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On residue currents and multivariable operator calculus

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Abstract

We prove that the Coleff-Herrera residue current, corresponding to a pair of holomorphic functions defining a complete intersection, can be obtained as the unrestricted weak limit of a natural smooth $(0,2)$-form depending on two parameters. Moreover, we prove that the rate of convergence i Hölder. This result is in contrast to the fact, first discovered by Passare and Tsikh, that the residue integral in general is discontinuous at the origin. We also generalize our regularization results to pairs of so called Bochner-Martinelli, or more generally, Cauchy-Fantappiè-Leray blocks in the case of a complete intersection.

We generalize the classical Cayley transform to tuples of unbounded operators by using Taylor's analytic functional calculus. We give necessary and sufficient conditions on an n-tuple a of closed unbounded operators in order that a can be transformed to an n-tuple of bounded commuting operators by a projective transformation of $\mathbb{CP}^n$. The components of such tuples need not all have non-empty resolvent sets. The construction gives an analytic functional calculus, supported by a closed subset of $\mathbb{CP}^n$, for each such a. This subset is then a natural candidate for a joint spectrum of a. We provide an integral representation for this functional calculus. We also study "all" tuples of unbounded operators admitting a smooth functional calculus by considering multiplicative operator valued distributions A with an additional property meaning, in a weak sense, that $A(1)$ is the identity operator.

Keywords: residue integral, Coleff-Herrera current, Cauchy-Fantappiè-Leray current, regularization, division problem, Cayley transform, Taylor spectrum, functional calculus, integral representation, projective space

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[i] H. Samuelsson,
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[ii] H. Samuelsson,
Regularizations of products of residue and principal value currents, *preprint CTH and Göteborg University, (2005).*

[iii] H. Samuelsson,
Multidimensional Cayley transforms and tuples of unbounded operators, *J. Operator Theory*, to appear.

[iv] M. Andersson, H. Samuelsson, S. Sandberg,
Operators with smooth functional calculi, *preprint CTH and Göteborg University, (2005).*

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ON RESIDUE CURRENTS AND MULTIVARIABLE OPERATOR CALCULUS
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INTRODUCTION

HÅKAN SAMUELSSON

1. Residue and principal value currents

1.1. **The case of one function.** Let X be a domain in $\mathbb{C}^n$, or more generally, a complex n-dimensional manifold and let $f\colon X \to \mathbb{C}$ be a holomorphic function. Of course, $1/f$ is not a function on X if $V_f := f^{-1}(0)$ is non-empty, but Schwartz found that there is at least a distribution U on X such that $fU = 1$, [21]. Let us consider the elementary case when $X = \mathbb{C}$. The function $z \mapsto 1/z$ is locally integrable in $\mathbb{C}$ and hence $(\partial^k/\partial z^k)(1/z)$ exists in the sense of distributions. We can realize it as the principal value distribution

$$(1)\quad \mathscr{D}(\mathbb{C}) \ni \varphi \mapsto (-1)^k k! \frac{i}{2}\Big[\frac{1}{z^{k+1}}\Big].\varphi := (-1)^k k! \lim_{\epsilon\to 0} \int_{|z|>\epsilon} \frac{\varphi}{z^{k+1}} \frac{idz\wedge d\bar{z}}{2}.$$

In fact, by integrating by parts we see that

$$(-1)^k k! \int_{|z|>\epsilon} \frac{\varphi}{z^{k+1}} \frac{idz\wedge d\bar{z}}{2} = (-1)^k \frac{i}{2} \sum_{j=0}^{k-1} j! \int_{|z|=\epsilon} \frac{1}{z^{j+1}} \frac{\partial^{k-j-1}\varphi}{\partial z^{k-j-1}} d\bar{z} + (-1)^k \int_{|z|>\epsilon} \frac{1}{z} \frac{\partial^k \varphi}{\partial z^k} \frac{idz\wedge d\bar{z}}{2},$$

and by Taylor expanding the functions $(\partial^\ell/\partial z^\ell)\varphi$ to appropriate orders and changing to polar coordinates it is not hard to see that the boundary integrals tend to zero as $\epsilon \to 0$. Now, if $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and not identically zero then f has a discrete zero set and moreover, $f(z) = (z-a)^k \tilde{f}(z)$ where $\tilde{f} \neq 0$ in some neighborhood of $a \in V_f$. Close to a we can thus choose a k:th root, $\tilde{f}^{1/k}$, of $\tilde{f}$ and make the change of variables $\zeta = (z-a)\tilde{f}^{1/k}(z)$. In the new coordinate, f is just ζ^k and so from the existence of this principal value distribution above it follows that one can define $1/f$ as a $(0,0)$-current, denoted $[1/f]$, in $\mathbb{C}$ as

$$\mathscr{D}(\mathbb{C}) \ni \varphi \mapsto \lim_{\epsilon\to 0}\int_{|f|^2>\epsilon} \frac{\varphi}{f}\, dz\wedge d\bar{z}. \tag{2}$$

The $\bar\partial$-image, $\bar\partial[1/f]$, of $[1/f]$ is of particular interest. It is a $(0,1)$-current, called the residue current associated to f, and by integrating by parts its action on a test form $\varphi\, dz$ is given by

$$\lim_{\epsilon\to 0}\int_{|f|^2=\epsilon} \frac{\varphi}{f}\, dz. \tag{3}$$

We note one feature of the residue current. Let (f) be the principal ideal generated by f in $\mathcal{O}(\mathbb{C})$ and assume that $h \in (f)$. Since $h = gf$ it is clear from (3) that h annihilates the residue current $\bar\partial[1/f]$. On the other hand, if h is an entire function such that $h\bar\partial[1/f] = 0$ then $h \in (f)$. This is so because the distribution $h[1/f]$ extends the function h/f across V_f and by hypotheses, $\bar\partial(h[1/f]) = h\bar\partial[1/f] = 0$. By elliptic regularity for the $\bar\partial$-operator it follows that $g = h/f \in \mathcal{O}(\mathbb{C})$ and hence $h \in (f)$. However, the duality

$$h \in (f) \;\Leftrightarrow\; h\,\bar\partial\Big[\frac{1}{f}\Big] = 0 \tag{4}$$

can be seen completely elementary since, by (3), an entire function h annihilates $\bar\partial[1/f]$ if and only if h vanishes where f does and to the same orders.

In higher dimensions several difficulties arise. In particular, the zero set of a holomorphic function f is no longer a discrete set of points. It is an analytic variety of codimension 1 and may have singularities. These can actually be resolved by Hironaka's famous desingularization theorem, [13]. It implies that if $\mathcal{U}$ is a sufficiently small neighborhood of a point $p \in V_f$ then there exists a complex manifold $\tilde{\mathcal{U}}$ and a proper holomorphic map $\Pi\colon \tilde{\mathcal{U}} \to \mathcal{U}$ such that Π i biholomorphic outside the hypersurface $\Pi^{-1}(V_f)$, and such that $\Pi^{-1}(V_f)$ has normal crossings. This means that locally around any point in $\tilde{\mathcal{U}}$ one can find coordinates, ζ, centered at the origin such that $\Pi^* f = \zeta^\alpha$ is a monomial. Using Hironaka's desingularization theorem Herrera and Lieberman proved in [12] that the principal value current $[1/f]$ defined by

$$\mathscr{D}_{n,n}(X) \ni \varphi \mapsto \lim_{\epsilon\to 0}\int_{|f|^2>\epsilon} \varphi/f \tag{5}$$

exists. Here, $\mathscr{D}_{n,n}(X)$ denotes the space of test forms of bidegree (n,n), i.e. the space of smooth compactly supported (n,n)-forms on X. As above, the residue current is the $\bar\partial$-image, $\bar\partial[1/f]$, of $[1/f]$ and its action on a test form $\varphi \in \mathscr{D}_{n,n-1}(X)$ is given by the limit as $\epsilon \to 0$ (along regular values for $|f|^2$) of the residue integral

$$I_f^\varphi(\epsilon) = \int_{|f|^2=\epsilon} \varphi/f. \tag{6}$$

One key fact discovered by Herrera and Lieberman is that if φ has bidegree $(n-1,n)$ then for each k there is a positive δ_k such that $I^\varphi_{f^k}(\epsilon) = \mathcal{O}(\epsilon^{\delta_k})$. This can be used to obtain regularizations of the currents $[1/f]$ and $\bar\partial[1/f]$. In fact, let φ be a test form of bidegree (n,n) and take a smooth function χ on $[0,\infty)$ such that $\chi(0) = 0$ and $\lim_{t\to\infty}\chi(t) = 1$. For some $k \geq 1$ we can write the test form $\varphi = \phi \wedge \partial f/f^k$, where ϕ is a test form of bidegree $(n-1,n)$ whose restriction to $|f|^2 = t$ is unique, for each $t > 0$. By Fubini's theorem we get

$$\begin{aligned}(7)\ \int \chi(|f|^2/\epsilon)\varphi/f &= \int \chi(|f|^2/\epsilon)\phi \wedge \partial f/f^{k+1} \\ &= \int \chi(|f|^2/\epsilon)\phi/f^k \wedge d|f|^2/|f|^2 = \int_0^\infty \chi(t/\epsilon)\Big(\int_{|f|^2=t} \phi/f^k\Big)dt/t.\end{aligned}$$

Herrera's and Lieberman's result now implies that we may pass to the limit in the last integral and we see that $\int \chi(|f|^2/\epsilon)\varphi/f \to \int_0^\infty \big(\int_{|f|^2=t} \phi/f^k\big)dt/t$ as $\epsilon \to 0^+$. This last expression can, by Fubini's theorem again, be seen to be equal to the value at zero of $\lambda \mapsto \int |f|^{2\lambda}\varphi/f$, which is the Mellin transform of the integral in (5). This is an alternative way of computing $[1/f].\varphi$, see e.g. [3], and hence, $\chi(|f|^2/\epsilon)/f \to [1/f]$ weakly. Since differentiation is a continuous operation on distributions it follows that $\bar\partial\chi(|f|^2/\epsilon)/f \to \bar\partial[1/f]$. With the natural choice $\chi(t) = t/(t+1)$ we thus get the well-known result that $\bar\partial[1/f]$ can be obtained as the weak limit of $\bar\partial(\bar f/(|f|^2+\epsilon))$. These regularization results are also consequences of Corollaries 4 and 5 in [ii]. For more historical accounts, see the survey article [7] by Björk.

1.2. **The case of two functions.** Let f and g be two holomorphic functions $X \to \mathbb{C}$ such that f and g define a complete intersection, i.e. the common zero set $V_{(f,g)}$ has codimension two. In order to generalize the duality (4), we will try to define the exterior product of the residue currents $\bar\partial[1/f]$ and $\bar\partial[1/g]$. In view of the previous section we consider the residue integral

$$I^\varphi_{f,g}(\epsilon_1,\epsilon_2) = \int_{\substack{|f|^2=\epsilon_1\\|g|^2=\epsilon_2}} \frac{\varphi}{fg}. \tag{8}$$

However, the unrestricted limit as $\epsilon_1,\epsilon_2 \to 0$ of the residue integral does not exist in general. This was first discovered by Passare and Tsikh in

[17], and Björk later found that this indeed is the typical case, [6]. See also [19]. We remark that if f and g do not define a complete intersection then it is easy to see that the residue integral is discontinuous at the origin. One can simply take $f = z_1$ and $g = z_1 z_2$, see [17]. Via Hironaka's desingularization theorem one may assume that the hypersurface $f \cdot g = 0$ has normal crossings, which means that there is a (finite) atlas of charts such that $f(\zeta) = \tilde{f}(\zeta)\zeta^\alpha$ and $g(\zeta) = \tilde{g}(\zeta)\zeta^\beta$ where α and β are multiindices (depending on the chart) and $\tilde{f}$ and $\tilde{g}$ are invertible holomorphic functions. It is actually the invertible factors which cause problems. We can always dispose of one of them by incorporating it in a coordinate, but in general we can not dispose of both. However, if the matrix A, whose two rows are the integer vectors α and β respectively, has rank two there is a change of variables $z = \tau(\zeta)$ such that $z^\alpha = \tilde{f}(\zeta)\zeta^\alpha$ and $z^\beta = \tilde{g}(\zeta)\zeta^\beta$, see e.g. [15]. Hence, when α and β are not linearly dependent we can make both the invertible factors disappear. Problems therefore arise in so called charts of resonance where α and β are linearly dependent. Coleff and Herrera realized that if one demands that ϵ_1 and ϵ_2 tend to zero in such a way that $\epsilon_1/\epsilon_2^k \to 0$ for all $k \in \mathbb{Z}_+$, along a so called admissible path, then one will get no contributions from the charts of resonance because one cannot have $|\tilde{f}(\zeta)\zeta^\alpha| << |\tilde{g}(\zeta)\zeta^\beta|$ if α and β are linearly dependent. They proved in [8] that the limit, along an admissible path, of the residue integral exists and defines the action of a $(0,2)$-current, the Coleff-Herrera residue current $[\bar\partial(1/f) \wedge \bar\partial(1/g)]$. This current actually has the property that its annihilator ideal equals the ideal generated by f and g. In [15] Passare smoothen the integration over the set $\{|f|^2 = \epsilon_1\} \cap \{|g|^2 = \epsilon_2\}$ by introducing functions χ as described in the previous section, and he studies possible weak limits of forms

$$\frac{\bar\partial\chi_1(|f|^2/\epsilon_1)}{f} \wedge \frac{\bar\partial\chi_2(|g|^2/\epsilon_2)}{g} \tag{9}$$

along parabolic paths $(\epsilon_1, \epsilon_2) = (\epsilon^{s_1}, \epsilon^{s_2})$, where $s = (s_1, s_2)$ belongs to the simplex $\Sigma_2(2) = \{(x, y) \in \mathbb{R}_+^2;\, s_1 + s_2 = 2\}$. He found that it is enough to impose finitely many linear conditions $(n_j, s) \neq 0$ to assure that (9) has a weak limit along the corresponding parabolic path. The linear conditions partition $\Sigma_2(2)$ into finitely many open segments and the weak limit of (9) along a parabolic path corresponding to an s in such a segment only depends on the segment. We say that (ϵ_1, ϵ_2) tends to zero inside a Passare sector. Moreover, as we assume that f and g define a complete intersection, the limit is even independent of the choice of segment. In this case it also coincides with the Coleff-Herrera current. One can obtain a $\bar\partial$-potential to the Coleff-Herrera current e.g. by changing the integration set in (8) to $\{|f|^2 > \epsilon_1\} \cap \{|g|^2 = \epsilon_2\}$ and pass to the limit along an admissible path or by removing the first $\bar\partial$ in (9) and pass to the limit inside a Passare sector. This $\bar\partial$-potential is denoted $[(1/f)\bar\partial(1/g)]$. The main result in [ii], Theorem 1.3 below, implies the following result.

Theorem 1.1. *Let $\chi_j \in C^\infty([0,\infty])$, $j = 1,2$, satisfy $\chi_j(0) = 0$ and $\chi_j(\infty) = 1$, then, in the sense of currents*

$$\lim_{\epsilon_1,\epsilon_2\to 0} \frac{\chi_1(|f|^2/\epsilon_1)}{f}\frac{\bar\partial\chi_2(|g|^2/\epsilon_2)}{g} = \Big[\frac{1}{f}\bar\partial\frac{1}{g}\Big].$$

Moreover, the integral of the smooth form on the left hand side against a test form is Hölder continuous for $(\epsilon_1,\epsilon_2) \in [0,\infty)^2$.

Taking $\bar\partial$ this implies that (9) tends unrestrictedly to the Coleff-Herrera current. For the particular case when $\chi_j(t) = t/(t+1)$ our result, apart from the Hölder continuity, was announced in [i]. Actually, it is possible to relax the smoothness assumption on one of the χ_j in Theorem 1.1. This is so because, as mentioned above, we can always arrange so that one of the invertible factors, say $\tilde f$, is trivial. Then, examining the proof one sees that one may take χ_1 to be the characteristic function of $[1,\infty]$, and hence,

$$\int_{|f|^2>\epsilon_1} \frac{\bar\partial\chi(|g|^2/\epsilon_2)}{fg}\wedge\varphi \to \Big[\frac{1}{f}\bar\partial\frac{1}{g}\Big].\varphi$$

unrestrictedly as $(\epsilon_1,\epsilon_2)\to 0$. It is worth noticing that if both χ_1 and χ_2 are the characteristic function of $[1,\infty]$ then the result is no longer true in view of the examples of Passare-Tsikh and Björk.

1.3. **Currents of the Bochner-Martinelli and Cauchy-Fantappiè-Leray type.** We now consider an m-tuple $f = (f_1,\dots,f_m)$ of holomorphic functions on X. The residue integral corresponding to f, $I_f^\varphi(\epsilon_1,\dots,\epsilon_m)$, is defined analogously to (8). If f defines a complete intersection, which now means that the common zero set, V_f, has codimension m, there is a well defined way of associating a residue current to f by letting $\epsilon = (\epsilon_1,\dots,\epsilon_m)$ tend to zero along an admissible path in the residue integral, see [8]. If f does not define a complete intersection one can proceed as in [18] and consider the mean value of the residue integral over $\epsilon \in \Sigma_m(\delta) := \{s \in \mathbb{R}^m_+;\ \sum s_j = \delta\}$. One then obtains

$$c_m\int_{|f|^2=\delta}\frac{\sum_{j=1}^m(-1)^{j+1}\bar f_j\bigwedge_{i\neq j}\bar\partial\bar f_i}{|f|^{2m}}\wedge\varphi, \tag{10}$$

where c_m is a constant only depending on m. Passare, Tsikh, and Yger prove in [18] that the limit of (10) as $\delta\to 0^+$ exists and defines the action of a $(0,m)$-current, which, in the case f defines a complete intersection, coincides with the Coleff-Herrera current, and also with the currents studied in [5] and [16]. We also mention that Passare proposed a way of associating a residue current to a non-complete intersection by taking the average of the currents corresponding to the various Passare sectors, see [15]. Based on the work in [18], Andersson introduces more general currents of the Cauchy-Fantappiè-Leray type in [3]. In short, to a holomorphic section f of some complex m-bundle over X he associates a singular form $u^f = \sum u^f_{k,k-1}$, where the terms $u^f_{k,k-1}$ are similar to the term in (10), and he shows that it is extendible to X as a current U^f, either as a principle value or by analytic continuation. The residue current, R^f, is derived from the current U^f and equals the Coleff-Herrera current locally if f defines a complete intersection. If f is a section of a line bundle, then, locally, we can write $f = f_e e$, where f_e is a holomorphic function and e is a local holomorphic frame of the bundle. We point out that, in this case, $U^f = [1/f_e]e$ and $R^f = \bar{\partial}[1/f_e] \wedge e$. We prove the following result in [ii].

Proposition 1.2. *Let f be a holomorphic section of an m-bundle over X and assume that $\chi \in C^\infty([0,\infty])$ vanishes to order $min(m,n)+1$ at zero and satisfies $\chi(\infty) = 1$. Then for any test form φ we have*

$$\lim_{\epsilon \to 0^+} \int \chi(|f|^2/\epsilon) u^f \wedge \varphi = U^f.\varphi$$

and

$$\lim_{\epsilon \to 0^+} \int \bar{\partial}\chi(|f|^2/\epsilon) \wedge u^f \wedge \varphi = R^f.\varphi.$$

We remark that this result is similar to Theorem 2.1 in [18]. However, their regularizations of the terms of R^f are in general only smooth for the term of top degree.

The main theme in [ii] is regularizations of products of two Cauchy-Fantappiè-Leray type currents. There is a natural way of defining the product of the Cauchy-Fantappiè-Leray type currents corresponding to two section f and g so that formal Leibnitz rules hold, see [25]. The following theorem is the main result in [ii].

Theorem 1.3. *Let f and g be holomorphic sections (locally non-trivial) of the holomorphic m_j-bundles $E_j^* \to X$, $j = 1, 2$, respectively. Assume that the section $f \oplus g$ of $E_1^* \oplus E_2^* \to X$ defines a complete intersection. Let $\chi_1, \chi_2 \in C^\infty([0,\infty])$ be any functions vanishing to orders m_1 and m_2 at zero respectively, and satisfying $\chi_j(\infty) = 1$. Then, for any test form φ we have*

$$\int \chi_1(|f|^2/\epsilon_1) u^f \wedge \bar{\partial}\chi_2(|g|^2/\epsilon_2) \wedge u^g \wedge \varphi \to U^f \wedge R^g.\varphi$$

as $\epsilon_1, \epsilon_2 \to 0^+$. Moreover, as a function of $\epsilon = (\epsilon_1, \epsilon_2) \in [0,\infty)^2$ the integral on the left hand side belongs to some Hölder class independently of φ.

As a consequence we get

Corollary 1.4. *With the same hypotheses as in Theorem 1.3 we have*

$$\int \bar\partial\chi_1(|f|^2/\epsilon_1)\wedge u^f\wedge\bar\partial\chi_2(|g|^2/\epsilon_2)\wedge u^g\wedge\varphi\to R^f\wedge R^g.\varphi,$$

$$\int \bar\partial\chi_1(|f|^2/\epsilon_1)\wedge u^f\chi_2(|g|^2/\epsilon_2)\wedge\varphi\to R^f.\varphi, \tag{11}$$

and

$$\int \chi_1(|f|^2/\epsilon_1)\wedge u^f\wedge\bar\partial\chi_2(|g|^2/\epsilon_2)\wedge\varphi\to 0$$

as $\epsilon_1,\epsilon_2\to 0^+$, *and as functions of* $\epsilon=(\epsilon_1,\epsilon_2)\in[0,\infty)^2$ *the integrals on the left hand sides belong to some Hölder classes independently of* φ.

Assume that f defines a complete intersection and pick a holomorphic function g such that $f\oplus g$ also defines a complete intersection and such that g is zero on the singular part of V_f. After resolving singularities we can always choose coordinates so that g is a monomial. Using this, one can repeat the proof of Theorem 1.3 to see that (11) holds for χ_2 equal to the characteristic function of $[1,\infty]$. It then follows, by letting ϵ_1 tend to zero before ϵ_2, that $\lim_{\epsilon_2\to 0^+}\chi_g(\epsilon_2)R^f=R^f$, where $\chi_g(\epsilon_2)$ is the characteristic function of $\{|g|^2=\epsilon_2\}$. Hence, in the complete intersection case, R^f does not have any mass concentrated on the singular part of V_f, and we have found the well-known result that R^f has the standard extension property. We remark that the product $\chi_g(\epsilon_2)R^f$ is well-defined since the wave front sets of $\chi_g(\epsilon_2)$ and R^f do not intersect for ϵ_2 sufficiently small, see e.g. [7].

2. Multivariable operator calculus

2.1. Taylor's functional calculus.

The purpose of this subsection is to give some background to Taylor's analytic functional and to point out some connections with residue currents. Let X be a Banach space and let $a=(a_1,\dots,a_n)$ be an n-tuple of bounded commuting operators on X. The tuple a induces a continuous bilinear pairing

$$\mathcal{O}(\mathbb{C}^n)\times X\ni(f,x)\mapsto f(a)x:=\sum_\alpha c_\alpha a^\alpha x\in X,$$

where $\sum_\alpha c_\alpha z^\alpha$ is the power series expansion of f. This gives us a continuous $\mathcal{O}(\mathbb{C}^n)$-module structure on X. Note that, for any open $\Omega\subseteq\mathbb{C}^n$, a continuous $\mathcal{O}(\Omega)$-module structure on X is equivalent to having a continuous algebra homomorphism $\mathcal{O}(\Omega)\to\mathcal{L}(X)$. Here $\mathcal{L}(X)$ is the Banach algebra of bounded linear operators on X. In this case one refers to the algebra homomorphism as an $\mathcal{O}(\Omega)$-functional calculus for a. Let $\Lambda_z^{p,q}$ denote

$X \otimes A_z^{p,q}(\mathbb{C}^n)$, where $A_z^{p,q}(\mathbb{C}^n)$ is the space of vectors in $\Lambda T_z^*(\mathbb{C}^n)$ of type (p,q), and let δ_{z-a} denote interior multiplication with the operator valued $(1,0)$-vector $\sum_1^n (z_j - a_j)(\partial/\partial z_j|_z)$. Since the a_j are commuting, $\delta_{z-a}^2 = 0$, and we get the complex

$$0 \leftarrow \Lambda_z^{0,0} \stackrel{\delta_{z-a}}{\longleftarrow} \Lambda_z^{1,0} \stackrel{\delta_{z-a}}{\longleftarrow} \dots \stackrel{\delta_{z-a}}{\longleftarrow} \Lambda_z^{n,0} \leftarrow 0. \tag{12}$$

The joint Taylor spectrum, $\sigma(a)$, of a is by definition the set of points $z \in \mathbb{C}^n$ such that (12) is not exact. Observe that if (12) is exact then the corresponding complex where $(p,0)$ is replaced by (p,q) is exact. It turns out, see [22], that $\sigma(a)$ is a compact non-empty subset of $\mathbb{C}^n$. In [22], Taylor also proves the following theorem.

Theorem 2.1 (Taylor). *There is a continuous algebra homomorphism $\mathcal{O}(\sigma(a)) \to \mathcal{L}(X)$ which extends the $\mathcal{O}(\mathbb{C}^n)$-functional calculus. The image, $f(a)$, of $f \in \mathcal{O}(\sigma(a))$ commutes with each $b \in \mathcal{L}(X)$ that commutes with all a_j. If $f = (f_1, \dots, f_m)$ is an analytic mapping, $f_j \in \mathcal{O}(\sigma(a))$, and $f(a) = (f_1(a), \dots, f_m(a))$, then $\sigma(f(a)) = f(\sigma(a))$.*

In [1] Andersson gives a realization of Taylor's functional calculus by means of Cauchy-Fantappiè-Leray type formulas, which we briefly describe. For any open set $\Omega \subseteq \mathbb{C}^n$ we let $\mathcal{E}_{p,q}(\Omega, X)$ be the space of smooth X-valued (p,q)-forms in Ω. We have $\delta_{z-a}\colon \mathcal{E}_{p+1,q}(\Omega, X) \to \mathcal{E}_{p,q}(\Omega, X)$ and as above, $\delta_{z-a}^2 = 0$. From the theory of parameterized complexes, see e.g. [23], it follows that if (12) is exact at z then it is also exact in a neighborhood of z. It also follows that the complexes $(\mathcal{E}_{\bullet,q}(\Omega, X), \delta_{z-a})$ are exact if (12) is exact for all $z \in \Omega$, i.e., if $\Omega \subseteq \mathbb{C}^n \setminus \sigma(a)$. It is straight forward to check that $\bar{\partial}\delta_{z-a} = -\delta_{z-a}\bar{\partial}$ and so $(\mathcal{E}_{\bullet,\bullet}(\Omega, X), \delta_{z-a}, \bar{\partial})$ is a double complex with exact rows if $\Omega \subseteq \mathbb{C}^n \setminus \sigma(a)$. We then define the corresponding total complex, $(\mathscr{L}^\bullet(\Omega), \nabla_{z-a})$, where $\mathscr{L}^r(\Omega) = \bigoplus_{q-p=r} \mathcal{E}_{p,q}(\Omega, X)$ and ∇_{z-a} is the coboundary operator $\delta_{z-a} - \bar{\partial}$. From standard homological algebra we know that if a double complex has exact rows (or columns) then the corresponding total complex is again exact. Hence, $(\mathscr{L}^\bullet(\Omega), \nabla_{z-a})$ is exact if $\Omega \subseteq \mathbb{C}^n \setminus \sigma(a)$. Denote by x the constant function $\mathbb{C}^n \to X$ with value $x \in X$. Then $\nabla_{z-a}x = 0$ and so there is a $u = \sum_1^n u_{k,k-1} \in \mathscr{L}^{-1}(\mathbb{C}^n \setminus \sigma(a))$ such that $x = \nabla_{z-a}u$. For degree reasons, $\bar{\partial}u_{n,n-1} = 0$, and if $\tilde{u}$ is another solution, $x = \nabla_{z-a}\tilde{u}$, then actually, $\tilde{u}_{n,n-1}$ and $u_{n,n-1}$ are $\bar{\partial}$-cohomologous in $\mathbb{C}^n \setminus \sigma(a)$. Hence, each $x \in X$ defines a $\bar{\partial}$-cohomology class $\omega_{z-a}x \in H_{\bar{\partial}}^{n,n-1}(\mathbb{C}^n \setminus \sigma(a), X)$ called the resolvent. Andersson proves in [1] that if Ω is a neighborhood of $\sigma(a)$ and $f \in \mathcal{O}(\Omega)$ then

$$f(a)x = \frac{1}{(2\pi i)^n} \int_{\partial D} f(z)\omega_{z-a}x, \tag{13}$$

where $\sigma(a) \subset D \Subset \Omega$ and D has smooth boundary. Note that by Stokes' theorem, the integral in (13) only depends on the homology class of ∂D. One can think of (13) as a residue formula. On a formal level we can write (13) as $(2\pi i)^n f(a)x = \bar{\partial}[\omega_{z-a}x].f$, and so if $f \mapsto f(a)$, defined by (13), extends to smooth f then $\bar{\partial}[\omega_{z-a}x]$ can be interpreted as a residue current with support $\sigma(a)$. We will discuss this connection between operator calculus and residue currents in more detail by considering two examples.

Example 2.2. Let M be an $m \times m$-matrix and denote by M' the matrix of all $(m-1)\times(m-1)$-minors with the property that $MM' = M'M = \det(M)$. Let also $P(z) = \det(z - M)$. Then the resolvent of M is $(z - M)'dz/P(z)$ and the analytic functional calculus of M is given by

$$\mathcal{O}(P^{-1}(0)) \ni f \mapsto \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|P|^2=\epsilon} f(z) \frac{(z-M)'dz}{P(z)} \in \mathcal{L}(X).$$

Note that we do not have to pass to the limit since the integral only depends on the homology class of $|P|^2 = \epsilon$. However, if we want to extend the analytic functional calculus to a smooth one, we will have to pass to the limit. This limit exists in view of the discussion in the beginning of Subsection 1.1 and equals the action of the operator valued residue current $\bar{\partial}[(z - M)'dz/(2\pi i P(z))]$ on the smooth function.

Example 2.3. Let A and B be two commuting 2×2-matrices. From elementary linear algebra it follows that if $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$ then either at least one of A and B is invertible or $\dim \operatorname{Ker} A = \dim \operatorname{Ker} B = 1$ and in addition,

(a) $\mathbb{C}^2 = \operatorname{Ker} A \oplus \operatorname{Ker} B$,
(b) $\operatorname{Ker} A = \operatorname{Im} B$,
(c) $\operatorname{Ker} B = \operatorname{Im} A$.

Using this simple fact we will compute the joint Taylor spectrum $\sigma(A, B)$. Exactness of (12) at $z = (z_1, z_2)$ means in this case that

(i) for all $v \in \mathbb{C}^2$ there are $v_1, v_2 \in \mathbb{C}^2$ such that $v = (z_1 - A)v_1 + (z_2 - B)v_2$,
(ii) if $(z_1 - A)v_1 = (z_2 - B)v_2$ then there is a $v \in \mathbb{C}^2$ such that $v_1 = -(z_2 - B)v$ and $v_2 = (z_1 - A)v$,
(iii) if $(z_1 - A)v = (z_2 - B)v$ then $v = 0$.

Note that (iii) is equivalent to $\operatorname{Ker}(z_1 - A) \cap \operatorname{Ker}(z_2 - B) = \{0\}$. If (iii) does not hold then $z \in \sigma(A, B)$. On the other hand, if (iii) holds then either one of $z_1 - A$ and $z_2 - B$ is invertible, in which case (i) and (ii) hold, or (a), (b), and (c) hold. But then clearly (i) holds by (a), (b), and (c), and if $(z_1 - A)v_1 = (z_2 - B)v_2$ then $(z_1 - A)v_1 \in \operatorname{Im}(z_1 - A) \cap \operatorname{Im}(z_2 - B) = \{0\}$ by (a), (b), and (c), and analogously, $(z_2 - B)v_2 = 0$. Hence,

$v_1 \in \operatorname{Ker}(z_1 - A) = \operatorname{Im}(z_2 - B)$ and $v_2 \in \operatorname{Ker}(z_2 - B) = \operatorname{Im}(z_1 - A)$ and so by (a) we can choose $\tilde{v}_1 \in \operatorname{Ker}(z_1 - A)$ such that $v_1 = (z_2 - B)\tilde{v}_1$ and $\tilde{v}_2 \in \operatorname{Ker}(z_2 - B)$ such that $v_2 = (z_1 - A)\tilde{v}_2$. Then $v = \tilde{v}_2 - \tilde{v}_1$ satisfies the requirement in (ii). Thus, $z \notin \sigma(A, B)$. In conclusion, $z \notin \sigma(A, B)$ if and only if (iii) holds which in turn is equivalent to $\operatorname{Ker}(z_1 - A) \cap \operatorname{Ker}(z_2 - B) = \{0\}$. We remark that in general, the vanishing of different homology groups of (12) are independent statements. Now, let $P_1(z) = \det(z_1 - A)$, $P_2(z) = \det(z_2 - B)$, $P_3(z) = \det(z_1 - A + z_2 - B)$, and $P = (P_1, P_2, P_3)$. Then $\sigma(A, B) = P^{-1}(0)$. In fact, if $z \in \sigma(A, B)$ then there is a non-zero $v \in \operatorname{Ker}(z_1 - A) \cap \operatorname{Ker}(z_2 - B)$ and so $P(z) = 0$. Conversely, assume that $z \notin \sigma(A, B)$, i.e. that $\operatorname{Ker}(z_1 - A) \cap \operatorname{Ker}(z_2 - B) = \{0\}$. Then, either at least one of $z_1 - A$ and $z_2 - B$ is invertible or (a), (b), and (c) hold. In the first case, either $P_1(z)$ or $P_2(z)$ is non-zero and in the second case, if $(z_1 - A + z_2 - B)v = 0$, then by arguing as above we find that $(z_1 - A)v = (z_2 - B)v = 0$, that is $v \in \operatorname{Ker}(z_1 - A) \cap \operatorname{Ker}(z_2 - B) = \{0\}$, and so $P_3(z) \neq 0$. We now construct the resolvent. Let

$$
\begin{aligned}
\sigma_{z-(A,B)} &= (\bar{P}_1(z_1 - A)' + \bar{P}_3(z_1 - A + z_2 - B)')dz_1 \\
&+ (\bar{P}_2(z_2 - B)' + \bar{P}_3(z_1 - A + z_2 - B)')dz_2
\end{aligned}
$$

and put $s_{z-(A,B)} = \sigma_{z-(A,B)}/|P|^2$. Recall that M' is the matrix of minors such that $MM' = M'M = \det(M)$. It is easily verified that $\delta_{z-(A,B)} s_{z-(A,B)}$ is equal to the identity matrix E and that

$$
\begin{aligned}
E &= \nabla_{z-(A,B)}(s_{z-(A,B)} + s_{z-(A,B)} \wedge \bar{\partial} s_{z-(A,B)}) \\
&= \nabla_{z-(A,B)}\Big(\frac{\sigma_{z-(A,B)}}{|P|^2} + \frac{\sigma_{z-(A,B)} \wedge \bar{\partial}\sigma_{z-(A,B)}}{|P|^4}\Big)
\end{aligned}
$$

outside $P^{-1}(0)$, i.e. outside $\sigma(A, B)$. By Andersson's theorem we have for any f holomorphic in a neighborhood of $P^{-1}(0)$ that

$$
f(A, B) = \lim_{\epsilon \to 0} \frac{1}{(2\pi i)^2} \int_{|P|^2 = \epsilon} f(z) \frac{\sigma_{z-(A,B)} \wedge \bar{\partial}\sigma_{z-(A,B)}}{|P|^4}. \tag{14}
$$

As before we see by Stokes' theorem that, for holomorphic f, the integral in (14) is independent of (sufficiently small) ϵ. From Subsection 1.2 we know that the limit in (14) also exists for smooth f defined in a neighborhood of $P^{-1}(0)$, and defines the action of a residue current. In this example Taylor's analytic functional calculus therefore is the pairing of a residue current and a function holomorphic in a neighborhood of its support.

2.2. **Cayley transforms of tuples of unbounded operators.** Let a be a closed, but not necessarily densely defined, linear operator on a Banach space X. The spectrum, $\sigma(a)$, of a is the set of points $z \in \mathbb{C}$ such that $z - a$ is a bijection from the domain, $\operatorname{Dom}(a)$, of a to X. The extended spectrum,

$\hat{\sigma}(a)$, is $\sigma(a)$ if a is bounded and $\sigma(a) \cup \{\infty\} \subseteq \widehat{\mathbb{C}}$ if a is unbounded. There is also the point spectrum, $\sigma_p(a)$, which is the set of $z \in \mathbb{C}$ such that $z - a$ is not injective. By the closed graph theorem the set $\mathbb{C} \setminus \sigma(a)$, known as the resolvent set of a, is the set of points $z \in \mathbb{C}$ such that $z - a$ has a bounded inverse, $(z-a)^{-1} \colon X \to \mathrm{Dom}(a)$. The mapping $z \mapsto (z-a)^{-1}$, for z in the resolvent set, is called the resolvent.

The original Cayley transform, $a \mapsto (a+i)(a-i)^{-1}$, introduced by von Neumann in [24], induces a one-to-one correspondence between the self-adjoint operators and the unitary operators such that 1 is not in the point spectrum. More generally, let $\phi(z) = (m_{11}z + m_{12})/(m_{21}z + m_{22})$ be any projective, or Möbius transformation of $\widehat{\mathbb{C}}$. Then $\phi(a) := (m_{11}a + m_{12})(m_{21}a + m_{22})^{-1}$ has a meaning as a closed operator on X if $\phi^{-1}(\infty) \notin \sigma_p(a)$. Moreover, it is not hard to see that $\phi(a)$ is bounded if and only if $\phi^{-1}(\infty) \notin \hat{\sigma}(a)$. See [23] or [iii]. We say that $\phi(a)$ is a Cayley transform of a, and we conclude that the closed operators on X which can be Cayley transformed to bounded operators are precisely those with a non-empty resolvent set. The spectral mapping property holds for these mappings, that is, for any projective transformation ϕ of $\widehat{\mathbb{C}}$ such that $\phi^{-1}(\infty) \notin \hat{\sigma}_p(a)$ it holds that $\phi(\hat{\sigma}(a)) = \hat{\sigma}(\phi(a))$. The preceding discussion suggests that the closed operator a defines some invariant object on $\mathbb{CP}^1 = \widehat{\mathbb{C}}$. In the canonical affine part of $\widehat{\mathbb{C}}$ this object becomes the operator a and in some other affine part, corresponding to a Möbius transformation ϕ of the canonical one, it becomes $\phi(a)$ and has spectrum $\phi(\hat{\sigma}(a))$.

Now, consider instead a tuple $a = (a_1, \ldots, a_n)$ of closed operators on X. If all a_j have non-empty resolvent sets and their respective resolvents commute, we can Cayley transform each a_j separately to obtain a tuple of bounded commuting operators on X, see [23] and [4]. A (Taylor) spectrum of a can then be naturally defined as a closed subset of $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$. In [iii] we have another approach to multivariable Cayley transforms. We describe a class of tuples of closed unbounded operators on X, which can be transformed to tuples of bounded commuting operators by a projective transformation of $\mathbb{CP}^n$. To achieve this we introduce an equivalence relation on the set of $n+1$-tuples of bounded commuting operators. It turns out that if the Taylor spectrum of one member of an equivalence class avoids 0 then the spectrum of any member in that class also does, and moreover, the projections to $\mathbb{CP}^n$ of the spectra of any two members in such a class coincide. We call such classes projective operators and denote by $[b] = [b_0, \ldots, b_n]$ the class corresponding to b, and by $\sigma[b]$ the image in $\mathbb{CP}^n$ of the spectrum of any of the members in $[b]$. We prove that there is an $\mathcal{O}(\sigma[b])$-functional calculus for the projective operator $[b]$ and that the spectral mapping property holds, see Theorem 3.8 in [iii]. If $\sigma[b]$ avoids some hyperplane in $\mathbb{CP}^n$ then we also provide integral formulas for this functional calculus. This construction is analogous to Andersson's realization of Taylor's functional calculus, see Section 7 in [iii]. Now, assume that $\sigma[b]$ avoids the hyperplane $\{[z];\ \langle z,\lambda^0\rangle = 0\} := [\lambda^0] \in \mathbb{CP}^{n*}$ and denote by ρ_λ the mapping

$$[z] \mapsto \big(\langle z,\lambda^1\rangle/\langle z,\lambda^0\rangle, \ldots, \langle z,\lambda^n\rangle/\langle z,\lambda^0\rangle\big)$$

where $\lambda^0,\ldots,\lambda^n$ are linearly independent. By the $\mathcal{O}(\sigma[b])$-functional calculus, $\rho_\lambda([b]) = (b'_1,\ldots,b'_n)$, $b'_j = \langle b,\lambda^j\rangle\langle b,\lambda^0\rangle^{-1}$, is a tuple of bounded commuting operators. It may happen, even if $\sigma[b]$ does not avoid the hyperplane $[\lambda^0]$, that $\rho_\lambda([b])$ has a meaning as a tuple of closed unbounded operators. We call these hyperplanes, together with the hyperplanes avoiding $\sigma[b]$, admissible. In the case of one operator the set of admissible hyperplanes corresponds to the complement of the point spectrum. One can show, see [iii], that if $[\lambda^0]$ is admissible then the tuple of closed operators, $(a_1,\ldots,a_n) = \rho_\lambda([b])$, $a_j = \langle b,\lambda^0\rangle^{-1}\langle b,\lambda^j\rangle$, satisfies the following conditions:

(1) There exists a $[\lambda] \in \mathbb{CP}^n$ such that the operator

$$a_0 := \lambda_0 + \sum_1^n \lambda_j a_j$$

with domain $\mathrm{Dom}(a_0) = \bigcap_1^n \mathrm{Dom}(a_j)$ is closed, injective and surjective.

(2) The operators $a_0, a_1, \ldots, a_n$ satisfy the following commutation conditions. If $x \in \mathrm{Dom}(a_j) \cap \mathrm{Dom}(a_j a_k)$ then $x \in \mathrm{Dom}(a_k a_j)$ and $a_j a_k x = a_k a_j x$ for $j,k = 0,1,\ldots,n$.

We call a tuple of closed operators satisfying these conditions, an *affine operator*. If a is just one single operator it is not hard to verify that a is affine if and only if a has non-empty resolvent set. We also remark that if $n \geq 2$ then an affine operator does not necessarily consist of operators all of which have non-empty resolvent sets. However, if all a_j have resolvents then conditions (1) and (2) imply that these commute. We prove in [iii] that any affine operator is the image of a unique projective operator in the following sense.

Theorem 2.4. *Fix $[\lambda^0],[\lambda^1],\ldots,[\lambda^n]$ linearly independent. Then to any affine operator $a = (a_1,\ldots,a_n)$ it corresponds a unique projective operator $[b]$, having $[\lambda^0]$ as an admissible hyperplane and with $\sigma[b]$ avoiding some hyperplane, such that $a_j = \langle b,\lambda^0\rangle^{-1}\langle b,\lambda^j\rangle$ for $j = 1,\ldots,n$.*

It follows, by choosing any hyperplane avoiding $\sigma[b]$ as the new hyperplane at infinity, that any affine operator a can be transformed to a tuple of bounded commuting operators by a projective transformation of $\mathbb{CP}^n$. We call the tuple of bounded commuting operators a Cayley transform of a. Conversely, any tuple of operators appearing as the (inverse) Cayley transform of a tuple of bounded commuting operators is affine. We also note that Theorem 2.4 implies that an affine operator has a natural spectrum $\sigma(a) \subseteq \mathbb{CP}^n$. We recall from above that if $a = (a_1, \dots, a_n)$ consists of operators, all of which have non-empty resolvent sets, then a also has a natural spectrum, $\tilde{\sigma}(a) \subseteq \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$. We prove in [iii] that if a is affine and consists of operators with non-empty resolvent sets then

$$\sigma(a) \cap \mathbb{C}^n \subseteq \tilde{\sigma}(a) \cap \mathbb{C}^n.$$

If $a = (a_1, a_2)$ is a pair we prove that the inclusion is an equality, but for n-tuples, $n > 2$, we do not know whether this is true or not.

