,e_{-\delta})=1$ if $\gamma=\delta$ and 0 otherwise; $Q((e_\gamma,e_\gamma),(e_{-\delta},0))=K(e_\gamma,e_{-\delta})$ and $Q((e_\gamma,e_\gamma),(0,-e_{-\delta}))=K(e_\gamma,e_{-\delta})$. We only have to change the elements $(h_\beta,-h_\beta)$, $\beta\in\Delta$ in order to make them dual to (h_θ,h_θ), $\theta\in\Delta$. For any $\beta\in\Delta$ we take

$$\widetilde{h}_\beta=\sum_{\lambda\in\Delta}a_{\beta\lambda}h_\lambda. \tag{3.5}$$

We determine $a_{\beta\lambda}$ by imposing (*) $Q((h_\theta,h_\theta),(\widetilde{h}_\beta,-\widetilde{h}_\beta))=1$ if $\beta=\theta$ and 0 otherwise. On the other hand,

$$Q((h_\theta,h_\theta),(\widetilde{h}_\beta,-\widetilde{h}_\beta))=2\sum_{\lambda\in\Delta}a_{\beta\lambda}K(h_\theta,h_\lambda). \tag{3.6}$$

Because the restriction of K to the Cartan subalgebra $\mathbf{h}$ is nondegenerate, the matrix $(K(h_\theta,h_\lambda))_{\theta,\lambda\in\Delta}$ is invertible and thus the linear system equivalent to condition (*) has a unique solution for any fixed $\beta\in\Delta$. Let us denote $\widetilde{H}^\beta=(\widetilde{h}_\beta,-\widetilde{h}_\beta)$ (which we determined above). We have proved the following result:

Lemma 3.2. *The systems (E_γ,G_β) and $(F^\gamma,\widetilde{H}^\beta)$ are dual bases in $d(P_S)$ and W respectively.*

The last step that we need in order to prove that the Lie algebras $D(P_S)$ and $\mathbf{g}\oplus\mathbf{Red}(P_S)$ are isomorphic is the following:

Lemma 3.3. *The Lie algebras W and P_S^* are isomorphic.*

Proof. We recall that P_S^* has a Lie algebra structure induced by the 1-cocycle δ_r. The linear spaces W and P_S^* have the same dimension, so they are isomorphic. Let us consider the dual basis (E_γ,G_β) and $(F^\gamma,\widetilde{H}^\beta)$ from the previous lemma. Take $\gamma>0$. We have the following computations:

$$[F^\gamma,\widetilde{H}^\beta]=[(e_{-\gamma},0),(\sum_{\lambda\in\Delta}a_{\beta\lambda}h_\lambda,-\sum_{\lambda\in\Delta}a_{\beta\lambda}h_\lambda)]=\sum_{\lambda\in\Delta}a_{\beta\lambda}([e_{-\gamma},h_\lambda],0).$$

Since $K(h_\lambda,h_\gamma)=K(h_\lambda,[e_\gamma,e_{-\gamma}])=K([h_\lambda,e_\gamma],e_{-\gamma})=\gamma(h_\lambda)K(e_\gamma,e_{-\gamma})=\gamma(h_\lambda)$, one has that $[e_{-\gamma},h_\lambda]=\gamma(h_\lambda)e_{-\gamma}=K(h_\gamma,h_\lambda)e_{-\gamma}$. Thus we have obtained

$$[F^\gamma,\widetilde{H}^\beta]=(\sum_{\lambda\in\Delta}a_{\beta\lambda}K(h_\gamma,h_\lambda))(e_{-\gamma},0). \tag{3.7}$$

On the other hand, if we write $h_\gamma=\sum c_{\gamma\theta}h_\theta$ (summation after all $\theta\in\Delta$) and we take into consideration that the constants $a_{\beta\lambda}$ verify the conditions

$$\sum_{\lambda\in\Delta}a_{\beta\lambda}K(h_\beta,h_\lambda)=\frac{1}{2} \tag{3.8}$$

$$\sum_{\lambda\in\Delta}a_{\beta\lambda}K(h_\theta,h_\lambda)=0, \theta\neq\beta, \tag{3.9}$$

we conclude that

$$[F^\gamma,\widetilde{H}^\beta]=(\sum_{\lambda,\theta\in\Delta}c_{\gamma\theta}a_{\beta\lambda}K(h_\theta,h_\lambda))F^\gamma=\frac{1}{2}c_{\gamma\beta}F^\gamma. \tag{3.10}$$

Analogously, when $\gamma>0$ does not contain simple roots from S, recalling that $F^{-\gamma}=(0,-e_\gamma)$, we obtain the following

$$[F^{-\gamma},\widetilde{H}^\beta]=\frac{1}{2}c_{\gamma\beta}F^{-\gamma}. \tag{3.11}$$

Now we have to analyse the bracket induced by δ_r on P_S^*. Using the computations from the second section, it results that for any $\gamma > 0$,

$$\delta_r(e_\gamma) = \frac{1}{2}e_\gamma \wedge h_\gamma = \frac{1}{2}\sum_{\theta\in\Delta} c_{\gamma\theta}e_\gamma \wedge h_\theta \tag{3.12}$$

and for any $\gamma > 0$ which does not contain simple roots from S in its decomposition,

$$\delta_r(e_{-\gamma}) = \frac{1}{2}e_{-\gamma} \wedge h_\gamma = \frac{1}{2}\sum_{\theta\in\Delta} c_{\gamma\theta}e_{-\gamma} \wedge h_\theta. \tag{3.13}$$

We can consider F^γ and $\widetilde{H}^\beta$as elements of P_S^* since the linear spaces W and P_S^* are already identified. Because the Lie algebra structure on P_S^* is induced by the 1-cocycle δ_r, we have the following computations

$$[F^\gamma, \widetilde{H}^\beta]_{P_S^*}(e_\lambda) = \delta_r^*(F^\gamma \otimes \widetilde{H}^\beta)(e_\lambda) = (F^\gamma \otimes \widetilde{H}^\beta)(\delta_r(e_\lambda)). \tag{3.14}$$

Suppose that $\gamma > 0$. By construction, the bases (E_γ, G_β) in $d(P_S)$ and $(F^\gamma, \widetilde{H}^\beta)$ in W are dual. Thus, if $\lambda \neq \gamma$, then $[F^\gamma, \widetilde{H}^\beta]_{P_S^*}(e_\lambda) = 0$. Otherwise, $[F^\gamma, \widetilde{H}^\beta]_{P_S^*}(e_\gamma) = \frac{1}{2}c_{\gamma\beta}$. In this way we see that

$$[F^\gamma, \widetilde{H}^\beta]_{P_S^*} = \frac{1}{2}c_{\gamma\beta}F^\gamma. \tag{3.15}$$

If $\gamma > 0$ and does not contain simple roots from S, we obtain analogously that

$$[F^{-\gamma}, \widetilde{H}^\beta]_{P_S^*} = \frac{1}{2}c_{\gamma\beta}F^{-\gamma}. \tag{3.16}$$

It is not necessary to consider brackets between other types of elements from the system $(F^\gamma, \widetilde{H}^\beta)$ because they are zero.

In conclusion, we have proved that the Lie bracket is in fact the same and thus $W \cong P_S^*$ as Lie algebras. □

We can now state the main result of this paper. The proof is straightforward from the above lemma.

Theorem 3.4. *Let $\mathbf{g}$ be a complex simple finite-dimensional Lie algebra. Consider a parabolic subalgebra $P_S \subseteq \mathbf{g}$ defined by a subset S of the set of simple roots. Then the classical double $D(P_S)$, induced by the 1-cocycle δ_r, is isomorphic to the Lie algebra $\mathbf{g} \oplus \mathbf{Red}(P_S)$.*

Example 3.5. Consider $\mathbf{g} = sl(3)$ and the parabolic subalgebra

$$P = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

corresponding to the root α, defined by $\alpha(ah_1 + bh_2) = a + 2b$, where $h_1 = e_{11} - e_{33}$, $h_2 = e_{22} - e_{33}$.

We put $r = \frac{1}{2}(e_{12}\wedge e_{21} + e_{23}\wedge e_{32} + e_{13}\wedge e_{31})$ and take the Lie bialgebra structure on P induced by δ_r. Since $P^\perp = \left\{ \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}$, $\mathbf{Red}(P) \cong \frac{P}{P^\perp} = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \cong gl(2)$.

Therefore $D(P) \cong sl(3) \oplus gl(2)$, where the direct sum of Lie algebras is equipped with the following nondegenerate invariant bilinear form:

$$Q((A,B),(C,D)) = Tr(AC) - Tr(BD) - TrB \cdot TrD.$$

Final remark. Our result has an infinite-dimensional analogue. Let us consider $\mathbb{C}[[u^{-1}]]$ the ring of formal power series in u^{-1} and $\mathbb{C}((u^{-1}))$ the field of its quotients. We set $\mathbf{g}[u] := \mathbf{g} \otimes \mathbb{C}[u]$, $\mathbf{g}[[u^{-1}]] := \mathbf{g} \otimes \mathbb{C}[[u^{-1}]]$ and $\mathbf{g}((u^{-1})) := \mathbf{g} \otimes \mathbb{C}((u^{-1}))$. Let us take $\{I_\mu\}$ an orthonormal basis in $\mathbf{g}$ with respect to the Killing form K and set $\Omega := \sum I_\mu \otimes I_\mu$. Let $r_0 = r + \frac{\Omega}{2}$ which satisfies the following equations:

$$r_0^{12} + r_0^{21} = \Omega \tag{3.17}$$

$$[r_0^{12}, r_0^{13}] + [r_0^{12}, r_0^{23}] + [r_0^{13}, r_0^{23}] = 0. \tag{3.18}$$

We consider the map δ: $\mathbf{g}[u] \to \mathbf{g}[u] \wedge \mathbf{g}[v]$ defined by

$$\delta(a(u)) = [\frac{u\Omega}{v-u} + r_0, a(u) \otimes 1 + 1 \otimes a(v)], \tag{3.19}$$

which is a 1-cocycle and therefore defines a Lie bialgebra structure on $\mathbf{g}[u]$.

According to the results in [6] (cf. [10]), the classical double $D(\mathbf{g}[u])$ induced by δ is isomorphic to the direct sum of Lie algebras $\mathbf{g}((u^{-1})) \oplus \mathbf{g}$, which is equipped with the following invariant bilinear form:

$$Q((f(u),a),(g(u),b)) = K(f(u),g(u))_0 - K(a,b), \tag{3.20}$$

where the index zero means that we have taken the free term in the series expansion.

We notice that $\mathbf{g}[u]$ is a parabolic subalgebra of $\mathbf{g}((u^{-1}))$ and the reductive part of it, $\mathbf{Red}(\mathbf{g}[u]) = \frac{\mathbf{g}[u]}{u\mathbf{g}[u]}$, is isomorphic to $\mathbf{g}$. Therefore $D(\mathbf{g}[u]) \cong \mathbf{g}((u^{-1})) \oplus \mathbf{Red}(\mathbf{g}[u])$, exactly as in the finite-dimensional case.

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References

[1] Belavin A. A., Drinfeld V. G., *On classical Yang-Baxter equations for simple Lie algebras*, Funct. An. Appl. **16** no. 3 (1982) 1-29.

[2] Belavin A. A., Drinfeld V. G., *Triangle equations and simple Lie algebras*, Soviet Scientific Reviews, Section C**4** (1984) 93-165 (2nd edition: *Classic Reviews in Mathematics And Mathematical Physics*, Harwood, Amsterdam, 1988; Vol. **1**).

[3] Drinfeld V. G., *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation*, Sov. Math. Dokl. **27** (1983) 68-71.

[4] Jimbo M., *Yang-Baxter equation in integrable systems*, Advanced Series in Mathematical Physics **10** (1989).

[5] Korogodski L. I., Soibelman Y. S., *Algebras of Functions on Quantum Groups, Part I*, Mathematical Surveys and Monographs **56** (1998).
[6] Montaner F., Zelmanov E., *Bialgebra structures on current Lie algebras*, preprint, University of Wisconsin, Madison (1993).
[7] Reshetikhin N., Semenov-Tian-Shansky M., *Quantum R-matrix and factorization problems*, J. Geom. Phys. **5** (1988) 533-550.
[8] Stolin A., *On rational solutions of the classical Yang-Baxter equation*, Doctoral Disertation. Dept. Mat. Univ. of Stockholm, 1991.
[9] Stolin A., *Some remarks on Lie bialgebra structures on simple complex Lie algebras*, Commun. Alg. **27**(9) (1999) 4289-4302.
[10] Stolin A., *Generalized trigonometric solutions of the classical Yang-Baxter equation*, in *Proceedings of the XXII International Colloquium on Group Theoretical Methods in Physics*, International Press (1999) 438-442.

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ON SOME LIE BIALGEBRA STRUCTURES ON POLYNOMIAL ALGEBRAS AND THEIR QUANTIZATION

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Abstract. We study classical twists of Lie bialgebra structures on the polynomial current algebra. We focus on the structures induced by so-called quasi-trigonometric solutions of the classical Yang-Baxter equation. We give complete classification for sl_2 and sl_3. For the sl_2 case we also emphasize quantization. We obtain a two-parameter twist of the quantum affine algebra and of the Yangian. Consequently, we determine the deformed quantum R matrices which correspond to quasi-trigonometric and rational solutions in sl_2.

Keywords: classical Yang-Baxter equation, Lie bialgebra, rational solution, trigonometric solution, twisting, quantization, quantum affine algebra, Yangian.
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1. Introduction

1.1. **General introduction.** Recall that, given a Lie algebra $\mathbf{g}$, the classical Yang-Baxter equation with one spectral parameter is the equation

$$[X^{12}(z), X^{13}(z+t)] + [X^{12}(z), X^{23}(t)] + [X^{13}(z+t), X^{23}(t)] = 0, \tag{1.1}$$

where $X(z)$ is a meromorphic function of one complex variable z, defined in a neighbourhood of 0, taking values in $\mathbf{g}\otimes\mathbf{g}$. In their outstanding paper [2], A. A. Belavin and V. G. Drinfeld investigated solutions of the CYBE for a simple complex Lie algebra $\mathbf{g}$. They considered so-called nondegenerate solutions (i.e. $X(z)$ has maximal rank for generic z). It was proved in [2] that nondegenerate solutions are of three types: rational, trigonometric and elliptic. Moreover the authors completely classified trigonometric and elliptic solutions, the last ones for the case $\mathbf{g} = sl_n$.

In literature there were several attempts to reconsider Belavin- Drinfeld classification. In [21] a connection between the CYBE and the A_∞-constraint was given. It was proved that all nondegenerate elliptic solutions for sl_n arise from certain triple Massey products on elliptic curve. Moreover all nondegenerate trigonometric solutions for sl_2 are produced in a similar manner by considering A_∞-categories of singular curves of arithmetic genus 1. It was also conjectured that this holds for sl_n.

The goal of the our paper is also to reconsider Belavin-Drinfeld list but from the point of view of Lie bialgebra structures on $\mathbf{g}[z]$. The classical Yang-Baxter equation is strongly related to the fundamental concepts of Lie bialgebra and classical double introduced by V. G. Drinfeld in [5,6]. We are interested in the description of the classical double corresponding to Lie bialgebra structures on $\mathbf{g}[z]$. In the work of

Montaner and Zelmanov [20] it was proved that there exist only four types of classical doubles. We will consider two of them: $\mathbf{g}((u^{-1}))$ and $\mathbf{g}((u^{-1}))\oplus\mathbf{g}$.

A natural question arises: which nondegenerate solutions, after a suitable change of variables, may be used to induce Lie bialgebra structures on $\mathbf{g}[z]$?

Let us consider rational solutions. It was proved in [23] that this type of solutions provide Lie bialgebra structures on $\mathbf{g}[z, z^{-1}]$ which can be reduced to $\mathbf{g}[z]$. The corresponding classical double was shown to be $\mathbf{g}((z^{-1}))$.

For trigonometric solutions, the situation is different. Any trigonometric solution has the form $f(e^{ku})$, where f is a rational function. After setting $e^{ku} = \frac{z}{t}$, this solution does not induce, generally speaking, a Lie bialgebra structure on $\mathbf{g}[z]$.

Therefore we are motivated to introduce a new class of solutions of "trigonometric" type that will induce Lie bialgebra structures on $\mathbf{g}[z]$. Let Ω denote the quadratic Casimir element of $\mathbf{g}$. We say that a solution X of the CYBE is *quasi-trigonometric* if it is of the form:

$$X(z,t) = \frac{t\Omega}{z-t} + p(z,t), \tag{1.2}$$

where $p(z,t)$ is a polynomial with coefficients in $\mathbf{g}\otimes\mathbf{g}$. We will prove that by applying a certain holomorphic transformation and a change of variables, any quasi-trigonometric solution becomes trigonometric, in the sense of Belavin-Drinfeld classification (see Appendix).

The study of the Lie bialgebra structures given by quasi-trigonometric solutions will be based on the description of the classical double. We show that all quasi-trigonometric solutions induce the same classical double $\mathbf{g}((z^{-1}))\oplus\mathbf{g}$.

Another question that arises is the quantization of the Lie bialgebras corresponding to quasi-trigonometric solutions. As it was proved by P. Etingof and D. Kazhdan in [9], any Lie bialgebra can be quantized and this quantization is given by an universal, functorial construction. However, to find explicit quantizations of classical r-matrices seems rather difficult. A very important step was made by P. Etingof, T. Schedler and O. Schiffmann in [11]. They obtained the explicit quantization of all non-skewsymmetric classical r-matrices from the Belavin-Drinfeld list [1]. Moreover a quantization of all the dynamical r-matrices for semisimple Lie algebras was given. Their method consists in constructing an appropriate (dynamical) twist in the tensor square of the Drinfeld-Jimbo quantum group $U_q(\mathbf{g})$.

Concerning the quantization of the classical r-matrices with spectral parameter, some results were obtained for rational solutions. The simplest rational solution of the CYBE for a simple complex Lie algebra $\mathbf{g}$ is $r(z) = \Omega/z$. The quantization of the corresponding Lie bialgebra structure on $\mathbf{g}[z]$ is the well-known Yangian, $Y(\mathbf{g})$. In [16] an explicit quantization of the simplest nonstandard rational r-matrix for sl_2, namely $r(z) = \Omega/z + h_\alpha \wedge e_{-\alpha}$, was presented. The quantization was obtained by twisting the usual Yangian $Y(sl_2)$. The deformed Yangian turned out to be a two-parametric deformation of the universal enveloping algebra $U(sl_2[z])$ of the polynomial current algebra $sl_2[z]$.

Regarding trigonometric solutions, the most typical ones among the trigonometric r-matrices are the classical solutions associated with the generalized Toda system. In

[14] M. Jimbo reported the explicit form of the quantum R-matrix in the fundamental representation for the generalized Toda system associated with non-exceptional affine Lie algebras.

However, the problem is far from being solved because there exist many nontrivial rational and trigonometric solutions. As we will see later, the first nontrivial examples are a rational and a quasi-trigonometric r-matrix with values in sl_2. These r-matrices can be obtained from the standard rational or trigonometric solutions by adding a polynomial of the first degree in the spectral parameters. In our paper we found the corresponding deformed quantum R matrices.

