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Bergman kernel asymptotics and holomorphic Morse inequalities

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From a purely analytical point of view this thesis is concerned with the study of global solutions of certain partial differential equations (the null space of the $\overline{\partial}-$Laplacian with $\overline{\partial}-$Neumann boundary conditions) in the "semi-classical limit" when the parameter k tends to infinity. The main motivation comes from certain geometric situations where a priori estimates are missing.

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This thesis consists of an introduction and the following papers:

Paper I: Berman, R., Bergman kernels and local holomorphic Morse inequalities *Math. Z. 248 (2004), no. 2, 325–344*

Paper II: Berman, R., Super Toeplitz operators on line bundles *Journal of Geometric Analysis, to appear*

Paper III: Berman, R., Holomorphic Morse inequalities on manifolds with boundary *Ann. Inst. Fourier (Grenoble) 55 (2005), no. 4, 1055–1103.*

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To my grandfather Zygmund Berman

BERGMAN KERNEL ASYMPTOTICS AND HOLOMORPHIC MORSE INEQUALITIES

ROBERT BERMAN

"Think globally, act locally" - John Lennon

Introduction

In complex analysis and complex geometry, as well as in related applications, it is crucial to be able to construct many complex analytical/holomorphic objects (holomorphic subvarieties, meromorphic functions etc). To this end one usually looks for *holomorphic sections* of a given *line bundle* L over a complex manifold X. Such sections can be considered as *twisted functions,* where the twist is described by the line bundle L. The basic example is a complex polynomial. A line bundle has *curvature* (measuring its twisting) and if the curvature is positive everywhere and the manifold X is *closed*, then there is a wealth of techniques, introduced by Kodaira, Hörmander and others, to produce many holomorphic sections. To get a rich and flexible theory one often considers high tensor powers L^k, focusing on the *asymptotic* properties of its holomorphic sections when k tends to infinity. This corresponds to studying properties of large degree polynomials. The flexibility that arises when k tends to infinity is somewhat similar to the way *any* continuous function on a segment of the *real* line can be obtained as the limit of a sequence of polynomials of increasing degree. Yet, polynomial functions of a *fixed* degree are very rigid objects.

In the case of a line bundle L with *positive* curvature over a *closed* manifold X, the study of the large k behaviour of the space $H^0(X, L^k)$ of holomorphic sections of L^k, especially its *Bergman kernel*, is closely related to various current research areas. For example, an important strategy in complex differential geometry is to use holomorphic sections of high powers L^k to approximate canonical metrics on the base manifold X (e.g. Kähler-Einstein metrics) by "polynomial" metrics, directly defined in terms of the Bergman kernel of L^k [39, 40, 99]. Other developments, using Bergman kernel asymptotics, include the study of *random holomorphic sections* [97, 13], extending the classical study of random polynomials with applications to to the study of quantum chaos [22] in physics and, very recently, the statistical study of the vacuum selection problem in string/M-theory [44]. A unifying theme in many of these developments is the notion of "quantization", which of course originates in physics, with $1/k$ playing the role of Planck's constant and the holomorphic sections the role of the wavefunctions [41]. The large k limit corresponds to the *semi-classical limit* in physics (which is mathematically studied in *semi-classical* and *microlocal analysis* in the context of general partial differential equations [36, 80]).

In general, the obstruction to construct holomorphic sections is measured by *cohomology groups* and it is hence vital to be able to decide when these groups vanish. The study of the corresponding *vanishing theorems* was introduced in the fifties by Serre in algebraic geometry for *ample* line bundles and by Kodaira in complex differential geometry, who showed that ample line bundles correspond to line bundles with *positive* curvature. In the general case of line bundles with varying curvature Demailly's holomorphic Morse inequalities, from the middle of the eighties, estimate the dimensions of the corresponding cohomology groups of high powers L^k in terms of curvature integrals, leading to *asymptotic* vanishing theorems for semi-positive line bundles. For example, such line bundles (as well as line bundles satisfying other weak notions of positivity) play an important role in the classification theory of higher dimensional complex algebraic varieties (Mori's minimal model program [66, 67]).

One of the main motivations for this thesis is to be able to produce and study holomorphic sections in situations where the curvature of the line bundle is *not strictly* positive or when the manifold X has a *boundary* whose curvature also has to be taken into account. In the latter situation, one of the most interesting cases is when the line bundle L has positive curvature, while the boundary has negative curvature (i.e. X is *pseudoconcave*). The starting point is a new approach (in Paper I) to Demailly's holomorphic Morse inequalities, based on Bergman kernel asymptotics, which gives point-wise versions od Demailly's inequalities.

In the following sections the main results of this thesis are presented in the light of previous results and developments. In the appendix some model examples of Bergman kernels are collected, comparing those occuring in this thesis with classical ones.

Contents

1. [1]Setup

A section of a holomorphic line bundle L over a complex manifold X (of complex dimension n) is *holomorphic* when it is in the null space of the $\overline{\partial}-$operator (i.e. when it is solution of the Cauchy-Riemann equations). The basic example is a *polynomial of degree* k on complex space $\mathbb{C}^n$ which extends to a section of the kth tensor power of the hyperplane line bundle $\mathcal{O}(1)$ over complex projective space $\mathbb{P}^n$. In the general case it is natural to consider *high tensor powers* L^k *and to treat* k *as an asymptotic parameter,* generalizing the degree of a polynomial. A general line bundle L may have very few or even no holomorphic sections at all. For example, the sections of the trivial holomorphic line bundle are just ordinary holomorphic functions which are all constant on a *closed* manifold (i.e compact without boundary). A line bundle L is called *big* if the dimension of the space $H^0(X, L^k)$ of holomorphic sections of L^k grows as a constant C times k^n, i.e. as in the case of polynomials of degree k. The leading constant C is called the *volume of* L :

$$\text{vol}(L) := \limsup_k k^{-n} \dim H^0(X, L^k), \quad \text{vol}(L) \neq 0 \Leftrightarrow L \text{ big} \tag{1.1}$$

and it has received much recent attention from various points of view, including algebraic geometry (see the recent book [70]), complex analytical geometry in the context of singular metrics on line bundles (see [21] and the survey [34]), representation theory (inspired by statistical physics) [85] and string/M-theory [64]).

Fix a Hermitian *fiber metric* on L (i.e. a smooth family of Hermitian norms on the fibers of L). Then its *curvature* is a two-form on X. The local picture is as follows: fixing a local holomorphic trivialization of L any holomorphic section of L is represented by a local holomorphic function α and its point-wise norm may be written as

$$|\alpha(z)|^2 e^{-\phi(z)}$$

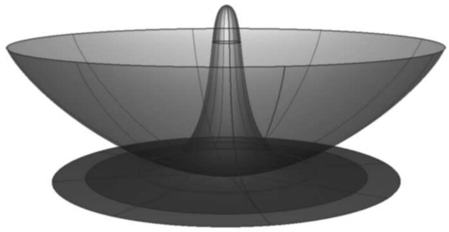
for some local function $\phi(z)$ representing the fiber metric on L. The fiber metric induced on L^k is locally given by $k\phi(z)$. The curvature of the fiber metric is the two-form of bidegree $(1,1)$ given by

$$\partial\overline{\partial}\phi = \sum_{i,j} \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j,$$

which turns out to be a *globally* defined form on X. The normalized curvature-form $\frac{i}{2\pi}\partial\overline{\partial}\phi$ represents the *first Chern class* $c_1(L)$ of L in the

[1]general references for this section are the books [56, 35].

FIGURE 1.1. Assume that the curvature form $\partial\overline{\partial}\phi > 0$ (e.g. $\phi(z) = |z|^2$). The bowl represents the local graph of the fiber metric $\phi(z)$ and the peak inside the bowl represents the graph of the point-wise norm $e^{-\phi(z)}$ of the local holomorphic section $\alpha \equiv 1$.



De Rham cohomology group $H^2(X,\mathbb{R})$[2] The curvature is said to be *positive* if the Hermitian matrix $(\frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j})$ is positive definite. This means that the curvature is positive when $\phi(z)$ is *plurisubharmonic*, i.e. subharmonic along complex lines. When the manifold X is $\mathbb{C}^n$, the corresponding weighted L^2-norms

$$\int_{\mathbb{C}^n} |\alpha(z)|^2 e^{-\phi(z)}$$

(integrating with respect to the Euclidean measure) were introduced by Hörmander in the sixties in his study of the $\overline{\partial}$-equation [60].

When X has a boundary ∂X we denote by ρ a defining function for ∂X, i.e. ρ is defined in a neighborhood of the boundary of X, vanishing on the boundary and negative on X and $d\rho \neq 0$. Then the (Levi) *curvature* of ∂X is the restriction of the two-form $\partial\overline{\partial}\rho$ to the maximal complex subbundle of the real tangentbundle of ∂X. The latter bundle is denoted by $T^{1,0}(\partial X)$. Given an Hermitian line bundle L, denote by D^* the unit disc bundle in the dual bundle L^*. Then L has positive curvature precisely when the boundary of D^* has has positive curvature (compare appendix A.6).

The $\overline{\partial}-$operator extends to a whole complex, the *Dolbeault complex*:
(1.2)

$$... \to \Omega^{0,q-1}(X,L^k) \overset{\overline{\partial}}{\to} \Omega^{0,q}(X,L^k) \overset{\overline{\partial}}{\to} \Omega^{0,q+1}(X,L^k) \to ...(\overline{\partial}^2 = 0)$$

Its cohomology groups are the *Dolbeault cohomology groups:*

$$H^q(X,L^k) := \frac{\ker \overline{\partial} : \Omega^{0,q}(X,L^k) \to \Omega^{0,q+1}(X,L^k)}{\mathrm{Im}\overline{\partial} : \Omega^{0,q-1}(X,L^k) \to \Omega^{0,q}(X,L^k)}$$

[2]Conversely, any closed $(1,1)-$form that represents an integral class can always be realized as the normalized curvature form of a fiber metric ϕ on a holomorphic line bundle L [56].

(when X has a boundary the cohomology will be assumed to be defined by forms that are smooth up to the boundary). Hence the zeroth cohomology group $H^0(X, L^k)$ is just the space of global holomorphic sections and the first cohomology group $H^1(X, L^k)$ measures the obstruction to solve the *inhomogeneous* $\overline{\partial}$-equation:

$$\overline{\partial} u = \alpha, \tag{1.3}$$

for a given form α satisfying the necessary condition $\overline{\partial}\alpha = 0$. The inhomogeneous $\overline{\partial}$-equation typically appears when trying to extend or modify local holomorphic sections into global ones.

Fix a hermitian metric on X (also referred to as the *base metric*). It may be identified with a $(1,1)-$form ω, locally given by $\sum_{i,j} h_{ij} dz_i \wedge \overline{dz_j}$, where h_{ij} is a positive definite Hermitian matrix. The fiber metric on L together with the base metric on X induce L^2-norms on the Dolbeault complex 1.2. When the manifold X is *closed* (i.e. compact without boundary) the classical Hodge theorem gives an isomorphism between the Dolbeault cohomology group $H^q(X, L^k)$ and the space $\mathcal{H}^q(X, L^k)$ of $\overline{\partial}-$harmonic forms

$$H^q(X, L^k) \simeq \mathcal{H}^q(X, L^k). \tag{1.4}$$

The latter space is the null space of the $\overline{\partial}-$Laplacian:

$$\Delta_{\overline{\partial}} := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}, \tag{1.5}$$

where $\overline{\partial}^*$ denotes the formal adjoint of $\overline{\partial}$, which depends on the fiber metric on L and the given metric ω on X. If X has a boundary $\mathcal{H}^q(X, L^k)$ denotes the space of $\overline{\partial}-$harmonic forms satisfying $\overline{\partial}$-Neumann boundary conditions [52] and it may be identified with $H^q(X, L^k)$, under certain curvature assumptions (compare section 4).

2. Line bundles over closed manifolds

As pointed out above the trivial line bundle has only constant holomorphic sections, which follows immediately from the maximum principle. This line bundle is *flat*, i.e. it admits a fiber metric with vanishing curvature form. Similarly, the maximum principle forces all holomorphic sections of a *negative* line bundle to vanish identically. However, when the curvature of L is *positive* the fundamental work of Kodaira, Hörmander established the existence of many holomorphic sections of high powers L^k of the line bundle. The key point is the presence of a positive uniform lower bound for the $\overline{\partial}$-Laplacian acting on $\Omega^{0,1}(X, L^k)$, for k sufficiently large (compare appendix A.1 for a simple example of such a bound). This is equivalent to the fact that the $\overline{\partial}$-equation 1.3 may be solved with an L^2-estimate which is independent of k. As a direct consequence, the obstruction group $H^1(X, L^k)$ vanishes if k is large. In fact, Kodaira's vanishing theorem gives vanishing in *all* positive degrees:

$$\dim H^{0,q}(X, L^k) = 0, \; q > 0, \; k >> 1 \tag{2.1}$$

and Kodaira showed that one can produce enough holomorphic sections to embed X in projective space. By Chow's lemma [56] this means the X is a projective *algebraic* manifold. The *Kodaira embedding* maps a point x to the projectivization of the corresponding evaluation functional on $H^0(X, L^{k_0})$, that may be identified with the projective hyperplane in $\mathbb{P}H^0(X, L^{k_0})$ of all sections vanishing at x :

$$X \hookrightarrow \mathbb{P}H^0(X, L^{k_0})^*, \;\; k_0 >> 1, \tag{2.2}$$

(dually, if (Ψ_i) is an orthonormal base for $H^0(X, L^{k_0})$, x is mapped to $(\Psi_1(x) : \Psi_2(x)... : \Psi_N(x))$. The Kodaira embedding theorem may also be formulated without any explicit reference to the line bundle L : if X admits an *integral Kähler metric*[3] then X is projective algebraic. The existence of a Kähler metric leads to very precise L^2-estimates. However, in many natural situations one has to allow the curvature to *degenerate*. In other words, the curvature is only semi-positive. For example, the pull-back of a positive line bundle is flat along the fibers of the map. Hence, if the map is generically one-to-one (e.g. the map obtained by blowing up a point) it gives rise to a semi-positive line bundle that is positive almost every where. There are remarkable recent examples of line bundles with such curvature properties where there is no uniform lower bound for the $\overline{\partial}$-Laplacian on $\Omega^{0,1}(X, L^k)$ [43]. Demailly's holomorphic Morse inequalities [32][33] handled the corresponding analytical difficulties in a new way. Their *weak* version estimate the dimensions of the qth cohomology groups in terms of the curvature of L:

$$\dim H^q(X, L^k) \leq k^n(-1)^q(\frac{i}{2\pi})^n \int_{X(q)} (\partial\overline{\partial}\phi)^n/n! + o(k^n), \tag{2.3}$$

i.e. $X(q)$ is the set where $i\partial\overline{\partial}\phi$ is non-degenerate with exactly q negative eigenvalues, i.e. where the curvature form has *index* q. Demailly's inspiration came from Witten's supersymmetry approach to the classical Morse inequalities on a *real* manifold X, where the role of the fiber-metric ϕ on L is played by a Morse function [101]. When L is semi-positive, the holomorphic Morse inequalities above replace Kodaira's vanishing theorem 2.1 by

$$\dim H^{0,q}(X, L^k) = o(k^n), \tag{2.4}$$

i.e. all cohomology groups of positive degree are "small". Given this result (which was first obtained by Siu [98][92], using a different method) it follows by a standard argument in complex geometry that L is *big* (compare formula 1.1) if it is semi-positive and positive somewhere. Indeed, the asymptotic version of the Riemann-Roch theorem expresses the Euler

[3]The $(1,1)-$form ω representing the metric on X is said to be *Kähler* if it is closed and *integer* if $\frac{i}{2\pi}[\omega] \in H^2(X, \mathbb{Z})$. Equivalently, $\omega = \partial\overline{\partial}\phi$, where ϕ is a fiber metric on an Hermitian holomorphic line bundle L [56].

characteristic of the Dolbeault complex 1.2 as

$$(2.5)\ \sum_{j=0}^{n}(-1)^{n-j}\dim H^j(X,L^k)=k^n(-1)^q(\frac{i}{2\pi})^n\int_X(\partial\overline{\partial}\phi)^n/n!+o(k^n).$$

But 2.4 shows that the contribution to the sum from positive degrees is negligible, giving that the volume of L (formula 1.1) may be expressed as

$$\mathrm{vol}(L)=(\frac{i}{2\pi})^n\int_X(\partial\overline{\partial}\phi)^n/n! \qquad (2.6)$$

if L is semi-positive. In particular, L is big if it is positive somewhere. Moreover, since big line bundles can also be characterized as the line bundles for which the map 2.2 is bimeromorphic [70, 33] this yields a weaker form of the Kodaira embedding theorem for complex manifolds with a semi-positive line bundle, as long as the line bundle is positive somewhere. This is the Grauert-Riemenschneider solved by Siu [98], saying that such line bundles are *Moishezon manifolds* [83](i.e. bimeromorphic to projective algebraic manifolds [83, 87]. There are examples of such manifolds that are not even *Kähler*[4]

Demailly's *strong* Morse inequalities can be seen as a refinement of the Riemann-Roch argument above. These inequalities give bounds on the *truncated* Euler characteristics of the Dolbeault complex:

(2.7)
$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(X,L^k)\leq k^n(-1)^q(\frac{i}{2\pi})^n\int_{X(\leq q)}(\partial\overline{\partial}\phi)^n/n!+o(k^n).$$

For example, the case when $q=1$ can be used to obtain *lower* bounds on the dimension of $H^0(X,L^k)$. To prove that the strong Morse inequalities hold, one first shows that the weak Morse inequalities 2.3 become *equalities* if $\mathcal{H}^q(X,L^k)$ is replaced with the space $\mathcal{H}^q_{\leq\nu_k}(X,L^k)$ spanned by all eigenforms of $\Delta_{\overline{\partial}}$ with eigenvalues bounded by ν_k, where $\nu_k=\mu_k k$ and μ_k is a certain sequence tending to zero:

$$\dim\mathcal{H}^q_{\leq\nu_k}(X,L^k)=k^n(-1)^q(\frac{i}{2\pi})^n\int_{X(q)}(\partial\overline{\partial}\phi)^n/n!+o(k^n) \qquad (2.8)$$

[5]The next point is that the cohomology groups of the Dolbeault complex 1.2 may be obtained as the cohomology groups of a subcomplex that Demailly calls the "Witten $\overline{\partial}-$complex":
(2.9)

$$...\rightarrow\ \mathcal{H}^{q-1}_{\leq\nu_k}(X,L^k)\ \overset{\overline{\partial}}{\rightarrow}\ \mathcal{H}^q_{\leq\nu_k}(X,L^k)\ \overset{\overline{\partial}}{\rightarrow}\ \mathcal{H}^{q+1}_{\leq\nu_k}(X,L^k)\ \rightarrow ...\ ,,$$

(this fact just uses that $\nu_k>0$ and can be seen as a generalization of the Hodge isomorphism 1.4). The fact that it is a subcomplex, follows from

[4]In fact, any Moishezon manifold that is Kähler is automatically projective [83].

[5]this is the approach taken in paper I. In [32] Demailly considers $\nu_k=\epsilon k$ and lets ϵ tendo to zero in the end of the argument.

the commutation relation

$$[\Delta_{\overline{\partial}}, \overline{\partial}] = 0, \tag{2.10}$$

showing that $\overline{\partial}$ preserves eigenforms of $\Delta_{\overline{\partial}}$. Since the dimensions of the components of the Witten $\overline{\partial}-$complex are given by 2.8, a basic homological algebra argument gives the strong Morse inequalities 2.7. Subsequently, proofs based on asymptotic estimates of the *heat* kernel of the $\overline{\partial}-$ Laplacian, were given by Demailly, Bouche and Bismut ([33], [20] and [11]).

2.1. **Paper I.** In this paper a new approach to holomorphic Morse inequalities is introduced, based on asymptotic Bergman kernel estimates. The *Bergman function* of the space $\mathcal{H}^q(X, L^k)$ is defined as

$$B_X^{q,k}(x) := \sum_i |\Psi_i(x)|^2 \tag{2.11}$$

where (Ψ_i) is an orthonormal base for $\mathcal{H}^q(X, L^k)$. This function can be seen as a "dimensional density" of the space $\mathcal{H}^q(X, L^k)$, since integration over X gives

$$\int_X B_X^{q,k} \omega_n = \dim \mathcal{H}^q(X, L^k).$$

(such functions are also called *density of states functions* in the physics literature). In this paper it is shown that $B_X^{q,k}(x)$ may be estimated in terms of *model Bergman functions.* The model Bergman function $B^q_{x,\mathbb{C}^n}$ associated to a point x is obtained by replacing the manifold X with $\mathbb{C}^n$ and the line bundle L with the constant curvature line bundle over $\mathbb{C}^n$ obtained by freezing the curvature of L at the point x. Since $\mathbb{C}^n$ is non-compact all forms are assumed to have finite L^2-norm. More concretely, one may always arrange so that locally around the fixed point x,

$$\phi(z) = \sum_{i=1}^n \lambda_i |z_i|^2 + ..., \quad \omega(z) = \frac{i}{2}\sum_{i=1}^n dz_i \wedge \overline{dz_i} + ... \tag{2.12}$$

where the dots indicate lower order terms and the leading terms are called *model metrics.* Hence, the corresponding model L^2-norm on $\mathbb{C}^n$ is given by

$$\int_{\mathbb{C}^n} |\alpha(z)|^2 e^{-\sum_{i=1}^n \lambda_i |z_i|^2}, \tag{2.13}$$

integrating with respect to the Euclidean measure on $\mathbb{C}^n$. The model harmonic space associated to the point x is just the L^2-null space of the $\overline{\partial}-$Laplacian defined with respect to the norm above.

Theorem 2.1. *Let X be a closed manifold with a given metric ω (more generally, if X is non-compact all forms are assumed to have finite norm). Then the following point-wise bound holds*

$$\limsup_k k^{-n} B_X^{q,k}(x) \leq B^q_{x,\mathbb{C}^n}(0) \tag{2.14}$$

and the left hand side is uniformly bounded by a constant. Moreover,

$$B^q_{x,\mathbb{C}^n}(0) = (\frac{1}{2\pi})^n 1_{X(q)}(x) \left|\det_\omega(\partial\overline{\partial}\phi)_x\right|,$$

where $1_{X(q)}$ is the characteristic function of the set $X(q)$ (compare formula 2.3 and below). In particular, outside $X(q)$:

$$B^{q,k}_X(x) = o(k^n). \tag{2.15}$$

Integrating the inequality over X we get Demailly's inequalities 2.3 (using the uniform bound to interchange the limits). The proof of the theorem is based on an extremal characterization of the Bergman function. For example, when $q = 0$ the following classical expression holds:

$$B^{0,k}_X(x) = \sup_\alpha |\alpha(x)|^2, \tag{2.16}$$

where the supremum is taken over all sections α in $H^0(X, L^k)$ of unit norm. A similar extremal characterization holds for higher degree forms, if the different components of the forms are taken into account. This gives a convenient way to localize the problem, since one just has to estimate the limiting value of a sequence of forms α_k at a fixed point. To this end one notes the the parameter k introduces a natural "length scale" into the problem of the order $k^{-1/2}$. Indeed, on a fixed ball of radius slightly larger than $k^{-1/2}$ the fiber metrics $k\phi$ on L^k tend, according to formula 2.12, to the model fiber metric (and similarly for the metric ω on X). Now restricting the sequence of forms to such shrinking balls one obtains after "magnification", i.e. after scaling the coordinates z by a factor $k^{1/2}$, a sequence of forms on expanding balls in $\mathbb{C}^n$. It is not hard[6] to see that the sequence converges to a model harmonic form in $\mathbb{C}^n$, which is hence a contender for the supremum in formula 2.16 in the model case. This proves the upper bound in the theorem. Finally, the model harmonic space is computed explicitly giving the last part of the theorem.

Similarly, a local version of Demailly's dimension formula 2.8, showing that the upper bound in theorem 2.1 becomes an asymptotic equality if the corresponding Bergman function of the space $\mathcal{H}^q_{\leq\nu_k}$ is used instead.

Theorem 2.2. *Let X be a closed manifold with a given Hermitian metric ω. Then the following limit holds point-wise*

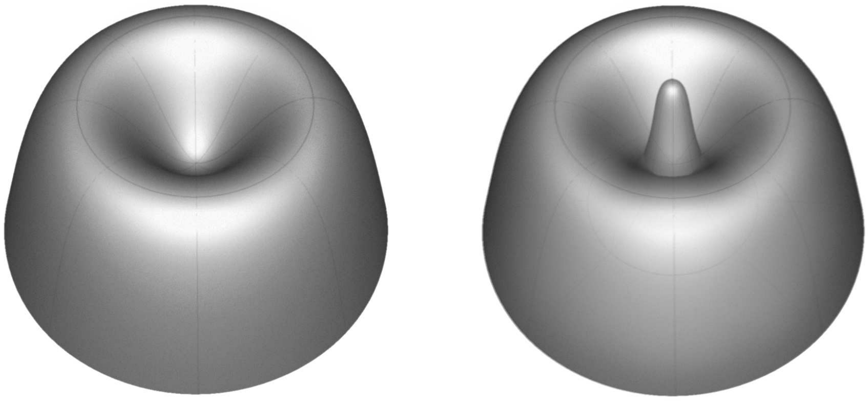
$$\lim_k k^{-n} B^{q,k}_{\leq\mu_k k}(x) = (\frac{1}{2\pi})^n 1_{X(q)}(x) \left|\det_\omega(\partial\overline{\partial}\phi)_x\right|,$$

for some sequence μ_k tending to zero.

The upper bound is obtained as before, since the scaled forms still converge to harmonic model forms as long as μ_k tends to zero. The lower bound is obtained by showing that an extremal model form gives rise to an extremal form of $\mathcal{H}^q_{\leq\nu_k}(X, L^k)$. The latter is obtained by first smoothly extending the former and then orthogonally projecting it on $\mathcal{H}^q_{\leq\nu_k}(X, L^k)$. The point is that there are no obstructions to produce elements of $\mathcal{H}^q_{\leq\nu_k}(X, L^k)$, as opposed to $\overline{\partial}-$harmonic forms.

[6]just using local elliptic estimates for the $\overline{\partial}$-Laplacian

FIGURE 2.1. Starting with a line bundle L of negative curvature a local deformation yields a new fiber metric on L with positive curvature close to a fixed point x as in the left picture below. The local situation then looks like in picture 1.1, where the section α now corresponds to the section saturating the equality 2.16 in the model case at x (compare the right picture below). However, α cannot be perturbed to a *global* holomorphic section (since $H^0(X, L^k)$ is trivial for a line bundle admitting a globally negatively curved metric).



Theorem 2.2 (and its proof) may be conveniently formulated in terms of the local scaling map

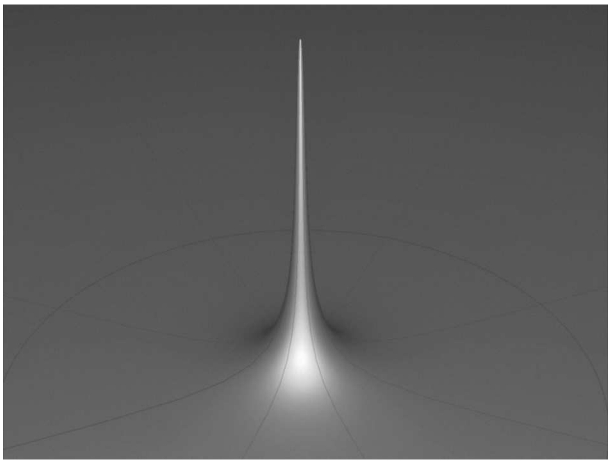
$$F_k(z) = k^{-1/2}z \tag{2.17}$$

centered at a fixed point x in X. Replace the base metric ω on X by $k\omega$ for each fixed k (this has the effect of multiplying the Bergman kernels by k^{-n}). The point is that the corresponding model metrics $k\omega_0$ and $k\phi_0$ are invariant under the pull-back of the scaling map F_k, as are the model Bergman functions. From this point of view the proof of theorem 2.2 shows that the general Bergman function $B^{q,k}_{\leq\mu_k k}$ tends to the corresponding scale-invariant model at each fixed point of X when k tends to infinity. As pointed out in [13], in the case of holomorphic sections of a positive line bundle, such fixed point formulations are related to a long tradition of (largely heuristic) scaling and universality results in statistical mechanics [51].

3. Bergman kernel asymptotics and Toeplitz operators

If L is positive then the lower bound for the $\overline{\partial}-$Laplacian on $\Omega^{0,1}(X, L^k)$ (compare the beginning of section 2) yields a *spectral gap* for the $\overline{\partial}$-Laplacian on $\Omega^{0,0}(X, L^k)$, saying that its positive eigenvalue is larger than a constant times k. Combining this fact with theorem 2.2 immediately gives an asymptotic equality for the Bergman function (or measure)

FIGURE 3.1. The Bergman kernel $K_x^{0,k}(y)$, gives rise to a sequence of *peak sections* of $H^0(X, L^k)$ peaking at the point x.



of $H^0(X, L^k)$ that may be formulated as

$$\lim_k k^{-n} B_X^{0,k} \omega^n = (\frac{i}{2\pi})^n (\partial\overline{\partial}\phi)^n \tag{3.1}$$

More generally, it is important to study the whole *Bergman kernel* $K_X^{0,k}(x, y)$ of $H^0(X, L^k)$, whose point-wise norm on the diagonal is the Bergman function $B_X^{0,k}(x)$. The Bergman kernel $K_X^{0,k}(x, y)$ takes values in $\bar{L}_x^k \otimes L_y^k$ and can be characterized either "intrinsically" as a reproducing kernel for the Hilbert space $H^0(X, L^k)$, i.e. $K_x^{q,k}$ represents point evaluation at x :

$$\alpha(x) = \left\langle \alpha, K_x^{0,k} \right\rangle \tag{3.2}$$

or "extrinsically" as the integral kernel of the orthogonal projection $P^{0,k}$ (the *Bergman projection*) from the smooth sections of L^k onto the holomorphic ones:

$$P^{0,k} : \Omega^{0,0}(X, L^k) \rightarrow H^0(X, L^k) \tag{3.3}$$

In terms of an orthonormal base (Ψ_i) for $H^0(X, L^k)$,

$$K_X^{0,k}(x, y) = \sum_i \overline{\psi_i(x)} \otimes \psi_i(y). \tag{3.4}$$

If L is positive the convergence 3.1 may be generalized to give the weak leading asymptotics of $K(x, y)$in terms of a delta function (or rather distribution) supported on the diagonal:

$$\lim_k k^{-n} \left| K_X^{0,k}(x, y) \right|^2 = (\frac{1}{2\pi})^n \left| \det_\omega (\partial\overline{\partial}\phi)_x \right| \delta(x - y) \tag{3.5}$$

Hence, $K_x^{0,k}$, gives rise to a sequence of *peak sections* peaking at the point x (compare figure 3.1). Peak sections were used by Tian [100] to show that the Kodaira map is "almost" an isometry/symplectomorphism in a suitable sense.[7] Similar peak sections were used by Donaldson [42, 41]

[7]The point is that if the curvature form $\partial\overline{\partial}\phi$ is positive it induces both an Hermitan mertric on X and a s*ymplectic form,* i.e. a non-degenerate real two-form.

in *symplectic geometry* and subsequently by Shiffman-Zelditch [96] to extend the Kodaira embedding theorem to the symplectic setting. The point is that the map 2.2 may be represented by mapping x to the section K_x.

In fact, much stronger results than 3.1 are known in the case when L has positive curvature, giving a *complete asymptotic expansion* in powers of k :

$$B_X^{0,k}(x) \sim k^n(b_0(x) + b_1(x)k^{-1} + ...) \tag{3.6}$$

and similarly, including a phase-function, for $K_X^{0,k}(x,y)$ (compare the references at the end of the section). The lower-order terms depend not only on the curvature of the fiber metric on L, but also on the curvature (and its covariant derivatives) of the metric ω on X. The expansion above can be seen as a point-wise version of the Riemann-Roch theorem, which combined with Kodaira's vanishing theorem, formula 2.1 gives an *exact* formula for the dimension of $H^0(X, L^k)$ (for k sufficiently large), refining formula 2.5:

$$\dim H^0(X, L^k) = \int_X Td(X) \wedge e^{kc_1(L)}, \tag{3.7}$$

where $c_1(L)$ is the first Chern class of L that may be represented by $\frac{i}{2\pi}\partial\overline{\partial}\phi$. The *Todd polynomial* $Td(X)$ is a characteristic class of the holomorphic tangent bundle of X and may be represented by a differential form constructed from the curvature of the metric ω on X, according to Chern-Weil theory [56][7]. The integral in formula 3.7 should be interpreted as the *super integral* of a differential form, i.e. the usual integral of its top degree component:

$$\int_X Td(X)\wedge e^{kc_1(L)} := \int_X k^n T_0 \frac{c_1(L)^n}{n!} + k^{n-1}T_2 \wedge \frac{c_1(L)^{n-1}}{(n-1)!} + ... + T_{2n} \tag{3.8}$$

where $Td(X)$ and $e^{kc_1(L)}$ have been expanded with respect to form degree.