2.3. **Unbounded operators with smooth functional calculi.** Let a be a closed operator on a Banach space X with real spectrum whose resolvent satisfies growth conditions $\|(z-a)^{-1}\| \lesssim |\mathfrak{Im}\, z|^{-M_K}$, $\mathfrak{Re}\, z \in K \Subset \mathbb{R}$, $M_K > 0$. Then a admits a $\mathscr{D}(\mathbb{R})$-functional calculus given by

$$\mathscr{D}(\mathbb{R}) \ni \varphi \mapsto \varphi(a) := \frac{1}{2\pi i} \int (z-a)^{-1} dz \wedge \bar{\partial}\tilde{\varphi} \in \mathcal{L}(X), \tag{15}$$

where $\tilde{\varphi}$ is an almost holomorphic extension of φ to $\mathbb{C}$ with compact support such that $|\bar{\partial}\tilde{\varphi}| = \mathcal{O}(|\mathfrak{Im}\, z|^{\infty})$. For bounded a this construction is due to Dynkin, [9], and it was generalized to unbounded operators by Helffer and Sjöstrand in [11]. If a is bounded then (15) extends to all smooth functions φ on $\mathbb{R}$ and it coincides with the holomorphic functional calculus if φ is holomorphic in a neighborhood of $\sigma(a)$. In general, (15) extends continuously to the algebra $\mathcal{G}$ of all smooth functions on $\mathbb{R}$ which are holomorphic at infinity in $\mathbb{C}$. In particular, it extends to all $t \mapsto 1/(z-t)$ for $z \in \mathbb{C} \setminus \mathbb{R}$, and the image under (15) equals $(z-a)^{-1}$. Conversely, it was proved in [4] that if there is a multiplicative mapping $A \colon \mathscr{D}(\mathbb{R}) \to \mathcal{L}(X)$, which extends continuously to $\mathcal{G}$, and such that $\cup_{\varphi \in \mathscr{D}(\mathbb{R})} \operatorname{Im} A(\varphi)$ is dense in X and $\cap_{\varphi \in \mathscr{D}(\mathbb{R})} \operatorname{Ker} A(\varphi) = \{0\}$, then there is a closed operator a on X, satisfying the growth conditions above, and giving A by (15). However, there are operators admitting a $\mathscr{D}(\mathbb{R})$-functional calculus although they do not have any resolvent. For example, the operator defined as multiplication with $t \mapsto t(2 + \sin t^3)$ on the Sobolev space $H^1(\mathbb{R})$ has empty resolvent set but nevertheless admits such a functional calculus. In [iv] we study abstract multiplicative $\mathcal{L}(X)$-valued distributions

$$A \colon \mathscr{D}(\mathbb{R}^n) \to \mathcal{L}(X),$$

with the extra properties that $D_A := \cup_{\varphi\in\mathscr{D}(\mathbb{R}^n)}\operatorname{Im} A(\varphi)$ is dense in X and $\cap_{\varphi\in\mathscr{D}(\mathbb{R}^n)}\operatorname{Ker} A(\varphi) = \{0\}$. We call these objects hyperoperators. The assumptions on the union of the images and the intersection of the kernels are a weak way of saying that $A(1)$ is the identity operator. We define the spectrum of A as the support of the distribution and we show that the class of hyperoperators is closed under tensor products and composition with proper maps. It also turns out that to any hyperoperator A it corresponds a unique closable operator (tuple of commuting closable operators), a, defined on D_A admitting an $\mathcal{E}(\mathbb{R}^n)$-functional calculus with respect to D_A, a so called weak hyperoperator. Roughly, this means that each $x \in D_A$ has a real and compact local spectrum with respect to D_A. We remark that if A_1 and A_2 are hyperoperators in $\mathbb{R}$ with corresponding weak hyperoperators a_1 and a_2 respectively, then, as mentioned above, $A = A_1 \otimes A_2$ is a hyperoperator in $\mathbb{R}^2$ and $a = (a_1, a_2)$. But all hyperoperators in $\mathbb{R}^2$ do not arise in this way and this gives support for the idea in e.g. [14], [23], and [iii] that a reasonable notion of a "tuple of commuting unbounded operators" should be considered as an object in its own.

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Paper I

A REGULARISATION OF THE COLEFF–HERRERA RESIDUE CURRENT

HÅKAN SAMUELSSON

Abstract. We prove that if a holomorphic mapping from some complex manifold to $\mathbb{C}^2$ defines a complete intersection then the corresponding Coleff–Herrera residue current can be smoothly regularised by a $(0,2)$-form depending on two parameters.

Résumé. **Une régularisation du courant résiduel de Coleff–Herrera** Nous démontrons que, si une application holomorphe d'une variété complexe à valeurs dans $\mathbb{C}^2$ définit une intersection complète, alors le courant résiduel de Coleff–Herrera correspondant peut être régularisé par une $(0,2)$-forme dépendant de deux paramètres.

Version français abrégée

Soit X une variété complexe et soit $f=(f_1,f_2):X\to\mathbb{C}^2$ une application holomorphe. Supposons que f définit une intersection complète, autrement dit que la variété $V_f=\{f_1=f_2=0\}$ est de codimension 2. Définissons, pour toute forme $\varphi\in\mathscr{D}_{n,n-2}(X)$, l'intégrale résiduelle

$$I_f^{\varphi}(\epsilon_1,\epsilon_2):=\int_{\substack{|f_1|^2=\epsilon_1\\ |f_2|^2=\epsilon_2}}\frac{\varphi}{f_1f_2}.$$

Le courant résiduel de Coleff–Herrera, noté $\left[\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}\right]$, est défini comme la limite de cette intégrale lorsque ϵ_1 et ϵ_2 tendent vers zéro le long d'un "chemin admissible", ce qui, dans ce cadre, signifie que, par exemple, ϵ_1 tend vers zéro plus vite que toute puissance de ϵ_2 (cf. [5]). Il est bien connu que l'integrale résiduelle est, en général, dicontinue à l'origine [8], [3]. Dans cette note, nous donnons une démonstration du résultat suivant.

Théorème. *Soit X une variété complexe et soit $f=(f_1,f_2):X\to\mathbb{C}^2$ une application holomorphe. Supposons que f définit une intersection complète. Alors*

$$\lim_{\epsilon_1,\epsilon_2\to 0}\int\bar\partial\frac{\bar f_1}{|f_1|^2+\epsilon_1}\wedge\bar\partial\frac{\bar f_2}{|f_2|^2+\epsilon_2}\wedge\varphi=\left[\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}\right].\varphi$$

pour toute forme $\varphi\in\mathscr{D}_{n,n-2}(X)$.

Par conséquent, le courant de Coleff–Herrera peut être obtenu comme la limite (au sens des courants) d'une $(0,2)$-forme régulière dépendant de deux paramètres, indépendamment de la façon dont on s'approche de l'origine.

1. Introduction and the Result

Let X be a complex manifold and let $f=(f_1,f_2):X\to\mathbb{C}^2$ be a holomorphic mapping. Assume that f defines a complete intersection, i.e. that $V_f=\{f_1=f_2=0\}$ has codimension 2 in X. The corresponding *Coleff–Herrera residue current* was originally defined as follows, [4]. Denote the *residue integral* by

$$I_f^{\varphi}(\epsilon_1,\epsilon_2):=\int\limits_{\substack{|f_1|^2=\epsilon_1\\|f_2|^2=\epsilon_2}}\frac{\varphi}{f_1f_2},$$

where φ is any test form of bidegree $(n,n-2)$. If we let ϵ_1 and ϵ_2 approach the origin along an "admissible path", which in this context means that ϵ_1 tends to zero faster then any power of ϵ_2 or vice versa, then the residue integral has a limit independently of the choice of admissible path and this limit defines the action of a $(0,2)$-current, the Coleff–Herrera residue current, on the test form φ. We will denote this current by $\left[\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}\right]$. It is known that an unrestricted limit of the residue integral does not exist in general. Passare and Tsikh showed in [8] that if we take $f_1=z_1^4$, $f_2=z_1^2+z_2^2+z_1^3$ and a test form which in a neighbourhood of the origin equals $\varphi(z)=\bar z_2f_2(z)dz_1\wedge dz_2$ then the residue integral has limit zero if we approach the origin along any path $\delta\mapsto(\delta^4,c\delta^2)$, $c\neq 1$ and a non zero limit if we approach the origin along the path $\delta\mapsto(\delta^4,\delta^2)$. Other examples disproving the continuity of the residue integral at the origin have been found by Björk, [3]. The aim of this note is to outline a proof of the following result saying that the Coleff–Herrera current can be obtained as the unrestricted (weak) limit of a smooth $(0,2)$-form depending on two parameters.

Theorem 1. *Let X be a complex manifold and let $f=(f_1,f_2):X\to\mathbb{C}^2$ be a holomorphic mapping. Assume that f defines a complete intersection in X. Then*

$$\lim_{\epsilon_1,\epsilon_2\to 0}\int\bar\partial\frac{\bar f_1}{|f_1|^2+\epsilon_1}\wedge\bar\partial\frac{\bar f_2}{|f_2|^2+\epsilon_2}\wedge\varphi=\left[\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}\right].\varphi$$

for all test forms φ of bidegree $(n,n-2)$.

Before we continue with the proof section we mention that a thorough study of the limits of the residue integral along paths of the form $\delta\mapsto(\delta^{s_1},\delta^{s_2})$ for $(s_1,s_2)\in\mathbb{R}_+$ has been done by Passare in [6]. He shows that as long as (s_1,s_2) avoids finitely many lines through the origin the corresponding limit of the residue integral equals the limit along an admissible path. We also mention, and will later use, an alternative approach to the Coleff–Herrera residue current proposed by Passare and Tsikh [7]. If we compute the (iterated) Mellin transform of the residue integral we get

$$\int\frac{\bar\partial|f_1|^{2\lambda_1}}{f_1}\wedge\frac{\bar\partial|f_2|^{2\lambda_2}}{f_2}\wedge\varphi,$$

at least for the real part of λ_1 and λ_2 large enough. Passare and Tsikh showed that the integral as a function of λ_1 and λ_2 has an analytic continuation to a neighbourhood of the origin in $\mathbb{C}^2$ and that the value at $\lambda_1 = \lambda_2 = 0$ equals the limit of the residue integral along an admissible path.

2. Outline of the Proof

We first present and indicate how to prove two technical results, Propositions 2 and 3, and then we finish the proof of Theorem 1 using these results.

Proposition 2. *Let Ψ and Φ be strictly positive smooth functions on $\mathbb{C}^n$. Then for any $\varphi \in \mathscr{D}_{n,n}(\mathbb{C}^n)$ we have*

$$\lim_{\epsilon_1,\epsilon_2\to 0+} \int \frac{\bar{\zeta}^\alpha \Psi}{|\zeta^\alpha|^2\Psi + \epsilon_1} \frac{\bar{\zeta}^\beta \Phi}{|\zeta^\beta|^2\Phi + \epsilon_2} \varphi = \Big[\frac{1}{\zeta^{\alpha+\beta}}\Big].\varphi.$$

Proposition 3. *Let Ψ and Φ be strictly positive smooth functions on $\mathbb{C}^n$. Then for any $\varphi \in \mathscr{D}_{n,n}(\mathbb{C}^n)$ we have*

$$\lim_{\epsilon_1,\epsilon_2\to 0+} \epsilon_2 \int \frac{\bar{\zeta}^{\alpha+\beta}}{(|\zeta^\alpha|^2\Psi + \epsilon_1)(|\zeta^\beta|^2\Phi + \epsilon_2)^2} \varphi = 0.$$

The key to understand Propositions 2 and 3 is the next lemma which maybe also has some independent interest. It is a version of Taylor's formula but unlike the usual one that gives us a polynomial approximation in a neighbourhood of the intersection of the coordinate hyperplanes our version provides us with an approximation in a neighbourhood of the union of the coordinate hyperplanes. Our approximation is in general not a polynomial though, but has enough similarities for our purposes. Define the linear operator $M_j^{r_j}$ on $C^\infty(\mathbb{C}^n)$ to be the operator that maps φ to the Taylor polynomial of degree r_j of the function $\zeta_j \mapsto \varphi(\zeta)$ (centered at $\zeta_j = 0$). A straight forward computation shows that $M_j^{r_j}$ and $M_i^{r_i}$ commute.

Lemma 4. *Let $K \subseteq \{1,\dots,n\}$ and $r = (r_{i_1},\dots,r_{i_{|K|}})$ and define the linear operator M_K^r on $C^\infty(\mathbb{C}^n)$ by*

$$M_K^r = \sum_{j\in K} M_j^{r_j} - \sum_{\substack{i,j\in K\\ i<j}} M_i^{r_i} M_j^{r_j} + \cdots + (-1)^{|K|+1} M_{i_1}^{r_{i_1}} \cdots M_{i_{|K|}}^{r_{i_{|K|}}}.$$

Then for any $\varphi \in C^\infty(\mathbb{C}^n)$ we have

$$\varphi - M_K^r\varphi = \mathcal{O}(\prod_{i\in K} |\zeta_i|^{r_i+1}).$$

Moreover, $M_K^r\varphi$ can be written as a (finite) sum of terms $\phi_{IJ}(\zeta)\zeta^I\bar{\zeta}^J$ where $I_i + J_i \le r_i$ for $i \in K$ and $\phi_{IJ}(\zeta)$ is independent of the coordinate ζ_i if $I_i + J_i > 0$, and also if L is the set of indices $i \in K$ such that $I_i + J_i = 0$ then $\phi_{IJ}(\zeta) = \mathcal{O}(\prod_{i\in L} |\zeta_i|^{r_i+1})$.

It is now quite easy to see that Propositions 2 and 3 hold in the case Ψ and Φ are constant (for simplicity equal to 1). We illustrate by considering Proposition 3. Choose $K = \{1,\dots,n\}$ and $r = \alpha + \beta - 1$ and add and subtract $M^r_K\varphi$. Then the integral in Proposition 3 splits into

$$(1) \qquad \epsilon_2 \int_\Delta \frac{\bar{\zeta}^{\alpha+\beta}}{(|\zeta^\alpha|^2+\epsilon_1)(|\zeta^\beta|^2+\epsilon_2)^2}\, M^r_K\varphi$$

$$(2) \qquad + \quad \epsilon_2 \int_\Delta \frac{\bar{\zeta}^{\alpha+\beta}}{(|\zeta^\alpha|^2+\epsilon_1)(|\zeta^\beta|^2+\epsilon_2)^2}\, (\varphi - M^r_K\varphi),$$

where Δ is a big polydisc containing the support of φ. The integral (1) is zero for all positive ϵ_1 and ϵ_2 by anti-symmetry since the terms in $M^r_K\varphi$ are polynomials in at least one of the variables. On the other hand, the integrand in (2) is locally integrable when $\epsilon_1 = \epsilon_2 = 0$, and so by the Dominated Convergence Theorem the limit of (2) equals the integral of the pointwise limit of the integrand and this is zero (almost everywhere). In the general case when Ψ and Φ are not constant we can not use anti-symmetry directly to see that certain integrals vanishes. However we can use the following two results to see that it actually is enough anti-symmetry left in the general case to deduce the same thing. With the notation from Lemma 4 we have

$$(3) \qquad \frac{\Psi}{\Psi + a/b}\frac{\Phi}{\Phi + c/d} = M^r_K\Big(\frac{\Psi}{\Psi + a/b}\frac{\Phi}{\Phi + c/d}\Big) + \prod_{i\in K} |\zeta_i^{r_i+1}| F(a,b,c,d,\zeta),$$

$$(4) \qquad \frac{c/d}{(\Psi + a/b)(\Phi + c/d)^2} = M^r_K\Big(\frac{c/d}{(\Psi + a/b)(\Phi + c/d)^2}\Big) + \prod_{i\in K} |\zeta_i^{r_i+1}| \tilde{F}(a,b,c,d,\zeta),$$

where F and $\tilde{F}$ are bounded on $(0,\infty)^4 \times D$ if $D \Subset \mathbb{C}^n$. The homogeneity in (3) and (4) enables us to re-write the integrals in Propositions 2 and 3 in such a way that we can use anti-symmetry but we skip the details. We can now finish the proof of Theorem 1 but first we need some terminology for multiindices. We say that two multiindices α and β with the same number of components are *disjoint* if $\alpha_i \neq 0$ implies that $\beta_i = 0$ and $\beta_i \neq 0$ implies that $\alpha_i = 0$.

Proof of Theorem 1. We prove the following slightly stronger statement

$$(5) \qquad \lim_{\epsilon_1,\epsilon_2\to 0} \int \frac{\bar{f}_1}{|f_1|^2+\epsilon_1}\bar{\partial}\frac{\bar{f}_2}{|f_2|^2+\epsilon_2} \wedge \varphi = \Big[\frac{1}{f_1}\bar{\partial}\frac{1}{f_2}\Big].\varphi,$$

where φ is any test form of bidegree $(n, n-1)$. We will use the analytic continuation definition of the right hand side of (5) ([7], [1]), that is we will use that

$$\Big[\frac{1}{f_1}\bar{\partial}\frac{1}{f_2}\Big].\varphi = \int \frac{|f_1|^{2\lambda}}{f_1}\frac{\bar{\partial}|f_2|^{2\lambda}}{f_2} \wedge \varphi\Big|_{\lambda=0}.$$

By Hironaka's theorem [5], for any sufficiently small open $U \subset X$ we can find a complex manifold $\tilde{U}$ and a proper holomorphic map $\pi : \tilde{U} \to U$ which is a biholomorphism outside the null-set $\pi^*\{f_1 \cdot f_2 = 0\}$ such that $\{\pi^* f_1 \cdot \pi^* f_2 = 0\}$ has normal crossings in $\tilde{U}$. Hence locally in $\tilde{U}$ we can choose coordinates such that $\pi^* f_1 = \mu_1 g_1$ and $\pi^* f_2 = \mu_2 g_2$ where μ_j are monomials

and g_j are non-vanishing holomorphic functions. After a partition of unity we may assume that φ has support in such a U, and so we see that in order to prove (5) it suffices to prove
(6)

$$\int \frac{\bar\mu_1 \bar g_1}{|\mu_1 g_1|^2+\epsilon_1}\bar\partial\frac{\bar\mu_2\bar g_2}{|\mu_2 g_2|^2+\epsilon_2}\wedge\rho\pi^*\varphi \to \int \frac{|\mu_1 g_1|^{2\lambda}}{\mu_1 g_1}\frac{\bar\partial|\mu_2 g_2|^{2\lambda}}{\mu_2 g_2}\wedge\rho\pi^*\varphi\Big|_{\lambda=0},$$

where ρ is a cut-off function in $\tilde U$. We write the monomials μ_j in local coordinates ζ on $\tilde U$ as $\mu_1=\zeta^\alpha\zeta^\gamma$ and $\mu_2=\zeta^\beta\zeta^\delta$ where the multiindices α, β and γ are pairwise disjoint and $\gamma_j=0$ if and only if $\delta_j=0$. Hence α, β and δ are also pairwise disjoint. Note that coordinate ζ_j divides both μ_1 and μ_2 if and only if $\gamma_j\neq 0$ or equivalently $\delta_j\neq 0$. The right hand side of (6) can be computed by integrations by parts as in e.g. [1] and the result can be written

$$\Big[\frac{1}{\zeta^{\alpha+\gamma+\delta}}\Big]\otimes\bar\partial\Big[\frac{1}{\zeta^\beta}\Big].\frac{\rho\pi^*\varphi}{g_1 g_2}. \tag{7}$$

Let K and L be the set of indices j such that $\beta_j\neq 0$ and $\gamma_j\neq 0$ respectively. Decompose the $\bar\partial$-operator as $\bar\partial=\bar\partial_K+\bar\partial_{K^c}$ where $\bar\partial_K$ and $\bar\partial_{K^c}$ are the parts corresponding to the variables ζ_j with $j\in K$ and $j\notin K$ respectively. Integrating by parts we see that the integral on the right-hand side of (6) equals

$$-\int\bar\partial_K\Big(\frac{\bar\zeta^{\alpha+\gamma}\bar g_1}{|\zeta^{\alpha+\gamma}|^2|g_1|^2+\epsilon_1}\rho\pi^*\varphi\Big)\frac{\bar\zeta^{\beta+\delta}\bar g_2}{|\zeta^{\beta+\delta}|^2|g_2|^2+\epsilon_2} \tag{8}$$

$$+\quad \epsilon_2\int\frac{\bar\zeta^{\alpha+\gamma}\bar g_1}{|\zeta^{\alpha+\gamma}|^2|g_1|^2+\epsilon_1}\frac{\bar\zeta^\beta\bar\partial_{K^c}(\bar\zeta^\delta\bar g_2)}{(|\zeta^{\beta+\delta}|^2|g_2|^2+\epsilon_2)^2}\wedge\rho\pi^*\varphi. \tag{9}$$

Let us first consider (8). When $\bar\partial_K$ falls on $\rho\pi^*\varphi$ we get an integral which can be handled by Proposition 2 and in the limit we get $-\big[1/\zeta^{\alpha+\beta+\gamma+\delta}\big].\bar\partial_K\frac{\rho\pi^*\varphi}{g_1g_2}$ which is precisely (7). On the other hand, when $\bar\partial_K$ falls on the quotient we get, since β is disjoint with both α and γ, an integral which by Proposition 3 tends to zero. It remains to see that also (9) tends to zero. When $\bar\partial_{K^c}$ falls on $\bar g_2$ we run into no problems and Proposition 3 says that this integral tends to zero. It is a bit more delicate when $\bar\partial_{K^c}$ falls on $\bar\zeta^\delta$ because then we get

$$\sum_{i\in L}\epsilon_2\delta_i\int\frac{\bar\zeta^{\alpha+\gamma}}{|\zeta^{\alpha+\gamma}|^2|g_1|^2+\epsilon_1}\frac{\bar\zeta^{\beta+\delta}}{(|\zeta^{\beta+\delta}|^2|g_2|^2+\epsilon_2)^2}\bar g_1\bar g_2\frac{d\bar\zeta_i}{\bar\zeta_i}\wedge\rho\pi^*\varphi.$$

Now for the first and only time we have to use that V_f has codimension 2. We use the Coleff–Herrera trick to see that $\pi^*\varphi$ is a sum of terms which are either divisible by $\bar\zeta_j$ or $d\bar\zeta_j$. In fact, if we let z be local coordinates on our original manifold X, then we can write

$$\varphi=\sum_{|I|=n-1}\varphi_I\wedge d\bar z^I,$$

where the φ_I are $(n,0)$-forms. Since V_f has codimension 2, the $(0,n-1)$-forms $d\bar z^I$ vanishes on V_f. Hence $\pi^*d\bar z^I$ vanishes on π^*V_f and in particular, since ζ_j divides both μ_1 and μ_2 for $j \in L$, it vanishes on $\{\zeta_j = 0\}$. Moreover, $\partial\pi^* d\bar{z}^I = \pi^*\partial d\bar{z}^I = 0$, and so if we write

$$\pi^* d\bar{z}^I = \sum_{|J|=n-1} C_J(\zeta) d\bar{\zeta}^J,$$

we see that the coefficients $C_J(\zeta)$ must be anti-holomorphic. Hence if $d\bar{\zeta}_i$ does not divide $d\bar{\zeta}^J$ then $\bar{\zeta}_j$ must divide $C_J(\zeta)$ since $C_J(\zeta)$ is anti-holomorphic and zero on $\{\zeta_j = 0\}$. Thus, for $j \in L$, the form $\frac{d\bar{\zeta}_j}{\zeta_j} \wedge \pi^*\varphi$ is actually smooth (and compactly supported) and so we can use Proposition 3 to see that all the integrals in the sum above tend to zero. □

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Paper II

REGULARIZATIONS OF PRODUCTS OF RESIDUE AND PRINCIPAL VALUE CURRENTS

HÅKAN SAMUELSSON

ABSTRACT. Let f_1 and f_2 be two functions on some complex n-manifold and let φ be a test form of bidegree $(n, n-2)$. Assume that (f_1, f_2) defines a complete intersection. The integral of $\varphi/(f_1 f_2)$ on $\{|f_1|^2 = \epsilon_1, |f_2|^2 = \epsilon_2\}$ is the residue integral $I^{\varphi}_{f_1,f_2}(\epsilon_1, \epsilon_2)$. It is in general discontinuous at the origin. Let χ_1 and χ_2 be smooth functions on $[0, \infty]$ such that $\chi_j(0) = 0$ and $\chi_j(\infty) = 1$. We prove that the regularized residue integral defined as the integral of $\bar\partial\chi_1 \wedge \bar\partial\chi_2 \wedge \varphi/(f_1 f_2)$, where $\chi_j = \chi_j(|f_j|^2/\epsilon_j)$, is Hölder continuous on the closed first quarter and that the value at zero is the Coleff-Herrera residue current acting on φ. In fact, we prove that if φ is a test form of bidegree $(n, n-1)$ then the integral of $\chi_1\bar\partial\chi_2 \wedge \varphi/(f_1 f_2)$ is Hölder continuous and tends to the $\bar\partial$-potential $[(1/f_1) \wedge \bar\partial(1/f_2)]$ of the Coleff-Herrera current, acting on φ. More generally, let f_1 and f_2 be sections of some vector bundles and assume that $f_1 \oplus f_2$ defines a complete intersection. There are associated principal value currents U^f and U^g and residue currents R^f and R^g. The residue currents equal the Coleff-Herrera residue currents locally. One can give meaning to formal expressions such as e.g. $U^f \wedge R^g$ in such a way that formal Leibnitz rules hold. Our results generalize to products of these currents as well.

1. INTRODUCTION

Consider a holomorphic function f defined on some complex n-manifold X and let $V_f = f^{-1}(0)$. Schwartz found that there is a distribution, or current, U on X such that $fU = 1$, [23]. The existence of the principal value current $[1/f]$ defined by

$$\mathscr{D}_{n,n}(X) \ni \varphi \mapsto \lim_{\epsilon\to 0} \int_{|f|^2>\epsilon} \varphi/f$$

was proved by Herrera and Lieberman in [11] using Hironaka's desingularization theorem, [12] and gives a realization of such a current U. The $\bar\partial$-image of the principal value current is the residue current associated to f. By Stokes' theorem its action on a test form of bidegree $(n, n-1)$ is given by the limit as $\epsilon \to 0$ (along regular values for $|f|^2$) of the residue integral

$$I^{\varphi}_f(\epsilon) = \int_{|f|^2=\epsilon} \varphi/f. \tag{1}$$

One main point discovered by Herrera and Lieberman is that if φ has bidegree $(n-1, n)$ then for each k, $I^{\varphi}_{f^k}(\epsilon) = \mathcal{O}(\epsilon^{\delta_k})$ for some positive δ_k. Using this, one can then smoothen the integration over $|f|^2 = \epsilon$ and regularize the residue current by using smooth functions χ defined on $[0, \infty)$ such that χ is 0 at zero and tends to 1 at infinity. In fact, we can make a Leray decomposition and write any (n,n)-test form φ as $\phi \wedge \partial f/f^k$ for some k, where ϕ is a test form of bidegree $(n-1,n)$ whose restriction to $|f|^2 = t$ is unique, for each $t > 0$. Then writing the integral of $\chi(|f|^2/\epsilon)\varphi/f$ as an integral over the level surfaces $|f|^2 = t$ and using Herrera's and Lieberman's result one sees that $\chi(|f|^2/\epsilon)/f$ is a regularization of the principal value current $[1/f]$. It follows that the residue current can be obtained as the weak limit of the smooth form $\bar\partial\chi(|f|^2/\epsilon)/f$. This is also a consequence of Corollary 5 below. A natural choice for χ is $\chi(t) = t/(t+1)$ and we see that we get the well known result that the residue current can be obtained as the weak limit of $\bar\partial(\bar f/(|f|^2+\epsilon))$. We also briefly mention the more general currents studied by Barlet, [3]. If we instead integrate over the fiber $f = s$ in (1) and let φ have bidegree $(n-1,n-1)$ then the integral has an asymptotic expansion in s with current coefficients. The constant term is Lelong's integration current on V_f and the residue current $\bar\partial[1/f]$ can be obtained from the coefficient of s^n.

We turn to the main focus of this paper which is the codimension two case. Let f and g be two holomorphic functions on X such that f and g define a complete intersection, that is, the common zero set $V_{f\oplus g}$ has codimension two. Consider the residue integral

$$I^{\varphi}_{f,g}(\epsilon_1,\epsilon_2) = \int_{\substack{|f|^2=\epsilon_1 \\ |g|^2=\epsilon_2}} \frac{\varphi}{fg}. \tag{2}$$

The unrestricted limit of the residue integral as $\epsilon_1,\epsilon_2 \to 0$ does not exist in general. The first example of this phenomenon was discovered by Passare and Tsikh in [19], and Björk later found that this indeed is the typical case, [6]. See also [21]. Via Hironaka's theorem on resolutions of singularities one may assume that the hypersurface $f \cdot g = 0$ has normal crossings, which means that there is a (finite) atlas of charts such that $f(\zeta) = \tilde f(\zeta)\zeta^\alpha$ and $g(\zeta) = \tilde g(\zeta)\zeta^\beta$ where α and β are multiindices (depending on the chart) and $\tilde f$ and $\tilde g$ are invertible holomorphic functions. It is actually the invertible factors which cause problems. One can always dispose of one of the factors, but in general not of both. However, if the matrix A, whose two rows are the integer vectors α and β respectively, has rank two there is a change of variables $z = \tau(\zeta)$ such that $z^\alpha = \tilde f(\zeta)\zeta^\alpha$ and $z^\beta = \tilde g(\zeta)\zeta^\beta$, see e.g. [16]. Hence, when α and β are not linearly dependent we can make both the invertible factors disappear. Problems therefore arise in so called charts of resonance where α and β are linearly dependent. Coleff and Herrera realized that if one demands that ϵ_1 and ϵ_2 tend to zero in such a way that $\epsilon_1/\epsilon_2^k \to 0$ for all $k \in \mathbb{Z}_+$, along a so called admissible path, then one will get no contributions from the charts of resonance because one cannot have $|\tilde f(\zeta)\zeta^\alpha| << |\tilde g(\zeta)\zeta^\beta|$ if α and β are linearly dependent. They proved in [8] that the limit, along an admissible path, of the residue integral exists and defines the action of a $(0,2)$-current, the Coleff-Herrera residue current $[\bar\partial(1/f) \wedge \bar\partial(1/g)]$. In [16] Passare smoothened the integration over the set $\{|f|^2 = \epsilon_1\} \cap \{|g|^2 = \epsilon_2\}$ by introducing functions χ as described above, and he studied possible weak limits of forms

$$\frac{\bar\partial\chi_1(|f|^2/\epsilon_1)}{f}\wedge\frac{\bar\partial\chi_2(|g|^2/\epsilon_2)}{g} \tag{3}$$

along parabolic paths $(\epsilon_1,\epsilon_2)=(\epsilon^{s_1},\epsilon^{s_2})$ where $s=(s_1,s_2)$ belongs to the simplex $\Sigma_2(2)=\{(x,y)\in\mathbb{R}^2_+;\, s_1+s_2=2\}$. He found that it is enough to impose finitely many linear conditions $(n_j,s)\neq 0$ to assure that (3) has a weak limit along the corresponding parabolic path. The linear conditions partition $\Sigma_2(2)$ into finitely many open segments and the weak limit of (3) along a parabolic path corresponding to an s in such a segment only depends on the segment. We say that (ϵ_1,ϵ_2) tends to zero inside a Passare sector. Moreover, as we assume that f and g define a complete intersection, the limit is even independent of the choice of segment. In this case it also coincides with the Coleff-Herrera current. One can obtain a $\bar\partial$-potential to the Coleff-Herrera current e.g. by changing the integration set in (2) to $\{|f|^2>\epsilon_1\}\cap\{|g|^2=\epsilon_2\}$ and pass to the limit along an admissible path or by removing the first $\bar\partial$ in (3) and pass to the limit inside a Passare sector. This $\bar\partial$-potential is denoted $[(1/f)\bar\partial(1/g)]$. The main result in this paper implies that if $\chi_j\in C^\infty([0,\infty])$ satisfy $\chi_j(0)=0$ and $\chi_j(\infty)=1$ then, in the sense of currents

$$\lim_{\epsilon_1,\epsilon_2\to 0}\frac{\chi_1(|f|^2/\epsilon_1)}{f}\frac{\bar\partial\chi_2(|g|^2/\epsilon_2)}{g}=[\frac{1}{f}\bar\partial\frac{1}{g}], \tag{4}$$

and the action of the smooth form on the left hand side on a test form depends Hölder continuously on $(\epsilon_1,\epsilon_2)\in[0,\infty)^2$. For the particular case when $\chi_j(t)=t/(t+1)$ our result, apart from the Hölder continuity, was announced in [22]. Actually, it is possible to relax the smoothness assumption on one of the χ_j in (4). As mentioned above, one can always dispose of one of the invertible factors. Say that we always arrange so that $\tilde f\equiv 1$. Then, examining the proof, one finds that one may take χ_1 to be the characteristic function of $[1,\infty]$. Hence,

$$\int_{|f|^2>\epsilon_1}\frac{\bar\partial\chi_2(|g|^2/\epsilon_2)}{fg}\wedge\varphi\to[\frac{1}{f}\bar\partial\frac{1}{g}].\varphi$$

with Hölder continuity. Note that if we let both χ_1 and χ_2 be the characteristic function of $[1,\infty]$ then this result is no longer true in view of the examples of Passare-Tsikh and Björk.

Our result also generalize to products of pairs of so called Bochner-Martinelli blocks. Consider a tuple $f=(f_1,\dots,f_m)$ of holomorphic functions on X. The residue integral corresponding to f, $I^\varphi_f(\epsilon_1,\dots,\epsilon_m)$, is defined analogously to (2). If we take the mean value of the residue integral over $\epsilon=(\epsilon_1,\dots,\epsilon_m)$ in the simplex $\Sigma_m(\delta)=\{s\in\mathbb{R}^m_+;\,\sum s_j=\delta\}$ we obtain

$$c_m\int_{|f|^2=\delta}\frac{\sum_{j=1}^m(-1)^{j+1}\bar f_j\bigwedge_{i\neq j}\bar\partial\bar f_i}{|f|^{2m}}\wedge\varphi, \tag{5}$$

where c_m is a constant only depending on m. It turns out, see [20], that the limit as δ tends to zero of (5) exists and defines the action of a $(0,m)$-current, which in the case f defines a complete intersection, coincides with the Coleff-Herrera current and also with the currents studied in [5] and [18]. Based on the work in [20] Andersson introduces more general currents of the Cauchy-Fantappiè-Leray type in [2]. We will briefly discuss Andersson's construction in Section 3. In short, he defines a singular form $u^f = \sum u^f_{k,k-1}$, where the terms $u^f_{k,k-1}$ are similar to the form in (5), and he shows that it is extendible to X as a current, U^f, either as principal values or by analytic continuation. The residue current, R^f, is derived from the current U^f and equals the Coleff-Herrera current locally if f defines a complete intersection. If g is also a tuple of functions there is a natural way of defining the product of the Cauchy-Fantappiè-Leray type currents corresponding to f and g so that formal Leibnitz rules hold, see [26]. If $f \oplus g$ defines a complete intersection and $\chi_1, \chi_2 \in C^\infty([0,\infty])$ vanish to high enough orders at zero and equals 1 at infinity then we prove that the smooth forms

$$\chi_1(|f|^2/\epsilon_1)u^f \wedge \bar\partial\chi_2(|g|^2/\epsilon_2) \wedge u^g \text{ and}$$

$$\bar\partial\chi_1(|f|^2/\epsilon_1) \wedge u^f \wedge \bar\partial\chi_2(|g|^2/\epsilon_2) \wedge u^g$$

are Hölder continuous as currents for $(\epsilon_1, \epsilon_2) \in [0,\infty)^2$ and tend to $U^f \wedge R^g$ and $R^f \wedge R^g$ respectively as $\epsilon_1, \epsilon_2 \to 0$; Theorem 21 and Corollary 23. If g is a function such that $f \oplus g$ defines a complete intersection, our techniques can also be used to prove that $\bar\partial\chi_1(|f|^2/\epsilon_1) \wedge u^f \chi_2(|g|^2/\epsilon_2) \to R^f$ when χ_2 equals the characteristic function of $[1,\infty]$. We use this to conclude that R^f has the standard extension property in the complete intersection case, Corollary 24. For more historical accounts we refer to the survey article [7] by Björk.

The disposition of the paper is as follows: In Section 2 we outline a proof of (4) since the proofs of the more general statements about Bochner-Martinelli or Cauchy-Fantappiè-Leray blocks are only more difficult to prove in the technical sense and to make it clear that it is not necessary to work through the constructions of Bochner-Martinelli or Cauchy-Fantappiè-Leray type currents in order to prove (4). In Section 3 we recall Andersson's construction and explain some useful notation. Section 4 contains some fairly well known regularization results about Cauchy-Fantappiè-Leray type currents. As Andersson's formalism makes the arguments a little smoother we also supply the proofs. Section 5 contains the technical core of this paper. We study regularizations of products of monomial currents which we then use in Section 6 to prove our main results; Theorem 21 and its corollaries 23, 25 and 26 and Theorem 27. In Section 7 we see by explicit computations that Corollary 26 holds for the example by Passare and Tsikh. This section is essentially self-contained.

2. Sketch of proof in the case of two functions

Let f and g be two holomorphic functions on X defining a complete intersection. We sketch how one can handle the difficulties arising in charts of resonance when proving (4). We study the integral

$$\int \frac{\chi_1(|f|^2/\epsilon_1)}{f}\frac{\bar\partial\chi_2(|g|^2/\epsilon_2)}{g} \wedge \varphi \tag{6}$$

where φ is a test form of bidegree $(n, n-1)$. By Hironaka's theorem we may assume that $f = \zeta^\alpha \tilde{f}$ and $g = \zeta^\beta \tilde{g}$ are monomials times non-vanishing functions. One of the non-zero factors can be incorporated in a variable and so we assume that $\tilde{f} \equiv 1$. We assume also that we are in a chart of resonance, i.e. that α and β are linearly dependent. After resolving singularities f and g no longer define a complete intersection in general, but on the other hand a degree argument shows that $d\bar{\zeta}_j/\bar{\zeta}_j \wedge \varphi$ becomes a test form for any ζ_j dividing both f and g. See the proof of Theorem 21 for more details. Since α and β are linearly dependent, $d\bar{\zeta}_j/\bar{\zeta}_j \wedge \varphi$ is a test form for all j such that $\alpha_j \neq 0$, or equivalently, $\beta_j \neq 0$. Now, (6) equals

$$\sum_j \beta_j \int \frac{\chi_1(|\zeta^\alpha|^2/\epsilon_1)}{\zeta^\alpha} \frac{\chi_2'(\Psi|\zeta^\beta|^2/\epsilon_2)}{\zeta^\beta} \frac{|\zeta^\beta|^2}{\epsilon_2} \wedge \frac{d\bar{\zeta}_j}{\bar{\zeta}_j} \wedge \varphi/\tilde{f}$$

where $\Psi = |\tilde{g}|^2$ is a strictly positive smooth function. It now follows from Corollary 15 that each term in this sum tends to zero as ϵ_1 and ϵ_2 tend to zero. Hence the charts of resonance do not give any contributions.

3. Preliminaries and notation

Assume that f is a section of the dual bundle E^* of a holomorphic m-bundle $E \to X$ over a complex n-manifold X. We will only deal with local problems and it is therefore no loss of generality in assuming that $E \to X$ is trivial. However, the formalism will run smoother with an invariant notation. As mentioned above, we will recall Andersson's construction in [2] and produce currents U^f and R^f and we emphasize that in the case $E \to X$ is the trivial line bundle then U^f and R^f are the currents $[1/f]$ and $\bar{\partial}[1/f]$ times some basis elements. On the exterior algebra ΛE of E, the section f induces mappings $\delta_f \colon \Lambda^{k+1}E \to \Lambda^k E$ of interior multiplication and $\delta_f^2 = 0$. We introduce the spaces $\mathcal{E}_{0,q}(X, \Lambda^k E)$ of the smooth sections of the exterior algebra of $E \oplus T^*_{0,1}X$ which are $(0,q)$-forms with values in $\Lambda^k E$. We also introduce the corresponding spaces of currents, $\mathscr{D}'_{0,q}(X, \Lambda^k E)$. The mappings δ_f extend to mappings $\delta_f \colon \mathscr{D}'_{0,q}(X, \Lambda^{k+1}E) \to \mathscr{D}'_{0,q}(X, \Lambda^k E)$ with $\delta_f^2 = 0$ and these mappings anti-commute with the $\bar{\partial}$-operator. Hence, $\mathscr{D}'_{0,q}(X, \Lambda^k E)$ is a double complex and the associated total complex is

$$\cdots \overset{\nabla_f}{\to} \mathcal{L}^{r-1}(X,E) \overset{\nabla_f}{\to} \mathcal{L}^r(X,E) \overset{\nabla_f}{\to} \cdots$$

where $\mathcal{L}^r(X,E) = \bigoplus_{q-k=r} \mathscr{D}'_{0,q}(X, \Lambda^k E)$ and $\nabla_f = \delta_f - \bar{\partial}$. We will refer to the total complex as the Andersson complex. The exterior product, $\wedge$, induces mappings

$$\wedge \colon \mathcal{L}^r(X,E) \times \mathcal{L}^s(X,E) \to \mathcal{L}^{r+s}(X,E)$$

when possible, and ∇_f is an antiderivation, i.e. $\nabla_f(\tau \wedge \sigma) = \nabla_f \tau \wedge \sigma + (-1)^r \tau \wedge \nabla_f \sigma$ if $\tau \in \mathcal{L}^r(X,E)$ and $\sigma \in \mathcal{L}^s(X,E)$. If $\tau \in \mathcal{L}^r(X,E)$ we write $\tau_{k,k+r}$ for the component of τ belonging to $\mathscr{D}'_{0,k+r}(X, \Lambda^k E)$. Note that functions define elements of $\mathcal{L}^0(X,E)$ of degree $(0,0)$ and sections of E define elements of $\mathcal{L}^{-1}(X,E)$ of degree $(1,0)$. One can show, see [2], that if X is Stein and the zero:th cohomology group of the Andersson complex vanishes then for any holomorphic function h there is a holomorphic section ψ of E such that $\delta_f \psi = h$. This means that if $f = (f_1, \dots, f_m)$ in some local holomorphic frame for E^* then the division problem $\sum f_j \psi_j = h$ has a holomorphic solution. This cannot hold for all h if f has zeros and the Andersson complex can therefore not be exact in this case. Still, we try to look for an element $u^f \in \mathcal{L}^{-1}(X, E)$ such that $\nabla_f u^f = 1$. To this end we assume that E is equipped with some Hermitian metric $|\cdot|$ and we let s_f be the section of E with pointwise minimal norm such that $\delta_f s_f = |f|^2$. Outside $V_f = f^{-1}(0)$ we may put

$$u^f = \frac{s_f}{\nabla_f s_f} = \frac{s_f}{\delta_f s_f - \bar\partial s_f} = \sum_k \frac{s_f \wedge (\bar\partial s_f)^{k-1}}{|f|^{2k}}.$$

Observe that $\nabla_f s_f$ has even degree so the expression $s_f / \nabla_f s_f$ has meaning outside V_f and it follows immediately that $\nabla_f u = 1$ there. The following theorem is proved in [2].

Theorem 1. *Assume that f is locally nontrivial. The forms $|f|^{2\lambda} u^f$ and $\bar\partial |f|^{2\lambda} \wedge u^f$ are locally bounded if $\mathfrak{Re}\,\lambda$ is sufficiently large and they have analytic continuations as currents to $\mathfrak{Re}\,\lambda > -\epsilon$. Let U^f and R^f denote the values at $\lambda = 0$. Then U^f is a current extension of u^f, R^f has support on V_f and*

$$\nabla_f U^f = 1 - R^f.$$

Moreover, $R^f = R^f_{p,p} + \cdots + R^f_{q,q}$ where $p = Codim(V_f)$ and $q = min(m, n)$.

Note that if $V_f = \emptyset$ then $\nabla_f U^f = 1$ on all of X, which implies that taking the exterior product with U^f is a homotopy operator for the Andersson complex. The current R^f is the Bochner-Martinelli, or more generally, the Cauchy-Fantappiè-Leray current associated to f, and if $f = (f_1, \dots, f_m)$ in some local holomorphic frame, $e_1, \dots, e_m$, of E then

$$R^f = [\bar\partial \frac{1}{f_1} \wedge \cdots \wedge \bar\partial \frac{1}{f_m}] \wedge e_1 \wedge \cdots \wedge e_m \tag{7}$$

if f defines a complete intersection, see [2].

Now if f_j, $j = 1, 2$, are sections of the dual bundles E_j^* of holomorphic Hermitian m_j-bundles $E_j \to X$ we can apply the above construction to the section $f = f_1 \oplus f_2$ of the bundle $E_1^* \oplus E_2^*$ and obtain the currents U^f and R^f. We could also try to combine the individual currents U^{f_j} and R^{f_j}. It is shown in [26] that the forms

$$|f_1|^{2\lambda} u^{f_1} \wedge |f_2|^{2\lambda} u^{f_2}, \ \ |f_1|^{2\lambda} u^{f_1} \wedge \bar\partial |f_2|^{2\lambda} \wedge u^{f_2} \ \text{ and } \ \bar\partial |f_1|^{2\lambda} \wedge u^{f_1} \wedge \bar\partial |f_2|^{2\lambda} \wedge u^{f_2},$$

which are locally bounded if $\mathfrak{Re}\,\lambda$ is large enough, have current extensions to $\mathfrak{Re}\,\lambda > -\epsilon$. The values at $\lambda = 0$ are denoted $U^{f_1} \wedge U^{f_2}$, $U^{f_1} \wedge R^{f_2}$, and $R^{f_1} \wedge R^{f_2}$, respectively, and formal computation rules such as e.g. $\nabla_f (U^{f_1} \wedge R^{f_2}) = (1 - R^{f_1}) \wedge R^{f_2} = R^{f_2} - R^{f_1} \wedge R^{f_2}$ hold. It is also shown in [26] that if f defines a complete intersection then $R^f = R^{f_1} \wedge R^{f_2}$.

We will use the names f and g, rather then f_1 and f_2, for the sections of the two bundles and the symbol ∇, without subscript, always denotes $\nabla_{f \oplus g}$. We will use multiindices extensively in the sequel. Multiindices will be denoted

α and β or I and J and sometimes also r and ρ. The number of variables will always be n but it will be convenient to define multiindices by expressions like $\alpha = (\alpha_j)_{j\in K}$ for $K \subseteq \{1,\dots,n\}$. By this we mean that $\alpha = (a_1,\dots,a_n)$ where $a_j = 0$ if $j \notin K$ and $a_j = \alpha_j$ if $j \in K$. Hence, if $z = (z_1,\dots,z_n)$ then $z^\alpha = \prod_{j\in K} z_j^{\alpha_j}$ and similarly for $\partial^\alpha/\partial z^\alpha$. Multiindices are added and multiplied by numbers as elements in $\mathbb{Z}^n$ and $\alpha \pm 1 = (\alpha_1 \pm 1,\dots,\alpha_n \pm 1)$. Also, $|\alpha|$ denotes the length of α as a vector in Euclidean space and $\#\alpha$ is the cardinality of the support of α.