1.2. **Structure of the paper.** We devote this paper to the study of the *quasi-trigonometric* r-matrices for a simple complex finite-dimensional Lie algebra $\mathbf{g}$. Our approach is from the point of view of *classical twists* (see Section 2). As it turns out in Section 3, quasi-trigonometric solutions are an infinite-dimensional example of *twisting* of a certain Lie bialgebra structure on $\mathbf{g}[z]$. We prove that the classical double corresponding to any quasi-trigonometric solution is isomorphic to $\mathbf{g}((z^{-1}))\oplus\mathbf{g}$. Moreover we construct a one-to-one correspondence between this type of solutions and a special class of Lagrangian subalgebras of the $\mathbf{g}((z^{-1}))\oplus\mathbf{g}$. By exploiting this correspondence, in Sections 4, 5 and 6 we investigate the case sl_n and compute all (up to gauge equivalence) quasi-trigonometric solutions for sl_2 and sl_3.

In the following sections, we determine the quantum R matrix which corresponds to the nontrivial quasi-trigonometric and rational solutions in sl_2. Our method is based on the following conjecture: Any classical twist can be extended to a quantum twist (Section 7). We support this conjecture by constructing a two-parameter twist of the quantum affine algebra $U_q(\widehat{sl_2})$ (Sections 8 and 9). Surprisingly, it has the simple form of a q-power function, but with q-commuting arguments. Moreover, its Yangian degeneration becomes the usual power function whose arguments belong to an additive variant of the Manin q-plane. In this setting, the q-power functions satisfy nontrivial generalizations of their standard properties, fact which guarantees the cocycle identity for the quantum twist. In Section 10 we compute the corresponding deformations of the quasi-trigonometric and rational R-matrices, including them into a single family. Finally, in Section 11 we give the two-parameter integrable deformations of the XXZ and XXX Heisenberg chains. As a particular case we get the deformed XXX chain discussed in [19].

2. Lie bialgebra structures and classical twists

Let $\mathbf{g}$ denote an arbitrary complex Lie algebra. We recall that a Lie bialgebra structure on $\mathbf{g}$ is a 1-cocycle $\delta:\mathbf{g}\to\wedge^2\mathbf{g}$ which satisfies the co-Jacobi identity. In other words, δ provides a Lie algebra structure for $\mathbf{g}^*$ compatible with the structure of $\mathbf{g}$.

To any Lie bialgebra one associates the so-called *classical double*. It is defined as the unique Lie algebra structure on the vector space $\mathbf{g}\oplus\mathbf{g}^*$ such that:

1) it induces the given Lie algebra structures on $\mathbf{g}$ and $\mathbf{g}^*$

2) the bilinear form Q defined by

$$Q(x_1+l_1,x_2+l_2)=l_1(x_2)+l_2(x_1)$$

is invariant with respect to the adjoint representation of $\mathbf{g}\oplus\mathbf{g}^*$.

Let us denote by $D(\mathbf{g},\delta)$ the classical double of $(\mathbf{g},\delta)$ and recall the following notion defined in [15]:

Definition 2.1. Two Lie bialgebra structures δ_1 and δ_2 on $\mathbf{g}$ are said to be *equivalent* if there is a Lie algebra isomorphism $f:D(\mathbf{g},\delta_1)\to D(\mathbf{g},\delta_2)$ satisfying the properties:

1) $Q_1(x,y)=Q_2(f(x),f(y))$ for any x, $y\in D(\mathbf{g},\delta_1)$, where Q_i denotes the canonical form on $D(\mathbf{g},\delta_i)$, $i=1,2$.

2) $f\circ j_1=j_2$, where j_i is the canonical embedding of $\mathbf{g}$ in $D(\mathbf{g},\delta_i)$.

If we regard any Lie bialgebra as a Lie quasi-bialgebra, there is a notion of *classical twist* according to [7]:

Definition 2.2. Let δ_1 be a Lie bialgebra structure on $\mathbf{g}$. Suppose $s\in\wedge^2\mathbf{g}$ satisfies

$$[s^{12},s^{13}]+[s^{12},s^{23}]+[s^{13},s^{23}]=Alt(\delta_1\otimes id)(s),$$

where $Alt(x):=x^{123}+x^{231}+x^{312}$ for any $x\in\mathbf{g}^{\otimes 3}$. Then

$$\delta_2(a):=\delta_1(a)+[a\otimes 1+1\otimes a,s]$$

defines a Lie bialgebra structure on $\mathbf{g}$. We say that δ_2 is obtained by *twisting via s* and s is a *classical twist*.

Remark 2.3. For a finite-dimensional $\mathbf{g}$, it was shown in [15] that two Lie bialgebra structures are equivalent if and only if one is obtained from the other by twisting via a classical twist.

Example 2.4. Let $\mathbf{g}$ be finite-dimensional. All Lie bialgebra structures induced by triangular r-matrices are equivalent. The classical double corresponding to any triangular r-matrix is isomorphic to the semidirect sum $\mathbf{g}\dotplus\mathbf{g}^*$ such that $\mathbf{g}^*$ is a commutative ideal and $[a,l]=\mathbf{ad}^*(a)(l)$ for any $a\in\mathbf{g}$ and $l\in\mathbf{g}^*$.

Another example of twisting is the following:

Example 2.5. Suppose $\mathbf{g}$ is simple and let δ_0 be the Lie bialgebra structure induced by the standard Drinfeld-Jimbo r-matrix. Then the entire Belavin-Drinfeld list [1] is obtained by twisting the standard structure δ_0. The classical double corresponding to any r-matrix from this list is isomorphic to $\mathbf{g}\oplus\mathbf{g}$.

Now, if we pass to the case of infinite-dimensional Lie bialgebra structures, we encounter more examples of twisting.

Let us recall several facts from the theory of rational solutions as it was developed in [23]. We let again $\mathbf{g}$ denote a simple Lie algebra. Denote by K the Killing form and let Ω be the corresponding Casimir element of $\mathbf{g}$. We look for functions X: $\mathbb{C}^2\to\mathbf{g}\otimes\mathbf{g}$ which satisfy

$$[X^{12}(z_1,z_2),X^{13}(z_1,z_3)]+[X^{12}(z_1,z_2),X^{23}(z_2,z_3)]+ \tag{2.1}$$

$$+[X^{13}(z_1,z_3),X^{23}(z_2,z_3)]=0,$$

$$X^{12}(z_1,z_2)=-X^{21}(z_2,z_1). \tag{2.2}$$

Remark 2.6. We will call these two equations the classical Yang-Baxter equation (CYBE). In the case of rational and quasi-trigonomtric solutions, the unitarity condition (2.2) can actually be dropped. We will prove in Appendix that (2.2) is a consequence of (2.1).

Definition 2.7. Let $X(z,t)=\frac{\Omega}{z-t}+p(z,t)$ be a function from $\mathbb{C}^2$ to $\mathbf{g}\otimes\mathbf{g}$, where $p(z,t)$ is a polynomial with coefficients in $\mathbf{g}\otimes\mathbf{g}$. If X satisfies the CYBE, we say that X is a *rational* solution.

Two rational solutions X_1 and X_2 are called *gauge equivalent* if there exists $\sigma(z)\in Aut(\mathbf{g}[z])$ such that $X_2(z,t)=(\sigma(z)\otimes\sigma(t))X_1(z,t)$, where $Aut(\mathbf{g}[z])$ denotes the group of automorphisms of $\mathbf{g}[z]$ considered as an algebra over $\mathbb{C}[z]$.

Remark 2.8. It was proved in [23] that any rational solution can be brought by means of a gauge transformation to the form:

$$X(z,t)=\frac{\Omega}{z-t}+p_{00}+p_{10}z+p_{01}t+p_{11}zt,$$

where p_{00}, p_{10}, p_{01}, $p_{11}\in\mathbf{g}\otimes\mathbf{g}$.

We recall that any rational solution induces a Lie bialgebra structure on the polynomial current algebra $\mathbf{g}[z]$. Let us consider a rational solution X and define the map $\delta_X\colon\ \mathbf{g}[z]\to\mathbf{g}[z]\wedge\mathbf{g}[z]$ by

$$\delta_X(a(z))=[X(z,t),a(z)\otimes 1+1\otimes a(t)], \tag{2.3}$$

for any $a(z)\in\mathbf{g}[z]$. Obviously δ_X is a 1-cocycle and therefore induces a Lie bialgebra structure on $\mathbf{g}[z]$.

The following result, proved in [23], shows that all Lie bialgebra structures corresponding to rational solutions are equivalent.

Let $\mathbb{C}[[z^{-1}]]$ be the ring of formal power series in z^{-1} and $\mathbb{C}((z^{-1}))$ its field of quotients. Consider the Lie algebras $\mathbf{g}[z]=\mathbf{g}\otimes\mathbb{C}[z]$, $\mathbf{g}[[z^{-1}]]=\mathbf{g}\otimes\mathbb{C}[[z^{-1}]]$ and $\mathbf{g}((z^{-1}))=\mathbf{g}\otimes\mathbb{C}((z^{-1}))$.

Theorem 2.9. *Let $D_X(\mathbf{g}[z])$ be the classical double corresponding to a rational solution X of the CYBE. Then $D_X(\mathbf{g}[z])$ and $\mathbf{g}((z^{-1}))$ are isomorphic as Lie algebras, with inner product which has the following form on $\mathbf{g}((z^{-1}))$:*

$$Q(f(z),g(z))=Res_{z=0}K(f(z),g(z)), \tag{2.4}$$

where $f(z)$, $g(z)\in\mathbf{g}((z^{-1}))$.

Remark 2.10. This result means that $\mathbf{g}((z^{-1}))$ can be represented as a Manin triple $\mathbf{g}((z^{-1}))=\mathbf{g}[z]\oplus W$, where W is a Lagrangian subalgebra with respect to the invariant form Q.

In the case $\mathbf{g}=sl_n$ all the rational solutions were described in the following way: Let $d_k=diag(1,...,1,z,...,z)$ (k many 1's), $0\le k\le[\frac{n}{2}]$. Then it was proved in [23] that every rational solution of the CYBE defines some Lagrangian subalgebra W contained in $d_k^{-1}sl_n[[z^{-1}]]d_k$ for some k. These subalgebras are in one-to-one correspondence with pairs (L,B) verifying:

(1) L is a subalgebra of sl_n such that $L+P_k=sl_n$, where P_k denotes the maximal parabolic subalgebra of sl_n not containing the root vector $e_{\alpha_k}$of the simple root α_k;

(2) B is a 2-cocycle on L which is nondegenerate on $L\cap P_k$.

In case of sl_2 one has just two non-standard rational r-matrices, up to gauge equivalence:

$$X_1(z,t)=\frac{\Omega}{z-t}+h_\alpha\wedge e_{-\alpha} \tag{2.5}$$

and

$$X_2(z,t)=\frac{\Omega}{z-t}+ze_{-\alpha}\otimes h_\alpha-th_\alpha\otimes e_{-\alpha}, \tag{2.6}$$

where $e_\alpha=e_{12}$, $e_{-\alpha}=e_{21}$ and $h_\alpha=e_{11}-e_{22}$ is the usual basis of sl_2.

3. Quasi-trigonometric solutions

We devote this section to studying another interesting case which provides infinite-dimensional Lie bialgebra structures on the polynomial current algebra $\mathbf{g}[z]$. We consider the simplest trigonometric solution and classical twists of it.

Let $\mathbf{g}$ be as before. We fix a Cartan subalgebra $\mathbf{h}$ and the associated root system. We choose a system of Chevalley-Weyl generators e_α, $e_{-\alpha}$ and h_α, where α is a positive root, such that $K(e_\alpha,e_{-\alpha})=1$.

Let r_0 denote the standard Drinfeld-Jimbo r-matrix

$$r_0=\frac{1}{2}(\sum_{\alpha>0}e_\alpha\wedge e_{-\alpha}+\Omega), \tag{3.1}$$

which satisfies the modified CYBE:

$$r_0+r_0^{21}=\Omega, \tag{3.2}$$

$$[r_0^{12},r_0^{13}]+[r_0^{12},r_0^{23}]+[r_0^{13},r_0^{23}]=0. \tag{3.3}$$

We consider the function X_0 defined by

$$X_0(z,t)=\frac{t\Omega}{z-t}+r_0 \tag{3.4}$$

It is easy to check that X_0 satisfies (2.1) and (2.2).

Let us make the remark that X_0 is related to the simplest trigonometric r-matrix. If we make the change of variables $z = e^\lambda$ and $t = e^\mu$, then

$$X_0(e^\lambda, e^\mu) = \frac{\Omega}{e^{\lambda-\mu} - 1} + r_0 \tag{3.5}$$

is a trigonometric solution, according to [2].

Definition 3.1. We say that a solution X of the CYBE is *quasi-trigonometric* if it is of the form:

$$X(z,t) = \frac{t\Omega}{z-t} + p(z,t), \tag{3.6}$$

where $p(z,t)$ is a polynomial with coefficients in $\mathbf{g} \otimes \mathbf{g}$.

Proposition 3.2. *Let $Aut(\mathbf{g}[z])$ denote the group of automorphisms of $\mathbf{g}[z]$ considered as an algebra over $\mathbb{C}[z]$. Let X_1 be a quasi-trigonometric solution and $\sigma(z) \in Aut(\mathbf{g}[z])$. Then*

$$X_2(z,t) = (\sigma(z) \otimes \sigma(t))X_1(z,t)$$

is also a quasi-trigonometric solution.

Proof. Let $X_1(z,t) = \frac{t\Omega}{z-t} + p(z,t)$. Since X_2 obviously satisfies the CYBE, it is enough to check that X_2 is quasi-trigonometric. We have the following:

$$X_2(z,t) = (\frac{t(\sigma(z)-\sigma(t))}{z-t} \otimes \sigma(t))\Omega + \frac{t}{z-t}(\sigma(t) \otimes \sigma(t))\Omega +$$
$$+(\sigma(z) \otimes \sigma(t))p(z,t).$$

Let $p_1(z,t) := (\frac{t(\sigma(z)-\sigma(t))}{z-t} \otimes \sigma(t))\Omega$ and $p_2(z,t) := (\sigma(z) \otimes \sigma(t))p(z,t)$. These are polynomial functions in z, t. Since $(\sigma(t) \otimes \sigma(t))\Omega = \Omega$, we obtain

$$X_2(z,t) = \frac{t\Omega}{z-t} + p_1(z,t) + p_2(z,t)$$

and this ends the proof. □

Definition 3.3. Two quasi-trigonometric solutions X_1 and X_2 are called *gauge equivalent* if there exists $\sigma(z) \in Aut(\mathbf{g}[z])$ such that

$$X_2(z,t) = (\sigma(z) \otimes \sigma(t))X_1(z,t). \tag{3.7}$$

Any quasi-trigonometric solution X of the CYBE induces a Lie bialgebra structure on $\mathbf{g}[z]$. Let δ_X be the 1-cocycle defined by

$$\delta_X(a(z)) = [X(z,t), a(z) \otimes 1 + 1 \otimes a(t)], \tag{3.8}$$

for any $a(z) \in \mathbf{g}[z]$.

It is expected that all Lie bialgebra structures corresponding to quasi-trigonometric solutions induce the same classical double. Let us first recall the description of the classical double corresponding to the solution X_0, according to [20]:

Theorem 3.4. *Let $D_{X_0}(\mathbf{g}[z])$ be the classical double of $\mathbf{g}[z]$ corresponding to X_0. Then $D_{X_0}(\mathbf{g}[z])$ is isomorphic to the direct sum of Lie algebras $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$, where we consider the following invariant bilinear form:*

$$Q((f(z),a),(g(z),b)) = K(f(z),g(z))_0 - K(a,b). \tag{3.9}$$

Here the index zero means that we have taken the free term in the series expansion.

Remark 3.5. The Lie algebra $\mathbf{g}[z]$ is embedded into $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ via $a(z) \longmapsto (a(z), a(0))$. The Lie algebra $\mathbf{g}[z]$ is naturally identified with

$$V_0 := \{(a(z), a(0)); a(z) \in \mathbf{g}[z]\}. \tag{3.10}$$

Moreover, the Lie algebra $(\mathbf{g}[z])^*$ is identified with the following Lie subalgebra of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$:

$$W_0 = z^{-1}\mathbf{g}[[z^{-1}]] \oplus \{(a,b) \in \mathbf{b}_+ \oplus \mathbf{b}_- : a_\mathbf{h} + b_\mathbf{h} = 0\}. \tag{3.11}$$

Here $\mathbf{h}$ is the fixed Cartan subalgebra of $\mathbf{g}$, $\mathbf{b}_\pm$ are the positive (negative) Borel subalgebras and $a_\mathbf{h}$ denotes the "Cartan part" of a.

In order to show that all quasi-trigonometric solutions induce the same classical double, we will prove the following result:

Theorem 3.6. *There exists a natural one-to-one correspondence between quasi-trigonometric solutions of the CYBE and linear subspaces W of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ such that*

1) W is a Lie subalgebra in $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ such that $W \supseteq z^{-N}\mathbf{g}[[z^{-1}]]$ for some $N > 0$;

2) $W \oplus V_0 = \mathbf{g}((z^{-1})) \oplus \mathbf{g}$;

3) W is a Lagrangian subspace with respect to the inner product of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$.

Proof. Let V_0 and W_0 be the Lie algebras given in Remark 3.5. We choose dual bases in V_0 and W_0 respectively. Let $\{k_j\}$ be an orthonormal basis in $\mathbf{h}$. The canonical basis of V_0 is formed by $e_\alpha z^k$, $e_{-\alpha} z^k$, $k_j z^k$ for any $\alpha > 0$, $k > 0$ and all j; $(e_{-\alpha}, e_{-\alpha})$, (e_α, e_α) for any $\alpha > 0$, and (k_j, k_j), for all j. The dual basis of W_0 is the following: $e_{-\alpha} z^{-k}$, $e_\alpha z^{-k}$, $k_j z^{-k}$ for any $\alpha > 0$, $k > 0$ and all j; $(e_\alpha, 0)$ and $(0, -e_{-\alpha})$ for all $\alpha > 0$, and $\frac{1}{2}(k_j, -k_j)$, for all j. Let us simply denote these dual bases by $\{v_i\}$ and $\{w_0^i\}$ respectively. We notice that the quasi-trigonometric solution X_0 given by (3.4) can be written as

$$X_0(z,t) = (\tau \otimes \tau)(\sum_i w_0^i \otimes v_i), \tag{3.12}$$

where τ denotes the projection of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ onto $\mathbf{g}((z^{-1}))$.

We denote by $Hom_{cont}(W_0, V_0)$ the space of those linear maps F: $W_0 \to V_0$ such that $KerF \supseteq z^{-N}\mathbf{g}[[z^{-1}]]$ for some $N > 0$. It is the space of linear maps F which are continuous with respect to the "z^{-1} - adic" topology.

Let us contruct a linear isomorphism Φ: $V_0 \otimes V_0 \to Hom_{cont}(W_0, V_0)$ in the following way:

$$\Phi(x \otimes y)(w_0) = Q(w_0, y) \cdot x, \tag{3.13}$$

for any $x,\ y \in V_0$ and any $w_0 \in W_0$. It is easy to check that Φ is indeed an isomorphism. The inverse mapping is Ψ: $Hom_{cont}(W_0, V_0) \to V_0 \otimes V_0$ defined by

$$\Psi(F) = \sum_i F(w_0^i) \otimes v_i. \tag{3.14}$$

We make the remark that this sum is finite since $F(w_0^i) \neq 0$ only for a finite number of indices i.