The convergence in the C^2-topology was first obtained by Tian [100], thereby proving a conjecture of Yau [103] and increasingly many terms where then computed by Ruan [91] and Lu [73], using Tian's method of peak sections. Subsequently, the existence of the complete asymptotic expansion was obtained independently by Zelditch [102] and Catlin [27], by first reducing the problem to the study of the singularity structure of the Szegö and Bergman kernels for *functions* on the circle and disc bundles in the total space of the dual bundle L^*. Then the microlocal analysis of Boutet de Monvel-Sjöstrand [25] (which in turn extended the seminal work [50] of Fefferman) yields the expansion 3.6. See also [89] [65] [28, 29] for relations to *star products* and *deformation quantization* and ([14, 15], [30] for relations to the problem of quantization of membranes in *M-theory*). Lately, many new results on asymptotic expansions of Bergman kernels in the *orbifold* and *symplectic setting* have been obtained stimulated by the spectacular applications to *Kähler geometry* introduced by Donaldson [39, 40], Tian and others (compare [99]). See for example

[31][75] for an approach to Bergman kernel expansions using heat kernels motivated by local index theory [12] and [16, 17][95][74] for other approaches. Very recently a new and simple proof of 3.6 and the corresponding result for $K_X^{0,k}(x,y)$ was obtained jointly with Berndtsson and Sjöstrand [8, 94], building on [93]. In particular, a new efficient algorithm to compute the coefficients of the expansion was obtained. However, all of these approaches to 3.6 use the existence of a positive lower bound for the $\overline{\partial}-$Laplacian on $\Omega^{0,1}(X,L^k)$, which is absent in the general case of a line bundle L whose curvature may degenerate on parts of the manifold X.

3.1. **Paper II.** In this paper the results on the leading Bergman kernel asymptotics for positive line bundles are extended in two directions: to line bundles with weaker curvature properties than positivity, such as semi-positivity and to $\overline{\partial}-$harmonic $(0,q)-$ forms with values in L^k. To emphasize the analogy between holomorphic sections and harmonic forms, rudiments of the theory of *super manifolds* [38][27] is used. The main point of the paper is to show that the method of local Morse inequalities from paper I may be used to obtain the leading asymptotics of Bergman kernels, in situations where there is no lower bound on the $\overline{\partial}-$Laplacian. For example, it is shown that if L is *semi-positive* the convergence 3.1 still holds on $X(0)$, i.e the part of X where the curvature of L is positive.

The "extrinsic" definition (preceding formula 2.16) of the Bergman kernel of the space carries over immediately to the space $\mathcal{H}^q(X,L^k)$.[8] Locally, the Bergman kernel $K_X^{q,k}(x,y)$ is a matrix and $B_X^{q,k}(x)$ is the trace of its restriction to the diagonal. In paper II the "intrinsic" definition 3.2 is extended to the space $\mathcal{H}^q(X,L^k)$, so that the Bergman kernel becomes a bundle valued *form* on $X\times X$, denoted by $\mathbb{K}^{q,k}(x,y)$ (which corresponds to replacing the tensor product in 3.4 by a wedge product). Its restriction to the diagonal gives rise to the *Bergman form* $\mathbb{B}_X^{q,k}(x)$ (using the fiber metric on L) which is a $(q,q)-$form on X. It turns out that $\mathbb{B}_X^{q,k}(x)$ is not only concentrated to $X(q)$ (compare formula 2.15) but all of its leading asymptotic contribution comes from a special "direction" (i.e. a special component of the form). The point is that in the model case (compare formula 2.12), when the eigenvalues λ_i are ordered so that precisely the first q eigenvalues of the curvature form are negative, any $\overline{\partial}-$harmonic $(0,q)-$ form may be written as

$$\beta(z)\overline{dz_1}\wedge\cdots\wedge\overline{dz_q},$$

for a *function* $\beta(z)$ (which is holomorphic with respect to a new complex structure on $\mathbb{C}^n$). To make this precise a (q,q)-form $\chi^{q,q}$ called the *direction form* with support on $X(q)$ is introduced. This form is naturally determined (using the metric ω on X) by the rank q subbundle of the

[8] stictly speaking, when $q>1$ this is a *generalized* Bergman kernel (compare the notation in [75]). In [9] the term Bergman-Hodge kernel was introduced for such a kernel.

holomorphic tangent bundle of X consisting of the "negative directions" of the curvature form of L. [9]It is shown that

$$k^{-n}\mathbb{B}_X^{q,k}(x) \to (\frac{1}{2\pi})^n \chi^{q,q} \left|det_\omega(\partial\overline{\partial}\phi)_x\right|$$

weakly if $X(q-1)$ and $X(q+1)$ are empty (generalizing the condition that L be positive in the case of holomorphic sections). The direction form $\chi^{q,q}$ can be used to transform a differential form f on X to a *function* f_χ with support on $X(q)$. Now the weak convergence of the whole Bergman kernel form may be formulated, using the super integral (compare formula 3.8), in the following way.

Theorem 3.1. *Let $\mathbb{K}_X^{q,k}$ be the Bergman kernel form of the Hilbert space $\mathcal{H}^q(X, L^k)$ and suppose that f and g are differential forms on X. If $X(q-1)$ and $X(q+1)$ are empty, then*

$$k^{-n}(\frac{i}{2})^n \int_{X\times X} f(x)\wedge g(y)\wedge \mathbb{K}_X^{q,k}(x,y)\wedge \mathbb{K}_X^{q,k\dagger}(x,y)\wedge e^{\Phi_k(x,y)} \tag{3.9}$$

tends to

$$(\frac{1}{\pi})^n \int_X f_X g_X (\partial\overline{\partial}\phi)^n/n!,$$

where $\Phi_k(x,y) = -k\phi(x) - k\phi(y) - 2i\omega(x) - 2i\omega(x)$, *using the super integral and "reversed complex conjugation" † (compare section 3.1 in paper II).*

When the curvature is non-degenerate, i.e. $X = X(q)$ for some q, the existence of a complete asymptotic expansion (including a phase-function) of $K_X^{q,k}(x,y)$ was obtained very recently, using semi-classical analysis, in joint work with Sjöstrand [9, 94], building on [81]). The expansion was obtained independently in [76] (without a phase-function). The relation to *almost holomorphic sections* [96] on symplectic manifolds, defined with respect to a new almost complex structure on X was also explored in [9].

In the case of holomorphic sections of *positive* line bundles, the Bergman kernel asymptotics, were used by Lindhom [72] to study Toeplitz operators, when the complex manifold X is $\mathbb{C}^n$ (see also [10] for the extension to the case when X is closed). Given a function f on X the *Toeplitz operator with symbol f*, is the operator acting on $H^0(X, L^k)$, defined by

$$T_f := P^{0,k} \circ f\cdot, \tag{3.10}$$

where $f\cdot$ denotes the usual multiplication operator and $P^{0,k}$ is the projection 3.3. The operator T_f is also called the *Berezin-Toeplitz quantization* of the function f on X (see for example [1, 28, 65]). In [72][10] it was shown how to obtain the asymptotic distribution of the eigenvalues of a Toeplitz operator, when k tends to infinity, in terms of the symbol of the operator (this was first shown by Boutet de Monvel-Guillemin [24],

[9]In the usual real Morse theory [82] this bundle corresponds to the tangent spaces of the unstable manifolds over the set of critical points of index q.

extending the classical limit theorem of Szegö [58]). In paper II the following generalization to *super* Toeplitz operators acting on $\mathcal{H}^q(X, L^k)$, whose symbol is a differential form on X, is obtained (see paper II, section 4, for the precise definitions):

Theorem 3.2. *Let (τ_i) be the eigenvalues of the super Toeplitz operator T_f with symbol f and denote by $d\xi_k$ the spectral measure of T_f divided by k^n, i.e.*

$$d\xi_k := k^{-n} \sum_i \delta_{\tau_i}, \tag{3.11}$$

Then $d\xi_k$ tends, in the weak-topology, to the push forward of the measure $(\frac{i}{2\pi})^n (\partial\overline{\partial}\phi)/n!$ under the map f_X.*

Similar super Toeplitz operators were studied in [23] in the context of *quantization of symplectic super manifolds*[10]. The theorem may also be expressed in terms of the spectral counting function $N(T_f > \gamma)$ which counts the number of eigenvalues of T_f (including multiplicity) that are greater than a given number γ. For example, when L is a semi-positive line bundle and T_f acts on its holomorphic sections one obtains (for almost all γ) :

$$\lim_k k^{-n} N(T_f > \gamma) = (\frac{i}{2\pi})^n \int_{\{f>\gamma\} \bigcap X(0)} (\partial\overline{\partial}\phi)^n/n! \tag{3.12}$$

Taking f as the characteristic function of a set U in $X(0)$ yields a large supply of holomorphic sections *concentrated on* U in the sense that

$$\|\alpha\|_\Omega^2 \geq (1-\varepsilon) \|\alpha\|_X^2$$

for a given positive (small) number ϵ. Indeed, if $\gamma = 1-\varepsilon$ then the eigensections of the corresponding Toeplitz operator satisfy the concentration property above.

Following [72][10] in the case of a positive line bundle (and originally [68, 69] in the case of the classical Paley-Wiener spaces on $\mathbb{R}^n$), such concentrated sections are used to study the problem of *sampling* of holomorphic sections, i.e. the problem of stably determine any section from its values at the sampling points. For each k one chooses a sampling set D_k in X. The sampling points are assumed to be uniformly separated by a distance of the order $k^{-1/2}$ (so that the total number of sampling points matches the order k^n of the dimension of $H^0(X, L^k)$). More precisely, the sequence of sets D_k is said to be *sampling* for the sequence of Hilbert spaces $H^0(X, L^k)$ if there exists a uniform constant A such that

$$A^{-1} k^{-n} \sum_{D_k} |\alpha(x)|^2 \leq \|\alpha\|^2 \leq A k^{-n} \sum_{D_k} |\alpha(x)|^2,$$

for any element α in $H^0(X, L^k)$. The model example of a sequence of sampling sets is a sequence of lattices $k^{-1/2}\lambda\mathbb{Z}^n + ik^{-1/2}\lambda\mathbb{Z}^n$ in $\mathbb{C}^n$.

[10]However, in [23] T_f acts on the space of all *holomorphic* forms with values in L^k (on certain symmetric spaces), rather than $\mathcal{H}^q(X, L^k)$ (compare remark 4.2 in Paper III).

Theorem 3.3. *Suppose that L is semi-positive (or more generally, that $X(1)$ is empty). Then a necessary condition for a sequence D_k to be sampling is that the number of sampling points in U is greater than*

$$k^n(\frac{i}{2\pi})^n \int_U (\partial\overline{\partial}\phi)^n/n! + o(k^n)$$

for any given open subset U of $X(0)$.

Equivalently, the sequence D_k gives rise to a sequence of (normalized) currents on X with support on D_k and the theorem says that the limiting current is bounded from below by $(\frac{i}{2\pi})^n(\partial\overline{\partial}\phi)^n/n!$ on $X(0)$, if D_k is sampling.

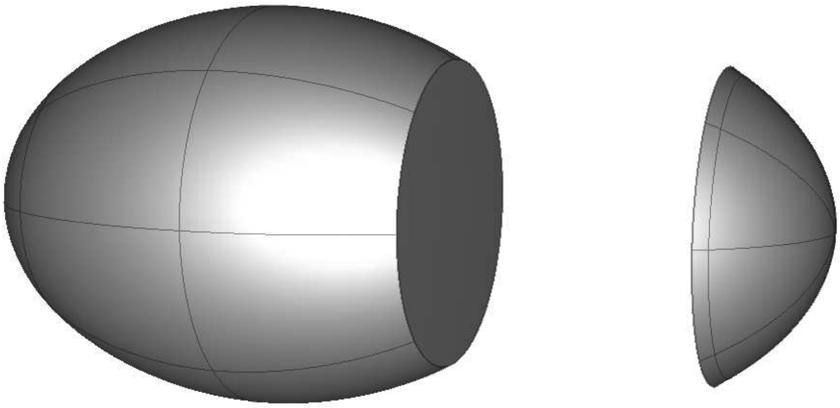
4. Line bundles over manifolds with boundary

Now consider the situation when the complex manifold X has a boundary ∂X. If the curvature $\partial\overline{\partial}\rho$ of ∂X is *positive* (a basic example is given by the unit ball in $\mathbb{C}^n$), then the dimension of the space of holomorphic functions on X is infinite. This is the classical theory of *pseudoconvex manifolds*, developed by Oka, Levi, Cartan and others. For example, the interior of such a manifold may be mapped bimeromorphically and properly into some $\mathbb{C}^N$ and even biholomorphically if X is a *Stein manifold*[11] (the Bishop-Remmert-Narasimhan theorem [61]). The study of the boundary behaviour of the Bergman kernel of the space of square integrable holomorphic functions on pseudoconvex manifolds has a long history. Its general leading behaviour was first determined by Hörmander [60]. The existence of the complete asymptotic expansion (in powers of the defining function of the boundary) was then obtained by Fefferman [50] on the diagonal and subsequently by Boutet de Monvel-Sjöstrand [25] complete generality.

In the opposite case when the curvature of ∂X is *negative*, i.e. X is *pseudoconcave*, one has to introduce a line bundle L over X to get non-trivial holomorphic sections. For example, such a situation appears if one starts with a positive line bundle over a closed manifold (of dimension at

[11]this is the case if ρ may be chosen such that $\partial\overline{\partial}\rho > 0$ on *all* of X. In general on open manifold is said to be a Stein manifold if it carries a strictly plurisubharmonic exhaustion function.

least two) and makes a "hole" in the manifold by removing a small ball.



This is the case when X is the complement in $\mathbb{P}^n$ of the unit ball and L is the restriction of the hyperplane line bundle $\mathcal{O}(1)$. In this particular case X may also be described as a neighbourhood of the hyperplane at infinity in $\mathbb{P}^n$, consisting of complex deformations of the hyperplane. More generally, any neighbourhood X of a complex hypersurface with positive normal bundle in an ambient manifold gives rise to a similar situation [55, 48].

However, in the general case, when the curvature of L is positive and the curvature of ∂X is negative, it is not even known if a sufficiently high power of L has any holomorphic sections at all. As in the case of *degenerate* curvature in section 2, the analytical problem - now coming from the presence of both positive and *negative* curvature - is the lack of a spectral gap for the $\overline{\partial}-$Laplacian. This problem is closely related to the well-known fact that in the global L^2- estimates for the $\overline{\partial}-$operator of Morrey-Kohn-Hörmander-Kodaira there is a curvature term from the line bundle as well from the boundary and, in general, it is difficult to control the sign of the total curvature contribution. Still, the following conjecture has been made by Marinescu [77] and Henkin [59].

Conjecture 4.1. *If X is pseudoconcave (i.e. its boundary has negative curvature) and of complex dimension at least three, then any positive line bundle over L is big, i.e.*

$$\dim H^0(X, L^k) \sim k^n.$$

The upper bound holds for *any* line bundle over X. according to a classical result of Andreotti [2]. By a well-known theorem of Rossi [90] any pseudoconcave manifold of dimension at least three may be *filled*, i.e. embedded as a manifold with boundary in a closed manifold $\widetilde{X}$ of the same dimension as X (in dimension two there are counterexamples[12]). However, the conjecture above is related to the problem of extending the line bundle L as a (semi-) positive line bundle to $\widetilde{X}$. This problem

[12]for example obtained by starting with the complement of the unit ball in $\mathbb{P}^2$ and deforming the standard complex structure close to the boundary three sphere [90, 54].

is similar to various filling problems in symplectic and contact geometry [45, 49]. The conjecture is also closely related to the problem of extending the Kodaira embedding theorem 2.2 to a pseudoconcave manifold (which turns out to be equivalent to the the problem whether $\widetilde{X}$ above may be chosen to be a projective algebraic manifold [2, 5]). Embedding and deformation problems of pseudoconcave manifolds and of *Cauchy-Riemann (CR) manifolds* (realized as the boundary of a pseudoconcave manifold) is a very active area of current research (the surface case is investigated by Epstein-Henkin in [48], see also [71]) extending the work of Andreotti, Tomassini, Siu, Grauert, Kuranishi and others from the beginning of the seventies (see for example [5, 4]). See also [86] for relations to Penrose's *twistor theory.*

The notion of a pseudoconcave manifold can also be defined for any *open* manifold Y (i.e. not realized as the interior of a manifold X with boundary), following Andreotti-Grauert [3, 54]. Essentially, Y is an (open) pseudoconcave manifold if it can be exhausted by pseudoconcave manifolds with boundary, i.e. if there exists an *exhaustion function* with certain concavity properties. The simplest example is obtained by removing a point (instead of a ball, as before) from a closed manifold. More generally, removing an analytic variety gives an open manifold with certain concavity or convexity properties [54]. If the original closed manifold is a projective manifold, the resulting open manifold is called *quasi-projective.* Conversely, it is natural to ask when a pseudoconcave manifold can be "compactified", for example realized as a quasi-projective manifold? A necessary assumption to be quasi-projective is the existence of a *complete Kähler metric* (see [84] for a survey on similar compatification problems, including the relations to the Baily-Borel-Satake compactifications of arithmetic quotients [23] and [37, 79] for the "hyperconcave" case)

The study of holomorphic Morse inequalities on manifolds with boundary was initiated by Bouche [20] and Marinescu [77] and they obtained the same curvature integral, formula 2.3, as in the case when X has no boundary. However, it was assumed that, close to the boundary, the curvature of the line bundle L is adapted to the curvature of the boundary, which excludes the interesting mixed curvature cases above. Morse inequalities over strictly pseudoconvex CR manifolds were obtained by Getzler [53], who also suggested that one should try to prove similar formulas for the $\overline{\partial}-$Neumann problem on a complex manifold with boundary. This is achieved in paper III. Morse inequalities on CR manifolds have also been announced by Ponge very recently [88].

4.1. **Paper III.** In this paper Demailly's weak holomorphic Morse inequalities are generalized to any line bundle L over a manifold X with (non-degenerate) boundary. To motivate the appearance of a boundary term in the inequality, consider the unit disc D in $\mathbb{C}$ and the Hilbert space

$H_k(D)$ of polynomials of degree at most k with the usual Euclidean norm

$$\frac{i}{2}\int_D |\alpha(\zeta)|^2 \, d\zeta \wedge d\bar{\zeta} \tag{4.1}$$

Hence, the space $H_k(D)$ is the restriction to D of the space $H^0(\mathbb{P}^1, \mathcal{O}(1)^k)$ with a particular norm. Denote by B_k the Bergman function of the Hilbert space $H_k(D)$ (note that the fiber-metric $\phi \equiv 0$). Clearly,

$$(i) \lim_k k^{-1} B_k(\zeta) = 0, \quad (ii) \lim_k \int_D k^{-1} B_k(\zeta) = 1 \tag{4.2}$$

Indeed, (i) essentially follows from theorem 2.1 (since $\partial\bar{\partial}\phi \equiv 0$) and (ii) follows from the fact that the dimension of $H_k(D)$ is equal to $k+1$. But the point is that the limit (i) is *not uniform* (as the boundary is approached k has to be increased in order for the balls of radius $k^{-1/2}$, used in the proof of theorem 2.1, to fit inside D). Hence, combining (i) and (ii) shows that the mass of the *Bergman measure* $B_k(\zeta)(\frac{i}{2} d\zeta \wedge d\bar{\zeta})$ divided by k, is pushed out towards the boundary. By rotational invariance it tends to the usual invariant measure supported on the boundary of D, i.e. on the unit circle (compare section A.3 in the appendix for further details on this example).

In order to state the next theorem, recall that $X(q)$ is the subset of X where index$(\partial\bar{\partial}\phi) = q$ (compare the definition following formula 2.3) and let $T(q)_{\rho,x}$ be the set of all positive numbers t such that $\partial\bar{\partial}\phi + t\partial\bar{\partial}\rho$ has index q along $T^{1,0}(\partial X)_x$.

Theorem 4.2. *Suppose that X is a compact complex manifold with boundary, such that the Levi curvature form is non-degenerate on the boundary. Then the dimension of $H^q(X, L^k)$ is bounded by*

$$k^n(-1)^q\left(\frac{i}{2\pi}\right)^n \frac{1}{n!}\left(\int_{X(q)} (\partial\bar{\partial}\phi)^n + n\int_{\partial X}\int_{T(q)_{\rho,x}} (\partial\bar{\partial}\phi + t\partial\bar{\partial}\rho)^{n-1} \wedge \partial\rho \wedge dt\right) \tag{4.3}$$

up to terms of order $o(k^n)$.

Morse inequalities on $q-$concave manifolds were also independently obtained by Marinescu in a recent preprint [78], but the boundary term was not as precise as the one above. Note that the boundary integral above is finite precisely when there is no point in the boundary where the Levi form $i\partial\bar{\partial}\rho$ has exactly q negative eigenvalues. Indeed, any sufficiently large t will then be in the complement of the set $T(q)_{\rho,x}$ (the case when the Levi form has exactly q negative eigenvalues *everywhere* on the boundary and the line bundle L is trivial was recently studied by Hörmander in [63]). The corresponding generalization of the *strong* holomorphic Morse inequalities (including the case of *open* manifolds) is also obtained in this paper, assuming as usual that suitable $q-$convexity or $q-$concavity holds.

The boundary term in theorem 4.2 expresses the interplay between the two curvature forms $\partial\bar{\partial}\phi$ and $\partial\bar{\partial}\rho$ of the line bundle L and the boundary ∂X, respectively, along $T^{1,0}(\partial X)$. As an application of theorem 4.2

(or rather of the strong version of the inequalities) pseudoconcave manifolds X of dimension $n \geq 3$ with a positive line bundle L are studied. It is shown that if it is further assumed that the curvature forms are *conformally equivalent* i.e.

$$\partial\overline{\partial}\rho = -f\partial\overline{\partial}\phi \tag{4.4}$$

along $T^{1,0}(\partial X)$ for some positive function f on ∂X, then, up to terms of order $o(k^n)$,

(4.5)
$$\dim H^0(X, L^k) = k^n(\frac{1}{2\pi})^n(\int_X (i\partial\overline{\partial}\phi)^n/n! + \frac{1}{n}\int_{\partial X}(i\partial\overline{\partial}\rho)^{n-1}\wedge\partial\rho)/(n-1)!$$

if the defining function ρ is chosen in an appropriate way. In particular, such a line bundle L is *big* (which confirms the conjecture 4.1 in this case) and the dimension formula above, generalizing formula 2.6, can be expressed as

$$\mathrm{Vol}(L) = \mathrm{Vol}(X) + \frac{1}{n}\mathrm{Vol}(\partial X)$$

in terms of the corresponding symplectic volume of X and contact volume of ∂X.

Examples are provided, showing that the leading constant in theorem 4.2 is sharp. In particular, in the generic examples, i.e. when the condition 4.4 does not hold,

$$\dim H^1(X, L^k) \sim Ck^n$$

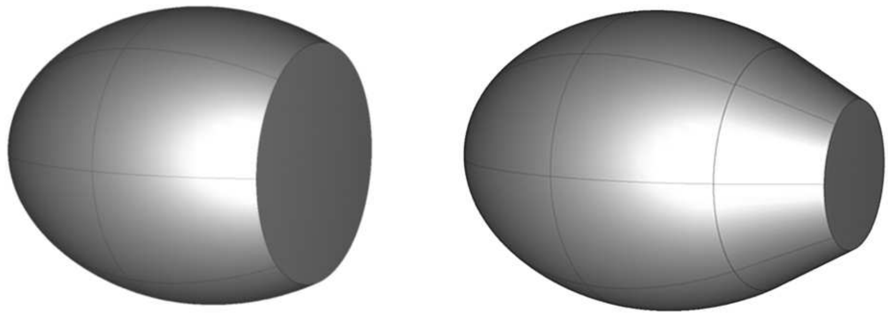
when L is a positive line bundle over a pseudoconcave manifold X, in contrast to the case when X is closed. The theorem is also shown to be sharp, as soon as the pair (X, L) may be *filled,* i.e. when L extends to a semi-positive line bundle over a *closed* manifold. The confirmation of this fact is based on the observation that the boundary integral in the theorem may also be expressed more intrinsically in terms of symplectic geometry as

$$\int_{X_+(q)} (d\gamma + i\partial\overline{\partial}\phi)^n/n!, \tag{4.6}$$

where $(X_+, d\gamma)$ is the *symplectification* of the *contact manifold* ∂X induced by the complex structure of X [6] (compare figure 4.1)

The proof of theorem 4.2 follows from local estimates for the corresponding Bergman *measure* $B_X^{q,k}\omega^n/n!$, where $B_X^{q,k}(x)$ is the Bergman function of the space $\mathcal{H}^{0,q}(X, L^k)$ of $\overline{\partial}-$harmonic $(0,q)-$forms satisfying $\overline{\partial}-$Neumann boundary conditions (which is isomorphic to $H^q(X, L^k)$ when the boundary integral in 4.3). It is shown that, for large k, the Bergman function (or more precisely the corresponding measure) is estimated by the sum of two model Bergman measures, giving rise to the bulk and the boundary integrals in theorem 4.2. The model at a point x in the interior of X is the same as in the case when X is closed (compare the overview of paper I above). Similarly, the model at a boundary point is obtained by replacing X with the unbounded domain X_0 in $\mathbb{C}^n$, whose

FIGURE 4.1. Assume that X is a pseudoconcave manifold (the left picture) with a positive line bundle L. In the right picture the subset $X_+(0)$ of the symplectification X_+ has been attached to the original manifold X along ∂X. The form $d\gamma + i\partial\overline{\partial}\phi$ on $X_+(0)$ coincides with i times the curvature form $\partial\overline{\partial}\phi$ of L along the contact distribution $T^{1,0}(\partial X)$ and $(d\gamma + i\partial\overline{\partial}\phi)^n$ vanishes on the boundary of $X \sqcup X_+(0)$.



constant Levi curvature is obtained by freezing the Levi curvature at the boundary point in X. The line bundle L is replaced by the constant curvature line bundle over X_0, obtained by freezing the curvature along the complex tangential directions, while making it flat in the complex normal direction. Concretely, the model at a boundary is given by the domain with defining function

$$\rho_0(z,w) = \mathrm{Im} w + \sum_{i=1}^{n-1} \mu_i |z_i|^2, \tag{4.7}$$

and with fiber metric ϕ_0 and metric ω_0 given by

$$\phi_0(z,w) = \sum_{i=1}^{n-1} \lambda_i |z_i|^2, \quad \omega_0 = \frac{i}{2}\sum_{i=1}^{n-1} dz_i \wedge \overline{dz_i} + 2i\partial\rho_0 \wedge \overline{\partial\rho_0}$$

respectively (more precisely, a slightly more general metric ω_0 is needed).

The method of proof is an elaboration of the technique introduced in paper I to handle Demailly's case of a manifold without boundary. The major new technical challenge that appears comes from the fact that the geometry close to the boundary introduces a new "length scale" of the order k^{-1} (in the normal direction) as opposed to the interior length scale $k^{-1/2}$. This corresponds to the two different homogeneity/scaling properties of the bulk and boundary models; at the boundary the scaling map 2.17 is replaced by

$$F_k(z,w) = (z/k^{1/2}, w/k).$$

It is shown how to interpolate between these two different length scales as the boundary is approached. The key point is to use a base metric that depends on k in the normal direction close to the boundary. In the model boundary case this corresponds to using the metric

$$\omega_{k,0} := \frac{i}{2}\sum_{i=1}^{n-1} dz_i \wedge \overline{dz_i} + k2i\partial\rho_0 \wedge \overline{\partial\rho_0}$$

which has the important property that $F_k^*(k\omega_{k,0}) = \omega_0$.

Appendix A. Model examples

In the following section some simple examples of Bergman kernels for spaces of holomorphic sections are collected. The aim is to illustrate the general theory and results in the simplest possible situations. The arguments are mostly just sketched (see paper III for detailed proofs in the more general case of $\overline{\partial}-$harmonic forms).

A.1. **Complex space with a constant curvature line bundle.** Consider the Hilbert space $H^0(\mathbb{C},\phi)$ of all holomorphic functions on the Euclidean complex plane $\mathbb{C}$ with the norm

$$\int_{\mathbb{C}} |\alpha(z)|^2 e^{-\phi(z)} \tag{A.1}$$

Taking $\phi(z) = \lambda |z|^2$ corresponds to a (holomorphically trivial) line bundle L_λ with *constant* curvature form $\lambda dz \wedge d\bar{z}$. First assume that λ is *positive* so that the constant function 1 is in $H^0(\mathbb{C},\phi)$ (compare figure 1.1). By symmetry the Bergman function $B(z)$ is constant and applying the reproducing property 3.2 to the constant function 1 fixes the value at the origin, giving

$$B(z) \equiv \frac{\lambda}{\pi}$$

Since, by definition $K(z,z) = B(z)e^{\phi(z)}$ and $K(z,z')$ is anti-holomorphic in the first variable and holomorphic in the second variable it follows that

$$K(z,z') = \frac{\lambda}{\pi}e^{\lambda\bar{z}z'}. \tag{A.2}$$

Note that if λ is replaced by $k\lambda$ then the corresponding Bergman kernels K_k scale as

$$k^{-1}K_k(k^{-1/2}z, k^{-1/2}z') = K_1(z,z')$$

If λ is non-positive the space $H^0(\mathbb{C},\phi)$ is trivial by the L^2-version of Liouville's theorem. If λ is *strictly negative*, the space $H^0(\mathbb{C},\phi)$ is trivial by the usual version of Liouville's theorem, saying that there are no bounded holomorphic functions on $\mathbb{C}$. In this case the $\overline{\partial}$-Laplacian $\Delta_{\overline{\partial}}$ is bounded from below by λ. To see this, first note that for a general fiber metric ϕ over Euclidean $\mathbb{C}$ the formal adjoint $\frac{\partial}{\partial\bar{z}}^*$ is given by

$$\frac{\partial}{\partial\bar{z}}^* = e^{\phi}(-\frac{\partial}{\partial z})e^{-\phi} = -\frac{\partial}{\partial z} + \frac{\partial\phi}{\partial z} \tag{A.3}$$

(which follows directly from the definition A.1 of the norm and a partial integration). In particular, the following important commutation relation holds

$$[\frac{\partial}{\partial\bar{z}}, \frac{\partial}{\partial\bar{z}}^*] = \frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial\bar{z}}^* - \frac{\partial}{\partial\bar{z}}^*\frac{\partial}{\partial\bar{z}} = \frac{\partial^2\phi}{\partial z\partial\bar{z}}.$$

Since the $\overline{\partial}$-Laplacian on functions is given by

$$\Delta_{\overline{\partial}} = \frac{\partial}{\partial\bar{z}}^*\frac{\partial}{\partial\bar{z}}$$

it follows that

$$\langle\Delta_{\overline{\partial}}\alpha, \alpha\rangle = \left\|\frac{\partial}{\partial\bar{z}}^*\alpha\right\|^2 + \left\langle -\frac{\partial^2\phi}{\partial z\partial\bar{z}}\alpha, \alpha\right\rangle$$

[13] proving that $\Delta_{\overline{\partial}}$ is uniformly bounded from below by a positive constant as soon as the curvature $\frac{\partial^2\phi}{\partial z\partial\bar{z}}$ is negative and bounded from above.

Also note that the study of the space $\mathcal{H}^1(\mathbb{C}, \phi)$ of all square integrable $\overline{\partial}$-harmonic $(0,1)$-forms is essentially "dual" to the previous case. The point is that $\mathcal{H}^1(\mathbb{C}, \phi)$ is the null space of $\frac{\partial}{\partial\bar{z}}^*$. For example, the commutation relation A.3 gives a positive lower bound on the corresponding Laplacian if the curvature is *positive* (illustrating Hörmander's and Kodaira's theorems, referred to in the beginning of section 2.

A.2. **The torus with a constant curvature line bundle.** Now replace Euclidean $\mathbb{C}$ by the flat torus $T = \mathbb{C}/(\sqrt{\pi}\mathbb{Z} + i\sqrt{\pi}\mathbb{Z})$. Note that the form $\lambda dz \wedge d\bar{z}$ descends to a closed $(1,1)$ -form on the torus. Moreover, multiplying the form by $i/2\pi$ and integrating over the torus gives λ. Hence if λ is an integer $\lambda dz \wedge d\bar{z}$ is the curvature form of a hermitian holomorphic line bundle L_λ over the torus. Note that $L_\lambda = L_1^\lambda$. When λ is *positive*, the space $H^0(T, L_\lambda)$ of holomorphic sections may be identified with the classical theta functions at level λ [56]. As in the case of the complex plane above

$$B(z) \equiv \frac{\lambda}{\pi} \tag{A.4}$$

The non-trivial part is to prove that L_1 has a non-trivial holomorphic section. If λ is *negative* then the space $H^0(T, L_\lambda)$ is trivial (essentially by the same argument as in the case of the complex plane above). See also remark 7.3 in paper III. Integrating A.4 gives

$$\dim H^0(T, L_\lambda) = \lambda \quad \text{if}\, \lambda > 0. \tag{A.5}$$

and it vanishes if $\lambda < 0$.

[13]Strictly speaking, one has to use a sequence of smooth cut-off functions χ_R such that the norms of the differential $d\chi_R$ are uniformly bounded, to perform the partial integration. In general such cut-off functions exist on manifolds with complete Kähler metrics [57].

A.3. **The unit disc.** Consider the Hilbert space $H^0(D)$ of all holomorphic functions on the unit disc with respect to the usual Euclidean norm (formula 4.1). Taylor expanding an element $\alpha(\zeta)$ around the origin gives

$$\alpha(\zeta) = \sum_{j\geq 0} a_j \zeta^j \tag{A.6}$$

Hence, the monomials ζ^j form an orthogonal bases for $H^0(D)$ with norms given by

$$\left\|\zeta^j\right\|^2 = 2\pi \int_0^1 (r^2)^j r dr = 2\pi(1/2)(j+1)^{-1} \tag{A.7}$$

Formula 3.4 then gives

$$K(\zeta, \zeta') = \frac{1}{\pi} \sum_{j\geq 0} \bar{\zeta}^j \zeta'^j (j+1)$$

Writing this is a differentiated geometric series (on the diagonal) yields the following classical formula

$$K(\zeta, \zeta') = \frac{1}{\pi}(1 - \bar{\zeta}\zeta')^{-2}.$$

In particular K blows up along the diagonal on the boundary. Note that writing

$$\zeta = e^{-\frac{i}{2}wt}$$

induces a local biholomorphism between the unit disc ($|\zeta|^2 < 1$) and the lower half plane ($\text{Im} w < 0$). In the new local coordinates

$$K(w, w') = \frac{1}{\pi} \sum_{j\geq 0} e^{\frac{i}{2}(\overline{w} - w')j}(j+1). \tag{A.8}$$

Note that $a_j = \widehat{\alpha}_j$ in the Taylor expansion A.6 are the Fourier coefficients of the restriction of α to the boundary (i.e. the unit circle) and the fact that α extends to the interior of the disc corresponds to the fact that $\widehat{\alpha}_j = 0$ when $j < 0$).