Integration over domains in $\mathbb{C}^n$ will always be with respect to the volume form $(i/2)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n := (i/2)^n dz \wedge d\bar{z}$ if nothing else is said. If Δ is a Reinhardt domain in $\mathbb{C}^n$ and φ is a function which only depends on the moduli of the variables and such that $z^\alpha\varphi(z)$ is integrable on Δ then

$$\int_\Delta z^\alpha \varphi(z) = 0$$

if α is a non-zero multiindex. This simple fact will play a fundamental role to us in what follows and we will refer to it as *anti-symmetry*.

Unless otherwise stated, the symbol χ with various subscripts will always denote a smooth function on $[0,\infty]$ which is zero to some order at 0 and such that $\chi(\infty) = 1$. By smooth at infinity we mean that $t \mapsto \chi(1/t)$ is smooth at zero.

4. Regularizations of Cauchy-Fantappiè-Leray type currents

Consider a function χ as above and let $\tilde{\chi}(s) = \chi(1/s)$. Then $\tilde{\chi}$ is differentiable at $s = 0$ and $\tilde{\chi}'(s) = -\chi'(1/s)/s^2$. Letting $t = 1/s$ we see that $\chi'(t) = \mathcal{O}(1/t^2)$ as $t \to \infty$. This simple observation will be frequently used in the sequel. It follows that for any continuous function φ with compact support in $[0,\infty)$ we have $|\varphi(\epsilon t)\chi'(t)| \le C(t+1)^{-2}$ for a constant independent of ϵ. Hence by the dominated convergence theorem we see that

$$\int_0^\infty \frac{d}{dt}\chi(t/\epsilon)\varphi(t)dt = \int_0^\infty \frac{d}{d\tau}\chi(\tau)\varphi(\epsilon\tau)d\tau \to \varphi(0)\int_0^\infty \frac{d}{d\tau}\chi(\tau)d\tau = \varphi(0),$$

and we have proved

Lemma 2. *Let $\chi \in C^1([0,\infty])$ satisfy $\chi(0) = 0$ and $\chi(\infty) = 1$. Then $(d/dt)\,\chi(t/\epsilon) \to \delta_0$ as measures on $[0,\infty)$.*

Proposition 3. *Assume $\chi \in C^\infty([0,\infty])$ vanishes to order ℓ at 0 and satisfies $\chi(\infty) = 1$. Then*

$$\lim_{\epsilon\to 0^+} \int \chi(|f|^2/\epsilon)u^f_{\ell,\ell-1} \wedge \varphi = U^f_{\ell,\ell-1}.\varphi$$

for any test form φ.

Proof. On the set $\Omega = \{(z,t) \in \mathbb{C}^n \times (0,\infty); |f(z)|^2 > t\}$ we have, for all fixed $\epsilon > 0$, that

$$\begin{aligned}\big|u^f_{\ell,\ell-1}\frac{d}{dt}\chi(t/\epsilon)\wedge\varphi\big| &\le C\frac{1}{|f|^{2\ell-1}}\big|\frac{d}{dt}\chi(t/\epsilon)\big| \le \\ &\quad C\frac{t^{1/2}}{t^\ell}\big|\frac{d}{dt}\chi(t/\epsilon)\big| \le C\frac{1}{t^{1/2}}\end{aligned}$$

since $\frac{d}{dt}\chi(t/\epsilon)=\mathcal{O}(t^{\ell-1})$. Hence we have an integrable singularity on Ω and by Fubini's theorem we get

$$\int_0^\infty \frac{d}{dt}\chi(t/\epsilon)\int_{|f|^2>t} u^f_{\ell,\ell-1}\wedge\varphi\,dt \quad=\quad \int u^f_{\ell,\ell-1}\wedge\varphi\int_0^{|f|^2}\frac{d}{dt}\chi(t/\epsilon)dt =$$

$$\int u^f_{\ell,\ell-1}\chi(|f|^2/\epsilon)\wedge\varphi. \tag{8}$$

But $J(t)=\int_{|f|^2>t}u^f_{\ell,\ell-1}\wedge\varphi$ is a continuous function with compact support in $[0,\infty)$ with $J(0)=U^f_{\ell,\ell-1}.\varphi$, see [20] or [2]. Hence by Lemma 2 the left hand side of (8) tends to $U^f_{\ell,\ell-1}.\varphi$ and the proof is complete. □

If we take $\chi(t)$ equal to appropriate powers of $t/(t+1)$ we obtain the following natural ways to regularize the currents U^f and R^f.

Corollary 4. *For any test form φ we have*

$$\lim_{\epsilon\to0^+}\int\sum_{\ell\geq1}\frac{s_f\wedge(\bar\partial s_f)^{\ell-1}}{(|f|^2+\epsilon)^\ell}\wedge\varphi=U^f.\varphi \tag{9}$$

and

$$\lim_{\epsilon\to0^+}\int\sum_{\ell\geq1}\epsilon\frac{(\bar\partial s_f)^{\ell}}{(|f|^2+\epsilon)^{\ell+1}}\wedge\varphi=R^f.\varphi. \tag{10}$$

Proof. Letting $\chi_\ell(t)=t^\ell/(t+1)^\ell$ we see that

$$u^f_{\ell,\ell-1}\chi_\ell(|f|^2/\epsilon)=\frac{s_f\wedge(\bar\partial s_f)^{\ell-1}}{(|f|^2+\epsilon)^\ell}$$

and so (9) follows from Proposition 3. To show that (10) holds we first note that

$$\sum_{\ell\geq1}\frac{s_f\wedge(\bar\partial s_f)^{\ell-1}}{(|f|^2+\epsilon)^\ell}=\frac{s_f}{\nabla_f s_f+\epsilon}.$$

Hence

$$\nabla_f\sum_{\ell\geq1}\frac{s_f\wedge(\bar\partial s_f)^{\ell-1}}{(|f|^2+\epsilon)^\ell}=\nabla_f\frac{s_f}{\nabla_f s_f+\epsilon}=\frac{\nabla_f s_f}{\nabla_f s_f+\epsilon}=1-\sum_{\ell\geq0}\epsilon\frac{(\bar\partial s_f)^\ell}{(|f|^2+\epsilon)^{\ell+1}}.$$

Since differentiation is a continuous operation on distributions it follows from (9) that

$$\lim_{\epsilon\to0^+}1-\sum_{\ell\geq0}\epsilon\frac{(\bar\partial s_f)^\ell}{(|f|^2+\epsilon)^{\ell+1}}=\nabla_f\lim_{\epsilon\to0^+}\sum_{\ell\geq1}\frac{s_f\wedge(\bar\partial s_f)^{\ell-1}}{(|f|^2+\epsilon)^\ell}=\nabla_f U^f=1-R^f$$

in the sense of currents. The term with $\ell=0$ in the sum on the left is easily seen to tend to zero in the sense of currents and hence (10) follows. □

Note that it is the difference

$$\bar\partial(\chi_\ell u^f_{\ell,\ell-1})-\delta_f(\chi_{\ell+1}u^f_{\ell+1,\ell})=\bar\partial\chi_\ell\wedge u^f_{\ell,\ell-1}+(\chi_\ell-\chi_{\ell+1})\delta_f u^f_{\ell+1,\ell} \tag{11}$$

which converges to the term of R^f of bidegree (ℓ,ℓ). It is only for the term of top degree, the last term in (11) is not present. This explains why the regularization result in [20], Theorem 2.1, coincides with our result for the top degree term but not for the terms of lower degree.

We can also take one χ which vanishes to high enough order at zero to regularize all terms of U^f and R^f.

Corollary 5. *Assume that* $\chi \in C^\infty([0,\infty])$*, vanishes to order* $min(m,n)+1$ *at zero and satisfies* $\chi(\infty)=1$*. Then for any test form* φ *we have*

$$\lim_{\epsilon\to 0^+}\int \chi(|f|^2/\epsilon)u^f\wedge\varphi = U^f.\varphi \tag{12}$$

$$\lim_{\epsilon\to 0^+}\int \bar\partial\chi(|f|^2/\epsilon)\wedge u^f\wedge\varphi = R^f.\varphi. \tag{13}$$

Proof. The first statement follows immediately from Proposition 3. For the second one we note that

$$\nabla\chi u^f = \nabla\chi\wedge u^f + \chi\nabla u^f = -\bar\partial\chi\wedge u^f + \chi\nabla u^f,$$

and since χ vanishes to high enough order at zero all terms are smooth. Outside $\{f=0\}$ we have $\nabla u^f = 1$ and hence $\chi\nabla u^f = \chi$ everywhere. Moreover, $\chi(|f|^2/\epsilon)$ tends to 1 in the sense of currents and hence

$$\bar\partial\chi\wedge u^f = \chi\nabla u^f - \nabla\chi u^f \to 1-(1-R^f) = R^f$$

in the sense of currents. □

5. Regularizations of products of monomial currents

This section contains the technical result about the normal crossing case needed to prove our main theorems in the next section. Of particular importance is Proposition 11. First we need a generalization of Taylor's formula. Lemma 6 enables us to approximate a smooth function defined on $\mathbb{C}^n$ in a neighborhood of the union of the coordinate hyperplanes instead of in a neighborhood of their intersection as in the usual Taylor's formula. The approximating functions are in our case not polynomials in general but have enough similarities for our purposes. For tensor products of one-variable functions this corresponds to multiplying the individual Taylor expansions. Lemma 6 appears as Lemma 2.3 in [22] but the formulation there is unfortunately not completely correct. We also remark that Lemma 6 is very similar to Lemma 2.4 in [8] and that very general Taylor expansions are considered in Chapter 1 in [13]. Define the linear operator $M_j^{r_j}$ on $C^\infty(\mathbb{C}^n)$ to be the operator that maps φ to the Taylor polynomial of degree r_j of the function $\zeta_j\mapsto\varphi(\zeta)$ (centered at $\zeta_j=0$). We note that $M_j^{r_j}$ and $M_i^{r_i}$ commute. To see this we only need to observe that

$$\frac{\partial}{\partial\tilde\zeta_i}\Big(\frac{\partial\varphi}{\partial\tilde\zeta_j}\Big|_{\zeta_j=0}\Big)\Big|_{\zeta_i=0} = \frac{\partial^2\varphi}{\partial\tilde\zeta_i\partial\tilde\zeta_j}\Big|_{\zeta_i=\zeta_j=0} = \frac{\partial}{\partial\tilde\zeta_j}\Big(\frac{\partial\varphi}{\partial\tilde\zeta_i}\Big|_{\zeta_i=0}\Big)\Big|_{\zeta_j=0}$$

where $\partial/\partial\tilde\zeta_j$ means that we do not specify whether we differentiate with respect to ζ_j or $\bar\zeta_j$.

Lemma 6. *Let $K \subseteq \{1,\dots,n\}$ have cardinality κ and let $r = (r_j)_{j\in K}$. Define the linear operator M_K^r on $C^\infty(\mathbb{C}^n)$ by*

$$M_K^r = \sum_{j\in K} M_j^{r_j} - \sum_{\substack{i,j\in K\\ i<j}} M_i^{r_i} M_j^{r_j} + \cdots + (-1)^{\kappa+1} M_{j_1}^{r_{j_1}} \cdots M_{j_\kappa}^{r_{j_\kappa}}.$$

Then for any $\varphi \in C^\infty(\mathbb{C}^n)$ we have

$$\varphi(\zeta) = M_K^r\varphi(\zeta) + \int_{[0,1]^\kappa} \frac{(1-t)^r}{r!} \frac{\partial^{r+1}}{\partial t^{r+1}} \varphi(t\zeta)\, dt \tag{14}$$

where $t\zeta$ should be interpreted as $(\xi_1,\dots,\xi_n)$, $\xi_j = t_j\zeta_j$ if $j\in K$ and $\xi_j = \zeta_j$ if $j \notin K$. In particular $\varphi - M_K^r\varphi = \mathcal{O}(|\zeta^{r+1}|)$. Moreover, $M_K^r\varphi$ can be written as a finite sum of terms, $\varphi_{IJ}(\zeta)\zeta^I\bar\zeta^J$, with the following properties:

(a) *$\varphi_{IJ}(\zeta)$ is independent of some variable and in particular of variable ζ_j if $I_j + J_j > 0$,*
(b) *$I_j + J_j \le r_j$ for $j \in K$,*
(c) *if L is the set of indices $j\in K$ such that $\zeta_j \mapsto \varphi_{IJ}(\zeta)$ is non-constant then $\varphi_{IJ}(\zeta) = \mathcal{O}(\prod_{j\in L} |\zeta_j|^{r_j+1})$.*

Proof. It is enough to prove the lemma when $K = \{1,\dots,n\}$. In case $n=1$, (14) is Taylor's formula. For $n \ge 2$, we write the integral in (14) as an iterated integral. Formula (14) then follows by induction. One can also show (14) by repeated integrations by parts. The difference $\varphi - M_K^r\varphi$ is seen to be of the desired size after performing the differentiations of $\varphi(t\zeta)$ with respect to t inside the integral. To see that $M_K^r\varphi$ can be written as a sum of terms $\varphi_{IJ}(\zeta)\zeta^I\bar\zeta^J$ with the properties (a), (b), and (c), we let $r_{\tilde K}$, for any $\tilde K \subseteq K$, denote the multiindex $(r_{j_1},\dots,r_{j_{|\tilde K|}})$, $r_{i_j} \in \tilde K$. A straight forward computation now shows that

$$\begin{aligned} M_K^r\varphi &= \sum_{j\in K} M_j^{r_j}(\varphi - M_{K\setminus\{j\}}^{r_{K\setminus\{j\}}}\varphi) \\ &+ \sum_{\substack{i,j\in K\\ i<j}} M_i^{r_i}M_j^{r_j}(\varphi - M_{K\setminus\{i,j\}}^{r_{K\setminus\{i,j\}}}\varphi) \\ &\vdots \\ &+ M_{j_1}^{r_{j_1}} \cdots M_{j_\kappa}^{r_{j_\kappa}}\varphi. \end{aligned}$$

From the first part of the proof (and the definition of $M_j^{r_j}$) it follows that every term on the right hand side is a finite sum of terms with the stated properties. □

Lemma 7. *Let α be a multiindex and let $M = M_K^r$ be the operator defined in Lemma 6 with K the set of indices j such that $\alpha_j \ge 2$ and $r_j = \alpha_j - 2$, $j\in K$. Then for any $\varphi \in \mathscr{D}(\mathbb{C}^n)$ we have*

$$\int_\Delta \frac{1}{\zeta^\alpha}(\varphi - M\varphi) = \Big[\frac{1}{\zeta^\alpha}\Big].\varphi\,(i/2)^n d\zeta\wedge d\bar\zeta$$

if Δ is a polydisc containing the support of φ.

Proof. Note that by Lemma 6 we have $\varphi - M\varphi = \mathcal{O}(|\zeta^{\alpha-1}|)$ and so $(\varphi - M\varphi)/\zeta^\alpha$ is integrable on Δ. Hence if we let $\Delta_\delta = \Delta \cap_j \{|\zeta_j| > \delta\}$ we get

$$\begin{aligned}\int_\Delta \frac{1}{\zeta^\alpha}(\varphi - M\varphi) &= \lim_{\delta\to 0}\int_{\Delta_\delta}\frac{1}{\zeta^\alpha}(\varphi - M\varphi)\\ &= \lim_{\delta\to 0}\int_{\Delta_\delta}\frac{1}{\zeta^\alpha}\varphi - \lim_{\delta\to 0}\int_{\Delta_\delta}\frac{1}{\zeta^\alpha}M\varphi.\end{aligned}$$

The first limit on the right hand side is the tensor product of the principal value currents $[1/\zeta_j^{\alpha_j}]$ (acting on $\varphi\,(i/2)^n d\zeta\wedge d\bar{\zeta}$) and hence it equals $[1/\zeta^\alpha].\varphi\,(i/2)^n d\zeta\wedge d\bar{\zeta}$. It follows by anti-symmetry that actually

$$\int_{\Delta_\delta}\frac{1}{\zeta^\alpha}M\varphi = 0$$

for all $\delta > 0$. In fact, $M\varphi$ is a sum of terms $\varphi_{IJ}(\zeta)\zeta^I\bar{\zeta}^J$ where $I_j + J_j \le \alpha_j - 2$ for all j and the coefficient $\varphi_{IJ}(\zeta)$ is at least independent of some variable. □

Lemma 8. *Let $\chi_1, \chi_2 \in C^\infty([0,\infty])$ and let Φ and Ψ be smooth strictly positive functions on $\mathbb{C}^n$. Let also M_K^r be the operator defined in Lemma 6 with K and r arbitrary. Then*

$$\chi_1(t_1\Phi)\chi_2(t_2\Psi) = M_K^r(\chi_1(t_1\Phi)\chi_2(t_2\Psi)) + |\zeta^{r+1}|B(t_1,t_2,\zeta),$$

where B is bounded on $(0,\infty)^2\times D$ if $D \Subset \mathbb{C}^n$.

Proof. If $D \Subset \mathbb{C}^n$ both Φ and Ψ have strictly positive infima and finite suprema on D and so there is a neighborhood U of $[0,\infty]^2$ in $\widehat{\mathbb{R}}\times\widehat{\mathbb{R}}$ such that the function $(t_1,t_2,\zeta)\mapsto \chi_1(t_1\Phi)\chi_2(t_2\Psi)$ is smooth on $U\times D$. From Lemma 6 it follows that

$$\chi_1(t_1\Phi)\chi_2(t_2\Psi) = M_K^r(\chi_1(t_1\Phi)\chi_2(t_2\Psi)) + \sum_{\substack{I,J\subseteq K\\ I_j+J_j=r_j+1}} G_{IJ}(t_1,t_2,\zeta)\zeta^I\bar{\zeta}^J$$

for some functions G_{IJ} which are smooth on $U\times D$, and the lemma readily follows. □

To prove Proposition 11 we will need the estimates of the following two elementary lemmas.

Lemma 9. *Let Δ be the unit polydisc in $\mathbb{C}^n$ and put $\Delta_\epsilon^\alpha = \{\zeta\in\Delta; |\zeta^\alpha|^2 \ge \epsilon\}$ and $\Delta_{\epsilon_1,\epsilon_2}^{\alpha,\beta} = \{\zeta\in\Delta; |\zeta^\alpha|^2\ge\epsilon_1, |\zeta^\beta|^2\ge\epsilon_2\}$. Then for all $\epsilon,\epsilon_j\le 1$ we have*

$$\int_{\Delta\setminus\Delta_\epsilon^\alpha}\frac{1}{|\zeta_1|\cdots|\zeta_n|} \lesssim \epsilon^{1/(2|\alpha|)}|\log\epsilon|^{n-1}$$

and

$$\int_{\Delta\setminus\Delta_{\epsilon_1,\epsilon_2}^{\alpha,\beta}}\frac{1}{|\zeta_1|\cdots|\zeta_n|} \lesssim |(\epsilon_1,\epsilon_2)|^\omega,\ \ 2\omega < min\{|\alpha|^{-1},|\beta|^{-1}\}.$$

Proof. On the set $\Delta \setminus \Delta^{\alpha,\beta}_{\epsilon_1,\epsilon_2}$, either $|\zeta^\alpha|^2 < \epsilon_1$ or $|\zeta^\beta|^2 < \epsilon_2$ and so it follows from the first inequality that the integral in the second inequality is less then or equal to (a constant times)

$$\begin{aligned} \epsilon_1^{1/(2|\alpha|)}|\log\epsilon_1|^{n-1} + \epsilon_2^{1/(2|\beta|)}|\log\epsilon_2|^{n-1} &\lesssim \epsilon_1^{1/(2|\alpha|)-\nu} + \epsilon_2^{1/(2|\beta|)-\nu} \\ &\lesssim |(\epsilon_1,\epsilon_2)|^{\omega_\nu}, \end{aligned}$$

for any $\nu > 0$ and $\omega_\nu \leq \min\{|\alpha|^{-1}, |\beta|^{-1}\}/2 - \nu$. Hence the second inequality follows from the the first one. To prove the first inequality we first integrate with respect to the angular variables and then we make the change of variables $x_j = \log|\zeta_j|$ to see that the integral in question equals

$$(4\pi)^n \int_{Q_\epsilon} e^{\sum x_j}\, dx, \tag{15}$$

where $Q_\epsilon = \{x \in (-\infty, 0]^n;\, 2\sum \alpha_j x_j < \log\epsilon\}$. Since all $x_j \leq 0$ on Q_ϵ we have $\exp(\sum x_j) \leq \exp(-|x|)$ here, and choosing $R = |\log\epsilon|/(2|\alpha|)$ we see that (15) is less then or equal to $\int_{\{|x|>R\}} \exp(-|x|)dx$. In polar coordinates this is easily seen to be of order $\epsilon^{1/(2|\alpha|)}|\log\epsilon|^{n-1}$. $\square$

Lemma 10. *Let Δ be the unit polydisc in $\mathbb{C}^n$ and put $\Delta^\alpha_\epsilon = \{\zeta \in \Delta; |\zeta^\alpha|^2 \geq \epsilon\}$ and $\Delta^{\alpha,\beta}_{\epsilon_1,\epsilon_2} = \{\zeta \in \Delta; |\zeta^\alpha|^2 \geq \epsilon_1, |\zeta^\beta|^2 \geq \epsilon_2\}$. Then, for $\epsilon, \epsilon_j \leq 1$, we have*

$$\int_{\Delta^\alpha_\epsilon} \frac{\epsilon}{|\zeta^\alpha|^2} \frac{1}{|\zeta_1|\cdots|\zeta_n|} \lesssim \epsilon^{1/(2|\alpha|)}|\log\epsilon|^{n-1},$$

$$\int_{\Delta^{\alpha,\beta}_{\epsilon_1,\epsilon_2}} \Big(\frac{\epsilon_1}{|\zeta^\alpha|^2} + \frac{\epsilon_2}{|\zeta^\beta|^2}\Big) \frac{1}{|\zeta_1|\cdots|\zeta_n|} \lesssim |(\epsilon_1,\epsilon_2)|^\omega$$

and

$$\int_{\Delta^{\alpha,\beta}_{\epsilon_1,\epsilon_2}} \frac{\epsilon_1\epsilon_2}{|\zeta^\alpha|^2|\zeta^\beta|^2} \frac{1}{|\zeta_1|\cdots|\zeta_n|} \lesssim |(\epsilon_1,\epsilon_2)|^\omega,$$

where $2\omega < min\{|\alpha|^{-1}, |\beta|^{-1}\}$.

Proof. The second and third inequality follow from the first one since it implies that the integral in the second one is of the size $\epsilon_1^{\tau+1/(2|\alpha|)} + \epsilon_2^{\tau+1/(2|\beta|)} \lesssim |(\epsilon_1,\epsilon_2)|^{\tau+\omega}$ for any $\tau > 0$ and that the integral in the third is of the size $\min\{\epsilon_1^{1/(2|\alpha|)}|\log\epsilon_1|^{n-1}, \epsilon_2^{1/(2|\beta|)}|\log\epsilon_2|^{n-1}\}$. To prove the first inequality we proceed as in the previous lemma and we see that the integral in question equals

$$\begin{aligned} (4\pi)^n\epsilon \int_{Q_\epsilon} \frac{e^{\sum x_j}}{e^{2\sum\alpha_j x_j}}dx &= (4\pi)^n\epsilon \int_{Q_\epsilon\cap\{|x|\leq R\}} \frac{e^{\sum x_j}}{e^{2\sum\alpha_j x_j}}dx \\ &+ (4\pi)^n\epsilon \int_{Q_\epsilon\cap\{|x|\geq R\}} \frac{e^{\sum x_j}}{e^{2\sum\alpha_j x_j}}dx, \end{aligned} \tag{16}$$

where $Q_\epsilon = \{x \in (-\infty, 0]^n;\, 2\sum\alpha_j x_j \geq \log\epsilon\}$. We choose $2R = |\log\epsilon|/|\alpha|$, and then $Q_\epsilon \cap \{|x| \leq R\} = \{x \in (-\infty, 0]^n;\, |x| \leq R\}$. If all $x_j \leq 0$ we have $\sum x_j \leq -|x|$ and by the Cauchy-Schwarz inequality we also have $-\sum\alpha_j x_j \leq |\alpha||x|$. Hence we may estimate the integrand in the second to last integral in (16) by $\exp((2|\alpha|-1)|x|)$. In the last integral we integrate where $\epsilon/\exp(2\sum\alpha_j x_j) \leq 1$ and so we see that the right hand side of (16) is less then or equal to

$$(4\pi)^n\epsilon\int_{\{|x|\leq R\}} e^{(2|\alpha|-1)|x|}dx + (4\pi)^n\int_{\{|x|\geq R\}} e^{-|x|}dx.$$

By changing to polar coordinates this is seen to be of the size $\epsilon^{1/(2|\alpha|)}|\log\epsilon|^{n-1}$.
□

The proof of the following proposition contains the technical core of this paper.

Proposition 11. *Assume that $\chi_1, \chi_2 \in C^\infty([0,\infty])$ vanish to orders $k \geq 0$ and $\ell \geq 0$ at 0, respectively, and that $\chi_1(\infty) = 1$. Then for any test form $\varphi \in \mathscr{D}_{n,n}(\mathbb{C}^n)$ we have*

$$\int \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1(\Phi|\zeta^\alpha|^2/\epsilon_1)\chi_2(\Psi|\zeta^\beta|^2/\epsilon_2)\varphi \to \begin{cases} [\frac{1}{\zeta^{k\alpha+\ell\beta}}].\varphi, & \chi_2(\infty)=1 \\ 0, & \chi_2(\infty)=0 \end{cases}$$

as $\epsilon_1, \epsilon_2 \to 0^+$. Moreover, as a function of $\epsilon = (\epsilon_1,\epsilon_2) \in [0,\infty)^2$, the integral belongs to all ω-Hölder classes with $2\omega < min\{|\alpha|^{-1}, |\beta|^{-1}\}$.

Remark 12. The values of the integral at points $(\epsilon_1, 0)$ and $(0,\epsilon_2)$, $\epsilon_j \neq 0$, are

$$\chi_2(\infty)\frac{\chi_1(\Phi|\zeta^\alpha|^2/\epsilon_1)}{\zeta^{k\alpha}}[\frac{1}{\zeta^{\ell\beta}}].\varphi \text{ and } \frac{\chi_2(\Phi|\zeta^\beta|^2/\epsilon_2)}{\zeta^{\ell\beta}}[\frac{1}{\zeta^{k\alpha}}].\varphi$$

respectively.

Remark 13. The modulus of continuity can be improved by sharpening the estimates in the Lemmas 9 and 10 but we will not bother about this. This is because the multiindices α and β will be implicitly given by Hironaka's theorem and so we can only be sure of the existence of some positive Hölder exponent when we prove our main theorems anyway.

Proof. We prove Hölder continuity for a path $(\epsilon_1,\epsilon_2) \to 0$, $\epsilon_j \neq 0$. For a general path (inside $[0,\infty)^2$) to an arbitrary point in $[0,\infty)^2$ one proceeds in a similar way. Let K be the set of indices j such that $k\alpha_j + \ell\beta_j \geq 2$ and let $M = M_K^r$ be the operator defined in Lemma 6 with $r_j = k\alpha_j + \ell\beta_j - 2$ for $j \in K$. Let also Δ be a polydisc containing the support of φ. In this proof we will identify φ with its coefficient function with respect to the volume form in $\mathbb{C}^n$. We make a preliminary decomposition

$$\int \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\chi_2\varphi = \int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\chi_2(\varphi - M\varphi) + \int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\chi_2 M\varphi. \tag{17}$$

Denote by Δ_ϵ the set $\{\zeta \in \Delta; |\zeta^\alpha|^2 \geq \epsilon_1, |\zeta^\beta|^2 \geq \epsilon_2\}$. Since $\varphi - M\varphi = \mathcal{O}(|\zeta^{r+1}|)$, according to Lemma 6, and $\chi_1(\infty) = 1$ we get

$$\begin{aligned}
&\Big|\int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\chi_2(\varphi - M\varphi) - \chi_2(\infty)\int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}}(\varphi - M\varphi)\Big| \\
&\quad\lesssim \int_\Delta \frac{1}{|\zeta_1|\cdots|\zeta_n|}|\chi_1\chi_2 - \chi_2(\infty)| \\
&\quad\lesssim \int_{\Delta_\epsilon} \frac{1}{|\zeta_1|\cdots|\zeta_n|}|\chi_1\chi_2 - \chi_2(\infty)| + \int_{\Delta\setminus\Delta_\epsilon} \frac{1}{|\zeta_1|\cdots|\zeta_n|}.
\end{aligned} \tag{18}$$

It follows from Lemma 9 that the last integral is of order $|\epsilon|^\omega$ as $\epsilon_1, \epsilon_2 \to 0^+$. On the other hand, for $\zeta \in \Delta_\epsilon$ both $|\zeta^\alpha|^2/\epsilon_1 \geq 1$ and $|\zeta^\beta|^2/\epsilon_2 \geq 1$ and by Taylor expanding at infinity we see that

$$\begin{aligned}\chi_1(\Phi|\zeta^\alpha|^2/\epsilon_1) &= \chi_1(\infty) + \frac{\epsilon_1}{|\zeta^\alpha|^2} B_1(\epsilon_1/|\zeta^\alpha|^2, \zeta),\\ \chi_2(\Psi|\zeta^\beta|^2/\epsilon_2) &= \chi_2(\infty) + \frac{\epsilon_2}{|\zeta^\beta|^2} B_2(\epsilon_2/|\zeta^\beta|^2, \zeta)\end{aligned}$$

where B_1 and B_2 are bounded. Using that $\chi_1(\infty) = 1$ we thus get that $|\chi_1\chi_2 - \chi_2(\infty)|$ is of the size $\epsilon_1/|\zeta^\alpha|^2 + \epsilon_2/|\zeta^\beta|^2$. Hence, by Lemma 10 the second to last integral in (18) is also of order $|\epsilon|^\omega$ as $\epsilon_1, \epsilon_2 \to 0^+$. In view of Lemma 7, we have thus showed that the first integral on the right hand side of (17) tends to $[1/\zeta^{k\alpha+\ell\beta}].\varphi$ if $\chi_2(\infty) = 1$ and to zero if $\chi_2(\infty) = 0$ and moreover, belongs to the stated Hölder classes. We will be done if we can show that the last integral in (17) is of order $|\epsilon|^\omega$. We know that $M\varphi = \sum_{IJ} \varphi_{IJ}\zeta^I\bar{\zeta}^J$ where each φ_{IJ} is independent of at least one variable and $I_j + J_j \leq k\alpha_j + \ell\beta_j - 2$ for $j \in K$. Hence, if Φ and Ψ are constants (or only depend on the modulus of the ζ_j) then the last integral in (17) is zero for all $\epsilon_1, \epsilon_2 > 0$ by anti-symmetry. For the general case, consider one term

$$\int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}} \chi_1\chi_2\varphi_{IJ}\zeta^I\bar{\zeta}^J \tag{19}$$

and let L be the set of indices $j \in K$ such that $\zeta_j \mapsto \varphi_{IJ}(\zeta)$ is constant. Let also $\mathscr{M} = M_L^\rho$ be the operator defined in Lemma 6 with $\rho_j = k\alpha_j + \ell\beta_j - I_j - J_j - 2$ for $j \in L$. We introduce the independent (real) variables, or "smoothing parameters", $t_1 = |\zeta^\alpha|^2/\epsilon_1$ and $t_2 = |\zeta^\beta|^2/\epsilon_2$. Below, $\mathscr{M}(\chi_1\chi_2)$ denotes the function we obtain by letting $\mathscr{M}$ operate on $\zeta \mapsto \chi_1(t_1\Phi(\zeta))\chi_2(t_2\Psi(\zeta))$ and then substituting $|\zeta^\alpha|^2/\epsilon_1$ and $|\zeta^\beta|^2/\epsilon_2$ for t_1 and t_2 respectively. We rewrite the integral (19) as

$$\begin{aligned}&\int_{\Delta_\epsilon} \frac{\varphi_{IJ}\zeta^I\bar{\zeta}^J}{\zeta^{k\alpha+\ell\beta}}(\chi_1\chi_2 - \mathscr{M}(\chi_1\chi_2)) + \int_{\Delta\setminus\Delta_\epsilon} \frac{\varphi_{IJ}\zeta^I\bar{\zeta}^J}{\zeta^{k\alpha+\ell\beta}}(\chi_1\chi_2 - \mathscr{M}(\chi_1\chi_2))\\ &\qquad + \int_\Delta \frac{\varphi_{IJ}\zeta^I\bar{\zeta}^J}{\zeta^{k\alpha+\ell\beta}}\mathscr{M}(\chi_1\chi_2).\end{aligned} \tag{20}$$

Now, $\mathscr{M}(\chi_1\chi_2)$ is a sum of terms which, at least for some $j \in L$, are monomials in ζ_j and $\bar{\zeta}_j$ times coefficient functions depending on $|\zeta_j|$ and the other variables. The degrees of these monomials do not exceed $\rho_j = k\alpha_j + \ell\beta_j - I_j - J_j - 2$ and since $\zeta_j \mapsto \varphi_{IJ}(\zeta)$ is constant for $j \in L$ we see, by counting exponents, that the last integral in (20) vanishes by anti-symmetry for all $\epsilon_1, \epsilon_2 > 0$. By Lemma 8 we have

$$\chi_1(t_1\Phi)\chi_2(t_2\Psi) - \mathscr{M}(\chi_1(t_1\Phi)\chi_2(t_2\Psi)) = |\zeta^{\rho+1}|B(t_1, t_2, \zeta), \tag{21}$$

where B is bounded on $(0,\infty)^2 \times \Delta$. We note also that by Lemma 6, $\varphi_{IJ}(\zeta) = \mathcal{O}(\prod_{j\in L\setminus K} |\zeta_j|^{r_j+1})$. From (21) we thus see that the modulus of the second integral in (20) can be estimated by

$$C\int_{\Delta\setminus\Delta_\epsilon} \frac{1}{|\zeta_1|\cdots|\zeta_n|},$$

which is of order $|\epsilon|^\omega$ by Lemma 9. It remains to consider the first integral in (20). On the set Δ_ϵ we have that $\Phi|\zeta^\alpha|^2/\epsilon_1$ and $\Psi|\zeta^\beta|^2/\epsilon_2$ are larger then some positive constant and so by multiplying the Taylor expansions of the functions $t_1 \mapsto \chi_1(t_1\Phi)$ and $t_2 \mapsto \chi_2(t_2\Psi)$ at infinity we get

$$\begin{aligned}\chi_1(\Phi|\zeta^\alpha|^2/\epsilon_1)\chi_2(\Psi|\zeta^\beta|^2/\epsilon_2) &= \chi_2(\infty) + \frac{\epsilon_2}{|\zeta^\beta|^2}\tilde{\chi}_2(|\zeta^\beta|^2/\epsilon_2,\zeta) \\ &+ \chi_2(\infty)\frac{\epsilon_1}{|\zeta^\alpha|^2}\tilde{\chi}_1(|\zeta^\alpha|^2/\epsilon_1,\zeta) \\ &+ \frac{\epsilon_1\epsilon_2}{|\zeta^\alpha|^2|\zeta^\beta|^2}\tilde{\chi}_1(|\zeta^\alpha|^2/\epsilon_1,\zeta)\tilde{\chi}_2(|\zeta^\beta|^2/\epsilon_2,\zeta)\end{aligned}$$

where $\tilde{\chi}_j$ are smooth on $[1,\infty]\times\Delta$. Now since $|\zeta^\alpha|^2/\epsilon_1 = t_1$ and $|\zeta^\beta|^2/\epsilon_2 = t_2$ are independent variables we conclude that

$$\begin{aligned}\chi_1\chi_2 - \mathscr{M}(\chi_1\chi_2) &= \frac{\epsilon_2}{|\zeta^\beta|^2}(\tilde{\chi}_2 - \mathscr{M}\tilde{\chi}_2) + \frac{\epsilon_1}{|\zeta^\alpha|^2}\chi_2(\infty)(\tilde{\chi}_1 - \mathscr{M}\tilde{\chi}_1) \\ &+ \frac{\epsilon_1\epsilon_2}{|\zeta^\alpha|^2|\zeta^\beta|^2}(\tilde{\chi}_1\tilde{\chi}_2 - \mathscr{M}(\tilde{\chi}_1\tilde{\chi}_2))\end{aligned}$$

for $\zeta \in \Delta_\epsilon$. By Lemmas 6 and 10 we see that the first integral in (20) also is of order $|\epsilon|^\omega$ as $\epsilon_1,\epsilon_2 \to 0^+$ and the proof is complete. □

Remark 14. Let us assume that the function Φ is identically 1 in the previous proposition. Then, instead of adding and subtracting $\mathscr{M}(\chi_1\chi_2)$ in (20), it is enough to add and subtract $\chi_1\mathscr{M}(\chi_2)$. This suggests that one can relax the smoothness assumption on χ_1. It is actually possible to take χ_1 to be the characteristic function of $[1,\infty]$. If we define the value of the integral in Proposition 11 at a point $(\epsilon_1, 0)$ to be

$$\int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1(|\zeta^\alpha|^2/\epsilon_1)(\varphi - M\varphi), \tag{22}$$

where Δ and M are as in the proof above, then the conclusions of Proposition 11 hold for this choice of χ_1. Only minor changes in the proof are needed to see this. One can also check that (22) is a way of computing

$$\chi_1(|\zeta^\alpha|^2/\epsilon_1)\left[\frac{1}{\zeta^{k\alpha+\ell\beta}}\right].\varphi.$$

The product $\chi_1(|\zeta^\alpha|^2/\epsilon_1)[1/\zeta^{k\alpha+\ell\beta}]$ is well defined because the wave front sets of the two currents behave in the right way, at least for almost all ϵ_1, see [7].

We make another useful observation. Since the function $\tilde{\chi}(s) = \chi(1/s)$ is smooth at zero and $\tilde{\chi}'(s) := -\frac{1}{s^2}\chi'(1/s)$, it follows that $s \mapsto \chi'(1/s)/s$ is smooth at zero and vanishes for $s = 0$. Hence, $t \mapsto \chi'(t)t$ is smooth on $[0,\infty]$, vanishes to the same order at zero as χ, and maps ∞ to 0. From Proposition 11 we thus see that we have

Corollary 15. *Assume that $\chi_1, \chi_2 \in C^\infty([0,\infty])$ vanish to orders k and ℓ at zero respectively, and satisfy $\chi_j(\infty) = 1$. For any smooth and strictly positive functions Φ and Ψ on $\mathbb{C}^n$ and any test form $\varphi \in \mathscr{D}_{n,n}(\mathbb{C}^n)$ we have*

$$\lim_{\epsilon_1,\epsilon_2\to 0^+}\int \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1(\Phi|\zeta^\alpha|^2/\epsilon_1)\chi_2'(\Psi|\zeta^\beta|^2/\epsilon_2)\frac{|\zeta^\beta|^2}{\epsilon_2}\varphi = 0, \tag{23}$$

and moreover, as a function of $\epsilon = (\epsilon_1, \epsilon_2) \in [0,\infty)^2$*, the integral belongs to all* ω*-Hölder classes with* $2\omega < min\{|\alpha|^{-1}, |\beta|^{-1}\}$*.*

6. Regularizations of products of Cauchy-Fantappiè-Leray type currents

We are now in a position to prove our main results. We start with a regularization of the product $U^f \wedge U^g$. Recall that if f is function then $U^f = [1/f]$ times some basis element.

Theorem 16. *Let f and g be holomorphic sections (locally non-trivial) of the holomorphic m_j-bundles $E_j^* \to X$, $j = 1, 2$, respectively. Let $\chi_1, \chi_2 \in C^\infty([0,\infty])$ be any functions vanishing to orders m_1 and m_2 at zero respectively, and satisfying $\chi_j(\infty) = 1$. Then, for any test form φ we have*

$$\int \chi_1(|f|^2/\epsilon_1) u^f \wedge \chi_2(|g|^2/\epsilon_2) u^g \wedge \varphi \to U^f \wedge U^g.\varphi,$$

as $\epsilon_1, \epsilon_2 \to 0^+$. Moreover, as a function of $\epsilon = (\epsilon_1, \epsilon_2) \in [0,\infty)^2$ the integral on the left hand side belongs to some Hölder class independently of φ.

Proof. Recall that $U^f \wedge U^g.\varphi$ is defined as the value at zero of the meromorphic function

$$\lambda \mapsto \int |f|^{2\lambda} u^f \wedge |g|^{2\lambda} u^g \wedge \varphi.$$

Assuming only that χ_1 and χ_2 vanish to orders $k \le m_1$ and $\ell \le m_2$ at zero respectively we will show that

$$\int \chi_1 u^f_{k,k-1} \wedge \chi_2 u^g_{\ell,\ell-1} \wedge \varphi \to \int |f|^{2\lambda} u^f_{k,k-1} \wedge |g|^{2\lambda} u^g_{\ell,\ell-1} \wedge \varphi\Big|_{\lambda=0} \qquad (24)$$

and that the left hand side belongs to some Hölder class. This will clearly imply the theorem. We may assume that φ has arbitrarily small support after a partition of unity. If φ has support outside $f^{-1}(0) \cup g^{-1}(0)$ it is easy to check that (24) holds and hence we can restrict to the case that φ has support in a small neighborhood $\mathcal{U}$ of a point $p \in f^{-1}(0) \cup g^{-1}(0)$. We may also assume that $\mathcal{U}$ is contained in a coordinate neighborhood and that all bundles are trivial over $\mathcal{U}$. We let $(f_1, \ldots, f_{m_1})$ and $(g_1, \ldots, g_{m_2})$ denote the components of f and g respectively, with respect to some holomorphic frames. It follows from Hironaka's theorem, possibly after another localization, that there is an n-dimensional complex manifold $\tilde{\mathcal{U}}$ and a proper holomorphic map $\Pi\colon \tilde{\mathcal{U}} \to \mathcal{U}$ such that Π is biholomorphic outside the nullset $\Pi^*\{f_1 \cdots f_{m_1} \cdot g_1 \cdots g_{m_2}\}$ and that this hypersurface has normal crossings in $\tilde{\mathcal{U}}$. Hence we can cover $\tilde{\mathcal{U}}$ by local charts, each centered at the origin, such that $\Pi^* f_j$ and $\Pi^* g_j$ are monomials times non-vanishing functions. The support of $\Pi^*\varphi$ is compact because Π is proper and hence, we can cover the support of $\Pi^*\varphi$ by finitely many of these charts. We let ρ_k be a partition of unity on supp$(\Pi^*\varphi)$ subordinate to this cover. Now, following [20] and [4], given monomials $\mu_1 \ldots, \mu_\nu$, one can construct an n-dimensional toric manifold $\mathcal{X}$ and a proper holomorphic map $\tilde{\Pi}\colon \mathcal{X} \to \mathbb{C}^n_t$ which is monoidal when expressed in local coordinates in each chart. Moreover, $\tilde{\Pi}$ is biholomorphic outside $\tilde{\Pi}^*\{t_1 \cdots t_n = 0\}$ and in each chart one of the monomials

$\tilde{\Pi}^*\mu_1, \dots, \tilde{\Pi}^*\mu_\nu$ divides all the others. By repeating this process, if necessary, and localizing with partitions of unity at each step, we may actually assume that $f_j = \mu_{f,j}\tilde{f}_j$ and $g_j = \mu_{g,j}\tilde{g}_j$ where $\tilde{f}_j$ and $\tilde{g}_j$ are non-vanishing and $\mu_{f,j}$ and $\mu_{g,j}$ are monomials with the property that μ_{f,ν_1} divides all $\mu_{f,j}$ and μ_{g,ν_2} divides all $\mu_{g,j}$ for some indices ν_1 and ν_2. Denote μ_{f,ν_1} by ζ^α and μ_{g,ν_2} by ζ^β. It follows that $|f|^2 = |\zeta^\alpha|^2\Phi$ and $|g|^2 = |\zeta^\beta|^2\Psi$ where Φ and Ψ are strictly positive functions. Moreover, $s_f = \bar{\zeta}^\alpha\tilde{s}_f$ and

$$u^f_{k,k-1} = \frac{s_f \wedge (\bar{\partial}s_f)^{k-1}}{|f|^{2k}} = \frac{1}{\zeta^{k\alpha}}\frac{\tilde{s}_f \wedge (\bar{\partial}\tilde{s}_f)^{k-1}}{\Phi^k} = \frac{1}{\zeta^{k\alpha}}\tilde{u}^f_{k,k-1}$$

where $\tilde{u}^f_{k,k-1}$ is a smooth form and similarly for $u^g_{\ell,\ell-1}$. In order to prove (24) it thus suffices to prove

$$(25)\qquad \int \frac{\chi_1(\Phi|\zeta^\alpha|^2/\epsilon_1)}{\zeta^{k\alpha}}\tilde{u}^f_{k,k-1} \wedge \frac{\chi_2(\Psi|\zeta^\beta|^2/\epsilon_2)}{\zeta^{\ell\beta}}\tilde{u}^g_{\ell,\ell-1} \wedge \tilde{\varphi} \\ \to \int \frac{|\zeta^\alpha|^{2\lambda}}{\zeta^{k\alpha}}\Phi^\lambda\tilde{u}^f_{k,k-1} \wedge \frac{|\zeta^\beta|^{2\lambda}}{\zeta^{\ell\beta}}\Psi^\lambda\tilde{u}^g_{\ell,\ell-1} \wedge \tilde{\varphi}\Big|_{\lambda=0}$$

where $\tilde{\varphi} = \rho_{k_j}\Pi_j^* \cdots \rho_{k_1}\Pi_1^*\varphi$ and that the integral on the left hand side belongs to some Hölder class. But by Proposition 11 it does belong to some Hölder class and tends to $[1/\zeta^{k\alpha+\ell\beta}].\tilde{u}^f_{k,k-1} \wedge \tilde{u}^g_{\ell,\ell-1} \wedge \tilde{\varphi}$. One can verify that this indeed is equal to the right hand side of (25) by integrations by parts as in e.g. [2]. □

Remark 17. This theorem can actually be generalized to any number of factors U^f. One first checks that the analogue of Proposition 11 holds for any number of functions χ_j and then reduces to this case just as in the proof above. In particular, if f_j, $j = 1, \dots, p$, are holomorphic functions and χ_j vanish at 0, we have

$$\int \frac{\chi_1(|f_1|^2/\epsilon_1)}{f_1} \cdots \frac{\chi_p(|f_p|^2/\epsilon_p)}{f_p}\varphi \to [\frac{1}{f_1} \cdots \frac{1}{f_p}].\varphi$$

unrestrictedly as all $\epsilon_j \to 0^+$. However, we focus on the two factor case since we do not know how to handle more than two residue factors.