The next step is to construct a bijection between $Hom_{cont}(W_0, V_0)$ and the set L of linear subspaces W of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ such that $W \oplus V_0 = \mathbf{g}((z^{-1})) \oplus \mathbf{g}$ and $W \supseteq z^{-N}\mathbf{g}[[z^{-1}]]$ for some $N > 0$. This can be done in a very natural way. For any $F \in Hom_{cont}(W_0, V_0)$ we take

$$W(F) = \{w_0 + F(w_0); w_0 \in W_0\}. \tag{3.15}$$

The inverse mapping associates to any W the linear function F_W such that for any $w_0 \in W_0$, $F_W(w_0) = -v$, uniquely defined by the decomposition $w_0 = w + v$ with $w \in W$ and $v \in V_0$.

Therefore we have a bijection between $V_0 \otimes V_0$ and L. By a straightforward computation, one can show that a tensor $r(z,t) \in V_0 \otimes V_0$ satisfies the condition $r(z,t) = -r^{21}(t,z)$ if and only if the linear subspace $W(\Phi(r))$ is Lagrangian with respect to Q.

Let us suppose now that $X(z,t) = X_0(z,t) + r(z,t)$ and $r(z,t) = -r^{21}(t,z)$. Then $X(z,t)$ satisfies (2.1) if and only if $W(\Phi(r))$ is a Lie subalgebra of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$. Indeed, since r is unitary, we have that $W(\Phi(r))$ is a Lagrangian subspace with respect to Q. It is enough to check that $X(z,t)$ satisfies (2.1) if and only if

$$Q([w_1 + \Phi(r)(w_1), w_2 + \Phi(r)(w_2)], w_3 + \Phi(r)(w_3)) = 0 \tag{3.16}$$

for any elements w_1, w_2 and w_3 of W_0. This follows by direct computations.

In conclusion, we see that a function $X(z,t) = \frac{t\Omega}{z-t} + p(z,t)$ is a quasi-trigonometric solution if and only if $W(\Phi(p - r_0))$ is a Lagrangian subalgebra of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$. This ends the proof. □

Corollary 3.7. *The Lie bialgebra structures on* $\mathbf{g}[z]$ *induced by quasi-trigonometric solutions are equivalent. The corresponding classical double is* $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ *together with the form* Q *given by (3.9).*

Proof. One can easily check that if W is a Lagrangian subalgebra of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$, corresponding to a quasi-trigonometric solution $X(z,t) = \frac{t\Omega}{z-t} + p(z,t)$, then W is isomorphic to $(\mathbf{g}[z])^*$ with the Lie algebra structure induced by X.

Indeed, with the notation introduced in Theorem 3.6, let $F := F_W$. It is enough to check that for any $v \in V_0$ and $w_1, w_2 \in W_0$ the following equality is satisfied:

$$Q(v, [w_1 + F(w_1), w_2 + F(w_2)]) =< \delta_X(v), w_1 \otimes w_2 >,$$

where $<,>$ denotes the pairing between $V_0^{\otimes 2}$ and $W_0^{\otimes 2}$ induced by Q. This equality is implied by the following identities:

$$Q(v, [w_1, w_2]) =< \delta_{X_0}(v), w_1 \otimes w_2 >,$$

$$Q(v, [F(w_1), w_2]) =< [p - r_0, 1 \otimes v], w_1 \otimes w_2 >,$$

$$Q(v, [w_1, F(w_2)]) =< [p - r_0, v \otimes 1], w_1 \otimes w_2 > .$$

□

Remark 3.8. If W is a Lagrangian subalgebra of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ satisfying the conditions of Theorem 3.6, then the corresponding solution $X(z,t)$ is constructed in the following way: take a basis $\{w^i\}$ in W which is dual to the canonical basis $\{v_i\}$ of V_0 and construct the tensor

$$\widetilde{r}(z,t) = \sum_i w^i \otimes v_i. \tag{3.17}$$

Let π denote the projection of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$ onto $\mathbf{g}[z]$ which is induced by the decomposition $\mathbf{g}((z^{-1})) \oplus \mathbf{g} = V_0 \oplus W_0$. Explicitly,

$$\pi(a_n z^n + ... + a_0 + a_{-1}z^{-1} + ..., b) = a_n z^n + ... + a_1 z + \tag{3.18}$$

$$+ \frac{1}{2}(a_{0\mathbf{h}} + b_{\mathbf{h}}) + a_{0-} + b_+ .$$

Here $a_0 = a_{0\mathbf{h}} + a_{0+} + a_{0-}$ and $b = b_{\mathbf{h}} + b_+ + b_-$ are the decompositions with respect to $\mathbf{g} = \mathbf{h} \oplus \mathbf{n}_+ \oplus \mathbf{n}_-$. Then

$$X(z,t) = X_0(z,t) + (\pi \otimes \pi)(\widetilde{r}(z,t)). \tag{3.19}$$

At this point we note the following fact that we will prove in Appendix:

Proposition 3.9. *Let* W *be a Lie subalgebra satisfying conditions 2) and 3) of Theorem 3.6. Let* $\widetilde{r}$ *be constructed as in (3.17). Assume* $\widetilde{r}$ *induces a Lie bialgebra structure on* $\mathbf{g}[z]$ *by* $\delta_{\widetilde{r}}(a(z)) = [\widetilde{r}(z,t), a(z) \otimes 1 + 1 \otimes a(t)]$*. Then* $W \supseteq u^{-N}\mathbf{g}[[u^{-1}]]$ *for some positive* N*.*

Theorem 3.10. *Let* X_1 *and* X_2 *be quasi-trigonometric solutions of the CYBE. Suppose that* W_1 *and* W_2 *are the corresponding Lagrangian subalgebras of* $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$*. Let* $\sigma(z) \in Aut(\mathbf{g}[z])$ *and* $\widetilde{\sigma}(z)$ *be the automorphism of* V_0 *induced by* $\sigma(z)$*. The following conditions are equivalent:*

1) $X_1(z,t) = (\sigma(z) \otimes \sigma(t))X_2(z,t)$*;*
2) $W_1 = \widetilde{\sigma}(z)W_2$*.*

Proof. 1) ⇒ 2). Let us begin by proving this for the particular case $X_1 = X_0$ and $X_2 = (\sigma(z) \otimes \sigma(t))X_0(z,t)$. The Lagrangian subalgebra corresponding to X_0 is W_0 given by (3.11). On the other hand, one can check the Lagrangian subalgebra W_2, corresponding to the solution X_2, consists of elements

$$\widetilde{f} := \sum_i (f, \widetilde{\sigma}(v_i)) \cdot \widetilde{\sigma}(w_0^i) = \sum_i (\widetilde{\sigma}^{-1}(f), v_i) \cdot \widetilde{\sigma}(w_0^i),$$

for any $f \in W_0$. Here $\{v_i\}$ and $\{w_0^i\}$ are the dual bases of V_0 and W_0 introduced in the proof of Theorem 3.6. We show that $W_2 = \widetilde{\sigma}(W_0)$.

Let g denote the projection of $\widetilde{\sigma}^{-1}(f)$ onto W_0 induced by the decomposition $V_0 \oplus W_0 = \mathbf{g}((z^{-1})) \oplus \mathbf{g}$. Then

$$g = \sum_i (\widetilde{\sigma}^{-1}(f), v_i) \cdot w_0^i$$

which implies that $\widetilde{f} = \widetilde{\sigma}(g)$. Therefore $W_2 \subseteq \widetilde{\sigma}(W_0)$. The other inclusion is similar.

Let us pass to the general case. If $X_1(z,t) = X_0(z,t) + r(z,t)$ is a quasi-trigonometric solution with $r(z,t) = \sum a_k z^k \otimes b_j t^j$, then the corresponding W_1 consists of elements of the form $f + \sum (f, b_j z^j) a_k z^k$, for any f in W_0. Now let $X_2(z,t) = (\sigma(z) \otimes \sigma(t)) X_1(z,t)$. The corresponding subalgebra W_2 is formed by elements of the form

$$\widetilde{f_r} := \sum_i (f, \widetilde{\sigma}(v_i)) \cdot \widetilde{\sigma}(w_0^i) + \sum (f, \widetilde{\sigma}(b_j z^j)) \widetilde{\sigma}(a_k z^k).$$

It is easy to see that $\widetilde{f_r} = \widetilde{\sigma}(h)$, where $h := g + \sum (g, b_j z^j) a_k z^k$ and g is the projection of $\widetilde{\sigma}^{-1}(f)$ onto W_0. These considerations prove that $\widetilde{\sigma}(W_1) = W_2$.

2) $\Rightarrow$ 1). Suppose that $W_2 = \widetilde{\sigma}(W_1)$. Let $\widetilde{X}_2 := (\sigma(z) \otimes \sigma(t)) X_1(z,t)$. It is a quasi-trigonometric solution which has a corresponding Lagrangian subalgebra $\widetilde{W_2}$. Because 1) $\Rightarrow$ 2) we obtain that $\widetilde{W_2} = \widetilde{\sigma}(W_1)$ and thus $W_2 = \widetilde{W_2}$. Since the correspondence between solutions and subalgebras is one-to-one, we get that $X_2 = \widetilde{X}_2$. □

Remark 3.11. We will say that W_1 and W_2 are *gauge equivalent* if condition 2) of Theorem 3.10 is satisfied.

Theorem 3.12. *Let $X(z,t) = \frac{t\Omega}{z-t} + p(z,t)$ be a quasi-trigonometric solution of the CYBE and W the corresponding Lagrangian subalgebra of $\mathbf{g}((z^{-1})) \oplus \mathbf{g}$. Then the following are equivalent:*
1) $p(z,t)$ is a constant polynomial;
2) W is contained in $\mathbf{g}[[z^{-1}]] \oplus \mathbf{g}$.

Proof. We keep the notations from the proof of Theorem 3.6 and also those from Remark 3.8. Let $r(z,t) = p(z,t) - r_0$ and $F = \Phi(r(z,t))$. If $p(z,t)$ is constant, then $F(w_0) \in \mathbf{g} \otimes \mathbf{g}$ for any $w_0 \in W_0$. Therefore $W(F) \subseteq \mathbf{g}[[z^{-1}]] \oplus \mathbf{g}$. Conversely, let us suppose that W is included in $\mathbf{g}[[z^{-1}]] \oplus \mathbf{g}$. The orthogonal of $\mathbf{g}[[z^{-1}]] \oplus \mathbf{g}$ with respect to Q is obviously $z^{-1}\mathbf{g}[[z^{-1}]]$. Since W is a Lagrangian subalgebra, it follows that W contains $z^{-1}\mathbf{g}[[z^{-1}]]$. According to the previous remark, $r(z,t) = (\pi \otimes \pi)(\widetilde{r}(z,t))$, where π is the projection onto $\mathbf{g}[z]$ induced by the decomposition $\mathbf{g}((z^{-1})) \oplus \mathbf{g} = V_0 \oplus W_0$ and $\widetilde{r}(z,t) = \sum_i w^i \otimes v_i$. Now it is clear that r is a constant. This ends the proof. □

Remark 3.13. A function $X(z,t) = \frac{t\Omega}{z-t} + r$, where $r \in \mathbf{g} \otimes \mathbf{g}$, satisfies the CYBE if and only if r is a solution of the modified CYBE.

Definition 3.14. A quasi-trigonometric solution $X(z,t) = \frac{t\Omega}{z-t} + p(z,t)$ is called *trivial* if $p(z,t)$ is a constant polynomial, and *nontrivial* otherwise.

Let us finally notice the following fact that we will use in the next section:

Remark 3.15. Any W satisfying conditions 1) and 3) of Theorem 3.6 is bounded, i.e. there exists $M \geq 0$ such that $W \subseteq z^M \mathbf{g}[[z^{-1}]] \oplus \mathbf{g}$. Conversely, if W is Lagrangian and bounded, then condition 1) is verified.

4. Quasi-trigonometric solutions for sl_n

In the case $\mathbf{g} = sl_n$ the result of Theorem 3.6 can be exploited further. We recall the following result [23]:

Theorem 4.1. *Any bounded subalgebra W of $sl_n((z^{-1}))$ is contained in $g^{-1} sl_n[[z^{-1}]] g$ for some $g \in GL_n(\mathbb{C}((z^{-1})))$.*

As a consequence, we obtain a similar result for the bounded subalgebras of $sl_n((z^{-1})) \oplus sl_n$:

Theorem 4.2. *Any bounded subalgebra W of $sl_n((z^{-1})) \oplus sl_n$ is contained in*

$$g^{-1} sl_n[[z^{-1}]] g \oplus sl_n$$

for some $g \in GL_n(\mathbb{C}((z^{-1})))$.

By the so-called "Sauvage Lemma" (see [23]), one obtains

Corollary 4.3. *Any bounded subalgebra of $sl_n((z^{-1})) \oplus sl_n$ is gauge equivalent to a subalgebra contained in $d^{-1} sl_n[[z^{-1}]] d \oplus sl_n$, where $d = diag(z^{m_1}, ..., z^{m_n})$, with m_i integers such that $m_1 \leq ... \leq m_n$.*

Now, quasi-trigonometric solutions correspond to bounded subalgebras that satisfy the conditions of Theorem 3.6. By Corollary 4.3 and condition 2) we get:

Corollary 4.4. *Let $W \subseteq g^{-1} sl_n[[z^{-1}]] g \oplus sl_n$ be a Lie subalgebra corresponding to a quasi-trigonometric solution. Then, up to a gauge equivalence, $g = d_k$, where $d_k = diag(1, ..., 1, z, ...z)$ (k-many 1's) and $0 \leq k \leq [\frac{n}{2}]$.*

Remark 4.5. In the above corollary, one could also consider $g = d_{n-k}$. However the corresponding solutions are not distinguished from those corresponding to $g = d_k$. They are equivalent via an outer automorphism induced by the automorphism of order 2 of the Dynkin diagram for sl_n.

Corollary 4.4 and Theorem 3.6 imply the following

Theorem 4.6. *Any quasi-trigonometric solution for sl_n is gauge equivalent to a quasi-trigonometric solution of the form*

$$X(z,t) = \frac{t\Omega}{z-t} + p_{00} + p_{10} z + p_{01} t + p_{11} zt,$$

where p_{00}, p_{10}, p_{01}, $p_{11} \in \mathbf{g} \otimes \mathbf{g}$.

Recall that a similar result was valid for rational solutions (see Remark 2.8).

Let us consider now

$$L_k := d_k^{-1} sl_n[[z^{-1}]]d_k \oplus sl_n. \tag{4.1}$$

We intend to investigate the quasi-trigonometric solutions for which the corresponding Lagrangian subalgebra W is contained in L_k. We make the remark that $L_k^{\perp} = z^{-1}d_k^{-1}sl_n[[z^{-1}]]d_k$, with respect to the inner product Q. It follows that

$$\frac{L_k}{L_k^{\perp}} \cong \frac{d_k^{-1}sl_n[[z^{-1}]]d_k}{z^{-1}d_k^{-1}sl_n[[z^{-1}]]d_k} \oplus sl_n \cong sl_n \oplus sl_n. \tag{4.2}$$

We identified the class of the matrix $\begin{pmatrix} A & Bz \\ Cz^{-1} & D \end{pmatrix}$ with $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and therefore the quotient that appears on the right-hand side is isomorphic to sl_n. Here A and D are matrix blocks of order k and $n-k$ respectively.

Let us also notice that $sl_n \oplus sl_n$ has an invariant bilinear form, $\overline{Q}$, induced by Q:

$$\overline{Q}((a,b),(c,d)) = K(a,c) - K(b,d). \tag{4.3}$$

We denote by p the canonical projection of L_k onto $sl_n \oplus sl_n$. We recall that we considered $V_0 = \{(a(z), a(0));\ a(z) \in sl_n[z]\}$. We immediately get that $L_k \cap V_0$ consists of all pairs

$$\left(\begin{pmatrix} A & Bz + \tilde{B} \\ 0 & D \end{pmatrix}, \begin{pmatrix} A & \tilde{B} \\ 0 & D \end{pmatrix}\right), \tag{4.4}$$

where A and D are matrix blocks of order k and $n-k$ respectively. Therefore its image $p(L_k \cap V_0)$ in $sl_n \oplus sl_n$ is

$$\Delta_k = \{\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \begin{pmatrix} A & \tilde{B} \\ 0 & D \end{pmatrix}\right)\}. \tag{4.5}$$

Theorem 4.7. *There is a bijection between the set of subalgebras W of L_k, corresponding to quasi-trigonometric solutions, and the set of Lagrangian subalgebras $\overline{W}$ of $sl_n \oplus sl_n$ such that $\overline{W} \oplus \Delta_k = sl_n \oplus sl_n$.*

Proof. Let $W \subseteq L_k$ be a Lagrangian subalgebra of $sl_n((z^{-1})) \oplus sl_n$ complementary to V_0. We set $p(W) = \overline{W}$. It is clear that $\overline{W}$ is isotropic with respect to $\overline{Q}$. Let us show that $\overline{W} + \Delta_k = sl_n \oplus sl_n$. For any $(a,b) \in sl_n \oplus sl_n$, there exists $l' \in L_k$ such that $p(l') = (a,b)$. But $l' = w + v$ with $w \in W$ and $v \in V_0$. It follows that $v \in L_k \cap V_0$ and thus $p(v) \in \Delta_k$. Thus $(a,b) = p(w) + p(v)$, $p(w) \in \overline{W}$, $p(v) \in \Delta_k$. Since $W \subseteq L_k$ and $W = W^{\perp}$ it follows that $L_k^{\perp} \subseteq W$. This inclusion and $W \cap V_0 = 0$ imply that $\overline{W} \cap \Delta_k = 0$. Thus $\overline{W}$ is a subalgebra complementary to Δ_k and isotropic with respect to $\overline{Q}$. Moreover $\overline{W}$ is Lagrangian because it is complementary to Δ_k and thus its dimension is $n^2 - 1$.

Conversely, suppose that $\overline{W}$ is a Lagrangian subalgebra satisfying $\overline{W} \oplus \Delta_k = sl_n \oplus sl_n$ and take $W = p^{-1}(\overline{W})$. It is obvious that $L_k^{\perp} \subseteq W$. We also have $L_k + V_0 = sl_n((z^{-1})) \oplus sl_n$. These facts imply $W + V_0 = sl_n((z^{-1})) \oplus sl_n$. Let $w \in W \cap V_0$, therefore $p(w) \in \overline{W} \cap \Delta_k$, so $p(w) = 0$. It follows that $w \in L_k^{\perp} \cap V_0$.

But one can check that $L_k^{\perp} \cap V_0 = 0$ and thus $w = 0$. In this way W is complementary to V_0. Since $\overline{W}$ is isotropic with respect to $\overline{Q}$ we have that W is isotropic with respect to Q. Let $\alpha \in W^{\perp}$. We have $\alpha = w + v$, where $w \in W$ and $v \in V_0$. Because W is isotropic with respect to Q, it follows that $v \in W^{\perp}$. On the other hand, V_0 is isotropic, so v is in the kernel of Q, which is 0. This implies $v = 0$ and $\alpha \in W$. In conclusion, W is Lagrangian. The condition $W \supseteq z^{-N} sl_n[[z^{-1}]]$ for some positive N is obviously satisfied because W is Lagrangian and bounded.

The correspondence which we established is a bijection: $p(p^{-1}(\overline{W})) = \overline{W}$ (p is surjective) and $p^{-1}(p(W)) = W$ (since $L_k^{\perp} \subseteq W$). □

This result enables us to replace the problem of finding Lagrangian subalgebras corresponding to quasi-trigonometric solutions and contained in L_k by a similar finite-dimensional problem. The question we have to answer is to determine the Lagrangian subalgebras $\overline{W}$ of $sl_n \oplus sl_n$ which are complementary to Δ_k given by (4.5).