If we consider the subspace $H_k(D)$ of all polynomials of degree at most k (compare the beginning of section 4.1), then its Bergman kernel $K_k(w, w')$ is simply obtained by by restricting the sum A.8 to all $j \leq k$. Writing

$$k^{-2} K_k(w/k, w'/k) = \frac{1}{\pi} \sum_{\frac{j}{k} \in [0,1]} e^{\frac{i}{2}(\overline{w} - w')\frac{j}{k}} (\frac{j+1}{k}) \frac{1}{k}$$

and letting k tend to infinity so that the sum converges to an integral (formally $j/k \to t$ and $1/k \to dt$), shows that the following scaling asymptotics for K_k hold

$$k^{-2} K_k(w/k, w'/k) \to \frac{1}{\pi} \int_0^1 e^{\frac{i}{2}(\overline{w} - w')t} t dt.$$

In particular, the corresponding Bergman function B_k grows as k^2 when the distance to the boundary is of the order k^{-1}, which is consistent with 4.2.

A.4. **The lower half plane.** Now consider the Hilbert space of all holomorphic functions on the lower half plane with respect to the usual Euclidean norm. Analogous arguments to the previous ones give

$$K(w,w') = \frac{1}{4\pi}\int_0^\infty e^{\frac{i}{2}(\overline{w}-w')}tdt,$$

One just replaces the Taylor expansion A.6, i.e. the corresponding Fourier series, by the "expansion"

$$\alpha(w) = \int_0^\infty \widehat{\alpha}_t e^{-\frac{i}{2}wt}dt,$$

using the Fourier-Laplace transform $\widehat{\alpha}_t$. Moreover, the norm calculation A.7 is now replaced by the formula

$$\langle \alpha, \alpha \rangle = 4\pi \int \langle \widehat{\alpha}_t, \widehat{\alpha}_t \rangle_t 1/tdt, \tag{A.9}$$

where 4π comes from integration with Imw fixed (using the Plancherel formula) and $1/t$ comes from

$$\int_{\mathrm{Im}w<0} e^{(\mathrm{Im}w)t}d(\mathrm{Im}w) = 1/t. \tag{A.10}$$

We now obtain the classical formula

$$K(w,w') = \frac{1}{2\pi i(\overline{w}-w')^2},$$

that may also be written as

$$K = \frac{1}{4\pi}\rho^{-2}, \tag{A.11}$$

using the suggestive notation ρ for the function $\rho(w,w')$, that is anti-holomorphic in the first variable and holomorphic in the second variable (like K) and such that $\rho(w,w)$ is the usual defining function.

A.5. **The model boundary case.** Consider the model boundary domain (compare formula 4.7) and assume for concreteness that the dimension $n=2$:

$$X = \{\rho < 0\} = \left\{(z,w) : \mathrm{Im}w + \mu\left|z\right|^2 < 0\right\},$$

and with fiber metric $\phi(z,w) = \lambda\left|z\right|^2$ on the (holomorphically trivial) line bundle L_λ. The corresponding norm is given by

$$\int_{\mathrm{Im}w+\mu|z|^2<0} \left|\alpha(z,w)\right|^2 e^{-\lambda|z|^2},$$

integrating with respect to the Euclidean measure. Note that X is a half plane bundle over (or rather under) $\mathbb{C}$ with fiber coordinate w and base coordinate z. For a fixed t we denote by K_t the Bergman kernel of the space $H^0(\mathbb{C}_z, (t\mu+\lambda)\left|z\right|^2)$ (section A.1) and by $\langle\cdot,\cdot\rangle_t$ the corresponding scalar product. Fixing z and applying A.9 now gives

$$\langle \alpha, \alpha \rangle_X = 4\pi \int \langle \widehat{\alpha}_t, \widehat{\alpha}_t \rangle_t 1/tdt$$

(the point is that the upper limit in the integral A.10 is replaced by $\mathrm{Im}w + \mu |z|^2$), which gives an extra $t-$dependent factor $e^{-\mu t|z|^2}$). Using this expression for the scalar product on X one can show that

$$K(z,w;z',w') = \frac{1}{4\pi}\int_0^\infty e^{\frac{i}{2}(\overline{w}-w')t} tK_t(z,z',)dt$$

Hence, formula A.2 (with λ replaced by $\mu t + \lambda$) applied to K_t gives

$$K(z,w;z',w') = \frac{1}{4\pi}\frac{1}{\pi}\int_{T(0)} e^{\frac{i}{2}(\overline{w}-w')t+\mu\overline{z}z't+\lambda\overline{z}z'} t(\lambda + t\mu)dt,$$

where $T(0)$ is the set of all positive t such that $\lambda + t\mu$ is positive, i.e. $T(0) = [0, -\lambda/\mu]$. Formulated in another way

$$K = \frac{1}{4\pi}\frac{1}{\pi}\int_{T(0)} e^{t\rho+\phi} t\det(\frac{i}{2}(\partial\overline{\partial}\phi + t\partial\overline{\partial}\rho))dt,$$

using the suggestive notation explained in connection to formula A.11.

Example A.1. *(Siegel's lower half plane).* Setting $\mu = 1$ and $\lambda = 0$ gives

$$K = \frac{1}{4\pi}\frac{1}{\pi}\int_0^\infty e^{t\rho}t^2 dt = \frac{1}{4\pi}\frac{1}{\pi}\int_0^\infty e^{t\rho}t^2 dt.$$

Since $\int_0^\infty e^{t\rho}dt = 1/\rho$ and the factor t^2 can be obtained by differentiating *two* times with respect to ρ we obtain

$$K = -\frac{1}{4\pi}\frac{1}{\pi}\rho^{-3},$$

which is the classical Bergman kernel of Siegel's lower half plane (which is biholomorphically equivalent to the interior of the unit ball in $\mathbb{C}^2$).

Example A.2. If $\mu < 0$, i.e. X is pseudoconcave, one has to take $\lambda > 0$, i.e. a line bundle with positive curvature to make sure that $T(0)$ is non-empty. For example, $(\mu, \lambda) = (-1, 1)$ gives

$$K = \frac{1}{4\pi}\frac{1}{\pi}\int_0^1 e^{t\rho+\phi}t(1-t)dt.$$

A.6. **Compact version of the model boundary case.** Let X be the manifold obtained as the total space of the *unit disc bundle* in the dual of the line bundle L_μ over the torus T, where L_μ is defined as in section A.2. We obtain an Hermitian holomorphic line bundle over X in the following way. Denote by π the natural projection from X onto the torus T. Then the pulled back line bundle $\pi^* L_\lambda$ is a line bundle over X. The construction is summarized by the following commutative diagram

$$\begin{array}{ccccc} \pi^* L_\lambda & & & & L_\lambda \\ \downarrow & & & & \downarrow \\ X & \hookrightarrow & L_\mu^* & \to & T \end{array}$$

The following local description of the situation is useful. The part of X that lies over a fundamental domain of the torus T can be represented in local holomorphic coordinates (z, ζ), where ζ is the fiber coordinate, as the set of all (z, ζ) such that

$$|\zeta|^2 \exp(+\mu |z|^2) \leq 1$$

and the fiber metric ϕ for the line bundle $\pi^* L_\lambda$ over X may be written as

$$\phi(z, \zeta) = \lambda |z|^2 .$$

Hence, X is locally biholomorphic to the non-compact model domain in the previous section.

Any section α of $H^0(X, \pi^* L_\lambda)$ may be locally Taylor expanded in the fiber variable:

$$\alpha(z, \zeta) := \sum_{j \geq 0} \alpha_j(z) \zeta^j. \tag{A.12}$$

Note that $\alpha_j(z)$ may be identified with an element of $H^0(T, L_\mu^j \otimes L_\lambda \pi^* L_\lambda)$. The reason is that the fiber coordinate ζ may be identified with a section of $\pi^* L_\mu^*$ over X (the "tautological section"). [14] Conversely, any sequence in $H^0(T, L_\mu^j \otimes L_\lambda \pi^* L_\lambda)$ determines an element of $H^0(X, \pi^* L_\lambda)$. This means that

$$H^0(X, \pi^* L_\lambda) = \bigoplus_{j \geq 0} H^0(T, L_\mu^j \otimes L_\lambda \pi^* L_\lambda).$$

Now the Bergman kernel of $H^0(X, \pi^* L_\lambda)$ may be obtained as in the previous section, using the Taylor expansion A.12 (compare paper III). In particular, when λ is replaced by $k\lambda$, i.e when the line bundle $\pi^* L_\lambda$ is replaced by its kth tensor power, one obtains the following scaling asymptotics for the corresponding Bergman kernels (using the local coordinate w along the fiber, as in section A.3):

$$k^{-3} K_k(k^{-1/2} z, k^{-1} w; k^{-1/2} z' k^{-1} w') \to (\frac{1}{\pi})^2 \int_{T(0)} e^{t\rho + \phi} t \det(\frac{i}{2}(\partial\overline{\partial}\phi + t\partial\overline{\partial}\rho)) dt,$$

where $T(0) = [0, -\lambda/\mu]$. Note that the dimension of $H^0(X, \pi^* L_\lambda)$ may be computed directly in the following way. Formula A.5 (and the corresponding vanishing for non-positive λ) gives

$$\dim H^0(X, \pi^* L_\lambda) = \sum_{j \in J(0)} (\lambda + j\mu), \tag{A.13}$$

where $J(0) = [0, -\lambda/\mu] \cap \mathbb{Z}$, which may also be written as the integral

$$(\frac{i}{2\pi})^2 \int_{\partial X} \sum_{j \in J(0)} (\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho) \wedge (-\partial\rho) \tag{A.14}$$

When λ is replaced by $k\lambda$ and k tends to infinity one obtains an asymptotic equality in theorem 4.2, since the sum becomes a Riemann sum as in section A.3. Note that the integral over X in theorem 4.2 vanishes since the pulled back line bundle is flat in the fiber-direction.

[14] Indeed, a point in X (locally given by (z, ζ)) is defined as a vector in the fiber $(L_\mu^*)_z$ (and the vector is locally given by ζ).

The preprints referred to in following references may be found at www.arXiv.org/abs/

References

[1] Ali, S. T; Englis, M: Quantization methods: a guide for physicists and analysts. Rev. Math. Phys. 17 (2005), no. 4, 391–490.

[2] Andreotti, A: Theoremes de dependence algebrique sur les espaces complexes pseudoconcaves. Bull. Soc. Math. France, 91, 1963, 1–38.

[3] Andreotti, A, Grauert, H: Theoremes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France, 90, 1962, 193–259.

[4] Andreotti, A; Siu, Y-T: Projective embedding of pseudoconcave spaces. Ann. Scuola Norm. Sup. Pisa (3) 24 1970 231–278.

[5] Andreotti, Aldo; Tomassini, Giuseppe Some remarks on pseudoconcave manifolds. 1970 Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham) pp. 85–104 Springer, New York

[6] Arnold, V.I: Symplectic Geometry. In Dynamical systems IV, 1–138 Encyclopaedia Math. Sci, 4, Springer, Berlin, 2001

[7] Berline, N; Getzler, E; Vergne, M: Heat kernels and Dirac operators. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 298. Springer-Verlag, Berlin, 1992. viii+369 pp.

[8] Berman R; Berndtsson B; Sjöstrand J: Asymptotics of Bergman kernels, preprint /math.CV/050636

[9] Berman R; Sjöstrand J: Asymptotics for Bergman-Hodge kernels for high powers of complex line bundles, preprint /math.CV/0511158

[10] Berndtsson, Bo: Bergman kernels related to hermitian line bundles over compact comlex manifolds. Explorations in complex and Riemannian geometry, 1–17, Contemp. Math, 332, Amer. Math. Soc, Providence, RI, 2003.

[11] Bismut, J.M: Demailly's asymptotic Morse inequalities, a heat kernel proof, J. Funct. Anal., 72(1987), 263–278.

[12] Bismut, J-M; Lebeau, G: Complex immersions and Quillen metrics. Inst. Hautes Études Sci. Publ. Math. No. 74 (1991), ii+298 pp. (1992).

[13] Bleher, Pl; Shiffman, B; Zelditch, S: Universality and scaling of correlations between zeros on complex manifolds. Invent. Math. 142 (2000), no. 2, 351–395.

[14] Bordemann, M; Hoppe, J; Schaller, P; Schlichenmaier, M: gl(\infty) and geometric quantization. Comm. Math. Phys. 138 (1991),

[15] Bordemann, M; Meinrenken, E; Schlichenmaier, M Toeplitz quantization of Kähler manifolds and ${\rm gl}(N)$, $N\to\infty$ limits. Comm. Math. Phys. 165 (1994), no. 2, 281–296

[16] Borthwick, D; Uribe, A: Nearly Kählerian embeddings of symplectic manifolds. Asian J. Math. 4 (2000), no. 3, 599–620

[17] Borthwick, D; Uribe, A: The semiclassical structure of low-energy states in the presence of a magnetic field. preprint /math.SP/0412223

[18] Borthwick, D;Klimek,S;Lesniewski, A; Rinaldi, M: Super Toeplitz operators and nonperturbative deformation quantization of supermanifolds. Comm. Math. Phys. 153 (1993)

[19] Bouche, T: Asymptotic results for Hermitian line bundles over complex manifolds: the heat kernel approach. Higher-dimensional complex varieties (Trento, 1994), 67–81, de Gruyter, Berlin, 1996.

[20] Bouche, T: Inegalite de Morse pour la d''-cohomologie sur une variete noncompacte. Ann. Sci. Ecole Norm. Sup. 22, 1989, 501-513

[21] Boucksom, S: On the volume of a line bundle. Internat. J. Math. 13 (2002), no. 10, 1043–1063.

[22] Bogomolny, E., Bohigas, O. and Leboeuf, P.: Quantum chaotic dynamics and random polynomials. J. Stat. Phys. 85, 639–679 (1996)

[23] Borel, A: Pseudo-concavité et groupes arithmétiques. (French) 1970 Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham) pp. 70–84 Springer, New York
[24] Boutet de Monvel, L-Guillemin, V: The spectral theory of Toeplitz operators. Annals of Mathematical Studies, 99. Princeton University Press, Princeton, NJ, University of Tokyo Press, Tokyo, 1981
[25] Boutet de Monvel,L-Sjöstrand, J: Sur la singularite des noyaux de Bergman et Szegö: Journees: Equations aux derivees partielles de Rennes (1975), 123–164, Asterisque, No. 34-35. Soc. Math. France, Paris, 1976
[26] Cartier, P: DeWitt-Morette, C; Ihl, M; Sämann, C: Supermanifolds - applications to supersymmetry.
[27] Catlin, D: The Bergman kernel and a theorem of Tian, Analysis and geometry in several complex variables (Katata, 1997), 1–23, Trends in Math. Birkhauser, Boston, MA, 1999
[28] Charles, L: Berezin-Toeplitz operators, a semi-classical approach. Comm. Math. Phys. 239 (2003), no. 1-2, 1–28
[29] Charles, L: Symbolic calculus for Toeplitz operators with half-forms. preprint /math.SG/0602167
[30] Cornalba, L; Taylor. W: Holomorphic curves from matrices. Nucl.Phys. B536 (1998) 513-552
[31] Dai, X; Liu, K- Ma, X: On the asymptotic expansion of Bergman kernel, J. Differential Geom. to appear
[32] Demailly, J-P: Champs magnetiques et inegalite de Morse pour la d''-cohomologie., Ann Inst Fourier, 355 (1985,185-229)
[33] Demailly, J-P: Holomorphic Morse inequalities. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 93-114
[34] Demailly, J-P On the geometry of positive cones of projective and Kähler varieties. The Fano Conference, 395–422, Univ. Torino, Turin, 2004.
[35] Demailly, J-P: Complex analytic and algebraic geometry. Available at www-fourier.ujf-grenoble.fr/~demailly/books.html
[36] Dimassi, M; Sjöstrand, J: Spectral asymptotics in the semi-classical limit. London Mathematical Society Lecture Note Series, 268.
[37] Dinh, T-C; Marinescu,G: On the compactification of hyperconcave ends and the theorems of Siu-Yau and Nadel. SFB 288 Preprint No. 565 preprint /math.CV/0210485
[38] DeWitt, B Supermanifolds. Second edition. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1992
[39] Donaldson, S. K: Scalar curvature and projective embeddings. I. J. Differential Geom. 59 (2001), no. 3, 479–522.
[40] Donaldson, S. K: Some numerical results in complex differential geometry. preprint /math.DG/0512625
[41] Donaldson, S. K: Planck's constant in complex and almost-complex geometry. XIIIth International Congress on Mathematical Physics (London, 2000), 63–72, Int. Press, Boston, MA, 2001.
[42] Donaldson, S. K: Symplectic submanifolds and almost-complex geometry. J. Differential Geom. 44 (1996), no. 4, 666–705.
[43] Donnelly, H: Spectral theory for tensor products of Hermitian holomorphic line bundles. Math. Z. 245 (2003) no. 1, 31–35
[44] Douglas, M. R.; Shiffman, B.; Zelditch, S: Critical points and supersymmetric vacua. I. Comm. Math. Phys. 252 (2004), no. 1-3, 325–358.
[45] Eliashberg, Y: A few remarks about symplectic filling. Geometry and topology, Vol 8 (2004) Nr 6, 277—293
[46] Englis, M: Weighted Bergman kernels and quantization. Comm. Math. Phys. 227 (2002), no. 2, 211–241.

[47] Epstein, C: Geometric bounds on the relative index. J. Inst. Math. Jussieu 1 (2002), no 3, 441–465.
[48] Epstein, C; Henkin, G: Stability of embeddings for pseudoconcave surfaces and their boundaries. Acta Math. 185 (2000), no 2, 161–237
[49] Etnyre, John B. On symplectic fillings. Algebr. Geom. Topol. 4 (2004), 73–80 (electronic)
[50] Fefferman, C: The Bergman kernel and biholomorphic mappings of pseudoconvex domains, Invent. Math. 26(1974), 1–65.
[51] Fernández, R; Fröhlich, J; Sokal, A. D: Random walks, critical phenomena, and triviality in quantum field theory. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.
[52] Folland, G.B, Kohn J.J: The Neumann problem for the Cauchy-Riemann complex. Annals of Math. Studies 75, Princeton University Press, 1972.
[53] Getzler, E: An analogue of Demailly's inequality for strictly pseudoconvex CR manifolds. J. Differential Geom. 29 (1989), no. 2, 231–244
[54] Grauert, H: Theory of q-convexity and q-concavity. Several complex variables, VII, 259–284, Encyclopaedia Math. Sci., 74, Springer, Berlin, 1994.
[55] Griffiths, P: The extension problem in complex analysis. II. Embeddings with positive normal bundle. Amer. J. Math. 88 1966
[56] Griffiths, P; Harris, J: Principles of algebraic geometry. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.
[57] Gromov, M: Kähler hyperbolicity and L^2-Hodge theory. J. Differential Geom. 33 (1991), no. 1, 263–292.
[58] Guillemin, V: Some classical theorems in spectral theory revisited. 219–259 in Seminar on Singularities of Solutions of Linear Partial Differential Equations. Annals of Mathematics Studies, 91. Princeton University Press, Princeton, N.J, 1979.
[59] Henkin, G;Marinescu, G: Unpublished manuscript
[60] Hörmander, L: L^2 estimates and existence theorems for the $\overline{\partial}-$operator. Acta Math. 113 1965 89–152.
[61] Hörmander, L: An introduction to complex analysis in several variables. Third edition. North-Holland Mathematical Library, 7. North-Holland Publishing Co., Amsterdam, 1990. xii+254 pp.
[62] Hörmander, L: A history of existence theorems for the Cauchy-Riemann complex in : L^2 spaces. J. Geom. Anal. 13 (2003), no. 2, 329–357.
[63] Hörmander, L: The null space of the $\overline{\partial}-$Neumann operator. Ann. Inst. Fourier (Grenoble) 54 (2004), no. 5, 1305–1369,
[64] A. Iqbal, N. Nekrasov, A. Okounkov, C. Vafa Quantum Foam and Topological Strings. preprint /hep-th/0312022
[65] Karabegov, A; Schlichenmaier, M: Identification of Berezin-Toeplitz deformation quantization. (English. English summary) J. Reine Angew. Math. 540 (2001), 49–76.
[66] Kollár, J: The structure of algebraic threefolds: an introduction to Mori's program. Bull. Amer. Math. Soc. (N.S.) 17 (1987), no. 2, 211–273.
[67] Kollár, J: Singularities of pairs. Algebraic geometry—Santa Cruz 1995, 221–287, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997.
[68] Landau, H. J: Necessary density conditions for sampling and interpolation of certain entire functions, Acta Math. 117 (1967), 37–52.
[69] Landau, H. J: Sampling, data transmission, and the Nyquist rate, Proc. IEEE 55 (1967), 1701–1706.
[70] Lazarsfeld, Robert: Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. A series of modern surveys in Mathematics, 48. Springer-Verlag, Berlin, 2004

[71] Lempert, L: Algebraic approximations in analytic geometry. Invent. Math. 121 (1995), no. 2, 335–353.
[72] Lindholm, N: Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel. J. Funct. Anal. 182 (2001), no. 2, 390–426
[73] Lu, Zhiqin On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch. Amer. J. Math. 122 (2000), no. 2, 235–273.
[74] Lu, Z; Tian, G: The log term of the Szegö kernel. Duke Math. J. 125 (2004), no. 2, 351–387.
[75] Ma, X; Marinescu, G: Generalized Bergman kernels on symplectic manifolds. C. R. Math. Acad. Sci. Paris 339 (2004), no. 7 preprint /math.DG/0411559
[76] Ma, X; Marinescu, G: The first coefficients of the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator. Xiaonan Ma, George Marinescu. preprint /math.CV/0511395
[77] Marinescu, G: Asymptotic Morse inequalities for Pseudoconcave manifolds. Ann. Scuola. Norm. Sup. Pisa CL Sci (4) 23 (1996), no 1, 27–55
[78] Marinescu, G: Existence of holomorphic sections and perturbation of positive line bundles over $q-$concave manifolds. preprint /math.CV/0402041)
[79] Marinescu, G: A criterion for Moishezon spaces with isolated singularities. Ann. Mat. Pura Appl. (4) 184 (2005), no. 1, 1–16
[80] Martinez, A: An introduction to semiclassical and microlocal analysis. Universitext. Springer-Verlag, New York, 2002. viii+190 pp.
[81] Menikoff, A.; Sjöstrand, J: On the eigenvalues of a class of hypoelliptic operators. Math. Ann. 235 (1978), no. 1, 55–85.
[82] Milnor, J: Morse theory. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51 Princeton University Press, Princeton, N.J. 1963
[83] Moishezon, B.G.: On n-dimensional compact varieties with n algebraically independent meromorphic functions. Amer. Math. Soc. Trans. 63, 51–177 (1967)
[84] Mok, N: Compactification of complete Kähler-Einstein manifolds of finite volume. Recent developments in geometry (Los Angeles, CA, 1987), 287–301,
[85] Okounkov, Andrei Why would multiplicities be log-concave? The orbit method in geometry and physics (Marseille, 2000), 329–347, Progr. Math., 213, Birkhäuser Boston, Boston, MA, 2003.
[86] Penrose, R: Physical space-time and nonrealizable CR-structures. Bull. Amer. Math. Soc. (N.S.) 8 (1983), no. 3, 427–448.
[87] Peternell, Th.: Modifications. Several complex variables, VII, 285–317, Encyclopaedia Math. Sci., 74, Springer, Berlin, 1994.
[88] Ponge, R: Announcement on JAMI 2004, Conference on Asymptotic and Effective Results in Complex Geometry (In Honor of Bernie Shiffman's 60th Birthday), March 15-21, 2004
[89] Reshetikhin, N.; Takhtajan, L. A. Deformation quantization of Kähler manifolds. L. D. Faddeev's Seminar on Mathematical Physics, 257–276, Amer. Math. Soc. Transl. Ser. 2, 201, Amer. Math. Soc., Providence, RI, 2000.Neumann
[90] Rossi, H: Attaching analytic spaces to an analytic space along a pseudoconcave boundary. Proc. Conf. Complex. Manifolds (Minneapolis), Springer-Verlag, New York 1965, 242–256
[91] Ruan.W: Canonical coordinates and Bergman metrics, Comm. Anal. Geom., 6(1998), 589–631.
[92] Siu, Y. T: Some recent results in complex manifold theory related to vanishing theorems for the semipositive case. Workshop Bonn 1984 (Bonn, 1984), 169–192, Lecture Notes in Math., 1111, Springer, Berlin, 1985.
[93] Sjöstrand, J; Singularités analytiques microlocales. (French) [Microlocal analytic singularities] Astérisque, 95, 1–166, Astérisque, 95, Soc. Math. France, Paris, 1982

[94] Sjöstrand, J: Asymptotics for Bergman kernels for high powers of complex line bundles, based on joint works with B. Berndtsson and R. Berman. Séminaire: Équations aux Dérivées Partielles. 2004–2005, Exp. No. XXIII, 10 pp., École Polytech., Palaiseau, 2005.
[95] Song, J: The Szegö kernel on an orbifold circle bundle. preprint /math.DG/0405071
[96] Shiffman, B; Zelditch S *Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds.* J. Reine Angew. Math. 544 (2002), 181–222
[97] Shiffman, Bernard; Zelditch, Steve Distribution of zeros of random and quantum chaotic sections of positive line bundles. Comm. Math. Phys. 200 (1999), no. 3, 661–683.
[98] Siu, Y.T: A vanishing theorem for semipositive line bundles over non-Kähler manifolds. J. Differential Geom. 19 (1984), no. 2, 431–452.
[99] Thomas, R.P: Notes on GIT and symplectic reduction for bundles and varieties. preprint /math.AG/0512411
[100] Tian, G: On a set of polarized Kähler metrics on algebraic manifolds. J. Differential Geom. 32 (1990), no. 1, 99–130
[101] Witten, E: Supersymmetry and Morse theory. J. Differential Geom. 17 (1982), no. 4, 661–692.
[102] Zelditch, S: Szegö kernels and a theorem of Tian. Internat. Math. Res. Notices 1998, no. 6, 317–331.
[103] Yau, S-T: Nonlinear analysis in geometry. Enseign. Math. (2) 33 (1987), no. 1-2, 109–158.

Paper I

BERGMAN KERNELS AND LOCAL HOLOMORPHIC MORSE INEQUALITIES.

ROBERT BERMAN

ABSTRACT. Let (X, ω) be a hermitian manifold and let L^k be a high power of a hermitian holomorphic line bundle over X. Local versions of Demailly's holomorphic Morse inequalities (that give bounds on the dimension of the Dolbeault cohomology groups associated to L^k), are presented - after integration they give the usual holomorphic Morse inequalities. The local *weak* inequalities hold on any hermitian manifold (X, ω), regardless of compactness and completeness. The proofs, which are elementary, are based on a new approach to pointwise Bergman kernel estimates, where the kernels are estimated by a model kernel in $\mathbb{C}^n$.

1. INTRODUCTION

Let X be an $n-$dimensional (possibly non-compact) complex manifold equipped with a hermitian metric two-form, denoted by ω. Furthermore, let L be a holomorphic line bundle over X. The hermitian fiber metric on L will be denoted by ϕ. In practice, ϕ is considered as a collection of local functions. Namely, let s be a local holomorphic trivializing section of L, then locally, $|s(z)|_\phi^2 = e^{-\phi(z)}$, and the canonical curvature two-form of L can be expressed as $\partial\overline{\partial}\phi$.[1] When X is compact Demailly's holomorphic Morse inequalities [6] give asymptotic bounds on the dimension of the Dolbeault cohomology groups associated to L^k :

$$\dim_{\mathbb{C}} H^{0,q}_{\overline{\partial}}(X, L^k \otimes E) \leq k^n \frac{(-1)^q}{\pi^n} \frac{1}{n!} \int_{X(q)} (\frac{i}{2}\partial\overline{\partial}\phi)^n + o(k^n), \tag{1.1}$$

where $X(q)$ is the subset of X where the curvature-two form $\partial\overline{\partial}\phi$ has exactly q negative eigenvalues and $n-q$ positive ones. These are the *weak* holomorphic Morse inequalities - they also have *strong* counterparts involving alternating sums of the dimensions. Demailly's inspiration came from Witten's analytical proof of the classical Morse inequalities for the Betti numbers of a *real* manifold [18], where the role of the fiber metric ϕ is played by a Morse function. Subsequently, proofs based on asymptotic estimates of the heat kernel of the $\overline{\partial}-$ Laplacian, were given by Demailly, Bouche and Bismut ([7], [5] and [3]). All of these proofs use quite delicate analytical arguments - heat kernel estimates and global estimates deduced from the Bochner-Kodaira-Nakano identity for non-Kähler manifolds. In the present paper it is shown that Demaillys inequalities may be obtained

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[1]The normalized curvature two-form $\frac{1}{\pi}\frac{i}{2}\partial\overline{\partial}\phi$ represents $c^1(L)$, the first Chern class of L, in real cohomology.

from comparatively elementary considerations. The starting point is the formula

$$\dim_{\mathbb{C}} H^{0,q}_{\overline{\partial}}(X, L^k) = \int_X \sum_i |\Psi_i(x)|^2 \,, \tag{1.2}$$

where $\{\Psi_i\}$ is any orthonormal base for the space of $\overline{\partial}-$harmonic $(0,q)-$forms with values in L^k), when X is compact (this is obvious for the dimension of the harmonic space - and by the Hodge theorem the dimensions coincide). It is shown that the integrand, called the *Bergman kernel function* $B^{q,k}_X(x)$, may be asymptotically estimated by a model kernel in $\mathbb{C}^n$. Integration then yields Demailly's weak inequalities and a similar argument gives the strong inequalities. The main point of the proof is to first show the corresponding localization property for the closely related *extremal function* $S^{q,k}_X(x)$ defined as

$$\sup \frac{|\alpha(x)|^2}{\|\alpha\|^2_X},$$

where the supremum is taken over all $\overline{\partial}-$harmonic $(0,q)-$forms with values in L^k. Since the estimates are purely local, they hold on any (possibly non-compact) complex manifold, and yield *local* weak holomorphic Morse inequalities for the corresponding L^2-objects. The main inspiration for the present paper comes from Berndtsson's recent article [2].

One final remark: it is fair to say that the formula 1.2 is the starting point for the previous writers' approaches to Demailly's inequalities, as well. The heat kernel approach is based on the observation that the term corresponding to the zero eigen value in the heat kernel on the diagonal $e^{q,k}(x,x;t)$ is precisely the Bergman kernel function $B^{q,k}_X(x)$ (if X is compact). Moreover, when t tends to infinity the contribution of the the other eigenvalues tends to zero. The main problem, then, is to obtain the asymptotic expression for the heat kernel in k and t and investigate the interchanging or the limits in k and t. The point of the present paper is to work directly with the Bergman kernel.

1.1. **Statement of the main result.** Recall that the $\overline{\partial}-$Laplacian is defined by $\Delta_{\overline{\partial}} := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ (where $\overline{\partial}^*$ denotes the formal adjoint of $\overline{\partial}$) and its L^2-kernel is the space of harmonic $(0,q)$-forms with values in L^k, denoted by $\mathcal{H}^{0,q}(X, L^k)$. By the well-known Hodge theorem this space is isomorphic to the Dolbeault cohomology group $H^{0,q}_{\overline{\partial}}(X, L^k)$, when X is compact. Now define the *Bergman kernel function* $B^{q,k}_X(x)$ and the *extremal function* $S^{q,k}_X(x)$ of $\mathcal{H}^q(X, L^k)$ by

$$B^{q,k}_X(x) := \sum_i |\Psi_i(x)|^2, \quad S^{q,k}_X(x) := \sup \frac{|\alpha(x)|^2}{\|\alpha\|^2_X},$$

where $\{\Psi_i\}$ is any orthonormal base in $\mathcal{H}^q(X, L^k)$ and the supremum is taken over all forms in $\mathcal{H}^q(X, L^k)$. To define the corresponding "model" functions at a point x in X, $B^q_{x,\mathbb{C}^n}$ and $S^q_{x,\mathbb{C}^n}$, proceed as before, replacing the manifold X with $\mathbb{C}^n$, the base metric in X with the Euclidean metric in $\mathbb{C}^n$ and the fiber metric ϕ on L with the fiber metric ϕ_0 on the trivial line bundle over $\mathbb{C}^n$, where

$$\phi_0(z) = \sum_{i=1}^{n} \lambda_{i,x} |z_i|^2 ,$$

and where the curvature two-form $\partial\overline{\partial}\phi_0$ in $\mathbb{C}^n$ is obtained by "freezing" the curvature two-form $\partial\overline{\partial}\phi$ on X, at the point x (with respect to on orthonormal frame at x).[2] Finally, denote by $X(q)$ the subset of X where the curvature-two form $\partial\overline{\partial}\phi$ has exactly q negative eigenvalues and $n-q$ positive ones. Its characteristic function is denoted by $1_{X(q)}$. Moreover, let $X(\leq q) := \bigcup_{0\leq i\leq q} X(i)$.

The first theorem we shall prove is a local version of Demailly's weak holomorphic Morse inequalities. Note that the manifold X is not assumed to be compact, neither is there any assumption, e.g. completeness, on the hermitian metric ω.

Theorem 1.1. *Let (X,ω) be a hermitian manifold. Then*

$$\limsup_k \tfrac{1}{k^n} B_X^{q,k}(x) \leq B^q_{x,\mathbb{C}^n}(0), \quad \limsup_k \tfrac{1}{k^n} S_X^{q,k}(x) \leq S^q_{x,\mathbb{C}^n}(0) \ ,$$

and

$$B^q_{x,\mathbb{C}^n}(0) = S^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n} 1_{X(q)} \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right|$$

Moreover,

$$\lim_k \frac{1}{k^n} B_X^{q,k}(x) = \lim_k \frac{1}{k^n} S_X^{q,k}(x)$$

if one of the limits exists.