To prove our regularization results for the currents $U^f \wedge R^g$ and $R^f \wedge R^g$ we have to structure the information obtained from an application of Hironaka's theorem more carefully and then use Proposition 11 and Corollary 15 in the right way. The technical part of this is contained in the following proposition.

Proposition 18. *Assume that $\chi_1, \chi_2 \in C^\infty([0,\infty])$ vanish to orders k and ℓ at zero, respectively, and satisfy $\chi_j(\infty) = 1$. Let α', α'', β' and β'' be multiindices such that α', α'' and β' have pairwise disjoint supports, and $\alpha''_j = 0$ if and only if $\beta''_j = 0$. Assume also that $\varphi \in \mathscr{D}_{n,n-1}(\mathbb{C}^n)$ has the property that $d\bar{\zeta}_j/\bar{\zeta}_j \wedge \varphi \in \mathscr{D}_{n,n}(\mathbb{C}^n)$ for all j such that $\alpha''_j \neq 0$. Then for any smooth and strictly positive functions Φ and Ψ on $\mathbb{C}^n$ we have*

$$\lim_{\epsilon_1,\epsilon_2 \to 0^+} \int \frac{1}{\mu_1^k\mu_2^\ell}\chi_1(\Phi|\mu_1|^2/\epsilon_1)\bar{\partial}\chi_2(\Psi|\mu_2|^2/\epsilon_2) \wedge \varphi = \Big[\frac{1}{\mu_1^k\zeta^{\ell\beta''}}\Big] \otimes \bar{\partial}\Big[\frac{1}{\zeta^{\ell\beta'}}\Big].\varphi,$$

where $\mu_1 = \zeta^{\alpha'+\alpha''}$ *and* $\mu_2 = \zeta^{\beta'+\beta''}$. *Moreover, as a function of* $\epsilon = (\epsilon_1, \epsilon_2) \in [0,\infty)^2$, *the integral belongs to all* ω*-Hölder classes with* $2\omega < min\{|\alpha' + \alpha''|^{-1}, |\beta' + \beta''|^{-1}\}$.

Remark 19. Note that the hypotheses on the multiindices imply that a factor ζ_j divides both the monomials μ_1 and μ_2 if and only if $\alpha_j'' \neq 0$ (or equivalently $\beta_j'' \neq 0$). In particular, the tensor product of the currents is well defined.

Remark 20. We may let k or ℓ or both of them be equal to zero and the conclusions of the proposition still hold. In case $\ell = 0$ one should interpret $\bar\partial[1/\zeta^{\ell\beta'}]$ as zero.

Proof. Let K, L and K^c be the set of indices j such that $\beta_j' \neq 0$, $\beta_j'' \neq 0$ and $\beta_j' = 0$ respectively. Clearly $L \subseteq K^c$. We write $\bar\partial = \bar\partial_K + \bar\partial_{K^c}$ and integrate by parts with respect to $\bar\partial_K$ to see that

$$
\begin{aligned}
(26)\qquad & \int \frac{1}{\mu_1^k\mu_2^\ell}\chi_1(\bar\partial_K + \bar\partial_{K^c})\chi_2 \wedge \varphi = \\
& - \int \frac{1}{\mu_1^k\mu_2^\ell}\chi_1'\frac{|\mu_1|^2}{\epsilon_1}\chi_2\bar\partial_K\Phi \wedge \varphi - \int \frac{1}{\mu_1^k\mu_2^\ell}\chi_1\chi_2\bar\partial_K\varphi \\
& + \int \frac{1}{\mu_1^k\mu_2^\ell}\chi_1\chi_2'\frac{|\mu_2|^2}{\epsilon_2}(\Psi\sum_{j\in L}\beta_j''\frac{d\bar\zeta_j}{\bar\zeta_j} + \bar\partial_{K^c}\Psi) \wedge \varphi.
\end{aligned}
$$

Note that $\bar\partial_K$ does not fall on $|\mu_1|^2$ because of the hypotheses on the multiindices. By assumption, $d\bar\zeta_j/\bar\zeta_j \wedge \varphi \in \mathscr{D}_{n,n}(\mathbb{C}^n)$ for $j \in L$ and so the first and the last integral on the right hand side of (26) tend to zero and has the right modulus of continuity by Corollary 15. The second to last integral in (26) tends to $-[1/(\mu_1^k\mu_2^\ell)].\bar\partial_K\varphi = [1/(\mu_1^k\zeta^{\ell\beta''})] \otimes \bar\partial[1/\zeta^{\ell\beta'}].\varphi$ and has the right modulus of continuity by Proposition 11. □

Theorem 21. *Let* f *and* g *be holomorphic sections (locally non-trivial) of the holomorphic* m_j*-bundles* $E_j^* \to X$, $j = 1, 2$, *respectively. Assume that the section* $f \oplus g$ *of* $E_1^* \oplus E_2^* \to X$ *defines a complete intersection. Let* $\chi_1, \chi_2 \in C^\infty([0,\infty])$ *be any functions vanishing to orders* m_1 *and* m_2 *at zero respectively, and satisfying* $\chi_j(\infty) = 1$. *Then, for any test form* φ *we have*

$$
(27)\qquad \int \chi_1(|f|^2/\epsilon_1)u^f \wedge \bar\partial\chi_2(|g|^2/\epsilon_2) \wedge u^g \wedge \varphi \to U^f \wedge R^g.\varphi
$$

as $\epsilon_1, \epsilon_2 \to 0^+$. *Moreover, as a function of* $\epsilon = (\epsilon_1, \epsilon_2) \in [0,\infty)^2$ *the integral on the left hand side belongs to some Hölder class independently of* φ.

Proof. We will assume that χ_1 and χ_2 only vanish to orders $k \le m_1$ and $\ell \le m_2$ respectively and show that

$$
(28)\ \int \chi_1 u^f_{k,k-1} \wedge \bar\partial\chi_2 \wedge u^g_{\ell,\ell-1} \wedge \varphi \to \int |f|^{2\lambda}u^f_{k,k-1} \wedge \bar\partial|g|^{2\lambda} \wedge u^g_{\ell,\ell-1} \wedge \varphi\Big|_{\lambda=0}.
$$

By arguing as in the proof of Theorem 16 we may assume that $|f|^2 = |\zeta^\alpha|^2\Phi$ and $|g|^2 = |\zeta^\beta|^2\Psi$ where Φ and Ψ are strictly positive functions and moreover, that $u^f_{k,k-1} = \tilde{u}^f_{k,k-1}/\zeta^{k\alpha}$ for a smooth form $\tilde{u}^f_{k,k-1}$ and similarly for $u^g_{\ell,\ell-1}$. What we have to prove is thus

$$\int \frac{\chi_1(\Phi|\zeta^\alpha|^2/\epsilon_1)}{\zeta^{k\alpha}}\tilde{u}^f_{k,k-1} \wedge \frac{\bar\partial\chi_2(\Psi|\zeta^\beta|^2/\epsilon_2)}{\zeta^{\ell\beta}}\tilde{u}^g_{\ell,\ell-1}\wedge\tilde\varphi \to \int \frac{|\zeta^\alpha|^{2\lambda}}{\zeta^{k\alpha}}\Phi^\lambda\tilde{u}^f_{k,k-1}\wedge\frac{\bar\partial(|\zeta^\beta|^{2\lambda}\Psi^\lambda)}{\zeta^{\ell\beta}}\tilde{u}^g_{\ell,\ell-1}\wedge\tilde\varphi\Big|_{\lambda=0} \tag{29}$$

where $\tilde\varphi = \rho_{k_j}\Pi^*_j\cdots\rho_{k_1}\Pi^*_1\varphi$. After the resolutions of singularities we can in general no longer say that the pull-back of $f\oplus g$ defines a complete intersection. On the other hand we claim that if ζ_j divides both ζ^α and ζ^β then $d\bar\zeta_j/\bar\zeta_j\wedge\tilde\varphi$ is smooth. In fact, let z be local coordinates on our original manifold. In order that the integrals in (28) should be non-zero, φ has to have degree $n-k-\ell+1$ in $d\bar z$ and so we can assume that

$$\varphi = \sum_{\#J=n-k-\ell+1}\varphi_J\wedge d\bar z_J.$$

Since the variety $V_{f\oplus g} = f^{-1}(0)\cap g^{-1}(0)$ has dimension $n-m_1-m_2 < n-k-\ell+1$ we see that $d\bar z_J$ vanishes on $V_{f\oplus g}$. The pull-back of $d\bar z_J$ through all the resolutions Π_j can be written $\sum_I C_I(\zeta)d\bar\zeta_I$ and it must vanish on the pull-back of $V_{f\oplus g}$. In particular it has to vanish on $\{\zeta_j=0\}$ if ζ_j divides both ζ^α and ζ^β. If $d\bar\zeta_j$ does not occur in $d\bar\zeta_I$ it must be that the coefficient function $C_I(\zeta)$ vanishes on $\{\zeta_j=0\}$. But these functions are anti-holomorphic and so $\bar\zeta_j$ must divide $C_I(\zeta)$. The claim is established. We now write $\zeta^\alpha=\zeta^{\alpha'+\alpha''}$ and $\zeta^\beta=\zeta^{\beta'+\beta''}$ where α', α'' and β' have pairwise disjoint supports and $\alpha''=0$ if and only if $\beta''=0$. Thus, ζ_j divides both ζ^α and ζ^β if and only if $\alpha''_j\neq 0$, or equivalently, $\beta''_j\neq 0$. According to Proposition 18 the left hand side of (29) belongs to some Hölder class and tends to

$$-\Big[\frac{1}{\zeta^{k\alpha+\ell\beta''}}\Big]\otimes\bar\partial\Big[\frac{1}{\zeta^{\ell\beta'}}\Big].\tilde{u}^f_{k,k-1}\wedge\tilde{u}^g_{\ell,\ell-1}\wedge\tilde\varphi.$$

One can compute the right hand side of (29) by integrations by parts as in e.g. [2] to see that it equals the same thing. □

Remark 22. The form $\bar\partial\chi_2(|g|^2/\epsilon_2)\wedge u^g$ is actually smooth even if χ_2 only vanishes to order m_2 at 0. The only possible problem is with the top degree term $\bar\partial\chi_2(|g|^2/\epsilon_2)\wedge u^g_{m_2,m_2-1}$. But we have

$$\begin{aligned} C^\infty(X)\ni\bar\partial(\chi_2(|g|^2/\epsilon_2)u^g_{m_2,m_2-1}) &= \bar\partial\chi_2(|g|^2/\epsilon_2)\wedge u^g_{m_2,m_2-1}\\ &+ \chi_2(|g|^2/\epsilon_2)\bar\partial u^g_{m_2,m_2-1},\end{aligned}$$

and since $u^g_{m_2,m_2-1}$ is $\bar\partial$-closed (outside V_g) it follows that $\bar\partial\chi_2(|g|^2/\epsilon_2)\wedge u^g_{m_2,m_2-1}$ is smooth as well.

Corollary 23. *With the same hypotheses as in Theorem 21 we have*

$$\int\bar\partial\chi_1(|f|^2/\epsilon_1)\wedge u^f\wedge\bar\partial\chi_2(|g|^2/\epsilon_2)\wedge u^g\wedge\varphi\to R^f\wedge R^g.\varphi,$$

$$\int\bar\partial\chi_1(|f|^2/\epsilon_1)\wedge u^f\chi_2(|g|^2/\epsilon_2)\wedge\varphi\to R^f.\varphi, \tag{30}$$

and

$$\int \chi_1(|f|^2/\epsilon_1) \wedge u^f \wedge \bar\partial\chi_2(|g|^2/\epsilon_2) \wedge \varphi \to 0 \tag{31}$$

as $\epsilon_1, \epsilon_2 \to 0^+$*, and as functions of* $\epsilon = (\epsilon_1, \epsilon_2) \in [0,\infty)^2$ *the integrals on the left hand sides belong to some Hölder classes independently of* φ*.*

Proof. We have the following equality of smooth forms:

$$\begin{aligned} \nabla(\bar\partial\chi_1 \wedge u^f \wedge \chi_2 u^g) &= -\bar\partial\chi_1 \wedge \chi_2 u^g - \bar\partial\chi_1 \wedge u^f \wedge \bar\partial\chi_2 \wedge u^g \\ &+ \bar\partial\chi_1 \wedge u^f \chi_2. \end{aligned} \tag{32}$$

The computation rules established in [26], and Theorem 21 now imply that, for any test form φ (of complementary total degree), we have

$$\begin{aligned} R^f.\varphi - R^f \wedge R^g.\varphi &= \nabla(R^f \wedge U^g).\varphi = -R^f \wedge U^g.\nabla\varphi \\ &= \lim -\int \bar\partial\chi_1 \wedge u^f \wedge \chi_2 u^g \wedge \nabla\varphi \\ &= \lim \int \nabla(\bar\partial\chi_1 \wedge u^f \wedge \chi_2 u^g) \wedge \varphi. \end{aligned}$$

The integral on the second row is Hölder continuous by Theorem 21 and so, also the integral on the third row is. By choosing φ of appropriate bidegrees the corollary now follows from (32). □

The statements (30) and (31) actually hold with no assumptions on the behavior of χ_2 at zero. This can be seen by using that we know this when $\chi_2 \equiv 1$ by Corollary 5, and when χ_2 vanishes to high enough order by the previous corollary.

Assume that f defines a complete intersection and pick a holomorphic function g such that $f \oplus g$ also defines a complete intersection and such that g is zero on the singular part of V_f. After resolving singularities in the proof of Theorem 21 we can find coordinates such that g is a monomial times a non-vanishing holomorphic function $\tilde g$. But $\tilde g$ can be incorporated in some coordinate and we can therefore assume that $\tilde g \equiv 1$. Repeating the proof of Theorem 21 and using Remark 14 one shows that (30) holds for χ_2 equal to the characteristic function of $[1,\infty]$. Then, if we first let ϵ_1 tend to zero, keeping ϵ_2 fixed, and after that let ϵ_2 tend to zero we get that

$$\lim_{\epsilon_2 \to 0^+} \chi_2(|g|^2/\epsilon_2) R^f = R^f.$$

We remark that the product $\chi_2(|g|^2/\epsilon_2)R^f$ is well defined since the wave front sets of $\chi_2(|g|^2/\epsilon_2)$ and R^f behave properly, see e.g. [7]. Since $\chi_2(|g|^2/\epsilon_2)$ equals the characteristic function of $\{|g|^2 > \epsilon_2\}$ we have

Corollary 24. *If f defines a complete intersection then the Cauchy-Fantappiè-Leray current R^f has the standard extension property.*

This is a well known result and follows from the fact that R^f equals the Coleff-Herrera current in the sense of (7). It is even true that $\chi_{\rho g}(\epsilon)R^f \to R^f$, $\epsilon \to 0^+$ where ρ is a positive smooth function and $\chi_{\rho g}(\epsilon)$ is the characteristic function of $\{|\rho g| > \epsilon\}$. In fact, via Hironaka and toric resolutions one reduces to the case of one function and then one can proceed as in [7].

We know from [26] that if $f \oplus g$ defines a complete intersection then $R^f \wedge R^g$ consists of one term of top degree. Hence, it is only the top degree term of $\bar\partial\chi_1 \wedge u^f \wedge \bar\partial\chi_2 \wedge u^g$ which gives a contribution in the limit. With the natural choices $\chi_1(t) = t^{m_1}/(t+1)^{m_1}$ and $\chi_2(t) = t^{m_2}/(t+1)^{m_2}$, Corollary 23 and Remark 22 thus give

Corollary 25. *Let f and g be holomorphic sections (locally non-trivial) of the holomorphic m_j-bundles $E_j^* \to X$, $j = 1, 2$, respectively. Assume that the section $f \oplus g$ of $E_1^* \oplus E_2^* \to X$ defines a complete intersection. Then, for any test form φ we have*

$$\int \bar\partial \frac{s_f \wedge (\bar\partial s_f)^{m_1-1}}{(|f|^2+\epsilon_1)^{m_1}} \wedge \bar\partial \frac{s_g \wedge (\bar\partial s_g)^{m_2-1}}{(|g|^2+\epsilon_2)^{m_2}} \wedge \varphi \to R^f \wedge R^g.\varphi$$

as $\epsilon_1, \epsilon_2 \to 0^+$, and the integral to the left belongs to some Hölder class independently of φ.

For sections f and g of the trivial line bundle we get the result announced in [22].

Corollary 26. *Let f and g be holomorphic functions defining a complete intersection. Then for any test form φ we have*

$$\int \bar\partial \frac{\bar f}{|f|^2+\epsilon_1} \wedge \bar\partial \frac{\bar g}{|g|^2+\epsilon_2} \wedge \varphi \to \left[\bar\partial\frac{1}{f} \wedge \bar\partial\frac{1}{g}\right].\varphi$$

as $\epsilon_1, \epsilon_2 \to 0^+$, and the integral to the left belongs to some Hölder class independently of φ.

Proof. We consider f and g as sections of (different copies of) the trivial line bundle $X \times \mathbb{C} \to X$ with the standard metric. Then, suppressing the natural global frame elements, we have $s_f = \bar f$ and $s_g = \bar g$. By Corollary 25 we are done since $R^f \wedge R^g$ is the Coleff-Herrera current. □

So far, in this section, we have used one function χ to regularize all terms of u^f. One could try to take different χ:s for different terms. We recall the natural choices $t^k/(t+1)^k$ from Corollary 4 and we let $u^f_\epsilon = s_f/(\nabla s_f + \epsilon) = \sum s_f \wedge (\bar\partial s_f)^{k-1}/(|f|^2+\epsilon)^k$. The next theorem says that, in the complete intersection case, the product of two such regularized currents goes unrestrictedly to the product, in the sense of [26], of the currents.

Theorem 27. *Let f and g be holomorphic sections (locally non-trivial) of the holomorphic m_j-bundles $E_j^* \to X$, $j = 1, 2$, respectively. Assume that the section $f \oplus g$ of $E_1^* \oplus E_2^* \to X$ defines a complete intersection. Then, for any test form φ we have*

$$\int u^f_{\epsilon_1} \wedge \nabla u^g_{\epsilon_2} \wedge \varphi = (U^f - U^f \wedge R^g).\varphi$$

as $\epsilon_1, \epsilon_2 \to 0^+$, and the integral to the left belongs to some Hölder class independently of φ.

Proof. We first note that

$$\nabla u^g_{\epsilon_2} = 1 - \epsilon_2 \sum_{\ell \ge 1} \frac{(\bar\partial s_g)^{\ell-1}}{(|g|^2+\epsilon_2)^\ell},$$

see the proof of Corollary 4. As $U^f \wedge R^f$ is defined as the value at zero of the analytic continuation (in the sense of currents) of $|f|^{2\lambda} u^f \wedge \bar\partial |g|^{2\lambda} \wedge u^g$, what we have to prove is

$$\int \frac{s_f \wedge (\bar\partial s_f)^{k-1}}{(|f|^2+\epsilon_1)^k} \wedge \epsilon_2 \frac{(\bar\partial s_g)^{\ell-1}}{(|g|^2+\epsilon_2)^\ell} \wedge \varphi \to \int |f|^{2\lambda} u^f_{k,k-1} \wedge \bar\partial |g|^{2\lambda} \wedge u^g_{\ell-1,\ell-2} \wedge \varphi \Big|_{\lambda=0} \tag{33}$$

and that the integral on the left belongs to some Hölder class. We first consider the case $\ell = 1$. The right hand side of (33) should then be interpreted as zero. We write the integrand on the left hand side of (33) as $\chi_1(|f|^2/\epsilon_1)\chi_2(|g|^2/\epsilon_2) u^f_{k,k-1} \wedge \varphi$ where $\chi_1(t) = t^k/(t+1)^k$ and $\chi_2(t) = 1/(t+1)$. As in the proof of Theorem 16 we may assume that $u^f_{k,k-1} = \tilde u^f_{k,k-1}/\zeta^{k\alpha}$, where $\tilde u^f_{k,k-1}$ is a smooth form, that $|f|^2 = |\zeta^\alpha|\Phi$ and that $|g|^2 = |\zeta^\beta|^2\Psi$, where Φ and Ψ are strictly positive smooth functions. Since $\chi_2(\infty) = 0$ the left hand side of (33) tends to zero and belongs to some Hölder class by Proposition 11. For $\ell \geq 2$ we proceed as in the proof of Theorem 21 and we see that we may assume that $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_{m_2})$ with $f_j = \zeta^{\alpha^j} f'_j$ and $g_j = \zeta^{\beta^j} g'_j$ where all f'_j and g'_j are non-vanishing and moreover, that for some indices ν_1 and ν_2 it holds that $\zeta^\alpha := \zeta^{\alpha^{\nu_1}}$ divides all ζ^{α^j} and $\zeta^\beta := \zeta^{\beta^{\nu_2}}$ divides all ζ^{β^j}. From the same proof we also see that we may assume that $d\bar\zeta_j/\bar\zeta_j \wedge \varphi$ is smooth (and compactly supported) for all ζ_j which divide both ζ^α and ζ^β, since $f \oplus g$ defines a complete intersection. We use the notation from the proof of Theorem 21, e.g. $|f|^2 = |\zeta^\alpha|^2\Phi = |\zeta^{\alpha'+\alpha''}|^2\Phi$, $u^f_{k,k-1} = \tilde u^f_{k,k-1}/\zeta^{k(\alpha'+\alpha'')}$ and $|g|^2 = |\zeta^\beta|^2\Psi = |\zeta^{\beta'+\beta''}|^2\Psi$ etc. We also introduce the notation $\chi_j(t)$ for the function $t^j/(t+1)^j$, and so, in particular, we can write $1/(t+\epsilon)^j = \chi_j(t/\epsilon)/t^j$. For $\ell \geq 2$, one can verify that

$$\begin{aligned} \epsilon_2 \frac{(\bar\partial s_g)^{\ell-1}}{(|g|^2+\epsilon_2)^\ell} &= \frac{1}{\zeta^{(\ell-1)\beta}} \bar\partial \chi_{\ell-1}(|\zeta^\beta|^2\Psi/\epsilon_2) \wedge \tilde u^g_{\ell-1,\ell-2} \\ &+ \frac{1}{\zeta^{(\ell-1)\beta}} \chi'_{\ell-1}(|\zeta^\beta|^2\Psi/\epsilon_2) \frac{|\zeta^\beta|^2}{\epsilon_2} \frac{\Psi}{\ell-1} \bar\partial \tilde u^g_{\ell-1,\ell-2}. \end{aligned} \tag{34}$$

Using this identity we see that the integral on the left hand side of (33) splits into two integrals. The integral corresponding to the last term in (34) tends to zero as $\epsilon_1, \epsilon_2 \to 0$ and belongs to some Hölder class according to Corollary 15. By Proposition 18, the integral corresponding to the first term on the right hand side of (34) also belongs to some Hölder class and tends to

$$-\Big[\frac{1}{\zeta^{k\alpha+(\ell-1)\beta''}}\Big] \otimes \bar\partial \Big[\frac{1}{\zeta^{(\ell-1)\beta'}}\Big] . \tilde u^f_{k,k-1} \wedge \tilde u^g_{\ell-1,\ell-2} \wedge \varphi \tag{35}$$

as $\epsilon_1, \epsilon_2 \to 0$. This is seen to be equal to the right hand side of (33) by using the methods in [26]. □

7. The Passare-Tsikh example

Let $f = z_1^4$, $g = z_1^2 + z_2^2 + z_1^3$ and $\varphi = \rho \bar{z}_2 g dz_1 \wedge dz_2$ where ρ has compact support and is identically 1 in a neighborhood of the origin. Since the common zero set of f and g is just the origin they define a complete intersection. In [19] Passare and Tsikh show that the residue integral

$$(\epsilon_1, \epsilon_2) \mapsto I^{\varphi}_{f,g}(\epsilon_1, \epsilon_2) = \int_{\substack{|f|^2=\epsilon_1 \\ |g|^2=\epsilon_2}} \frac{\varphi}{fg}$$

is discontinuous at the origin. More precisely, they show that for any fixed positive number $c \neq 1$ one has $\lim_{\epsilon\to 0} I^{\varphi}_{f,g}(\epsilon^4, c\epsilon^2) = 0$ but $\lim_{\epsilon\to 0} I^{\varphi}_{f,g}(\epsilon^4, \epsilon^2) \neq 0$. On the other hand, by Fubini's theorem we have

$$\int\limits_{[0,\infty)^2} \frac{\epsilon_2\epsilon_2 I^{\varphi}_{f,g}(t_1,t_2)dt_1dt_2}{(t_1+\epsilon_1)^2(t_2+\epsilon_2)^2} = \int \frac{\epsilon_1 d|f|^2}{(|f|^2+\epsilon_1)^2} \wedge \frac{\epsilon_2 d|g|^2}{(|g|^2+\epsilon_2)^2} \wedge \frac{\varphi}{fg} =$$

$$\int \bar{\partial}\frac{\bar{f}}{|f|^2+\epsilon_1} \wedge \bar{\partial}\frac{\bar{g}}{|g|^2+\epsilon_2} \wedge \varphi. \tag{36}$$

Hence, this average of the residue integral is continuous at the origin by Corollary 26. In this section we will examine the last integral in (36) as $\epsilon_1, \epsilon_2 \to 0$ explicitly. We will see that it is continuous at the origin with Hölder exponent at least 1/8 and that it tends to zero. Morally, the value of $I^{\varphi}_{f,g}(\epsilon_1, \epsilon_2)$ at 0 should be the Coleff-Herrera current associated to f and g multiplied by $\bar{z}_2 g$ acting on $\rho dz_1 \wedge dz_2$. But both g and $\bar{z}_2$ annihilate the Coleff-Herrera current since g belongs to the ideal generated by f and g, and z_2 belongs to the radical of this ideal. We will thus verify Corollary 26 explicitly in this special case.

Our first objective is to resolve singularities to obtain normal crossings. This is accomplished by a blow-up of the origin. The map $\pi\colon \mathcal{B}_0\mathbb{C}^2 \to \mathbb{C}^2$ looks like $\pi(u,v) = (u, uv)$ and $\pi(u',v') = (u'v', u')$ in the two standard coordinate systems on $\mathcal{B}_0\mathbb{C}^2$. The exceptional divisor, E, corresponds to the sets $\{u = 0\}$ and $\{u' = 0\}$ and π is a biholomorphism $\mathcal{B}_0\mathbb{C}^2 \setminus E \to \mathbb{C}^2 \setminus \{0\}$. In the (u,v)-coordinates we have $\pi^* f = u^4$ and $\pi^* g = u^2(1 + v^2 + u)$. The function $1 + v^2 + u$ has non-zero differential and its zero locus intersects E normally in the two points $v = i$ and $v = -i$. Moreover, in the (u',v')-coordinates we have $\pi^* f = u'^4 v'^4$ and $\pi^* g = u'^2(v'^2 + 1 + u'v'^3)$. The zero locus of $v'^2+1+u'v'^3$ intersects E normally in the points $v' = -i$ and $v' = i$, which we already knew, and it does not intersect $v' = 0$. Also, the differential of $v'^2 + 1 + u'v'^3$ is non-zero on the zero locus of $v'^2 + 1 + u'v'^3$. Hence, $\{\pi^* f \cdot \pi^* g = 0\}$ has normal crossings. We assume that φ has support so close to the origin that $\text{supp}(\pi^*\varphi) \cap \{1+v^2+u = 0\}$ has two (compact) components, K_1 and K_2, and that these components together with the compacts $K_3 = \text{supp}(\pi^*\varphi) \cap \{v = 0\}$ and $K_4 = \text{supp}(\pi^*\varphi) \cap \{v =' 0\}$ are pairwise disjoint. We can then choose a partition of unity $\{\rho_j\}_1^4$ such that $\sum \rho_j \equiv 1$ on the support of $\pi^*\varphi$ and for each $j = 1, 2, 3, 4$, the support of ρ_j intersects only one of the compacts K_1, K_2, K_3 and K_4. We choose the numbering such that the support of ρ_j intersects K_j. The last integral in (36) now equals

$$\sum_1^4 \int \bar\partial \frac{\pi^*\bar f}{|\pi^* f|^2+\epsilon_1} \wedge \bar\partial \frac{\pi^*\bar g}{|\pi^* g|^2+\epsilon_2} \wedge \rho_j \pi^*\varphi := I_1 + I_2 + I_3 + I_4. \tag{37}$$

In fact, it is only in I_3 we have resonance and we start by considering the easier integrals I_1, I_2 and I_4. The integrals I_1 and I_2 are similar and we only consider I_1. The support of ρ_1 is contained in a neighborhood of $p_1 = (0, i)$ in the (u, v)-coordinates and $\rho_1 \pi^*\varphi = \rho_1 \pi^*\rho \bar u \bar v \pi^* g u du \wedge dv$. Integrating by parts we thus see that

$$I_1 = -\int \bar\partial \frac{\pi^*\bar f}{|\pi^* f|^2+\epsilon_1} \frac{|\pi^* g|^2}{|\pi^* g|^2+\epsilon_2} \wedge u\bar\partial(\bar u \bar v \rho_1 \pi^*\rho du \wedge dv).$$

Since $\pi^* f = u^4$ depends on u only, the term of $\bar\partial(\bar u \bar v \rho_1 \pi^*\rho)$ involving $d\bar u$ does not give any contribution to I_1. Hence we can replace $\bar\partial(\bar u \bar v \rho_1 \pi^*\rho)$ by $\bar u \varphi_1$ where φ_1 is smooth and supported where ρ_1 is. We put $\zeta_1 = u$ and $\zeta_2 = 1 + v^2 + u$, which defines a change of variables on the support of ρ_1. In these coordinates $\pi^* f = \zeta_1^4$ and $\pi^* g = \zeta_1^2\zeta_2$ and so we get

$$I_1 = -\int \frac{1}{\zeta_1^3} \bar\partial\chi(|\zeta_1^4|^2/\epsilon_1)\chi(|\zeta_1^2\zeta_2|^2/\epsilon_2) \wedge \bar\zeta_1 \varphi_1$$

where $\chi(t) = t/(t+1)$. We also write $\bar\partial\chi(|\zeta_1^4|^2/\epsilon_1) = 4\tilde\chi(|\zeta_1^4|^2/\epsilon_1) d\bar\zeta_1/\bar\zeta_1$, where $\tilde\chi(t) = t/(t+1)^2$. To proceed we replace (the coefficient function of) $d\bar\zeta_1/\bar\zeta_1 \wedge \bar\zeta_1\varphi_1$ by its Taylor expansion of order one, considered as a function of ζ_1 only, plus a remainder term $|\zeta_1|^2 B(\zeta)$, with B bounded. The terms corresponding to the Taylor expansion do not give any contribution to I_1 since we have anti-symmetry with respect to ζ_1 for these terms. Hence, we obtain

$$|I_1| \lesssim \int_\Delta \Big| \frac{|\zeta_1|^2 B(\zeta)}{\zeta_1^3} \tilde\chi(|\zeta_1^4|^2/\epsilon_1)\chi(|\zeta_1^2\zeta_2|^2/\epsilon_2)\Big|, \tag{38}$$

where Δ is a polydisc containing the support of φ_1. We estimate $|B(\zeta)|$ and $\chi(|\zeta_1^2\zeta_2|^2/\epsilon_2)$ by constants, and on the sets $\Delta_\epsilon = \{\zeta \in \Delta;\, |\zeta_1^4|^2 \geq \epsilon_1\}$ and $\Delta \setminus \Delta_\epsilon$ we use that $\tilde\chi(|\zeta_1^4|^2/\epsilon_1) \lesssim \epsilon_1/|\zeta_1^4|^2$ and $\tilde\chi(|\zeta_1^4|^2/\epsilon_1) \lesssim |\zeta_1^4|^2/\epsilon_1$ respectively, to see that the right hand side of (38) is of the size $|\epsilon|^{1/8}$.

To deal with I_4 we proceed as follows. The support of ρ_4 is contained in a neighborhood of $p_4 = (0, 0)$ in the (u', v')-coordinates and $\pi^* f = u'^4 v'^4$ and $\pi^* g = u'^2(1 + v'^2 + u'v'^3) := u'^2\tilde g$. On the support of ρ_4 we have $\tilde g \neq 0$. The multiindices $(4, 4)$ and $(2, 0)$ are linearly independent and so we can make the factor $\tilde g$ disappear. Explicitly, choose a square root $\tilde g^{1/2}$ of $\tilde g$ and put $\zeta_1 = u'\tilde g^{1/2}$ and $\zeta_2 = v'\tilde g^{-1/2}$. In these coordinates $\pi^* f = \zeta_1^4\zeta_2^4$ and $\pi^* g = \zeta_1^2$. One also checks that $\rho_4\pi^*\varphi = |\zeta_1|^2\pi^* g\varphi_4$ where φ_4 is a test form of bidegree $(2, 0)$. After an integration by parts we see that

$$I_4 = \int \frac{\pi^*\bar f}{|\pi^* f|^2+\epsilon_1} \bar\partial \frac{|\pi^* g|^2}{|\pi^* g|^2+\epsilon_2} \wedge \bar\partial(|\zeta_1|^2\varphi_4). \tag{39}$$

Since $\pi^*g = \zeta_1^2$ only depends on ζ_1 we may replace $\bar\partial(|\zeta_1|^2\varphi_4)$ by $|\zeta_1|^2\bar\partial\varphi_4$ in (39). Computing $\bar\partial(|\pi^*g|^2/(|\pi^*g|^2+\epsilon_2))$ we find that

$$I_4 = 2\int \frac{1}{\zeta_1^3\zeta_2^4}\chi(|\zeta_1^4\zeta_2^4|^2/\epsilon_1)\tilde\chi(|\zeta_1^2|^2/\epsilon_2)d\bar\zeta_1\wedge\bar\partial\varphi_4.$$

With abuse of notation we write the test form $d\bar\zeta_1\wedge\bar\partial\varphi_4$ as $\varphi_4 d\zeta\wedge d\bar\zeta$. Let $M = M^{1,2}_{1,2}$ be the operator defined in Lemma 6. Explicitly, we have

$$\begin{aligned} M\varphi_4 &= M_1^1\varphi_4 + M_2^2\varphi_4 - M_1^1M_2^2\varphi_4 \\ &= M_1^1(\varphi_4 - M_2^2\varphi_4) + M_2^2(\varphi_4 - M_1^1\varphi_4) + M_1^1M_2^2\varphi_4. \end{aligned}$$

All of the following properties will not be important for this computation but to illustrate Lemma 6 we note that the second expression of $M\varphi$ reveals that $M\varphi_4$ can be written as a sum of terms $\phi_{IJ}(\zeta)\zeta^I\bar\zeta^J$ with $I_1+J_1\le 1$ and $I_2+J_2\le 2$ and moreover, that ϕ_{IJ} is independent of at least one variable and is of the size $\mathcal{O}(|\zeta_1|^2)$ if it depends on ζ_1 and of the size $\mathcal{O}(|\zeta_2|^3)$ if it depends on ζ_2. By Lemma 6 we also have $\varphi_4 = M\varphi_4 + |\zeta_1|^2|\zeta_2|^3B(\zeta)$ for some bounded function B and so

$$I_4 = \int_\Delta \frac{1}{\zeta_1^3\zeta_2^4}\chi\tilde\chi M\varphi_4 + \int_\Delta \frac{1}{\zeta_1^3\zeta_2^4}\chi\tilde\chi|\zeta_1|^2|\zeta_2|^3B(\zeta) =: I_{4.1} + I_{4.2},$$

where Δ is a polydisc containing the support of φ_4. By anti-symmetry $I_{4.1}=0$. To estimate $I_{4.2}$ we use that $|\chi B|$ is bounded by a constant and that $\tilde\chi(\Psi|\zeta_1^2|^2/\epsilon_2)\lesssim \epsilon_2/|\zeta_1^2|^2$ and $\tilde\chi(\Psi|\zeta_1^2|^2/\epsilon_2)\lesssim|\zeta_1^2|^2/\epsilon_2$ on the sets $\Delta_\epsilon = \{\zeta\in\Delta;\, |\zeta_1^2|^2\ge\epsilon_2\}$ and $\Delta\setminus\Delta_\epsilon$ respectively. Hence,

$$|I_{4.2}| \lesssim \int_{\Delta_\epsilon}\frac{\epsilon_2}{|\zeta_1^2|^2|\zeta_1||\zeta_2|} + \int_{\Delta\setminus\Delta_\epsilon}\frac{|\zeta_1^2|^2}{\epsilon_2|\zeta_1||\zeta_2|}, \tag{40}$$

which is seen to be of the size $|\epsilon|^{1/4}$.

It remains to take care of I_3. We are now working close to $u=v=0$ and $\pi^*f = u^4$ and $g = u^2(1+v^2+u) := u^2\tilde g$. The multiindices are linearly dependent and we cannot dispose of the non-zero factor $\tilde g$. We rename our variables $(u,v)=(\zeta_1,\zeta_2)$ and proceed in precisely the same way as we did when we were considering I_1. We get

$$I_3 = -4\int\frac{1}{\zeta_1^3}\tilde\chi(|\zeta_1^4|^2/\epsilon_1)\chi(\Phi|\zeta_1^2|^2/\epsilon_2)\varphi_3 d\zeta\wedge d\bar\zeta,$$

where $\Phi = |\tilde g|^2$ is a strictly positive smooth function and φ_3 is smooth with compact support. As before, we replace φ_3 by $M^1_{\zeta_1}\varphi_3 + |\zeta_1|^2B(\zeta)$. The integral corresponding to $|\zeta_1|^2B(\zeta)$ satisfies the same estimate as the one in (38) and hence is of the size $|\epsilon_1|^{1/8}$. We cannot use anti-symmetry directly to conclude the the integrals corresponding to the other terms in the Taylor expansion tend to zero since the factor $\tilde g$ is present. We illustrate why this is true anyway by considering the integral corresponding to the term $\varphi_3(0,\zeta_2)$. Let Δ be a polydisc containing the support of φ_3 and consider

$$\int_\Delta\frac{1}{\zeta_1^3}\tilde\chi(|\zeta_1^4|^2/\epsilon_1)\chi(\Phi|\zeta_1^2|^2/\epsilon_2)\varphi_3(0,\zeta_2). \tag{41}$$

We introduce the smoothing parameter $t = |\zeta_1^2|^2/\epsilon_2$ as an independent variable and write

$$\chi(\Phi t) = \chi(\Phi t) - M^1_{\zeta_1}\chi(\Phi t) + M^1_{\zeta_1}\chi(\Phi t) := |\zeta_1|^2 B(t,\zeta) + M^1_{\zeta_1}\chi(\Phi t).$$

Here B is bounded on $[0,\infty] \times \Delta$. Substituting into (41) we obtain one integral corresponding to $|\zeta_1|^2 B(|\zeta_1^2|^2/\epsilon_2, \zeta)$, which satisfies an estimate like (38), while the integral corresponding to $M^1_{\zeta_1}\chi(\Phi|\zeta_1^2|^2/\epsilon_2)$ is zero since we have anti-symmetry with respect to ζ_1. Hence $|I_3| \lesssim |\epsilon|^{1/8}$.

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Paper III

MULTIDIMENSIONAL CAYLEY TRANSFORMS AND TUPLES OF UNBOUNDED OPERATORS

HÅKAN SAMUELSSON

Abstract. We generalize the Cayley transform to tuples of unbounded operators. To achieve this we introduce intrinsically defined objects, with spectrum in projective space, which admit an analytic functional calculus. We also provide an integral representation for this functional calculus.

1. Introduction

The Cayley transform, $a \mapsto (a+i)(a-i)^{-1}$, introduced by von Neumann in [13] induces a one-to-one correspondence between the self-adjoint operators and the unitary operators such that 1 is not in the point spectrum. More generally, one can consider any automorphism of $\widehat{\mathbb{C}}$ and apply it to an arbitrary closed operator provided that the point mapped to infinity is outside the point spectrum of the operator. In case this point is outside all of the spectrum then the image is a bounded operator. In this way we get an intrinsic object with spectrum in $\mathbb{CP}^1$ which for any choice of point at infinity, outside the point spectrum, and any linear coordinate gives rise to a closed operator. One possible generalization to higher dimensions, i.e. to tuples of operators, is to take the Cayley transform of each of the operators. This is possible if all the operators have nonempty resolvent sets, and if the operators commute in the strong sense we obtain a tuple of bounded commuting operators in this way. Vasilescu used this technique in [11] to prove spectral theorems for unbounded self-adjoint operators. In a more general setting this has recently been studied by Andersson and Sjöstrand in [2]. There is also a notion of Quaternionic Cayley transform introduced in [12] but we will not consider it here.

In this paper we are concerned with another generalization of the Cayley transform. We characterize the n-tuples of closed unbounded operators which by a projective transformation of $\mathbb{CP}^n$ can be mapped to tuples of

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bounded commuting operators. This is what we will call a *multidimensional Cayley transform*. One point to be made is that these tuples of unbounded operators may consist of operators with empty resolvent sets. The characterization is in terms of an algebraic relation linking the operators together and a commutation condition stronger but similar to the notion of permutability described in [5]. Tuples of closed unbounded operators satisfying these conditions will be called *affine operators*. We define a Taylor spectrum for the affine operators and we show that the spectral mapping property holds. In case all the operators making up the affine operator have resolvents we can also consider the tuple of one-dimensional Cayley transforms as mentioned above. This tuple has a well defined Taylor spectrum and as in e.g. [11] and [2] we can define a joint spectrum for the original tuple by claiming that the spectral mapping property should hold. We show that the spectrum we define is contained in this spectrum and that we have equality in the case of pairs of operators. To carry out our idea we introduce *projective operators*, an intrinsic object in $\mathbb{CP}^n$ with an invariant spectrum and admitting an analytic functional calculus. From the abstract point of view a projective operator is an $\mathscr{O}_{\mathbb{CP}^n}$-module as described in [4]; see also Section 7. More concretely, we realize projective operators as certain equivalence classes of $n+1$-tuples of bounded commuting operators. The spectrum for the projective operator can be described from the Taylor spectrum for a representative and via integral formulas inspired by [1] we can also describe the module structure from a representative of the equivalence class. In this paper we will only consider projective operators having a spectrum avoiding some hyperplane in $\mathbb{CP}^n$. In an affinization where we take such a hyperplane to be the hyperplane at infinity the projective operator corresponds to a tuple of bounded commuting operators and the module structure is Taylor's functional calculus.

The disposition of the paper is as follows. In Section 2 we briefly review Taylor's functional calculus and we state the basic facts about one-dimensional Cayley transforms. In Section 3 we define projective operators and study their fundamental properties. In Section 4 we study the behavior of projective operators under various projections from $\mathbb{CP}^n$ to $\mathbb{C}^n$ and we define affine operators. In Section 5 we define a Taylor spectrum for affine operators and relate it to some other existing definitions. In Section 6 we summarize our results and interpret them on the affine level. In Section 7 we provide an integral representation for the analytic functional calculus obtained in Section 3.

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topics in this paper and on general questions about tuples of unbounded operators.

2. Preliminaries

If a is an operator on some space X then $\mathscr{D}(a)$ is the domain of definition for a and $\mathscr{R}(a)$ is the range, i.e. the set of all ax such that $x \in \mathscr{D}(a)$. The set of $x \in \mathscr{D}(a)$ such that $ax = 0$, the nullspace of a, is denoted $\mathscr{N}(a)$. If also b is an operator on X then $a \subseteq b$ means that the graph of a is included in the graph of b in $X \times X$. In particular $a = b$ means that a and b have the same domain of definition.

2.1. **Taylor's functional calculus.** Let X be a Banach space, $L(X)$ the algebra of bounded operators on X and let E be an n-dimensional complex vector space with a non-sense basis $\{e_1, \dots, e_n\}$. We write $\Lambda^k X$ for the tensor product $X \otimes \Lambda^k E$ of X and the k:th exterior product of E. Let $b = (b_1, \dots, b_n)$ be an n-tuple of bounded commuting operators on X. On $\Lambda X = \bigoplus_1^n \Lambda^k X$ we have the natural operation of interior multiplication with the operator-valued co-vector $\sum_1^n (z_j - b_j)e_j^*$. We denote this operation δ_{z-b}. Since b is commuting $\delta_{z-b} \circ \delta_{z-b} = 0$ and so we have the Koszul complex

$$(1) \qquad 0 \longleftarrow \Lambda^0 X \xleftarrow{\delta_{z-b}} \Lambda^1 X \xleftarrow{\delta_{z-b}} \cdots \xleftarrow{\delta_{z-b}} \Lambda^n X \longleftarrow 0\,,$$

or $K_\bullet(\delta_{z-b}, \Lambda^\bullet X)$ for short. The joint Taylor spectrum $\sigma(b)$ for b is defined as the complement in $\mathbb{C}^n$ of the set of points z such that $K_\bullet(\delta_{z-b}, \Lambda^\bullet X)$ is exact, [9]. Taylor's fundamental result in [9] and [8] is that the natural algebra homomorphism $\mathscr{O}(\mathbb{C}^n) \to L(X)$ given by $\sum_\alpha c_\alpha z^\alpha \mapsto \sum_\alpha c_\alpha b^\alpha$ extends to an algebra homomorphism $\mathscr{O}(\sigma(b)) \to L(X)$.

Theorem 2.1 (Taylor, 1970). *There is an extension of the natural continuous algebra homomorphism $\mathscr{O}(\mathbb{C}^n) \to L(X)$ to a continuous algebra homomorphism*

$$f \mapsto f(b)\colon\ \mathscr{O}(U) \to L(X)$$

for all open sets U such that $\sigma(b) \subseteq U$. If $f = (f_1, \dots, f_m) \in \mathscr{O}(U, \mathbb{C}^m)$, then $f(\sigma(b)) = \sigma(f(b))$, where $f(b) = (f_1(b), \dots, f_n(b))$.