Remark 4.8. All subalgebras W which induce trivial quasi-trigonometric solutions are in one-to-one correspondence with Lagrangian subalgebras of $sl_n \oplus sl_n$ which are complementary to $\Delta_0 = \{(a,a); a \in sl_n\}$.

On the other hand, it is known that these Lagrangian subalgebras are in one-to-one correspondence with solutions of the modified CYBE. We obtain again (see Theorem 3.12) that the trivial quasi-trigonometric solutions are exactly of the form $X(z,t) = \frac{t\Omega}{z-t} + r$, where r satisfies the mCYBE.

It is useful to recall also the classical description of Lagrangian subalgebras in terms of triples:

Lemma 4.9. *There exists a one-to-one correspondence between the set of Lagrangian subalgebras of $sl_n \oplus sl_n$ and the set of triples (S_1, S_2, Φ) such that S_i are subalgebras of sl_n, $S_i \supseteq S_i^{\perp}$, $i = 1, 2$, and $\Phi : \frac{S_1}{S_1^{\perp}} \to \frac{S_2}{S_2^{\perp}}$ is a Lie algebras isomorphism preserving the induced bilinear form.*

5. Application 1: quasi-trigonometric solutions for sl_2

In this section we will compute all (up to gauge equivalence) quasi-trigonometric solutions for sl_2 using the results obtained in the previous section. Let $e_{\alpha} = e_{12}$, $e_{-\alpha} = e_{21}$, $h_{\alpha} = e_{11} - e_{22}$ be the usual basis of sl_2. According to the definition of X_0, in this case we have

$$X_0(z,t) = \frac{z+t}{4(z-t)} h_{\alpha} \otimes h_{\alpha} + \frac{z}{z-t} e_{\alpha} \otimes e_{-\alpha} + \frac{t}{z-t} e_{-\alpha} \otimes e_{\alpha}. \tag{5.1}$$

The main result of this section is the following:

Theorem 5.1. *Up to gauge equivalence, there exists a unique nontrivial quasi-trigonometric solution for sl_2, given by*

$$X_1(z,t) = X_0(z,t) + (z-t) e_{\alpha} \otimes e_{\alpha}. \tag{5.2}$$

Let us begin to investigate the subalgebras W of $sl_2((u^{-1}))\oplus sl_2$ which correspond to nontrivial quasi-trigonometric solution. Up to a gauge equivalence, such W is embedded into $L_1 = d_1^{-1}sl_2[[z^{-1}]]d_1 \oplus sl_2$, where $d_1 = diag(1,z)$. Theorem 4.7 implies the following

Corollary 5.2. *There is a bijection between the set of subalgebras W of L_1, corresponding to quasi-trigonometric solutions, and the set of Lagrangian subalgebras $\overline{W}$ of $sl_2\oplus sl_2$ such that $\overline{W}\oplus\Delta_1 = sl_2\oplus sl_2$, where*

$$\Delta_1 = \{(\begin{pmatrix} a & b \\ 0 & -a\end{pmatrix}, \begin{pmatrix} a & c \\ 0 & -a\end{pmatrix}; a,b,c\in\mathbb{C}\}. \tag{5.3}$$

Let us recall a result from [23] which is important for the proof of the main theorem of this section.

Lemma 5.3. *Let L be a Lie algebra, R_1, R_2 and N be its subalgebras such that $L = R_1+N$ and $L = R_2+N$. Assume there exists $X\in G(L)$ such that $R_1 = \mathbf{Ad}(X)R_2$. Then there exists $Y\in G(N)$ such that $R_1 = \mathbf{Ad}(Y)R_2$.*

Proposition 5.4. *There exists only one family of Lagrangian subalgebras $\overline{W}$ of $sl_2\oplus sl_2$ such that $\overline{W}\oplus\Delta_1 = sl_2\oplus sl_2$ and which provide nontrivial quasi-trigonometric solutions:*

$$\overline{W}_T = \{(x, TxT^{-1}); x\in sl_2\}, \tag{5.4}$$

where $T = (t_{ij})\in SL_2$ with $t_{21}\neq 0$.

Proof. Recall by Lemma 4.9 that any Lagrangian subalgebra $\overline{W}$ of $sl_2\oplus sl_2$ is constructed from the following data: two subalgebras S_1 and S_2 of sl_2 such that $S_i\supseteq S_i^\perp$ (with respect to the bilinear form on sl_2) and a Lie algebras isomorphism Φ: $\frac{S_1}{S_1^\perp}\to\frac{S_2}{S_2^\perp}$ which is an isometry. More precisely, if (S_1,S_2,Φ) is a triple as above, then the corresponding Lagrangian subalgebra is

$$\overline{W} = S_1^\perp\oplus S_2^\perp + \{(x,\Phi(x)); x\in\frac{S_1}{S_1^\perp}\}. \tag{5.5}$$

On the other hand, the only subalgebras of sl_2 which contain their orthogonal are sl_2 and, up to conjugation, the Borel subalgebra $\mathbf{b}_-$. The subalgebras S_i which provide a Lagrangian subalgebra $\overline{W}$ are either both sl_2 or both Borel.

Case 1. $S_1 = S_2 = sl_2$. It follows that $\frac{S_i}{S_i^\perp} = sl_2$. Any automorphism of sl_2 is inner and preserves the Killing form. Therefore we have a triple (sl_2, sl_2, Φ) for any automorphism Φ of sl_2. If we impose the requirement that the induced Lagrangian subalgebra is complementary to Δ_1, we obtain a family of triples (sl_2, sl_2, Φ_T), where Φ_T is defined by $\Phi_T(x) = TxT^{-1}$, $T=(t_{ij})\in SL_2$ with $t_{21}\neq 0$. The triples of this family induce exactly the subalgebras $\overline{W}_T$.

Case 2. We suppose that S_1 and S_2 are Borel subalgebras. There exist X and Y in SL_2 such that $\mathbf{Ad}(X)(S_1) = \mathbf{b}_-$ and $\mathbf{Ad}(Y)(S_2) = \mathbf{b}_-$. We obtain that $(\mathbf{Ad}(X)\oplus\mathbf{Ad}(Y))(\overline{W})\subseteq\mathbf{b}_-\oplus\mathbf{b}_-$. It results that $\overline{W}$ is contained in a Borel subalgebra B conjugated to $\mathbf{b}_-\oplus\mathbf{b}_-$. If $\overline{W}$ is complementary to Δ_1, then $B+\Delta_1 = sl_2\oplus sl_2$. On

the other hand, $(\mathbf{b}_-\oplus\mathbf{b}_-)+\Delta_1 = sl_2\oplus sl_2$. These facts imply (see Lemma 5.3) that there exists a conjugation preserving Δ_1 which transforms B into $\mathbf{b}_-\oplus\mathbf{b}_-$.

In conclusion, by means of a gauge transformation, one may suppose that the subalgebras S_i are both $\mathbf{b}_-$. It follows that $\frac{S_i}{S_i^\perp}$ is isomorphic to the Cartan subalgebra $\mathbf{h}$. An automorphism of $\mathbf{h}$ preserves the bilinear form only if it is $\pm id_{\mathbf{h}}$.

If we impose the requirement that the induced Lagrangian subalgebra is complementary to Δ_1, we obtain one possible triple: $(\mathbf{b}_-,\mathbf{b}_-,-id_{\mathbf{h}})$. The Lagrangian subalgebra corresponding to this triple is

$$\overline{W}_0 = \{(\begin{pmatrix} a & 0 \\ b & -a\end{pmatrix}, \begin{pmatrix} -a & 0 \\ c & a\end{pmatrix}); a,b,c\in\mathbb{C}\}. \tag{5.6}$$

The Lagrangian subalgebra $\overline{W}_0$ induces the trivial solution X_0. Indeed, let W_0 be the corresponding Lagrangian subalgebra of $sl_2((z^{-1}))\oplus sl_2$ (via the bijection from Corollary 5.2). One may check that W_0 is exactly the subalgebra given by (3.11) which corresponds to the solution X_0. This ends the proof. □

Remark 5.5. All the subalgebras $\overline{W}_T$ are conjugate to each other, particularly to

$$\overline{W}_1 = \{(\begin{pmatrix} a & b \\ c & -a\end{pmatrix}, \begin{pmatrix} -a & -c \\ -b & a\end{pmatrix}); a,b,c\in\mathbb{C}\}, \tag{5.7}$$

which corresponds to $T_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$.

In the following lemma we will construct a conjugation which preserves Δ_1 and transforms $\overline{W}_1$ into $\overline{W}_T$. Moreover we will extend this conjugation to a gauge equivalence of the corresponding Lagrangian subalgebras, W_1 and W_T, in $sl_2((z^{-1}))\oplus sl_2$. Therefore the induced quasi-trigonometric solutions will be gauge equivalent.

Lemma 5.6. *Let $\overline{W}_1$ and $\overline{W}_T$, where $T=(t_{ij})\in SL_2$, $t_{21}\neq 0$, be as above. There exist constants a, b and λ such that*

$$(\mathbf{Ad}(T_{a,\lambda})\oplus\mathbf{Ad}(T_{b,\lambda}))(\overline{W}_1) = \overline{W}_T, \tag{5.8}$$

where $T_{a,\lambda} = \begin{pmatrix} \lambda & a \\ 0 & \lambda^{-1}\end{pmatrix}$ and $T_{b,\lambda} = \begin{pmatrix} \lambda & b \\ 0 & \lambda^{-1}\end{pmatrix}$.

Proof. By imposing the condition that $\overline{W}_1$ is transformed into $\overline{W}_T$ we obtain that a, b and λ must satisfy the following property

$$TT_{a,\lambda}xT_{a,\lambda}^{-1}T^{-1} = T_{b,\lambda}T_1xT_1^{-1}T_{b,\lambda}^{-1}, \tag{5.9}$$

for any $x\in sl_2$. Equivalently, $TT_{a,\lambda} = \pm T_{b,\lambda}T_1$. In the first case, we obtain $\lambda^2 = -t_{21}^{-1}$, $a = -\lambda^{-1}t_{22}/t_{21}$ and $b = -\lambda t_{11}$. In the second case, one gets $\lambda^2 = t_{21}^{-1}$, $a = -\lambda^{-1}t_{22}/t_{21}$ and $b = \lambda t_{11}$.

With these choices of a, b and λ, $\mathbf{Ad}(T_{a,\lambda})\oplus\mathbf{Ad}(T_{b,\lambda})$ transforms $\overline{W}_1$ into $\overline{W}_T$. This ends the proof. □

We will reconstruct now the corresponding Lagrangian subalgebras of $sl_2((z^{-1}))\oplus sl_2$ which are complementary to V_0 and are included in L_1, according to Corollary 5.2.

Lemma 5.7. *The Lagrangian subalgebras of $sl_2((z^{-1}))\oplus sl_2$ which correspond to $\overline{W}_1$ and $\overline{W}_T$ respectively, via the bijection from Corollary 5.2, are the following:*

$$W_1=\{(\begin{pmatrix} a+f(z)z^{-1} & bz+g(z)\\ cz^{-1}+h(z)z^{-2} & -a-f(z)z^{-1}\end{pmatrix},\begin{pmatrix} -a & -c\\ -b & a\end{pmatrix}); \quad a,b,c\in\mathbb{C}, f,g,h\in\mathbb{C}[[z^{-1}]]\},\tag{5.10}$$

$$W_T=\{(\begin{pmatrix} a+f(z)z^{-1} & bz+g(z)\\ cz^{-1}+h(z)z^{-2} & -a-f(z)z^{-1}\end{pmatrix},T\begin{pmatrix} a & b\\ c & -a\end{pmatrix}T^{-1}); \quad a,b,c\in\mathbb{C}, f,g,h\in\mathbb{C}[[z^{-1}]]\}.\tag{5.11}$$

Proposition 5.8. *The subalgebras W_1 and W_T given by Lemma 5.7 are gauge equivalent.*

Proof. In Lemma 5.6 we constructed an automorphism of $sl_2\oplus sl_2$ preserving Δ_1 and sending $\overline{W}_1$ to $\overline{W}_T$. We check that this automorphism can be lifted to a gauge equivalence between W_1 and W_T. Indeed, keeping the notations of Lemma 5.6, let us recall that there exist a, b and λ such that

$$(\mathbf{Ad}(T_{a,\lambda})\oplus\mathbf{Ad}(T_{b,\lambda}))(\overline{W}_1)=\overline{W}_T.\tag{5.12}$$

We consider the following automorphism Ψ of $sl_2((z^{-1}))\oplus sl_2$ defined by

$$\Psi=\mathbf{Ad}(T(z))\oplus\mathbf{Ad}(T(0)),\tag{5.13}$$

where

$$T(z)=\begin{pmatrix}\lambda & az+b\\ 0 & \lambda^{-1}\end{pmatrix}.\tag{5.14}$$

A simple computation shows that

$$\Psi(W_1)=W_T.\tag{5.15}$$

It is not difficult to check that Ψ preserves V_0. Therefore Ψ is a gauge equivalence. □

Remark 5.9. By the same procedure, we can see that any automorphism of $sl_2\oplus sl_2$ preserving Δ_1 can be lifted to an automorphism of $sl_2((z^{-1}))\oplus sl_2$ which preserves V_0.

By Theorem 3.10 and the previous result we obtain:

Corollary 5.10. *The quasi-trigonometric solutions corresponding to W_1 and W_T are gauge equivalent.*

Therefore in sl_2 there exists a unique nontrivial quasi-trigonometric solution, up to gauge equivalence.

We proceed now to the determination of the quasi-trigonometric solution corresponding to the Lagrangian subalgebra W_1.

Proposition 5.11. *The quasi-trigonometric solution corresponding to W_1 is the following:*

$$X_1(z,t)=X_0(z,t)+(z-t)e_\alpha\otimes e_\alpha.\tag{5.16}$$

Proof. We keep the notations of Theorem 3.6 and we apply the constructive procedure given in Remark 3.8. Let us first take two dual bases in W_1 and V_0. We recall that the canonical basis of V_0 is: $e_\alpha z^k$, $e_{-\alpha}z^k$, $\frac{1}{\sqrt{2}}h_\alpha z^k$ for all $k\geq 2$, $e_{-\alpha}z$, $e_\alpha z$, $\frac{1}{\sqrt{2}}h_\alpha z$, (e_α,e_α), $(e_{-\alpha},e_{-\alpha})$ and $\frac{1}{2\sqrt{2}}(h_\alpha,h_\alpha)$. The dual basis of W_1 is formed by the following elements: $e_{-\alpha}z^{-k}$, $e_\alpha z^{-k}$, $\frac{1}{\sqrt{2}}h_\alpha z^{-k}$ for all $k\geq 2$, $e_\alpha z^{-1}$, $(e_{-\alpha}z^{-1}-e_\alpha,-e_\alpha)$, $\frac{1}{\sqrt{2}}h_\alpha z^{-1}$, $(e_\alpha z,-e_{-\alpha})$, $(e_\alpha,0)$, $\frac{1}{\sqrt{2}}(h_\alpha,-h_\alpha)$.

Let us simply denote these dual bases in W_1 and V_0 by $\{w^i\}$ and $\{v_i\}$ respectively and construct the tensor

$$\widetilde{r}(z,t)=\sum_i w^i\otimes v_i.\tag{5.17}$$

It follows that

$$\widetilde{r}(z,t)=\frac{t\Omega}{z-t}-(e_\alpha,e_\alpha)\otimes e_\alpha t+(e_\alpha z,-e_{-\alpha})\otimes(e_\alpha,e_\alpha) +(e_\alpha,0)\otimes(e_{-\alpha},e_{-\alpha})+\frac{1}{4}(h_\alpha,-h_\alpha)\otimes(h_\alpha,h_\alpha).\tag{5.18}$$

Let us take, as in Remark 3.8, the projection π onto $sl_2[z]$ induced by the decomposition $V_0\oplus W_0=sl_2((z^{-1}))\oplus sl_2$. Let

$$r(z,t)=(\pi\otimes\pi)(\widetilde{r}(z,t)).\tag{5.19}$$

A simple computation gives

$$r(z,t)=(z-t)(e_\alpha\otimes e_\alpha).\tag{5.20}$$

Therefore the solution corresponding to W_1 is exactly X_1 given by (5.16). □

The above result and Corollary 5.10 prove Theorem 5.1.

Remark 5.12. We will show in Appendix that the solution X_1 is related to the following trigonometric solution obtained via Belavin-Drinfeld classification [2]:

$$X_1^{BD}(u)=\frac{e^u+1}{4(e^u-1)}h_\alpha\otimes h_\alpha+\frac{e_\alpha\otimes e_{-\alpha}+e_{-\alpha}\otimes e_\alpha}{e^{u/2}-e^{-u/2}}+ +(e^{u/2}-e^{-u/2})e_\alpha\otimes e_\alpha.\tag{5.21}$$

In the following sections concerning quantization, we are not going to quantize directly X_1 but another quasi-trigonometric solution which is gauge equivalent to it. Therefore we need the following:

Corollary 5.13. *The quasi-trigonometric solution X_1 is gauge equivalent to the following solution:*

$$X_{a,b}(z,t)=\frac{t\Omega}{z-t}+e_{-\alpha}\otimes e_\alpha+\frac{1}{4}h_\alpha\otimes h_\alpha+\tag{5.22}$$

$$+a(ze_{-\alpha} \otimes h_\alpha - th_\alpha \otimes e_{-\alpha}) + b(e_{-\alpha} \otimes h_\alpha - h_\alpha \otimes e_{-\alpha}),$$

for any nonzero constants a and b.

Proof. One can easily check that $X_{a,b}$ is obtained from X_1 by applying the gauge transformation $\sigma(z) = \mathbf{Ad}(T(z))$ with

$$T(z) = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda & -(\mu z + \nu) \end{pmatrix},$$

where λ, μ and ν are chosen such that $\lambda^4 = 4ab$, $\mu = -2a\lambda^{-1}$, $\nu = 2b\lambda^{-1}$. $\square$

6. Application 2: Quasi-trigonometric solutions for sl_3

We compute the nontrivial quasi-trigonometric solutions of the CYBE for sl_3 (up to gauge equivalence).

The Lagrangian subalgebras of $sl_3((z^{-1})) \oplus sl_3$ corresponding to nontrivial quasi-trigonometric solutions may be embedded, after a gauge transformation, into $L_2 = d_2^{-1} sl_3[[z^{-1}]] d_2 \oplus sl_3$, where $d_2 = diag(1, 1, z)$.