This seems to be a new result. The inequality for $S_X^{q,k}(x)$ generalizes results of Bouche [5] and Gillet, Soulé, Gromov [9], that are non-local and concern *compact* manifolds X. Integration of the inequality for the Bergman kernel gives Demailly's weak inequalities 1.1.

When X is compact the local Morse inequalities can be extended to an asymptotic *equality* as follows. Let $B^{q,k}_{\leq \nu_k}$ be the Bergman kernel function of the space spanned by all the eigenforms of the $\overline{\partial}-$Laplacian, whose eigenvalues are bounded by ν_k.

Theorem 1.2. *Let (X,ω) be a compact hermitian manifold. Then*

$$\lim_k k^{-n} B^{q,k}_{\leq \mu_k k}(x) = \frac{1}{\pi^n} 1_{X(q)} \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right| ,$$

for some sequence μ_k tending to zero.

Again, integrating this yields, with the help of a well-known homological algebra argument Demailly's *strong* holomorphic Morse inequalities[3].

[2]Equivalently: the $\lambda_{i,x}$ are the eigenvalues of the curvature two-form of L with respect to on orthonormal frame at x.

[3]Just as in the papers of Demailly, Bouche and Bismut ([7],[5] and [3]). In fact, the idea of using the "low-energy spectrum" was introduced in Witten's seminal paper [18]. A similar technique is used in the heat equation proof of Atiyah-Singer's index theorem.

When the curvature of the line bundle L is strictly positive the asymptotic equality holds for the usual Bergman kernel for the space of holomorphic sections of L^k. This was first proved by Tian [19]) with a certain control on the lower order terms in k. A complete asymptotic expansion was given in [20] using microlocal analysis. See also [14] where the manifold is $\mathbb{C}^n$ and [4] where the complex structure is non-integrable.

Remark 1.3. The two theorems have straight forward generalizations to the case where the forms take values in $L^k \otimes E$, for a given rank r hermitian holomorphic vector bundle E over X. The estimate for the extremal function $S_X^{q,k}$ is unaltered, while the results for the Bergman kernel function $B_X^{q,k}$ are modified by a factor r in the right hand side.

Notation 1.4. The notation $a \sim (\lesssim) b$ will stand for $a = (\leq) C_k b$, where C_k tends to one when k tends to infinity.

1.2. **A sketch of the proof of the local weak holomorphic Morse inequalities.** First we will show how to obtain the estimate

$$\limsup_k k^{-n} S_X^{q,k}(x) \leq S_{x,\mathbb{C}^n}^q(0) \tag{1.3}$$

By definition, there is a unit norm sequence α_k of harmonic forms with values in L^k such that

$$\limsup_k k^{-n} S_X^{q,k}(x) = \limsup_k k^{-n} \left|\alpha_k(x)\right|^2$$

Now consider the restriction of the form $\alpha_k(x)$ to a ball B_{R_k} with center in the point x and with radius R_k decreasing to zero with k. The main point of the proof is that the form α_k is asymptotically harmonic, with respect to a *model* fiber metric, on a ball of radius slightly larger than $\frac{1}{\sqrt{k}}$ and is then a candidate for the corresponding model extremal function. Indeed, we can arrange that the fiber metric $k\phi$ on the line bundle L^k can be written as

$$k\phi(z) = k(\textstyle\sum_{i=1}^n \lambda_i \left|z_i\right|^2 + kO(\left|z\right|^3))$$

in local coordinates around x. The form $\beta^{(k)} := k^{-\frac{1}{2}n}\alpha^{(k)}$, defined on the scaled ball $B_{\sqrt{k}R_k}$, where $\alpha^{(k)}(z)$ denotes the (component wise) scaled form $\alpha(\frac{z}{\sqrt{k}})$, satisfies

$$\limsup_k k^{-n} S^{q,k}(x) = \limsup_k \left|\beta^{(k)}(0)\right|^2.$$

Moreover, $\beta^{(k)}$ is harmonic with respect to the scaled Laplacian $\Delta_{\overline{\partial}}^{(k)}$, taken with respect to the scaled fiber metric $(k\phi)^{(k)}$on L^k :

$$(k\phi)^{(k)}(z) = \sum_{i=1}^n \lambda_i \left|z_i\right|^2 + \frac{1}{\sqrt{k}} O(\left|z\right|^3)$$

The point is that the scaled fiber metric converges to the quadratic model fiber metric ϕ_0 in $\mathbb{C}^n$, with an appropriate choice of the radii R_k. As a consequence the operator $\Delta_{\overline{\partial}}^{(k)}$ converges to the model Laplacian $\Delta_{\overline{\partial},\phi_0}$. Standard techniques for elliptic operators then yields a subsequence of forms $\beta^{(k_j)}$ converging to a form β defined in all of $\mathbb{C}^n$, which is harmonic with respect to the model Laplacian. This means that

$$\limsup_k k^{-n} S^{q,k}(x) = |\beta(0)|^2 \leq S^q_{x,\mathbb{C}^n}(0),$$

proving 1.3. Finally, lemma 2.1, relating $S^{q,k}(x)$ and the Bergman kernel function $B^{q,k}(x)$, is used to deduce the corresponding estimate for the Bergman kernel function. All that remains is to compute the Bergman kernel and the extremal function in the model case (section 4).

2. The Bergman kernel function $B(x)$ and the extremal function $S(x)$.

In section 1.1 the Bergman kernel function $B^{q,k}_X$ and the extremal function $S^{q,k}_X$ were defined. We will also have use for component versions of $S^{q,k}_X$. For a given orthonormal frame e^I_x in $\bigwedge^{0,q}_x(X, L^k)$ let

$$S^{q,k}_{X,I}(x) := \sup \frac{|\alpha_I(x)|^2}{\|\alpha\|^2_X},$$

where $\alpha_I(x)$ denotes the component of α along e^I_x. It will be clear from the context what frame is being used. These functions are closely related according to the following, simple, yet very useful lemma. Its statement generalizes a lemma used in [2] (see also [13]).

Lemma 2.1. *Let L be a hermitian holomorphic line bundle over X. With notation as above*

$$S^{q,k}_X(x) \leq B^{q,k}_X(x) \leq \sum_I S^{q,k}_{X,I}(x)$$

Proof. To prove the first inequality in the statement, take any α in $\mathcal{H}^{0,q}(X, L^k)$ of unit norm. Since α is contained in an orthonormal base, obviously $|\alpha(x)|^2 \leq B(x)$, which proves the first inequality. For the second inequality, let $\{\Psi^i\}$ be an orthonormal base for $\mathcal{H}^{0,q}(X, L^k)$, so that $B^{q,k}_X(x) := \sum_i |\Psi^i(x)|^2 = \sum_{i,J} |\Psi^i_J(x)|^2$. Fix a J and let $c_i := \overline{\Psi^i_J(x)}$. Then, summing only over i, gives

$$\sum_i |\Psi^i_J(x)|^2 = \sum_i c_i \Psi^i_J(x) = \alpha_J(x) \left(\sum_i |c_i|^2,\right)^{1/2}$$

where $\alpha = \sum_i \frac{c_i}{(\sum_i |c_i|^2,)_i^{1/2}} \Psi^i$ lies in $\mathcal{H}^q(X, L^k)$ and is of unit norm. Thus,

$$\left(\sum_i |\Psi^i_J(x)|^2\right)^{1/2} = \alpha_J(x) = \frac{|\alpha_J(x)|}{\|\alpha\|} \leq S_J(x)^{1/2}.$$

Finally, squaring the last relation and summing over J proves the second inequality. □

3. The Weak Holomorphic Morse Inequalities

In this section we prove the local version of Demailly's holomorphic Morse inequalities over *any* complex manifold X. The usual version is obtained as a corollary. Around each point x in X, fix local complex coordinates $\{z_i\}$ and a holomorphic trivializing section s of L such that[4]

$$\omega(z) = \frac{i}{2}\sum_{i,j} h_{ij}(z)dz_i \wedge \overline{dz_j}, \ \ h_{ij}(0) = \delta_{ij}$$

and

$$\left|s(z)\right|^2 = e^{-\phi(z)}, \ \ \phi(z) = \sum_{i=1}^n \lambda_{i,x} \left|z_i\right|^2 + O(\left|z\right|^3),$$

where the quadratic part of ϕ is denoted by ϕ_0. Note that in the local coordinates the base metric ω coincides with the Euclidean metric at the origin. Moreover, we have the identities,

$$\lambda_{1,x}\lambda_{2,x}\cdots\lambda_{n,x}Vol_{(X,\omega)} = det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x Vol_{(X,\omega)} = \frac{1}{n!}(\frac{i}{2}\partial\overline{\partial}\phi)^n.$$

The following notation will be useful. Let $B_R := \{z : |z| < R\}$ in $\mathbb{C}^n$ and let $R_k := \frac{\ln k}{\sqrt{k}}$. Using the local chart around x, a (small) ball B_R is identified with a subset of X. Given a function f on the ball B_{R_k}, we define a *scaled* function on $B_{\sqrt{k}R_k}$ by

$$f^{(k)}(z) := f(\frac{z}{\sqrt{k}})$$

Forms are scaled by scaling the components. When scaling an object that is indexed by k, e.g. $\alpha_k^{(k)}$, we will write $\alpha^{(k)}$ to simplify the notation. Observe that scaling the fiber metric on L^k gives

$$(k\phi)^{(k)}(z) = \sum_{i=1}^n \lambda_i \left|z_i\right|^2 + \frac{1}{\sqrt{k}}O(\left|z\right|^3) \tag{3.1}$$

which motivates the choice of scaling. The radius $R_k := \frac{\ln k}{\sqrt{k}}$ has been chosen to make sure that the fiber metric on L^k tends to the quadratic model fiber metric ϕ_0 with all derivatives on scaled balls, i.e.

$$\sup_{|z|\leq\sqrt{k}R_k} \left|\partial^\alpha((k\phi)^{(k)} - \phi_0)(z)\right| \to 0, \tag{3.2}$$

where the convergence is of the order $\frac{1}{\sqrt{k}}$ to some power, which follows immediately from the expansion 3.1. Moreover, $\sqrt{k}R_k$ tends to infinity, so that the sequence of scaled balls $B_{\sqrt{k}R_k}$ exhausts $\mathbb{C}^n$. Let us denote by $\Delta_{\overline{\partial}}^{(k)}$ the Laplacian, taken with respect to the scaled fiber metric $(k\phi)^{(k)}$ and the scaled base metric $\omega^{(k)}$. One can check that

$$\Delta_{\overline{\partial}}^{(k)}\alpha = \frac{1}{k}(\Delta_{\overline{\partial}}\alpha)^{(k)}. \tag{3.3}$$

Hence, if α_k is a form with values in L^k, which is harmonic with respect to the global Laplacian, i.e $\Delta_{\overline{\partial}}\alpha_k = 0$, then the scaled form $\alpha^{(k)}$ satifies

$$\Delta_{\overline{\partial}}^{(k)}\alpha^{(k)} = 0,$$

[4] this is always possible; see [17].

on the scaled ball $B_{\sqrt{k}R_k}$. Moreover, because of the convergence property 3.2 it is not hard to check that

$$\Delta^{(k)}_{\overline{\partial}} = \Delta_{\overline{\partial_{\phi_0}}} + \epsilon_k \mathcal{D}_k, \tag{3.4}$$

where $\mathcal{D}_k$ is a second order partial differential operator with bounded variable coeffiecients on the scaled ball $B_{\sqrt{k}R_k}$ and ϵ_k is a sequence tending to zero with k (in fact, all the derivatives of the coefficients of $\mathcal{D}_k$ are uniformly bounded). It also follows from 3.2 that for any form α^k with values in L^k,

$$\|\alpha_k\|_{B_{R_k}} \sim k^{-n} \left\|\alpha^{(k)}\right\|_{\phi_0,\sqrt{k}R_k}, \tag{3.5}$$

where the factor k^{-n} comes from the change of variables $w = \frac{z}{\sqrt{k}}$ in the integral.

The proof of the following lemma is based on standard techniques for elliptic operators.

Lemma 3.1. *For each k, suppose that $\beta^{(k)}$ is a smooth form on the ball $B_{\sqrt{k}R_k}$ such that $\Delta^{(k)}_{\overline{\partial}}\beta^{(k)} = 0$. Identify $\beta^{(k)}$ with a form in $L^2_{\phi_0}(\mathbb{C}^n)$ by extending with zero. Then there is constant C independent of k such that*

$$\sup_{z\in B_1} \left|\beta^{(k)}(z)\right|^2_{\phi_0} \leq C \left\|\beta^{(k)}\right\|^2_{\phi_0, B_2}$$

Moreover, if the sequence of norms $\left\|\beta^{(k)}\right\|^2_{\phi_0,\mathbb{C}^n}$ is bounded, then there is a subsequence of $\left\{\beta^{(k)}\right\}$ which converges uniformly with all derivatives on any ball in $\mathbb{C}^n$ to a smooth form β, where β is in $L^2_{\phi_0}(\mathbb{C}^n)$.

Proof. Fix a ball B_R, of radius R in $\mathbb{C}^n$. By Gårding's inequality for the elliptic operator $(\Delta^{(k)}_{\overline{\partial}})^m$, we have the following estimates for the Sobolev norm of $\beta^{(k)}$on the ball B_R with $2m$ derivatives in L^2 :

$$\left\|\beta^{(k)}\right\|^2_{B_R,2m} \leq C_{R,k} \left(\left\|\beta^{(k)}\right\|^2_{B_{2R}} + \left\|(\Delta^{(k)}_{\overline{\partial}})^m \beta^{(k)}\right\|^2_{B_{2R}} \right), \tag{3.6}$$

for all positive integers m. Since $\Delta^{(k)}_{\overline{\partial}}$ converges to $\Delta_{\overline{\partial},\phi_0}$ on the ball B_2 it is straight forward to see that $C_{R,k}$ may be taken to be independent of k. Hence, for all positive integers m,

$$\left\|\beta^{(k)}\right\|^2_{B_1,2m} \leq C \left\|\beta^{(k)}\right\|^2_{B_2}$$

and the continuous injection $L^{2,k} \hookrightarrow C^0$, $k > n$, provided by the Sobolev embedding theorem, proves the first statement in the lemma. To prove the second statement assume that $\left\|\alpha^{(k)}\right\|^2_{\phi_0,\mathbb{C}^n}$ is uniformly bounded in k. Then 3.6 shows that

$$\left\|\beta^{(k)}\right\|^2_{B_R,2,2m} \leq D_R$$

Since this holds for any $m \geq 1$, Rellich's compactness theorem yields, for each R, a subsequence of $\left\{\beta^{(k)}\right\}$, which converges in all Sobolev spaces $L^{2,k}(B_R)$ for $k \geq 0$. The compact embedding $L^{2,k} \hookrightarrow C^l$, $k > n + \frac{1}{2}l$, shows that the sequence converges in all $C^l(B_R)$. Choosing a diagonal sequence, with respect to a sequence of balls exhausting $\mathbb{C}^n$, finishes the proof of the lemma. ∎

Before turning to the proof of the local weak holomorphic Morse inequalities, theorem, we state the following facts about the model case, that will be proved in the following section:

$$B^q_{x,\mathbb{C}^n}(0) = S^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n}1_{X(q)}(x)\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|.$$

Moreover, suppose that the first q eigenvalues of the quadratic form ϕ_0 are negative and the rest are positive (which corresponds to the case when x is in $X(q)$). Then

$$S^q_{I,x,\mathbb{C}^n}(0) = 0, \tag{3.7}$$

unless $I = (1, 2, ..., q)$.

Theorem 3.2. *Let (X, ω) be a hermitian manifold. Then the Bergman kernel function $B^{q,k}_X$ and the extremal function $S^{q,k}_X$ of the space of global $\overline{\partial}-$harmonic $(0, q)$ forms with values in L^k, satisfy*

$$\limsup_k k^{-n}B^{q,k}_X(x) \leq B^q_{x,\mathbb{C}^n}(0), \quad \limsup k^{-n}S^{q,k}_X(x) \leq S^q_{x,\mathbb{C}^n}(0)\ ,$$

where

$$B^q_{x,\mathbb{C}^n}(0) = S^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n}1_{X(q)}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$$

and $\lim_k \frac{1}{k^n}B^{q,k}_X(x) = \lim_k \frac{1}{k^n}S^{q,k}_X(x)$ *if one of the limits exists.*

Proof. First we will prove that

$$\limsup_k k^{-n}S^{q,k}_X(x) \leq S^q_{x,\mathbb{C}^n}(0)$$

By definition, there is a unit norm sequence α_k in $\mathcal{H}^{0,q}(X, L^k)$ such that

$$\limsup_k k^{-n}S^{q,k}_X(x) = \limsup_k k^{-n}\left|\alpha_k(x)\right|^2.$$

Now consider the sequence $\beta^{(k)} := k^{-\frac{1}{2}n}\alpha^{(k)}$, where $\beta^{(k)}$ is a form on the ball $B_{\sqrt{k}R_k}$ that we identify with a form in $L^2_{\phi_0}(\mathbb{C}^n)$, by extending with zero (we write $\alpha^{(k)}$ instead of $\alpha^{(k)}_k$) to simplify the notation). Note that

$$\limsup_k \left\|\beta^{(k)}\right\|^2_{\phi_0} = \limsup_k k^{-n}\left\|\alpha^{(k)}\right\|^2_{\phi_0,B_{\sqrt{k}R_k}} \leq \limsup_k \left\|\alpha_k\right\|^2_X = 1,$$

where we have used the norm localization 3.5. According to the previous lemma there is a subsequence of $\left\{\beta^{(k_j)}\right\}$ that converges uniformly with all derivatives to β on any ball in $\mathbb{C}^n$, where β is smooth and $\left\|\beta\right\|^2_{\phi_0} \leq 1$. Hence, we have that $\Delta_{\overline{\partial},\phi_0}\beta = 0$, which follows from the expansion 3.4, showing that

$$\limsup_k k^{-n}S^{q,k}(x) = \lim_j\left|\beta^{(k_j)}(0)\right|^2 = \left|\beta(0)\right|^2 \leq \frac{\left|\beta(0)\right|^2}{\left\|\beta\right\|^2_{\phi_0}} \leq S^q_{x,\mathbb{C}^n}(0),$$

Moreover, by proposition 4.3,

$$S^q_{x,\mathbb{C}^n}(0) = B^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n}1_{X(q)}(x)\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|.$$

Lemma 2.1, then shows that $\lim_k k^{-n} B_X^{q,k}(x) = 0$ outside the set $X(q)$.

Next, if x is in $X(q)$ we may assume that λ_1 up to λ_q are the negative eigenvalues. By 3.7 we then have that $\beta^I = 0$ if $I \neq (1, 2, ..., q)$. We deduce that for $I \neq (1, 2, ..., q)$:

$$\lim k_j^{-n} S_I^{k_j,q}(0) = \lim k_j^{-n} \left|\alpha_{k_j}^I(0)\right|^2 = \left|\beta^I(0)\right|^2 = 0.$$

This proves that

$$\lim k^{-n} S_I^{k,q}(0) = 0,$$

if $I \neq (1, 2, ..., q)$. Finally, lemma 2.1 shows that

$$\limsup_k k^{-n} B_X^{k,q}(x) \leq 0 + 0 + ... + S_{x,\mathbb{C}^n}^q(0) = B_{x,\mathbb{C}^n}^q(0),$$

which finishes the proof of the theorem. □

As a corollary we obtain Demailly's weak holomorphic Morse inequalities:

Corollary 3.3. *Suppose that X is compact. Then*

$$\dim_{\mathbb{C}} H_{\overline{\partial}}^{0,q}(X, L^k) \leq k^n \frac{(-1)^q}{\pi^n} \frac{1}{n!} \int_{X(q)} (\frac{i}{2} \partial\overline{\partial}\phi)^n + o(k^n).$$

Proof. Let us first show that the sequence $k^{-n} S_X^{q,k}(x)$ is dominated by a constant if X is compact. Since X is compact it is enough to prove this for a sufficiently small neighbourhood of a fixed point x_0. Now, for a given form α_k in $\mathcal{H}^{0,q}(X, L^k)$ consider its restriction to a ball of radius $\frac{1}{\sqrt{k}}$ with center in x_0. Using Gårding's inequality 3.6 as in lemma 3.1, we see that there is a constant $C(x_0)$, depending continuosly on x_0 such that

$$|\alpha_k(x_0)|^2 \leq C(x_0) \left\|\alpha^{(k)}\right\|_{\phi_0, B_1}^2$$

for k larger than $k_0(x_0)$, say. Morever, we may assume that the same $k_0(x_0)$ works for all x sufficiently close to x_0. Using the norm localization 3.5 we get that

$$k_{-n} |\alpha_k(x_0)|^2 \leq 2C(x_0) \|\alpha_k\|_X^2$$

for k larger than $k_1(x_0)$ and the same k_1 works for all x sufficiently close to x_0. This proves that $k^{-n} S_X^{q,k}(x)$ is dominated by a constant if X is compact. By 2.1 and the fact that X has finite volume, this means that the sequence $k^{-n} B_X^{q,k}(x)$ is dominated by an L^1–function. Finally, the Hodge theorem shows that

$$\limsup_k \dim_{\mathbb{C}} k^{-n} H_{\overline{\partial}}^{0,q}(X, L^k) = \limsup_k \int_X k^{-n} B_X^{q,k}$$

and Fatou's lemma yields, since the sequence $k^{-n} B_X^{q,k}$ is L^1- dominated,

$$\int \limsup k^{-n} B_X^{q,k} \leq \frac{(-1)^q}{\pi^n} \int_{X(q)} (\frac{i}{2} \partial\overline{\partial}\phi)^n,$$

where we have used the previous theorem and the fact that

$$\frac{1}{n!} (\frac{i}{2} \partial\overline{\partial}\phi)^n = (-1)^q \left|det_\omega(\frac{i}{2} \partial\overline{\partial}\phi)_x\right| Vol_X$$

on $X(q)$. □

Remark 3.4. If there is no point where the curvature two-form is non-degenerate and has exactly one negative eigenvalue, then Demailly observed (see [7]) that combining his weak inequalities with their strong counterparts, gives that

$$\dim_{\mathbb{C}} H^{0,q}_{\overline{\partial}}(X, L^k) = k^n \frac{(-1)^q}{\pi^n} \frac{1}{n!} \int_{X(0)} (\frac{i}{2}\partial\overline{\partial}\phi)^n + o(k^n).$$

Now, if one also uses theorem 1.1[5] one gets the following asymptotic result for the Bergman kernel function:

$$\frac{1}{k^n} B^{0,k}_X(x) \to \frac{1}{\pi^n} \left| det_\omega (\frac{i}{2}\partial\overline{\partial}\phi)_x \right|$$

in $L^1(X, \omega^n/n!)$. In particular there is a subsequence of $k^{-n}B^{0,k}_X(x)$ that converges to $\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$ almost everywhere on the part of X where the curvature two-form is strictly positive (on the complement the limit is zero). This seems to be a new result (in case L is stricly positive on all of X a complete asymptotic expansion of $B^{0,k}_X$ is known [20]). The result is used in [1] to obtain asymptotic results on the Bergman kernel $K_k(x, y)$ associated to the space of holomorphic section with values in L^k, when for example L is semi-positive (the Bergman kernel function B_k is the pointwise norm of the restriction of the Bergman kernel K_k to the diagonal).

4. The model case

In this section we will be concerned with $\mathbb{C}^n$ with its standard metric. Any smooth function ϕ defines a hermitian metric on the trivial line bundle and associated bundles, via $|1|^2_\phi(z) = e^{-\phi(z)}$. Explicitly, this means that if $\alpha^{0,q} = \sum_I f_I d\overline{z}^I$ is a $(0, q)-$form on $\mathbb{C}^n$, then

$$\left|\alpha^{0,q}\right|^2_\phi(z) = \sum_I |f_I(z)|^2 e^{-\phi(z)}.$$

The standard differential operators on smooth functions are extended to operators on forms, by letting them act componentwise. We denote by $\frac{\partial}{\partial\overline{z_i}}^*$ the formal adjoint of $\frac{\partial}{\partial\overline{z_i}}$ with respect to the norm induced by ϕ. A partial integration shows that

$$\frac{\partial}{\partial\overline{z_i}}^* = e^{\phi}(-\frac{\partial}{\partial z_i})e^{-\phi} = -\frac{\partial}{\partial z_i} + \phi_{z_i} \tag{4.1}$$

The following classical commutation relations ([12]) are essential for what follows:

$$\frac{\partial}{\partial\overline{z}_i}\frac{\partial}{\partial\overline{z_j}}^* - \frac{\partial}{\partial\overline{z}_j}^*\frac{\partial}{\partial\overline{z}_i} = \phi_{\overline{z_i}z_j} \tag{4.2}$$

In this section $\phi = \phi_0 := \sum_{i=1}^n \lambda_i |z_i|^2$, so that the right hand side simplifies to $\delta_{ij}\lambda_{ii}$.

[5]Only the case when $q = 0$ is needed. This case is considerably more elementary than the other cases. Indeed, all sections are holomorphic (independently of k) and one can use the submean inequality for holomorphic functions, without invoking the scaled Laplacian.

The next lemma gives an explicit expression for $\Delta_{\overline{\partial},\phi_0}$, that will enable us to compute the model Bergman kernel function.

Lemma 4.1. *We have that*

$$\Delta_{\overline{\partial}}(f d\overline{z}^I) = (\sum_{i\in I} \frac{\partial}{\partial \overline{z}_i}\frac{\partial}{\partial \overline{z}_i}^* + \sum_{i\in I^c} \frac{\partial}{\partial \overline{z}_i}^* \frac{\partial}{\partial \overline{z}_i}) f d\overline{z}^I =: (\Delta_{\overline{\partial},I} f) d\overline{z}^I),$$

with respect to the fiber metric ϕ_0 on the trivial line bundle.

Proof. All adjoints and formal adjoints are taken with respect to ϕ_0. The following notation will be used. We let $d\overline{z}^i$ act on forms by wedge multiplication, and denote the adjoint by $d\overline{z}^{i*}$. Then we have the anti-commutation relations

$$d\overline{z^i}\,\overline{dz^j}^* + \overline{dz^j}^*\,\overline{dz^i} = 0 \text{ if } i \neq j \tag{4.3}$$

Also,

$$(\overline{dz^i}^*\,\overline{dz^i})\overline{dz^J} = \begin{cases} 1, & i \notin J \\ 0, & i \in J \end{cases} \quad \overline{dz^i}\,\overline{dz^i}^*\,\overline{dz^J} = \begin{cases} 0, & i \notin J \\ 1, & i \in J \end{cases} \tag{4.4}$$

$\overline{\partial}$ can now be expressed as

$$\overline{\partial} = \sum_i \frac{\partial}{\partial \overline{z}_i}\overline{dz^i} = \sum_i \overline{dz^i}\frac{\partial}{\partial \overline{z}_i.}$$

Applying $*$ to this relation immediately yields

$$\overline{\partial}^* = \sum_i \frac{\partial^*}{\partial \overline{z}_i}\overline{dz}^{i*} = \sum_i \overline{dz}^{i*}\frac{\partial^*}{\partial \overline{z}_i.}.$$

Now we compute $\Delta_{\overline{\partial}} := \overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial}:$

$$\overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial} = \sum_{i,j} \frac{\partial}{\partial \overline{z}_i}\frac{\partial^*}{\partial \overline{z}_j}\overline{dz^i}\,\overline{dz}^{j*} + \sum_{i,j}\frac{\partial^*}{\partial \overline{z}_j}\frac{\partial}{\partial \overline{z}_i}\overline{dz}^{j*}\,\overline{dz^i}.$$

By applying 4.3 to second term and splitting the sum this equals

$$\sum_i \frac{\partial}{\partial \overline{z}_i}\frac{\partial^*}{\partial \overline{z}_i}\overline{dz}^i\overline{dz}^{i*} + \sum_i \frac{\partial^*}{\partial \overline{z}_i}\frac{\partial}{\partial \overline{z}_i}\overline{dz^i}\,\overline{dz^i}^* + \sum_{i\neq j}(\frac{\partial}{\partial \overline{z}_i}\frac{\partial^*}{\partial \overline{z}_j} - \frac{\partial^*}{\partial \overline{z}_j}\frac{\partial}{\partial \overline{z}_i})\overline{dz^i}\,\overline{dz^j}^*.$$

According to the commutation relation 4.2 the second sum vanishes and 4.4 finally gives that

$$(\overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial})(f\overline{dz^I}) = (\sum_{i\in I} \frac{\partial}{\partial \overline{z}_i}\frac{\partial}{\partial \overline{z}_i}^{*\phi_0} + \sum_{i\in I^c}\frac{\partial}{\partial \overline{z}_i}^{*\phi_0}\frac{\partial}{\partial \overline{z}_i}) f\overline{dz^I}.$$

□

In order to be able to integrate partially in $\mathbb{C}^n$ without getting boundary terms, the following lemma is useful. See [11] for a slightly more general proof. The point is to choose $\chi_R(z) = (\chi(\frac{z}{R}))^2$, where χ is a smooth compact supported function, that equals 1 on the unit ball, say.

Lemma 4.2. *There is an exhaustion sequence χ_R such that for any smooth function f and smooth $(0,q)-$ form α with f, $\frac{\partial}{\partial \overline{z_i}}f$, α and $(\overline{\partial}+\overline{\partial}^*)\alpha$ in $L^2_{\phi_0}(\mathbb{C}^n)$*

$$\lim_R \langle \Delta_{\overline{\partial}}\alpha, \chi_R\alpha\rangle = \left\|(\overline{\partial}+\overline{\partial}^*)\alpha\right\|^2_{\phi_0}, \ \lim_R \left\langle \frac{\partial}{\partial \overline{z}_i}^* \frac{\partial}{\partial \overline{z_i}} f, \chi_R f\right\rangle = \left\|\frac{\partial}{\partial \overline{z_i}} f\right\|^2$$

and similarly for $\left\|\frac{\partial^*}{\partial \overline{z_i}} f\right\|^2$.

Next, we turn to the Bergman kernel function $B^q_{\phi_0,\mathbb{C}^n}(z)$ and the extremal function $S^q_{\phi_0,\mathbb{C}^n}(z)$. They are defined as in section 2 but with respect to the space $\mathcal{H}^{0,q}_{\phi_0}(\mathbb{C}^n)$ of all $(0,q)-$forms in $\mathbb{C}^n$, that are $\overline{\partial}-$harmonic with respect to the fiber metric ϕ_0 and have finite L^2 norm with respect to ϕ_0. Note that with notation as in section 1.1,

$$B^q_{\phi_0,\mathbb{C}^n}(z) = B^q_{x,\mathbb{C}^n}(z)$$

if $\phi(z) = \phi_0(z) + O(|z|^3)$ is the local expression of the fiber metric of L at the point x in X.

The proof of the following proposition is based on a reduction to the case when $q = 0$ and we are considering holomorphic functions in the so called Fock space. Then it is well-known that

$$B^q_{x,\mathbb{C}^n}(z) = S^q_{\mathbb{C}^n}(0) = \frac{1}{\pi^n}|\lambda_1||\lambda_2|\cdots|\lambda_n|. \tag{4.5}$$

Indeed, if f is holomorphic, then $|f|^2$ is subharmonic. Hence,

$$\int_{\Delta_R} |f(0)|^2 e^{-k\phi_0(z)} \leq \int_{\Delta_R} |f(z)|^2 e^{-k\phi_0(z)}, \tag{4.6}$$

where Δ_R is a polydisc of radius R and where we have used that ϕ_0 is radial in each variable. Letting R tend to infinity, shows 4.5 for $S^q_{\mathbb{C}^n}(0)$.

Proposition 4.3. *Assume that q of the numbers λ_i are negative and the rest are positive. Then*

$$B^q_{\phi_0,\mathbb{C}^n}(0) = S^q_{\phi_0,\mathbb{C}^n}(0) = \frac{1}{\pi^n}|\lambda_1||\lambda_2|\cdots|\lambda_n|,$$

Otherwise there are no $\overline{\partial}-$harmonic $(0,q)-$forms in $L^2_{\phi_0}(\mathbb{C}^n)$, i.e.

$$B^q_{\mathbb{C}^n}(0) = S^q_{\mathbb{C}^n}(0) = 0.$$

With notation as in section 1.1 this means that

$$B^q_{x,\mathbb{C}^n}(0) = S^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n} 1_{X(q)}(x)\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$$

Proof. Suppose that $\alpha^{0,q} = \sum_I f_I d\overline{z}^I, \alpha^{0,q} \in L^2_{\phi_0}(\mathbb{C}^n)$ and $\Delta_{\overline{\partial},\phi_0}\alpha^{0,q} = 0$. Then $\langle\Delta_{\overline{\partial}}\alpha^{0,q}, \chi_R\alpha^{0,q}\rangle_{\phi_0} = 0$ and we get that

$$\sum_I \langle \Delta_I f_I, \chi_R f_I\rangle_{\phi_0} = 0.$$

Letting $R \to \infty$ and using lemma 4.2 shows that

$$\left\|\frac{\partial^{*\phi_0}}{\partial \overline{z_i}} f_I\right\|^2_{\phi_0} = 0,\ i \in I \quad \left\|\frac{\partial}{\partial \overline{z_i}} f_I\right\|^2_{\phi_0} = 0,\ i \in I^c\ . \tag{4.7}$$

Let $F_I(\zeta) := e^{-\sum_{i\in I}\lambda_i|z_i|^2} f_I$, where $\zeta_i = \overline{z}_i$ if $i \in I$ and $\zeta_i = z_i$ if $i \in I^c$. Then 4.7 and 4.1 implies that F_I is holomorphic:

$$\frac{\partial}{\partial \overline{z}_i} F_I = 0 \text{ for all } i.$$

Moreover,

$$|f_I|^2_{\phi_0} = |F_I|^2_{\Phi_I}, \ \Phi_I(z) := \sum_{i\in I} -\lambda_i |z_i|^2 + \sum_{i\in I^c} \lambda_i |z_i|^2 . \tag{4.8}$$

Now assume that it is not the case that q of the numbers λ_i are negative and the rest are positive. Then $\Phi_I(z) = \sum_{i=1}^n \Lambda_i^I |z_i|^2$, with some $\Lambda_i^I := \Lambda_{i_I}^I \leq 0$. By assumption, $\int |f_I|^2 e^{-\phi} < \infty$. Now 4.8 and Fubini-Tonelli's theorem give that

$$\int |F_I(0,..,z_{i_I},0,..)|^2_{\phi_I} e^{-\Lambda_{i_I}^I |z_{i_I}|^2} < \infty.$$

Since $\frac{\partial}{\partial \overline{z}_{i_I}} F_I = 0$ and $\Lambda_{i_I}^I \leq 0$ this forces $F_I(0,..,z_{i_I},0,..) \equiv 0$ and in particular $f_I(0) = 0$,[6]which proves that $B^q_{\mathbb{C}^n}(0) = S^q_{\mathbb{C}^n}(0) = 0$.