The statement $f(\sigma(b)) = \sigma(f(b))$ will be referred to as the Spectral mapping theorem.

2.2. **The one-dimensional Cayley transform.** Let X be a Banach space and let $\mathscr{C}(X)$ be the set of closed, but not necessarily densely defined operators on X. For any linear operator a on X the spectrum of a, $\sigma(a)$, is the complement in $\mathbb{C}$ of the set of points λ such that $\lambda - a$ is a bijection $\mathscr{D}(a) \to X$. The point spectrum, $\sigma_p(a) \subseteq \sigma(a)$, is the set of $\lambda \in \mathbb{C}$ such that $\lambda - a$ is not injective. For $a \in \mathscr{C}(X)$ we have by the Closed Graph Theorem that $\lambda \notin \sigma(a)$ if and only if $\lambda - a$ has a bounded inverse. We let $\widehat{\mathbb{C}}$ denote the extended complex plane; $\mathbb{C} \cup \{\infty\}$ and we define the extended spectrum $\hat{\sigma}(a)$ as $\sigma(a)$ if a is bounded and $\sigma(a) \cup \{\infty\}$ if a is not bounded.

Let ϕ be a projective-, or Möbius transformation of $\widehat{\mathbb{C}}$. We claim that $\phi(a)$ has meaning as an element in $\mathscr{C}(X)$ if $\phi^{-1}(\infty) \notin \sigma_p(a)$. Given the projective transformation ϕ we let $M_\phi \in \mathrm{GL}(2, \mathbb{C})$ be the corresponding 2×2-matrix. If $M_\phi = \{m_{j,k}\}_{1 \leq j,k \leq 2}$ and $\phi^{-1}(\infty) \notin \sigma_p(a)$ we may put

$$\phi(a) = (m_{1,1}a + m_{1,2})(m_{2,1}a + m_{2,2})^{-1}. \tag{2}$$

The matrix M_ϕ acts naturally as a homeomorphism of $X \times X$ and it is straight forward to verify that $M_\phi \mathrm{Graph}(a) = \mathrm{Graph}(\phi(a))$ and hence $\phi(a)$ is closed if a is. Moreover, it is not hard to see that $\phi(a)$ is bounded if and only if $\phi^{-1}(\infty) \notin \hat{\sigma}(a)$. We conclude that the closed operators on X which can be Cayley transformed to bounded operators are precisely those with a non-empty resolvent set. The spectral mapping property holds for these mappings, that is for any closed operator a on X and projective transformation ϕ of $\widehat{\mathbb{C}}$ such that $\phi^{-1}(\infty) \notin \hat{\sigma}_p(a)$ it holds that $\phi(\hat{\sigma}(a)) = \hat{\sigma}(\phi(a))$. For a more thorough treatment of the one-dimensional Cayley transform, see [11] and [7].

The preceding discussion suggests that the closed operator a defines some invariant object on $\mathbb{CP}^1 = \widehat{\mathbb{C}}$ if $\infty \notin \sigma_p(a)$. In the canonical affine part of $\widehat{\mathbb{C}}$ this object becomes the operator a and in some other affine part, corresponding to a Möbius transformation ϕ of the canonical one, it becomes $\phi(a)$ and has spectrum $\phi(\hat{\sigma}(a))$.

3. Projective operators and analytic functional calculus

In analogy with the construction of projective space we consider an equivalence relation on a subset of the $n+1$-tuples of bounded commuting operators on a Banach space and define a *projective operator* as an equivalence class. We will see that a projective operator has a well defined invariant Taylor spectrum in $\mathbb{CP}^n$ and that it admits an analytic functional calculus.

Definition 3.1. Let $b = (b_0, \dots, b_n)$ and $\tilde{b} = (\tilde{b}_0, \dots, \tilde{b}_n)$ be tuples of bounded commuting operators on a Banach space X. We define $b \sim \tilde{b}$ if there are finitely many bounded commuting tuples b^j, $j = 1, \dots, m$ such that $b^1 = b$ and $b^m = \tilde{b}$ and for $j = 1, \dots, m-1$ we have $b^{j+1} = c_j b^j$ for some invertible $c_j \in (b^j)'$; the commutant of b^j.

Lemma 3.2. *The relation $\sim$ of definition 3.1 is an equivalence relation.*

Proof. We note that the relation R on bounded commuting $n+1$-tuples defined by $bR\tilde{b}$ if $\tilde{b} = cb$ for some invertible $c \in (b)'$ is reflexive and symmetric. Reflexivity is obvious since $e \in (b)'$. It is symmetric because if $\tilde{b} = cb$ for some invertible $c \in (b)'$ then $b = c^{-1}\tilde{b}$ and letting $\tilde{b} = (\tilde{b}_0, \ldots, \tilde{b}_n)$ and $b = (b_0, \ldots, b_n)$ we see

$$c^{-1}\tilde{b}_j = c^{-1}cb_j = b_j = b_j cc^{-1} = cb_j c^{-1} = \tilde{b}_j c^{-1}$$

so $c^{-1} \in (\tilde{b})'$. The relation $\sim$ is defined as the transitive closure of R so it is by definition transitive and it inherits reflexivity and symmetry from R. □

Remark 3.3. We will see later on, Remark 4.6, that for the tuples we will be interested in there is a simpler description of the relation $\sim$. For these tuples it will also turn out, see Remark 4.5, that even though $\sim$ is defined as the transitive closure of R, any two representatives for an equivalence class are not more than two steps from each other.

We denote the equivalence class containing b by $[b]$ and we let π denote the canonical mapping $\mathbb{C}^{n+1} \to \mathbb{CP}^n$.

Proposition 3.4. *Let $b = (b_0, \ldots, b_n)$ be a commuting tuple of bounded operators on X and let $c \in (b)'$ be invertible. If $0 \notin \sigma(b)$ then $0 \notin \sigma(cb)$ and*

$$\pi\sigma(b_0, \ldots, b_n) = \pi\sigma(cb_0, \ldots, cb_n).$$

Proof. Define ψ and $\phi : \mathbb{C}^{n+2} \to \mathbb{C}^{n+1}$ by

$$\begin{aligned} \psi(z, z_0, \ldots, z_n) &= (zz_0, \ldots, zz_n) \\ \phi(z, z_0, \ldots, z_n) &= (z_0, \ldots, z_n) \end{aligned}$$

respectively. The hyperplane in $\mathbb{C}^{n+2}$ orthogonal to the vector $(1, 0, \ldots, 0)$ does not intersect $\sigma(c, b_0, \ldots, b_n)$ since c is invertible and we have

$$\sigma(c, b_0, \ldots, b_n) \subseteq \sigma(c) \times \sigma(b_0, \ldots, b_n) \tag{3}$$

according to [9]. Moreover from (3) and the assumption that $0 \notin \sigma(b)$ we see that $\sigma(c, b_0, \ldots, b_n)$ also avoids the coordinate axis $(z, 0, \ldots, 0)$. Hence we may take a neighborhood U of $\sigma(c, b_0, \ldots, b_n)$ such that U does not intersect neither the hyperplane orthogonal to $(1, 0, \ldots, 0)$ nor the coordinate axis $(z, 0, \ldots, 0)$. Then the images V_1 and V_2 of U under ψ and ϕ respectively do not contain the origin and so the diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V_1 \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\pi} \\ V_2 & \xrightarrow[\pi]{} & \mathbb{CP}^n \end{array} \tag{4}$$

must commute. By the Spectral mapping theorem

$$\sigma(b_0,\dots,b_n) = \sigma\,\phi(c,b_0,\dots b_n) = \phi\,\sigma(c,b_0,\dots,b_n)$$

$$\sigma(cb_0,\dots,cb_n) = \sigma\,\psi(c,b_0,\dots b_n) = \psi\,\sigma(c,b_0,\dots,b_n)$$

and since the diagram (4) commutes we conclude that $\pi\,\sigma(cb_0,\dots,cb_n) = \pi\,\sigma(b_0,\dots,b_n)$. □

It follows immediately from this proposition that we have

Corollary 3.5. *Let $b \sim \tilde{b}$ and assume $0 \notin \sigma(b)$. Then $0 \notin \sigma(\tilde{b})$ and $\pi\sigma(b) = \pi\sigma(\tilde{b})$.*

Hence if $0 \notin \sigma(b)$ then $0 \notin \sigma(\tilde{b})$ for any $\tilde{b} \in [b]$ and $\pi\sigma(b) = \pi\sigma(\tilde{b})$ and so we can make the following definitions.

Definition 3.6. Let b be a commuting tuple of bounded operators on a Banach space X such that $0 \notin \sigma(b)$. We define the *projective operator* $[b]$ as the equivalence class containing b.

Definition 3.7. Let $[b]$ be a projective operator. The *spectrum* $\sigma[b] \subseteq \mathbb{CP}^n$ of the projective operator $[b]$ is defined by

$$\sigma[b] = \pi\sigma(b).$$

We now construct the analytic functional calculus for the projective operators. The main theorem of this section is the following.

Theorem 3.8. *If $[b]$ is a projective operator, then there is a unique $\mathscr{O}(\sigma[b])$-module structure on X given by*

$$\mathscr{O}(\sigma[b]) \times X \to X, \qquad (f,x) \mapsto f([b])x$$

and if $f = (f_1,\dots,f_m) \in \mathscr{O}(\sigma[b],\mathbb{C}^m)$ then $\sigma(f([b]) = f(\sigma[b])$ where $f([b]) = (f_1([b]),\dots,f_m([b]))$.

Proof. We construct the module-structure as follows. Given some $f \in \mathscr{O}(\sigma[b])$ we consider the canonical lift $\tilde{f}$ of f to $\mathbb{C}^{n+1}$. Then $\tilde{f}$ is holomorphic in a neighborhood of $\sigma(b)$ for any representative $b \in [b]$ and $\tilde{f}$ is constant on the complex lines through the origin (with the origin deleted). From Taylor's analytic functional calculus we get for each $b \in [b]$ an operator $\tilde{f}(b) \in L(X)$. We will see that in fact $\tilde{f}(b)$ is independent of representative b and our desired pairing $\mathscr{O}(\sigma[b]) \times X \to X$ will be $(f,x) \mapsto \tilde{f}(b)x$ where b is any representative of $[b]$.

Let $b \in [b]$ and let $c \in (b)'$ be invertible. Put $\tilde{b} = cb$ and let ϕ and ψ be the mappings defined in Proposition 3.4. Let U_1 be a neighborhood of $\sigma(b)$ in which $\tilde{f}$ is holomorphic and let V be a neighborhood of $\sigma(c)$ such that $\overline{D(0,r)} \cap V = \emptyset$ for some $0 < r < 1$. Since c is invertible $0 \notin \sigma(c)$ and such a neighborhood exists. Let U be the union over $\lambda \notin \overline{D(0,r)}$ of λU_1. Then $\tilde{f} \circ \phi$ and $\tilde{f} \circ \psi$ are holomorphic in $V \times U$. Moreover since $r < 1$ we have $\sigma(b) \subseteq U$ and so $\sigma(c, b) \subseteq \sigma(c) \times \sigma(b) \subseteq V \times U$. Now since $\tilde{f}$ is constant on the complex lines through the origin we have $\tilde{f} \circ \phi \,|_{V\times U} = \tilde{f} \circ \psi \,|_{V\times U}$ and we conclude from the composition rule that $\tilde{f}(b) = \tilde{f} \circ \phi \,|_{V\times U}\, (c, b) = \tilde{f} \circ \psi \,|_{V\times U}\, (c, b) = \tilde{f}(cb) = \tilde{f}(\tilde{b})$. It follows inductively that $\tilde{f}(b) = \tilde{f}(\tilde{b})$ for any two representatives b and $\tilde{b}$ for $[b]$. Thus f is well defined on $[b]$ and we write $f([b])$ for the operator $\tilde{f}(b)$.

To prove the spectral mapping property we proceed as follows. Since $\tilde{f}$ is constant on the complex lines through the origin $\tilde{f}(\sigma(b))$ is the same for all $b \in [b]$ and so from Theorem 2.1 we get

$$f(\sigma[b]) = \tilde{f}(\sigma(b)) = \sigma(\tilde{f}(b)) = \sigma(f([b])).$$

Uniqueness follows from the spectral mapping property. See [4]. □

Let M be a complex manifold and assume $f : U \supseteq \sigma[b] \to M$ is holomorphic. We obtain a $\mathscr{O}(M)$-module structure $\mathscr{M}$ on X by

$$\mathscr{O}(M) \times X \to X, \qquad (g, x) \mapsto g \circ f([b])x.$$

In [4] Eschmeier and Putinar define the spectrum $\sigma(M, \mathscr{M}) \subseteq M$ of the module $\mathscr{M}$ and show that the $\mathscr{O}(M)$-module structure extends uniquely to an $\mathscr{O}(\sigma(M, \mathscr{M}))$-module structure on X. Moreover they show a Spectral mapping theorem which in our case implies

$$\sigma(M, \mathscr{M}) = f(\sigma[b]).$$

It is shown that if $M = \mathbb{C}^m$ we can realize the extended module structure as the analytic functional calculus for an m-tuple of commuting bounded operators c on X by choosing coordinates on $\mathbb{C}^m$ and that the spectrum of the abstract module is precisely $\sigma(c)$. The composition rule in Taylor's functional calculus is therefore built into the construction.

To stress the independence of coordinates in our study of projective operators we adopt an invariant notation. For a subset M of $\mathbb{CP}^n$ we denote by M^* the *dual complement* of M, that is

$$M^* = \{[\lambda] \in \mathbb{CP}^{n*}; \langle z, \lambda\rangle \neq 0 \ \forall [z] \in M\}.$$

Geometrically M^* is the set of hyperplanes in $\mathbb{CP}^n$ which do not intersect M. The correspondence between hyperplanes in $\mathbb{CP}^n$ and points in $\mathbb{CP}^{n*}$ is the usual duality correspondence. To $[\lambda] \in \mathbb{CP}^{n*}$ we associate the hyperplane $\{[z]; \langle z, \lambda\rangle = 0\}$. We will not make any distinction between points in $\mathbb{CP}^{n*}$ and their corresponding hyperplanes and we will freely allow ourselves to speak about "the hyperplane $[\lambda]$" if $[\lambda] \in \mathbb{CP}^{n*}$.

Lemma 3.9. *Let* $[b]$ *be a projective operator. Then*

$$\sigma[b]^* = \{[\lambda] \in \mathbb{CP}^{n*}; \langle b, \lambda\rangle \text{ is invertible}\}.$$

Proof. Since any two representatives of $[b]$ differ by an invertible operator we see that the statement in the lemma only depends on $[b]$. For the inclusion $\subseteq$ assume $[\mu] \in \sigma[b]^*$. Then from the definition we have $\langle z, \mu\rangle \neq 0$ for all $[z] \in \sigma[b]$. Thus the function $z \mapsto 1/\langle z, \mu\rangle$ from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ is holomorphic in a neighborhood of $\sigma(b)$. Hence from the functional calculus we see that $\langle b, \mu\rangle$ is invertible.

For the other inclusion assume $\langle b, \mu\rangle$ is invertible. We shall show that $\sigma(b)$ does not intersect the hyperplane μ. If $\mu = (1, 0, \dots, 0)$ we have to show that if b_0 is invertible then $\sigma(b)$ does not intersect the hyperplane orthogonal to $(1, 0, \dots, 0)$. But if b_0 is invertible then $0 \notin \sigma(b_0)$ and since

$$\sigma(b) \subseteq \sigma(b_0) \times \sigma(b_1, \dots, b_n),$$

see [9], $\sigma(b)$ can not intersect the hyperplane in question. For the general case let L be an invertible linear transformation sending μ to $(1, 0, \dots, 0)$. By the Spectral mapping theorem, to show that $\sigma(b)$ does not intersect μ is equivalent to show that $\sigma(L^{*-1}b)$ does not intersect $L\mu = (1, 0, \dots, 0)$. But the first component in $L^{*-1}b$ is $\langle L^{*-1}b, L\mu\rangle = \langle b, \mu\rangle$ which is invertible by assumption and so the lemma follows. □

Remark 3.10. From now on we will always assume that $\sigma[b]$ avoids some hyperplane in $\mathbb{CP}^n$. We will do this because it will make it possible to realize the projective operator as an ordinary n-tuple of bounded operators. Since the objective of this paper is to construct multidimensional Cayley transforms of tuples of unbounded operators into tuples of bounded operators the assumption is natural.

If we fix some $[\tilde{\lambda}] \in \sigma[b]^*$ then the function $[z] \mapsto \langle z, \tilde{\lambda}\rangle/\langle z, \lambda\rangle$ is holomorphic in a neighborhood of $\sigma[b]$ if also $[\lambda] \in \sigma[b]^*$. Theorem 3.8 then implies that we get a holomorphic mapping from $\sigma[b]^*$ to the algebra generated by b for $b \in [b]$ given by $[\lambda] \mapsto \langle b, \tilde{\lambda}\rangle/\langle b, \lambda\rangle$. This is the *Fantappiè transform* of the $L(X)$-valued analytic functional $\mathscr{O}(\sigma[b]) \to L(X)$, $f \mapsto f([b])$ given by Theorem 3.8.

4. Affine operators

We extend the Fantappiè transform to a larger set $\sigma[b]^*_{adm}$, called the set of admissible hyperplanes, and get instead a $\mathscr{C}(X)$-valued mapping. We will define *affine operators* to be tuples of closed operators with certain commutation properties. We will show that affine operators are precisely the tuples obtained by projecting a projective operator from an admissible hyperplane. In order to keep track of the various domains of definition that turn up we start with some technical results.

In what follows we will often make implicit use of the following easily checked fact.

Proposition 4.1. *Let a be any closed operator on X and let b be bounded. Then, the operator ab with domain $\mathscr{D}(ab) = \{x \in X; bx \in \mathscr{D}(a)\}$ is closed.*

The following lemma generalizes the fact that if b and c are bounded operators and b is invertible, then $bc = cb$ if and only if $b^{-1}c = cb^{-1}$.

Lemma 4.2. *Let b and c be bounded operators on X and assume that b is injective. Then $bc = cb$ if and only if $cb^{-1} \subseteq b^{-1}c$. If this condition is fulfilled and in addition c is invertible then actually $cb^{-1} = b^{-1}c$.*

Proof. Assume $bc = cb$ and let $x \in \mathscr{D}(cb^{-1}) = \mathscr{D}(b^{-1})$. Then $x = by$ for some $y \in X$. Since b and c commute we get $cx = cby = bcy$ and so we must have $cx \in \mathscr{D}(b^{-1})$. Hence $\mathscr{D}(cb^{-1}) \subseteq \mathscr{D}(b^{-1}c)$ and

$$cb^{-1}x = cb^{-1}by = cy = b^{-1}bcy = b^{-1}cby = b^{-1}cx.$$

It follows that $cb^{-1} \subseteq b^{-1}c$. Conversely assume $cb^{-1} \subseteq b^{-1}c$. Then if $x \in \mathscr{D}(b^{-1})$ we have $cx \in \mathscr{D}(b^{-1})$ and so $b^{-1}cb \in L(X)$. By assumption $b^{-1}cb \supseteq cb^{-1}b = c$ and because $c \in L(X)$ we must have equality. Multiplying by b from the left we obtain $cb = bc$.

For the last statement assume c is invertible and commutes with b. Then c^{-1} also commutes with b. To show $cb^{-1} = b^{-1}c$ it is enough to show $\mathscr{D}(b^{-1}c) \subseteq \mathscr{D}(cb^{-1})$ by the proof this far. Take $x \in \mathscr{D}(b^{-1}c)$, i.e. such that $cx \in \mathscr{D}(b^{-1})$. Then $cx = by$ for some $y \in X$. We get $x = c^{-1}by = bc^{-1}y$ and so $x \in \mathscr{D}(b^{-1}) = \mathscr{D}(cb^{-1})$. □

The next lemma and the remarks following it shed some light on the equivalence classes $[b]$.

Lemma 4.3. *Let $[b]$ be a projective operator and assume that $[\lambda] \in \sigma[b]^*$. Then there is a representative b' for $[b]$ such that*

$$\langle b', \lambda \rangle = \sum_0^n \lambda_j b'_j = e.$$

Proof. If $\lambda \in \sigma[b]^*$ Lemma 3.9 says that $B = \langle b, \lambda \rangle$ is invertible. Then clearly $[B^{-1}b] = [b]$ and $b' = B^{-1}b$ is the desired representative. □

Remark 4.4. There is no loss of generality in assuming that $\lambda_0 \neq 0$ because we may perturbate $[\lambda]$ a little and still belong to $\sigma[b]^*$.

Remark 4.5. We have defined the equivalence relation on commuting tuples as the transitive closure of a symmetric and reflexive relation R. The proof of Lemma 4.3 shows that given a class $[b]$ such that $\sigma[b]^*$ is nonempty, any representative is not more then one step from the representative b' with $\langle b', \lambda\rangle = e$. Hence if b and $\tilde{b}$ are any two representatives for $[b]$ then they are not more then two steps from each other.

Remark 4.6. Lemma 4.3 also enable us to to give an alternative description of the equivalence relation $\sim$ if we restrict ourselves to look at commuting $n+1$-tuples of operators with the additional property that their spectrum avoid some hyperplane through the origin in $\mathbb{C}^{n+1}$. In fact for such tuples, b and $\tilde{b}$, we have $b \sim \tilde{b}$ if and only if $\tilde{b} = cb$ for some invertible c. The only if part is clear. Conversely assume that $\tilde{b} = cb$ for some invertible c. The assumption on the spectrum for $\tilde{b}$ says precisely that $\sigma[\tilde{b}]^*$ is nonempty and so from Lemma 4.3 we see that we may assume that $\langle \tilde{b}, \lambda\rangle = e$ for some $[\lambda]$. Hence $\langle b, \lambda\rangle = c^{-1}$ so $c \in (b)'$ and therefore $[b] = [\tilde{b}]$.

Definition 4.7. Let $[b]$ be a projective operator. We define $\sigma[b]^*_{adm}$, the set of *admissible* hyperplanes for $[b]$, by saying that $[\alpha] \in \sigma[b]^*_{adm}$ if $\langle b, \alpha\rangle\langle b, \lambda\rangle^{-1}$ is injective, where $[\lambda]$ is some hyperplane in $\sigma[b]^*$.

The definition clearly does not depend on the representative b for $[b]$ and also not on the choice of $[\lambda]$ because if $[\tilde{\lambda}] \in \sigma[b]^*$ is some other choice, then $\langle b, \alpha\rangle\langle b, \tilde{\lambda}\rangle^{-1} = \langle b, \alpha\rangle\langle b, \lambda\rangle^{-1}\langle b, \lambda\rangle\langle b, \tilde{\lambda}\rangle^{-1}$ and $\langle b, \lambda\rangle\langle b, \tilde{\lambda}\rangle^{-1}$ is invertible by the functional calculus.

Remark 4.8. Observe that $\sigma[b]^*_{adm}$ is not defined as the dual complement of some set in $\mathbb{CP}^n$. It is defined directly as a subset of $\mathbb{CP}^{n*}$. However, in the one variable case $\sigma[b]^*_{adm}$ corresponds to the point spectrum in the following sense. If $[\lambda] \in \sigma[b]^* \subseteq \mathbb{CP}^1$ and P_λ a projection from the hyperplane (point) $[\lambda]$ onto $\mathbb{C}$ then

$$\sigma[b]^*_{adm} = \big(P_\lambda^{-1}\sigma_p(P_\lambda([b]))\big)^*.$$

Proposition 4.9. *Let $[b]$ be a projective operator and let $[\lambda] \in \sigma[b]^*$ and $[\alpha] \in \sigma[b]^*_{adm}$. Then $\langle b, \alpha\rangle^{-1}\langle b, \lambda\rangle$ is a closed operator which does not depend on the particular representative $b \in [b]$. Moreover $\langle b, \alpha\rangle^{-1}\langle b, \lambda\rangle = \langle b, \lambda\rangle\langle b, \alpha\rangle^{-1}$ and we denote this operator $\langle b, \lambda\rangle/\langle b, \alpha\rangle$. Its domain of definition $\mathscr{D}_\alpha := \mathscr{D}(\langle b, \lambda\rangle/\langle b, \alpha\rangle)$ does not depend on the choice of $[\lambda] \in \sigma[b]^*$. Finally if $[\beta_1], \dots, [\beta_n]$ are any points such that $[\alpha], [\beta_1], \dots, [\beta_n]$ are in general position then*

$$\mathscr{D}_\alpha = \bigcap_{j=1}^{n} \mathscr{D}(\langle b, \alpha\rangle^{-1}\langle b, \beta_j\rangle).$$

Proof. It is clear that $\langle b,\alpha\rangle^{-1}\langle b,\lambda\rangle$ is a closed linear operator on X. Since $[\lambda]\in\sigma[b]^*$ we have that $\langle b,\lambda\rangle$ is invertible and so it follows from Lemma 4.2 that $\langle b,\alpha\rangle^{-1}\langle b,\lambda\rangle = \langle b,\lambda\rangle\langle b,\alpha\rangle^{-1}$. From this we immediately obtain that

$$\langle b,\lambda\rangle/\langle b,\alpha\rangle = (\langle b,\alpha\rangle/\langle b,\lambda\rangle)^{-1} \tag{5}$$

in the set theoretical sense and hence $\langle b,\lambda\rangle/\langle b,\alpha\rangle$ does not depend on the representative $b\in[b]$ since the right hand side of (5) does not. Moreover since $\mathscr{D}(\langle b,\lambda\rangle\langle b,\alpha\rangle^{-1}) = \mathscr{D}(\langle b,\tilde{\lambda}\rangle\langle b,\alpha\rangle^{-1})$ for any other $[\tilde{\lambda}]\in\sigma[b]^*$ the domain $\mathscr{D}_\alpha$ can not depend on the choice of $[\lambda]\in\sigma[b]^*$. For the last statement we first assume that $[\alpha]=[1,0,\dots,0]$ and $[\beta_j]=[0,\dots,1,\dots,0]$ where the 1 is in the j:s position. Then what we have to show is $\mathscr{D}(\langle b,\lambda\rangle/b_0) = \bigcap_1^n \mathscr{D}(b_0^{-1}b_j)$. But from Lemma 4.2 we see that $\mathscr{D}(b_0^{-1}b_j)\supseteq\mathscr{D}(b_jb_0^{-1}) = \mathscr{D}(\langle b,\lambda\rangle b_0^{-1})$ and so $\mathscr{D}(\langle b,\lambda\rangle/b_0)\subseteq\bigcap_1^n\mathscr{D}(b_0^{-1}b_j)$. On the other hand $\bigcap_1^n\mathscr{D}(b_0^{-1}b_j)\subseteq\mathscr{D}(b_0^{-1}\langle b,\lambda\rangle)$ so we are done. We reduce the general case to this one by considering the projective transformation P defined by $[z]\mapsto[\langle z,\alpha\rangle,\langle z,\beta_1\rangle,\dots,\langle z,\beta_n\rangle]$. Then $P^{*-1}[\alpha]=[1,0,\dots,0]$ and $P^{*-1}[\beta_j] = [0,\dots,1,\dots,0]$. We want to show the equality

$$\mathscr{D}(\langle b,\lambda\rangle/\langle b,\alpha\rangle) = \bigcap_{j=1}^n \mathscr{D}(\langle b,\alpha\rangle^{-1}\langle b,\beta_j\rangle)$$

but this is equivalent to

$$\mathscr{D}(\langle Pb,P^{*-1}\lambda\rangle/\langle Pb,P^{*-1}\alpha\rangle) = \bigcap_{j=1}^n \mathscr{D}(\langle Pb,P^{*-1}\alpha\rangle^{-1}\langle Pb,P^{*-1}\beta_j\rangle).$$

Hence the proposition follows from the special case above. □

Remark 4.10. We saw in the proof that there was no loss of generality in assuming that the hyperplanes were of a special kind because we could reduce to this case by a projective transformation of $\mathbb{CP}^n$. In order to simplify calculations in the proofs below we will often make such assumptions and it is supposed to be understood that there is no loss of generality in doing it.

Let us fix an $[\alpha]\in\sigma[b]^*_{adm}$ and $[\beta_1],\dots,[\beta_n]\in\mathbb{CP}^{n*}$ such that $[\alpha],[\beta_1],\dots,[\beta_n]$ are in general position. We denote the closed operator $\langle b,\alpha\rangle^{-1}\langle b,\beta_j\rangle$ by a_j.

Proposition 4.11. *With the hypothesis of the preceding proposition, if* $x\in\mathscr{D}(a_j)\cap\mathscr{D}(a_k)$ *then the following conditions are equivalent*

$$\begin{aligned} a_jx &\in \mathscr{D}(a_k), \\ a_kx &\in \mathscr{D}(a_j). \end{aligned}$$

If any of these conditions are satisfied then also $a_k a_j x = a_j a_k x$.

Proof. We may assume $[\alpha] = [1, 0, \dots, 0]$ and $[\beta_j] = [0, \dots, 1, \dots, 0]$ and hence $a_j = b_0^{-1} b_j$. Suppose $x \in \mathscr{D}(b_0^{-1} b_j) \cap \mathscr{D}(b_0^{-1} b_k)$. Then from Lemma 4.2 we get $b_k b_0^{-1} b_j x = b_j b_0^{-1} b_k x$. Hence $b_0^{-1} b_j x \in \mathscr{D}(b_0^{-1} b_k)$ precisely when $b_0^{-1} b_k x \in \mathscr{D}(b_0^{-1} b_j)$ and $b_0^{-1} b_k b_0^{-1} b_j x = b_0^{-1} b_j b_0^{-1} b_k x$. □

Definition 4.12. A tuple $(a_1, \dots, a_n)$ of closed operators on X is called an *affine operator* if

(i) it exists a $[\lambda] \in \mathbb{CP}^n$ such that the operator
$$a_0 := \lambda_0 + \sum_1^n \lambda_j a_j$$
with domain $\mathscr{D}(a_0) = \bigcap_1^n \mathscr{D}(a_j)$ is closed, injective and surjective,

(ii) the operators $a_0, a_1, \dots, a_n$ satisfy the following commutation conditions; if $x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_j a_k)$ then $x \in \mathscr{D}(a_k a_j)$ and $a_j a_k x = a_k a_j x$ for $j, k = 0, 1, \dots, n$.

Remark 4.13. In the one variable case Definition 4.12 just means that $\sigma(a)$ is not all of $\mathbb{C}$. In fact, if $\lambda_1 \neq 0$ then $\lambda_0 + \lambda_1 a$ is injective and surjective if and only if $-\lambda_0/\lambda_1 \notin \sigma(a)$ and if $\lambda_1 = 0$ then $\mathscr{D}(a) = X$, i.e. a is bounded and therefore $\sigma(a) \neq \mathbb{C}$. The commutation conditions are clearly satisfied in the one variable case and so from Section 2 we see that a closed operator is affine if and only if it can be Cayley transformed to a bounded operator.

Remark 4.14. Morally what condition (i) should mean is that no matter how we may define the spectrum of a, the hyperplane $[\lambda]$ should avoid its closure in $\mathbb{CP}^n$. For instance if $[\lambda] = [1, 0, \dots, 0]$, that is the spectrum of a does not intersect the hyperplane at infinity, then one should expect that all the a_j are bounded. In fact if $[1, 0, \dots, 0]$ works as $[\lambda]$ in Definition 4.12 then condition (i) says that the domain of the identity is $\bigcap_1^n \mathscr{D}(a_j)$, that is $\mathscr{D}(a_j) = X$ for all j and so all the a_j are bounded by the Closed Graph Theorem.

Remark 4.15. We do not demand that each a_j has a non-empty resolvent set. We will see in Example 4.18 that there are affine operators such that some of the components have all of $\mathbb{CP}^1$ as spectrum.

Remark 4.16. Condition (ii) of Definition 4.12 implies that affine operators are pemutable multioperators in the sense of [5]. It also implies that the operators $a_1, \dots, a_n$ commute with the bounded operator a_0^{-1} in the sense that $a_0^{-1} a_j \subseteq a_j a_0^{-1}$. In fact, let $x \in \mathscr{D}(a_j)$. Then clearly $a_0^{-1} x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_j a_0)$ and so condition (ii) implies that $a_0^{-1} x \in \mathscr{D}(a_0 a_j)$ and

$a_0a_ja_0^{-1}x = a_ja_0a_0^{-1}x = a_jx$. Hence $a_ja_0^{-1}x = a_0^{-1}a_jx$ for all $x \in \mathscr{D}(a_j)$. It will follow that if all a_j have resolvents then these commute, see Corollary 4.19.

The operators we get when we project a projective operator from an admissible hyperplane are affine, and these are the only affine operators as we now show.

Theorem 4.17. *A tuple $a = (a_1, \dots, a_n)$ of closed operators on X is affine if and only if there is a projective operator $[b]$ with $\sigma[b]^*$ nonempty, an $[\alpha] \in \sigma[b]^*_{adm}$ and $[\beta_1], \dots, [\beta_n] \in \mathbb{CP}^{n*}$ in general position together with $[\alpha]$, such that*

$$a_j = \langle b, \alpha\rangle^{-1}\langle b, \beta_j\rangle, \quad j = 1, \dots, n.$$

Proof. We may assume that $\alpha = [1, 0, \dots, 0]$, $\beta_j = [0, \dots, 0, 1, 0, \dots 0]$ where 1 is in the j:s place. First assume that $a_j = \langle b, \alpha\rangle^{-1}\langle b, \beta_j\rangle$, $j = 1, \dots, n$ for some projective operator $[b]$, that is $a_j = b_0^{-1}b_j$. Let $[\lambda] \in \sigma[b]^*$ so that $B = \langle b, \lambda\rangle$ is invertible. From Proposition 4.2 we get that $b_0^{-1}B = Bb_0^{-1}$ and so we see that

$$a_0 := b_0^{-1}B = b_0^{-1}\sum_0^n \lambda_j b_j = \sum_0^n \lambda_j b_0^{-1}b_j = \lambda_0 + \sum_1^n \lambda_j a_j$$

has domain $\mathscr{D}(a_0) = \mathscr{D}(b_0^{-1}) = \bigcap_1^n \mathscr{D}(a_j)$ by Proposition 4.9, is closed, injective and surjective. Hence a satisfies condition (i) in Definition 4.12. Moreover Proposition 4.11 implies that if $x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_ja_k)$ then $x \in \mathscr{D}(a_ka_j)$ and $a_ja_kx = a_ka_jx$ for $j, k = 1, \dots, n$. To see that this is also satisfied for $j = 0$ and $k = 0$ respectively we first assume that $x \in \mathscr{D}(a_0) \cap \mathscr{D}(a_0a_k)$. Then since $x \in \mathscr{D}(a_0a_k)$ we have that $b_0^{-1}b_kx \in \mathscr{D}(b_0^{-1})$ and since also $x \in \mathscr{D}(b_0^{-1})$ Lemma 4.2 implies that $b_0^{-1}b_kx = b_kb_0^{-1}x$. Hence $b_kb_0^{-1}x \in \mathscr{D}(b_0^{-1})$, that is $x \in \mathscr{D}(a_ka_0)$, and $b_0^{-1}b_0^{-1}b_kx = b_0^{-1}b_kb_0^{-1}x$ that is $a_0a_kx = a_ka_0x$. Now assume that $x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_ja_0)$ which just means that $x \in \mathscr{D}(b_0^{-1})$ and $b_jb_0^{-1}x \in \mathscr{D}(b_0^{-1})$. From Lemma 4.2 we see that $b_jb_0^{-1}x = b_0^{-1}b_jx$ so $b_0^{-1}b_jx \in \mathscr{D}(b_0^{-1})$ and $b_0^{-1}b_0^{-1}b_jx = b_0^{-1}b_jb_0^{-1}x$. Hence $x \in \mathscr{D}(a_0a_j)$ and $a_0a_jx = a_ja_0x$ so a also satisfies condition (ii) and thus a is affine.

Conversely assume that a is affine and take $[\lambda] \in \mathbb{CP}^n$ such that the operator $a_0 = \lambda_0 + \sum_1^n \lambda_j a_j$ satisfies the requirements of condition (i) in Definition 4.12. Then

$$b_0 := (\lambda_0 + \sum_1^n \lambda_j a_j)^{-1}, \;\; b_j := a_j(\lambda_0 + \sum_1^n \lambda_j a_j)^{-1} \;\; j = 1, \dots, n$$

are bounded operators by the Closed Graph Theorem. We claim that $b = (b_0, \dots, b_n)$ is commutative, that $\langle b, \lambda\rangle$ is invertible and that $a_j = b_0^{-1}b_j$.

We start by showing commutativity. In Remark 4.16 we saw that it followed from condition (ii) that $a_0^{-1}a_j \subseteq a_j a_0^{-1}$, that is $b_0 a_j \subseteq a_j b_0$ for $j = 1, \dots, n$. Hence for any $x \in X$ we have $a_k b_0^2 x = b_0 a_k b_0 x \in \bigcap_1^n \mathscr{D}(a_j)$. So we see from condition (ii) that for any $x \in X$ we have $a_l b_0 a_k b_0 x = a_l a_k b_0^2 x = a_k a_l b_0^2 x = a_k b_0 a_l b_0 x$. Thus b is commutative. To see that $a_k = b_0^{-1} b_k$ we assume $x \in \mathscr{D}(a_k)$, then condition (ii), via Remark 4.16, implies that $a_k b_0 x = b_0 a_k x \in \bigcap_1^n \mathscr{D}(a_j)$. Hence $a_l a_k b_0 x = a_k a_l b_0 x$ for all l by condition (ii), and we obtain $b_0^{-1} a_k b_0 x = a_k x$. Thus $a_k \subseteq b_0^{-1} b_k$. To show equality it suffices to show $\mathscr{D}(b_0^{-1} b_k) \subseteq \mathscr{D}(a_k)$. Therefore assume $x \in \mathscr{D}(b_0^{-1} b_k)$, that is $a_k b_0 x \in \mathscr{D}(b_0^{-1})$ and so, again by condition (ii), we have $a_l a_k b_0 x = a_k a_l b_0 x$. Hence $a_l b_0 x \in \mathscr{D}(a_k)$ for all l and this gives us $x = b_0^{-1} b_0 x \in \mathscr{D}(a_k)$. Finally we observe that

$$\langle b, \lambda \rangle = \sum_0^n \lambda_j b_j = \lambda_0 b_0 + \sum_1^n \lambda_j a_j b_0 = (\lambda_0 + \sum_1^n \lambda_j a_j) b_0 = e.$$

Hence $\sigma(b)$ avoids the hyperplane λ through the origin in $\mathbb{C}^{n+1}$ and hence $[b]$ is a projective operator with $\sigma[b]^*$ nonempty. □

Example 4.18. Let K be the compact subset of $\mathbb{C}^3$ defined by

$$K = \{(1, z_1, 0); |z_1| \leq 1\} \cup \{(1/z_1, 1, 1/z_1); |z_1| \geq 1\} \cup \{(0,1,0)\}.$$

Let $X = C(K)$ be the Banach space of continuous functions on K and let b_j denote the operator on X of multiplication with the coordinate function z_j, $j = 0, 1, 2$. Then $b = (b_0, b_1, b_2)$ defines a projective operator $[b]$ and $\sigma[b] = \pi(K)$, the projection of K on $\mathbb{CP}^2$. Moreover, one checks that the hyperplane $[2, 1, -3/2]$ avoids $\sigma[b]$. Clearly b_0 is injective and so the hyperplane $[1, 0, 0]$ is admissible. We get the affine operator $(a_1, a_2) = (b_0^{-1} b_1, b_0^{-1} b_2)$. We claim that $\sigma(a_1) = \mathbb{C}$. Let $w \in \mathbb{C}$ be arbitrary and take a point $(z_0, z_1, z_2) \in K$ such that $z_1/z_0 = w$. If $f \in C(K)$ is such that $f(z_0, z_1, z_2) \neq 0$ then f is not in the range of $w - a_1$ and therefore $w \in \sigma(a_1)$.

Corollary 4.19. *If $(a_1, \dots, a_n)$ is affine and each a_j has resolvents then these commute.*

Proof. Let $[b]$ be a projective operator such that $a_j = b_0^{-1} b_j$. We consider the case when each a_j has a bounded inverse. The general case is completely analogous. We first check that $a_j^{-1} = b_j^{-1} b_0$. Actually, b_j has to be injective since otherwise $b_j x = 0$ for some $x \neq 0$, but then $x \in \mathscr{D}(a_j)$ and $a_j x = 0$ which is impossible. Also, b_j has to be surjective onto $\mathscr{D}(b_0^{-1})$ and hence $\mathscr{R}(b_0) \subseteq \mathscr{D}(b_j^{-1})$. The closed operator $b_j^{-1} b_0$ therefore has to be bounded. It follows that $b_0^{-1} b_j b_j^{-1} b_0$ is the identity on X and that $b_j^{-1} b_0 b_0^{-1} b_j$ is the identity on $\mathscr{D}(a_j)$ and so $a_j^{-1} = b_j^{-1}b_0$. Now we use Lemma 4.2 to see that a_j^{-1} and a_k^{-1} commute. Let $y = a_j^{-1}a_k^{-1}x = b_j^{-1}b_0b_k^{-1}b_0x = b_j^{-1}b_k^{-1}b_0^2x$. Then $b_0^2x = b_kb_jy = b_jb_ky$ and hence $y = b_k^{-1}b_0b_j^{-1}b_0x = a_k^{-1}a_j^{-1}x$ since $\mathscr{R}(b_0) \subseteq \mathscr{D}(b_j^{-1})$. □

Corollary 4.20. *If $(a_1, \dots, a_n)$ is affine then affine combinations of the a_j are closable.*

Proof. To any affine map of $\mathbb{C}^n$ it corresponds a projective transformation of $\mathbb{CP}^n$. Substituting a projective operator, representing $(a_1, \dots, a_n)$, into this map and projecting the result back to $\mathbb{C}^n$ we obtain a closed extension of the affine combination. □

The correspondence between affine- and projective operators is one-to-one in the following sense.

Theorem 4.21. *Fix $[\alpha], [\beta_1], \dots, [\beta_n] \in \mathbb{CP}^n$ in general position. Then to any affine operator $a = (a_1, \dots, a_n)$ it corresponds a unique projective operator $[b]$ with nonempty $\sigma[b]^*$ and with $[\alpha] \in \sigma[b]^*_{adm}$ such that $a_j = \langle b, \alpha\rangle^{-1}\langle b, \beta_j\rangle$ for $j = 1, \dots, n$.*

Proof. The existence of a projective operator $[b]$ and $[\tilde{\alpha}]$ and $[\tilde{\beta}_j]$, $j = 1, \dots, n$, in general position such that $a_j = \langle b, \tilde{\alpha}\rangle^{-1}\langle b, \tilde{\beta}_j\rangle^{-1}$ is part of Theorem 4.17. Let L be an invertible projective transformation sending $[\tilde{\alpha}]$ to $[\alpha]$ and $[\tilde{\beta}_j]$ to $[\beta_j]$. Then $L^{*-1}[b]$ is a projective operator with $a_j = \langle L^{*-1}b, \alpha\rangle^{-1}\langle L^{*-1}b, \beta_j\rangle^{-1}$. For uniqueness we assume that $\alpha = [1, 0, \dots, 0]$, $\beta_j = [0, \dots, 0, 1, 0, \dots, 0]$ and that $[b]$ and $[\tilde{b}]$ are two projective operators corresponding to a, i.e. we assume that $b_0^{-1}b_j = a_j = \tilde{b}_0^{-1}b_j$, $j = 1, \dots, n$. We may also assume that b is the representative for $[b]$ such that $e = \langle b, \lambda\rangle$ by Lemma 4.3. We show that $[b] = [\tilde{b}]$. From Proposition 4.9 we get

$$\mathscr{D}(b_0^{-1}) = \cap \mathscr{D}(b_0^{-1}b_j) = \cap \mathscr{D}(\tilde{b}_0^{-1}\tilde{b}_j) = \mathscr{D}(\tilde{b}_0^{-1}).$$

Hence $c := \tilde{b}_0^{-1}b_0$ is an invertible bounded operator. Moreover from Lemma 4.2 and the assumption we see that $\tilde{b}_jc = \tilde{b}_j\tilde{b}_0^{-1}b_0 = \tilde{b}_0^{-1}\tilde{b}_jb_0 = b_0^{-1}b_jb_0 = b_j$ and so $b = \tilde{b}c$. It remains to show that $c \in (\tilde{b})'$. But $e = \langle b, \lambda\rangle = \sum_0^n \lambda_jb_j$ so $c^{-1} = \sum_0^n \lambda_j\tilde{b}_j$ and hence $c \in (\tilde{b})'$. □

Definition 4.22. Let $[\alpha], [\beta_1], \dots, [\beta_n] \in \mathbb{CP}^n$ be fixed in general position. We define $\rho_{\alpha,\beta}$ to be the mapping

$$[z] \mapsto (\langle z, \alpha\rangle^{-1}\langle z, \beta_1\rangle, \dots, \langle z, \alpha\rangle^{-1}\langle z, \beta_n\rangle).$$

The one-to-one correspondence can now be stated by saying that the mapping $\rho_{\alpha,\beta}\colon \{[b]; \sigma[b]^* \neq \emptyset,\ [\alpha] \in \sigma[b]^*_{adm}\} \to \{a; a \text{ is affine}\}$ is one-to-one and onto.

5. Spectra of affine operators

We define the spectrum of an affine operator a, corresponding to a projective operator $[b]$ via $\rho_{\alpha,\beta}([b]) = a$, and show that $\rho_{\alpha,\beta}(\sigma[b]) = \sigma(a)$. Throughout this section we will assume that $\alpha = [1, 0, \dots, 0]$ and $\beta_j = [0, \dots, 1, \dots, 0]$ in the proofs.