Proposition 6.1. *There is a bijection between the set of subalgebras W of L_2, corresponding to quasi-trigonometric solutions, and the set of Lagrangian subalgebras $\overline{W}$ of $sl_3 \oplus sl_3$ such that $\overline{W} \oplus \Delta_2 = sl_3 \oplus sl_3$, where*

$$\Delta_2 = \{(\begin{pmatrix} A & b \\ 0 & -TrA \end{pmatrix}, \begin{pmatrix} A & \tilde{b} \\ 0 & -TrA \end{pmatrix}), A \in gl_2, b, \tilde{b} \in \mathbb{C}^2\}. \tag{6.1}$$

Theorem 6.2. *Up to gauge equivalence, in sl_3 there exist two nontrivial quasi-trigonometric solutions. They correspond to the following Lagrangian subalgebras of $sl_3 \oplus sl_3$:*

$$\overline{W}_1 = \{(\begin{pmatrix} -a-d & 0 & 0 \\ * & a & b \\ * & c & d \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ * & * & -a-d \end{pmatrix})\}, \tag{6.2}$$

$$\overline{W}_2 = \{(x, TxT^{-1}); x \in sl_3\}, \tag{6.3}$$

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Proof. We will prove the theorem in several steps:

Step 1. By Proposition 6.1, the quasi-trigonometric solutions are given by the Lagrangian subalgebras $\overline{W}$ of $sl_3 \oplus sl_3$ complementary to Δ_2. On the other hand, there exists a one-to-one correspondence between the set of Lagrangian subalgebras of $sl_3 \oplus sl_3$ and the set of triples (S_1, S_2, Φ) such that S_i are subalgebras of sl_3, $S_i \supseteq S_i^\perp$ and $\Phi : \frac{S_1}{S_1^\perp} \to \frac{S_2}{S_2^\perp}$ is a Lie algebras isomorphism preserving the induced bilinear form. Let us also recall that the trivial quasi-trigonometric solutions correspond to Lagrangian subalgebras which are included in $sl_3[[z^{-1}]] \oplus sl_3$.

Step 2. From the previous step, we see that the subalgebras S_i which provide a Lagrangian subalgebra $\overline{W}$ are simultaneously irreducible or reducible. Let us consider the irreducible case. It is clear that both $S_i = sl_3$. Any automorphism of sl_3 preserves the Killing form and therefore it induces a Lagrangian subalgebra of $sl_3 \oplus sl_3$ of the following form:

$$\overline{W}_\Phi = \{(x, \Phi(x); x \in sl_3\}. \tag{6.4}$$

On the other hand, any automorphism of sl_3 has one of the following forms:

$$\Phi(x) = TxT^{-1}, \tag{6.5}$$

$$\Phi(x) = -Tx^tT^{-1}, \tag{6.6}$$

where $T \in SL_3$ and therefore we have two families of Lagrangian subalgebras:

$$\overline{W}_T = \{(x, TxT^{-1}); x \in sl_3\}, \tag{6.7}$$

$$\widetilde{W}_T = \{(x, -Tx^tT^{-1}); x \in sl_3\}. \tag{6.8}$$

All the Lagrangian subalgebras in the first family are conjugate to each other and the same is true for the second family.

One can easily check that the Lagrangian subalgebra $\overline{W}_T$ is complementary to Δ_2 if and only if $T = (t_{ij}) \in SL_3$ is such that the following matrix has rank 8:

$$\begin{pmatrix}
0 & -t_{21} & t_{12} & 0 & 0 & 0 & -t_{31} & 0 \\
-t_{12} & t_{11} - t_{22} & 0 & t_{12} & 0 & 0 & -t_{32} & 0 \\
-2t_{13} & -t_{23} & 0 & -t_{13} & t_{11} & t_{12} & -t_{33} & 0 \\
t_{21} & 0 & t_{22} - t_{11} & -t_{21} & 0 & 0 & 0 & -t_{31} \\
0 & t_{21} & -t_{12} & 0 & 0 & 0 & 0 & -t_{32} \\
-t_{23} & 0 & -t_{13} & -2t_{23} & t_{21} & t_{22} & 0 & -t_{33} \\
2t_{31} & 0 & t_{32} & t_{31} & 0 & 0 & 0 & 0 \\
t_{32} & t_{31} & 0 & 2t_{32} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & t_{31} & t_{32} & 0 & 0
\end{pmatrix}.$$

All these Lagrangian subalgebras induced by various T satisfying the above condition are conjugate to each other. Lemma 5.3 insures the fact that there exists a conjugation preserving Δ_2 which transforms one into the other and therefore the induced quasi-trigonometric solutions are gauge equivalent.

In conclusion, up to gauge equivalence, there is only one quasi-trigonometric solution which corresponds to the following:

$$\overline{W}_2 = \{(x, TxT^{-1}); x \in sl_3\}, \tag{6.9}$$

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{6.10}$$

Now, let us analyse the Lagrangian subalgebras of the second type, $\widetilde{W}_T$. The condition that $\widetilde{W}_T \cap \Delta_2 = 0$ is equivalent to the fact the following matrix has rank

8:

$$\begin{pmatrix} -2t_{11} & -t_{12}-t_{21} & 0 & 0 & -t_{13} & 0 & -t_{31} & 0 \\ -t_{12} & -t_{22} & -t_{11} & -t_{12} & 0 & -t_{13} & -t_{32} & 0 \\ 0 & -t_{23} & 0 & t_{13} & 0 & 0 & -t_{33} & 0 \\ -t_{21} & -t_{22} & -t_{11} & -t_{21} & -t_{23} & 0 & 0 & -t_{31} \\ 0 & 0 & -t_{12}-t_{21} & -2t_{22} & 0 & -t_{23} & 0 & -t_{32} \\ t_{23} & 0 & -t_{13} & 0 & 0 & 0 & 0 & -t_{33} \\ 0 & -t_{32} & 0 & t_{31} & -t_{33} & 0 & 0 & 0 \\ t_{32} & 0 & -t_{31} & 0 & 0 & -t_{33} & 0 & 0 \\ t_{33} & 0 & 0 & t_{33} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By direct computations, one can prove that the rank of this matrix is at most 7. In conclusion, the Lagrangian subalgebra $\widetilde{W}_T$ induced by the second type of automorphisms of sl_3 is never complementary to Δ_2, for any choice of the matrix $T \in SL_3$.

Step 3. Both S_i determining $\overline{W}$ are reducible and each preserve a line in $\mathbb{C}^3$. Then S_1 and S_2 can be embedded, by means of some conjugations (not necessarily the same) into the following maximal parabolic subalgebra:

$$P_2 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}. \tag{6.11}$$

Thus there exist X, Y in SL_3 such that

$$(\mathbf{Ad}(X) \oplus \mathbf{Ad}(Y))(\overline{W}) \subseteq P_2 \oplus P_2. \tag{6.12}$$

It results that $\overline{W}$ is contained in a maximal parabolic subalgebra $\widetilde{P}$ conjugate to $P_2 \oplus P_2$. Since $\overline{W} \oplus \Delta_2 = sl_3 \oplus sl_3$, it follows that $\widetilde{P} + \Delta_2 = sl_3 \oplus sl_3$. On the other hand, $(P_2 \oplus P_2) + \Delta_2 = sl_3 \oplus sl_3$. Lemma 5.3 implies that there is a conjugation preserving Δ_2 which transforms $P_2 \oplus P_2$ into $\widetilde{P}$. In conclusion, by a gauge transformation, $\overline{W}$ may always be embedded into $P_2 \oplus P_2$. One can easily see that such $\overline{W}$ can only provide trivial solutions, if there are any. The simple reason is that the corresponding subalgebra in $sl_3((z^{-1})) \oplus sl_3$ is included in $sl_3[[z^{-1}]] \oplus sl_3$.

Step 4. Each of S_i preserves some plane in $\mathbb{C}^3$. By the same method as in step 3, if there is some solution, then by means of a gauge transformation, we may suppose that S_1 and S_2 are included in

$$P_1 = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}. \tag{6.13}$$

Actually, either both coincide with P_1 or they are both solvable and in this situation we are back to step 3. If S_1 and S_2 coincide with P_1, then obviously (e_{21}, e_{21}) is in the intersection $\overline{W} \cap \Delta_2$. Therefore we have no solution at all in this case.

Step 5. S_1 preserves a line and S_2 preserves a plane in $\mathbb{C}^3$. By means of a gauge transformation, we may suppose that $S_1 \subseteq P_2$ and $S_2 \subseteq P_1$. We see again that if there exist some solution, then this is trivial because the corresponding subalgebra of $sl_3((u^{-1})) \oplus sl_3$ is contained actually in $sl_3[[z^{-1}]] \oplus sl_3$.

Step 6. S_1 preserves a plane and S_2 preserves a line in $\mathbb{C}^3$. Again we may reduce the problem to the analysis of the case $S_1 \subseteq P_1$ and $S_2 \subseteq P_2$. Moreover, since the solvable cases have been treated earlier, we may even suppose that $S_1 = P_1$ and $S_2 = P_2$. In this case,

$$\frac{S_1}{S_1^{\perp}} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \tag{6.14}$$

$$\frac{S_2}{S_2^{\perp}} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}. \tag{6.15}$$

Both these subalgebras are isomorphic to gl_2. Any isometry between the subalgebras $\frac{S_1}{S_1^{\perp}}$ and $\frac{S_2}{S_2^{\perp}}$ is therefore given by an automorphism φ of gl_2 which preserves the bilinear form Q defined as

$$Q(A, B) = Tr(AB) + TrA \cdot TrB. \tag{6.16}$$

It is not difficult to see that any automorphism of gl_2 which preserves Q has one of the following forms:

$$\varphi_U \begin{pmatrix} a & b \\ c & d \end{pmatrix} = U \begin{pmatrix} a & b \\ c & d \end{pmatrix} U^{-1}, \tag{6.17}$$

$$\psi_U \begin{pmatrix} a & b \\ c & d \end{pmatrix} = U \begin{pmatrix} -d & b \\ c & -a \end{pmatrix} U^{-1}, \tag{6.18}$$

where $U \in SL_2$. The Lagrangian subalgebras which correspond to the triples (S_1, S_2, φ_U) and (S_1, S_2, ψ_U) respectively are the following:

$$\overline{W}_U = \{(\begin{pmatrix} -TrA & 0 \\ * & A \end{pmatrix}, \begin{pmatrix} \varphi_U(A) & 0 \\ * & TrA \end{pmatrix})\}. \tag{6.19}$$

$$\widetilde{W}_U = \{(\begin{pmatrix} -TrA & 0 \\ * & A \end{pmatrix}, \begin{pmatrix} \psi_U(A) & 0 \\ * & TrA \end{pmatrix})\}. \tag{6.20}$$

A simple computation shows that $\overline{W}_U$ is complementary to Δ_2 if and only if $u_{11} \neq 0$. On the other hand, one can check that the Lagrangian subalgebra $\widetilde{W}_U$ is never complementary to Δ_2, for any choice of U.

In conclusion, we have one family of Lagrangian subalgebras $\overline{W}_U$ that provides nontrivial quasi-trigonometric solutions. All these subalgebras are conjugate to each other, in particular to

$$\overline{W}_1 = \{(\begin{pmatrix} -a-d & 0 & 0 \\ * & a & b \\ * & c & d \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ * & * & -a-d \end{pmatrix})\}, \tag{6.21}$$

Moreover, Lemma 5.3 implies that all these subalgebras induce gauge-equivalent quasi-trigonometric solutions. □

Lemma 6.3. *The Lagrangian subalgebras of $sl_3((z^{-1}))\oplus sl_3$ which correspond to $\overline{W}_1$ and $\overline{W}_2$ respectively, via the bijection from Corollary 5.2, are the following:*

$$W_1 = diag(z^{-1},z^{-1},z^{-2})sl_3[[z^{-1}]]diag(1,1,z)\oplus \tag{6.22}$$

$$\{(\begin{pmatrix} -a-d & 0 & 0 \\ g & a & bz \\ hz^{-1} & cz^{-1} & d \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & -a-d \end{pmatrix})\},$$

$$W_2 = diag(z^{-1},z^{-1},z^{-2})sl_3[[z^{-1}]]diag(1,1,z)\oplus \tag{6.23}$$

$$\{(\begin{pmatrix} a_{11} & a_{12} & a_{13}z \\ a_{21} & a_{22} & a_{23}z \\ a_{31}z^{-1} & a_{32}z^{-1} & -a_{11}-a_{22} \end{pmatrix}, T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & -a_{11}-a_{22} \end{pmatrix}T^{-1})\}.$$

Let $X_0 = \frac{t\Omega}{z-t} + r_0$ be the simplest quasi-trigonometric solution in sl_3. Here

$$\Omega = \frac{1}{3}\sum_{i<j}(e_{ii}-e_{jj})\otimes(e_{ii}-e_{jj}) + \sum_{i\neq j} e_{ij}\otimes e_{ji} \tag{6.24}$$

and

$$r_0 = \frac{1}{6}\sum_{i<j}(e_{ii}-e_{jj})\otimes(e_{ii}-e_{jj}) + \sum_{i<j} e_{ij}\otimes e_{ji}. \tag{6.25}$$

Corollary 6.4. *The quasi-trigonometric solutions induced by the Lagrangian subalgebras W_1 and W_2 given by (6.22) and (6.23) are respectively the following:*

$$X_1(z,t) = X_0(z,t) + te_{21}\otimes e_{23} - ze_{23}\otimes e_{21} \tag{6.26}$$
$$-\frac{1}{6}(e_{11}-e_{22})\wedge(e_{22}-e_{33}),$$

$$X_2(z,t) = X_0(z,t) + ze_{13}\otimes(e_{12}+e_{21}) - t(e_{12}+e_{21})\otimes e_{13} \tag{6.27}$$
$$+ze_{23}\otimes(\frac{1}{3}e_{11}+\frac{1}{3}e_{22}-\frac{2}{3}e_{33}) - t(\frac{1}{3}e_{11}+\frac{1}{3}e_{22}-\frac{2}{3}e_{33})\otimes e_{23}$$
$$+e_{23}\wedge(e_{12}+e_{21}) + e_{13}\wedge(\frac{1}{3}e_{11}+\frac{1}{3}e_{22}-\frac{2}{3}e_{33}) - \frac{1}{6}(e_{11}-e_{22})\wedge(e_{22}-e_{33}).$$

Proof. Let $V_0 = \{(a(z),a(0));\ a(z)\in sl_3[z]\}$. In order to construct the quasi-trigonometric solution corresponding to W_i, we apply the standard procedure of choosing convenient dual bases in V_0 and W_i.

For W_1 we choose $(e_{32}z^{-1}+e_{21},e_{21})$, $(e_{23}z,e_{12})$, $(-e_{11}+e_{22},e_{11}-e_{33})$, $(-e_{11}+e_{33},e_{22}-e_{33})$ and the rest are standard. The dual basis of V_0 is $e_{23}z$, $-(e_{21},e_{21})$, $(-\frac{1}{3}e_{11}+\frac{1}{3}e_{22},-\frac{1}{3}e_{11}+\frac{1}{3}e_{22})$, $(-\frac{1}{3}e_{22}+\frac{1}{3}e_{33},-\frac{1}{3}e_{22}+\frac{1}{3}e_{33})$ and the rest are standard. The induced quasi-trigonometric solution is $X_1(z,t)$ given by (6.26).

For W_2 we take the following: $(e_{31}z^{-1}-e_{12}-e_{21},-e_{12}-e_{21})$, $(e_{32}z^{-1}-(\frac{1}{3}e_{11}+\frac{1}{3}e_{22}-\frac{2}{3}e_{33}),-(\frac{1}{3}e_{11}+\frac{1}{3}e_{22}-\frac{2}{3}e_{33}))$, $(e_{12}+e_{23}+e_{13}z,e_{23})$, $(e_{23}+e_{13}z,e_{23}-e_{21})$, $(e_{13},0)$, $(-\frac{1}{3}e_{11}-\frac{1}{3}e_{22}+\frac{2}{3}e_{33},-\frac{1}{3}e_{11}-\frac{1}{3}e_{22}+\frac{2}{3}e_{33}-e_{31})$ $(e_{23},0)$, $(-e_{12}-e_{21},-e_{12}-e_{21}-e_{32})$, $(e_{11}-e_{22}-e_{13}-e_{23}z,e_{22}-e_{33}-e_{13})$, $(-\frac{2}{3}e_{11}+\frac{1}{3}e_{22}+\frac{1}{3}e_{33},\frac{1}{3}e_{11}-\frac{2}{3}e_{22}+\frac{1}{3}e_{33})$ and the rest are standard. The corresponding dual elements in V_0 are: $e_{13}z$, $e_{23}z$, (e_{21},e_{21}), (e_{12},e_{12}), (e_{31},e_{31}), (e_{13},e_{13}), (e_{32},e_{32}), (e_{23},e_{23}), $(-\frac{1}{3}e_{11}-\frac{1}{3}e_{22}+\frac{2}{3}e_{33},-\frac{1}{3}e_{11}-\frac{1}{3}e_{22}+\frac{2}{3}e_{33})$, $(-e_{11}+e_{33},-e_{11}+e_{33})$ and the rest are standard. The corresponding solution is $X_2(z,t)$ given by (6.27). This ends the proof. □

Remark 6.5. (i) We will discuss in Appendix the relationship between the quasi-trigonometric and the trigonometric solutions in the Belavin-Drinfeld classification for the sl_3 case [2, p. 173].

(ii) The form of the quasi-trigonometric solutions appears to be also related to that of the rational solutions in sl_3 (see Table 3 in [20, p.64]). This fact gives hope that quasi-trigonometric and rational solutions in sl_3 might be quantized simultaneously.

7. Quantum twists

Let us recall (see [10, p. 84-85]) that a method for constructing new quantized enveloping algebras from old ones is the quatization by twists. Let $H = (H,m,\Delta,i,\varepsilon,S)$ be a QUE-algebra.

Definition 7.1. An invertible element $F\in H\otimes H$ is called a *quantum twist* if
1) $F\equiv 1 (\mathrm{mod}\ \hbar)$,
2) $(\varepsilon\otimes id)F = (id\otimes\varepsilon)F = 1\otimes 1$,
3) $F^{12}(\Delta\otimes id)F = F^{23}(id\otimes\Delta)F\cdot\Phi$, for some H-invariant element Φ of $H^{\otimes 3}$.

Here we use the standard notation: $F^{12} = \sum a_i\otimes b_i\otimes 1$, $F^{23} = \sum 1\otimes a_i\otimes b_i$ if $F = \sum a_i\otimes b_i$.

The following proposition is well-known:

Proposition 7.2. *Any quantum twist F defines a new QUE-algebra*

$$H_F = (H,m,\Delta_F,i,\varepsilon,S_F)$$

if we set

$$\Delta_F = F\Delta F^{-1}, \tag{7.1}$$

$$S_F = QSQ^{-1}, \tag{7.2}$$

where $Q = m((S\otimes 1)(F))$.

The classical limit of H_F is a Lie bialgebra $(\mathbf{g},\delta_F)$, where δ_F is not necessarily the cobracket δ we started with. It turns out that δ_F can be obtained by twisting δ, in the classical meaning. More precisely, according to [10, p. 147], we have:

Proposition 7.3. *Let H be a QUE-algebra and F a quantum twist. Let H_F denote the twist of H by F. Set $s\equiv\frac{F-F^{21}}{\hbar}$ (mod $\hbar$). Then the classical limit of H_F is obtained from the classical limit of H by twisting via s.*

In this way, any quantum twist induces a classical twist. Concerning the converse statement, there is a general opinion that the following conjecture is true.

Conjecture 7.4. *Any classical twist can be extended to a quantum twist.*

This conjecture is supported by some examples. Let us recall the following cases that we considered in Section 2:

Example 7.5. In [8], V.G. Drinfeld proved that any triangular r-matrix can be quantized, by constructing an appropiate quantum twist of the universal enveloping algebra.

Another illustration of the conjecture is the quantization of the Belavin-Drinfeld list [1], in the way it was performed in [11]:

Example 7.6. Let $\mathbf{g}$ be a finite-dimensional simple complex Lie algebra. Denote by r_0 the "standard" Drinfeld-Jimbo r-matrix. We have seen in Example 2.5 that by twisting the standard structure one obtains the entire Belavin-Drinfeld list [1]. Let $q = e^{\hbar/2}$ and $U_q(\mathbf{g})$ denote the Drinfeld-Jimbo quantum universal enveloping algebra, which is a quantization of the Lie bialgebra $(\mathbf{g}, \delta_{r_0})$. It was proved in [11] that any twist s of r_0 can be extended to a quantum twist F_s.