Finally, assume that q of the numbers λ_i are negative and the rest are positive. We may assume that λ_1 up to λ_q are the negative ones. Then the same argument as the one above gives that $f_I(0) = 0$ if $I \neq I_0 := (1,2,...,q)$. Now since $\Lambda_i^{I_0} > 0$ for all i we get

$$S^q_{\mathbb{C}^n}(0) = \sup \frac{|\alpha(x)|^2}{\|\alpha\|^2_{\phi_0}} = \sup \frac{|f_{I_0}(x)|^2}{\|f_{I_0}\|^2_{\phi_0}} = \sup \frac{|F_{I_0}(x)|^2}{\|F_{I_0}\|^2_{\Phi_I}} = \frac{\Lambda_1\Lambda_2 \cdots \Lambda_n}{\pi^n},$$

where we have used 4.5 in the last step. Since $\Lambda_i = |\lambda_i|$this proves the statement about $S^q_{\mathbb{C}^n}(0)$. The proof is finished by observing that $S^q_{\mathbb{C}^n}(0) = B^q_{\mathbb{C}^n}(0)$. This follows from lemma 2.1, but it is also easy to see directly in this special case, since all the components F_I vanish if $I \neq (1,2,...,q)$. □

Remark 4.4. The statement 3.7 also follows from the previous proof.

A similar argument to the one in the previous proof shows that if $\partial\overline{\partial}\phi_0$ is non-degenerate and if it is *not* the case that q of the numbers λ_i are negative and $n-q$ of the numbers are positive, then there is an apriori estimate of the form

$$\|\alpha^q\|^2 \leq C_q(\left\|\overline{\partial}\alpha^q\right\|^2 + \left\|\overline{\partial}^*\alpha^q\right\|^2),$$

where the norms are taken with respect to ϕ_0. The result appears already in Hörmander's seminal paper [12] and it can be used to give a direct proof of the fact that the global Bergman kernel function $B_X^{q,k}$ vanishes (modulo terms of order $o(k^n)$ at a point outside $X(q)$, where the curvature two-form is non-degenerate.

[6]this follows for example from the submean inequality 4.6

5. The strong holomorphic Morse inequalities

In this section X is assumed to be compact. While the weak holomorphic Morse inequalities give estimates on individual cohomology groups, their strong counter parts say that

$$(5.1)\quad \sum_{j=0}^{q}(-1)^{q-j}\dim_{\mathbb{C}} H^j(X,L^k) \leq k^n\frac{(-1)^q}{\pi^n}\frac{1}{n!}\int_{X(\leq q)}(\frac{i}{2}\partial\overline{\partial}\phi)^n+o(k^n).$$

Let $\mathcal{H}^q_{\leq\nu_k}(X,L^k)$ denote the space spanned by the eigenforms of $\Delta_{\overline{\partial}}$ whose eigenvalues are bounded by ν_k and denote by $B^{q,k}_{\leq\nu_k}$ the Bergman kernel function of the space $\mathcal{H}^q_{\leq\nu_k}(X,L^k)$. Extending the previous methods (section 3) we will show the asymptotic equality

$$\lim_k k^{-n}B^{q,k}_{\leq\nu_k}(x)=\frac{1}{\pi^n}1_{X(q)}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|,$$

where $\{\nu_k\}$ is a judicially chosen sequence. From its integrated version

$$\dim_{\mathbb{C}}\mathcal{H}^q_{\leq\nu_k}(X,L^k)=k^n\frac{(-1)^q}{\pi^n}\frac{1}{n!}\int_{X(q)}(\frac{i}{2}\partial\overline{\partial}\phi)^n+o(k^n)$$

one can then deduce the strong inequalities 5.1 as in [6]. The basic idea is as follows. First we obtain an *upper* bound on the corresponding Bergman kernel function, by a direct generalizaton of the harmonic case: the local weak Morse inequalities. We just have to make sure that terms of the form $\left\|(\Delta^{(k)}_{\overline{\partial}})^m\beta^{(k)}\right\|^2_{\sqrt{k}R_k}$, which are now non-zero, tend to zero with k. Recall, that $\Delta^{(k)}_{\overline{\partial}}$ denotes the $\overline{\partial}$-Laplacian with respect to the scaled metrics on the ball $B_{\sqrt{k}R_k}$. In fact, for scaling reasons they give contributions which are polynomial in $\frac{\nu_k}{k}$. This dictates the choice $\nu_k=\mu_k k$ with μ_k tending to zero with k. The last step is to get a *lower* bound on the spectral density function on $X(q)$, which amounts to proving the existence of a unit norm sequence $\{\alpha_k\}$ in $\mathcal{H}^q_{\leq\nu_k}(X,L^k)$ with

$$|\alpha_k(x)|^2=k^n\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|+o(k^n).$$

To this end we first take a sequence $\{\alpha_k\}$ of $(0,q)-$forms on $\mathbb{C}^n$, which are harmonic with respect to the *flat* metric and the fiber metric $k\phi_0$, having the corresponding property. The point is that the mass of these forms concentrate around 0, when k tends to infinity. Hence, by cutting down their support to small decreasing balls we obtain global forms $\widetilde{\alpha_k}$ on X, that are of unit norm in the limit. Moreover, their Laplacians are "small". After projection on $\mathcal{H}^q_{\leq\nu_k}(X,L^k)$, we finally get the sought after sequence.

We now proceed to carry out the details of the argument sketched above.

Proposition 5.1. *Assume that $\mu_k\to 0$. Then the following estimate holds:*

$$B^{q,k}_{\leq\mu_k k}(x)\leq k^n\frac{1}{\pi^n}1_{X(q)}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|+o(k^n).$$

Proof. The proof is a simple modification of the proof of the local holomorphic Morse inequalities and in what follows these modifications will be presented. The difference is that α_k is in $\mathcal{H}^{0,q}_{\leq\mu_k k}(X,L^k)$ and we have to make sure that the all terms of the form $(\Delta^{(k)}_{\overline{\partial}})^m(k^{-\frac{n}{2}}\alpha^{(k)}) = (\Delta^{(k)}_{\overline{\partial}})^m(\beta^{(k)})$ vanish in the limit. But for any ball B,

$$\left\|(\Delta^{(k)}_{\overline{\partial}})^m\beta^{(k)}\right\|^2_{\phi_0,B} \leq k^{-n}\left\|(\Delta^{(k)}_{\overline{\partial}})^m\alpha^{(k)}\right\|^2_{\phi_0,B_{\sqrt{k}R_k}} \lesssim k^{-2m}\left\|(\Delta_{\overline{\partial}})^m\alpha_k\right\|^2_X$$

and the last term is bounded by a sequence tending to zero:

$$k^{-2m}(\mu_k k)^{2m} \to 0,$$

since by assumption α_k is of unit norm and in $\mathcal{H}^{0,q}_{\leq\mu_k k}(X,L^k)$ and $\mu_k \to 0$. Gårding's inequality as in 3.6 gives that

$$\left\|\beta^{(k)}\right\|^2_{\phi_0,B,2,m} \leq C\left(\left\|\beta^{(k)}\right\|^2_{\phi_0,B} + \left\|(\Delta^{(k)}_{\overline{\partial}})^m\beta^{(k)}\right\|^2_{\phi_0,B}\right) \lesssim (C+(\mu_k)^{2m}) \lesssim C,$$

which shows that the conclusion of lemma 3.1 is still valid. Finally, $\Delta_{\overline{\partial}}\beta = 0$ as before and the rest of the argument goes through word by word. □

The next lemma provides the sequence that takes the right values at a given point x in $X(q)$, with "small" Laplacian, that was referred to in the beginning of the section.

Lemma 5.2. *Let* $c_\phi(x) := \frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$. *For any point* x_0 *in* $X(q)$ *there is a sequence* $\{\alpha_k\}$ *such that* a_k *is in* $\Omega^{0,q}(X,L^k)$ *with*

$$\begin{array}{ll} (i) & |a_k(x)|^2 = k^n c_\phi(x) \\ (ii) & \lim_k \|\alpha_k\|^2 = 1 \\ (iii) & \|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_k\|^2_X = 0 \end{array}$$

Moreover, there is a sequence δ_k , *independent of* x_0 *and tending to zero, such that*

$$(iv)\ \left\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k, \alpha_k\right\rangle_X \leq \delta_k$$

Proof. We may assume that the first q eigenvalues λ_{i,x_0} are negative, while the remaining eigenvalues are positive. Define the following form in $\mathbb{C}^n$:

$$\beta(w) = \left(\frac{|\lambda_1||\lambda_2|\cdots|\lambda_n|}{\pi^n}\right)^{\frac{1}{2}} e^{+\sum_{i=1}^q \lambda_i|w_i|^2} d\overline{w_1}\wedge d\overline{w_2}\wedge \ldots \wedge d\overline{w_q},$$

so that[7] $|\beta|^2_{\phi_0} = \frac{|\lambda_1||\lambda_2|\cdots|\lambda_n|}{\pi^n}e^{-\sum_{i=1}^n|\lambda_i||w_i|^2}$ and $\|\beta\|_{\phi_0,\mathbb{C}^n} = 1$. Observe that β is in $L^{2,m}_{\phi_0}$, the Sobolev space with m derivatives in $L^2_{\phi_0}$, for all m Now define α_k on X by

$$\alpha_k(z) := k^{\frac{n}{2}}\chi_k(\sqrt{k}z)\beta(\sqrt{k}z),$$

where $\chi_k(w) = \chi(\frac{w}{\sqrt{k}R_k})$ and χ is a a smooth function supported on the unit ball, which equals one on the ball of radius $\frac{1}{2}$. Thus $|a_k(x_0)|^2 = k^n c_\phi(x)$, showing (i). To see (ii) note that

$$\|\alpha_k\|^2_X = \|\chi_k\beta\|^2_{\phi_0,\mathbb{C}^n} = \|\beta\|^2_{\phi_0,\frac{1}{2}\sqrt{k}R_k} + \|\chi_k\beta\|^2_{\phi_0,\geq\frac{1}{2}\sqrt{k}R_k} \tag{5.2}$$

[7] compare the proof of proposition 4.3.

and the "tail" $\|\beta\|^2_{\phi_0,\geq\frac{1}{2}\sqrt{k}R_k}$ tends to zero, since β is in $L^2_{\phi_0}(\mathbb{C}^n)$ and $\sqrt{k}R_k$ tends to infinity.

Next, we show (iii). Changing variables (proposition 3.3) and using 3.4 gives

$$\left\|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_k\right\|^2_X \lesssim \left\|(\Delta^{(k)}_{\overline{\partial}})^{m-1}(\Delta_{\overline{\partial},\phi_0}+\epsilon_k\mathcal{D}_k)(\chi_k\beta))\right\|^2_{\phi_0,\sqrt{k}R_k}, \tag{5.3}$$

where $\mathcal{D}_k$ is a second order PDO, whose coefficients have derivatives that are uniformly bounded in k. To see that this tends to zero first observe that

$$\left\|(\Delta^{(k)}_{\overline{\partial}})^{m-1}(\Delta_{\overline{\partial},\phi_0}\chi_k\beta)\right\|^2_{\phi_0,\sqrt{k}R_k} \tag{5.4}$$

tends to zero. Indeed, β has been chosen so that $\Delta_{\overline{\partial},\phi_0}\beta = 0$. Moreover, $\Delta_{\overline{\partial},\phi_0}$ is the square of the first order operator $\overline{\partial}+\overline{\partial}^{*,\phi_0}$, which also annihilates β and obeys a Leibniz like rule, showing that

$$\Delta_{\overline{\partial},\phi_0}\chi_k\beta = \gamma_k\beta$$

where γ_k is a function, uniformly bounded in k, and supported outside the ball $B_{\frac{1}{2}\sqrt{k}R_k}$ (γ_k contains second derivatives of χ_k). Now using 3.4 again we see that 5.4 is bounded by the norm of

$$\gamma_k p(w,\overline{w})\beta,$$

where p is a polynomial. Thus 5.3 can be estimated by the "tail" of a convergent integral, as in 5.2 - the polynomial does not affect the convergence - which shows that 5.4 tends to zero. To finish the proof of (iii) it is now enough to show that

$$\left\|(\Delta^{(k)}_{\overline{\partial}})^{m-1}\mathcal{D}_k(\chi_k\beta)\right\|^2_{\phi_0,\sqrt{k}R_k},$$

is uniformly bounded. As above one sees that the integrand is bounded by the norm of

$$q(w,\overline{w})\beta,$$

for some polynomial q, which is finite as above.

To prove (iv) observe that, as above,

$$\left\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k,\alpha_k\right\rangle = \left\|\frac{1}{\sqrt{k}}(\overline{\partial}+\overline{\partial}^*)\alpha_k\right\|^2_X \sim \left\|(\overline{\partial}+\overline{\partial}^{*(k)})(\chi_{\sqrt{k}R_k}\beta)\right\|^2_{\sqrt{k}R_k}.$$

Hence, by Leibniz' rule

$$\left\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k,\alpha_k\right\rangle \lesssim \left\|(\chi_{\sqrt{k}R_k}(\overline{\partial}+\overline{\partial}^{*(k)})\beta\right\|^2_{\sqrt{k}R_k} + \frac{C}{(\sqrt{k}R_k)^2}\|\beta\|^2_{\mathbb{C}^n}.$$

Clearly, there is an expansion for the first order operator $(\overline{\partial}+\overline{\partial}^{*(k)})$ as in 3.4, giving

$$\left\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k,\alpha_k\right\rangle \lesssim \varepsilon_k\left(\|\beta\|^2+\sum_{i=1}^{2n}\|\partial_i\beta\|^2\right)+\frac{C}{(\sqrt{k}R_k)^2}\|\beta\|^2.$$

Note that even if $\|\beta\|^2$ is independent of the eigenvalues λ_{i,x_0}, the norms $\|\partial_i\beta\|^2$do depend on the eigenvalues, and hence on the point x_0. But the dependence amounts to a factor of eigenvalues and since X is compact, we deduce that $\|\partial_i\beta\|^2$is bounded by a constant independent of the point x_0. This shows that $\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k, \alpha_k\rangle_X \leq \delta_k$. Note that ϵ_k also can be taken to be independent of the point x_0, by a similar argument. □

Proposition 5.3. *Assume that the sequence μ_k is such that $\mu_k \neq 0$ and $\frac{\delta_k}{\mu_k} \to 0$, where δ_k is the sequence appearing in lemma 5.2. Then, for any point x in $X(q)$, the following holds*

$$\liminf k^{-n} B^{q,k}_{\leq \mu_k k}(x) \geq \frac{1}{\pi^n}\left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right|$$

Proof. Let $\{\alpha_k\}$ be the sequence that lemma 5.2 provides and decompose it with respect to the orthogonal decomposition $\Omega^{0,q}(X, L^k) = \mathcal{H}^q_{\leq\mu_k k}(X, L^k)\oplus \mathcal{H}^q_{>\mu_k k}(X, L^k)$, induced by the spectral decomposition of the elliptic operator $\Delta_{\overline{\partial}}$:

$$\alpha_k = \alpha_{1,k} + \alpha_{2,k}$$

First, we prove that

$$\lim_k k^{-n}\left|\alpha_2^{(k)}(0)\right|^2 = 0. \tag{5.5}$$

As in the proof of lemma 3.1 we have that

$$k^{-n}\left|\alpha_2^{(k)}(0)\right|^2 \leq C(x)\left(k^{-n}\left\|\alpha_2^{(k)}\right\|^2_{B_1} + k^{-n}\left\|(\Delta^{(k)}_{\overline{\partial}})^m\alpha_2^{(k)}\right\|^2_{,B_1}\right).$$

To see that the first term tends to zero, observe that by the spectral decomposition of $\Delta_{\overline{\partial}}$:

$$\|\alpha_{2,k}\|^2_X \leq \frac{1}{\mu_k k}\langle\Delta_{\overline{\partial}}\alpha_{2,k}, \alpha_{2,k}\rangle_X \leq \frac{1}{\mu_k}\left\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k, \alpha_k\right\rangle_X \leq \frac{\delta_k}{\mu_k}$$

Furthermore, the second term also tends to zero:

$$k^{-n}\left\|(\Delta^{(k)}_{\overline{\partial}})^m\alpha_2^{(k)}\right\|^2_{B_1} \leq \left\|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_{2,k}\right\|^2_X \leq \left\|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_k\right\|^2_X \to 0.$$

by (iii) in lemma 5.2. Finally,

$$k^{-n}S^{q,k}_{\mu_k k}(x) \geq k^{-n}\frac{|\alpha_{k,1}(0)|^2}{\|\alpha_{k,1}\|^2_X} \geq k^{-n}|\alpha_{1,k}(0)|^2 = k^{-n}|\alpha_k(0) - \alpha_{2,k}(0)|^2.$$

By 5.5 this tends to the limit of $k^{-n}|\alpha_k(0)|^2$, which proves the proposition according to (i) in lemma 5.2 and lemma 2.1. □

Now we can prove the following asymptotic equality:

Theorem 5.4. *Let (X, ω) be a compact hermitian manifold. Then*

$$\lim k^{-n}B^{q,k}_{\leq\mu_k k}(x) = \frac{1}{\pi^n}1_{X(q)}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|,$$

for some sequence μ_k tending to zero.

Proof. Let $\mu_k := \sqrt{\delta_k}$. The theorem then follows immediately from proposition 5.1 and proposition 5.3 if x is in $X(q)$. If x is outside of $X(q)$ then the upper bound given by proposition 5.1 shows that $\lim_k k^{-n}B^{q,k}_{\leq\mu_k k}(x) = 0$, which finishes the proof of the theorem. □

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References

[1] Berman, R: Work in progres.
[2] Berndtsson, B: An eigenvalue estimate for the $\overline{\partial}-$ Laplacian. J. Differential Geom. 60 (2002), no. 2, 295–313
[3] Bismut, J-M: Demailly's asymptotic Morse inequalities: a heat equation proof. J. Funct. Anal. 72 (1987), no. 2, 263–278.
[4] Borthwick, D; Uribe, A: Nearly Kählerian embeddings of symplectic manifolds. Asian J. Math. 4 (2000), no. 3, 599–620.
[5] Bouche, T: Asymptotic results for Hermitian line bundles over complex manifolds: the heat kernel approach. Higher-dimensional complex varieties (Trento, 1994), 67–81, de Gruyter, Berlin, 1996.
[6] Demailly, J-P: Champs magnetiques et inegalite de Morse pour la d''-cohomologie., Ann Inst Fourier, 355 (1985,185-229)
[7] Demailly, J-P: Holomorphic Morse inequalities. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 93-114
[8] Demailly, J-P: Introduction à la théorie de Hodge. In "Transcendental methods in algebraic geometry. Lectures given at the 3rd C.I.M.E. Session held in Cetraro, July 4–12, 1994." Lecture Notes in Mathematics, 1646. Springer-Verlag, 1996.
[9] Gillet, H; Soulé, C: Amplitude arithmétique. C. R. Acad. Sci. Paris Sér. I Math. 307 (1988), no. 17, 887–890.
[10] Griffiths, P; Harris, J: Principles of algebraic geometry. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.
[11] Gromov, M: Kähler hyperbolicity and L^2-Hodge theory. J. Differential Geom. 33 (1991), no. 1, 263–292.
[12] Hörmander, L: L^2 estimates and existence theorems for the $\overline{\partial}-$operator. Acta Math. 113 1965 89–152.
[13] Li, P: On the Sobolev constant and the p-spectrum of a compact Riemannian manifold. Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 4, 451–468.
[14] Lindholm, N: Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel. J. Funct. Anal. 182 (2001), no. 2, 390–426
[15] Siu, Y. T: Some recent results in complex manifold theory related to vanishing theorems for the semipositive case. Workshop Bonn 1984 (Bonn, 1984), 169–192, Lecture Notes in Math., 1111, Springer, Berlin, 1985.
[16] Siu, Y.T: A vanishing theorem for semipositive line bundles over non-Kähler manifolds. J. Differential Geom. 19 (1984), no. 2, 431–452.
[17] Wells, R. O., Jr.: Differential analysis on complex manifolds. Graduate Texts in Mathematics, 65. Springer-Verlag, New York-Berlin, 1980.
[18] Witten, E: Supersymmetry and Morse theory. J. Differential Geom. 17 (1982), no. 4, 661–692.
[19] Tian, G: On a set of polarized Kähler metrics on algebraic manifolds. J. Differential Geom. 32 (1990), no. 1, 99–130
[20] Zelditch, S: Szegö kernels and a theorem of Tian. Internat. Math. Res. Notices 1998, no. 6, 317–331.

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Paper II

SUPER TOEPLITZ OPERATORS ON LINE BUNDLES

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ABSTRACT. Let L^k be a high power of a hermitian holomorphic line bundle over a complex manifold X. Given a differential form f on X, we define a super Toeplitz operator T_f acting on the space of harmonic $(0, q)$-forms with values in L^k, with symbol f. The asymptotic distribution of its eigenvalues, when k tends to infinity, is obtained in terms of the symbol of the operator and the curvature of the line bundle L, given certain conditions on the curvature. For example, already when $q = 0$, i.e. the case of holomorphic sections, this generalizes a result of Boutet de Monvel and Guillemin to semi-positive line bundles. The asympotics are obtained from the asymptotics of the Bergman kernels of the corresponding harmonic spaces, which have independent interest. Applications to sampling are also given.

1. INTRODUCTION

Let (X, ω) be an $n-$dimensional compact hermitian manifold and let L be a hermitian holomorphic line bundle over X. The fiber metric on L will be denoted by ϕ. It can be thought of as a collection of local functions: let s be a local holomorphic trivializing section of L, then locally, $|s(z)|_{\phi}^{2} = e^{-\phi(z)}$ and the canonical curvature two-form of L is $\partial\overline{\partial}\phi$. Denote by $X(q)$ be the subset of X where the curvature two-form of L is non-degenerate and has exactly q negative eigenvalues. The notation $\eta_p := \eta^p/p!$ will be used in the sequel, so that the volume form on X may be written as ω_n.

The spaces $H^0(X, L^k)$, consisting of global holomorphic sections with values in high powers of L, appear naturally in complex and algebraic geometry, as well as in mathematical physics. In many applications the line bundle L is positive i.e. its curvature two-form is positive and the asymptotic properties of the sequence of Hilbert spaces $H^0(X, L^k)$ have been studied thoroughly in this case. For example the asymptotic behaviour of the corresponding Bergman kernels is known and can be used to study asymptotic properties of Toeplitz operators acting on $H^0(X, L^k)$ as well as asymptotic conditions on the density of the distribution of sampling points on the manifold X (see [3] for a recent survey from this point of view). The aim of the present article is to extend these results in two directions: to line bundles with weaker curvature properties than positivity, such as semi-positivity (part 1) and to harmonic $(0, q)-$ forms with values in L^k (part 2). To emphasize the analogy between holomorphic sections and harmonic forms, some rudiments of the theory of super manifolds is recalled. The super formalism also offers a compact notation.

In part 1, following [16] and [3], everything is reduced to knowing the leading asymptotic behaviour of the Bergman kernel $K(x,y)$. The asymptotics, in turn, are obtained using a new and comparatively elementary approach based on the method used in [2] to prove local holomorphic Morse inequalities. The main application is a generalization of a theorem of Boutet de Monvel and Guillemin that expresses the asymptotic distribution of the eigenvalues of a Toeplitz operator in terms of the symbol of the operator [6], [14]. When L is positive the associated dual disc bundle over X is strictly pseudoconvex. One can then profit from the knowledge of the Bergman kernel on a strictly pseudoconvex manifold [7]. However, when L is only semi-positive one would have to use the corresponding result on a weakly pseudoconvex manifold, which is not available. In fact, a recent counter example of Donnelly [12] to a conjecture due to Siu, shows that the tangential Cauchy-Riemann operator on the boundary of the dual disc bundle does not have closed range. This property is essential to the previous approaches to the asymptotics of the Bergman kernel.

In part 2 the approach in part 1 is extended to study the Bergman kernel of the space of harmonic $(0,q)-$ forms with values in L^k, considered as a bundle valued form on $X\times X$. The main application is a generalization of the theorem of Boutet de Monvel and Guillemin to Toeplitz operators, whose symbol is a differential form on X. These operators are called super Toeplitz operators and they are closely related to the operators introduced in [4] in the context of Berezin-Toeplitz quantization of symplectic super manifolds.

It should be added that part one is just a special case of part two (when q is equal to zero), except for the applications to sampling. However, it has been included to motivate the more general discussion given in the second part.

Part 1. Holomorphic sections

Let (ψ_i) be an orthonormal base for $H^0(X,L)$. Denote by π_1 and π_2 the projections on the factors of $X\times X$. The *Bergman kernel* of the Hilbert space $H^0(X,L)$ is defined by

$$K(x,y)=\sum_i \psi_i(x)\otimes\overline{\psi_i(y)}.$$

Hence, $K(x,y)$ is a section of the pulled back line bundle $\pi_1^*(L)\otimes\overline{\pi_2^*(L)}$ over $X\times X$. For a fixed point y we identify $K_y(x):=K(x,y)$ with a section of the hermitian line bundle $L\otimes L_y$, where L_y denotes the line bundle over X, whose constant fiber is the fiber of L over y, with the induced metric. The definition of K is made so that K satisfies the following reproducing property

$$\alpha(y) = (\alpha, K_y) \tag{1.1}$$

[1]for any element α of $H^0(X, L)$, which also shows that K is well-defined. In other words K represents the orthogonal projection onto $H^0(X, L)$ in $L^2(X, L)$. The restriction of K to the diagonal is a section of $L \otimes \overline{L}$ and we let $B(x) = |K(x, x)|$ be its point wise norm:

$$B(x) = \sum_i |\psi_i(x)|^2 .$$

We will refer to $B(x)$ as the *Bergman function* of $H^0(X, L)$. It has the following extremal property:

$$B(x) = \sup |\alpha(x)|^2 , \tag{1.2}$$

where the supremum is taken over all normalized elements α of $H^0(X, L)$. An element realizing the extremum, is called an *extremal at the point x* and is determined up to a complex constant of unit norm. In order to estimate $K(x, y)$ we will have great use for a more general identity. It is just a reformulation of the fact that, by the reproducing property 1.1, $K_x/\sqrt{B(x)}$ may be identified with an extremal at the point x.

Proposition 1.1. *Let α be an extremal at the point x. Then*

$$|K(x, y)|^2 = |\alpha(y)|^2 B(x)$$

Proof. First fix the point x and take a local holomorphic trivialization of L around x. Then we may identify $K(x, y)$ with an element K_x of $H^0(X, L)$. Now we may assume that $\|K_x\| \neq 0$ - it will be clear that otherwise the statement is trivially true. By the reproducing property 1.1 of K the normalized element $K_x/\|K_x\|$ is an extremal of B at x. Furthermore, the reproducing property 1.1 also shows that the squared norm of K_x is given by $(K_x, K_x) = K_x(x) = K(x, x)$. Hence

$$|\alpha(y)|^2 = |K_x(y)|^2 / K(x, x),$$

for any other extremal α of B at x. Since $|K(x, y)|^2 = |K_x(y)|^2 e^{-\phi(x)}$ and since by definition $K(x, x)e^{-\phi(x)} = B(x)$ this proves the proposition. □

Next, we will define certain operators on $H^0(X, L)$. Given a complex-valued bounded measurable function f on X we define T_f, the so called *Toeplitz operator with symbol f*, by

$$T_f := P \circ f\cdot,$$

where $f\cdot$ denotes the usual multiplication operator on $L^2(X, L)$ and P is the orthogonal projection onto $H^0(X, L)$. Equivalently:

$$(T_f\alpha, \beta) = (f\alpha, \beta), \tag{1.3}$$

for all elements α and β of $H^0(X; L)$. Note that the operator T_f is hermitian if f is real-valued.

[1]We are abusing notation here: the scalar product $(\cdot, \cdot)$ on $H^0(X, L)$ determines a pairing of K_y with any element of $H^0(X, L)$, yielding an element of L_y.

When studying asymptotic properties of L^k, all objects introduced above will be defined with respect to the line bundle L^k.

2. Asymptotic results for Bergman kernels and Toeplitz operators.

Let us first see how to prove the following upper bound on the Bergman kernel function $B(x)$:

$$B^k(x) \leq k^n \frac{1}{\pi^n} 1_{X(0)}(x) \left| det_\omega (\frac{i}{2} \partial\overline{\partial}\phi)_x \right| + o(k^n), \tag{2.1}$$

where we have identified the two-form $i\partial\overline{\partial}\phi$ with an endomorphism, using the metric ω, so that its determinant is well-defined. Integrating this over all of X gives an upper bound on the dimension of the space of holomorphic sections:

$$\dim_{\mathbb{C}} H^0(X, L^k) \leq k^n \frac{(-1)^q}{\pi^n} \frac{1}{n!} \int_{X(0)} (\frac{i}{2} \partial\overline{\partial}\phi)^n + o(k^n), \tag{2.2}$$

which is precisely Demailly's holomorphic Morse-inequalities for $(0,q)-$forms when $q = 0$. In [2] the inequality 2.1 and it's generalization to harmonic $(0,q)-$ forms were called *local* holomorphic Morse-inequalities. As we will see 2.1 follows from the submean property of holomorphic functions and a simple localization argument. Fix a point x in X and choose complex coordinates z and a holomorphic trivialization s of L around x, such the metric ω is Euclidean with respect to z at 0 and the fiber metric $\phi(z) = \phi_0(z) + O(|z|^3)$, where $\phi_0(z) = \sum_{i=1}^n \lambda_i |z_i|^2$ and λ_i are the eigenvalues of the curvature two-form $\partial\overline{\partial}\phi$, with respect to the base metric ω, at the point x. According to the extremal property 1.2 of $B(x)$ we have to estimate the point wise norm of section α at x in terms of the global L^2 norm. Let B_{R_k} be balls centered at x of radius $R_k \to 0$. By first restricting the global norm to the ball B_{R_k} and than making the change of variables $z = \frac{w}{\sqrt{k}}$ in the integral we get

$$\frac{|\alpha_k(x)|^2}{\|\alpha_k\|_X^2} \leq \frac{|\alpha_k(x)|^2}{\|\alpha_k\|_{R_k}^2} = \rho_k k^n |f_k(0)|^2 / \int_{B_{\sqrt{k}R_k}} |f_k(w)|^2 e^{-\phi_0(w)},$$

where the holomorphic functions f_k represent α_k in the local frame. The factor ρ_k comes from the base manifold metric and the terms of order $O(|z|^3)$ in the fiber metric on L. If we now choose e.g. $R_k = \frac{lnk}{\sqrt{k}}$ then the factor $\rho_k \to 1$ and the scaled radii $\sqrt{k}R_k \to \infty$ so that the integration in the variable w is over all of $\mathbb{C}^n$ in the limit. Furthermore, since $|f_k|^2$ is plurisubharmonic the quotient in the right hand side can be estimated by the inverse of the Gaussian

$$1/\int_{B_{\sqrt{k}R_k}} e^{-\phi_0(w)}$$

which tends to $(1/\pi)^n\lambda_1\lambda_2\cdots\lambda_n$ if all eigenvalues are positive and is equal to zero in the limit otherwise. This proves the upper bound on the Bergman kernel function 2.1. In fact, what we have proved is the stronger statement that for any sequence (α_k), where α_k is in $H^0(X,L^k)$,

$$\limsup_k k^{-n}\left|\alpha_k(x)\right|^2/\left\|\alpha_k\right\|^2_{B_{R_k}} \leq \frac{1}{\pi^n}1_{X(0)}(x)\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| \tag{2.3}$$

which will be important in the proof of theorem 2.4.

It is well-known that 2.1 is actually an asymptotic *equality* when L is positive on all of X. One can e.g. use Hörmander's celebrated L^2 estimates to obtain the equality [17],[16], [3] (for complete asymptotic expansions see [19] , which is based on the micro local analysis in [7] and [1] for a simple and direct approach). But these methods break down if the curvature of L is only semi-positive. On the other hand Demailly proved, using his holomorphic Morse inequalities, that 2.1 is an asymptotic equality under the more general condition that $X(1)$ is empty. Combining Demailly's result with the upper bound 2.1 we obtain the following theorem:

Theorem 2.1. *Suppose that $X(1)$ is empty. Then*

$$k^{-n}B^k(x)\to\frac{1}{\pi^n}1_{X(0)}(x)\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$$

in $L^1(X,\omega_n)$. In particular, the measure $B^k\omega_n/k^n$ converges to $\pi^{-n}1_{X(0)}(\partial\overline{\partial}\phi)_n$ in the weak-topology.*

Proof. The upper bound 2.1 says that

$$\limsup_k\ k^{-n}B^k(x)\leq\tfrac{1}{\pi^n}1_{X(0)}(x)\left|det_\omega(\tfrac{i}{2}\partial\overline{\partial}\phi)_x\right|\ ,$$

for any line bundle L. Moreover, if the curvature of the line bundle L is such that $X(1)$ is empty, then

$$\lim_k k^{-n}\int_X B^k(x)\omega_n=\frac{1}{\pi^n}\int 1_{X(0)}(x)\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|\omega_n.$$

To see this, note that the left hand side is the dimension of the space $H^0(X,L^k)$. In this form the statement was first shown by Demailly in [9]. See also proposition 3.2 in the present paper. Finally the theorem follows from the following simple lemma: □

Lemma 2.2. *Assume that (X,μ) is a finite measure space and that f and f_k are bounded functions, where the sequence f_k is uniformly bounded. If*

$$(i)\lim_k\int_X f_k d\mu=\int_X f d\mu,\quad and\quad (ii)\limsup f_k\leq f.$$

Then the sequence f_k converges to f in $L^1(X,\mu)$.