Let $a = (a_1, \dots, a_n)$ be an affine operator. For $z \in \mathbb{C}^n$ we let δ_{z-a} denote interior multiplication with $\sum_1^n (z_j - a_j)e_j^*$ and the domain of definition $\mathscr{D}(\delta_{z-a})$ for this operator is all forms with coefficients in $\bigcap_1^n \mathscr{D}(a_j)$. See Section 2 for notation.

Definition 5.1. Let $a = (a_1, \dots, a_n)$ be an affine operator. We define $\sigma(a) \subseteq \mathbb{C}^n$ by specifying its complement: $z \notin \sigma(a)$ if and only if for any k-form $f^k \in \mathscr{N}(\delta_{z-a})$ it exists a $k+1$-form f^{k+1} with coefficients in $\bigcap_{j,k=1}^n \mathscr{D}(a_j a_k)$ such that $f^k = \delta_{z-a} f^{k+1}$.

Remark 5.2. Affine operators are permutable multioperators and as such they also have a joint Ionaşcu-Vasilescu spectrum, [5]. If all components in an affine operator have resolvents we can also consider the joint spectrum associated to an iterated one-dimensional Cayley transform as in [11] and [2]. It is shown in [5] that this spectrum equals the Ionaşcu-Vasilescu spectrum in this case. We will see in Theorem 5.5 that our spectrum is contained in the spectrum obtained by an iterative one-dimensional Cayley transform in case both spectra are defined.

We denote the set of all forms with coefficients in $\bigcap_{j,k=1}^n \mathscr{D}(a_j a_k)$ by $\mathscr{D}^2$.

Lemma 5.3. *Let $[b]$ be a projective operator and assume that $[1, 0, \dots, 0]$ is an admissible hyperplane and that $\sigma[b]^*$ is nonempty. Put $b' = (b_1, \dots, b_n)$ and let $a = (b_0^{-1}b_1, \dots, b_0^{-1}b_n)$. Then $K_\bullet(\delta_{b'}, X)$ is exact if and only if for any $f^k \in \mathscr{N}(\delta_a)$ it exists an f^{k+1} with coefficients in $\mathscr{D}(b_0^{-2}) = \mathscr{R}(b_0^2)$ such that $f^k = \delta_a f^{k+1}$.*

Proof. Note that $\mathscr{D}(b_0^{-1}) = \bigcap_1^n \mathscr{D}(b_0^{-1}b_j)$ by Proposition 4.9. Assume that $K_\bullet(\delta_{b'}, X)$ is exact and let $f^k \in \mathscr{N}(\delta_a)$. Then $\delta_{b'} b_0^{-1} f^k = 0$ and so there is an $\tilde{f}^{k+1}$ such that $b_0^{-1} f^k = \delta_{b'} \tilde{f}^{k+1}$. But then $f^k = \delta_{b'} b_0 \tilde{f}^{k+1} = \delta_a b_0^2 \tilde{f}^{k+1}$. Thus $f^{k+1} := b_0^2 \tilde{f}^{k+1}$ has coefficients in $\mathscr{D}(b_0^{-2})$ and $f^k = \delta_a f^{k+1}$.

Now assume that if $f^k \in \mathscr{N}(\delta_a)$ it exists an f^{k+1} with coefficients in $\mathscr{D}(b_0^{-2})$ such that $f^k = \delta_a f^{k+1}$. If $\delta_{b'} \tilde{f}^k = 0$ then clearly $b_0 \tilde{f}^k \in \mathscr{N}(\delta_a)$ and so there is an $\tilde{f}^{k+1}$ with coefficients in $\mathscr{D}(b_0^{-2})$ such that $b_0 \tilde{f}^k = \delta_a \tilde{f}^{k+1} = \delta_{b'} b_0^{-1} \tilde{f}^{k+1}$. Hence $\tilde{f}^k = \delta_{b'} b_0^{-2} \tilde{f}^{k+1}$ and so $K_\bullet(\delta_{b'}, X)$ is exact. □

Theorem 5.4. *Let a be an affine operator and let $[b]$ be a projective operator with nonempty $\sigma[b]^*$. If $[\alpha] \in \sigma[b]^*_{adm}$ has the property that $a = \rho_{\alpha,\beta}([b])$ then $\sigma(a) = \rho_{\alpha,\beta}(\sigma[b])$.*

Proof. Under our assumptions on $[\alpha]$ and $[\beta]$ we have that $\rho_{\alpha,\beta}$ is the mapping $[z] \mapsto (z_1/z_0, \dots, z_n/z_0)$. We will show that $[1, 0, \dots, 0] \notin \sigma[b]$ if and only if $0 \notin \sigma(a)$. By the Spectral mapping theorem we get that the line through the origin and $(1, 0, \dots, 0)$ in $\mathbb{C}^{n+1}$ does not intersect $\sigma(b)$ if and only if $0 \notin \sigma(b_1, \dots, b_n)$. Thus what we have to show is that $0 \notin \sigma(b_1, \dots, b_n)$ if and only if $0 \notin \sigma(a)$. But this is exactly the statement in Lemma 5.3 and so the only thing left in order to prove the theorem is to check that $\mathscr{D}(b_0^{-2}) = \bigcap_{j,k=1}^n \mathscr{D}(a_j a_k)$. Since $a_j a_k = b_0^{-1} b_j b_0^{-1} b_k \supseteq b_j b_k b_0^{-2}$ the inclusion $\subseteq$ is clear. Conversely assume $x \in \bigcap_{j,k=1}^n \mathscr{D}(a_j a_k)$. Then, at least $x \in \bigcap_j^n \mathscr{D}(a_j) = \mathscr{D}(b_0^{-1})$ by Proposition 4.9. Thus $x = b_0 y$ for some y. The assumption on x now implies that $b_k y = b_0^{-1} b_k x \in \bigcap_j^n \mathscr{D}(a_j) = \mathscr{D}(b_0^{-1})$ for $k = 1, \dots, n$. Since we may assume that $e = \sum_0^n \lambda_k b_k$ we get $y = \sum_0^n \lambda_k b_k y \in \mathscr{D}(b_0^{-1})$. Thus $x = b_0 y \in \mathscr{D}(b_0^{-2})$ and we are done. □

Theorem 4.21 implies that to an affine operator a we have a unique projective operator $[b]$ such that $a = \rho_{\alpha,\beta}([b])$ for some fixed choice of $[\alpha], [\beta_1], \dots, [\beta_n]$ in general position. So applying Theorem 5.4 we see that $\sigma(a)$ has a well defined, invariant and closed extension $\hat{\sigma}(a) \subseteq \mathbb{CP}^n$ defined by

$$\hat{\sigma}(a) = \sigma[b].$$

Now suppose that $a = (a_1, \dots, a_2)$ is affine and assume in addition that each a_j has a resolvent. As we have seen, Example 4.18, affine operators need not have this property but may off course have it, see Example 5.7 below. After an affine transformation we may assume that each a_j has a bounded inverse $a_j^{-1} \in L(X)$. Then as in e.g. [11] and [2] we can define the spectrum for a as the inverse image of the spectrum of $(a_1^{-1}, \dots, a_n^{-1})$ under the mapping $(z_1, \dots, z_n) \mapsto (1/z_, \dots, 1/z_n)$. We will denote this spectrum by $\tilde{\sigma}(a)$.

Theorem 5.5. *Let $a = (a_1, \dots, a_n)$ be an affine operator and assume that each a_j has a resolvent. Then $\sigma(a) \subseteq \tilde{\sigma}(a)$ and in the case $n = 2$ we have equality.*

Since $(a_1, \dots, a_n)$ is affine there is a unique projective operator $[b] = [b_0, \dots, b_n]$ such that $a_j = b_0^{-1} b_j$ and $j = 1, \dots, n$. Before we prove Theorem 5.5 we prove a lemma.

Lemma 5.6. *A point $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is outside $\sigma(a)$ if and only if $0 \in \mathbb{C}^n$ is outside $\sigma(b_1 - \lambda_1 b_0, \dots, b_n - \lambda_n b_0)$.*

Proof. First note that from Theorem 5.4 $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is outside $\sigma(a)$ if and only if $[1, \lambda_1, \dots, \lambda_n] \in \mathbb{CP}^n$ is outside $\sigma[b]$ which in turn precisely means that the line through the origin and $(1, \lambda_1, \dots, \lambda_n)$ in $\mathbb{C}^{n+1}$ does not intersect $\sigma(b_0, \dots, b_n)$. The lemma now follows by applying the Spectral mapping theorem to the mapping $(z_0, \dots, z_n) \mapsto (z_1 - \lambda_1 z_0, \dots, z_n - \lambda_n z_0)$. □

Proof of Theorem 5.5. We may assume without loss of generality that $0 \notin \sigma(a_j)$ for $j = 1, \dots, n$, i.e. that each a_j^{-1} is bounded. It follows from the proof of Corollary 4.19 that $a_j^{-1} = b_j^{-1} b_0$ and thus in particular that $\mathscr{D}(b_0^{-1}) \subseteq \mathscr{D}(b_j^{-1})$ for all j. Let $(\lambda_1, \dots, \lambda_n)$ be any point in $\mathbb{C}^n$. We define two boundary operators δ and $\tilde{\delta}$ on $\Lambda X = \bigoplus_1^n \Lambda^j X$ by letting δ and $\tilde{\delta}$ be interior multiplication with $\sum_1^n (b_j - \lambda_j b_0) e_j^*$ and $\sum_{j \in I_0} e_j^* + \sum_{j \in I_1} (1/\lambda_j - b_j^{-1} b_0) e_j^*$ respectively, where I_0 and I_1 is the set of indices with $\lambda_j = 0$ and $\lambda_j \neq 0$ respectively. By the previous lemma $(\lambda_1, \dots, \lambda_n)$ is outside $\sigma(a_1, \dots, a_n)$ if and only if the complex $K_\bullet(\delta, \Lambda^\bullet X)$ is exact. If some $\lambda_j = 0$ then we are automatically outside $\tilde{\sigma}(a_1, \dots, a_n)$ and if all λ_j are non zero we are outside this spectrum if and only if the complex $K_\bullet(\tilde{\delta}, \Lambda^\bullet X)$ is exact. We now define morphisms of complexes $\Psi\colon K_\bullet(\delta, \Lambda^\bullet X) \to K_\bullet(\tilde{\delta}, \Lambda^\bullet X)$ and $\Phi\colon K_\bullet(\tilde{\delta}, \Lambda^\bullet X) \to K_\bullet(\delta, \Lambda^\bullet X)$. Let $\mathscr{A}$ be a commutative unital subalgebra of $L(X)$ containing $b_0, b_j, b_j^{-1} b_0$, $j = 1, \dots, n$. Note that δ and $\tilde{\delta}$ are natural mappings $E \to \mathscr{A}$ extended as anti-derivations to $\mathscr{A} \otimes \Lambda E \to \mathscr{A} \otimes \Lambda E$. We define the injective mapping $\Psi\colon \mathscr{A} \otimes \Lambda E \to \mathscr{A} \otimes \Lambda E$ inductively by setting $\Psi(e_j) = \lambda_j b_j e_j$ and $\Psi(f \wedge g) = \Psi(f) \wedge \Psi(g)$. The operator $b_1 \cdots b_n$ is naturally a mapping $\mathscr{A} \otimes \Lambda E \to \mathscr{A} \otimes \Lambda E$ and if we are in the image of this mapping we are in the image of Ψ. We may thus define $\Phi := \Psi^{-1} \lambda_1 \cdots \lambda_n b_1 \cdots b_n$ and get a mapping $\Phi\colon \mathscr{A} \otimes \Lambda E \to \mathscr{A} \otimes \Lambda E$. It is straight forward to check that $\tilde{\delta}\Psi = \Psi\delta$ and $\delta\Phi = \Phi\tilde{\delta}$ as mappings $\mathscr{A} \otimes \Lambda E \to \mathscr{A} \otimes \Lambda E$ and so we obtain our morphisms of complexes $\Psi\colon K_\bullet(\delta, \Lambda^\bullet X) \to K_\bullet(\tilde{\delta}, \Lambda^\bullet X)$ and $\Phi\colon K_\bullet(\tilde{\delta}, \Lambda^\bullet X) \to K_\bullet(\delta, \Lambda^\bullet X)$. We now claim that the quotient complex

$$(6) \qquad 0 \longleftarrow \Lambda^0 X / \Lambda^0 \mathscr{R}(b_0) \longleftarrow \cdots \longleftarrow \Lambda^n X / \Lambda^n \mathscr{R}(b_0) \longleftarrow 0$$

with boundary operator δ is exact. Indeed, since a is affine there is some point $\mu = (\mu_1, \dots, \mu_n)$ outside $\sigma(a)$. Let δ_μ be interior multiplication with $\sum_1^n (b_j - \mu_j b_0) e_j^*$. From the previous lemma we know that $K_\bullet(\delta_\mu, \Lambda^\bullet X)$ is exact. Since δ and δ_μ are equal modulo elements with coefficients in $\mathscr{R}(b_0)$ it suffices to see that (6) is exact with δ_μ as boundary operator. Assume that $\delta_\mu(f^k) \in \Lambda^{k-1}\mathscr{R}(b_0)$, i.e. $\delta_\mu(f^k) = b_0 f^{k-1}$. Then $b_0 \delta_\mu(f^{k-1}) = \delta_\mu(b_0 f^{k-1}) = \delta_\mu^2(f^k) = 0$ and since b_0 is injective $\delta_\mu(f^{k-1}) = 0$. Hence $f^{k-1} = \delta_\mu(g^k)$ and so $\delta_\mu(f^k - b_0 g^k) = 0$. Thus $f^k - b_0 g^k = \delta_\mu(f^{k+1})$ which precisely means that the equivalence class containing f^k is in the image of δ_μ. Now suppose that the complex $K_\bullet(\tilde{\delta}, \Lambda^\bullet X)$ is exact. Assume that

$\delta(f^k) = 0$. Since the quotient complex (6) with boundary operator δ is exact there are g_0^k and f_0^{k+1} such that $f^k = b_0 g_0^k + \delta(f_0^{k+1})$. Since b_0 is injective it follows that $\delta(g_0^k) = 0$ and so, again, there are g_1^k and f_1^{k+1} such that $g_0^k = b_0 g_1^k + \delta(f_1^{k+1})$ and $\delta(g_1^k) = 0$. Repeating this process we see that we can write

$$f^k = b_0^{k+1} g_k^k + \delta(\sum_{j=0}^{k} b_0^j f_j^{k+1}) \tag{7}$$

and $\delta(g_k^k) = 0$. Now, $0 = \Psi\delta(g_k^k) = \tilde{\delta}\Psi(g_k^k)$ and so by assumption, $\Psi(g_k^k) = \tilde{\delta}(h^{k+1})$ and thus $\Psi(b_0^{k+1} g_k^k) = \tilde{\delta}(b_0^{k+1} h^{k+1})$. But the range of b_0 is contained in the domain of every b_j^{-1} and so $b_0^{k+1} h^{k+1} \in \mathscr{R}(\Psi)$. Hence

$$\Psi(b_0^{k+1} g_k^k) = \tilde{\delta}(b_0^{k+1} h^{k+1}) = \tilde{\delta}\Psi\Psi^{-1}(b_0^{k+1} h^{k+1}) = \Psi\delta\Psi^{-1}(b_0^{k+1} h^{k+1})$$

and since Ψ is injective we get $b_0^{k+1} g_k^k = \delta\Psi^{-1}(b_0^{k+1} h^{k+1})$. According to (7) we obtain

$$f^k = \delta(\Psi^{-1}(b_0^{k+1} h^{k+1}) + \sum_{j=0}^{k} b_0^j f_j^{k+1})$$

and $K_\bullet(\delta, \Lambda^\bullet X)$ is exact. Note that in case some $\lambda_j = 0$ then automatically $K_\bullet(\tilde{\delta}, \Lambda^\bullet X)$ is exact. Thus $\sigma(a) \subseteq \tilde{\sigma}(a)$.

We now show that if $K_\bullet(\delta, \Lambda^\bullet X)$ is exact then $H_k(\tilde{\delta}, \Lambda X) = 0$ for $k = 0$, $k = n$ and for $k = n-1$, thus implying that $\sigma(a) = \tilde{\sigma}(a)$ for $a = (a_1, a_2)$. We may of course assume that $\lambda_j \neq 0$ for all j. If $\tilde{\delta}(f^n) = 0$ then $0 = \Phi\tilde{\delta}(f^n) = \delta\Phi(f^n)$. By assumption then $\Phi(f^n) = 0$ and since Φ is injective (at this level the identity) we have $f^n = 0$. If instead $\tilde{\delta}(f^{n-1}) = 0$ we conclude that $0 = \Phi\tilde{\delta}(f^{n-1}) = \delta\Phi(f^{n-1})$. Since Φ is the identity at the top level we get

$$\Phi(f^{n-1}) = \delta(f^n) = \delta\Phi(f^n) = \Phi\tilde{\delta}(f^n),$$

which implies that $f^{n-1} = \tilde{\delta}(f^n)$. Finally, given any $f^0 \in \Lambda^0 X$ we can write $f^0 = \delta(f^1)$. At the lowest level Ψ is the identity and we see that $f^0 = \Psi(f^0) = \Psi\delta(f^1) = \tilde{\delta}\Psi(f^1)$ finishing the proof. □

Example 5.7. Let $X = L^2(\mathbb{R})$ and let b_0 and b_1 be multiplication with $1/(i+\xi)^2$ and $1/(i+\xi)$ on X and let b_2 be the identity. Then $[b_0, b_1, b_2]$ is a projective operator and $(a_1, a_2) = (b_0^{-1} b_1, b_0^{-1} b_2)$ is affine and has the property that each a_j has a bounded inverse. It is straight forward to check explicitly that $\sigma(a_1, a_2) = \tilde{\sigma}(a_1, a_2) = \{(i+x, (i+x)^2) \in \mathbb{C}^2; x \in \mathbb{R}\}$, i.e. the (essential) range of the multiplication operator (a_1, a_2).

6. Cayley transforms

We summarize our results to see that the affine operators are precisely those operators which are Cayley transforms of bounded ones and that the Spectral mapping theorem holds.

Let $a = (a_1, \dots, a_n)$ be affine and let $[\lambda] \in \mathbb{CP}^n$ be such that condition (i) in Definition 4.12 is fulfilled. Then if $a_0 = \lambda_0 + \sum_1^n \lambda_j a_j$, the projective operator $[b] = [a_0^{-1}, a_1 a_0^{-1}, \dots, a_n a_0^{-1}]$ projects to a and $[\lambda] \in \sigma[b]^*$ by Theorem 4.17 and its proof. Let $[\beta_1], \dots, [\beta_n]$ be points in $\mathbb{CP}^n$ such that $[\lambda], [\beta_1], \dots, [\beta_n]$ are in general position. Applying the projection $\rho_{\lambda,\beta}$ to $[b]$ we get the bounded commuting tuple

$$\rho_{\lambda,\beta}([b]) = ((\beta_{1,0} + \sum_1^n \beta_{1,j} a_j) a_0^{-1}, \dots, (\beta_{n,0} + \sum_1^n \beta_{n,j} a_j) a_0^{-1})$$

and $\sigma(\rho_{\lambda,\beta}([b])) = \rho_{\lambda,\beta}(\sigma[b])$ by Theorem 3.8. Hence if ϕ is the corresponding rational fractional transformation we see that $\phi(a) = \rho_{\lambda,\beta}([b])$ is a bounded commuting tuple and by Theorem 5.4 we have $\sigma(\phi(a)) = \phi(\hat{\sigma}(a))$ naturally interpreted.

Conversely assume that a tuple of closed operators $a = (a_1, \dots, a_n)$ is the Cayley transform of a bounded commuting tuple $(b_1, \dots, b_n)$, that is

$$a_k = (\lambda_{0,0} + \sum_1^n \lambda_{0,j} b_j)^{-1} (\lambda_{k,0} + \sum_1^n \lambda_{k,j} b_j),$$

where $(\lambda_{j,k})$ is an invertible matrix and $\lambda_{0,0} + \sum_1^n \lambda_{0,j} b_j$ is injective, i.e. the affine hyperplane $\{z \in \mathbb{C}^n; \langle z, \lambda_0 \rangle = 0\}$ is admissible. Then clearly $[e, b_1, \dots, b_n]$ is a projective operator and $[1, 0, \dots, 0] \in \sigma[e, b_1, \dots, b_n]^*$. Moreover, the hyperplane $[\lambda_{0,0}, \dots, \lambda_{0,n}]$ has to be admissible and so a is the projection of a projective operator from an admissible hyperplane. Since the spectrum of the projective operator also has a nonempty dual complement it follows from Theorem 4.17 that a is affine.

7. Integral formulas for the analytic functional calculus of projective operators

We provide integral formulas realizing the functional calculus described in Section 3. Analogously to [1] we will construct a $\bar{\partial}$-closed $(n, n-1)$-form, $\omega_b^n x$, with values in $X \otimes L^n$, defined in $U \setminus \sigma[b]$, where L^{-1} is the tautological line bundle and U is $\mathbb{CP}^n$ minus some hyperplane, such that if $f \in \mathscr{O}(\sigma[b])$, then

$$f([b])x = \int_{\partial D} f \frac{\langle b, \lambda \rangle^n}{\langle z, \lambda \rangle^n} \omega_b^n x$$

where $\lambda \in \sigma[b]^*$ and D is a suitable neighborhood of $\sigma[b]$.

We let δ_z denote interior multiplication with the vector field $\sum_0^n z_j \frac{\partial}{\partial z_j}$. Letting f be a k-homogeneous $(p, 0)$-form in some cone in $\mathbb{C}^{n+1}$ then f is the pullback of an L^k-valued $(p, 0)$-form in the projection of the cone in $\mathbb{CP}^n$ if and only if $\delta_z f = 0$. The statement is local and we may verify it when $z_0 \neq 0$. If f is the pullback of an L^k-valued $(p, 0)$-form then f is k-homogeneous and can be written as $f = \sum_I f_I d(z_{I_1}/z_0) \wedge \cdots \wedge d(z_{I_p}/z_0)$. Since $\delta_z d(z_i/z_0) = \delta_z(dz_i/z_0 - z_i/z_0^2 dz_0) = z_i/z_0 - z_0 z_i/z_0^2 = 0$ we have $\delta_z f = 0$. Conversely, a straight-forward calculation shows that if $f = \sum f_I dz_I$ is any k-homogeneous $(p, 0)$-form then

$$f = z_0^p \sum_{0 \notin I} f_I d(z_{I_1}/z_0) \wedge \cdots \wedge d(z_{I_p}/z_0) + \frac{(-1)^{p-1}}{z_0} (\delta_z f) \wedge dz_0.$$

So if $\delta_z f = 0$ then clearly f is the pullback of a $(p, 0)$-form which has to have values in L^k since f is k-homogeneous. In what follows we will identify the space of $X \otimes L^k$ valued $(p, 0)$-forms on some subset of $\mathbb{CP}^n$ with the space of k-homogeneous X-valued δ_z-closed $(p, 0)$-forms on the cone over this subset in $\mathbb{C}^{n+1}$. Also if we are in e.g. $U = \mathbb{CP}^n \setminus \{z_0 = 0\}$ we will identify sections of L^k with functions via the natural trivialization of L^k over U given by putting $z_0 = 1$ in the k-homogeneous polynomials representing L^k.

We let δ_b denote interior multiplication with $\sum_0^n b_j \frac{\partial}{\partial z_j}$. This operator commutes with δ_z so it maps δ_z-closed X-valued forms to δ_z-closed X-valued forms. However, δ_b reduces the homogeneity one step and therefore δ_b maps k-homogeneous k-forms to $k-1$-homogeneous $k-1$-forms. Moreover b is commuting so we have $\delta_b \circ \delta_b = 0$, and we get the complex

$$K_\bullet(\delta_b, X \otimes L^\bullet \otimes \Lambda^{\bullet,0} T^* \mathbb{CP}^n_{[z]}). \tag{8}$$

The operator δ_b depends on the choice of representative for $[b]$ but nevertheless we have the following proposition.

Proposition 7.1. *Let $[b]$ be a projective operator and b any representative. Then $[z] \notin \sigma[b]$ if and only if the complex (8) is exact.*

Proof. We may assume that $[z] = [1, 0, \ldots, 0]$. We first claim that $[1, 0, \ldots, 0] \notin \sigma[b]$ if and only if $0 \notin \sigma(b_1, \ldots, b_n)$. Actually, if $0 \notin \sigma(b_1, \ldots, b_n)$, that is $(b_1, \ldots, b_n)$ is nonsingular, then $(z_0 - b_0, b_1, \ldots, b_n)$ is nonsingular for all $z_0 \in \mathbb{C}$, see [9]. Hence $(z_0, 0, \ldots, 0) \notin \sigma(b_0, \ldots, b_n)$ for all $z_0 \in \mathbb{C}$, which means that $[1, 0, \ldots, 0] \notin \sigma[b]$. On the other hand, if $[1, 0, \ldots, 0] \notin \sigma[b]$ then $(z_0, 0, \ldots, 0) \notin \sigma(b_0, \ldots, b_n)$ for all $z_0 \in \mathbb{C}$. From the projection property for the Taylor spectrum, [9], we conclude that $0 \notin \sigma(b_1, \ldots, b_n)$.

To finish the proof we show that $0 \notin \sigma(b_1, \ldots, b_n)$ if and only if the complex (8) is exact at $[z] = [1, 0, \ldots, 0]$. Note that for any $f \in X \otimes L^k \otimes \Lambda^{k,0} T^* \mathbb{CP}^n_{[1,0,\ldots,0]}$ we have $\delta_{[1,0,\ldots,0]} f = z_0 \frac{\partial}{\partial z_0} f = 0$ so f does not contain any

dz_0. Hence δ_b acts just as interior multiplication with $\sum_1^n b_j \frac{\partial}{\partial z_j}$, which we denote by $\delta_{b'}$, and we can identify the complex (8) with the complex

$$0 \longleftarrow \Lambda^0 X \xleftarrow{\delta_{b'}} \Lambda^1 X \xleftarrow{\delta_{b'}} \cdots \xleftarrow{\delta_{b'}} \Lambda^n X \longleftarrow 0\,.$$

However, by definition, this complex is exact precisely when $0 \notin \sigma(b_1, \ldots, b_n)$, and we are done. □

Assume $[1, 0, \ldots, 0] \in \sigma[b]^*$ and let $(\zeta_1, \ldots, \zeta_n) = (z_1/z_0, \ldots, z_n/z_0)$ be local coordinates around $[1, 0, \ldots, 0]$. In these local coordinates δ_b is interior multiplication with

$$b_0 \sum_1^n (b_0^{-1} b_j - \zeta_j) \frac{\partial}{\partial \zeta_j}$$

if we work in the natural trivialization of L^k around $[1, 0, \ldots, 0]$. We abbreviate this operator $b_0 \delta_{b_0^{-1} b - \zeta}$.

Proposition 7.2. *Let $[b]$ be a projective operator with $\sigma[b]^*$ nonempty and let U be a neighborhood of $\sigma[b]$ which does not intersect a hyperplane. Then for any q the following complex is exact:*

$$K_\bullet(\delta_b, \mathcal{E}_{\bullet,q}(U \setminus \sigma[b], X \otimes L^\bullet)).$$

Proof. We may assume that U does not intersect the hyperplane $[1, 0, \ldots, 0]$. We know that pointwise for $[z] \in U \setminus \sigma[b]$ the complex (8) is exact. In the local coordinates $(\zeta_1, \ldots, \zeta_n) = (z_1/z_0, \ldots, z_n/z_0)$ this means that the complex $K_\bullet(X \otimes \Lambda^{\bullet,0} T^* \mathbb{C}^n, b_0 \delta_{b_0^{-1} b - \zeta})$ is exact for $\zeta \in U \setminus \sigma[b]$. From the theory of parameterized complexes it follows that

$$K_\bullet(\mathcal{E}_{\bullet,0}(U \setminus \sigma[b], X), b_0 \delta_{b_0^{-1} b - \zeta})$$

is exact, see e.g. [11]. But this is the statement in the proposition (in local coordinates) for $q = 0$. Taking exterior products with barred differentials does not affect exactness since δ_b commutes with this operation. Hence the statement is true for any q. □

We now construct the integral representation of the functional calculus. Let $f \in \mathscr{O}(U)$ where U is a neighborhood of $\sigma[b]$ that avoids a hyperplane. Let x be the function which is identically x in $U \setminus \sigma[b]$. From Proposition 7.2 we see that there is a form $\omega_b^1 x \in \mathcal{E}_{1,0}(U \setminus \sigma[b], X \otimes L^1)$ such that $x = \delta_b \omega_b^1 x$. Now δ_b and $\bar\partial$ anti-commute and so $\delta_b \bar\partial \omega_b^1 x = -\bar\partial \delta_b\, \omega_b^1 x = -\bar\partial x = 0$. Hence by Proposition 7.2 there is a form $\omega_b^2 \in \mathcal{E}_{2,1}(U \setminus \sigma[b], X \otimes L^2)$ such that $\bar\partial \omega_b^1 x = \delta_b \omega_b^2 x$. Continuing in this way and successively solving the equations $\bar\partial \omega_b^j x = \delta_b \omega_b^{j+1} x$ we finally arrive at a form $\omega_b^n x \in \mathcal{E}_{n,n-1}(U \setminus \sigma[b], X \otimes L^n)$. This form is $\bar\partial$-closed because, as above $\delta_b \bar\partial \omega_b^n x = 0$ and since δ_b is injective on this level we must have $\bar\partial \omega_b^n x = 0$. If we start with another solution $x = \delta_b \tilde{\omega}_b^1 x$ and solve the equations $\bar{\partial}\tilde{\omega}_b^j x = \delta_b \tilde{\omega}_b^{j+1} x$ then $\omega_b^n x$ and $\tilde{\omega}_b^n x$ define the same $\bar{\partial}$-cohomology class. In fact, since $\delta_b(\omega_b^2 x - \tilde{\omega}_b^2 x) = \bar{\partial}(\omega_b^1 x - \tilde{\omega}_b^1 x)$ and $\delta_b(\omega_b^1 x - \tilde{\omega}_b^1 x) = 0$ we get from Proposition 7.2 that $\delta_b(\omega_b^2 x - \tilde{\omega}_b^2 x) = \bar{\partial}\delta_b w^1 = -\delta_b \bar{\partial}\, w^1$, that is $\delta_b(\omega_b^2 x - \tilde{\omega}_b^2 x + \bar{\partial} w^1) = 0$, for some w^1. Inductively we obtain $\delta_b(\omega_b^n x - \tilde{\omega}_b^n x + \bar{\partial} w^{n-1}) = 0$ and since δ_b is injective on that level we get $\omega_b^n x - \tilde{\omega}_b^n x + \bar{\partial} w^{n-1} = 0$. Hence we get a well defined mapping (depending on the representative b)

$$x \mapsto [\omega_b^n x]_{\bar{\partial}}.$$

From the construction it is clear that this map is linear in x.

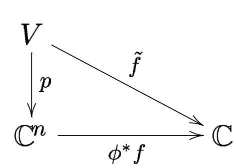
Proposition 7.3. *Let b be a projective operator and assume $[\lambda] \in \sigma[b]^*$. Then the $\bar{\partial}$-cohomology class of $\langle b, \lambda\rangle^n \langle z, \lambda\rangle^{-n} \omega_b^n x$ does not depend on the representative for $[b]$.*

Proof. Clearly $\langle z, \lambda\rangle \langle b, \lambda\rangle^{-1} \delta_b$ does not depend on the representative. Let $\tilde{\omega}^j$, $j = 1, \dots, n$ be solutions to the equations $x = \langle z, \lambda\rangle \langle b, \lambda\rangle^{-1} \delta_b\, \tilde{\omega}^1$, $\bar{\partial}\tilde{\omega}^j = \langle z, \lambda\rangle \langle b, \lambda\rangle^{-1} \delta_b\, \tilde{\omega}^{j+1}$ in $U \setminus \sigma[b]$. Then $\tilde{\omega}^j$ can not depend on the representative. Moreover $\omega^j := \langle z, \lambda\rangle^j \langle b, \lambda\rangle^{-j} \tilde{\omega}^j$, $j = 1, \dots, n$ must satisfy the equations $x = \delta_b \omega^1$, $\bar{\partial}\omega^j = \delta_b \omega^{j+1}$ in $U \setminus \sigma[b]$. Hence we get that $\langle b, \lambda\rangle^n \langle z, \lambda\rangle^{-n}\, \omega_b^n x$ defines the same $\bar{\partial}$-cohomology class as $\tilde{\omega}^n$ and we are done. □

Theorem 7.4. *Let $[b]$ be a projective operator with $[\lambda] \in \sigma[b]^*$. Assume that $f \in \mathscr{O}(\sigma[b])$ and let D be a neighborhood of $\sigma[b]$ such that its closure is contained in an open set, which avoids some hyperplane and in which f is holomorphic. Then*

$$f([b])x = \int_{\partial D} f \frac{\langle b, \lambda\rangle^n}{\langle z, \lambda\rangle^n} \omega_b^n x.$$

Proof. After a projective transformation we can assume that $[\lambda] = [1, 0, \dots, 0]$ and since the $\bar{\partial}$-cohomology class of $\langle b, \lambda\rangle^n \langle z, \lambda\rangle^{-n} \omega_b^n x$ does not depend on the representative we may assume that b is the representative such that $e = \langle b, \lambda\rangle = b_0$ given by Proposition 4.3. We recapitulate the definition of $f([b])$. Let $\tilde{f}$ be the canonical lift of f to $\mathbb{C}^{n+1}$. Then $f([b])x = \tilde{f}(b)x$. Let p denote the mapping $V = \{z \in \mathbb{C}^{n+1}; z_0 \neq 0\} \to \mathbb{C}^n$ given by $(z_0, \dots, z_n) \mapsto (z_1/z_0, \dots, z_n/z_0)$ and let ϕ be the local chart $(\zeta_1, \dots, \zeta_n) \mapsto [1, \zeta_1, \dots, \zeta_n]$. Then



must commute. From the composition rule in Taylor's functional calculus we get that $\tilde{f}(b) = \phi^* f(b_1, \ldots, b_n)$. We will show that

$$\int_{\partial D} f \omega_b^n x = \phi^* f(b_1, \ldots, b_n) x.$$

In the local chart ϕ and in the natural trivialization over it, δ_b is the operator $\delta_{b'-\zeta}$ where $b' = (b_1, \ldots, b_n)$ because of our choice of b. So our solutions $\omega_b^j x$ to the δ_b-equations must satisfy

$$\begin{aligned} x &= \delta_{b'-\zeta} \phi^*(\omega_b^1 x) \\ \bar{\partial} \phi^*(\omega_b^1 x) &= \delta_{b'-\zeta} \phi^*(\omega_b^2 x) \\ &\vdots \\ \bar{\partial} \phi^*(\omega_b^{n-1} x) &= \delta_{b'-\zeta} \phi^*(\omega_b^n x) \end{aligned}$$

in $\phi^{-1}(U \setminus \sigma[b])$. But from the Spectral mapping theorem $\phi^{-1}(U \setminus \sigma[b]) = \phi^{-1}(U) \setminus \sigma(b')$. Hence $[\phi^*(\omega_b^n x)]_{\bar{\partial}}$ must be the same $\bar{\partial}$-cohomology class as the resolvent class Andersson defines in [1] corresponding to b'. Moreover it is shown in [1] that integrating against this resolvent realizes the functional calculus. Thus we obtain

$$\phi^* f(b_1, \ldots, b_n) x = \int \phi^* f \phi^*(\omega_b^n x) = \int \phi^*(f \omega_b^n x) = \int f \omega_b^n x. \qquad \square$$

We have seen that the resolvent, that is the $\bar{\partial}$-cohomology class determined by $\langle b, \lambda \rangle^n \langle z, \lambda \rangle^{-n} \omega_b^n x$, does not depend on the representative for $[b]$ and that the functional calculus is realized by integrating against it. Actually, the resolvent is even independent of the choice of $[\lambda] \in \sigma[b]^*$ in the following sense.

Theorem 7.5. *Let $[b]$ be a projective operator and assume that $[\lambda], [\tilde{\lambda}] \in \sigma[b]^*$. Let U be a pseudoconvex neighborhood of $\sigma[b]$ such that none of the hyperplanes $[\lambda]$ and $[\tilde{\lambda}]$ intersect U. Then $\langle b, \lambda \rangle^n \langle z, \lambda \rangle^{-n} \, \omega_b^n x$ and $\langle b, \tilde{\lambda} \rangle^n \langle z, \tilde{\lambda} \rangle^{-n} \, \omega_b^n x$ are $\bar{\partial}$-cohomologous in $U \setminus \sigma[b]$.*

In order to prove Theorem 7.5 we have to look more closely at the relation between the homological construction of the functional calculus and the integral construction. We recapitulate the homological construction. Let $c = (c_1, \ldots, c_n)$ be a commuting tuple of bounded operators on X. We let $\mathcal{E}_{p,q}(U, X)$ denote the set of smooth X-valued (p,q)-forms in $U \subseteq \mathbb{C}^n$ and we put

$$\mathscr{L}^k(U, X) = \bigoplus_{q-p=k} \mathcal{E}_{p,q}(U, X).$$

The operator $\nabla_{z-c} = \delta_{z-c} - \bar{\partial}$ is an anti-derivative on $\bigoplus_k \mathscr{L}^k(U,X)$ and maps $\mathscr{L}^k(U,X)$ to $\mathscr{L}^{k+1}(U,X)$. Moreover $\nabla_{z-c} \circ \nabla_{z-c} = 0$ and we get the complex $Tot\mathscr{L}(U,X)$:

$$\cdots \xrightarrow{\nabla_{z-c}} \mathscr{L}^{k-1}(U,X) \xrightarrow{\nabla_{z-c}} \mathscr{L}^k(U,X) \xrightarrow{\nabla_{z-c}} \mathscr{L}^{k+1}(U,X) \xrightarrow{\nabla_{z-c}} \cdots.$$

This complex is exact if U is disjoint with $\sigma(c)$ since the Koszul complex is exact outside of $\sigma(c)$. The crucial part of the homological construction of the functional calculus for c is to show that for any neighborhood U of $\sigma(c)$ we have that X and $H^0(Tot\mathscr{L}(U,X))$ are isomorphic as $\mathscr{O}(\mathbb{C}^n)$-modules. Since $H^0(Tot\mathscr{L}(U,X))$ has a natural $\mathscr{O}(U)$-module structure, which extends the $\mathscr{O}(\mathbb{C}^n)$-module structure, the isomorphism yields a $\mathscr{O}(U)$-module structure on X extending the $\mathscr{O}(\mathbb{C}^n)$-module structure. Furthermore one shows that if $U' \subseteq U$ are neighborhoods of $\sigma(c)$ then the $\mathscr{O}(U')$-module structure on X extends the $\mathscr{O}(U)$-module structure. Hence we get a $\mathscr{O}(\sigma(c))$-module structure on X and this is our functional calculus. Given a function $f \in \mathscr{O}(U)$ (U a neighborhood of $\sigma(c)$) the X-valued function $z \mapsto xf(z)$ determines an element in $H^0(Tot\mathscr{L}(U,X))$ and the isomorphism maps this element to $f(c)x$ by definition. This construction is due to Taylor see [9] and [8].

The integral construction of $f(c)x$ is first to solve the equation $\nabla_{z-c}\omega_{z-c}x = x$ in $U\setminus\sigma(c)$, then identifying the component, $\omega^n_{z-c}x$, of $\omega_{z-c}x$ of bidegree $(n, n-1)$, and put

$$f(c)x = \int_{\partial D} f(z)\,\omega^n_{z-c}x.$$

Note that for bidegree reasons, solving $\nabla_{z-c}\omega_{z-c}x = x$ is exactly the same as solving the equations $x = \delta_{z-c}\,\omega^1_{z-c}x$, $\bar{\partial}\,\omega^k_{z-c}x = \delta_{z-c}\,\omega^{k+1}_{z-c}x$, $k = 1,\dots,n-1$. In [1] Andersson shows that the two definitions of $f(c)x$ coincide. The crucial step in proving Theorem 7.5 is the following lemma.

Lemma 7.6. *Let $c = (c_1,\dots,c_n)$ be bounded commuting operators on X and let U be a pseudoconvex neighborhood of $\sigma(c)$. If $f \in \mathscr{O}(U)$ and $f(c) = 0$ then $[f(z)\,\omega^n_{z-c}x]_{\bar{\partial}} = 0$, where $\omega^n_{z-c}x$ is the component of bidegree $(n, n-1)$ of a solution $\omega_{z-c}x$ to $\nabla_{z-c}\,\omega_{z-c}x = x$ in $U \setminus \sigma(c)$.*

Proof. Clearly we have $\nabla_{z-c}\,f(z)\omega_{z-c}x = f(z)x$ in $U \setminus \sigma(c)$. From the homological construction we see that $xf(z)$ must be ∇_{z-c}-exact in U since $f(c)x = 0$. Hence, $xf(z) = \nabla_{z-c}\,u(z)$ for some $u \in \mathscr{L}^{-1}(U,X)$. Thus, $u - f(z)\omega_{z-c}x$ is ∇_{z-c}-closed in $U \setminus \sigma(c)$. Since $Tot\mathscr{L}(U \setminus \sigma(c), X)$ is exact there is a $v \in \mathscr{L}^{-2}(U \setminus \sigma(c), X)$ such that $u(z) - f(z)\omega_{z-c}x = \nabla_{z-c}\,v(z)$ in $U \setminus \sigma(c)$. Identifying terms of bidegree $(n, n-1)$ we see that

$$u_{n,n-1} - f(z)\omega^n_{z-c}x = \bar{\partial}v_{n,n-2} \tag{9}$$

in $U\setminus\sigma(c)$. Moreover, $\nabla_{z-c}u = xf(z)$ so for bidegree reasons $\bar\partial\, u_{n,n-1} = 0$. Since U is pseudoconvex $u_{n,n-1}$ is actually $\bar\partial$-exact and letting $u_{n,n-1} = \bar\partial\,\tilde v_{n,n-2}$ we get from (9) that

$$f(z)\omega^n_{z-c}x = \bar\partial(\tilde v_{n,n-2} - v_{n,n-2})$$

in $U\setminus\sigma(c)$ which is what we wanted to show. □

We proceed and prove Theorem 7.5.

Proof of Theorem 7.5. From Theorem 7.4 we know that both the forms $\langle b,\lambda\rangle^n\langle z,\lambda\rangle^{-n}\omega^n_b x$ and $\langle b,\tilde\lambda\rangle^n\langle z,\tilde\lambda\rangle^{-n}\omega^n_b x$ represent the functional calculus. We have to show that they are $\bar\partial$-cohomologous in $U\setminus\sigma[b]$. We let ρ be a projection from $[\lambda]$. From the proof of Theorem 7.4 we see that $\rho_*(\langle b,\lambda\rangle^n\langle z,\lambda\rangle^{-n}\omega^n_b x)$ defines the resolvent class $\omega_{\zeta-\rho([b])}$ corresponding to $\rho([b])$ if we choose $b\in[b]$ such that $\langle b,\lambda\rangle = e$. Hence in the local coordinates $\zeta=\rho([z])$ the difference between the two forms has to be on the form

$$(1-f(\zeta))\omega_{\zeta-\rho([b])}$$

where f is holomorphic in $\rho(U)$. Now since both of the forms realize the functional calculus we must have $1(\rho([b])) - f(\rho([b])) = 0$. Hence from Lemma 7.6 we see that in the local coordinates, the two forms has to be $\bar\partial$-cohomologuos in $\rho(U)\setminus\sigma(\rho([b]))$. □

The function $f(\zeta)$ is the function $\langle b,\tilde\lambda\rangle^n\langle z,\tilde\lambda\rangle^{-n}$ in the local coordinates ζ. Hence we see that making a change of variables by a rational fractional transform of $\mathbb{C}^n$, computing the resolvent in the new coordinates and pulling it back, we get $\langle b,\tilde\lambda\rangle^n\langle z(\zeta),\tilde\lambda\rangle^{-n}$ times the resolvent we get if we compute it directly. Theorem 7.5 implies that the two forms are $\bar\partial$-cohomologuos in suitable domains.

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Paper IV

OPERATORS WITH SMOOTH FUNCTIONAL CALCULI

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ABSTRACT. We introduce a class of (tuples of commuting) unbounded operators on a Banach space, admitting smooth functional calculi, that contains all operators of Helffer-Sjöstrand type and is closed under the action of smooth proper mappings. Moreover, the class is closed under tensor product of commuting operators. In general an operator in this class has no resolvent in the usual sense so the spectrum must be defined in terms of the functional calculus. We also consider invariant subspaces and spectral decompositions.

1. INTRODUCTION

In this paper we study unbounded operators on a Banach space X that admit smooth functional calculi, although they do not necessarily have resolvents. Throughout this paper X is a complex Banach space, $\mathcal{L}(X)$ is the space of bounded linear operators on X, and e_X denotes the identity operator.