8. q-Power function over q-commuting variables

In this section we generalize the well known Faddeev-Volkov identity [12], which will be used later in the construction of a quantum twist for $U_q(\widehat{sl_2})$.

Let us consider the following q-binomial series of an indeterminate u (see [13]):

$$F_a(u) = (1-u)_q^{(a)} = 1 + \sum_{k>0} \frac{(-a)_q(-a+1)_q ... (-a+k-1)_q}{(k)_q!} u^k, \tag{8.1}$$

where $(a)_q = (q^a - 1)/(q-1)$. We recall that if $|q| < 1$ then $F_a(u)$ is uniquely characterized by the difference equation

$$F_a(u) = \frac{1-q^{-a}u}{1-u} F_a(qu). \tag{8.2}$$

Proposition 8.1. *The unital formal power series $F_a(u)$ satisfies the following additive properties:*

$$(1-u)_q^{(a)}(1-q^{-a}u)_q^{(b)} = (1-u)_q^{(a+b)} \tag{8.3}$$

$$(1-u)_q^{(a)}(1-v)_q^{(a)} = (1-u-v+q^{-a}uv)_q^{(a)} \tag{8.4}$$

$$(1-v)_q^{(a)}(1-u)_q^{(a)} = (1-u-v+uv)_q^{(a)}, \tag{8.5}$$

where the variables u and v in (8.4) and (8.5) q-commute, i.e. $vu = quv$.

Proof. These properties may be deduced from the presentation of $F_a(u)$ as a ratio of q-exponential functions and from the corresponding properties of q exponents. More precisely, let

$$\exp_q(u) = 1 + \sum_{n>0} \frac{u^n}{(n)_q!} \tag{8.6}$$

$$(u;q)_\infty = (1-u)(1-qu).... \tag{8.7}$$

It follows that

$$F_a(u) = \frac{\exp_q(\frac{u}{1-q})}{\exp_q(\frac{uq^{-a}}{1-q})} = \frac{(q^{-a}u;q)_\infty}{(u;q)_\infty} \tag{8.8}$$

since the right-hand side of (8.8) satisfies the same difference equation (8.2) under the assumption $|q| < 1$. Clearly, under this assumption, the solution $F_a(u)$ of (8.2) is unique if $F_a(0) = 1$.

Let us note that (8.3) is a direct corollary of (8.8). The relations (8.4) and (8.5) follow from the addition law for the q-exponents ([17])

$$\exp_q(u)\exp_q(v) = \exp_q(u+v) \tag{8.9}$$

and the Faddeev-Volkov identity(see [12]):

$$\exp_q(v)\exp_q(u) = \exp_q(u+v+(q-1)vu). \tag{8.10}$$

□

Remark 8.2. We refer to the identities (8.4) and (8.5) also as to Faddeev-Volkov identities. We will prove below a more general result and get (8.4) and (8.5) as consequences.

Let us consider now the q-power series as a function of a sum of two q-commuting variables u and v, $vu = quv$:

$$F_a(u+v) = (1-u-v)_q^{(a)}. \tag{8.11}$$

Proposition 8.3. *The formal power series $F_a(u+v)$ has the following properties:*

$$(1-q^{-b}v-u)_q^{(a)}(1-v-q^{-a}u)_q^{(b)} = (1-u-v)_q^{(a+b)}, \tag{8.12}$$

$$(1-w(1-q^{-a}v-q^{-1}u)^{-1})_q^{(a)}(1-u-v)_q^{(a)} = (1-u-v-w)_q^{(a)}, \tag{8.13}$$

$$(1-u-v)_q^{(a)}(1-(1-q^{-1}v-q^{-a}u)^{-1}w)_q^{(a)} = (1-u-v-w)_q^{(a)}, \tag{8.14}$$

where $vu = quv$ everywhere and $vw = qwv$, $uw = q^{-1}wu$ in (8.13) and (8.14).

Proof. The proof is based on the following observation:

$$(1-q^a v-u)(1-q^b v-q^{-1}u) = (1-q^b v-u)(1-q^a v-q^{-1}u) \tag{8.15}$$

for q-commuting variables u and v.

In order to prove (8.12), let us note that it is sufficient to show this identity only for positive integers a and b. Indeed, both sides are infinite power series and they are equal for any q-commuting u and v if and only if their coefficients of the ordered monomials are equal. But these coefficients are rational functions of $q^a$and q^b, so if they are equal for all positive integers a and b, then they are equal identically.

From (8.3) we know that for any positive integer n the following relation holds:

$$(1-u)_q^{(n)} = (1-q^{-1}u)(1-q^{-2}u)...(1-q^{-n}u). \tag{8.16}$$

Then, by means of the identity (8.15), we can reorder the factors of the product

$$(1-u-v)_q^{(n)} = (1-q^{-1}u-q^{-1}v)...(1-q^{-n}u-q^{-n}v) \tag{8.17}$$

and get another presentation:

$$\begin{aligned}(1-u-v)_q^{(n)} &= (1-q^{-n}v-q^{-1}u)\\ &\cdot(1-q^{-(n-1)}v-q^{-2}u)...(1-q^{-1}v-q^{-n}u).\end{aligned} \tag{8.18}$$

This implies now the identity (8.12).

Similarly, let us check that (8.13) is satisfied for any positive integer a. Denote the left-hand side of (8.13) by $F_a(u,v,w)$ and the right-hand side by $G_a(u,v,w)$. Clearly, $F_1(u,v,w) = G_1(u,v,w)$. Next, we see from (8.12) that the function $F_n(u,v,w)$ satisfies the recurrence relation

$$\begin{aligned}F_{n+1}(u,v,w) &= (1-w(1-q^{-(n+1)}v-q^{-1}u)^{-1})\cdot\\ &\cdot F_n(u,q^{-1}v,q^{-1}w)(1-q^{-1}v-q^{-n-1}u).\end{aligned} \tag{8.19}$$

So it remains to show that the same recurrence is satisfied by $G_n(u,v,w)$. For this, we note that we can, analogously to (8.18), prove the following identity:

$$\begin{aligned}&(1-q^{-1}v-u-q^{-1}w)_q^{(n)}(1-q^{-1}v-q^{-n-1}u) =\\ &= (1-q^{-n-1}v-q^{-1}u-q^{-2}w)...(1-q^{-2}v-q^{-n}u-q^{-n-1}w)(1-q^{-1}v-q^{-n-1}u).\end{aligned} \tag{8.20}$$

Then one obtains

$$\begin{aligned}&(1-q^{-1}v-u-q^{-1}w)_q^{(n)}(1-q^{-1}v-q^{-n-1}u) =\\ &= (1-q^{-n-1}v-q^{-1}u)(1-q^{-n-2}v-q^{-2}u-q^{-2}w)...\\ &...(1-q^{-1}v-q^{-n-1}u-q^{-n-1}w)\end{aligned} \tag{8.21}$$

and this proves the desired recurrence for $G_n(u,v,w)$. Thus (8.13) holds. □

Remark 8.4. We will call the identities of the above proposition *the generalized Faddeev-Volkov identities.*

Moreover we can obtain a rational degeneration of the generalized Faddeev-Volkov identities by the procedure established in [4].

Set $x = u + \frac{\eta}{q^{-1}-1}v$, $y = v$ and $z = w$. Then the q-commutativity relations $vu = quv$, $vw = qwv$ and $uw = q^{-1}wu$ are transformed respectively into

$$xy - q^{-1}yx = -\eta y^2 \tag{8.22}$$

$$yz = qzy \tag{8.23}$$

$$xz - q^{-1}zx = -\eta(2)_{\bar q}yz. \tag{8.24}$$

Here $\bar q = q^{-1}$. Therefore we can rewrite the equalities (8.12)-(8.14) in the variables x, y and z:

$$(1-x-\eta(c)_{\bar q}y)_q^{(a+b)} = (1-x-\eta(c+b)_{\bar q}y)_q^{(a)}(1-q^{-a}x-q^{-a}\eta(c-a)_{\bar q}y)_q^{(b)},$$

$$\begin{aligned}(1-z(1-\bar q x-\eta\bar q(c+a-1)_{\bar q}y)^{-1})_q^{(a)}(1-x-\eta(c)_{\bar q}y)_q^{(a)} &=\\ = (1-x-\eta(c)_{\bar q}y-z)_q^{(a)},&\end{aligned}$$

$$\begin{aligned}(1-x-\eta(c)_{\bar q}y)_q^{(a)}(1-(1-q^{-a}x-\eta q^{-a}(c-a+1)_{\bar q}y)^{-1}z)_q^{(a)} &=\\ = (1-x-\eta(c)_{\bar q}y-z)_q^{(a)}.&\end{aligned}$$

All these relations make sense for the Yangian limit $q = 1$. In this case, the q-power series becomes the usual geometric series for the power function $(1-x)^a$ considered now as a function of linear combinations of the Yangian variables x and y verifying $[x,y] = -\eta y^2$. We obtain the following

Corollary 8.5. *Suppose* $[x,y] = -\eta y^2$, $[y,z] = 0$ *and* $[x,z] = -2\eta yz$. *Then the following identities hold:*

$$(1-x-\eta cy)^{a+b} = (1-x-\eta(c+b)y)^a(1-x-\eta(c-a)y)^b, \tag{8.25}$$

$$(1-z(1-x-\eta(c+a-1)y)^{-1})^a(1-x-\eta cy)^a = (1-x-\eta cy-z)^a, \tag{8.26}$$

$$(1-x-\eta cy)^a(1-(1-x-\eta(c-a+1)y)^{-1}z)^a = (1-x-\eta cy-z)^a. \tag{8.27}$$

9. Twisting cocycles

In this section we construct a proper quantum twist of the quantum affine algebra $U_q(\widehat{sl_2})$.

Let $e_{\pm\alpha}$, $e_{\pm(\delta-\alpha)}$ and $q^{h_{\pm\alpha}} = q^{\pm h}$ be the generators of $U_q(\widehat{sl_2})$ with zero central charge, satisfying the relations

$$q^h e_{\pm\alpha} q^{-h} = q^{\pm 2} e_{\pm\alpha}, \tag{9.1}$$

$$q^h e_{\pm(\delta-\alpha)} = q^{\mp 2} e_{\pm(\delta-\alpha)}, \tag{9.2}$$

$$[e_\alpha, e_{-\alpha}] = \frac{q^h - q^{-h}}{q-q^{-1}}, \tag{9.3}$$

$$[e_{\delta-\alpha}, e_{-\delta+\alpha}] = \frac{q^{-h} - q^h}{q-q^{-1}}, \tag{9.4}$$

$$[e_{\pm\alpha}, e_{\mp(\delta-\alpha)}] = 0, \tag{9.5}$$

together with q-Serre relations, which we do not use here. The comultiplication is given by the following formulae:

$$\Delta(e_\alpha) = e_\alpha \otimes 1 + q^{-h} \otimes e_\alpha \tag{9.6}$$

$$\Delta(e_{\delta-\alpha}) = e_{\delta-\alpha} \otimes 1 + q^h \otimes e_{\delta-\alpha} \tag{9.7}$$

$$\Delta(e_{-\alpha}) = e_{-\alpha} \otimes q^h + 1 \otimes e_{-\alpha} \tag{9.8}$$

$$\Delta(e_{-\delta+\alpha}) = e_{-\delta+\alpha} \otimes q^{-h} + 1 \otimes e_{-\delta+\alpha}. \tag{9.9}$$

Let a and b be two constants and set

$$u = (2)_{q^2} a e_{\delta-\alpha} \tag{9.10}$$

$$v = (2)_{q^2} b q^{-h} e_{-\alpha}. \tag{9.11}$$

These elements satisfy the relations $vu = q^2 uv$, $uq^h = q^{h+2}u$ and $vq^h = q^{h+2}v$.

Let us also make the following notation: $x_1 = x \otimes 1 \otimes 1$, $x_2 = 1 \otimes x \otimes 1$, $x_3 = 1 \otimes 1 \otimes x$ for any $x \in U_q(\widehat{sl_2})$.

Lemma 9.1. *The following identities hold:*

$$(1 - v_3 - q^{h_2}u_3)_{q^2}^{(\frac{h_2}{2})}(q^{-h_1}v_2 + u_2)^n = \tag{9.12}$$

$$= (q^{-h_1}v_2 + u_2)^n (1 - v_3 - q^{h_2-2n}u_3)_{q^2}^{(\frac{h_2}{2}-n)},$$

$$(q^{-h_1}v_2 + u_2)^n (1 - q^{-h_2}v_3 - q^{h_2-2n}u_3)_{q^2}^{(-n)} = \tag{9.13}$$

$$= ((q^{-h_1}v_2 + u_2)(1 - q^{-h_2}v_3 - q^{h_2-2}u_3)^{-1})^n.$$

Proof. Let us check the first identity. The right-hand side may be written as

$$(q^{-h_1}v_2 + u_2)^n \sum_{m \geq 0} D_{mn}(v_3 + q^{h_2-2n}u_3)^m, \tag{9.14}$$

where $D_{mn} = \frac{(-\frac{h_2}{2}+n)_{q^2} \dots (-\frac{h_2}{2}+n+m-1)_{q^2}}{(m)_{q^2}!}$. Since

$$(q^{-h_1}v_2 + u_2)^n (-\frac{h_2}{2} + n + m)_{q^2} = (-\frac{h_2}{2} + m)_{q^2} (q^{-h_1}v_2 + u_2)^n,$$

we obtain

$$(q^{-h_1}v_2 + u_2)^n (1 - v_3 - q^{h_2-2n}u_3)_{q^2}^{(\frac{h_2}{2}-n)} = \tag{9.15}$$

$$= \sum_{m \geq 0} E_m (q^{-h_1}v_2 + u_2)^n (v_3 + q^{h_2-2n}u_3)^m,$$

where $E_m = \frac{(-\frac{h_2}{2})_{q^2} \dots (-\frac{h_2}{2}+m-1)_{q^2}}{(m)_{q^2}!}$.

On the other hand, one can easily check by induction that

$$(q^{-h_1}v_2 + u_2)^n (v_3 + q^{h_2-2n}u_3)^m = (v_3 + q^{h_2}u_3)^m (q^{-h_1}v_2 + u_2)^n \tag{9.16}$$

and the identity (9.12) follows now directly.

We will prove the second identity. We proceed by induction. The equality is obvious for $n = 1$. We assume that the identity is true for some n. By the first generalized Faddeev-Volkov identity (8.12),

$$(1 - q^{-h_2}v_3 - q^{h_2-2n-2}u_3)_{q^2}^{(-n-1)} = (1 - q^{-h_2+2n}v_3 - q^{h_2-2n-2}u_3)_{q^2}^{(-1)} \cdot$$

$$\cdot (1 - q^{-h_2}v_3 - q^{h_2-2n}u_3)_{q^2}^{(-n)}.$$

The left-hand side of the identity we intend to prove for $n+1$ equals

$$(q^{-h_1}v_2 + u_2)^{n+1} (1 - q^{-h_2+2n}v_3 - q^{h_2-2n-2}u_3)^{-1} \cdot (1 - q^{-h_2}v_3 - q^{h_2-2n}u_3)_{q^2}^{(-n)}.$$

On the other hand, the following equality holds

$$(q^{-h_1}v_2 + u_2)^n (1 - q^{-h_2+2n}v_3 - q^{h_2-2n-2}u_3)^{-1} = \tag{9.17}$$

$$= (1 - q^{-h_2}v_3 - q^{h_2-2}u_3)^{-1} (q^{-h_1}v_2 + u_2)^n.$$

By the inductive assumption and the above identity, if follows that the equality (9.13) is true for $n+1$. This ends the proof. □

Theorem 9.2. *The element*

$$\widetilde{F} = (1 - (2)_{q^2}(a \cdot 1 \otimes e_{\delta-\alpha} + bq^{-h} \otimes q^{-h}e_{-\alpha}))_{q^2}^{(-\frac{h \otimes 1}{2})} \tag{9.18}$$

is a quantum twist of $U_q(\widehat{sl_2})$ for any constants a and b.

Proof. Set $u = (2)_{q^2} a e_{\delta-\alpha}$, $v = (2)_{q^2} b q^{-h} e_{-\alpha}$ and keep the notation as in Lemma 9.1. The cocycle identity

$$\widetilde{F}^{12}(\Delta \otimes id)\widetilde{F} = \widetilde{F}^{23}(id \otimes \Delta)\widetilde{F} \tag{9.19}$$

is equivalent to the following identity:

$$(1 - q^{-h_1}v_2 - u_2)_{q^2}^{(-\frac{h_1}{2})} (1 - q^{-h_1-h_2}v_3 - u_3)_{q^2}^{(-\frac{h_1+h_2}{2})} = \tag{9.20}$$

$$= (1 - q^{-h_2}v_3 - u_3)_{q^2}^{(-\frac{h_2}{2})} (1 - q^{-h_1}v_2 - q^{-h_1-h_2}v_3 - u_2 - q^{h_2}u_3)_{q^2}^{(-\frac{h_1}{2})}.$$

By the first generalized Faddeev-Volkov identity (8.12) we obtain

$$(1 - v_3 - q^{h_2}u_3)_{q^2}^{(\frac{h_2}{2})} (1 - q^{-h_2}v_3 - u_3)_{q^2}^{(-\frac{h_2}{2})} = 1 \tag{9.21}$$

and therefore the following equality remains to be proved

$$(1 - v_3 - q^{h_2}u_3)_{q^2}^{(\frac{h_2}{2})} (1 - q^{-h_1}v_2 - u_2)_{q^2}^{(-\frac{h_1}{2})} \cdot \tag{9.22}$$

$$\cdot (1 - q^{-h_1-h_2}v_3 - u_3)_{q^2}^{(-\frac{h_1+h_2}{2})} = (1 - q^{-h_1}v_2 - q^{-h_1-h_2}v_3 - u_2 - q^{h_2}u_3)_{q^2}^{(-\frac{h_1}{2})}.$$

Let us write the second factor of the left-hand side as a series and permute each term with the factor $(1-v_3-q^{h_2}u_3)_{q^2}^{(\frac{h_2}{2})}$. We have

$$(9.23)\qquad (1-v_3-q^{h_2}u_3)_{q^2}^{(\frac{h_2}{2})}(1-q^{-h_1}v_2-u_2)_{q^2}^{(-\frac{h_1}{2})}=$$

$$=(1-v_3-q^{h_2}u_3)_{q^2}^{(\frac{h_2}{2})}\sum_{n\geq 0}C_n(q^{-h_1}v_2+u_2)^n,$$

where $C_n=\frac{(\frac{h_1}{2})(\frac{h_1}{2}+1)_{q^2}\ldots(\frac{h_1}{2}+n-1)_{q^2}}{(n)_{q^2}!}$. By the first identity of Lemma 9.1, we get

$$(9.24)\qquad (1-v_3-q^{h_2}u_3)_{q^2}^{(\frac{h_2}{2})}(1-q^{-h_1}v_2-u_2)_{q^2}^{(-\frac{h_1}{2})}=$$

$$=\sum_{n\geq 0}C_n(q^{-h_1}v_2+u_2)^n(1-v_3-q^{h_2-2n}u_3)_{q^2}^{(\frac{h_2}{2}-n)}.$$