Proof. By the assumption (i)

$$\limsup_k \int_X |f_k - f| \, d\mu = 2 \limsup_k \int_X \chi_k (f_k - f) d\mu,$$

where χ_k is the characteristic function of the set where $f_k - f$ is non-negative. The right hand side can be estimated by Fatou's lemma, which is equivalent to the inequality

$$\limsup_k \int_X g_k d\mu \leq \int_X \limsup_k g_k d\mu,$$

if the sequence g_k is dominated by an L^1-function. Taking $g_k = \chi_k(f_k - f)$ and using the assumption (ii), finishes the proof of the lemma. □

The weak convergence of the previous theorem can be reformulated in terms of Toeplitz operators:

Corollary 2.3. *Suppose that $X(1)$ is empty. Then for any bounded function f on X*

$$\lim_k \mathrm{k}^{-n} \mathrm{Tr} T_f = (2\pi)^{-n} \int_{X(0)} f (i\partial\overline{\partial}\phi)_n$$

Proof. From the definition 1.3 of a Toeplitz operator $\mathrm{Tr} T_f = \sum_i (f\Psi_i, \Psi_i)$, which is equal to $\int_X f B_k(x) \omega_n$. The corollary now follows from the L^1 convergence in the previous theorem. □

We have seen how to obtain the leading asymptotics of $B(x)$, the norm of the Bergman kernel K on the diagonal.The main point of the present paper is that the argument presented actually also shows that $|K(x,y)|^2$ tends to zero off the diagonal to the leading order. The idea is that combining the upper bound 2.3 with the asymptotics for $B(x)$, one sees that any sequence of extremals α_k at a given point x, becomes localized around x in the large k limit. Since $K_x(y)$ is essentially equal to $\alpha_k(y)$, this will show that $K(x,y)$ localizes to the diagonal when k tends to infinity. A similar argument has been used by Bouche [5] to construct holomorphic peak sections when L is positive.

Theorem 2.4. *Suppose that $X(1)$ is empty. Denote by Δ the current of integration on the diagonal in $X \times X$. Then*

$$\lim_k k^{-n} \left| K^k(x,y) \right|^2 \omega_n(x) \wedge \omega_n(y) = (2\pi)^{-n} 1_{X(0)} \Delta \wedge (i\partial\overline{\partial}\phi)_n \ ,$$

*as measures on $X \times X$, in the weak *-topology.*

Proof. First note that the mass of the measures $\mu_k := \left| K^k(x,y) \right|^2 / k^n \omega_n(x) \wedge \omega_n(y)$ are uniformly bounded in k: first integrating over y and using the reproducing property 1.1 gives

$$\mu_k(X \times X) = k^{-n} \int_{X_x} \left| (K_x^k, K_x^k) \right| \omega_n(x) = k^{-n} \int_{X_x} B_k(x) \omega_n = k^{-n} \dim H^0(X, L^k),$$

which clearly is bounded by 2.2. Moreover, the mass of $X(0)^c \times X$ tends to zero:

$$\mu_k(X(0)\times X) = k^{-n}\int_{X(0)} \left|(K_x^k, K_x^k)\right| = k^{-n}\int_{X(0)} B_k(x)\omega_n = 0,$$

where we have used the bound 2.1.

Hence it is enough to prove the convergence on $X(0)\times X$. Moreover since the mass of the measures μ_k is bounded, it is enough to show that any subsequence of μ_k has another subsequence that converges to $\pi^{-n}1_{X(0)}\Delta\wedge(\partial\overline{\partial}\phi)_n$, in the weak*-topology. To simplify the notation the first subsequence will be indexed by k in the following.

According to theorem 2.1 and standard integration theory there is a subsequence of $k^{-n}B_k$ that converges to $\pi^{-n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$ almost everywhere on $X(0)$. Fix a point x in $X(0)$ where $k^{-n}B_k(x)$ converges. Take a sequence of sections α_k, where α_k is a normalized extremal at the point x. Then according to proposition 1.1:

$$\left|K_x^k(y)\right|^2 = \left|\alpha_k(y)\right|^2 B^k(x) \tag{2.4}$$

We will now show that there is a subsequence of α_k such that

$$\lim_k \left\|\alpha_k\right\|^2_{B_{R_k}(x)} = 1 \tag{2.5}$$

where $B_{R_k}(x)$ is a ball centered in x of radius k of radius $R_k := \ln k/\sqrt{k}$ with respect to the "normal" coordinates around x used in section 2. Since the global norm of α_k is equal to one and the radii R_k tend to zero, it follows that the function $\left|\alpha_k(y)\right|^2$ on X converges to the Dirac measure at x in the weak*-topology. From this convergence we will be able to deduce the statement of the theorem. To prove the claim 2.5 first observe that there is a subsequence of α_k such that at the point x :

$$\pi^{-n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| = \lim_j k_j^{-n}\left|\alpha_{k_j}(x)\right|^2$$

Indeed, since α_k is normalized 1.2 says that the right hand side is equal to k^{-n} times $B_k(x)$, the Bergman function at the point x, which in turn tends to the left hand side according to theorem 2.1 and by the assumption on the point x.

Furthermore, since α_k is normalized the restricted norm $\left\|\alpha_k\right\|^2_{B_{R_k}(x)}$ is less than one. Hence the right hand side is trivially estimated by

$$\limsup_j k_j^{-n}\left|\alpha_{k_j}(x)\right|^2 / \left\|\alpha_{k_j}\right\|^2_{B_{R_{k_j}}(x)}.$$

According to 2.3 this in turn may be estimated by $\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$. All in all this shows that

$$\pi^{-n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| = \pi^{-n}\lim_j\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| / \left\|\alpha_{k_j}\right\|^2_{B_{R_{k_j}}(x)}$$

Since the left hand side is non-zero on $X(0)$ this proves the claim 2.5. Now to prove the theorem we take a test function $f(x,y)$ and consider the integral

$$k_j^{-n}\int_{X(0)\times X} f(x,y)\left|K^{k_j}(x,y)\right|^2\omega_n(x)\wedge\omega_n(y).$$

We may restrict the integration over the first factor in $X(0)\times X$ to the set of all x satisfying the assumption used above, since the the complement is of measure zero. Using the identity 2.4 the integral over y (for a fixed point x) equals

$$k_j^{-n}B^{k_j}(x)\int_X f(x,y)\left|\alpha_{k_j}(y)\right|^2\omega_n(y).$$

Since by 2.5 the function $\left|\alpha_k(y)\right|^2$ converges to the Dirac measure at x in the weak*-topology, the integral is equal to $f(x,x)$ in the limit and the first factor is equal to $\pi^{-n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$ in the limit (by theorem 2.1). Hence the previous integral is equal to

$$(2\pi)^{-n}\int_{X(0)} f(x,x)(i\partial\overline{\partial}\phi)_n$$

in the limit, which finishes the proof of the theorem. □

As before the convergence may be formulated in terms of Toeplitz operators (a more general statement will be proved in part II (corollary 5.5)).

Corollary 2.5. *Suppose that $X(1)$ is empty. Then*

$$\lim_k k^{-n}TrT_{fg}=\lim_k k^{-n}TrT_fT_g\ .$$

We will now use the results on the asymptotics of the Bergman kernel K to express asymptotic spectral properties of Toeplitz operators in terms of their symbol. Denote by

$$N(T_f>\gamma)$$

the number of eigenvalues of T_f that are greater than the number γ (counted with multiplicity). Furthermore, $N(T_f<\gamma)$ is defined similarly.

Theorem 2.6. *Suppose that $X(1)$ is empty and that f is a real-valued bounded function. Then for all γ except possibly countably many the following holds:*

$$\lim_k k^{-n}N(T_f>\gamma)=(2\pi)^{-n}\int_{\{f>\gamma\}\bigcap X(0)}(i\partial\overline{\partial}\phi)_n \tag{2.6}$$

and similarly for $N(T_f<\gamma)$.

Proof. Given the asymptotic behaviour of $K(x,y)$ in theorem 2.4, the proof can be adapted word by word from [16],[3]. But for completeness we give a proof here, that slightly simplifies the proof in [16]. We first prove the statement when f is the characteristic function for a given set $\Omega : f=1_\Omega$. Let us denote by T_Ω the corresponding operator. We may assume that $1>\gamma>0$. By corollary 2.3 the right hand side of 2.6 is then equal to the limit of $k^{-n}\mathrm{Tr}\ \mathrm{T}_\Omega$. Moreover, by corollary 2.5 this limit in turn is equal to the limit of $k^{-n}\mathrm{Tr}\mathrm{T}_\Omega^2$. We will now see that this can happen only if $\lim_k k^{-n}N(T_\Omega > \gamma) = \lim_k k^{-n}\mathrm{Tr}\ \mathrm{T}_\Omega$, which proves the statement with this special choice of f. Indeed, since if we denote by τ_j the eigenvalues of T_Ω,

$$\lim_k k^{-n} \sum \tau_j(1-\tau_j) = 0,$$

it follows from estimating $(1-\tau_j)$ from below that

$$\lim_k k^{-n} \sum_{\tau_j \leq \gamma} \tau_j = 0.$$

Hence

$$\lim_k k^{-n} \sum_{\tau_j > \gamma} \tau_j = \lim_k k^{-n}\mathrm{Tr}\ \mathrm{T}_\Omega = \int_{\Omega \bigcap X(0)} (\partial\overline{\partial}\phi)_n$$

Now it is not hard to deduce that $\lim_k k^{-n}N(T_\Omega > \gamma) = \int_{\Omega \bigcap X(0)} (\partial\overline{\partial}\phi)_n$.

By comparing an arbitrary function f with a characteristic function we will now finish the proof of the theorem. Let us first prove the lower bound

$$\liminf_k k^{-n}N(T_f > \gamma) \geq (2\pi)^{-n} \int_{\{f>\gamma\} \bigcap X(0)} (i\partial\overline{\partial}\phi)_n. \tag{2.7}$$

First note that we may assume that f is non-negative by adding an appropriate constant to f. By the max-min principle applied to the operator T_f and by 1.3

$$N(T_f > \gamma) = \max\left\{\dim V :\ (f\alpha, \alpha) > \gamma(\alpha,\alpha)\, \forall \alpha \in V\right\},$$

where V is a linear subspace of H^0. Hence we have to find a sequence of subspaces V_k with

$$\dim V_k = k^n (2\pi)^{-n} \int_{\{f>\gamma\} \bigcap X(0)} (i\partial\overline{\partial}\phi)_n + o(k^n), \tag{2.8}$$

such that for any normalized α in V_k $(f\alpha,\alpha) > \gamma$. To this end denote by Ω the set where $f > \gamma$ and denote by χ the corresponding characteristic function. Since we have already proved the theorem for characteristic functions, there is, for any given small positive ε, a sequence of subspaces V_k with the correct dimension 2.8 such that

$$(\chi\alpha, \alpha) > 1 - \varepsilon$$

for all α in V_k, Since by definition $f > \gamma\chi$ it follows that

$$(f\alpha,\alpha) > \gamma(1-\varepsilon),$$

for all α in V_k. By symmetry this means that

$$\liminf_k k^{-n}N(T_f > \gamma) \geq (2\pi)^{-n} \int_{\{f>\gamma+\varepsilon\} \bigcap X(0)} (i\partial\overline{\partial}\phi)_n.$$

By letting ϵ tend to zero in the right hand side we obtain the desired lower bound 2.7. If we now apply this result to the function $-f$ we obtain the following equivalent result:

$$\liminf_{k} k^{-n} N(T_f < \delta) \geq (2\pi)^{-n} \int_{\{f<\delta\} \bigcap X(0)} (i\partial\overline{\partial}\phi)_n. \tag{2.9}$$

Now note that for all except countably many numbers γ, the set $\{f = \gamma\}$ is of measure zero with respect to the measure $(i\partial\overline{\partial}\phi)_n$ on X. Indeed, the function $g(\gamma) = \int_{\{f\leq\gamma\} \bigcap X(0)} (i\partial\overline{\partial}\phi)_n$ on the real line is increasing and it is well-known that an increasing function is continuous accept on a countable set. This forces the measure of the set $\{f = \gamma\}$ to be zero for all γ accept those in the countable set. Finally, since the total number of eigenvalues for any operator T_f is equal to the dimension of $H^0(X, L^k)$, we get, when k tends to infinity, that the sum

$$\lim_{k} k^{-n}(N(T_f > \gamma) + \lim_{k} k^{-n} N(T_f \leq \gamma)$$

is equal to the asymptotic dimension

$$(2\pi)^{-n}(\int_{\{f>\gamma\} \bigcap X(0)} (i\partial\overline{\partial}\phi)_n + (2\pi)^{-n} \int_{\{f<\gamma\} \bigcap X(0)} (i\partial\overline{\partial}\phi)_n)$$

for all γ such that the measure of the set $\{f = \gamma\}$ is zero. Combing this with the lower bounds 2.7 and 2.9, we see that we must have equality in 2.7, which proves the theorem. □

The main application of the previous theorem is to show that there is a large supply of holomorphic sections concentrated on any given set Ω in $X(0)$, in the following sens:

$$\|\alpha\|_{\Omega}^2 \geq (1 - \varepsilon) \|\alpha\|_X^2$$

for any given positive (small) ϵ. To see this, denote by χ the characteristic function of the set Ω, and note that if α is a linear combination of eigen-sections of the Toeplitz operator T_χ then α will be concentrated on Ω, as long as the eigenvalues are bounded from below by $(1-\varepsilon)$. The number of such sections α is precisely the spectral counting function $N(T_\chi \geq 1-\varepsilon)$. Hence, the previous theorem shows that there is a subspace of dimension

$$k^n(2\pi)^{-n} \int_{\Omega} (i\partial\overline{\partial}\phi)_n + o(k^n),$$

consisting of concentrated sections, a result that will be useful when studying sampling sequences in the next section.

The following equivalent formulation of theorem 2.6 can be obtained by standard methods in spectral theory. It generalizes a theorem of Boutet de Monvel and Guillemin [6], [14], valid when L is positive, to the case when $X(1)$ is empty.

Theorem 2.7. *Suppose that $X(1)$ is empty. Let (τ_i) be the eigenvalues of T_f and denote by $d\xi_k$ the spectral measure of T_f divided by k^n, i.e.*

$$d\xi_k := k^{-n} \sum_i \delta_{\tau_i},$$

where δ_{τ_i} is the Dirac measure centered at τ_i. Then $d\xi_k$ tends, in the weak-topology, to the push forward of the measure $1_{X(0)}(2\pi)^{-n}(i\partial\overline{\partial}\phi)_n$ under the map f, i.e.*

$$\lim_i \sum_i a(\tau_i) = (2\pi)^{-n} \int_{X(0)} a(f(x)(i\partial\overline{\partial}\phi)_n$$

for any measurable function a on the real line.

3. Sampling

Let D_k be a finite set of points in X. We say that the sequence of sets D_k is *sampling* for the sequence of Hilbert spaces $H^0(X, L^k)$ if there exists a uniform constant A such that

$$A^{-1}k^{-n} \sum_{D_k} |\alpha(x)|^2 \leq \|\alpha\|^2 \leq Ak^{-n} \sum_{D_k} |\alpha(x)|^2,$$

for any element α in $H^0(X, L^k)$. The points in D_k will assumed to be *separated* in the following sense: the distance between any two points in D_k is bounded from below by a uniform constant times $k^{-1/2}$. Consider the measures $d\nu_k$ on X corresponding to the sets D_k :

$$d\nu_k := k^{-n} \sum_{D_k} \delta_x.$$

Because of the separability assumption their mass is uniformly bounded in k. Hence any subsequence has a subsequence that is weak *-convergent. Denote by $d\nu$ such a limit measure. It is natural to ask how dense the sampling points should be, for large k, in order to be sampling. i.e. we ask for asymptotic density conditions on the measure $d\nu$. The model case is sampling on lattices for the Fock space, i.e. X is taken to be $\mathbb{C}^n$ with its standard Euclidean metric form $\omega$and L is the line bundle with constant positive curvature $-2i\omega$. If the sequence D_k is a sequence of lattices generated over $\mathbb{Z}$ by $k^{-1/2}(a_1, ..., a_{2n})$, where the a_i are positive numbers, then a necessary condition for this sequence to be sampling is that $a_1 \cdots a_{2n} \leq \pi^n$. In ([16], [3]) this necessary condition was generalized to any positive line bundle. Namely, the limit measure has to satisfy

$$d\nu \geq (2\pi)^{-n}(i\partial\overline{\partial}\phi)_n.$$

The next theorem shows that in order to sample $H^0(X, L^k)$ when $X(1)$ is empty, the sampling points have to satisfy the same necessary density conditions in $X(0)$ (the part of X where L is positive) as in the case when the curvature is positive everywhere on X.

Theorem 3.1. *Assume that the sequence of sets D_k is sampling for the sequence of Hilbert spaces $H^0(X, L^k)$. If $X(1)$ is empty then the following necessary condition holds:*

$$d\nu \geq (2\pi)^{-n}(i\partial\overline{\partial}\phi)_n$$

on $X(0)$. In other words

$$\liminf_k \#(D_k \bigcap \Omega)/k^n \geq (2\pi)^{-n} \int_\Omega (i\partial\overline{\partial}\phi)_n \tag{3.1}$$

for any smooth domain Ω contained in $X(0)$.

Proof. Given theorem 2.6 (applied to the characteristic function for a set Ω) the proof can be given word by word as in [16], [3]. For completeness we give the argument. Suppose that the sequence D_k is sampling and consider a set Ω in $X(0)$. As was explained as a comment to theorem 2.6, the theorem shows that there is subspace of dimension

$$k^n(2\pi)^{-n}\int_\Omega (i\partial\overline{\partial}\phi)_n + o(k^n)$$

consisting of functions satisfying the concentration property

$$\|\alpha\|_\Omega^2 \geq (1-\varepsilon)\|\alpha\|_X^2 \tag{3.2}$$

(we fix some small ϵ).Now the claim is that for any such concentrated α the sampling property of the sequence D_k yields

$$\|\alpha\|_X^2 \leq Ak^{-n} \sum_{D_k \bigcap \Omega_k} |\alpha(x)|^2, \tag{3.3}$$

where Ω_k consist of all points with distance smaller than $1/\sqrt{k}$ to Ω. Accepting this for a moment it is easy to see how the theorem follows. First note that it is enough to prove the theorem with Ω replaced with the larger set Ω_k, since the number of points in $D_k \bigcap(\Omega_k - \Omega)$ is of the order $o(k^n)$. To get a contradiction we now assume that the number of points in $D_k \bigcap \Omega_k$ is strictly less than $k^n(2\pi)^{-n}\int_\Omega(i\partial\overline{\partial}\phi)_n + o(k^n)$ (meaning that condition 3.1 in the theorem does not hold). But then we can find a non-trivial element α concentrated on Ω and vanishing in all the points in $D_k \bigcap \Omega_k$. Indeed, α can be chosen in a space of dimension of order $k^n(2\pi)^{-n}\int_\Omega(i\partial\overline{\partial}\phi)_n$ and by assumption there are sufficiently few linear conditions to find such an α vanishing in all the points in $D_k \bigcap \Omega_k$. But then 3.3 forces α to vanish on all of X, which is a contradiction.

Finally, we just have to show how 3.3 follows. For any point x a simple submean inequality gives as in the beginning of section 2:

$$k^n |\alpha(x)|^2 \leq C\|\alpha\|_{B_{1/\sqrt{k}}(x)}^2.$$

By the separation property of D_k we may thus estimate the sum over $D_k \bigcap \Omega_k^c$ to obtain

$$k^{-n}\sum_{D_k \bigcap \Omega_k^c} |\alpha(x)|^2 \leq C\|\alpha\|_{\Omega^c}^2 \leq \varepsilon,$$

where we have used the concentration property 3.2 in the last step. Hence, we have proved 3.3, which finishes the proof of the theorem. □

Part 2. Harmonic forms

The aim of the second part of the paper is to generalize the results in part one to $\overline{\partial}-$*harmonic* $(0,q)-$ forms with values in L^k. We denote the corresponding spaces by $\mathcal{H}^q(X,L^k)$, which by Hodge's theorem are isomorphic to the Dolbeault cohomology groups $H^q(X,L^k)$. The first part was based on the observation 2.1 that there is always an asymptotic *upper* bound on the Bergman kernel function $B_k(x)$ of the space of holomorphic sections with values in L^k. Furthermore, if the line bundle L is such that $X(1)$ is empty, then it was shown, using Demailly's strong Morse inequalities, that the estimate is actually an asymptotic *equality* (at least in the sense of L^1-convergence).

Let us recall the approach to Demailly's inequalities presented in [2]. First one shows that

$$\dim_{\mathbb{C}} \mathcal{H}^q_{\leq \nu_k}(X,L^k) = k^n \frac{(-1)^q}{\pi^n} \int_{X(q)} (\frac{i}{2}\partial\overline{\partial}\phi)_n + o(k^n), \tag{3.4}$$

for the space $\mathcal{H}^q_{\leq \nu_k}(X,L^k)$ spanned by all eigenforms of $\Delta_{\overline{\partial}}$ with eigenvalues bounded by ν_k, where $\nu_k = \mu_k k$ and μ_k is a certain sequence tending to zero. We will refer to the elements of the previous space as *low-energy forms.* The dimension formula 3.4 is deduced from the following point wise asymptotics for the corresponding Bergman kernel functions:

$$B^q_{\leq \nu_k}(x) = \frac{k^n}{\pi^n} 1_{X(q)} \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right| + o(k^n) \tag{3.5}$$

The method of proof is a generalization of the argument used to prove the upper bound on $B_k(x)$ in section 2 and will not be repeated here. From 3.5 one immediately gets an upper bound on the Bergman kernel functions for the space of all harmonic forms:

$$B^q_k(x) \leq \frac{k^n}{\pi^n} 1_{X(q)} \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right| + o(k^n) \tag{3.6}$$

(in [2] these bounds were called *local holomorphic Morse inequalities*). However, the equality 3.5 captures much more of the asymptotic information of the Dolbeault complex as can be seen in the following way. First observe that the complex

$$\left(\mathcal{H}^*_{\mu_k}(X,L^k), \overline{\partial}\right), \tag{3.7}$$

consisting of all eigenforms of $\Delta_{\overline{\partial}}$ with eigenvalue μ_k, forms a finite dimensional sub complex of the Dolbeault complex $\left(\Omega^*(X,L^k), \overline{\partial}\right)$. Indeed, $\overline{\partial}$ commutes with $\Delta_{\overline{\partial}}$. Moreover, the complex is exact in positive degrees for non-zero μ_k. The Witten $\overline{\partial}-$complex is now defined as the direct sum of all the complexes 3.7 with eigenvalue μ_k less than ν_k. By the Hodge theorem the inclusion of the Witten $\overline{\partial}-$complex

$$\left(\mathcal{H}^*_{\leq\nu_k}(X,L^k),\overline{\partial}\right)\hookrightarrow\left(\Omega^*(X,L^k),\overline{\partial}\right),$$

is a quasi-isomorphism, i.e. the cohomologies of the complexes are isomorphic and 3.4 gives the dimension of the components of the Witten $\overline{\partial}-$complex. A homological argument now yields Demailly's *strong* Morse inequalities for the truncated Euler characteristics of the Dolbeault complex [9].

The main point of this is approach to Demailly's inequalities is to first prove the Bergman kernel asymptotics 3.5 - the rest of the proof is more or less as Demailly's original proof, which in turn was inspired by Witten's analytical approach to the classical real Morse inequalities [18].

Let us now turn to the study of harmonic $(0,q)-$ forms for a fixed q, i.e. the space $\mathcal{H}^q(X,L^k)$. As above this space can be identified with the cohomology groups at degree q of the Witten $\overline{\partial}-$complex:

(3.8)

$$...\rightarrow\ \mathcal{H}^{q-1}_{\leq\nu_k}(X,L^k)\ \overset{\overline{\partial}}{\rightarrow}\ \mathcal{H}^{q}_{\leq\nu_k}(X,L^k)\ \overset{\overline{\partial}}{\rightarrow}\ \mathcal{H}^{q+1}_{\leq\nu_k}(X,L^k)\ \rightarrow ...$$

The natural condition on the line bundle L that generalizes the condition that $X(1)$ is empty, which was used in the study of holomorphic sections in part one, is that $X(q-1)$ and $X(q+1)$ both are empty.

Proposition 3.2. *Suppose that $X(q-1)$ and $X(q+1)$ are empty. Then*

$$\dim_{\mathbb{C}}\mathcal{H}^q(X,L^k)=k^n\frac{(-1)^q}{\pi^n}\frac{1}{n!}\int_{X(q)}(\frac{i}{2}\partial\overline{\partial}\phi)^n+o(k^n).$$

Proof. Consider the orthogonal decomposition

$$\mathcal{H}^q_{\leq\nu_k}(X,L^k)=\mathcal{H}^q(X,L^k)\oplus\mathcal{H}^q_+(X,L^k),$$

where $\mathcal{H}^q_+(X,L^k)$ denotes the eigenspaces corresponding to positive eigenvalues. According to the dimension formula 3.4 we just have to show that the dimension of $\mathcal{H}^q_+(X,L^k)$ is of order $o(k^n)$. Since the operator $\overline{\partial}+\overline{\partial}^*$ maps $\mathcal{H}^q_+(X,L^k)$ invectively into $\mathcal{H}^{q+1}_{\leq\nu_k}(X,L^k)\oplus\mathcal{H}^{q-1}_{\leq\nu_k}(X,L^k)$ the proposition now follows by applying the dimension formula 3.4 again. □

Combining the previous proposition with the upper bound on the Bergman kernel function $B^q_k(x)$ of the space of harmonic $(0,q)-$forms shows that

$$k^{-n}B^q_k(x)\rightarrow\frac{1}{\pi^n}1_{X(q)}(x)\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| \tag{3.9}$$

in $L^1(X,\omega_n)$, exactly as in the proof of theorem 2.4. When q is zero $B^q_k(x)$ is the point wise norm of the restriction of the Bergman kernel $K(x,y)$ to the diagonal. For q positive, $K(x,x)$ is (locally) a matrix and $B^q_k(x)$ is its trace. But this means that there is now a larger gap between 3.9 and the behaviour of $K(x,y)$ - one would rather like to obtain generalizations of theorem 2.4 to the matrix elements of $K(x,y)$. To achieve this in an invariant way it will be very convenient to think of $K(x,y)$ as a bundle valued form on $X \times X$. It will turn out that $K(x,y)$ is not only localized on the subset corresponding to $X(q)$ of the diagonal in $X \times X$, but all of the contribution to $K(x,x)$ comes from a special direction. This is already clear from Demailly´s work. In [2] this fact was shown by reducing the problem to a model case in $\mathbb{C}^n$ where it can be checked explicitly. In order to treat $K(x,y)$ as a form and to define the special direction the following formalism will be useful.

3.1. **The super integral and the dagger † (reversed complex conjugation).** Consider first a real $m-$dimensional manifold X. Let f be a differential form on X, i.e. f is an element of $\Omega^*(X,\mathbb{C})$. Then the super integral of f is defined to be the usual integral of the top degree form of f, i.e

$$\int_X f_1 + f_2 + ... + f_m := \int_X f_m,$$

where we have decomposed the form f with respect to the degree grading of $\Omega^*(X,\mathbb{C})$. It is often convenient to think of the super integral of a form as a double integral in the following way. Suppose that we are given a volume element, that we write as ω_m, on X. Fix a point x in X. Then $f(x)$ is element of the exterior algebra over x and we define an "integral" of $f(x)$ by

$$\int_{X^{0|m}} f(x) := f_m(x)/\omega_m(x). \tag{3.10}$$

Next, for a function f_0 on X, we let

$$\int_{X^{m|0}} f(x) := \int_X f(x)\omega_n,$$

i.e. the usual integral over X of f_0 with respect to the volume form ω_n. Then the super integral of a form can be written as

$$\int_X f = \int_{X^{m|0}} \int_{X^{0|m}} f(x).$$

A word on the notation: In the mathematics literature the integral 3.10 is called the Berezin integral. In the physics literature one often thinks of a differential form as a "super" function of m commuting ("bosonic") variables x_i and m anti-commuting ("fermionic") variables dx_i. Taylor-expanding the function $f(x_1,...,x_m,dx_1,...,dx_m)$ in the anti-commuting variables yields the usual expression of a differential form. This has been formalized in the theory of super manifolds, where the super integral corresponds to the integral of a function over the super manifold $X^{m|m}$, that can be obtained from the manifold TX^* by changing the parity along the fibers [11],[8].

Now assume that X is an n-dimensional complex manifold with a hermitian metric ω. Consider a $(0,q)-$form f on X. It is well-known that the squared norm of f may be written as

$$\|f\|_X^2 = c_{n,q} \int_X f \wedge \overline{f} \wedge \omega^{n-q}/(n-q)!,$$

in terms of the usual integral, where $c_{n,q}$ is complex constant needed to make the right hand side real and positive. Using the super integral we may write the squared norm of *any* form f in $\Omega^{0,*}(X,\mathbb{C})$ as

$$\|f\|_X^2 = (\frac{i}{2})^n \int_X f \wedge f^\dagger \wedge e^{\omega'},$$

where $\omega' = -2i\omega$ and where the dagger † is the linear operator on $\Omega^*(X,\mathbb{C})$ that coincides with the usual complex conjugation on $\Omega^1(X,\mathbb{C})$ and satisfying

$$(\alpha \wedge \beta)^\dagger = \beta^\dagger \wedge \alpha^\dagger,$$

for any elements α and β in $\Omega^*(X,\mathbb{C})$. In particular $\omega'^\dagger = \omega'$ and if if $I = (i_1, ..., i_q)$, then

$$dz^I \wedge dz^{I\dagger} = (dz^{i_1} \wedge \overline{dz}^{i_1}) \wedge \cdots \wedge (dz^{i_q} \wedge \overline{dz}^{i_q})$$

If there is also given a hermitian line bundle L over X, then the squared norm of an element α of $\Omega^{0,*}(X,\mathbb{C})$ may be written as

$$\|\alpha\|_X^2 = (\frac{i}{2})^n \int_X \alpha \wedge \alpha^\dagger \wedge e^{-\phi+\omega'}.$$

Note that we are abusing notation here: the function $e^{-\phi}$ representing the fiber metric on L is only defined locally and $\alpha \wedge \alpha^\dagger$ is a $(q,q)-$ form with values in $L \otimes \overline{L}$ and may not be canonically integrated. However, the combination $\alpha \wedge \alpha^\dagger \wedge e^{-\phi}$ yields a well-defined global $(q,q)-$form on X. When X is $\mathbb{C}^n$ with its standard Euclidean metric form $\omega$and L is the line bundle with constant positive curvature ω', the exponent in the norm above, is equal to

$$\sum_i (-z_i\overline{z_i} + dz_i \wedge \overline{dz_i})$$

and from the point of view of super manifolds the corresponding Hilbert space is the space of super functions on $\mathbb{C}^{n|n}$ that are holomorphic in the even variables and anti-holomorphic in the odd variables.

3.2. **The direction form $\chi^{q,q}$.** Fix a point x in $X(q)$. Using the metric ω we can identify the curvature two-form $\partial\overline{\partial}\phi_x$ at x with a hermitian endomorphism of the fiber over x of the holomorphic tangent bundle $TX^{1,0}$ in the usual way [13]. By the definition of $X(q)$, $\partial\overline{\partial}\phi$ has precisely q negative eigenvalues at x and we denote the complex subspace spanned by the corresponding eigenvectors by $V(q)_x$. [2] This defines a sub bundle $V(q)$ of $TX^{1,0}$over $X(q)$. Denote the corresponding inclusion map by i

[2]In the usual real Morse theory the space $V(q)_x$ corresponds to the linearisation of the unstable manifold at a critical point.

and let π be the orthogonal projection of $TX^{1,0}$ onto $V(q)$. On $X(q)$ we define the *direction form* $\chi^{q,q}$ by

$$\chi^{q,q} := \pi^* i^* \omega'_q$$

and extend it by zero to all of X. Locally, $\chi^{q,q}$ can be expressed in the following way on $X(q)$. Let (e_i) be a local orthonormal frame of $TX^{1,0}$ such that $e_1, ..., e_q$ is a local frame for $V(q)$ and denote by e^i the dual $(1,0)-$ forms. Then

$$\chi^{q,q} := e^{I_0} \wedge e^{I_0 \dagger} \tag{3.11}$$

In the sequel, when working with local frames over $X(q)$, we will assume that $e_1, ..., e_q$ are as above. Given a a form f in $\Omega^*(X, \mathbb{C})$ let f_χ be the function defined by

$$f_\chi(x) := (\frac{i}{2})^n \int_{X^{0|n}} \chi^{q,q}(x) \wedge f(x) \wedge e^{\omega'(x)},$$

which is just a compact way of saying that on $X(q)$ f_χ is a sum of $f^{0,0}$and all coefficients f_{JJ} such that $J \bigcap I_0 = \emptyset$,where f_{IJ} denotes components of the form f with respect to the local base elements $e^I \wedge e^{J\dagger}$.

One final remark: in the notation of the previous section one can think of $\chi^{q,q}$ as a cut-off function on the super manifold $X^{n|n}$.