Let a be a closed (densely defined) operator with real spectrum and with the property that for each compact set $K \subset\subset \mathbb{C}$ there are N_K and C_K such that

$$\|\omega_{z-a}\| \leq C_K |\mathrm{Im}\, z|^{-N_K}, \quad z \in K \setminus \mathbb{R}, \tag{1.1}$$

where ω_{z-a} is the resolvent form $\omega_{z-a} = (z-a)^{-1} dz/2\pi i$. Then there is a continuous multiplicative mapping $[a]\colon \mathcal{D}(\mathbb{R}) \to \mathcal{L}(X)$, defined by

$$[a](\phi) = \int \omega_{z-a} \wedge \bar{\partial}\tilde{\phi}, \tag{1.2}$$

where $\tilde{\phi}$ is an almost holomorphic extension to $\mathbb{C}$ of ϕ with compact support. This was done by Dynkin, [7], for bounded operators a and for unbounded operators by Helffer and Sjöstrand, [10]. If a is bounded, $[a]$ acts on all smooth functions ϕ on $\mathbb{R}$ and it coincides with the holomorphic functional calculus if ϕ is holomorphic in a neighborhood of the spectrum. In general, $[a]$ has a continuous extension to the algebra $\mathcal{G}$ of all smooth functions on $\mathbb{R}$ that are holomorphic at infinity, in particular, to each $r_z(\xi) = 1/(z-\xi)$ for $z \in \mathbb{C} \setminus \mathbb{R}$, and $[a](r_z) = (z-a)^{-1}$. Conversely, it was proved in [3] that if there exists such a multiplicative mapping $[a]$ such that, in addition,

$$\cup_{\phi \in \mathcal{D}(\mathbb{R})} \mathrm{Im}\, [a](\phi) \text{ is dense}, \quad \cap_{\phi \in \mathcal{D}(\mathbb{R})} \mathrm{Ker}\, [a](\phi) = \{0\}, \tag{1.3}$$

and $[a]$ extends continuously to $\mathcal{G}$, then there is a closed operator a satisfying (1.1) and such that (1.2) holds, see Theorem 6.3 for the precise statement.

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However, in many cases there exists such a smooth functional calculus although the resolvent does not exist at all. For example, let a be multiplication with $\xi \mapsto \xi(2+\sin\xi^3)$ on $X = H^1(\mathbb{R})$. Then the resolvent set is empty, but nevertheless a admits a smooth functional calculus $\mathcal{D}(\mathbb{R}) \to \mathcal{L}(X)$, and (1.3) holds.

We take the existence of a smooth functional calculus as our starting point, and introduce the notion of a *hyperoperator*, (with respect to smooth functions). It is a multiplicative $\mathcal{L}(X)$-valued distribution A on $\mathbb{R}$ such that (1.3) holds. This additional requirement means that $A(1) = e_X$ in a weak sense.

The spectrum of A is defined as the support of the distribution. A closable operator (tuple of commuting closable operators) defined on a dense subspace D is a *weak hyperoperator*, who, if a admits an $\mathcal{E}$ functional calculus with respect to D, i.e., a multiplicative continuous mapping $\mathcal{E}(\mathbb{R}^n) \to \mathcal{L}(D)$, where $\mathcal{L}(D)$ is the set of closable operators mappings $D \to D$. Roughly speaking this means that each $x \in D$ has real and compact local spectrum with respect to D. If a is a who and f is any smooth mapping then $f(a)$ is again a who. It turns out that for any hyperoperator A there is an associated who a. If f is proper, then the push-forward $B = f_*A$ of A is a hyperoperator and $b = f(a)$ is the who associated to B. Conversely, a who a is (or corresponds to) a hyperoperator if and only if for each $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\phi(a)$ extends to a bounded operator on X. Moreover, a is bounded (extends to a bounded operator) if and only if for each $f \in \mathcal{E}(\mathbb{R}^n)$, $f(a)$ extends to a bounded operator on X.

It is a well-known problem to find a suitable definition of commutativity for unbounded operators to get a reasonable theory. We will consider hyperoperators on $\mathbb{R}^n$ as well, with a completely analogous definition. For instance, if A_1 and A_2 are hyperoperators in $\mathbb{R}$, with associated whos a_1 and a_2, commuting in the functional calculus sense, then $A = A_1 \otimes A_2$ is a new hyperoperator in $\mathbb{R}^2$, and $a = (a_1, a_2)$ is the associated who. However, it is not true that each hyperoperator in $\mathbb{R}^2$ appears in this way. Similar phenomena hold for the unbounded analogs of a commuting tuple of bounded operators that are studied in e.g., [12], [15], [22], and [23]. This gives support for the idea that a reasonable notion of "commuting tuple of unbounded operators" must be considered as an object in its own. Weaker forms of commutativity of unbounded operators are studied in [17], [18], [19], and [20].

One can think of (1.2) as meaning that

$$\bar{\partial}\omega_{z-a} = [a], \tag{1.4}$$

where $[a]$ is the operator-valued distribution $\phi \mapsto \phi(a)$. For a general hyperoperator the resolvent form does not exist, but we present other solutions to (1.4) such that representations like (1.2) still hold.

Contents

2. Notation and some preliminaries

Any closed (densely defined) operator a on X, has a well-defined resolvent set $\rho(a)$ which is an open (possibly empty) subset of the extended plane $\widehat{\mathbb{C}}$. The spectrum of a is the set $\sigma(a) = \widehat{\mathbb{C}} \setminus \rho(a)$. Moreover, the operator a is bounded if and only if its spectrum is contained in $\mathbb{C}$. For any automorphism $\phi(\zeta)$ of $\widehat{\mathbb{C}}$ such that $\phi^{-1}(\infty)$ is not in the point spectrum of a, $\phi(a)$ is a well-defined closed operator, and the spectral mapping property $\phi(\sigma(a)) = \sigma(\phi(a))$ holds. The automorphism

$$C(\zeta) = \frac{\zeta + i}{\zeta - i}, \quad C^{-1}(\tau) = i\frac{\tau + 1}{\tau - 1}, \tag{2.1}$$

maps $\widehat{\mathbb{R}}$ bijectively onto to the unit circle $\mathbb{T}$. It induces the Cayley transform which establishes a one-to-one correspondence between closed operators with spectrum contained in $\mathbb{R}$, and bounded operators b with spectrum contained in $\mathbb{T}$ such that $b - e_X$ is injective.

If a is a densely defined operator on X, then it is closable if there is a closed operator a' such that $a \subset a'$, i.e., that the graph of a is contained in the graph of a'. In that case the closure of the graph of a is the graph of a (closed) operator called the closure $\bar{a}$ of a. If a has a bounded extension, then it is equal to $\bar{a}$.

We let $H^k(\mathbb{R}^n)$ denote the Sobolev space consisting of all functions in $L^2(\mathbb{R}^n)$ such that all derivatives up to order k belongs to $L^2(\mathbb{R}^n)$ as well.

2.1. **The Dynkin-Helffer-Sjöstrand functional calculus.** For any $\phi \in \mathcal{D}(\mathbb{R})$ one can find an extension $\tilde{\phi}$ to $\mathbb{C}$ such that

$$\bar{\partial}\tilde{\phi}(\zeta) = \mathcal{O}(|\mathrm{Im}\,\zeta|^\infty);$$

such a $\tilde{\phi}$ is called an almost holomorphic extension of ϕ. Moreover, if $\tilde{K}$ is a complex neighborhood of $\mathrm{supp}\,\phi$, one may assume that $\tilde{\phi}$ has support in $\tilde{K}$. Now let a be a closed operator with real spectrum such that (1.1) holds, such an operator will be referred to as an HS operator. Then clearly the integral in (1.2) converges, and it turns out to be independent of the choice of almost holomorphic extension. The multiplicativity follows from an application of the resolvent identity

$$\frac{1}{w-a}\frac{1}{z-a} = \frac{1}{z-w}\frac{1}{w-a} + \frac{1}{w-z}\frac{1}{z-a}.$$

It is easy to see that $[a]$ is continuous in the sense that $[a](\phi_j) \to 0$ in operator norm if $\phi_j \to 0$ in $\mathcal{D}(\mathbb{R})$. It also follows that the support of $[a]$ coincides with $\sigma(a) \cap \mathbb{C}$. Moreover, we claim that

$$[a](\xi\phi)x = [a](\phi)ax, \quad x \in \mathrm{Dom}\,(a). \tag{2.2}$$

This is of course well-known, but for further reference we sketch a proof. From the resolvent identity we have, assuming that $-i$ is outside the support of $\tilde{\phi}$,

$$\int \Big(\frac{1}{a+i} - \frac{1}{z+i}\Big)\frac{dz}{z-a} \wedge \bar\partial\tilde\phi(z) = \int \frac{dz}{(a+i)(z+i)} \wedge \bar\partial\tilde\phi(z) = -\int \bar\partial\Big(\frac{dz}{(a+i)(z+i)} \wedge \tilde\phi(z)\Big) = 0,$$

where the last equality follows from Stokes' theorem. Thus we have

$$\frac{1}{a+i}[a](\phi) = [a](\phi)\frac{1}{a+i} = [a](\frac{1}{\xi+i}\phi(\xi)). \tag{2.3}$$

Replacing $\phi(\xi)$ by $(\xi+i)\phi(\xi)$ we get

$$\frac{1}{a+i}[a]((\xi+i)\phi) = [a]((\xi+i)\phi)\frac{1}{a+i} = [a](\phi). \tag{2.4}$$

If $x \in \mathrm{Dom}\,(a)$ we therefore have $[a]((\xi+i)\phi)x = [a](\phi)(a+i)x$, which implies (2.2).

Example 1. Let a be a closed operator with spectrum equal to $\{\infty\}$. For instance one can take the inverse of the Volterra operator. Then clearly (1.1) holds, but the resulting multiplicative mapping $[a]$ is identically 0. □

If $a_1, \ldots, a_n$ is a tuple of HS operators such that their resolvents (anti-) commute, i.e., $\omega_{\zeta_j - a_j} \wedge \omega_{\zeta_k - a_k} = -\omega_{\zeta_k - a_k} \wedge \omega_{\zeta_j - a_j}$, for $\zeta_j, \zeta_k \in \mathbb{C} \setminus \mathbb{R}$, then $[a] = [a_1] \otimes [a_2] \cdots \otimes [a_n] \in \mathcal{D}'(\mathbb{R}^n, \mathcal{L}(X))$ is multiplicative. This follows by simple abstract considerations, but it can also be realized explicitly as

$$[a](\phi) = \int \omega_{\zeta_1 - a_1} \wedge \ldots \wedge \omega_{\zeta_n - a_n} \wedge \bar\partial_{\zeta_n} \cdots \bar\partial_{\zeta_1} \tilde\phi(\zeta),$$

where $\tilde\phi$ is a special almost holomorphic extension to $\mathbb{C}^n$ with compact support as in [3], i.e., such that

$$\bar\partial_{\zeta_1} \cdots \bar\partial_{\zeta_m} \tilde\phi(\zeta) = \mathcal{O}(|\mathrm{Im}\,\zeta_1|^\infty \cdots |\mathrm{Im}\,\zeta_m|^\infty). \tag{2.5}$$

2.2. **Commuting bounded operators.** Let $a = (a_1, \ldots, a_n)$ be a commuting tuple of bounded operators on X. If the Taylor spectrum $\sigma(a)$ is contained in $\mathbb{R}^n$, then it coincides with the spectrum of a with respect to the commutative Banach algebra (a) generated by a. If the tuple a has real spectrum, then we say that a admits a smooth functional calculus if the real-analytic functional calculus $[a]\colon \mathcal{O}(\mathbb{R}^n) \to \mathcal{L}(X)$ has a continuous extension to a mapping $[a]\colon \mathcal{E}(\mathbb{R}^n) \to \mathcal{L}(X)$. Since $C^\omega(\mathbb{R}^n)$ is dense in $\mathcal{E}(\mathbb{R}^n)$, the extension is then unique and multiplicative, and in fact it extends to $\mathcal{E}(\sigma(a)) \to \mathcal{L}(X)$. The existence of such an extension is equivalent to that $\exp(iat)$ has polynomial growth in $t \in \mathbb{R}^n$, see, e.g., [1]; it is also equivalent to that the resolvent satisfies

$$\|\omega_{z-a}\| \le C|\mathrm{Im}\,z|^{-M},$$

for some $M > 0$.

If a has non-real (Taylor) spectrum $\sigma(a)$, then there is in general no unique extension of the holomorphic functional calculus. For instance, let b be a nilpotent operator and let $A(\phi) = \tilde{\phi}(b, 0)$ and $B(\phi) = \tilde{\phi}(b, b)$ respectively, where $\tilde{\phi}(z, \bar{z}) = \phi(z)$ for real-analytic ϕ (only a finite Taylor expansion is needed). Then A and B extend to two different multiplicative mappings $\mathcal{E}(\mathbb{C}) \to \mathcal{L}(X)$ which both extend the holomorphic functional calculus. In general, a possible smooth functional calculus is uniquely determined by the image of $\bar{z}$ (or $\bar{z}_j$ if we have an n-tuple of commuting operators). In our situation the bounded (tuples of) operators that appear are like $b = A(\phi)$ for a possibly complex-valued ϕ, and then we have a natural conjugated operator, namely $b^* = A(\bar{\phi})$. A smooth functional calculus for such an operator b is then understood to map $\bar{z}$ to b^*. If $f(z) = \tilde{f}(z, \bar{z}) = \hat{f}(\operatorname{Re} z, \operatorname{Im} z)$, then $f(b) = \tilde{f}(b, b^*) = \hat{f}((b + b^*)/2, (b - b^*)/2i)$, and therefore we can reduce to the case of real-valued functions ϕ.

We conclude this section with the following useful observation.

Lemma 2.1. *If A is a linear and multiplicative mapping $\mathcal{D}(\mathbb{R}) \to \mathcal{L}(X)$ then, for any $\chi \in \mathcal{D}(\mathbb{R})$, $z \mapsto A(\chi(\xi)/(z - \xi))$ is strongly holomorphic in $\mathbb{C} \setminus \mathbb{R}$.*

Proof. Let $\tilde{\chi} \in \mathcal{D}(\mathbb{R})$ be identically 1 on $\operatorname{supp} \chi$. From linearity and multiplicativity we get

$$A(\frac{\chi(\xi)}{z - \xi}) = A(\frac{\chi(\xi)}{z_0 - \xi}) - (z - z_0) A(\frac{\chi(\xi)}{z - \xi}) A(\frac{\tilde{\chi}(\xi)}{z_0 - \xi}). \tag{2.6}$$

Letting $\|A(\chi(\xi)/(z_0 - \xi))\| = C$ and $\|A(\tilde{\chi}(\xi)/(z_0 - \xi))\| = \tilde{C}$ we see that

$$\|A(\chi(\xi)/(z - \xi))\| \leq C + |z - z_0| \tilde{C} \|A(\chi(\xi)/(z - \xi))\|$$

and so $\|A(\chi(\xi)/(z - \xi))\| \leq C/(1 - |z - z_0|\tilde{C})$. Thus $\|A(\chi(\xi)/(z - \xi))\|$ is locally uniformly bounded in z. From (2.6) it now follows that $A(\chi(\xi)/(z - \xi))$ is strongly continuous at z_0. With this fact in mind it follows immediately from (2.6) that

$$\frac{1}{z - z_0}(A(\frac{\chi(\xi)}{z - \xi}) - A(\frac{\chi(\xi)}{z_0 - \xi})) \to -A(\frac{\chi(\xi)}{(z_0 - \xi)^2}), \quad z \to z_0,$$

in operator norm. $\square$

3. Definition and basic properties

We say that a linear mapping $A \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{L}(X)$ is continuous, $A \in \mathcal{D}'(\mathbb{R}^n, \mathcal{L}(X))$, if $A(\phi_j) \to 0$ in operator norm when $\phi_j \to 0$ in $\mathcal{D}(\mathbb{R}^n)$. As for ordinary distributions it follows immediately that A has finite order on compact subsets, i.e., for any compact $K \subset \mathbb{R}^n$ there is a constant C_K and a non-negative integer M_K such that

$$\|A(\phi)\| \leq C_K \sum_{|\alpha| \leq M_K} \sup_K |\partial^\alpha \phi|$$

for all $\phi \in \mathcal{D}(\mathbb{R}^n)$ with support in K.

Definition 1. A continuous multiplicative mapping $A\colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{L}(X)$ is a hyperoperator on $\mathbb{R}^n$, $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$, if

$$(i) \quad D_A = \cup \mathrm{Im}\, A(\phi) \text{ is dense in } X, \qquad \text{and}$$

$$(ii) \quad N = \cap \mathrm{Ker}\, A(\phi) = \{0\}.$$

If a is an HS operator such that $[a]$ satisfies (i) and (ii), then $[a]$ is a hyperoperator. If a is bounded (or a commuting tuple of bounded operators), then $[a](\phi) = \phi(a)$. It is readily checked that the operator (tuple of operators) 0_X gives rise to the hyperoperator $[0_X]$, defined by $[0_X](\phi) = \phi(0)e_X$. In the same way, $[e_X](\phi) = \phi(1)e_X$.

Remark 1. Let $A\colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{L}(X)$ be a continuous multiplicative mapping. If A has compact support, i.e., A has a continuous extension to $\mathcal{E}(\mathbb{R}^n)$, then (i) and (ii) hold if and only if $A(1) = e_X$. In fact, let χ_N be a sequence in $\mathcal{D}(\mathbb{R}^n)$ that tends to 1 in $\mathcal{E}(\mathbb{R}^n)$. If now $A(1) = e_X$, then for any $x \in X$ we have that $x = A(1)x = \lim A(\chi_N)x$, and hence (i) holds. In the same way, if $A(\chi_N)x = 0$ for all N, then $x = A(1)x = 0$ so that (ii) holds as well. Conversely, if $x \in D_A$, then $x = A(\phi)z$ and therefore $A(1)x = A(1)A(\phi)z = A(1 \cdot \phi)z = A(\phi)z = x$. If D_A is dense it follows that $A(1) = e_X$. Therefore it is natural to think of (i) and (ii) as a weak form of saying that $A(1) = e_X$. □

We say that $\chi_N \in \mathcal{D}(\mathbb{R}^n)$ is an exhausting sequence if $0 \le \chi_N \le 1$, $\chi_N \nearrow 1$, and the compact sets $K_N = \{\chi_N = 1\}$ form an exhausting sequence of compact sets; i.e., $K_N \subset int(K_{N+1})$ and $\cup K_N = \mathbb{R}^n$.

Lemma 3.1. *Suppose that χ_N is an exhausting sequence in $\mathbb{R}^n$ and $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$. Then $\cup \mathrm{Im}\, A(\chi_N) = D_A$.*

Proof. If $\phi \in \mathcal{D}(\mathbb{R}^n)$, then $\chi_N \phi = \phi$ if N is large enough, and therefore

$$A(\chi_N)A(\phi) = A(\chi_N \phi) = A(\phi),$$

which shows that $A(\chi_N)$ is the identity on $\mathrm{Im}\, A(\phi)$. Thus $\mathrm{Im}\, A(\chi_N) \supset \mathrm{Im}\, A(\phi)$. □

Proposition 3.2. *Assume that A_1 and A_2 are hyperoperators in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and that they are commuting, i.e.,*

$$A_1(\phi)A_2(\psi) = A_2(\psi)A_1(\phi), \quad \phi \in \mathcal{D}(\mathbb{R}^n),\ \psi \in \mathcal{D}(\mathbb{R}^m).$$

Then $A = A_1 \otimes A_2$ is a hyperoperator in $\mathbb{R}^{n+m}$ and $D_A = D_{A_1} \cap D_{A_2}$.

In particular it follows that $D_{A_1} \cap D_{A_2}$ is dense as soon as A_1 and A_2 are commuting.

Proof. The tensor product A is defined as usual for distributions; thus $A(\phi \otimes \psi) = A_1(\phi)A_2(\psi)$, and it is extended to $\mathcal{D}(\mathbb{R}^{n+m})$ by linearity and continuity. The assumption on commutativity implies that A is multiplicative. If $0 = A(\phi \otimes \psi)x = A_1(\phi)A_2(\psi)x$ for all ϕ and ψ it follows from condition (ii) for A_1 and A_2 that $x = 0$. Thus (ii) holds for A. Given $x \in X$ we can find y and ϕ such that $\|x - A_1(\phi)y\| < \epsilon/2$. In the same way we can find z and ψ such that $\|y - A_2(\psi)z\| < \epsilon/(2\|A_1(\phi)\|)$. It follows that $\|x - A(\phi \otimes \psi)z\| < \epsilon$. Thus D_A is dense in X. On the other hand, since $\chi_N \otimes \chi'_M$ is an exhausting sequence in $\mathbb{R}^{n+m}$ if χ_N and χ'_M are exhausting sequences in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, it follows that $x \in D_A$ if and only if $A(\chi_N \otimes \chi_M)x = x$ for sufficiently large N and M, and this in turn holds if and only if $x \in D_{A_1} \cap D_{A_2}$. □

If a hyperoperator A in $\mathbb{R}^{n+m}$ is the tensor product $A_1 \otimes A_2$ of two commuting, multiplicative $\mathcal{L}(X)$-valued distributions in $\mathbb{R}^n$ and $\mathbb{R}^m$, then each A_j is indeed a hyperoperator. In fact, since $\chi_N \otimes \chi'_M$ is exhausting in $\mathbb{R}^{n+m}$, $\cup \mathrm{Im}\, A(\chi_N \otimes \chi'_M) = \cup \mathrm{Im}\, A_1(\chi_N)A_2(\chi'_M) = \cup \mathrm{Im}\, A_2(\chi'_M)A_1(\chi_N)$ is dense, so A_j satisfy condition (i). If $A_j(\phi)x = 0$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$, then $A(\phi \otimes \phi')x = 0$ for all ϕ, ϕ'. Therefore $A(\psi)x = 0$ for all $\psi \in \mathcal{D}(\mathbb{R}^{n+m})$, so $x = 0$. Hence A_j satisfies (ii).

Proposition 3.3. *If $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$ and $f \in \mathcal{E}(\mathbb{R}^n, \mathbb{R}^m)$ is a proper mapping, then the push-forward $B = f_*A \in \mathcal{D}'(\mathbb{R}^m, \mathcal{L}(X))$ is a hyperoperator, and $D_B = D_A$.*

Proof. Since f is proper, $f^*\colon \mathcal{D}(\mathbb{R}^m) \to \mathcal{D}(\mathbb{R}^n)$ and hence f_*A, defined by $f_*A(\phi) = A(f^*\phi) = A(\phi \circ f)$, is a multiplicative distribution. If χ_N is an exhausting sequence in $\mathbb{R}^m$, since f is proper, then $\chi_N \circ f$ is an exhausting sequence in $\mathbb{R}^n$. Therefore,

$$D_B = \cup_N \mathrm{Im}\, f_*A(\chi_N) = \cup_N \mathrm{Im}\, A(\chi_N \circ f) = D_A$$

according to Lemma 3.1. Thus f_*A satisfies (i). Finally, suppose that $f_*A(\psi)y = 0$ for all $\psi \in \mathcal{D}(\mathbb{R}^m)$. For fixed $\phi \in \mathcal{D}(\mathbb{R}^n)$ and large N, then

$$A(\phi)y = A\big(\phi(\chi_N \circ f)\big)y = A(\phi)A(\chi_N \circ f)y = 0,$$

and since ϕ is arbitrary, we conclude that $y = 0$. Thus f_*A is a hyperoperator. □

It is easy to check that any hyperoperator A extends to a multiplicative mapping on the algebra $\mathcal{D}(\mathbb{R}^n) \oplus \mathbb{C}$ of smooth functions that are constant outside some compact set, just by letting $A(h) = h(\infty)e_X + A(h - h(\infty))$. If ϕ has compact support, then $h = f \circ \phi$ is in this algebra, and therefore we have

Proposition 3.4. *Assume that $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$ and $\phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^m)$. Then the bounded operator $\phi(a) = A(\phi)$ admits a $\mathcal{E}$-functional calculus that extends the holomorphic (real-analytic) functional calculus, defined by $f \mapsto f(0)e_X + A(f \circ \phi - f(0))$.*

4. Weak hyperoperators

We shall now see that for each hyperoperator A there is an associated closable operator a on D_A. We will use the operator a to model the definition of a *weak hyperoperator*, see Definition 2 below.

Let A be a hyperoperator in $\mathbb{R}^n$ and let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be any smooth mapping. If $x \in D_A$ and $x = A(\phi)y$ we define $f(a)x = A(f\phi)y$. If $\chi = 1$ in a neighborhood of $\operatorname{supp}\phi$, then $f(a)x = A(f\chi\phi)y = A(f\chi)A(\phi)y = A(f\chi)x$; thus $f(a)x = A(f\chi)x$ and in particular $f(a)$ is a well-defined densely defined operator. Also observe that if $\phi \in \mathcal{D}(\mathbb{R}^n)$, then $\phi(a)x = A(\phi)x$ for all $x \in D_A$.

For any $x \in X$ we let $\sigma_x(A)$ be the support of the X-valued distribution $\phi \mapsto A(\phi)x$; this is the local spectrum at x. If $K \subset \mathbb{R}^n$ is compact, we let

$$D_{A,K} = \{x \in X;\ \sigma_x(A) \subset K\}.$$

It is readily checked that $D_A = \cup_K D_{A,K}$.

Proposition 4.1. *Assume that A is a hyperoperator in $\mathbb{R}^n$.*
(a) If $f \in \mathcal{E}(\mathbb{R}^n, \mathbb{R})$, then $f(a)$ maps $D_A \to D_A$, and if $g \in \mathcal{E}(\mathbb{R}^n, \mathbb{R})$, then $g(a)f(a) = (fg)(a)$ on D_A.
(b) If $f \in \mathcal{E}(\mathbb{R}^n, \mathbb{R}^m)$, then $f(a)$ is a closable operator (tuple of operators).
(c) If $f_k \to f$ in $\mathcal{E}(\mathbb{R}^m, \mathbb{R}^p)$, then $f_k(a)x \to f(a)x$ for all $x \in D_A$.
(d) If $x_j \in D_{A,K}$ for some fixed compact set K and $x_j \to x$ in X, then $f(a)x_j \to f(a)x$.

Proof. If $x \in D_A$ and $y = f(a)x$, then $y = A(\chi f)x$ for an appropriate χ and hence by definition $y \in D_A$. Moreover,

$$g(a)f(a)x = g(a)A(\chi f)x = A(\tilde{\chi} g)A(\chi f)x = A(\tilde{\chi}\chi f g)x = A(\chi f g)x = (fg)(a)x,$$

if $A(\chi)x = x$ and $\tilde{\chi} = 1$ on the support of χ. Thus (a) holds.

We can always take the closure of the graph of $f(a)$ in $X^n \times X^m$. If $x_k \in D_A$, $x_k \to x$, and $f(a)x_k \to y$, then for any $\psi \in \mathcal{D}(\mathbb{R}^n)$, $A(\psi)f(a)x_k \to A(\psi)y$; but also $A(\psi)f(a)x_k = A(\psi f)x_k \to A(\psi f)x$, so $A(\psi)y = A(\psi f)x$. Because of condition (ii) we have that y is then uniquely determined by x, and hence the closure is a graph. Thus (b) is proved.

Given $x \in D_A$, take χ such that $A(\chi)x = x$. Since $f_k\chi \to f\chi$ in $\mathcal{D}(\mathbb{R}^n)$, we have that $f_k(a)x = A(f_k\chi)x \to A(\chi f)x = f(a)x$. Thus (c) holds. For the last statement, observe that $A(\chi)x_k = x_k$ if $\chi = 1$ in a neighborhood of K. Hence $f(a)x_k = A(f\chi)x_k \to A(f\chi)x = f(a)x$. □

Notice that if $\phi \in \mathcal{D}(\mathbb{R}^n)$, then the closure of $\phi(a)$ is equal to the bounded operator $A(\phi)$. Moreover, the closure of $1(a)$ is equal to e_X and the closure of $0(a)$ is equal to 0_X. Applying Proposition 4.1 to the mapping $f(\xi) = \xi$, we find that a has a meaning as a densely defined closable operator (tuple of closable operators $a_j = f_j(a)$, where $f_j(\xi) = \xi_j$, that commute on D_A).

In view of this proposition it is natural to introduce a special class of densely defined linear operators. If D is a dense subspace, let $\mathcal{L}(D)$ be the set of closable linear operators $D \to D$.

Definition 2. Let $c = (c, D)$ be a linear operator mapping the dense subspace D of X into itself. Moreover, assume that c is closable, and that there is a linear and multiplicative mapping $\mathcal{E}(\mathbb{R}^n) \to \mathcal{L}(D)$, that extends the trivial one on polynomials, and such that $h_k(c)x \to h(c)x$, for $x \in D$ if $h_k \to h$ in $\mathcal{E}(\mathbb{R}^n)$. Then we say that c, or rather (c, D), is a weak hyperoperator, a who.

The sum of two closable operators is not necessarily closable (so $\mathcal{L}(D)$ is not a space), so part of the requirement is that each polynomial $p(c)$ in c is closable. Moreover, since the polynomials are dense in $\mathcal{E}$ the extension to $\mathcal{E}(\mathbb{R}^n) \to \mathcal{L}(D)$ is unique if it exists.

Assume that A is a hyperoperator and f is any smooth mapping. If $h \in \mathcal{E}(\mathbb{R}^m, \mathbb{R}^p)$ then we can define $h(f(a)) = (h \circ f)(a)$ on D_A. Therefore (a, D_A) as well as $(f(a), D_A)$ are whos. Also notice that if f is proper, $B = f_* A$, and a and b are the associated whos, then $b = f(a)$.

We say that a who $c = (c, D)$ is (extendable to) a hyperoperator A if $D \subset D_A$ and $\phi(c)x = A(\phi)x$ for $x \in D$. If such an A exists it is unique in view of Proposition 4.2 below. In the sequel we will therefore often talk about the hyperoperator a, meaning that a is the who associated to some hyperoperator A.

Proposition 4.2. *Suppose that A and A' are in $H_{\mathcal{D}(\mathbb{R}^n)}(X)$ and that $D_A \cap D_{A'}$ is dense. Moreover, assume that there is a dense subspace D of $D_A \cap D_{A'}$ such that $a_j = a'_j$ on D and map $D \to D$. Then $A = A'$.*

Proof. If $x \in D$ and χ is identically 1 on a large enough set, then

$$A(\xi_j \chi)x = a_j x = a'_j x = A'(\xi_j \chi)x.$$

Moreover, if $\tilde{\chi}$ is 1 in a neighborhood of $\operatorname{supp} \phi$, (recall that $x \in D$ implies that $a_j x \in D$)

$$A(\xi_k \xi_j \chi)x = A(\xi_k \xi_j \chi \tilde{\chi})x = A(\xi_k \chi)A(\xi_j \tilde{\chi})x = A(\xi_k \chi)a_j x = a_k a_j x = a'_k a'_j x = \ldots = A'(\xi_k \xi_j \chi)x,$$

and so on, so we get $A(p\chi)x = A'(p\chi)x$ for all polynomials p. If ϕ is a test function it follows by the Weierstrass approximation theorem that $A(\phi)x = A'(\phi)x$, and hence $A(\phi) = A'(\phi)$ since D is dense. □

Corollary 4.3. *If A is a hyperoperator and f is proper, then $f_* A = [0_X]$ if and only if $f(a) = 0$. In particular, $A = [0_X]$ if and only if $a = 0$.*

In fact, if $B = f_* A$, then $b = f(a)$, so $bx = 0 = 0_X x$ for all $x \in D_B = D_A$. Hence, by the previous proposition, $B = [0_X]$.

Corollary 4.4. *If A, A' are commuting hyperoperators and $a = a'$ on $D_A \cap D_{A'}$, then $A = A'$.*

This is just because $D_A \cap D_{A'}$ is dense if A and A' commute, cf., Proposition 3.2.

Assume that A is a hyperoperator and let h be smooth and constant outside a compact set. It is easily checked that the bounded operator $A(h) = h(\infty)e_X + A(h - h(\infty)$ is the closure of the densely defined operator $h(a)$. Therefore, cf., Proposition 3.4,

$$f(A(\phi))x = (f \circ \phi)(a)x, \quad x \in D_A,$$

for any smooth f if ϕ has compact support.

Proposition 4.5. *Let a_0 be an HS operator such that $[a_0]$ satisfies (i) and (ii) so that $A = [a_0]$ is a hyperoperator. If a is the associated who, then $\bar{a} = a_0$.*

Proof. Since by assumption $a_0 + i$ has a bounded inverse, we have that $\operatorname{Dom}(a_0) = \operatorname{Dom}(a_0 + i) = \operatorname{Im}(a_0 + i)^{-1}$. If $x \in D_A$, then $x = A(\chi)x$ so by (2.4)

$$x = A(\chi)x = \frac{1}{a_0 + i} A\big((\xi + i)\chi\big)x,$$

and hence $x \in \mathrm{Dom}\,(a_0)$ and by (2.2), $ax = a_0x$. Thus $a \subset a_0$.

Now, if $x \in \mathrm{Dom}\,(a_0)$ there is some y such that $x = (a_0+i)^{-1}y$. Take $y_k \in D_A \subset \mathrm{Dom}\,(a_0)$ such that $y_k \to y$. Then $x_k = (a_0+i)^{-1}y_k = (a+i)^{-1}y_k \in D_A$ according to (2.3). Thus

$$ax_k = a_0x_k = \frac{a_0}{a_0+i}y_k \to \frac{a_0}{a_0+i}y = a_0x,$$

since $a_0/(a_0+i)$ is bounded. Therefore, (x, a_0x) belongs to the closure of (the graph of) a. □

It is now easy to see that there exist non-trivial hyperoperators.

Example 2. Let a be the unbounded operator defined as multiplication with ξ on $X = H^1(\mathbb{R}_\xi)$. It defines a hyperoperator $A = [a]$ and the associated who is (a, D_A) where $D_A = \{x \in X;\, \mathrm{supp}\, x \subset\subset \mathbb{R}\}$. The mapping $f(\xi) = \xi(2+\sin\xi^3)\colon \mathbb{R} \to \mathbb{R}$ is proper and so $B := f_*A$ is a hyperoperator with $D_B = D_A$. By definition $B(\phi)$ is multiplication with $\phi \circ f$. The who b associated to B is just multiplication with f because if $x \in D_B$ and χ is chosen so that $\mathrm{supp}\, x \subset \{\chi \circ f = 1\}$ then $bx = B(\xi\chi(\xi))x = A(f\chi \circ f)x = f(a)x = fx$. We claim that B is not $[b_0]$ for any HS operator b_0. If there were such a b_0, then by Proposition 4.5, $\bar{b} = b_0$ and therefore there would be a bounded operator c such that $c(\bar{b}+i)x = (\bar{b}+i)cx = x$ for all $x \in D_B = \mathrm{Dom}(b)$. However, then c would have to be multiplication with $(f(\xi)+i)^{-1}$ on the image of D_B under $\bar{b}+i$ which again is D_B, but this is impossible since multiplication with $(f(\xi)+i)^{-1}$ has no bounded extension to all of X. □

Example 3. If B is a hyperoperator in $\mathbb{R}^2$ then the associated who b is equal to (b_1, b_2) where $b_j = \pi_j b$ are whos as well. However it may happen that none of the b_j are hyperoperators. Let $f_1(\xi)$ be equal to ξ for $\xi > 1$ and $\sin\xi^2$ for $\xi < -1$, and let $f_2(\xi) = -f_1(-\xi)$. Then $F = (f_1, f_2)\colon \mathbb{R} \to \mathbb{R}^2$ is proper and therefore $B := F_*A$ is a hyperoperator, if $A \in H_{\mathcal{D}(\mathbb{R})}(H^1(\mathbb{R}))$ is the hyperoperator that sends ϕ to multiplication with ϕ. In this case $b_1 = f_1(a)$ and $b_2 = f_2(a)$. Now, $\phi(b_j)$ is multiplication with $\phi \circ f_j$ and this operator has in general no bounded extension to $H^1(\mathbb{R})$, so b_j is not a hyperoperator. Take for instance $\phi \in \mathcal{D}(\mathbb{R})$ such that $\phi'(\xi) = 1$ for $-1 \le \xi \le 1$; then $(\phi \circ f_j)'(\xi)$ is unbounded. □

Example 4. Let (M, μ) be a finite measure space and let h be a real or complex valued measurable function (tuple of functions) defined a.e. with respect to μ. The operator defined as multiplication with h on $L^p(M, \mu)$, $1 \le p < \infty$, is then a hyperoperator and $\sigma(a)$ (see Section 5) is the essential range of h. Composing with smooth maps and/or taking tensor products will not take us outside this class of multiplication operators. By basic spectral theory any normal operator (tuple of normal commuting operators) can be viewed as such an operator (tuple of operators) on some $L^2(M, \mu)$. Therefore, our theory does not add anything to the usual theory of self-adjoint operators. □

We conclude this section with a result which together with Proposition 4.1 characterizes those whos that are hyperoperators.

Proposition 4.6. *Let $a = (a, D)$ be a who such that the closure of $\phi(a)$ is bounded on X for all $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then the mapping A defined by $A(\phi) =$*

$\overline{\phi(a)}$ *is a hyperoperator with* $D_A = \cup_{\phi\in\mathcal{D}} Im\, \overline{\phi(a)} \supseteq D$. *Moreover if* a' *is the who associated to* A *then* $\overline{a'} = \overline{a}$.

Let X and a be as in Example 2. Then $(a, \mathcal{D}(\mathbb{R}))$ is a who satisfying the hypotheses of the proposition. The induced hyperoperator is $A = [a]$ and D_A is the space of all f in X with compact support.

Proof. We first show that A so defined is a continuous mapping $\mathcal{D}(\mathbb{R}^n) \to \mathcal{L}(X)$. To this end, we take a compact set $K \subset \mathbb{R}^n$, and a cut-off function χ that is 1 in a neighborhood of K. For each $x \in X$ we can define a mapping $A_x\colon \mathcal{E}(\mathbb{R}^n) \to X$ by $A_x f = \overline{(\chi f)(a)}x$. For $x \in D_1 = D \cap \{|x| \le 1\}$ the mapping A_x is continuous, since (a, D) is a who. By the Banach-Steinhaus theorem it follows that $\{A_x\}_{x\in D_1}$ is equi-continuous, which means that

$$|A_x f| \le C \sum_{|\alpha|\le M} \sup_{K'} |\partial^\alpha f|, \tag{4.1}$$

for some C, M and K' independent of $x \in D_1$. Applying to ϕ with support in K, and using that D is dense, we get

$$\|\overline{\phi(a)}\| \le C \sum_{|\alpha|\le M} \sup |\partial^\alpha \phi|.$$

Thus A is continuous. The multiplicativity $A(\phi\psi) = A(\phi)A(\psi)$ now follows by continuity, since it holds when applied to $x \in D$. Moreover, for any $x \in D$ the map $\mathcal{E}(\mathbb{R}^n) \ni f \mapsto f(a)x \in X$ is continuous and therefore has compact support, $\sigma_x(a)$. If $\chi = 1$ in a neighborhood of $\sigma_x(a)$ it follows that $\chi(a)x = 1(a)x = x$. Hence A is a hyperoperator with $D_A = \cup_{\phi\in\mathcal{D}} \mathrm{Im}\, \overline{\phi(a)} \supseteq D$.

It remains to see that $\overline{a'} = \overline{a}$. If χ_N is an exhausting sequence, then $\psi_N = \xi\chi_N \to \xi$ in $\mathcal{E}(\mathbb{R}^n)$ and so for $x \in D \subseteq D_A$ we have

$$ax = \lim_{N\to\infty} \psi_N(a)x = \lim_{N\to\infty} A(\psi_N)x = a'x.$$

Hence $a \subseteq a'$ and so $\overline{a} \subseteq \overline{a'}$. To obtain the converse inclusion it suffices to show that $\overline{\mathrm{Graph}(a)} \supseteq \mathrm{Graph}(a')$. Let $(x, a'x) \in \mathrm{Graph}(a')$. Since $x \in D_A$, there is an N_0 such that $A(\chi_{N_0})x = x$. Take any sequence y_j in D converging to x and put $x_j = \chi_{N_0}(a)y_j$. Then x_j is a sequence in D and it also converges to x since $\chi_{N_0}(a)$ has a bounded extension. It follows that

$$ax_j = \lim_{N\to\infty} \psi_N(a)x_j = \lim_{N\to\infty} \psi_N(a)\chi_{N_0}(a)y_j = \psi_{N_0}(a)y_j \to \overline{\psi_{N_0}(a)}x,$$

as $j \to \infty$. However, $\overline{\psi_{N_0}(a)}x = a'x$ and hence $(x_j, ax_j) \to (x, a'x)$, that is, $(x, a'x) \in \overline{\mathrm{Graph}(a)}$. □

Remark 2. Let (a, D) be a who. For each $x \in D$ the mapping $\phi \mapsto \phi(a)x$ is a continuous mapping $A_x\colon \mathcal{E}(\mathbb{R}^n) \to X$, and hence it has compact support. As for a hyperoperator, we can define the local spectrum $\sigma_x(a)$ as this support. If $D_K = \{x \in D;\ \sigma_x(a) \subset K\}$, then clearly $D = \cup D_K$. For each $x \in D_K$ we have an estimate like (4.1), where K' is a compact neighborhood of K. However, in general this estimate cannot be uniform in x for $|x| \le 1$, since otherwise $\phi(a)$ would have a bounded extension to X.

To see how this lack of uniformity may appear, assume that $a = f(b)$ for some hyperoperator b, where f takes values in $K \subset\subset \mathbb{R}^m$. Then $D_{K,a} =$

$D_b = D_a$ because if $x \in D_b$ then $\chi(b)x = x$ and since b has finite order on $\operatorname{supp}\chi$ we get

$$|\phi(a)x| = |(\phi\circ f\cdot\chi)(b)x| \leq C\sum_{|\alpha|\leq N_\chi}\sup|\partial^\alpha(\phi\circ f\cdot\chi)||x|$$
$$\leq C\sum_{|\alpha+\beta+\gamma|\leq N_\chi}\sup|\partial^\beta f||\partial^\gamma\chi|\sup_K|\partial^\alpha\phi||x|.$$

However, N_χ and $\sup|\partial^\beta f||\partial^\gamma\chi|$ may blow up as $\chi\to 1$. □

5. Spectrum of a hyperoperator

We first recall

Proposition 5.1. *Suppose that $a = (a_1, \dots, a_n)$ is a tuple of bounded commuting operators with real spectra and resolvents with temperate growths, and A is the corresponding hyperoperator on $\mathbb{R}^n$. Then* $\operatorname{supp} A$ *is equal to the (Taylor) spectrum of a.*

For a proof, see [3]. In view of this result the following definition is natural.

Definition 3. For $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$, the spectrum $\sigma(A)$ is the support of A as a distribution.

When A is identified with the who a we often write $\sigma(a)$ instead of $\sigma(A)$. Notice that $A(\phi)$ only depends on the values of ϕ in a small neighborhood of $\sigma(A)$. If the spectrum of A is compact, then clearly A has a continuous extension to a multiplicative mapping $\mathcal{E}(\mathbb{R}^n)\to\mathcal{L}(X)$. For such an A and $f\in\mathcal{E}(\mathbb{R}^n)$, we have that $f(a)x = \lim A(f\chi_N)x = A(f)x$ for $x\in D$, and thus the closure of $f(a)$ is equal to the bounded operator $A(f)$. Applying to the identity mapping $\xi\mapsto\xi$ on $\mathbb{R}^n$ we get

Proposition 5.2. *Suppose that $A\in H_{\mathcal{D}(\mathbb{R}^n)}(X)$ and $\sigma(A)$ is compact in $\mathbb{R}^n$. Then the closure $\bar{a}$ of a is bounded, and $[a] = A$. Moreover, $\sigma(A)$ coincides with the Taylor spectrum of $\bar{a}$.*

If $f\in\mathcal{E}(\mathbb{R}^n)$ has its support in the complement of $\sigma(A)$, then $f(a)x = 0$ for all $x\in D$, so the closure of $f(a)$ is 0_X.

Definition 4. For a who $b = (b, D)$ we introduce the weak spectrum $\sigma_w(b)$ defined as the intersection of all closed sets F such that $\phi(b)x = 0$ for all $x\in D$ and ϕ with support in $\mathbb{R}^n\setminus F$.

Thus a point p is outside $\sigma_w(b)$ if and only if for all ϕ with support sufficiently close to p we have $\phi(b)x = 0$ for all $x\in D$. It follows that if b happens to be a hyperoperator then $\sigma_w(b) = \sigma(b)$. In particular, if $bx = 0$ for all $x\in D$, then $\sigma_w(b) = \sigma(0) = \{0\}$.

Proposition 5.3. *Let $b = (b, D)$ be a who and let $f\in\mathcal{E}(\mathbb{R}^n,\mathbb{R}^m)$. Then $\sigma_w(f(b)) = f(\sigma_w(b))$.*

Proof. If $h\in\mathcal{D}(\mathbb{R}^m)$ has its support outside $f(\sigma(b))$, then $h\circ f$ vanishes in a neighborhood of $\sigma(b)$ so $(h\circ f)(b)x = 0$ for $x\in D$, i.e, by definition, $h(f(b))x = 0$. This means that $\sigma_w(f(b))\subset f(\sigma_w(b))$. For the converse inclusion, take any point p outside $\sigma_w(f(b))$ and let h be a function identically equal to 1 in a neighborhood of p and with support outside $\sigma_w(f(b))$. Then if $y \in f^{-1}(p)$ we have $h \circ f$ identically equal to 1 in a neighborhood of y. Hence for any ϕ with support in this neighborhood $\phi \cdot (h \circ f) = \phi$. Since h has support outside $\sigma_w(f(b))$ we have $h \circ f(b)x = h(f(b))x = 0$ for $x \in D$ and so $\phi(b)x = \phi \cdot (h \circ f)(b)x = \phi(b)h \circ f(b)x = 0$ for $x \in D$. Thus $f^{-1}(p) \cap \sigma_w(b) = \emptyset$, i.e. $f(\sigma_w(b)) \subset \sigma_w(f(b))$. □

Noting that $\sigma_w(b) = \sigma(b)$ when b is a (strong) hyperoperator we immediately get

Corollary 5.4. *If $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$ and $f \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^m)$, or $f \in \mathcal{E}(\mathbb{R}^n, \mathbb{R}^m)$ is proper, then $\sigma(f(a)) = f(\sigma(a))$.*

Since $\sigma_w(0_{D_A}) = \sigma(0_X) = \{0\}$ we have

Corollary 5.5. *If $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$ and $f \in \mathcal{E}(\mathbb{R}^n, \mathbb{R}^m)$ and $f(a)x = 0$ for all $x \in D_A$, then $\sigma(a) \subset f^{-1}(0)$.*

It is not true in general that $f(\sigma(a))$ bounded implies that $f(a)$ is bounded (if f is neither proper nor compactly supported). For instance, take $f(\xi) = \sin \xi^m$ and $a \sim \xi$ on $X = H^1(\mathbb{R})$. Then $|f| \leq 1$ on $\sigma(a)$ but $f(a)$, i.e., multiplication with $\sin \xi^m$ is not bounded on X. However we have

Lemma 5.6. *If a is a hyperoperator, $f \in \mathcal{E}(\mathbb{R}^n)$, and $b = f(a)$ is bounded, then $f(\sigma(a)) \subset \sigma(f(a))$.*

Proof. We know that $p \circ f(a) = p(b)$ for all polynomials. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ have support outside $\sigma(b)$ and take p_j such that $p_j \to \phi$ in $\mathcal{E}(\mathbb{R}^n)$. Then $p_j \to 0$ uniformly in a neighborhood of $\sigma(b)$; we may even assume that this holds in a complex neighborhood; thus we can conclude that $p_j(b) \to 0$ (even though we do not know whether b admits a smooth functional calculus or not!).