By using the first generalized Faddeev-Volkov identity (8.12), the above identity is equivalent to

$$(9.25)\qquad (1-v_3-q^{h_2}u_3)_{q^2}^{(\frac{h_2}{2})}(1-q^{-h_1}v_2-u_2)_{q^2}^{(-\frac{h_1}{2})}=$$

$$=\sum_{n\geq 0}C_n(q^{-h_1}v_2+u_2)^n(1-q^{-h_2}v_3-q^{h_2-2n}u_3)_{q^2}^{(-n)}(1-v_3-q^{h_2}u_3)_{q^2}^{(\frac{h_2}{2})}.$$

The second identity of Lemma 9.1 implies that the right-hand side of the above identity is equal to

$$(1-(q^{-h_1}v_2+u_2)(1-q^{-h_2}v_3-q^{h_2-2}u_3)^{-1})_{q^2}^{(-\frac{h_1}{2})}(1-v_3-q^{h_2}u_3)^{(\frac{h_2}{2})}.$$

It results that the desired equality (9.22) is equivalent to

$$(9.26)\qquad (1-(q^{-h_1}v_2+u_2)(1-q^{-h_2}v_3-q^{h_2-2}u_3)^{-1})_{q^2}^{(-\frac{h_1}{2})}(1-v_3-q^{h_2}u_3)^{(\frac{h_2}{2})}.$$

$$\cdot(1-q^{-h_1-h_2}v_3-u_3)_{q^2}^{(-\frac{h_1+h_2}{2})}=(1-q^{-h_1}v_2-q^{-h_1-h_2}v_3-u_2-q^{h_2}u_3)_{q^2}^{(-\frac{h_1}{2})}$$

and moreover to

$$(9.27)\qquad (1-(q^{-h_1}v_2+u_2)(1-q^{-h_2}v_3-q^{h_2-2}u_3)^{-1})_{q^2}^{(-\frac{h_1}{2})}.$$

$$\cdot(1-q^{-h_1-h_2}v_3-q^{h_2}u_3)_{q^2}^{(-\frac{h_1}{2})}=(1-q^{-h_1}v_2-q^{-h_1-h_2}v_3-u_2-q^{h_2}u_3)_{q^2}^{(-\frac{h_1}{2})}.$$

This is exactly the second generalized Faddeev-Volkov identity (8.13) and therefore the cocycle identity is satisfied by $\widetilde{F}$. The proof is now complete. □

Further, as in the previous section, we can make a change of variables (see [25], [26]) such that the twisting element $\widetilde{F}$ has a limit when $q\to 1$. Let us consider the following elements:

$$(9.28)\qquad f_1=e_{\delta-\alpha}+\frac{\eta}{q^{-2}-1}q^{-h}e_{-\alpha},$$

$$(9.29)\qquad f_0=q^{-h}e_{-\alpha}.$$

The elements f_1, f_0 and h generate a Hopf subalgebra of $U_q(\widehat{sl_2})$, considered now as an algebra over $\mathbb{C}[[\eta]](q)$ (see [25], [26]):

$$(9.30)\qquad [h,f_1]=-2f_1$$

$$(9.31)\qquad [h,f_0]=-2f_0$$

$$(9.32)\qquad f_1f_0-q^{-2}f_0f_1=-\eta f_0^2$$

$$(9.33)\qquad \Delta(f_0)=f_0\otimes 1+q^{-h}\otimes f_0$$

$$(9.34)\qquad \Delta(f_1)=f_1\otimes 1+q^h\otimes f_1+\eta q^h(h)_{q^{-2}}\otimes f_0.$$

Corollary 9.3. *Let $a=\xi$ and $b=\frac{\xi\eta}{q^{-2}-1}$. The twisting element $\widetilde{F}$ given by (9.18) has the form*

$$(9.35)\qquad \widetilde{F}=(1-(2)_{q^2}\xi(1\otimes f_1+\eta(\frac{h}{2})_{q^{-2}}\otimes f_0)_{q^2}^{(-\frac{h\otimes 1}{2})}.$$

Moreover, in the Yangian limit $q=1$, $\widetilde{F}$ has the following form:

$$(9.36)\qquad \widetilde{F}=(1-2\xi(1\otimes f_1+\eta\frac{h}{2}\otimes f_0))^{(-\frac{h\otimes 1}{2})}.$$

10. Quantization of quasi-trigonometric and rational solutions

In this section we use the quantum twist $\widetilde{F}$ of the quantum affine algebra $U_q(\widehat{sl_2})$ to construct the corresponding twisted R-matrix. Let $\pi_{1/2}(z)$ be the two-dimensional vector representation of $U_q(\widehat{sl_2})$. In this representation, the generator $e_{-\alpha}$ acts as a matrix unit e_{21}, $e_{\delta-\alpha}$ as ze_{21} and h_α as $e_{11}-e_{22}$. In this section and the following we will also use the notation $\sigma^+=e_{12}$, $\sigma^-=e_{21}$ and $\sigma^z=e_{11}-e_{22}$.

According to [14], the quantum R-matrix of $U_q(\widehat{sl_2})$ in the tensor product $\pi_{1/2}(z_1)\otimes\pi_{1/2}(z_2)$ is the following:

$$(10.1)\qquad R_0(z_1,z_2)=e_{11}\otimes e_{11}+e_{22}\otimes e_{22}+\frac{z_1-z_2}{q^{-1}z_1-qz_2}(e_{11}\otimes e_{22}+e_{22}\otimes e_{11})$$

$$+\frac{q^{-1}-q}{q^{-1}z_1-qz_2}(z_2e_{12}\otimes e_{21}+z_1e_{21}\otimes e_{12}).$$

Let us consider the quantum twist $\widetilde{F}$ constructed in the previous section. We note that the image of $\widetilde{F}$ in the representation $\pi_{1/2}(z)$ of $U_q(\widehat{sl_2})$ is

$$F = 1 + \frac{q^h - 1}{q-1}(az_2 + bq^{-h+1}) \otimes e_{21} \tag{10.2}$$

$$= 1 + ((az_2 + b)e_{11} - (q^{-1}az_2 + qb)e_{22}) \otimes e_{21}.$$

As a consequence we obtain the following

Proposition 10.1. *In the above representation, the quantum R-matrix of the twisted quantum affine algebra by $\widetilde{F}$ is the following:*

$$R^F(z_1, z_2) = F^{21}R_0F^{-1} = R_0(z_1, z_2) + \frac{z_1 - z_2}{q^{-1}z_1 - qz_2}((b + az_2)\sigma^z \otimes \sigma^- \tag{10.3}$$

$$+(q^{-1}az_1 + qb)\sigma^- \otimes \sigma^z + (b + az_2)(q^{-1}az_1 + qb)\sigma^- \otimes \sigma^-).$$

Corollary 10.2. *The R-matrix given by (10.3) is a quantization of the following quasi-trigonometric solution of the CYBE:*

$$X_{a,b}(z_1, z_2) = \frac{z_2\Omega}{z_1 - z_2} + \sigma^- \otimes \sigma^+ + \frac{1}{4}\sigma^z \otimes \sigma^z \tag{10.4}$$

$$+a(z_1\sigma^- \otimes \sigma^z - z_2\sigma^z \otimes \sigma^-) + b(\sigma^- \otimes \sigma^z - \sigma^z \otimes \sigma^-).$$

Remark 10.3. This is exactly the solution we obtained in the end of Section 5, given by (5.22).

In order to get a deformation of Yang's R-matrix, we use again the realization (9.28) -(9.29) of $U_q(\widehat{sl_2})$ and the evaluation homomorphism

$$\pi_{1/2}(u)(f_1) = (u + \eta(\frac{h}{2})_{q^2})q^h f_0 \tag{10.5}$$

$$\pi_{1/2}(u)(f_0) = q\sigma^-, \tag{10.6}$$

which corresponds to a shift of the spectral parameter

$$z = u - \eta/(q^{-2} - 1). \tag{10.7}$$

In this notation, the non-twisted R-matrix $R_0(u_1, u_2)$ has the following form:

$$R_0(u_1, u_2) = \frac{1}{2}(1 + \sigma^z \otimes \sigma^z) + \frac{u_1 - u_2}{2(q^{-1}u_1 - qu_2 - q\eta)}(1 - \sigma^z \otimes \sigma^z) \tag{10.8}$$

$$+\frac{(q^{-1} - q)u_2 - q\eta}{q^{-1}u_1 - qu_2 - q\eta}\sigma^+ \otimes \sigma^- + \frac{(q^{-1} - q)u_1 - q\eta}{q^{-1}u_1 - qu_2 - q\eta}\sigma^- \otimes \sigma^+.$$

As a consequence one obtains the following

Proposition 10.4. *With the above realization of $U_q(\widehat{sl_2})$, the twisted R-matrix given by (10.3) has the form*

$$R^F(u_1, u_2) = R_0(u_1, u_2) + \frac{u_1 - u_2}{q^{-1}u_1 - qu_2 - q\eta}(-\xi u_2\sigma^z \otimes \sigma^- + \tag{10.9}$$

$$+\xi(q^{-1}u_1 - q\eta)\sigma^- \otimes \sigma^z + \xi^2 u_2(q^{-1}u_1 - q\eta)\sigma^- \otimes \sigma^-).$$

Moreover, in the Yangian limit $q = 1$, a deformation of Yang's R-matrix is obtained:

$$R^F(u_1, u_2) = \frac{u_1 - u_2}{u_1 - u_2 - \eta}(1 - \eta\frac{P_{12}}{u_1 - u_2} - \xi u_2\sigma^z \otimes \sigma^- \tag{10.10}$$

$$+\xi(u_1 - \eta)\sigma^- \otimes \sigma^z + \xi^2 u_2(u_1 - \eta)\sigma^z \otimes \sigma^z).$$

Here P_{12} denotes the permutation of factors in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Corollary 10.5. *The rational degeneration $R^F(u_1, u_2)$ given above is a quantization of the following rational solution of the CYBE:*

$$r(u_1, u_2) = \frac{\Omega}{u_1 - u_2} + \xi(u_1\sigma^- \otimes \sigma^z - u_2\sigma^z \otimes \sigma^-). \tag{10.11}$$

Thus we answer the question of quantization of all the rational solutions of the CYBE for sl_2 (see also [16]).

11. Deformed Hamiltonians

In this section we compute the Hamiltonians of the periodic chains related to the twisted R-matrix we found in the previous section. Let us firstly notice that $R^F(z_1, z_2)$ satisfies the basic property $R^F(z, z) = P_{12}$.

Let us compute the deformed Hamiltonians. We consider

$$t(z) = Tr_0 R_{0N}(z, z_2)R_{0N-1}(z, z_2)...R_{01}(z, z_2) \tag{11.1}$$

a family of commuting transfer matrices for the corresponding homogeneous periodic chain, $[t(z'), t(z")] = 0$, where we treat z_2 as a parameter of the theory and $z = z_1$ as a spectral parameter. Then the Hamiltonian

$$H_{a,b,z_2} = (q^{-1} - q)z\frac{d}{dz}t(z)\mid_{z=z_2} t^{-1}(z_2) \tag{11.2}$$

can be computed by a standard procedure:

$$H_{a,b,z_2} = H_{XXZ} + \sum_k (C(\sigma_k^z\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^z) + D\sigma_k^-\sigma_{k+1}^-. \tag{11.3}$$

Here $C = ((q-1)/2)(b - az_2q^{-1})$, $D = (az_2 + b)(q^{-1}az_2 + qb)$, $\sigma^+ = e_{12}$, $\sigma^- = e_{21}$, $\sigma^z = e_{11} - e_{22}$ and

$$H_{XXZ} = \sum_k (\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+ + \frac{q + q^{-1}}{2}\sigma_k^z\sigma_{k+1}^z). \tag{11.4}$$

We see that, by a suitable choice of parameters a, b and z_2, we can add to the XXZ Hamiltonian an arbitrary linear combination of the terms $\sum_k \sigma_k^z\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^z$ and $\sum_k \sigma_k^-\sigma_{k+1}^-$ and the model will remain integrable.

In the realization (9.28)-(9.29) of $U_q(\widehat{sl_2})$, the R-matrix $R^F(u_1,u_2)$ satisfies again the property $R^F(u,u) = P_{12}$ and the Hamiltonian

$$H_{\eta,\xi,u_2} = ((q^{-1}-q)u - q^{-1}\eta)\frac{d}{du}t(u)\mid_{u=u_2} t^{-1}(u_2) \tag{11.5}$$

for

$$t(u) = Tr_0 R_{0N}(u,u_2)R_{0N-1}(u,u_2)...R_{01}(u,u_2), \tag{11.6}$$

is given by the same formula (11.2), where $C = \xi((q^{-1}-1)/2)u_2 - (q^{-1}\xi\eta)/2$ and $D = \xi^2 u_2(q^{-1}u_2 - q\eta)$. Now it also makes sense in the XXX limit $q=1$:

$$H_{\eta,\xi,u_2} = H_{XXX} + \sum_k (C(\sigma_k^z\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^z) + D\sigma_k^-\sigma_{k+1}^-), \tag{11.7}$$

where $C = -\xi\eta/2$ and $D = \xi^2 u_2(u_2-\eta)$.

Remark 11.1. It would be interesting to study the spectra and the eigenstates of the above Hamiltonians. The particular case of (11.7) with $C=0$ was studied in [19]. The study was based on a quantization of a simpler r-matrix suggested in [16]. It was shown that in this case the spectrum of the Hamiltonian remains unchanged after the deformation. However, the deformed Hamiltonian has Jordanian blocks and thus it is not diagonalizable. Therefore one can expect that at least the deformed XXZ chains (11.7) are not equivalent to the undeformed one.

12. Appendix

In this appendix we give the proofs of the following results mentioned in the text:

Proposition 12.1. *Let X be a rational or quasi-trigonometric solution of (2.1). Then X satisfies the unitarity condition (2.2).*

Proof. The proof is almost a word to word transcription of the proof of [2, Prop. 4.1]. Interchanging z_1 and z_2 and also the first and second factors in $\mathbf{g}^{\otimes 3}$ in equation (2.1), we obtain

$$\begin{aligned}[X^{21}(z_2,z_1), X^{23}(z_2,z_3)] + [X^{21}(z_2,z_1), X^{13}(z_1,z_3)] + \\ +[X^{23}(z_2,z_3), X^{13}(z_1,z_3)] = 0.\end{aligned} \tag{12.1}$$

Adding (12.1) and (2.1), we get

$$[X^{12}(z_1,z_2) + X^{21}(z_2,z_1), X^{13}(z_1,z_3) + X^{23}(z_2,z_3)] = 0. \tag{12.2}$$

a) Suppose X is rational, i.e. $X(z,t) = \frac{\Omega}{z-t} + p(z,t)$, where p is a polynomial. For $z_1$and z_2 fixed, let us multiply (12.2) by $z_2 - z_3$ and let $z_3 \to z_2$. It follows that

$$[X^{12}(z_1,z_2) + X^{21}(z_2,z_1), \Omega^{23}] = 0. \tag{12.3}$$

It is known that if a tensor $r \in \mathbf{g}\otimes\mathbf{g}$ satisfies $[r\otimes 1, \Omega^{23}] = 0$, then $r=0$. It follows that $X^{12}(z_1,z_2) + X^{21}(z_2,z_1) = 0$.

b) Suppose X is quasi-trigonometric, i.e. $X(z,t) = \frac{t\Omega}{z-t} + q(z,t)$ where q is a polynomial function. By the same procedure we get

$$[X^{12}(z_1,z_2) + X^{21}(z_2,z_1), z_2\Omega^{23}] = 0 \tag{12.4}$$

which also implies the unitarity condition. □

Proposition 12.2. *Let W be a Lie subalgebra satisfying conditions 2) and 3) of Theorem 3.6. Let $\widetilde{r}$ be constructed as in (3.17). Assume $\widetilde{r}$ induces a Lie bialgebra structure on $\mathbf{g}[z]$ by $\delta_{\widetilde{r}}(a(z)) = [\widetilde{r}(z,t), a(z)\otimes 1 + 1\otimes a(t)]$. Then $W \supseteq u^{-N}\mathbf{g}[[u^{-1}]]$ for some positive N.*

Proof. Since W is Lagrangian subalgebra, it is enough to prove that W is bounded. Let us write

$$\widetilde{r}(z,t) = X_0(z,t) + \sum_m \Gamma_m \tag{12.5}$$

where Γ_m is the homogeneous polynomial of degree m with coefficients in $\mathbf{g}\otimes\mathbf{g}$:

$$\Gamma_m = \sum_{n+k=m} a_{mnk} z^n t^k. \tag{12.6}$$

It is enough to prove that there exists a positive integer N such that $\Gamma_m = 0$ for $m \geq N$.

We know that $\delta_{\widetilde{r}}(a)$ should belong to $\mathbf{g}[z]\otimes\mathbf{g}[t]$ for any element a of $\mathbf{g}$. On the other hand, one can see that $[\Gamma_m, a\otimes 1 + 1\otimes a]$ is either 0 or has degree m. This implies that $[\Gamma_m, a\otimes 1 + 1\otimes a] = 0$ for m large enough. Therefore

$$\Gamma_m = P_m(z,t)\Omega \tag{12.7}$$

with $P_m(z,t) \in \mathbb{C}[[z,t]]$. Let us compute the following:

$$\begin{aligned}[\Gamma_m, az\otimes 1 + 1\otimes at] &= P_m(z,t)(z-t)[\Omega, a\otimes 1] + P_m(z,t)t[\Omega, a\otimes 1 + 1\otimes a] \\ &= P_m(z,t)(z-t)[\Omega, a\otimes 1].\end{aligned}$$

We choose an element a such that $[\Omega, a\otimes 1] \neq 0$. We obtain that if $P_m(z,t)$ is not identically zero then $P_m(z,t)(z-t)[\Omega, a\otimes 1]$ is a homogeneous polynomial of degree $m+1$. Consequently,

$$\delta_{\widetilde{r}}(az) = \sum_m P_m(z,t)(z-t)[\Omega, a\otimes 1] \tag{12.8}$$

cannot belong to $\mathbf{g}[z]\otimes\mathbf{g}[t]$ unless $P_m(z,t) = 0$ for m large enough. □

Theorem 12.3. *Let $X(z_1,z_2)$ be a quasi-trigonometric solution of the CYBE. There exists a holomorphic transformation and a change of variables such that $X(z_1,z_2)$ becomes a trigonometric solution, in the sense of Belavin-Drinfeld classification.*

Proof. The proof follows the ideas of [3]. Let us consider $X(z_1,z_2) = \frac{z_2\Omega}{z_1-z_2} + p(z_1,z_2)$, where p is a polynomial. After applying the change of variables $z_1 = e^u$, $z_2 = e^v$ we obtain the function

$$\widetilde{X}(u,v) = \frac{\Omega}{e^{u-v}-1} + p(e^u, e^v) \tag{12.9}$$

which satisfies the equation

$$[\widetilde{X}^{12}(u_1,u_2),\widetilde{X}^{13}(u_1,u_3)]+[\widetilde{X}^{12}(u_1,u_2),\widetilde{X}^{23}(u_2,u_3]+ \\ +[\widetilde{X}^{13}(u_1,u_3),\widetilde{X}^{23}(u_2,u_3)]=0. \tag{12.10}$$

Let us decompose the solution $\widetilde{X}$ as

$$\widetilde{X}(u,v)=\frac{\Omega}{u-v}+g(u,v), \tag{12.11}$$

where

$$\begin{aligned} g(u,v) &= \frac{\Omega}{e^{u-v}-1}-\frac{\Omega}{u-v}+p(e^u,e^v) \\ &= \sum_{n=1}^{\infty}\frac{B_n(u-v)^{2n-1}}{(-1)^{n-1}(2n)!}\Omega+p(e^u,e^v) \end{aligned} \tag{12.12}$$

is a holomorphic function in a neighbourhood of $(0,0)$ (here B_n denote the Bernoulli numbers).