4. Bergman kernels and Toeplitz operators

Let (ψ_i) be an orthonormal base for a finite dimensional Hilbert space $\mathcal{H}^{0,q}$ of $(0,q)-$ forms with values in L. Denote by π_1 and π_2 the projections on the factors of $X \times X$. The *Bergman kernel form* of the Hilbert space $\mathcal{H}^{0,q}$ is defined by

$$\mathbb{K}(x,y) = \sum_i \psi_i(x) \wedge \psi_i(x)^\dagger$$

Hence, $\mathbb{K}(x,y)$ is a form on $X \times X$ with values in the pulled back line bundle $\pi_1^*(L) \otimes \overline{\pi_2^*(L)}$. For a fixed point y we identify $\mathbb{K}_y(x) := \mathbb{K}(x,y)$ with a $(0,q)-$form with values in $L \otimes \Omega^{0,q}(X, \overline{L})_y$. The definition of $\mathbb{K}$ is made so that $\mathbb{K}$ satisfies the following reproducing property:

$$\alpha(y) = (\frac{i}{2})^n \int_X \alpha \wedge \mathbb{K}_y^\dagger \wedge e^{-\phi+\omega'} \tag{4.1}$$

for any element α in $\mathcal{H}^{0,q}$. The restriction of $\mathbb{K}$ to the diagonal can be identified with a $(q,q)-$form on X with values in $L \otimes \overline{L}$. The *Bergman form* is defined as $\mathbb{K}(x,x)e^{-\phi(x)}$, i.e.

$$\mathbb{B}(x) = \sum_i \psi_i(x) \wedge \psi_i(x)^\dagger e^{-\phi(x)} \tag{4.2}$$

and it is a globally well-defined $(q,q)-$form on X. Note that the Bergman function B is the trace of $\mathbb{B}^q$, i.e.

$$B\omega_n = c_{n,q} \mathbb{B} \wedge \omega_{n-q}.$$

For a given form α in $\Omega^{0,q}(X,L)$ and a decomposable form θ in $\Omega^{0,q}(X)_x$ of unit norm, let $\alpha_\theta(x)$ denote the element of $\Omega^{0,0}(X,L)_x$ defined as

$$\alpha_\theta(x) = \langle \alpha, \theta \rangle_x$$

where the scalar product takes values in L_x. We call $\alpha_\theta(x)$ the *value of α at the point x, in the direction θ*. Similarly, let $B_\theta(x)$ denote the function obtained by replacing 4.2 by the sum of the squared point wise norms of $\psi_{i,\theta}(x)$. Then $B_\theta(x)$ has the following useful extremal property:

$$B_\theta(x) = \sup_\alpha |\alpha_\theta(x)|^2 , \tag{4.3}$$

where the supremum is taken over all elements α in $\mathcal{H}^q$ of unit norm. An element α realizing the supremum will be referd to as an *extremal form for the space $\mathcal{H}^{0,q}$ at the point x, in the direction θ*. The reproducing formula 4.1 may now be written as

$$\alpha_\theta(y) = (\alpha, \mathbb{K}_{y,\theta}) \tag{4.4}$$

and we have the following extremal characterization of the Bergman kernel (which also gives 4.3).

Lemma 4.1. *Let α be an an extremal at the point x in the normalized direction θ. Then*

$$|\mathbb{K}_{x,\theta}(y)|^2 = |\alpha(y)|^2 B_\theta(x)$$

Proof. Fix the point x and the form θ in $\Lambda^{n-q,0}(X)_x$ and take frames around x such that $\theta = \overline{e^I}$.Then the pair (x,θ) determines a functional on $\mathcal{H}^q$:

$$\alpha \ \mapsto \ \alpha_I(x) \ .$$

By the reproducing property 4.4

$$\alpha_I(x) = (\frac{i}{2})^n \int_X \alpha \wedge \mathbb{K}_{x,I} \wedge e^{-\phi+\omega'} = (\alpha, \mathbb{K}_{x,I})$$

for any element α in $\mathcal{H}^{0,q}$, where $\mathbb{K}_{x,I} := \sum_i \psi_i^I(x)\overline{\psi_i}$ is an element of $\mathcal{H}^q$. In terms of a frame at y we can write

$$\mathbb{K}_{x,I}(y) := \sum_J K_{IJ}(x,y)\overline{e^J}.$$

By the reproducing property 4.4 $\mathbb{K}_{x,I}/\left\|\mathbb{K}_{x,I}\right\|$ is an extremal at the point x in the direction $\overline{e^I}$. This means that if α is another extremal at the point x in the direction $\overline{e^I}$ then

$$|\alpha(y)|^2 = |\mathbb{K}_{x,I}(y)/\left\|\mathbb{K}_{x,I}\right\||^2 ,$$

The previous equality may be written as

$$B_I(x,x)\,|\alpha(y)|^2 = |\mathbb{K}_{x,I}(y)|^2\, e^{-\phi(x)}, \tag{4.5}$$

since the reproducing property 4.4 shows that $\left\|\mathbb{K}_{x,I}\right\|^2 = K_{II}(x,x)$ and by definition $K_{II}(x,x)e^{-\phi(x)} := B_I(x)$. This proves the lemma. $\square$

Next, denote by $\Omega^{(0)}(X,\mathbb{C})$ the commutative sub algebra $\bigoplus_p \Omega^{p,p}(X,\mathbb{C})$ of $\Omega^*(X,\mathbb{C})$. For an element f in $\Omega^{(0)}(X,\mathbb{C})$, we define T_f, the so called *super Toeplitz operator with form symbol* f, by

$$(T_f\alpha)(y) = (\frac{i}{2})^n \int_X f \wedge \alpha \wedge \mathbb{K}_y^\dagger \wedge e^{-\phi+\omega'}. \tag{4.6}$$

Equivalently,

$$(T_f\alpha,\beta) = (\frac{i}{2})^n \int_X f \wedge \alpha \wedge \beta^\dagger \wedge e^{-\phi+\omega'} \tag{4.7}$$

for all elements α and β of $\mathcal{H}^q$. Note that the operator T_f is hermitian if f is a real form with respect to the real structure on $\bigoplus_p \Omega^{p,p}(X,\mathbb{C})$ defined by $\dagger$,i.e. if $f^\dagger = f$. This means that if f is real-valued in the usual sense i.e. f is an element of $\Omega^*(X,\mathbb{R})$, then $i^m T_f$ is hermitian for some integer m. In the following we will only consider symbols f that are real with respect to $\dagger$.

When studying asymptotic properties of L^k, all objects introduced above will be defined with respect to the line bundle L^k.

Remark 4.2. The term super Toeplitz operator was used in [4] in a closely related context. However, the most natural global setting corresponding to [4] is obtained by taking the sequence of Hilbert spaces to be the spaces $\mathcal{H}^{*,0}(X,L^k)$, i.e the direct sum of all harmonic $(q,0)-$forms with values in L^k, where $q=0,1,..n$ and where L is a *positive* line bundle. Then $\mathcal{H}^{*,0}(X,L^k)$ is actually the space of all holomorphic forms with values in L^k. In particular, the space $\mathcal{H}^{q,0}(X,L^k)$ may be written as $H^0(X,L^k\otimes E_q)$, where E_q is a holomorphic vector bundle, so that the analysis for the corresponding Bergman kernels is reduced to the situation studied in part 1 (twisting with a fixed vector bundle has only minor effects on the analysis).

5. Asymptotic results for Bergman kernels and Toeplitz operators.

The next theorem generalizes the bound 2.3 in part 1 to low-energy forms on X, i.e. elements of $\mathcal{H}^q_{\leq\nu_k}(X,L^k)$. It is a refined formulation of the local weak holomorphic Morse inequalities obtained in [2].

Theorem 5.1. *Fix a point x in X and a direction form θ in $\Lambda^{0,q}(X)_x$. Then the following inequality holds:*

$$\limsup_k \left(k^{-n} \sup \frac{\left|\alpha_\theta(x)\right|^2}{\|\alpha\|^2_{B_{R_k}}} \right) \leq \frac{1}{\pi^n} \left\langle \chi^{q,q}, \theta\wedge\theta^\dagger \right\rangle_x \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right|$$

where the supremum is taken over all elements α of $\mathcal{H}^q_{\leq\nu_k}(X,L^k)$ and $R_k = \frac{lnk}{\sqrt{k}}$.

Proof. In the statement of the theorem concerning local holomorphic Morse inequalities in [2] the global norm $\|\alpha_k\|_X^2$ was considered. However, the proof given there actually yields the stronger statement involving $\|\alpha_k\|_{B_{R_k}}^2$. An outline of the argument is as follows. It is enough to prove the statement for $\theta = \overline{e^I}$. Consider the restriction of the normalized form α_k to the ball B_{R_k} centered at the point x. Let $\beta^{(k)}(z) := k^{-\frac{1}{2}n}\alpha(k^{-\frac{1}{2}}z)$ and extend it by zero to a form on all of $\mathbb{C}^n$. In [2] it was shown that one may assume that the sequence $\beta^{(k)}(z)$ tends to a form β weakly in $L^2(\mathbb{C}^n)$. Moreover, the sequence convergences uniformly with all derivatives on the the unit ball. This entails that β is harmonic with respect to the fiber metric ϕ_0 (section 2) on the trivial line bundle in $\mathbb{C}^n$. Thus,

$$\limsup_k \left|\beta_I^{(k)}(0)\right|^2 / \left\|\beta^{(k)}\right\|_{B_{lnk}(0)}^2 \leq \left|\beta_I(0)\right|^2 / \left\|\beta\right\|_{\mathbb{C}^n}^2 \tag{5.1}$$

where we have used that $\|\beta\|_{\mathbb{C}^n}^2 \leq \limsup \left\|\beta^{(k)}\right\|_{B_{lnk}(0)}^2$ thanks to the weak L^2- convergence in $\mathbb{C}^n$. In [2] the right hand side was shown to be bounded by $\pi^{-n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$ if $I = I_0$ and equal to zero otherwise. Since the limit in the left hand side in 5.1 equals the limit of $k^{-n}\left|\alpha_k^I(x)\right|^2 / \|\alpha_k\|_{B_{R_k}(x)}^2$, this proves the theorem for $\theta = \overline{e^I}$. □

In [2] the asymptotics of $B_k(x)$, the trace of the Bergman kernel form $\mathbb{B}_k$ associated with the Hilbert spaces $\mathcal{H}^q_{\leq\nu_k}(X, L^k)$, was deduced from the previous theorem. In fact, the proof given there actually yields the asympototics of the Bergman kernel form itself. As in part 1 the convergence holds for the Hilbert spaces $\mathcal{H}^q(X, L^k)$ as well, under special conditions on the curvature of L.

Theorem 5.2. *Let $\mathbb{B}_k$ be the Bergman $(q,q)-$ form of the Hilbert space $\mathcal{H}^q_{\leq\nu_k}(X, L^k)$. Then*

$$k^{-n}\mathbb{B}_k(x) \to \frac{1}{\pi^n}\chi^{q,q}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$$

point wise. If $X(q-1)$ and $X(q+1)$ are empty than the convergence holds in $L^1(X, \omega_n)$ for $\mathbb{B}_k^q$ associated to the Hilbert space $\mathcal{H}^q(X, L^k)$.

Proof. Using the extremal property 4.3 of $\mathbb{B}_k$, the upper bound follows immediately from the previous theorem. In particular $k^{-n}B_I(x)$ tends to zero unless $I = I_0$ (using frames as in section 3.2). Hence, it is enough to prove the lower bound for the trace $B_k(x)$ of $\mathbb{B}_k$. But this is contained in the asymptotics 3.9 proved in [2] by constructing a sequence of low-energy forms that become sufficiently large at the point x, when k tends to infinity. The corresponding result for $\mathcal{H}^q(X, L^k)$ follows just as in the proof of theorem 2.1 in part 1, now using proposition 3.2. □

The following corollary is obtained just as in part 1:

Corollary 5.3. *Let T_f be the super Toeplitz operator on the Hilbert space $\mathcal{H}^q_{\leq\nu_k}(X, L^k)$ with symbol form f. Then*

$$\lim_k k^{-n}\, TrT_f = \pi^{-n} \int_X f \wedge \chi^{q,q} \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right| \wedge e^{\omega'}.$$

The right hand side may also be written as

$$(2\pi)^{-n} \int_X f_\chi (i\partial\overline{\partial}\phi)_n.$$

If $X(q-1)$ and $X(q+1)$ are empty then the corresponding result holds for the Hilbert space $\mathcal{H}^q(X, L^k)$.

Now we can give the weak convergence of the Bergman kernel stated in an invariant way.

Theorem 5.4. *Let $\mathbb{K}_k$ be the Bergman kernel form of the Hilbert space $\mathcal{H}^q_{\leq\nu_k}(X, L^k)$ and suppose that f and g are forms in $\Omega^{(0)}(X, \mathbb{C})$. Then*

(5.2)
$$k^{-n} \int_{X\times X} f(x)\wedge g(y)\wedge \mathbb{K}_k(x,y)\wedge \mathbb{K}(x,y)_k^\dagger \wedge e^{\Phi_k(x,y)} \to (2\pi)^{-n} \int_X f_\chi g_\chi (i\partial\overline{\partial}\phi)_n,$$

where $\Phi_k(x,y) = -k\phi(x) - k\phi(y) + \omega'(x) + \omega'(y)$. If $X(q-1)$ and $X(q+1)$ are empty then the corresponding result holds for the Hilbert spaces $\mathcal{H}^q(X, L^k)$.

Proof. Let us first assume that the Hilbert space is $\mathcal{H}^q_{\leq\nu_k}(X, L^k)$. Consider a subset $U\times V$ of X, where U and V are open sets in $X(q)$ with associated local frames e^I_U and e^J_V. By using a partition of unity it is enough to prove the convergence with $X \times X$ replaced by $U \times V$ for any such product. Moreover, by linearity we may assume that $f(x) = F(x)e^{L,L}(x)$ and $g(y) = G(y)e^{M,M}(y)$, where F and G are functions and $e^{I,I}$ is an abbreviations for $e^I \wedge e^{I\dagger}$. Note that f_χ is equal to F if $L \bigcap I_0$ is empty and vanishes otherwise. Using the special form of f and g, the integral 5.2 can be written as

$$\sum k^{-n} \int_{U\times V} F(x)G(y) \left|K_{IJ}(x,y)\right|^2 e^{-k\phi(x)-k\phi(y)} \omega_n(x) \wedge \omega_n(y), \tag{5.3}$$

where the sum is over all (I, J) such that $L \bigcap I$ and $M \bigcap J$ are empty. Let us now show that

$$\lim_k k^{-n} \int_{U\times V} \left|K_{IJ}(x,y)\right|^2 e^{-k\phi(x)-k\phi(y)} \omega_n(x) \wedge \omega_n(y) = 0, \tag{5.4}$$

unless $U \times V$ is contained in $X(q) \times X(q)$ and $(I, J) = (I_0, J_0)$ for the special indices related to the direction form $\chi^{q,q}$ as in 3.11. To see this, assume for example that U is in the complement of $X(q)$ or $I \neq I_0$ on U. The integral above is trivially estimated by the limit of

$$k^{-n} \sum_J \int_{U\times X} \left|K_{IJ}(x,y)\right|^2 e^{-k\phi(x)-k\phi(y)} \omega_n(x) \wedge \omega_n(y),$$

Note that the latter integral may be written as the integral of $\|\mathbb{K}_{x,J}\|^2 e^{-k\phi(x)}$ over all x in U. Now, by the reproducing property 4.4 this integral equals $\int_U B_I\omega_n$, which vanishes if U is in the complement of $X(q)$ or $I \neq I_0$, by theorem 5.2. This proves 5.4. Using 5.4 we may now write the limit of 5.3 as

$$\lim_k k^{-n} \int_{U\times V} f_X(x)g_X(y) \left|K_{I_0J_0}(x,y)\right|^2 e^{-k\phi(x)-k\phi(y)}\omega_n(x)\wedge\omega_n(y).$$

Hence to prove the theorem it is is enough to show that if $U \times V$ is contained in $X(q) \times X(q)$, the following holds:

$$\lim_k k^{-n} \sum_J \int_{U\times V} h(x,y) \left|K_{I_0J}(x,y)\right|^2 = \int_{U\bigcap X(q)} h(x,x)(\partial\overline{\partial}\phi)_n, \tag{5.5}$$

for any test function $h(x,y)$, integrating with respect to $e^{-k\phi(x)-k\phi(y)}\omega_n(x)\wedge \omega_n(y)$ in the left hand side (using 5.4 again). To this end, recall the relation between K and an extremal α at the point x in the direction $\overline{e^{I_0}}$:

$$\sum_J \left|K_{I_0J}(x,y)\right|^2 e^{-k\phi(x)-k\phi(y)} = \left|\alpha(y)\right|^2 e^{-k\phi(y)}B_{I_0}(x),$$

given in lemma 4.1. Now the proof of 5.5, just as the proof of theorem 2.4, is based on the observation that a sequence of extremals α_k at the point x in the direction I_0 satisfies the localization property

$$\lim_k \|\alpha_k\|^2_{B_{R_k}} = 1. \tag{5.6}$$

To show this, note that by theorem 3.9

$$\lim_k k^{-n} \left|\alpha_k^{I_0}(x)\right|^2 e^{-k\phi(x)} = \lim_k k^{-n} B_{I_0}(x) = \pi^{-n} \left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| \tag{5.7}$$

and by theorem 5.1

$$\limsup k^{-n} \left|\alpha_k^{I_0}(x)\right|^2 e^{-k\phi(x)} / \|\alpha\|^2_{B_{R_k}(x)} \leq \pi^{-n} \left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|. \tag{5.8}$$

Now 5.6 follows from 5.7 together with 5.8 just as in the proof of theorem 2.4. Indeed, for a fixed point x the mass of

$$k^{-n}\sum_J \left|K_{I_0J}(x,y)\right|^2 e^{-k\phi(x)-k\phi(y)}$$

considered as a function of y becomes localized close to $y = x$,when k tends to infinity.

Finally, assume that $X(q-1)$ and $X(q+1)$ are empty and consider the Hilbert space $\mathcal{H}^q(X, L^k)$. Using the L^1-convergence in 3.9 one sees that 5.6 holds almost everywhere on X for a subsequence of (α_k). As in the proof of theorem 2.4 this is enough to prove the convergence of $\mathbb{K}(x,y)$ stated in the theorem. □

To formulate the convergence in terms of the local matrix elements of the Bergman kernel, let e^I be a local frame as in section 3.2 on the open set U in X. Then we may write the convergence on $U \times U$ in the following suggestive way

$$\lim_k k^{-n} \left|K_{IJ}(x,y)\right|^2 e^{-k\phi(x)-k\phi(y)} = \delta(x-y)\delta_{IJ}1_{X(q)}(x)1_q(I)\pi^{-n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$$

where $1_q(I) = 1$ if $I = I_0$ and zero otherwise. As in part 1 the convergence may also be formulated in terms of (super) Toeplitz operators:

Corollary 5.5. *Let T_f and T_g be the super Toeplitz operators on the Hilbert space $\mathcal{H}^q_{\leq\nu_k}(X, L^k)$ with symbol forms f and g, respectively. Then*

$$\lim_k k^{-n}\, Tr(T_f T_g) = \lim_k k^{-n}\, Tr(T_{f_X g_X}).$$

If $X(q-1)$ and $X(q+1)$ are empty then the corresponding result holds for the Hilbert spaces $\mathcal{H}^q(X, L^k)$.

Proof. By definition we have that for any α and β in $\mathcal{H}^{0,q}$

$$(T_f\alpha, \beta) = \int_X f \wedge \alpha \wedge \beta^\dagger e^{-k\phi+\omega'}.$$

Choosing $\alpha = T_g\psi$ and $\beta = \Psi$ and and expressing T_f in terms of the Bergman kernel form 4.6 gives

$$(T_f T_g \Psi, \Psi) = \int_{X\times X} f(x) \wedge g(x) \wedge \Psi(x) \wedge \mathbb{K}_k(x,y) \wedge \Psi(y)^\dagger \wedge e^{\Phi_k(x,y)},$$

where $\Phi_k(x,y) = -k\phi(x) - k\phi(y) + \omega'(x) + \omega'(y)$. Finally, if we let Ψ be an orthonormal base element Ψ_i and sum over all i the corollary follows from the previous theorem. □

Finally, the following theorem expresses the asymptotic distribution of the eigenvalues of a super Toeplitz operator in terms of the symbol of the operator and the curvature of the line bundle L.

Theorem 5.6. *Let T_f be the Toeplitz operator with form symbol f on the Hilbert space $\mathcal{H}^q_{\leq\nu_k}(X, L^k)$. Let (τ_i) be the eigenvalues of T_f and denote by $d\xi_k$ the spectral measure of T_f divided by k^n, i.e.*

$$d\xi_k := k^{-n}\sum_i \delta_{\tau_i},$$

where δ_{τ_i} is the Dirac measure centered at τ_i. Then $d\xi_k$ tends, in the weak-topology, to the push forward of the measure $(2\pi)^{-n}(i\partial\overline{\partial}\phi)_n$ under the map f_X, i.e.*

$$\lim_i k^{-n}\sum_i a(\tau_i) = (2\pi)^{-n}\int a(f_X(x)(i\partial\overline{\partial}\phi)_n$$

for any measurable function a on the real line. If $X(q-1)$ and $X(q+1)$ are empty then the corresponding result holds for the Hilbert spaces $\mathcal{H}^q(X, L^k)$.

Proof. As in part 1 we just have to prove the theorem for a equal to the characteristic function of a half interval, i.e. for the counting function of T_f. Using a partition of unity and the max-min-principle we may assume that f is supported in a small open set U and is of the form $Fe^{I,I}$, where F is a function on U. By a comparison argument it is enough to prove the theorem for F a characteristic function 1_Ω, just as in the proof of theorem 2.6 in part 1. Furthermore, as previously we just have to show that

$$\lim_k k^{-n} TrT_f^2 = \lim_k k^{-n} TrT_f.$$

To this end, observe that for $f = 1_\Omega e^{I,I}$ clearly $f_\chi^2 = f_\chi$. Hence, corollary 5.5 shows that $\lim_k TrT_f^2 = \lim_k TrT_{f_\chi}$, which finally is equal to $\lim_k TrT_f$, by corollary 5.3. □

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References

[1] Berman R; Berndtsson B; Sjöstrand J: Asymptotics of Bergman kernels (arXiv.org/abs/math.CV/050636)

[2] Berman, R: Bergman kernels and local holomorphic Morse inequalities. Math Z., Vol 248, Nr 2 (2004), 325–344 (arXiv.org/abs/math.CV/0211235)

[3] Berndtsson, Bo: Bergman kernels related to hermitian line bundles over compact comlex manifolds. Explorations in complex and Riemannian geometry, 1–17, Contemp. Math, 332, Amer. Math. Soc, Providence, RI, 2003.

[4] Borthwick, D;Klimek,S;Lesniewski, A; Rinaldi, M: Super Toeplitz operators and nonperturbative deformation quantization of supermanifolds. Comm. Math. Phys. 153 (1993)

[5] Bouche, T: Asymptotic results for Hermitian line bundles over complex manifolds: the heat kernel approach. Higher-dimensional complex varieties (Trento, 1994), 67–81, de Gruyter, Berlin, 1996.

[6] Boutet de Monvel, L-Guillemin, V: The spectral theory of Toeplitz operators. Annals of Mathematical Studies, 99. Princeton University Press, Princeton, NJ, University of Tokyo Press, Tokyo, 1981

[7] Boutet de Monvel,L-Sjöstrand, J: Sur la singularite des noyaux de Bergman et Szegö: Journees: Equations aux derivees partielles de Rennes (1975), 123–164, Asterisque, No. 34-35. Soc. Math. France, Paris, 1976

[8] Cartier, P: DeWitt-Morette, C; Ihl, M; Sämann, C: Supermanifolds - applications to supersymmetry.

[9] Demailly, J-P: Champs magnetiques et inegalite de Morse pour la d''-cohomologie., Ann Inst Fourier, 355 (1985,185-229)

[10] Demailly, J-P: Holomorphic Morse inequalities. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 93-114

[11] DeWitt, B Supermanifolds. Second edition. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1992

[12] Donnelly, H: Spectral theory for tensor products of Hermitian holomorphic line bundles. Math. Z. 245 (2003) no. 1, 31–35

[13] Griffiths, P; Harris, J: Principles of algebraic geometry. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[14] Guillemin, V: Some classical theorems in spectral theory revisited. 219–259 in Seminar on Singularities of Solutions of Linear Partial Differential Equations.

Annals of Mathematics Studies, 91. Princeton University Press, Princeton, N.J, 1979.

[15] Hörmander, L: L^2 estimates and existence theorems for the $\overline{\partial}-$operator. Acta Math. 113 1965 89–152.

[16] Lindholm, N: Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel. J. Funct. Anal. 182 (2001), no. 2, 390–426

[17] Tian, G: On a set of polarized Kähler metrics on algebraic manifolds. J. Differential Geom. 32 (1990), no. 1, 99–130

[18] Witten, E: Supersymmetry and Morse theory. J. Differential Geom. 17 (1982), no. 4, 661–692.

[19] Zelditch, S: Szegö kernels and a theorem of Tian. Internat. Math. Res. Notices 1998, no. 6, 317–331.

Paper III

HOLOMORPHIC MORSE INEQUALITIES ON MANIFOLDS WITH BOUNDARY

ROBERT BERMAN

ABSTRACT. Let X be a compact complex manifold with boundary and let L^k be a high power of a hermitian holomorphic line bundle over X. When X has no boundary, Demailly's holomorphic Morse inequalities give asymptotic bounds on the dimensions of the Dolbeault cohomology groups with values in L^k, in terms of the curvature of L. We extend Demailly's inequalities to the case when X has a boundary by adding a boundary term expressed as a certain average of the curvature of the line bundle and the Levi curvature of the boundary. Examples are given that show that the inequalities are sharp.

Soit X une varieté complexe compacte à bord et soit L^k une grande puissance d'un fibré en droites hermitien holomorphe sur X. Quand X n'a pas de bord, les inégalités de Morse holomorphes de Demailly donnent des estimations asymptotiques des dimensions des groupes de cohomologie de Dolbeault à valuers dans L^k, en termes de la courbure de X. On étend les inégalités de Demailly au cas où X a un bord, en ajoutant un terme au bord exprimé comme une certaine moyenne de la courbure du fibré et de la courbure de Levi du bord. Des exemples sont donnés qui montrent que les inégalités sont optimales.

1. INTRODUCTION

Let X be a compact $n-$dimensional complex manifold with boundary. Let ρ be a defining function of the boundary of X, i.e. ρ is defined in a neighborhood of the boundary of X, vanishing on the boundary and negative on X. We take a hermitian metric ω on X such that $d\rho$ is of unit-norm close to the boundary of X. The restriction of the two-form $i\partial\overline{\partial}\rho$ to the maximal complex subbundle $T^{1,0}(\partial X)$ of the tangent bundle of ∂X, is the Levi curvature form of the boundary ∂X. It will be denoted by $\mathcal{L}$. Furthermore, let L be a hermitian holomorphic line bundle over X with fiber metric ϕ, so that $i\partial\overline{\partial}\phi$ is the curvature two-form of L. It will be denoted by Θ. The line bundle L is assumed to be smooth up to the boundary of X. Strictly speaking, ϕ is a collection of local functions. Namely, let s_i be a local holomorphic trivializing section of L, then locally, $|s_i(z)|^2 = e^{-\phi_i(z)}$. The notation $\eta_p := \eta^p/p!$ will be used in the sequel, so that the volume form on X may be written as ω_n.

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When X is a compact manifold without boundary Demailly's (weak) holomorphic Morse inequalities [9] give asymptotic bounds on the dimension of the Dolbeault cohomology groups associated to the k :th tensor power of the line bundle L :

$$\dim_{\mathbb{C}} H^{0,q}(X, L^k) \leq k^n(-1)^q(\frac{1}{2\pi})^n \int_{X(q)} \Theta_n + o(k^n), \tag{1.1}$$

where $X(q)$ is the subset of X where the curvature-two form Θ has exactly q negative eigenvalues, i.e the set where index$(\Theta) = q$. Demailly's inspiration came from Witten's analytical proof of the classical Morse inequalities for the Betti numbers of a real manifold [28], where the role of the fiber metric ϕ is played by a Morse function. Subsequently, holomorphic Morse inequalities on manifolds with boundary where studied. The cases of $q-$convex and $q-$concave boundary were studied by Bouche [7], and Marinescu [21], respectively, and they obtained the same curvature integral as in the case when X has no boundary. However, it was assumed that, close to the boundary, the curvature of the line bundle L is adapted to the curvature of the boundary. For example, on a pseudoconcave manifold (i.e the Levi form is negative on the boundary) it is assumed that the curvature of L is non-positive close to the boundary. This is related to the well-known fact that in the global L^2- estimates for the $\overline{\partial}-$operator of Morrey-Kohn-Hörmander-Kodaira there is a curvature term from the line bundle as well from the boundary and, in general, it is difficult to control the sign of the total curvature contribution. Morse inequalities over strictly pseudoconvex CR manifolds have been obtained by Getzler [15], who also suggested that one should try to prove similar formulas for the $\overline{\partial}-$Neumann problem on a complex manifold with boundary. This will be done in the present paper.

We will consider an arbitrary holomorphic line bundle L over a complex manifold with boundary and extend Demailly's inequalities to this situation. We will write $h^q(L^k)$ for the dimension of $H^{0,q}(X, L^k)$, the Dolbeault cohomology group of $(0, q)-$forms with values in L^k. The cohomology groups are defined with respect to forms that are smooth up to the boundary. Recall that $X(q)$ is the subset of X where index$(\Theta) = q$ and we let

$$T(q)_{\rho,x} = \left\{t > 0 : \text{ index}(\Theta + t\mathcal{L}) = q \text{ along } T^{1,0}(\partial X)_x\right\}.$$

The main theorem we will prove is the following generalization of Demailly's weak holomorphic Morse inequalities.

Theorem 1.1. *Suppose that X is is a compact complex manifold with boundary, such that the Levi form is non-degenerate on the boundary. Then, up to terms of order $o(k^n)$,*

(1.2)
$$h^q(L^k) \leq k^n(-1)^q(\frac{1}{2\pi})^n(\int_{X(q)} \Theta_n + \int_{\partial X}\int_{T(q)_{\rho,x}} (\Theta + t\mathcal{L})_{n-1} \wedge \partial\rho \wedge dt),$$

The boundary integral above may also be expressed more directly in terms of symplectic geometry as

$$\int_{X_+(q)} (\Theta + d\alpha)_n \tag{1.3}$$

where $(X_+, d\alpha)$ is the symplectification of the contact manifold ∂X induced by the complex structure of X (section 7.1).

Examples will be presented that show that the leading constants in the bounds of the theorem are sharp. We will also obtain the corresponding generalization of the strong holomorphic Morse inequalities. The most interesting case is when the manifold is a strongly pseudoconcave manifold X of dimension $n \geq 3$ with a positive line bundle L. Then, if the curvature forms of L and ∂X are conformally equivalent along the complex tangential directions of ∂X, we will deduce that

$$h^0(L^k) = k^n(\int_X \Theta_n + \frac{1}{n}\int_{\partial X} (i\partial\overline{\partial}\rho)_{n-1} \wedge i\partial\rho)) + o(k^n), \tag{1.4}$$

if the defining function ρ is chosen in an appropriate way. In particular, such a line bundle L is *big* and 1.4 can be expressed as

$$\mathrm{Vol}(L) = \mathrm{Vol}(X) + \frac{1}{n}\mathrm{Vol}(\partial X)$$

in terms of the corresponding symplectic volume of X and contact volume of ∂X. Examples are provided that show that theorem 6.5 is sharp and also compatible with "hole filling".

The proof of theorem 6.5 will follow from local estimates for the corresponding Bergman function $B_X^{q,k}$ where $B_X^{q,k}$ is the Bergman function of the space $\mathcal{H}^{0,q}(X, L^k)$ of $\overline{\partial}-$harmonic $(0,q)-$forms satisfying $\overline{\partial}-$Neumann boundary conditions (simply referred to as the harmonic forms in the sequel). The point is that the integral of the Bergman function is the dimension of $\mathcal{H}^{0,q}(X, L^k)$. It is shown that, for large k, the Bergman function (or more precisely the corresponding measure) is estimated by the sum of two model Bergman functions, giving rise to the bulk and the boundary integrals in theorem 6.5. The model at a point x in the interior of X is obtained by replacing the manifold X with flat $\mathbb{C}^n$ and the line bundle L with the constant curvature line bundle over $\mathbb{C}^n$ obtained by freezing the curvature of the line bundle at the point x. Similarly, the model at a boundary point is obtained by replacing X with the unbounded domain X_0 in $\mathbb{C}^n$, whose constant Levi curvature is obtained by freezing the Levi curvature at the boundary point in X. The line bundle L is replaced by the constant curvature line bundle over X_0, obtained by freezing the curvature along the complex tangential directions, while making it flat in the complex normal direction.

The method of proof is an elaboration of the, comparatively elementary, technique introduced in [4] to handle Demailly's case of a manifold without boundary.

Remark 1.2. The boundary integral in 1.2 is finite precisely when there is no point in the boundary where the Levi form $i\partial\overline{\partial}\rho$ has exactly q negative eigenvalues. Indeed, any sufficiently large t will then be in the complement of the set $T(q)_{\rho,x}$. Since, we have assumed that the Levi form $i\partial\overline{\partial}\rho$ is non-degenerate, this condition coincides with the so called condition $Z(q)$ [14]. However, for an arbitrary Levi form the latter condition is slightly more general: it holds if the Levi form has at least $q+1$ negative eigen values or at least $n-q$ positive eigen values everywhere on ∂X. In fact, the proof of theorem 6.5 only uses that ∂X satisfies condition $Z(q)$ and is hence slightly more general than stated. Furthermore, a function ρ is said to satisfy condition $Z(q)$ at a point x if x is not a critical point of ρ and if $\partial\overline{\partial}\rho$ satisfies the curvature condition at x along the level surface of ρ passing through x.