Moreover, $p_j \circ f \to \phi \circ f$ in $\mathcal{E}(\mathbb{R}^n)$ so $p_j \circ f(a)x \to \phi \circ f(a)x$ for $x \in D$. Since $p_j \circ f(a) = p_j(b) \to 0$ we conclude that $\phi \circ f(a) = 0$. From Corollary 5.5 we get $f(\sigma(a)) \subset \{\phi = 0\}$ and we conclude that $f(\sigma(a)) \subset \sigma(b)$. □

Proposition 5.7. *Assume that $a = (a, D)$ is a who and that the closure of $r_z(a)$ is bounded for each $r_z(\xi)$, $z \in \mathbb{C} \setminus \mathbb{R}$. Then $\bar{a}$ has real spectrum in the usual sense.*

Proof. We first prove that the closure b of $r(a) = r_i(a)$ is the inverse of $\bar{a} + i$. We know that $(a + i)bx = x = b(a + i)x$ for $x \in D$. Suppose that $x \in \text{Dom}\,(\bar{a} + i) = \text{Dom}\,(\bar{a})$. Then there are $x_j \in D$ such that $x_j \to x$ and $(a + i)x_j \to (\bar{a} + i)x$. Since b is bounded we have

$$x \leftarrow x_j = b(a + i)x_j \to b(\bar{a} + i)x$$

so $b(\bar{a} + i)x = x$ for $x \in \text{Dom}\,(\bar{a} + i)$. Moreover, if x is arbitrary and $x_j \in D$ and $x_j \to x$, then $bx_j \to bx$ and $(a + i)bx_j = x_j \to x$ so by definition bx is in the domain of $\bar{a} + i$ and $(\bar{a} + i)bx = x$. □

6. Representation by pseudoresolvents

We first consider the case $n = 1$. If a is an HS operator, then we have the representation (1.2) of $\phi(a)$. For a general $a \in H_{\mathcal{D}(\mathbb{R})}(X)$ such a representation cannot hold simply because the resolvent is not defined. We will discuss various ways to obtain formulas that will replace (1.2). The simplest way is to use cut-off functions χ and define

$$\omega^{\chi}_{\zeta-a} = \chi(\xi)\omega_{\zeta-\xi}|_{\xi=a}, \quad \omega_{\zeta-\xi} = d\zeta/(\zeta-\xi)2\pi i.$$

Proposition 6.1. *Suppose that $a \in H_{\mathcal{D}(\mathbb{R})}(X)$. Then $\omega^{\chi}_{\zeta-a}$ is holomorphic for $|\operatorname{Im}\zeta| > 0$ and*

$$\|\omega^{\chi}_{\zeta-a}\| = \mathcal{O}(|\operatorname{Im}\zeta|^{-M}) \tag{6.1}$$

for some M. If $\phi \in \mathcal{D}(\mathbb{R})$ and $\operatorname{supp}\phi \subset\subset \{\chi = 1\}$, then

$$\phi(a) = \int \omega^{\chi}_{\zeta-a} \wedge \bar\partial\tilde\phi(\zeta). \tag{6.2}$$

Proof. By Lemma 2.1 ω^{χ}_{z-a} is strongly holomorphic in $\mathbb{C}\setminus\mathbb{R}$. Since A has finite order on $K \supset\supset \operatorname{supp}\chi$, $A(\psi)$ only depends on a finite number of derivatives of ψ if $\operatorname{supp}\psi \subset K$ and so we get (6.1). If

$$\phi_\epsilon(\xi) = \frac{1}{2\pi i}\int_{|\operatorname{Im}\zeta|>\epsilon} \frac{\chi(\xi)d\zeta}{\zeta-\xi} \wedge \bar\partial\tilde\phi(\zeta),$$

it is readily checked, for instance by approximating by Riemann sums, that

$$\phi_\epsilon(a) = \int_{|\operatorname{Im}\zeta|>\epsilon} \omega^{\chi}_{\zeta-a} \wedge \bar\partial\tilde\phi(\zeta). \tag{6.3}$$

Moreover, $\phi_\epsilon \to \phi\chi = \phi$ in $\mathcal{D}(\mathbb{R})$, and hence $\phi_\epsilon(a) \to \phi(a)$. Because of (6.1) it follows that the right hand side of (6.2) is absolutely convergent and equal to the limit of the right hand side of (6.3). □

Proposition 6.2. *Each ω^{χ}_{z-a} has a holomorphic continuation to the set $\mathbb{C}\setminus\sigma(a)$; more precisely, $\mathbb{C}\setminus\sigma(a)$ is precisely the set where all $\omega^{\chi}_{\zeta-a}$ are strongly holomorphic.*

Proof. The first statement is proved analogously to Lemma 2.1. If $x \in \mathbb{R}\setminus\sigma(a)$, let $\tilde\chi$ be a cut-off function that is equal to χ in a neighborhood of $\sigma(a)$ and zero in a neighborhood of x. Then $\omega^{\chi}_{z-a} = A(\tilde\chi/(z-\xi))$ and imitating the proof of Lemma 2.1 we see that ω^{χ}_{z-a} is strongly holomorphic close to x. For the converse, assume ϕ has its support where $\omega^{\chi}_{\zeta-a}$ is holomorphic and χ identically 1 in a neighborhood of $\operatorname{supp}\phi$. Then by Proposition 6.1,

$$A(\phi) = \int \omega^{\chi}_{z-a} \wedge \bar\partial\tilde\phi(z) = -\int \bar\partial(\tilde\phi(z)\omega^{\chi}_{z-a}) = 0$$

by Stokes' theorem and thus we are done. □

The advantage with the usual representation (1.2) is of course that a priori we only have to compute $\phi(a)$ for $\phi(\xi) = 1/(\zeta-\xi)$. For the general hyperoperator we must insert various functions χ as well. However, if we impose growth restrictions on $[a]$, one single formula will do. In Section 7 we will consider the case with polynomial growth restrictions.

If a is a hyperoperator or even just a who, then for each $x \in D$, the resolvent $\omega_{\zeta-a}x$ is holomorphic outside the compact set $\sigma_x(a) \subset \mathbb{R}$, and from (4.1) we have that $|\omega_{\zeta-a}x| \leq C|\mathrm{Im}\,\zeta|^{-M}$. With a similar argument as above we therefore have the representation

$$\phi(a)x = \int \omega_{\zeta-a}x \wedge \bar{\partial}\tilde{\phi}(\zeta), \quad \phi \in \mathcal{E}(\mathbb{R}).$$

Recall that $\mathcal{G}(\mathbb{R})$ is the algebra of functions on $\widehat{\mathbb{R}}$ that are holomorphic in a complex neighborhood of ∞. Convergence in $\mathcal{G}(\mathbb{R})$ of a sequence f_j means that f_j converges in $\mathcal{E}(\widehat{\mathbb{R}})$ and moreover, that all f_j are holomorphic in a fixed complex neighborhood of ∞ and converge uniformly on compacts in this neighborhood.

Theorem 6.3. *A hyperoperator $A \in H_{\mathcal{D}(\mathbb{R})}(X)$ corresponds to an HS operator if and only if $A\colon \mathcal{D}(\mathbb{R}) \to \mathcal{L}(X)$ has a multiplicative continuous extension to a mapping $\mathcal{G}(\mathbb{R}) \to \mathcal{L}(X)$.*

This result was more or less proved in [3]; one part is contained in the proof of Proposition 7.2 in [3] and the other part is stated in Proposition 11.4 in the same paper, but for the reader's convenience we supply a proof here.

Proof. First we notice that such an extension of A must be unique if it exists at all. In fact, for any $\psi \in \mathcal{G}(\mathbb{R})$ and $x \in D_A$ we have $A(\psi\chi)x = \psi(a)x$ if χ is chosen so that $A(\chi)x = x$. On the other hand if $\hat{A}$ is a multiplicative extension of A we get $\hat{A}(\psi)x = \hat{A}(\psi)A(\chi)x = A(\psi\chi)x$ Hence $\hat{A}(\psi)$ coincides with $\psi(a)$ on D_A and since D_A is dense and $\hat{A}(\psi)$ is bounded this uniquely determines $\hat{A}(\psi)$. Here a denotes the who associated to A.

For the "only if"-part we first assume that (the closure of) a is an HS operator, cf., Proposition 4.5. Then the action of A is given by (1.2) and we want to extend this formula to any function f in $\mathcal{G}(\mathbb{R})$. Let F be the holomorphic extension to a complex neighborhood O of ∞, and let χ be a cut-off function in $\mathbb{R}$ that is equal to 1 in a neighborhood of $K = \mathbb{R} \setminus (\mathbb{R} \cap O)$. One can find an almost holomorphic extension $\tilde{\chi}$ which is 0 in a complex neighborhood of ∞ and 1 in a complex neighborhood of K. Then $\tilde{f} = (1-\tilde{\chi})F + \widetilde{\chi f}$ is an almost holomorphic extension of f to a complex neighborhood of $\widehat{\mathbb{R}}$ in $\widehat{\mathbb{C}}$ which is holomorphic in a neighborhood of ∞. Let ψ be a function identically equal to 1 in a neighborhood of $\widehat{\mathbb{R}}$ in $\widehat{\mathbb{C}}$ and with support in a slightly larger neighborhood avoiding the point i. Then

$$\frac{1}{2\pi i}\int \frac{a-i}{\zeta-i}\frac{d\zeta}{\zeta-a} \wedge \bar{\partial}(\tilde{f}\psi(\zeta)) \tag{6.4}$$

provides the desired extension. In fact, if f has compact support then $\tilde{f}\psi$ is an almost holomorphic extension of f with compact support avoiding i. It follows by Stokes' theorem that formula (6.4) yields the same operator as (1.2). Moreover (6.4) is continuous and multiplicative on $\mathcal{G}(\mathbb{R})$. This is perhaps most easily seen by pulling back to the unit circle $\mathbb{T}$. The Cayley transform $b = C(a)$, cf., (2.1), is a bounded operator with spectrum contained in the unit circle $\mathbb{T}$, and

$$\frac{1}{2\pi i}\int \frac{a-i}{\zeta-i}\frac{d\zeta}{\zeta-a}\wedge\bar\partial(\tilde f\psi(\zeta)) = \frac{1}{2\pi i}\int \frac{dw}{w-b}\wedge\bar\partial(\tilde f\psi(C^{-1}(w))).$$

The right hand side is a continuous extension of the holomorphic functional calculus for b to the space of smooth functions on $\mathbb{T}$ which are analytic in a neighborhood of 1 since $\|(w-b)^{-1}\|$ has tempered growth in $\mathbb{T}\setminus\{1\}$. Since the analytic functions are dense in this space, the multiplicativity follows automatically.

Conversely, assuming that A is a hyperoperator that admits an extension to $\mathcal{G}(\mathbb{R})$, we want to prove that $\bar a$ is an HS operator. Since A now operates on all $r_z(\xi) = 1/(z-\xi)$ it follows from Proposition 5.7 that $\bar a$ has spectrum in $\mathbb{R}$ in the usual sense. Clearly then $r_z(a)dz/2\pi i$ is the resolvent of $\bar a$. Given a compact $K\subset\mathbb{R}$ take χ and $\tilde\chi$ as above. As A has finite order m on $K' = \operatorname{supp}\chi$ it follows that

$$\left\|\frac{\chi(a)}{z-a}\right\| \le C_K|\operatorname{Im} z|^{-(m+1)} \tag{6.5}$$

for any $z\in\mathbb{C}\setminus\mathbb{R}$. For z in a small neighborhood of K, the functions

$$g_z(\xi) = \frac{\tilde\chi(z)-\chi(\xi)}{z-\xi}.$$

are uniformly bounded in $\mathcal{G}(\mathbb{R})$, and by (6.5) so are $\|g_z(a)\| = \|A(g_z)\|$. Thus (1.1) follows by the triangle inequality. □

Remark 3. Let $a\in H_{\mathcal{D}(\mathbb{C})}(X)$. Then we can define $\omega^\chi_{\zeta-a}$ as a $\mathcal{L}(X)$-valued distribution ((1,0)-current) in $\mathbb{C}$ by

$$\omega^\chi_{\zeta-a}.\psi d\bar\zeta = \frac{1}{2\pi i}\int_\zeta \frac{\chi(\xi)\psi(\zeta)d\zeta\wedge d\bar\zeta}{\zeta-\xi}\Big|_{\xi=a}, \quad \psi\in\mathcal{D}(\mathbb{C}). \tag{6.6}$$

If we apply to $\bar\partial\psi$ we get $\omega^\chi_{\zeta-a}.\bar\partial\psi = \chi(\xi)\psi(\xi)|_{\xi=a} = \chi(a)\psi(a)$. Thus $\bar\partial\omega^\chi_{\zeta-a} = \chi[a]$. If in fact $a\in H_{\mathcal{D}(\mathbb{R})}(X)$ and we choose $\psi=\tilde\phi$ in (6.6), then we can move a inside the integral and thus get back (6.2). However, in general it is not possible to put a inside the integral. □

If we want an absolutely convergent integral representation for $\phi(a)$ when $a\in H_{\mathcal{D}(\mathbb{R}^n)}(X)$ we can use the Bochner-Martinelli form

$$\omega_\xi = b(\xi)\wedge(\bar\partial b(\xi))^{n-1}, \quad b(\xi) = \frac{\sum\bar\xi_j d\xi_j}{2\pi i|\xi|^2},$$

and define $\omega^\chi_{\zeta-a} = \chi(\xi)\omega_{\zeta-\xi}|_{\xi=a}$. Then $\omega_{\zeta-a}$ is $\bar\partial$-closed in $\mathbb{C}^n\setminus\mathbb{R}^n$ and the analogue of Proposition 6.1 holds. Proposition 6.2 also has a generalization to the $\mathbb{R}^n$ case; $\mathbb{C}^n\setminus\sigma(a)$ is precisely the set where ω^χ_{z-a} is strongly $\bar\partial$-closed. If we consider a hyperoperator $a\in H_{\mathcal{D}(\mathbb{R}^{2n})}(X)$ as an element in $a\in H_{\mathcal{D}(\mathbb{C}^n)}(X)$, the analog of Remark 3 also holds.

Tensor products of hyperoperators can also be defined by integral formulas. Assume that $A_1,\dots,A_m$ are in $H_{\mathcal{D}(\mathbb{R}^{n_j})}$ but not necessarily commuting. Then we can form the tensor product $A = A_1\otimes\cdots\otimes A_m$, and obtain a linear continuous, though not multiplicative, operator $\mathcal{D}(\mathbb{R}^n)\to\mathcal{L}(X)$, where $n = n_1+\cdots+n_m$. For $\phi\in\mathcal{D}(\mathbb{R}^n)$ we can find an almost holomorphic extension $\tilde{\phi}$ such that (2.5) holds. In [3] this is only proved when all $n_j = 1$ but the general case follows along the same lines. Then

$$(A_1 \otimes \cdots \otimes A_m)(\phi) = \int \omega^{\chi_1}_{\zeta_1 - a_1} \wedge \ldots \wedge \omega^{\chi_m}_{\zeta_m - a_m} \wedge \bar{\partial}_{\zeta_m} \cdots \bar{\partial}_{\zeta_1} \tilde{\phi}(\zeta), \tag{6.7}$$

if the support of ϕ is contained in the set where $\chi_1 \otimes \cdots \otimes \chi_m = 1$. To see this, first notice that the integral makes sense in view of the assumption (2.5) and the estimates (6.1) of $\omega^{\chi_j}_{\zeta_j - a_j}$. Since (6.7) clearly holds for ϕ of the form $\phi = \phi_1 \otimes \cdots \otimes \phi_m$, the general case follows by continuity. One can also prove directly that (6.7) is independent of the choice of special almost analytic extension $\tilde{\phi}$ along the lines in [3], and then use this as the definition of the tensor product.

Remark 4. We can also generalize Theorem 6.3 to several variables, and we illustrate by considering a hyperoperator $A \in H_{\mathcal{D}(\mathbb{R}^2)}(X)$. First we define $\mathcal{G}(\mathbb{R}^2)$ as the union (direct limit) of the spaces $\mathcal{G}_U(\mathbb{R}^2)$, U a complex neighborhood of ∞ in $\widehat{\mathbb{C}}$, defined as all smooth functions f on $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ which are holomorphic on $U \times U$ and such that $x \mapsto f(x, y)$ is holomorphic in U for any y and $y \mapsto f(x, y)$ is holomorphic in U for any x. A sequence f_j in $\mathcal{G}(\mathbb{R}^2)$ converges if all f_j are in some fixed $\mathcal{G}_U(\mathbb{R}^2)$ and converges in $\mathcal{E}((\widehat{\mathbb{R}} \cup U) \times (\widehat{\mathbb{R}} \cup U))$. The analog of Theorem 6.3 is: A has a continuous extension to $\mathcal{G}(\mathbb{R}^2)$ if and only if the closures of $a_j = A(\pi_j)$, $j = 1, 2$, are of HS type and commute strongly, i.e., their resolvents commute. Notice however that this condition highly depends on the choice of coordinates on $\mathbb{R}^2$, whereas the notion of general hyperoperator is coordinate invariant. □

7. Temperate hyperoperators

We say that $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$ is temperate, $A \in H_{\mathcal{S}(\mathbb{R}^n)}(X)$, if it extends to a (necessarily multiplicative) mapping $\mathcal{S}(\mathbb{R}^n) \to \mathcal{L}(X)$.

Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ it follows that a continuous multiplicative map $\mathcal{S}(\mathbb{R}^n) \to \mathcal{L}(X)$ satisfies (*i*) and (*ii*) in Definition 1 if and only if it holds with $\mathcal{D}(\mathbb{R}^n)$ replaced by $\mathcal{S}(\mathbb{R}^n)$ (but the corresponding dense domain may be larger).

For standard functional analysis reasons it follows that for any temperate A there is an integer M such that

$$|A(\phi)| \leq C \sum_{|\alpha|, |\beta| \leq M} \sup_{\mathbb{R}^n} |\xi^\beta \partial^\alpha \phi|, \tag{7.1}$$

which in particular means that $A(\phi)$ is defined for ϕ such that its derivatives up to order M as least have decay like $1/|\xi|^M$.

Example 5. Let X be the set of functions $\phi(\xi)$ on $\mathbb{R}$ with norm $\|\phi\| = \sum_\ell \|\phi\|_{C^\ell(K_{\ell+1} \setminus int K_{\ell-1})}$. Then multiplication with $\xi(2 + \sin \xi^3)$ is a hyperoperator that is not temperate. □

The multiplication hyperoperator $f(\xi) = \xi(2 + \sin \xi^3)$ on $H^1(\mathbb{R})$ from Example 2 is a tempered hyperoperator, which has no ordinary resolvent. Notice, though, that

$$\frac{i + \zeta}{i + f(\xi)} \frac{1}{\zeta - f(\xi)}$$

is bounded for all $\zeta \in \mathbb{C} \setminus \mathbb{R}$. More generally, if $A \in H_{\mathcal{S}(\mathbb{R})}(X)$ and m is a large enough integer we can define, in view of (7.1),

$$\omega^m_{\zeta-a} = \left(\frac{i+\zeta}{i+\xi}\right)^m \omega_{\zeta-\xi}\big|_{\xi=a},$$

for $\zeta \in \mathbb{C} \setminus \mathbb{R}$. If $A \in H_{\mathcal{S}(\mathbb{R}^n)}(X)$ we can take instead

$$\omega^m_{\zeta-a} = \left(\frac{1+\zeta\cdot\xi}{1+|\xi|^2}\right)^m \omega_{\zeta-\xi}\big|_{\xi=a}.$$

for $\zeta \in \mathbb{C}^n \setminus \mathbb{R}^n$.

Proposition 7.1. *The form $\omega^m_{\zeta-a}$ is $\bar\partial$-closed in $\mathbb{C}^n \setminus \mathbb{R}^n$ and admits a $\bar\partial$-closed extension to $\mathbb{C}\setminus\sigma(a)$. Moreover, if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\tilde\phi$ is an appropriate almost holomorphic extension, then*

$$A(\phi) = \int \omega^m_{\zeta-a} \wedge \bar\partial\tilde\phi. \tag{7.2}$$

This means, cf., Remark 3, that $\bar\partial\omega^m_{\zeta-a} = [a]$. Moreover, if $\omega^m_{\zeta-a}$ has a $\bar\partial$-closed extension to $\mathbb{C}^n \setminus F$, then $\sigma(a) \subset F$.

Sketch of proof. First notice that

$$\sup_{\xi\in\mathbb{R}^n} |\xi^\beta \partial^\alpha_\xi \omega^m_{\zeta-\xi}| \le C\frac{(1+|\zeta|)^m}{|\mathrm{Im}\,\zeta|^{2n-1+|\alpha|}}$$

if just $|\beta| < m$. If A satisfies (7.1), therefore $\omega^m_{\zeta-a}$ is well-defined if $m \ge M$, and

$$\|\omega^m_{\zeta-a}\| \le C\frac{(1+|\zeta|)^m}{|\mathrm{Im}\,\zeta|^{2n-1+|\alpha|}}. \tag{7.3}$$

Given $\phi \in \mathcal{S}(\mathbb{R}^n)$ we let

$$\tilde\phi(\zeta) = \int_t e^{it\cdot\zeta}\hat\phi(t)\chi\big(\sqrt{1+|t|^2}|\mathrm{Im}\,\zeta|\big),$$

where $\chi(s)$ smooth, supported in the unit ball in $\mathbb{R}^n$ and identically 1 in a neighborhood of the origin. One easily checks that $\tilde\phi(\zeta)$ is smooth, and equal to ϕ on $\mathbb{R}^n$, and that moreover,

$$\bar\partial\tilde\phi(\zeta) = \mathcal{O}_{M_1,M_2}(|\mathrm{Im}\,\zeta|^{M_1}(1+|\zeta|)^{-M_2}), \quad M_1, M_2 > 0. \tag{7.4}$$

In view of (7.3), therefore, the integral in (7.2) is well-defined. Moreover, from (7.4) it is easily seen that $\int \omega_{\zeta-\xi} \wedge \bar\partial\tilde\phi(\zeta) = \phi(\xi)$, and replacing $\tilde\phi(\zeta)$ by

$$\tilde\phi(\zeta)\Big(\frac{1+\zeta\cdot\xi}{1+|\xi|^2}\Big)^m$$

which satisfies a similar estimate, we get that

$$\int \omega^m_{\zeta-\xi} \wedge \bar\partial\tilde\phi(\zeta) = \phi(\xi).$$

One then proves (7.2) along the same lines as Proposition 6.1. □

For tempered hyperoperators the theory for tempered distributions is at our disposal. We will use this to prove a new form of Stone's theorem. We first recall a simple known variant.

Example 6. If $v \in C^1(\mathbb{R}^n, \mathcal{L}(X))$ and

$$v(t+s) = v(t)v(s), \quad v(0) = e_X, \tag{7.5}$$

then $v(t) = e^{ia\cdot t}$, for the commuting tuple $a_k = (\partial v/\partial t_k)(0)/i$ in $\mathcal{L}(X)$. If in addition $|v(t)| = \mathcal{O}(|t|^m)$, when $|t| \to \infty$, then $\sigma(a) \subset \mathbb{R}^n$. If we only assume that $v(t)$ is continuous and satisfies (7.5), then the conclusion is not true. (For instance, if $n=1$ and a is multiplication with ξ on $L^2(\mathbb{R}_\xi)$, then $v(t) = e^{iat}$ is multiplication by $e^{i\xi t}$ and thus a bounded operator, but $v'(0)$ is not bounded.) However, v is generated by a hyperoperator $A \in H_{\mathcal{S}(\mathbb{R}^n)}(X)$, i.e., $v(t) = \exp ia\cdot t$.

In fact, assume that $v(t)$ is continuous in the weak sense that $v(t)x$ is continuous for each $x \in X$. It then follows from the Banach-Steinhaus theorem that $\|v(t)\|$ is uniformly bounded on compact sets. Therefore, $v.\phi = \int_t v(t)\phi(t)dt$ is a bounded operator for each $\phi \in \mathcal{S}(\mathbb{R}^n)$. Moreover, the condition $v(0) = e_X$ implies that $\cap \mathrm{Ker}\, v(\phi) = \{0\}$ and $\cup \mathrm{Im}\, v(\phi)$ is dense. In fact, let $\phi_j \to \delta_0$. Then $v(\phi_j)x \to x$ since $\varphi \mapsto v(\varphi)x$ is continuous and we easily see that $\cap \mathrm{Ker}\, v(\phi) = \{0\}$ and that $\cup \mathrm{Im}\, v(\phi)$ is dense. The existence of the generator A now follows from Proposition 7.2 below. □

Let A be a tempered hyperoperator and let $D = \cup_{\phi \in \mathcal{S}(\mathbb{R}^n)} \mathrm{Im}\, A(\phi)$. If $f \in \mathcal{E}(\mathbb{R}^n)$ is a multiplier on $\mathcal{S}(\mathbb{R}^n)$, i.e., $f\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, we can define $f(a)x$ for $x \in D$ as $A(f\phi)y$ if $x = A(\phi)y$. To see that this is well-defined, assume that also $x = A(\phi')y'$. By the multiplicativity, we then have that $A(\chi_N f\phi)y = A(\chi_N f\phi')y'$ since $\chi_N f$ is in $\mathcal{S}$. When $N \to \infty$, $\chi_N f\phi \to f\phi$ in $\mathcal{S}$, and hence $A(f\phi)y = A(f\phi')y'$. It is readily checked that $f(a)$ maps $D \to D$ and that $(fg)(a)x = f(a)g(a)x$.

Observe that $f(\xi) = \exp(i\xi\cdot t)$ is a multiplier on $\mathcal{S}$, so $\exp(ia\cdot t)x$ is defined for all $x \in D$. Moreover, $x = A(\phi)y$ so $\exp(ia\cdot t)x = A(\phi(\xi)\exp(i\xi\cdot t))y$, and therefore (7.1) implies that

$$|e^{ia\cdot t}x| \le C_x|t|^M. \tag{7.6}$$

We claim that

$$A(\hat{\psi})x = \int_t \psi(t)e^{-ia\cdot t}x dt, \quad \psi \in \mathcal{S}, x \in D. \tag{7.7}$$

In fact, the integral is convergent in view of (7.6) and it is easy to see that it is equal to $A(\hat{\psi})x$ since $\int_{|t|<R} \psi(t)e^{-i\xi\cdot t}dt \to \hat{\psi}(\xi)$ in $\mathcal{S}$. In particular, the integral in (7.7) has a continuous extension to X. Since A is in $\mathcal{S}'$ it has a Fourier transform $\hat{A}$, defined by $\hat{A}(\psi) = A(\hat{\psi})$, and thus we have the suggestive formula $\hat{A}(t) = \exp(-ia\cdot t)$. If we let $v(t) = \exp(-ia\cdot t) = \hat{A}(t)$ then clearly $v(t+s)x = v(t)v(s)x$ for $x \in D$. Moreover, clearly v defined by $v.\psi = \hat{\psi}(a)$ satisfies

$$\int_s \int_t v(t+s)\phi(t)\psi(s) = \int_t v(t)\phi(t) \int_s v(s)\psi(s), \quad \phi, \psi \in \mathcal{S} \tag{7.8}$$

and

$$\cap \mathrm{Ker}\, v(\phi) = \{0\}, \quad \cup \mathrm{Im}\, v(\phi) = D \text{ dense}. \tag{7.9}$$

We have the following variant of Stone's theorem.

Proposition 7.2. *Assume that* $v\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{L}(X)$ *is linear, and continuous in the sense that for fixed* $x \in X$, $v.\phi_j x \to 0$ *whenever* $\phi_j \to 0$ *in* $\mathcal{S}(\mathbb{R}^n)$. *Moreover, assume that* $v(t)$ *is group of operators in the sense of* (7.8) *and* $v(0) = e_X$ *in the sense of* (7.9). *Then* v *is generated by a hyperoperator* $A \in H_{\mathcal{S}(\mathbb{R}^n)}(X)$ *in the sense that* $v(t)x$ *is smooth for* $x \in D_A$ *and* $(\partial v/\partial t_k)(0)x = ia_k x$.

Proof. Define $A(\phi) = v.\hat{\phi}$. By the Banach-Steinhaus theorem the pointwise continuity of v implies strong continuity and so A is a continuous map $\mathcal{S}(\mathbb{R}^n) \to \mathcal{L}(X)$. Moreover, the weak multiplicativity of v implies that $v(\phi * \psi) = v(\phi)v(\psi)$ and hence

$$A(\phi\psi) = v(\widehat{\phi\psi}) = v(\hat{\phi} * \hat{\psi}) = v(\hat{\phi})v(\hat{\psi}) = A(\phi)A(\psi).$$

Since the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^n)$ we get that $\cap \mathrm{Ker}\, A(\phi) = \{0\}$ and $\cup \mathrm{Im}\, A(\phi) = D$ is dense. Thus A is a tempered hyperoperator. For $x \in D$ we can define $u(t)x = e^{iat}x$ and since A satisfies an estimate like (7.1) it is easy to see that $|u(t)x| \le C|t|^M$ and so $u(t)x$ defines an element in $\mathcal{S}'(\mathbb{R}^n, X)$. We also see that $t \mapsto u(t)x$ is in C^1 (even in C^∞) and $u'(t)x = iax$. In fact, if $\phi \in \mathcal{S}$ then $\phi(\xi)(e^{i\xi t} - 1)/t \to i\xi\phi(\xi)$ in $\mathcal{S}$ as $t \to 0$, and hence if $x = A(\phi)y$ we get

$$\frac{e^{iat} - e_X}{t} x = \frac{\phi(a)e^{iat} - \phi(a)}{t} y \to ia\phi(a)y = iax.$$

We finally check that $u(t)x = v(t)x$ as tempered distributions. If, as before, $x = A(\phi)y$, then for any $\psi \in \mathcal{S}$ we have

$$\begin{aligned}\int_t \psi(t)u(t)x &= \int_t \psi(t)A_\xi(\phi(\xi)e^{i\xi t})y = A_\xi(\phi(\xi)\int_t \psi(t)e^{i\xi t})y \\ &= A_\xi(\phi(\xi)\hat{\psi}(\xi))y = A(\hat{\psi})x = \int_t \psi(t)v(t)x.\end{aligned}$$

□

8. Operators with ultradifferentiable functional calculus

Let $h(t) = H(|t|)$ where $H(0) = 0$ and H increasing and concave on $[0, \infty)$. Then h is subadditive. We also assume that $\lim_{|t|\to\infty} h(t)/|t| = 0$ and that

$$\limsup_{|t|\to\infty} \frac{\log(1 + |t|)}{h(t)} = 0. \tag{8.1}$$

Let $\mathcal{A}_h$ be the space of tempered distributions f on $\mathbb{R}^n$ such that $\hat{f}$ is a measure and

$$\|f\|_{\mathcal{A}_h} = \int_t |\hat{f}(t)|e^{h(t)}dt < \infty. \tag{8.2}$$

Because of (8.1), $\mathcal{A}_h$ is contained in $C^\infty(\mathbb{R}^n)$. Clearly $\mathcal{A}_h$ is a Banach space of functions that is closed under translations, and since h is subadditive it follows, see e.g., [2], that $\mathcal{A}_h$ actually is a Banach algebra under pointwise multiplication. These algebras were introduced by Beurling, [4]. If $h(t) = |t|^\alpha$, $0 < \alpha < 1$, then $G_\alpha = \cup_{c>0}\mathcal{A}_{ch}$ is the classical Gevrey algebra, see [11]. We say that the class $\mathcal{A}_h$ is non-quasianalytic if for each compact set E and open neighborhood $U \supset E$ there is a function $\chi \in \mathcal{A}_h$ with support in U which is identically 1 in some neighborhood of E. We recall the following version of the Denjoy-Carleman theorem.

Theorem 8.1. *The class $\mathcal{A}_h$ is non-quasianalytic if and only if*

$$\int_1^\infty \frac{H(s)ds}{s^2} < \infty. \tag{8.3}$$

Assume now that $h(t) = H(|t|)$ satisfies the condition (8.3). Let B_h be the algebra of all functions on $\mathbb{R}^n$ which are locally in $\mathcal{A}_{ch}$ for some $c > 1$, and let $B_{h,0}$ be the subalgebra of functions with compact support. There is an associated convex decreasing function $G(s) = \sup_t(H(t) - ts)$ on $(0, \infty)$. Let $H_c(s) = H(cs)$ and let G_c be the corresponding decreasing function.

Proposition 8.2. *A function $\phi \in B_h$ if and only if it admits an almost holomorphic extension $\tilde{\phi}$ such that for each compact $K \subset \mathbb{R}^n$, for some $c > 1$ we have*

$$\sup_{\operatorname{Re}\zeta \in K} |\bar{\partial}\tilde{\phi}| e^{g_c(\operatorname{Im}\zeta)} < \infty.$$

If ϕ has compact support and U is a complex neighborhood of $\operatorname{supp}\phi$ *we can choose $\tilde{\phi}$ with support in U.*

For a proof, see, e.g., [2]. It follows that composition of functions in B_h stays in B_h. In a completely analogous way as before we can now define a hyperoperator $A \in H_{B_{h,0}}(X)$ as a continuous multiplicative mapping $B_{h,0}(\mathbb{R}^n) \to \mathcal{L}(X)$ such that $\cup_{\phi \in B_{h,0}} \operatorname{Im} A(\phi) = D$ is dense and $\cap_{\phi \in B_{h,0}} \operatorname{Ker} A(\phi) = \{0\}$. Everything that is done in Sections 3,4, and 5 carry over directly to these ultrahyperoperators; for instance, D is the set of $x \in X$ such that $x = A(\chi)x$ for some cut-off function χ in B_h. If $A \in H_{B_{h,0}}(X)$, then $\|A(\phi)\| \le C_c \sup_{K'_c} |\phi|_{\mathcal{A}_{ch}}$ for each $c > 1$. If we define $\omega^\chi_{\zeta - z} = \chi(\xi)\omega_{\zeta-\xi}|_{\xi=a}$ it turns out that $\|\omega^\chi_{\zeta-z}\| \le C_c \exp g_c(\operatorname{Im}\zeta)$ for each $c > 1$. If $\operatorname{supp}\phi \subset \{\chi = 1\}$ we thus have the representation

$$A(\phi) = \int \omega^\chi_{\zeta - a} \wedge \bar{\partial}\tilde{\phi}(\zeta).$$

9. Invariant subspaces and spectral decomposition

Precisely as for a bounded operator (tuple of commuting bounded operators) that admits a smooth functional calculus, for a hyperoperator a there is a rich structure of invariant subspaces as well as spectral decompositions.

Proposition 9.1. *Assume that $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$, $f \in \mathcal{E}(\mathbb{R}^n, \mathbb{R}^m)$, and let*

$$X' = \{x \in D_A;\ f(a)x = 0\}.$$

Then $Y = \overline{X'}$ is an a-invariant subspace of X, and $a' = a|_Y$ is a hyperoperator. Moreover, $D_{a'} = X'$ and

$$\operatorname{int}\{f = 0\} \cap \sigma(a) \subset \sigma(a') \subset \{f = 0\} \cap \sigma(a). \tag{9.1}$$

If $\{f = 0\}$ contains some open subset of $\sigma(a)$, then Y has nontrivial vectors.

Proof. Since $f(a)$ and $\phi(a)$ commute, X' and hence Y are a-invariant. If ϕ has compact support, then $\phi(a)$ is bounded, and hence $\phi(a')$ extends to a bounded operator on $\overline{X'}$. Moreover, the continuity with respect to ϕ is clear. Since $1(a')x = x$ for all $x \in X'$, the properties (i) and (ii) in Definition 1 are satisfied, so a' is indeed a hyperoperator on the Banach space Y.

By definition, $X' \subset D_a$. If $x \in D_{a'}$, then $x = \chi(a')x$ for some $x \in Y$. This means that $x = \chi(a)x$ and so $x \in D_a$, and moreover $f(a)x = 0$. Thus $D_{a'} = X'$.

If $\phi(a) = 0$ for all $\phi \in \mathcal{D}(\omega)$ then $\phi(a') = 0$ for all such ϕ, and hence $\sigma(a') \subset \sigma(a)$. If p is any point outside $\{f = 0\}$ then $f_j(p) \neq 0$ for some f_j $(f = (f_1, \ldots, f_m))$. We may assume that $f_j(p) = 1$. If $\omega \ni p$ is small enough, $|f_j - 1| \leq 1/2$ in ω. For $\phi \in \mathcal{D}(\omega)$ we have that

$$\phi(a)x = \phi(a)(1 - f_j)^N(a)x = (\phi(1 - f_j)^N)(a)x, \quad x \in X',$$

and since $\phi(1 - f_j)^N \to 0$ in $\mathcal{D}(\omega)$ when $N \to \infty$ we can conclude that $\phi(a)x = 0$. Thus ω is contained in the complement of $\sigma(a')$ and so we have proved the second inclusion in (9.1). To see the first one, take $p \in \operatorname{int}\{f = 0\} \cap \sigma(a)$ and a neighborhood ω such that $p \in \omega \subset \{f = 0\}$. Since ω intersects $\sigma(a)$ there exists some $\phi \in \mathcal{D}(\omega)$ and $z \in X$ such that $x = \phi(a)z \neq 0$. However, then $x \in D$ and $f(a)x = (f\phi)(a)z = 0$ since $f\phi = 0$, so $x \in X'$. Thus ω intersects $\sigma(a')$. Since $\omega \ni p$ can be chosen arbitrarily small, we conclude that $p \in \sigma(a')$. If $\sigma(a')$ is nonempty, then Y is nontrivial, and so the last statement follows from (9.1). □

If p is an isolated point in $\sigma(a)$ and $f = 0$ in a neighborhood of p, then X' is non-trivial. There are also non-trivial a-invariant subspaces as soon as $\sigma(a)$ contains more than one point. Notice that a' is bounded if $\{f = 0\} \cap \sigma(a)$ is compact.

It is easy to make spectral decompositions. Let $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$ be a hyperoperator and let $\{\Omega_j\}$ be a locally finite open cover of $\sigma(a)$. Moreover, choose $\phi_j \in \mathcal{E}(\mathbb{R}^n)$ such that $\Omega_j \subset \{\phi_j = 1\}$, and let

$$X_j = \{x \in D_A;\ \phi_j(a)x = x\}.$$

If Ω_j is bounded, we can choose ϕ_j in $\mathcal{D}(\mathbb{R}^n)$ and then $X_j = \operatorname{Ker}(e_X - A(\phi))$ is a closed subspace of D_A. Then X_j are a-invariant subspaces, $\sigma(a|_{X_j}) \subset \Omega_j \cap \sigma(a)$, and

$$\sum_1^\infty X_j = D_A. \tag{9.2}$$

All these statements but the last one follows from Proposition 9.1. To see (9.2), choose a smooth partition of unity χ_j subordinate to the cover $\{\Omega_j\}$. Then, since $\sum \chi_j = 1$, for each $x \in D_A$ we have $x = \sum_1^M \chi_j(a)x$ for some M. However, $(1 - \phi_j)\chi_j = 0$ so $\chi_j(a)x$ belongs to X_j. Hence, (9.2) follows.

In general the sum (9.2) is not direct. However, if $\sigma(a)$ is a disjoint union of closed sets F_j, we can find ϕ_j with disjoint supports such that $\{\phi_j = 1\}$ contain a neighborhood of F_j. If $x \in X_j \cap X_k$, then $x = \phi_j(a)x = \phi_j(a)\phi_k(a)x = (\phi_j\phi_k)(a)x = 0$, and hence we get

$$D_A = \oplus_1^\infty X_j.$$

Example 7. Let $A \in H_{\mathcal{D}(\mathbb{R}^n)}(X)$ and let $f \in \mathcal{E}(\mathbb{R}^n, \mathbb{R}^m)$ be a mapping such that $f(a) = 0$. From Corollary 5.5 (or (9.1)) we know that $\sigma(a) \subset \{f = 0\}$. Let us also assume that the zero set $\{f = 0\} = \{\alpha^j\}$ is discrete. Then we have the decomposition $D_A = \oplus_1^\infty X_j$ where $\sigma(a|_{X_j}) = \{\alpha^j\}$. For each j, let $g_1^j, \dots, g_{\ell_j}^j$ be functions in the local ideal generated by f at α^j, and let $Y_j = \{x \in D_A;\ g_\ell^j(a)x = 0, \ell = 1, \dots, \ell_j\}$. If $x \in X_j$, then $x = \phi_j(a)x$, and since moreover $g_\ell^j \phi_j = \sum_k h_k f_k \phi_j$ for some h_k, it follows that $x \in Y_j$. Thus $X_j \subset Y_j$. Furthermore, if for each j the common zero set of g_ℓ^j is just the point α_j, then by (9.1), $\sigma(a, \overline{Y}_j) \subset \{\alpha_j\}$. If $x \in Y_j \cap Y_i$, therefore $\sigma_x(a) = \emptyset$, and hence $x = 0$ since a is a hyperoperator. It therefore follows that $X_j = Y_j$.

If all zeros of f are of first order, i.e., the local ideal at α^j is generated by $\xi_i - \alpha_i^j$, $i = 1, \dots, n$, then X_j is the eigenspace

$$X_j = \{x \in D_A;\ a_i x = \alpha_i^j x,\ i = 1, \dots, n\}.$$

If $A \in H_{\mathcal{D}(\mathbb{C})}(X)$ and f is holomorphic in a neighborhood of $\sigma(a)$ and the zeros α^j have multiplicities r_j, then $X_j = \{x \in D_A;\ (a - \alpha_j)^{r_j} x = 0\}$. □

The situation in this example appears naturally when we consider homogeneous solutions to an equation like $f(a)x = 0$.

Example 8. Let $\mathcal{A}_h(\mathbb{R}^n)$ be a Beurling algebra, cf., Section 8, containing cut-off functions, and let X be the space of inverse Fourier transforms of the dual space $\mathcal{A}_h'$. Then the tuple of commuting operators $a_j = i\partial/\partial\xi_j$ on X admits an $\mathcal{A}_h$ functional calculus (since $\mathcal{A}_h$ is an algebra). Then D_a is the space of (inverse) Fourier transforms of elements with compact supports in $\mathcal{A}_h'$. Notice that $\mathcal{A}_h'$ contains all distributions with compact support, but also some hyperfunctions of infinite order. Let f be a $\mathcal{A}_h$-smooth mapping and consider the space $\{x \in D_a;\ f(a)x = 0\}$. If x is the inverse Fourier transform of u, then $f(t)u(t) = 0$, which means that u has support on $Z = \{t \in \mathbb{R}^n;\ f(t) = 0\}$. It follows that we have the representation

$$x(\xi) = \int_{\mathbb{R}^n} e^{i\xi \cdot t} u(t) dt, \tag{9.3}$$

meaning the action of u on $t \mapsto \exp i\xi \cdot t$. Since u has support on the set $Z = \{f = 0\}$, x is expressed as a combination of exponentials with frequencies in Z. □

Even if f is a polynomial, only solutions generated by real frequencies can appear as long as we have restricted to non-quasianalytic classes. To get an operator-theoretic frame of this kind for the general fundamental principle of Ehrenpreis and Palamodov, [9] and [14], one must consider operators that only admit a holomorphic functional calculus.

10. Non-commuting hyperoperators

Assume that $A_1, \dots, A_m$ are in $H_{\mathcal{D}(\mathbb{R}^{n_j})}$ but not necessarily commuting. Then we can form the tensor product $A = A_1 \otimes \cdots \otimes A_m$, and obtain a linear continuous, though not multiplicative, operator $\mathcal{D}(\mathbb{R}^n) \to \mathcal{L}(X)$, where $n = n_1 + \cdots + n_m$. This can also be done explicitly by the formula

(6.7). We also write this operator of course as $\phi(a_1,\ldots,a_m)$. In case when all $n_j=1$ and a_j are HS operators, we get back the definition in [3]. Now the order of the operators is crucial. Therefore it is convenient to use Feynman notation, see, e.g., [13]. Then this operator $\phi(a_1,\ldots,a_m)$ can be written $\phi(\overset{m}{a_1}$ $,\ldots,\overset{1}{a_m})$ indicating that the operator a_m is to be applied first, then a_{m-1} etc and finally a_1, and the order is reflected by the order of the resolvents. Therefore, if b is a bounded operator one can easily define for instance

$$\phi(\overset{3}{a_1},\overset{1}{a_2})\overset{2}{b}=\int\int(\partial_{\bar{\zeta}_1}\partial_{\bar{\zeta}_2})\tilde{\phi}(\zeta_1,\zeta_2)\wedge\omega^{\chi_1}_{\zeta_1-a_1}\wedge b\omega^{\chi_1}_{\zeta_2-a_2}.$$

Notice that this is *not* an ordinary composition of $f(\overset{3}{a_1},\overset{1}{a_2})$ and b, while for instance $f(\overset{2}{a_1},\overset{1}{a_2})\overset{3}{b}=bf(\overset{2}{a_1},\overset{1}{a_2})$.

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