Let $\{I_i\}$ be an orthonormal basis in $\mathbf{g}$ with respect to the Killing form and $\{c_{ij}^k\}$ denote the structure constants with respect to $\{I_i\}$. Let us write

$$g(u,v)=\sum_{i,j}g^{ij}(u,v)I_i\otimes I_j.$$

Following the method from [3], we construct the function

$$h(u):=\sum_{i,j}g^{ij}(u,u)[I_i,I_j].$$

We notice that the term $\frac{\Omega}{e^{u-v}-1}-\frac{\Omega}{u-v}$ has no contribution to $h(u)$. In fact, the only contribution comes from $p(e^u,e^v)$. If we put

$$p(e^u,e^v)=\sum_{i,j}a^{ij}(u,v)I_i\otimes I_j, \tag{12.13}$$

then

$$h(u)=\sum_{i,j}a^{ij}(u,u)[I_i,I_j]=\sum_{i,j,k}a^{ij}(u,u)c_{ij}^kI_k. \tag{12.14}$$

Suppose $\Psi(u)$ is a function with values in $Aut(\mathbf{g})$ which satisfies the differential equation

$$\frac{d\Psi(u)}{du}=(\mathbf{ad}h(u))\cdot\Psi(u). \tag{12.15}$$

According to [3], the function $Y(u,v)$ defined as

$$Y(u,v)=(\Psi(u)^{-1}\otimes\Psi(v)^{-1})\widetilde{X}(u,v) \tag{12.16}$$

has the property

$$\frac{\partial Y(u,v)}{\partial u}+\frac{\partial Y(u,v)}{\partial v}=0 \tag{12.17}$$

and therefore $Y(u,v)$ is a function of $u-v$.

Let us prove that $Y(u,v)$ is a trigonometric solution, in the sense of Belavin-Drinfeld classification. Obviously $Y(u,v)$ satisfies the CYBE since $\widetilde{X}$ has this property. We have to check that the set of poles is a 1-dimensional lattice.

It is enough to show that the equation (12.15) admits a holomorphic solution $\Psi(u)$ defined on the entire complex plane. Let $U(u)$ be the matrix of the unknown operator $\Psi(u)$ in the basis $\{I_i\}$. Equation (12.15) is equivalent to

$$\frac{dU(u)}{du}=H(u)U(u), \tag{12.18}$$

where $H(u)$ is the matrix with elements

$$h_{ij}(u)=\sum_{s,r,t}c_{sj}^ic_{rt}^sa^{rt}(u,u). \tag{12.19}$$

Since the matrix function $H(u)$ is holomorphic in $\mathbb{C}$, the matrix equation (12.18) admits a unique solution satisfying $U(0)=E$. This solution is holomorphic in $\mathbb{C}$ because $U(u)=P\exp(\int_0^u H(v)dv)$ (ordered exponential) and

$$\begin{aligned} \|U(u)\| &= \left\|1+\int_0^u H(v)dv+\int_0^u(\int_0^{v_1}H(v_1)H(v_2)dv_2)dv_1+...\right\| \leq \\ &\leq \exp(\int_0^u \|H(v)\|\,dv. \end{aligned}$$

Moreover, according to [3], the linear operator $\Psi(u)$, corresponding to $U(u)$, is an automorphism of $\mathbf{g}$. This ends the proof. □

Remark 12.4. The following converse question arises: can trigonometric solutions $X(u)$ from the Belavin-Drinfeld list be changed into quasi-trigonometric solutions by setting $e^u=\frac{z}{t}$ and applying some non-holomorphic transformation? For sl_2 and sl_3 we found positive answers.

(i) For sl_2 case, we have two nonequivalent trigonometric solutions [2, p. 172]:

$$\begin{aligned} X_0^{BD}(u) &= \frac{e^u+1}{4(e^u-1)}(e_{11}-e_{22})\otimes(e_{11}-e_{22})+ \\ &+\frac{e_{12}\otimes e_{21}+e_{21}\otimes e_{12}}{e^{u/2}-e^{-u/2}}, \end{aligned} \tag{12.20}$$

$$X_1^{BD}(u)=X_1(u)+(e^{-u/2}-e^{u/2})(e_{12}\otimes e_{12}). \tag{12.21}$$

Let $X_0^{BD}(z,t)$ and $X_1^{BD}(z,t)$ be obtained from the above solutions by making the substitution $e^u=\frac{z}{t}$. Let $\varphi:\mathbb{C}\to Aut(\mathbf{g})$ be defined by $\varphi(z)=\mathbf{Ad}(T(z))$, where $T(z)=diag(1,z^{-1/2})$. Then

$$X_0(z,t):=(\varphi(z)\otimes\varphi(t))X_0^{BD}(z,t), \tag{12.22}$$

$$X_1(z,t):=(\varphi(z)\otimes\varphi(t))X_1^{BD}(z,t) \tag{12.23}$$

are the two quasi-trigonometric solutions for sl_2 given respectively by (5.1) and (5.2).

(ii) Let us consider now the sl_3 case, where we have four nonequivalent trigonometric solutions, according to [2, p. 173]. Three of them correspond to one Coxeter automorphism denoted by C_1 and one corresponds to the second automorphism C_2.

By setting $e^u = \frac{z}{t}$ and applying the singular transformation $\varphi(z) = \mathbf{Ad}(T(z))$ with $T(z) = diag(1, z^{-1/3}, z^{-2/3})$, all three solutions corresponding to C_1 lead to quasi-trigonometric solutions. One can check that cases a) and b) of [2, p. 173] provide two trivial quasi-trigonometric solutions:

$$X_1(z,t) = \frac{t\Omega}{z-t} + r_0 + \sum_{i,j=1}^{3} \sigma_{ij} e_{ii} \otimes e_{jj}, \tag{12.24}$$

with $(\sigma_{ij}) = \begin{pmatrix} 0 & a & -a \\ -a & 0 & a \\ a & -a & 0 \end{pmatrix}$, $a \in \mathbb{C}$ and r_0 is given by (6.25);

$$X_2(z,t) = X_1(z,t) + a(e_{11} - e_{22}) \wedge (e_{22} - e_{33}) + e_{12} \wedge e_{32}. \tag{12.25}$$

Case c) of [2, p. 173] gives a nontrivial quasi-trigonometric solution:

$$\begin{aligned} X_3(z,t) = X_2(z,t) - z(e_{13} \otimes (e_{12} + e_{23}) + e_{12} \otimes e_{13}) + \\ +t((e_{12} + e_{23}) \otimes e_{13} + e_{13} \otimes e_{12}). \end{aligned} \tag{12.26}$$

We conjecture that the remaining solution which corresponds to the Coxeter automorphism C_2 also provides a nontrivial quasi-trigonometric solution by applying a singular transformation.

In this way, for sl_2 and sl_3 we obtain a one-to-one correspondence between trigonometric solution of the Belavin-Drinfeld list [2] and quasi-trigonometric solutions.

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References

[1] Belavin A.A., Drinfeld V.G., *Triangle equation and simple Lie algebras*, Soviet Sci. reviews, Section C **4** (1984) 93-165.
[2] Belavin A.A., Drinfeld V.G., *On classical Yang-Baxter equation for simple Lie algebras*, Funct. An. Appl., **16** (1982) 1-29.
[3] Belavin A.A., Drinfeld V.G., *On classical Yang-Baxter equations for simple Lie algebras.* Funct. Anal. and Appl. **17** no. 3 (1983) 69-70.
[4] Benaoum H.B., *(q,h)-analogue of Newton's binomial formula*, J.Phys. A **32** (1999), 2037-2040.
[5] Drinfeld V.G., *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations.* Soviet Math. Dokl. **27** (1983) 68-71.
[6] Drinfeld V.G., *Quantum groups*, Proceedings ICM (Berkeley 1986) **1** (1987), AMS 798-820.
[7] Drinfeld V.G., *Quasi-Hopf algebras*, Leningrad Math. Journal **1** (1990) 1419-1457.
[8] Drinfeld V.G., *On constant, quasiclassical solutions of the Yang-Baxter quantum equation*, Soviet Math. Dokl. **28** (1983), no. 3, 667-671.
[9] Etingof P., Kazhdan, D. *Quantization of Lie bialgebras I.* Selecta Math. **2**, no.1 (1986) 1-41.
[10] Etingof P., Schiffmann O., *Lectures on Quantum Groups*, International Press (1998).
[11] Etingof P., Schedler T., Schiffmann O., *Explicit quantization of dynamical R-matrices for finite-dimensional semisimple Lie algebras*, J. AMS **13** (2000) 595-609.

[12] Faddeev L.D., Volkov A.Yu., *Abelian current algebra of the Virasoro algebra and gauged WZW models*, Phys. Lett. B **315**, no. 3-4 (1993) 311-318.
[13] Gasper G., Rahman M., *Basic Hypergeometric Series*, Cambridge Univ.Press (1990).
[14] Jimbo M., *Quantum R-matrix for the generalized Toda system*, Commun. Math. Phys. **102**, no.1 (1986) 537-547.
[15] Karolinsky E., Stolin A.A., *Classical dynamical r-matrices, Poisson homogeneous spaces and Lagrangian subalgebras*, preprint, Dep. Math. Gothenburg University (2001).
[16] Khoroshkin S.M., Stolin A.A., Tolstoy V.N., *Deformation of the Yangian $Y(sl_2)$*, Commun. Alg. **26**, no.3 (1998) 1041-1055.
[17] Khoroshkin S.M., Tolstoy V.N., *Universal R-matrix for quantized (super) algebras*, Commun. Math. Phys. **141**, (1991) 599-617.
[18] Khoroshkin S.M., Stolin A.A., Tolstoy V.N., *q-Power function over q-commuting variables and deformed XXX and XXZ chains*, Phys. Atomic Nuclei **64**, no.12 (2001) 2173-2178.
[19] Kulish P.P., Stolin A.A., *New rational solutions of Yang-Baxter equation and deformed Yangians*, Czech. J. Phys. **47** (1997) 123-129.
[20] Montaner F., Zelmanov E., *Bialgebra structures on current Lie algebras*, preprint, University of Wisconsin, Madison (1993).
[21] Polishchuk A., *Classical Yang-Baxter equation and the A_∞-constraint.* Adv. Math. **168** (2002) 56-95.
[22] Pop I.I., *Solutions of the classical Yang-Baxter equation and Lagrangian subalgebras*, Licentiate thesis, Dep. Math. Gothenburg University (2002).
[23] Stolin A.A., *On rational solutions of Yang-Baxter equation for $sl(n)$*, Math. Scand. **69** (1991) 57-80.
[24] Stolin A.A., *Generalized trigonometric solutions of the classical Yang-Baxter equation*, in *Proceedings of the XXII International Colloquium on Group Theoretical Methods in Physics*, International Press (1999) 438-442.
[25] Tolstoy V.N., *Drinfeldians*, in *Lie Theory and Its Applications in Physics II, Clausthal*, 1997, World Sci., River Edge, NJ, 1998.
[26] Tolstoy V.N., *From quantum affine Kac-Moody algebras to Drinfeldians and Yangians*, in *Kac-Moody Lie algebras and related topics*, Contemporary Mathematics **343**, AMS, Providence, RI (2004) 349-370.

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Discussion

Finally, let us formulate some open problems that might be of interest for future work:

1. Classification of the nontrivial quasi-trigonometric solutions for sl_n.
As we have seen in paper IV, this problem reduces to the question of finding, for all $k = 1, ..., \left[\frac{n}{2}\right]$, the Lagrangian subalgebras $\overline{W}$ of $sl_n \oplus sl_n$ such that $\overline{W} \oplus \Delta_k = sl_n \oplus sl_n$.
So far we have not been able to find a general method to classify the Lagrangian subalgebras satisfying this condition (unfortunately, the results in [5] do not apply here).

2. Any classical twist can be extended to a quantum twist.
In paper IV we formulated the "refined" Drinfeld conjecture that any classical twist can be quantized. It would be worthwhile to verify it.

3. An explicit quantization of the quasi-trigonometric solutions for sl_3.
In paper IV we presented an explicit quantization of the rational and quasi-trigonometric solutions for sl_2. Unfortunately so far we have not obtained a similar result for sl_3. Hopefully the construction of an appropriate quantum twist could lead to the quantization of both quasi-trigonometric and rational solutions for sl_3.

4. To find nontrivial rational solutions in exceptional Lie algebras.
As we have seen, the theory of rational solutions for a simple complex Lie algebra $\mathbf{g}$ is based on the study of so-called *orders* in $\mathbf{g}((u^{-1}))$. In this theory the central part is played by the maximal orders corresponding to the vertices of the extended Dynkin diagram of $\mathbf{g}$. A vertex of the extended Dynkin diagram is either *singular* (i.e., there exists an automorphism of the extended Dynkin diagram sending this vertex to $-\alpha_{max}$) or *regular* (i.e., there is no such automorphism). In the decomposition of the maximal root, singular vertices appear with coefficient 1 and regular vertices with coefficient > 1.
In paper II of the thesis we found examples of rational solutions for $o(n)$ by investigating orders corresponding to singular vertices.
For exceptional Lie algebras, a way of obtaining rational solutions could be the analysis of orders corresponding to regular vertices with coefficient 3. In [9] the following result was given (we refer to [9] for notation):

Theorem. [9] *Let $\mathbf{g}$ be a simple complex finite-dimensional Lie algebra and α be a simple root whose coefficient in the decomposition of the maximal root with respect to simple ones is 3. Then there is a one-to-one correspondence between isotropic orders $W \subseteq O_\alpha$ such that $W \oplus \mathbf{g}[u] = \mathbf{g}((u^{-1}))$ and the set of pairs (S, B), where:*
(1) S is a subalgebra of $(L_\alpha, V_{\alpha,1})$ such that $S + (P_\alpha^-, P_{\alpha,1}^-) = (L_\alpha, V_{\alpha,1})$.
(2) B is a skewsymmetric bilinear form on S, nondegenerate on $S \cap (P_\alpha^-, P_{\alpha,1}^-)$ and such that

$$B([x,y]_{af}, z) + B([z,x]_{af}, y) + B([y,z]_{af}, x) = ([x,y], z)$$

for any elements x, y, z of S. Here in the expression $([x,y],z)$ we consider x, y, z as elements of $L_\alpha + \varepsilon V_{\alpha,1} + \varepsilon^2 V_{\alpha,2} + \varepsilon^3 L_\alpha$.

For each exceptional Lie algebra $\mathbf{g}$, the data $(L_\alpha, V_{\alpha,1}, P_\alpha^-, P_{\alpha,1}^-)$ has been determined in [9]. It remains to find the rational solutions themselves, in case they exists, i.e., to find pairs (S, B) satisfying the conditions of the above theorem.

5. Rational solutions of CYBE for simple real Lie algebras.
In the first paper of the thesis we presented the theory of rational solutions of CYBE for simple compact Lie algebras. One could develop a theory of rational solutions for simple real Lie algebras also based on the study of orders. The correspondence between solutions and orders is valid so the next task would be to describe the maximal orders in terms of real root system.

6. Poisson homogeneous $G(\mathbb{C}[u])$-spaces and relations with classical dynamical Yang-Baxter equation with spectral parameter.
The notion of Poisson homogeneous space was formulated by Drinfeld as a generalization of the notion of homogeneous spaces to the Poisson-Lie group context (see [1]). He gave a general approach to the classification of Poisson homogeneous spaces showing that if G is a Poisson-Lie group, $\mathbf{g}$ is the corresponding Lie algebra, then Poisson homogeneous G-spaces are esentially in a one-to-one correspondence with G-orbits in the set of Lagrangian subalgebras of the classical double $D(\mathbf{g})$. For the case when $\mathbf{g}$ is a simple complex Lie algebra with strictly quasitriangular structure, the classification of such Lagrangian subalgebras was given in [5].
On the other hand, the classical dynamical Yang-Baxter equation (CDYBE) for a pair $(\mathbf{g}, \mathbf{u})$, consisting of a finite-dimensional complex Lie algebra $\mathbf{g}$ and a subalgebra $\mathbf{u}$, was defined by G. Felder as a generalization of the CYBE (see [4]). The first classification results for the solutions of the CDYBE were obtained by P. Etingof, A. Varchenko and O. Schiffmann, for the pair $(\mathbf{g}, \mathbf{h})$, where $\mathbf{g}$ is a complex simple Lie algebra and $\mathbf{h}$ a Cartan subalgebra (see [2, 3, 8]). In [7] J.-H. Lu found a connection, which is esentially a one-to-one correspondence, between dynamical r-matrices for the pair $(\mathbf{g}, \mathbf{h})$ and Poisson homogeneous G-structures on G/H, where H is the Cartan subgroup with Lie algebra $\mathbf{h}$ and G is equipped with the standard strictly quasitriangular Poisson-Lie structure. This result was generalized by E. Karolinsky and A. Stolin who gave some general conditions under which the Lu correspondence is one-to-one (see [6]).
So far the Lie bialgebra $\mathbf{g}$ has been considered finite-diminesional. It might be worthwhile to discuss the above problems for some infinite-dimensional cases. In the fourth paper of the thesis we discussed Lie bialgebra structures on $\mathbf{g}[u]$ which are induced by rational and quasi-trigonometric solutions of the CYBE. Consequently, $G(\mathbb{C}[u])$ is equipped with a triangular Poisson-Lie structure. In this context, it would be perhaps interesting to study Poisson homogeneous $G(\mathbb{C}[u])$-spaces and to find a relationship with solutions of the CDYBE with spectral parameter.

References

[1] Drinfeld V.G., *On Poisson homogeneous spaces of Poisson-Lie groups*, Theor. Math. Phys., **95** (1993) 226-227.

[2] Etingof P., Varchenko A., *Geometry and classification of solutions of the classical dynamical Yang-Baxter equation*, Commun. Math. Phys., **196** (1998) 591-640.

[3] Etingof P., Schiffmann O., *On the moduli space of classical dynamical r-matrices*, Math. Res. Letters, **8** (2001) 157-170.

[4] Felder G., *Conformal field theory and integrable systems associated to elliptic curves*, in *Proceedings of the International Congress of Mathematicians, Zurich 1994, 1247-1255.*

[5] Karolinsky E., *A classification of Poisson homogeneous spaces of complex reductive Poisson-Lie groups*, in *Poisson geometry (Warsaw, 1998)*, 103-108, Banach Center Publ., **51**, Polish Acad. Sci., Warsaw, 2000.

[6] Karolinsky E., Stolin A., *Classical dynamical r-matrices, Poisson homogeneous spaces, and Lagrangian subalgebras*, Lett. Math. Phys. **60** (2002), no. 3, 257-274.

[7] Lu J.-H., *Classical dynamical r-matrices and homogeneous Poisson structures on G/H and K/T*, Commun. Math. Phys., **212** (2000) 337-370.

[8] Schiffmann O., *On classification of dynamical r-matrices*, Math. Res. Letters, **5** (1998) 13-30.

[9] Stolin A. A., *On rational solutions of Yang-Baxter equation. Maximal orders in loop algebra.* Comm. Math. Phys., **141** (1991) 533-548.