One final remark about the extension of the Morse inequalities to open manifolds:

Remark 1.3. The cohomology groups $H^{0,*}(X, L^k)$ associated to the manifold with boundary X occurring in the weak Morse inequalities, theorem 2.1, are defined with respect to forms that are smooth up to the boundary. Removing the boundary from X we get an open manifold, that we denote by $\dot{X}$. By the Dolbeault theorem [16] the usual Dolbeault cohomology groups $H^{0,*}(\dot{X}, L^k)$ of $\dot{X}$ are isomorphic to the cohomology groups $H^*(\dot{X}, \mathcal{O}(L^k))$ of the sheaf $\mathcal{O}(L^k)$ of germs of holomorphic sections on $\dot{X}$ with values in L^k. Moreover, if we assume that condition $Z(q)$ *and* $Z(q+1)$ hold then $H^{0,q}(X, L^k)$ and $H^{0,q}(\dot{X}, L^k)$ are isomorphic [14]. Furthermore, consider a given *open* manifold Y with a smooth exhaustion function ρ, i.e a function such that the open sublevel sets of ρ are relatively compact in Y for every real number c. Then, if for a fixed regular value c_0, the curvature conditions $Z(q)$ and $Z(q+1)$ hold for ρ when $\rho \geq c_0$, the group $H^{0,q}(Y, L^k)$ is isomorphic to $H^{0,q}(X_{c_0}, L^k)$[19], where X_{c_0} is the corresponding closed sublevel set of ρ. In this way one gets Morse inequalities on certain open manifolds Y.

Notation 1.4. The notation $a_k \sim (\lesssim) b_k$ will stand for $a_k = (\leq) C_k b_k$, where C_k tends to one when k tends to infinity. The $\overline{\partial}-$Laplacian [16] will be called just the Laplacian. It is the differential operator defined by $\Delta := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ (where $\overline{\partial}^*$ denotes the formal adjoint of $\overline{\partial}$) acting on smooth forms on X with values in L^k. Similarly, we will call an element in the kernel of Δ harmonic, instead of $\overline{\partial}-$harmonic.

The paper is organized in two parts. In the first part we will state and prove the *weak* holomorphic Morse inequalities. In the second part the *strong* holomorphic Morse inequalities are obtained. Finally, the weak Morse inequalities are shown to be sharp and the relation to hole filling investigated.

Contents

Part 1. The weak Morse inequalities

2. Setup and a sketch of the proof

In the first part we will show how to obtain weak holomorphic Morse inequalities for $(0,q)-$ forms, with values in a given line bundle L over a manifold with boundary X. In other words we will estimate the dimension of $H^{0,q}(X,L^k)$ in terms of the curvature of L and the Levi curvature of the boundary of X. With notation as in the introduction of the article the theorem we will prove is as follows.

Theorem 2.1. *Suppose that X is is a compact complex manifold with boundary, such that the Levi form is non-degenerate on the boundary. Then, up to terms of order $o(k^n)$,*

$$h^q(L^k) \leq k^n(-1)^q(\frac{1}{2\pi})^n(\int_{X(q)} \Theta_n + \int_{\partial X}\int_{T(q)_{\rho,x}} (\Theta + t\mathcal{L})_{n-1} \wedge \partial\rho \wedge dt),$$

Note that the last integral is independent of the choice of defining function. Indeed, if $\rho' = f\rho$ is another defining function, where f is a positive function, the change of variables $s = ft$ shows that the integral is unchanged. A more intrinsic formulation of the last integral will be given in section 7.1. Let us now fix the grade q. Since, the statement of the theorem is vacuous if the Levi form $i\partial\overline{\partial}\rho$ has exactly q negative eigenvalues somewhere on ∂X (compare remark 1.2) we may assume that this is not the case. Then it is well-known that the Dolbeault cohomology group $H^{0,q}(X,E)$ is finite dimensional for any given vector bundle E over X. The cohomology groups are defined using forms that are smooth up to the boundary. Moreover, the Hodge theorem, in this context, says that $H^{0,q}(X,E)$ is isomorphic to the space $\mathcal{H}^{0,q}(X,E)$, consisting of harmonic $(0,q)-$forms, that are smooth up to the boundary, where they satisfy $\overline{\partial}-$Neumann boundary conditions ([14]). The space $\mathcal{H}^{0,q}(X,E)$ is defined with respect to given metrics on X and E.

The starting point of the proof of theorem 2.1 is the fact that the dimension of the space $\mathcal{H}^{0,q}(X,L^k)$ may be expressed as an integral over X of the so called Bergman function $B_X^{q,k}$ defined as

$$B_X^{q,k}(x) = \sum_i |\Psi_i(x)|^2 ,$$

where $\{\Psi_i\}$ is any orthonormal base for $\mathcal{H}^{0,q}(X, L^k)$. Indeed, the integral of each term in the sum is equal to one. Note that the Bergman function $B_X^{q,k}$ depends on the metric ω on X. It is convenient to use $k\omega$ as metric for a given k. The point is that the volume of X measured with respect to $k\omega$ is of the order k^n. Hence, the dimension bound in theorem 2.1 will follow from a point-wise estimate of the corresponding Bergman function $B_X^{q,k}$. Another reason why $k\omega$ is a natural metric on X is that, since $k\phi$ is the induced fiber metric on L^k, the norms of forms on X with values in L^k become more symmetrical with respect to the base and fiber metrics. In fact, we will have to let the metric ω itself depend on k (and on a large parameter R) close to the boundary and we will estimate the Bergman function of the space $\mathcal{H}^{0,q}(X, k\omega_k, L^k)$ in terms of model Bergman functions and compute the model cases explicitly. The sequence of metrics ω_k will be of the following form. First split the manifold X in an *inner region* X_ε, with defining function $\rho + \varepsilon$ and its complement, the *boundary region,* given a small positive number ϵ. The level sets where $\rho = -Rk^{-1}$ and $\rho = -k^{-1/2}$ divide the boundary region into three regions. The one that is closest to the boundary of X will be called the *first region* and so on. Next, define ω_T, the complex tangential part of ω close to the boundary by

$$\omega_T := \omega - 2i\partial\rho \wedge \overline{\partial\rho}$$

(recall that we assumed that $d\rho$ is of unit-norm with respect to ω close to the boundary of X). The metric ω_k is of the form

$$\omega_k = \omega_T + a_k(\rho)^{-1} 2i\partial\rho \wedge \overline{\partial\rho}, \tag{2.1}$$

where the sequence of smooth functions a_k will be chosen so that, basically, the distance to the boundary, when measured with respect to $k\omega_k$, in the three different regions is independent of k. The properties of ω_k that we will use in the two regions will be stated in the proofs below, while the precise definition of ω_k is deferred to section 5.4.

2.1. **A sketch of the proof of the weak Morse inequalities.** To make the sketch of the proof cleaner, we will just show how to estimate the *extremal function*

$$S_X^{q,k}(x) = \sup_{\alpha_k} |\alpha_k(x)|^2 \tag{2.2}$$

closely related to $B_X^{q,k}$, where the supremum is taken over all normalized elements of the space $\mathcal{H}^{0,q}(X, \omega_k, L^k)$. When $q = 0$, i.e the case of holomorphic sections, it is a classical fact that they are actually equal and the general relation is given in section 3. Let us first see how to get the following bound in the inner region X_ε defined above:

$$S_X^{q,k}(x) \lesssim S_{\mathbb{C}^n,x}^q(0), \tag{2.3}$$

where the right hand side is the extremal function for the model case defined below. Moreover, the left hand side is uniformly bounded by a constant, which is essential when integrating the estimate to get an estimate on the dimension of $\mathcal{H}^{0,q}(X, k\omega_k, L^k)$. The proof of 2.3 proceeds exactly as in the case when X is a compact manifold without boundary [4]. Let us recall the argument, slightly reformulated. Fix a point x in X_ε. We may take local holomorphic coordinates centered at x and a local trivialization of L such that

$$\phi(z) = \sum_{i=1}^{n} \lambda_i z_i \overline{z_i} + ..., \quad \omega(z) = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge \overline{dz_i} + ... \tag{2.4}$$

where the dots indicate lower order terms and the leading terms are called model metrics and denoted by ϕ_0 and ω_0, respectively. Hence, the model situation is a line bundle of constant curvature on flat $\mathbb{C}^n$. Note that the unit ball at x with respect to the metric $k\omega_k$ corresponds approximately to the coordinate ball at 0 of radius $k^{-1/2}$. To make this more precise, define a scaling map

$$F_k(z) = k^{-1/2} z$$

and consider a sequence of expanding balls centered at 0 in $\mathbb{C}^n$ of radius r_k, slowly exhausting all of $\mathbb{C}^n$. We will call $F_k^*(k\phi)$ and $F_k^*(k\omega)$ the scaled metrics on the expanding balls. The point is that they converge to the model metrics ϕ_0 and ω_0. This follows immediately from the expressions 2.4 and the fact that the model metrics are invariant when kF_k^* is applied. Next, given a $(0, q)$ form on X with values in L^k, we denote by $\alpha^{(k)}$ the scaled form defined by $\alpha^{(k)} = F_k^* \alpha_k$. Then, by the convergence of the scaled metrics,

$$\|\alpha_k\|^2_{F_k(B_{r_k})} \sim \left\|\alpha^{(k)}\right\|^2_{B_{r_k}}, \tag{2.5}$$

using the norms induced by the model metrics in the right hand side above. Now, if α_k is a normalized sequence of extremals (i.e realizing the extremum in 2.2) we have

$$S_X^{q,k}(x) = \left|\alpha^{(k)}(0)\right|^2.$$

By 2.5, the norms of the scaled sequence $\alpha^{(k)}$ are less than one, when k tends to infinity. Moreover, $\alpha^{(k)}$ is harmonic with respect to the scaled metrics and since these converge to model metrics, inner elliptic estimates for the Laplacian show that there is a subsequence of $\alpha^{(k)}$ that converges to a model harmonic form β in $\mathbb{C}^n$. In fact, we may assume that the whole sequence $\alpha^{(k)}$ converges. Hence,

$$\limsup_k \left|\alpha^{(k)}(0)\right|^2 = |\beta(0)|^2$$

which in turn is bounded by the model extremal function $S_{X_0,x}(0)$. Moreover, since X_ε may be covered by coordinate balls of radius $k^{-1/2}$, staying inside of X for large k, one actually gets a uniform bound.

Let us now move on to the boundary region $X - X_\varepsilon$ that we split into three regions as above. Fix a point σ in the boundary of X. We may take local coordinates centered at σ and orthonormal at σ, so that

$$\rho(z,w)=v-\sum_{i=1}^{n-1}\mu_i\left|z_i\right|^2+...$$

where v is the imaginary part of w [8]. The leading term of ρ will be denoted by ρ_0, and will be referred to as the defining function of the *model domain* X_0 in $\mathbb{C}^n$. Observe that the model domain X_0 is invariant under the holomorphic anisotropic scaling map

$$F_k(z,w)=(z/k^{1/2},w/k).$$

Moreover, the scaled fiber metric on L^k now tends to the new model fiber metric $\phi_0(z,0)$, since the terms in ϕ involving the coordinate w are suppressed by the anisotropic scaling map F_k. Now, the bound 2.3 is replaced by

$$S_X^{q,k}(0,iv/k)\lesssim S(0,iv), \tag{2.6}$$

in terms of the new model case. To see this one replaces the balls of decreasing radii used before with $F_k(D_k)$ intersected with X, where D_k is a sequence of slowly expanding polydiscs. Moreover, we have to let the initial metric ω on X depend on k in the normal direction in order that the scaled metric converge to a non-degenerate model metric. In the first region we will essentially let

$$\omega_k=\omega_T+k2i\partial\rho\wedge\overline{\partial\rho}.$$

As a *model metric* in X_0 we will essentially use

$$\omega_0=\frac{i}{2}\partial\overline{\partial}\left|z\right|^2+2i\partial\rho_0\wedge\overline{\partial\rho_0}.$$

Then clearly

$$F_k^*(k\omega_k)=\omega_0 \tag{2.7}$$

in the model case and it also holds asymptotically in k, in the general case. Replacing the inner elliptic estimates used in the inner part X_ε with subelliptic estimates for the $\overline{\partial}-$Laplacian close to the boundary one gets the bound 2.6 more or less as before. Finally, using similar scaling arguments, one shows that the contribution from the second and third region to the total integral of $B_X^{q,k}$ is negligible when k tends to infinity.

This gives the bound

$$\int_X B_X^{q,k}(k\omega_k)_n\lesssim k^n(\int_X B^q_{\mathbb{C}^n,x}\omega_n+\int_{\partial X}\int_{-\infty}^0 B^q_{X_0,\sigma}(iv)dvd\sigma)$$

integrating over an infinite ray in the model region X_0 in the second integral (after letting R tend to infinity). Computing the model Bergman functions explicitly then finishes the proof of the theorem.

3. Bergman kernel forms

Let us now turn to the detailed proof of theorem 2.1. First we introduce Bergman kernel forms to relate the Bergman function $B_X^{q,k}$ to extremal functions taking account of the components of a form (see [5] for proofs). Let (ψ_i) be an orthonormal base for a finite dimensional Hilbert space $\mathcal{H}^{0,q}$ of $(0,q)-$ forms with values in L. Denote by π_1 and π_2 the projections on the factors of $X \times X$. The *Bergman kernel form* of the Hilbert space $\mathcal{H}^{0,q}$ is defined by

$$\mathbb{K}^q(x,y) = \sum_i \overline{\psi_i(x)} \wedge \psi_i(y)$$

Hence, $\mathbb{K}^q(x,y)$ is a form on $X \times X$ with values in the pulled back line bundle $\overline{\pi_1^*(L)} \otimes \pi_2^*(L)$. For a fixed point x we identify $\mathbb{K}_x^q(y) := \mathbb{K}^q(x,y)$ with a $(0,q)-$form with values in $L \otimes \Lambda^{0,q}(X,\overline{L})_x$. The definition of $\mathbb{K}^q$ is made so that $\mathbb{K}^q$ satisfies the following reproducing property:

$$\alpha(x) = c_{n,q} \int_X \alpha \wedge \overline{\mathbb{K}_x^q} \wedge e^{-\phi}\omega_{n-q}, \tag{3.1}$$

for any element α in $\mathcal{H}^{0,q}$, using a suggestive notation and where $c_{n,q}$ is a complex number of unit norm that ensures that 3.1 may be interpreted as a scalar product. Properly speaking, $\alpha(x)$ is equal to the push forward $\pi_{2*}(c_{n,q}\alpha \wedge \mathbb{K}^q \wedge \omega_{n-q}e^{-\phi})(x)$. The restriction of $\mathbb{K}^q$ to the diagonal can be identified with a $(q,q)-$form on X with values in $\overline{L} \otimes L$. The *Bergman form* is defined as $\mathbb{K}^q(x,x)e^{-\phi(x)}$, i.e.

$$\mathbb{B}^q(x) = \sum_i \overline{\psi_i(x)} \wedge \psi_i(x)e^{-\phi(x)} \tag{3.2}$$

and it is a globally well-defined $(q,q)-$form on X. The following notation will turn out to be useful. For a given form α in $\Omega^{0,q}(X,L)$ and a decomposable form in $\Omega^{0,q}(X)_x$ of unit norm, let $\alpha_\theta(x)$ denote the element of $\Omega^{0,0}(X,L)_x$ defined as

$$\alpha_\theta(x) = \langle \alpha, \theta \rangle_x$$

where the product takes values in L_x. We call $\alpha_\theta(x)$ the *value of α at the point x, in the direction θ*. Similarly, let $B_\theta^q(x)$ denote the function obtained by replacing 3.2 by the sum of the squared pointwise norms of $\psi_{i,\theta}(x)$. Then $B_\theta^q(x)$ has the following useful extremal property:

$$B_\theta^q(x) = \sup_\alpha |\alpha_\theta(x)|^2, \tag{3.3}$$

where the supremum is taken over all elements α in $\mathcal{H}^{0,q}$ of unit norm. The supremum will be denoted by $S_\theta^q(x)$ and an element α realizing the supremum will be referred to as an *extremal form for the space $\mathcal{H}^{0,q}$ at the point x, in the direction θ*. The reproducing formula 3.1 may now be written as

$$\alpha_\theta(x) = (\alpha, \mathbb{K}_{x,\theta}^q).$$

Finally, note that the Bergman function B is the trace of $\mathbb{B}^q$, i.e.

$$B^q\omega_n = c_{n,q}\mathbb{B}^q \wedge \omega_{n-q}.$$

Using the extremal characterization 3.3 we have the following useful expression for B :

$$B(x) = \sum_{\theta} S_\theta(x), \tag{3.4}$$

where the sum is taken over any orthonormal base of direction forms θ in $\Lambda^{0,q}(X)_x$.

4. The model boundary case

In the sketch of the proof of the weak holomorphic Morse inequalities (section 2.1) it was explained how to bound the Bergman function on X by model Bergman functions. In this section we will compute the Bergman kernel explicitly in the model boundary case. Consider $\mathbb{C}^n$ with coordinates (z, w), where z is in $\mathbb{C}^{n-1}$ and $w = u + iv$. Let X_0 be the domain with defining function

$$\rho_0(\mathrm{z}, w) = v + \psi_0(z) := v + \sum_{i=1}^{n-1} \mu_i \left| z_i \right|^2,$$

and with the metric

$$\omega_0 = \frac{i}{2} \partial\overline{\partial} \left| z \right|^2 + a(\rho)^{-1} 2i \partial\rho_0 \wedge \overline{\partial\rho_0}.$$

Note that the corresponding volume element $(\omega_0)_n$ is given by $a(\rho)^{-1}$ times the usual Euclidean volume element on $\mathbb{C}^n$. We will take $a(\rho_0)$ to be comparable to $(1-\rho_0)^2$ (compare section 5.4) but we will only use that the corresponding metric ω_0 is "relatively complete" (compare section 4.1). We fix the q and assume that condition $Z(q)$ holds on ∂X_0, i.e. that at least $q+1$ of the eigen values μ_i are negative or that at least $n-q$ of them are positive.

Let $\mathcal{H}^{0,q}(X_0, \phi_0)$ be the space of all $(0, q)-$ forms on X that have finite L^2- norm with respect to the norms defined by the metric ω_0 and the weight $e^{-\phi_0(z)}$, where ϕ_0 is quadratic, and that are harmonic with respect to the corresponding Laplacian. Moreover, we assume the the forms are smooth up to the boundary of X_0 where they satisfy $\overline{\partial}-$Neumann boundary conditions (in fact, the regularity properties are automatic, since we have assumed that condition $Z(q)$ holds [14]). The Bergman kernel form of the Hilbert space $\mathcal{H}^{0,q}(X_0, \phi_0)$ will be denoted by $\mathbb{K}^q_{X_0}$. We will show how to expand $\mathbb{K}^q_{X_0}$ in terms of Bergman kernels on $\mathbb{C}^{n-1}$, and then compute it explicitly. Note that the metric ω_0 is chosen so that the pullback of any form on $\mathbb{C}^{n-1}$ satisfies $\overline{\partial}-$Neumann boundary conditions. Conversely, we will show that any form in $\mathcal{H}^{0,q}(X_0, \phi_0)$ can be written as a superposition of such pulled-back forms.

By the very definition of the metric ω_0, the forms dz_i and $a^{-1/2}\partial\rho_0$ together define an orthogonal frame of $(1,0)-$ forms. Any $(0, q)-$ form α on X may now be uniquely decomposed in a *tangential* and a *normal* part:

$$\alpha = \alpha_T + \alpha_N,$$

where $\alpha = \alpha_T$ modulo the algebra generated by $\overline{\partial \rho_0}$. A form α without normal part will be called *tangential.* The proof of the following proposition is postponed till the end of the section.

Proposition 4.1. *Suppose that α is in $\mathcal{H}^{0,q}(X_0, \phi_0)$. Then α is tangential, closed and coclosed (with respect to $\overline{\partial}$).*

By the previous proposition any form α in $\mathcal{H}^{0,q}(X_0, \phi_0)$ may be written as

$$\alpha(z, w) = \sum_I f_I d\overline{z_I}$$

Moreover, since α is in $L^2(X_0, \phi_0)$ and $\overline{\partial}-$ closed, the components f_I are in $L^2(X_0, \phi_0)$ and holomorphic in the $w-$ variable. We will have use for the following basic lemma:[1]

Lemma 4.2. *Let $m(v)$ be a positive function on $[0, \infty[$ with polynomial growth at infinity. If $f(w)$ is a a holomorphic function in $\{v < c\}$ with finite L^2-norm with respect to the measure $m(v)dudv$, then there exists a function $\widehat{f(t)}$ on $]0, \infty[$ such that*

$$f(w) = \int_0^\infty \widehat{f(t)} e^{-\frac{i}{2}wt} dt$$

Moreover,

$$\int_{v<c} \int_{u=-\infty}^{\infty} |f(w)|^2 m(v) dudv = 4\pi \int_{v<c} \int_{t=0}^{\infty} \left|\widehat{f}(t)\right|^2 e^{vt} m(v) dtdv \tag{4.1}$$

We will call $\widehat{f(t)}$ the Fourier transform of $f(w)$. Now, fix z in $\mathbb{C}^{n-1}$ and take $c = -\psi_0(z)$ and $m(v) = a(\rho_0)^{-1} = a(v + \psi_0(z))^{-1}$. Then f_I, as a function of w, must satisfy the requirements in the lemma above for almost all z. Fixing such a z we write $\widehat{f_{I,t}(z)}$ for the function of t obtained by taking the Fourier transformation with respect to w.

Hence, we can write

$$\alpha(z, w) = \int_0^\infty \widehat{\alpha_t}(z) e^{-\frac{i}{2}wt} dt \tag{4.2}$$

for almost all z, if we extend the Fourier transform and the integral to act on forms coefficient wise. Note that the equality 4.2 holds in $L^2(X_0, \phi_0)$. The following proposition describes the space $\mathcal{H}^{0,q}(X_0, \phi_0)$ in terms of the spaces $\mathcal{H}^{0,q}(\mathbb{C}^{n-1}, t\psi_0 + \phi_0)$, consisting of all harmonic $(0, q)-$ forms in $L^2(\mathbb{C}^{n-1}, t\psi_0 + \phi_0)$ (with respect to the Euclidean metric in $\mathbb{C}^n$). The corresponding scalar products over $\mathbb{C}^{n-1}$ are denoted by $(\cdot, \cdot)_t$.

Proposition 4.3. *Suppose that α is a tangential $(0, q)-$form on X_0 with coefficients holomorphic with respect to w. Then $\widehat{\alpha_t}$ is in $L^2(\mathbb{C}^{n-1}, t\psi_0 + \phi_0)$ for almost all t and*

$$(\alpha, \alpha)_{X_0} = 4\pi \int (\widehat{\alpha_t}, \widehat{\alpha_t})_t b(t) dt, \tag{4.3}$$

[1]This lemma can be reduced to the Payley-Wiener theorem 19.2 in [24].

where $b(t)=\int_{s<0}e^{st}a(\rho_0)^{-1}ds$. *Moreover, if* α *is in* $\mathcal{H}^{0,q}(X_0,\phi_0)$, *then* $\widehat{\alpha}_t$ *is in* $\mathcal{H}^{0,q}(\mathbb{C}^{n-1},t\psi_0+\phi_0)$ *for almost all* t.

Proof. It is clearly enough to prove 4.3 for the components f_I of α, i.e for a function f in X_0 that is holomorphic with respect to w. When evaluating the norm $(f,f)_{X_0}$ over X_0 we may first perform the integration over u, using 4.1, giving

$$(f,f)_{X_0}=4\pi\int_{\rho_0<0}\left|\widehat{f}_t(z)\right|^2 e^{vt}e^{-\phi(z)}a(\rho_0)^{-1}dzdtdv,$$

where dz stands for the Euclidean volume form $(\frac{i}{2}\partial\overline{\partial}\left|z\right|^2)_{n-1}$ on $\mathbb{C}^{n-1}$. If we now fix z and make the change of variables $s:=v+\psi_0(z)$ and integrate with respect to s we get

$$4\pi\int\left|\widehat{f}_t(z)\right|^2 b(t)e^{-(t\psi(z)+\phi(z))}dtdz=4\pi\int(\widehat{f}_t,\widehat{f}_t)_t b(t)dt.$$

Since this integral is finite, it follows that $(\widehat{f}_t,\widehat{f}_t)$ is finite for almost all t.

Next, assume that α is in $\mathcal{H}^{0,q}(X_0,\phi_0)$. By proposition 4.1 α is $\overline{\partial}-$closed, so that 4.2 gives that $\widehat{\alpha}_t$ is $\overline{\partial}-$closed for almost all t. Let us now show that $\widehat{\alpha}_t$ is $\overline{\partial}-$coclosed with respect to $L^2(\mathbb{C}^{n-1},t\psi_0+\phi_0)$ for almost all t. Fix an interval I in the positive half-line and let β be a form in X_0 that can be written as

$$\beta(z,w)=\int_{t\,\in I}\eta_t(z)e^{-\frac{i}{2}wt}dt$$

where η_t is a smooth $(0,q-1)-$form with compact support on $\mathbb{C}^{n-1}$ for a fixed t (and measurable with respect to t for z fixed). In particular β is a smooth form in $L^2(X_0,\phi_0)$ that is tangential and holomorphic with respect to w. According to 4.2 $\widehat{\beta}_t$ is equal to η_t for $t\in I$ and vanishes otherwise. By proposition 4.1 α is $\overline{\partial}-$coclosed (with respect to $L^2(X_0,\phi_0)$). Using 4.3 we get that

$$0=(\overline{\partial}^*\alpha,\beta)=(\alpha,\overline{\partial}\beta)=4\pi\int_{t\in I}(\widehat{\alpha}_t,\overline{\partial}\eta_t)_t b(t)dt,$$

where we have used lemma 4.6 proved in the next section to get the second equality. Since this holds for any choice of form β and interval I as above we conclude that $\overline{\partial}^*\widehat{\alpha}_t=0$ for almost all t. Hence $\widehat{\alpha}_t$ is in $\mathcal{H}^{0,q}(\mathbb{C}^{n-1},t\Psi_0+\phi_0)$ for almost all t. □

Denote by $\mathbb{K}_t^q$ the Bergman kernel of the Hilbert space $\mathcal{H}^{0,q}(\mathbb{C}^{n-1},t\Psi_0+\phi_0)$.

Lemma 4.4. *The Bergman kernel* $\mathbb{K}_{X_0}^q$ *may be expressed as*

$$\mathbb{K}_{X_0}^q(z,w,z',w')=\frac{1}{4\pi}\int_0^\infty\mathbb{K}_t^q(z,z')e^{\frac{i}{2}(\overline{w}-w')t}b(t)^{-1}dt$$

In particular, the Bergman form $\mathbb{B}_{X_0}^q$ *is given by*

$$\mathbb{B}_{X_0}^q(z,w)=\frac{1}{4\pi}\int_0^\infty\mathbb{B}_t^q(z,z)e^{\rho_0 t}b(t)^{-1}dt.$$

Proof. Take a form α is in $\mathcal{H}^{0,q}(X_0,\phi_0)$ and expand it in terms of its Fourier transform as in 4.2. According to the previous lemma $\widehat{\alpha_t}$ is in $\mathcal{H}^{0,q}(\mathbb{C}^{n-1},t\Psi_0+\phi_0)$ for almost all t. Hence, we can express it in terms of the corresponding Bergman kernel $\mathbb{K}_t^q$, giving

$$f_I(z,w)=\int \widehat{f_{I,t}}(z)e^{-\frac{i}{2}wt}dt=\int(\widehat{\alpha_t}(z),\mathbb{K}_{t,z,I}^q)e^{\frac{i}{2}wt}dt,$$

where $\mathbb{K}_{t,z,I}^q$ denotes the Bergman kernel form $\mathbb{K}_t^q$ at the point z in the direction $d\overline{z_I}$ (see section 3) and where have used the reproducing property 3.1 of the Bergman kernel. Now, using the relation between the different scalar products in the previous lemma we get

$$f_I(z,w)=\frac{1}{4\pi}(\alpha(z',w'),\int_t \mathbb{K}_{t,z,I}^q(z')e^{\frac{i}{2}\overline{w}t}e^{-\frac{i}{2}w't}b(t)^{-1}dt)_{X_0}$$

where (z',w') are the integration variables in the scalar product. But this means exactly that $\mathbb{K}_{X_0}^q$ as defined in the statement of the lemma is the Bergman kernel form of the space $\mathcal{H}^{0,q}(X_0,\phi_0)$ since α was chosen arbitrarily. Finally, by definition we have that

$$\mathbb{B}_{X_0}^q(z,w)=\mathbb{K}_{X_0}^q(z,w,z,w)e^{-\phi_0(z)},\ \mathbb{B}_t^q(z,z)=\mathbb{K}_t^q(z,z')e^{-(t\Psi_0+\phi_0)(z)}.$$

Hence, the expression for $\mathbb{B}_{X_0}^q$ is obtained. □

Now we can give an explicit expression for the Bergman kernel form and the Bergman function. In the formulation of the following theorem we consider X_0 as a fiber bundle of infinity rays $]-\infty,0]$ over (or rather under) the boundary ∂X_0. Then we can consider the fiber integral over 0, i.e. the push forward at 0, of forms on X_0. Moreover, given a real-valued function η on $\mathbb{C}^{n-1}$ such that $\frac{i}{2}\partial\overline{\partial}\eta$ has exactly q negative eigenvalues, we define an associated $(q,q)-$ form $\chi^{q,q}$ by

$$\chi^{q,q}:=(i/2)^q e^1\wedge\cdots\wedge e^q\wedge\overline{e^1}\wedge\cdots\wedge\overline{e^q}$$

where e^i is a orthonormal $(1,0)-$frame that is dual to a base e_i of the direct sum of eigen spaces corresponding to negative eigenvalues of $i\partial\overline{\partial}\eta$ (compare [5]). The $(q,q)-$ form associated to $\frac{i}{2}\partial\overline{\partial}\phi_0+t\frac{i}{2}\partial\overline{\partial}\rho_0$ is denoted by $\chi_t^{q,q}$ in the statement of the following theorem.

Theorem 4.5. *The Bergman form $\mathbb{B}_{X_0}^q$ can be written as an integral over a parameter t :*

$$\mathbb{B}_{X_0}^q(0,u+iv)=\frac{1}{4\pi}\frac{1}{\pi^{n-1}}\int_{T(q)}\chi_t^{q,q}\det(\frac{i}{2}\partial\overline{\partial}\phi_0+t\frac{i}{2}\partial\overline{\partial}\rho_0)e^{vt}b(t)^{-1}dt,$$

where $b(t)=\int_{\rho<0}e^{\rho t}a(\rho)^{-1}d\rho$. In particular, the fiber integral over 0 of the Bergman function B_{X_0} times the volume form is given by

(4.4)
$$\int_{v=-\infty}^{0}B_{X_0}^q(0,iv)(\omega_0)_n=(\frac{i}{2\pi})^n(-1)^q\int_{T(q)}(\partial\overline{\partial}\phi_0+t\partial\overline{\partial}\rho_0)_{n-1}\wedge\partial\rho\wedge dt.$$

Proof. Let us first show how to get the expression for $\mathbb{B}^q_{X_0}$. Using the previous proposition we just have to observe that in $\mathbb{C}^m$, with η a quadratic weight function, the Bergman form is given by

$$\mathbb{B}^q_\eta = \frac{1}{\pi^m} 1_{X(q)} \chi^{q,q} \det(\frac{i}{2}\partial\overline{\partial}\eta), \tag{4.5}$$

where the constant function $1_{X(q)}$ is equal to one if $\frac{i}{2}\partial\overline{\partial}\eta$ has precisely q negative eigenvalues and is zero otherwise (see [4], [5], [6]). Next, from section 3 we have that $B^q_{X_0}(\omega_0)_n$ is given by $\mathbb{B}^q_{X_0}(\omega_0)_{n-q}$. Note that

$$\chi^{q,q} \wedge (\omega_0)_{n-q} = (\frac{i}{2}\partial\overline{\partial}\left|z\right|^2)_{n-1} \wedge a(\rho_0)^{-1}(2i)\partial\rho_0 \wedge d\rho_0,$$

Thus, the fiber integral of $B^q_{X_0}(\omega_0)_n$ over 0 reduces to 4.4 since the factor $b(t)$ is cancelled by the integral of $e^{vt}a(v)^{-1}$. □

Let us finally prove proposition 4.1 .

4.1. **The proof of proposition 4.1: all harmonic forms are tangential, closed and coclosed.** We may write

$$\omega_0 = \frac{i}{2}\partial\overline{\partial}\left|z\right|^2 + 2i\partial\rho' \wedge \overline{\partial\rho'}$$

for a certain function ρ' of ρ_0. The forms $2^{-1/2}dz_i$ and $2^{1/2}\partial\rho'$ together define a orthonormal frame of $(1,0)-$ forms. However, we will use the orthogonal frame consisting of all dz_i and $\partial\rho'$ in order not to clutter the formulas. A dual frame of $(1,0)-$ vector fields is obtained as

$$Z_i := \frac{\partial}{\partial z_i} - 2i\mu_i\overline{z_i}\frac{\partial}{\partial w}, i = 1, 2, ..., n \quad N := ia^{1/2}\frac{\partial}{\partial w}, \tag{4.6}$$

where Z_i is tangential to the level surfaces of ρ, while N is a complex normal vector field. We decompose any form α as

$$\alpha = \alpha_T + \alpha_N = \sum f_I d\overline{z_I} + \overline{\partial\rho}' \wedge g^{0,q-1}$$

Similarly, we decompose the $\overline{\partial}-$ operator acting on the algebra of forms $\Omega^{0,*}(X_0)$ as

$$\overline{\partial} = \overline{\partial_T} + \overline{\partial_N} = \sum_{i=1}^{n-1} \overline{Z_i} d\overline{z_i} \wedge + \overline{N}\overline{\partial\rho'}\wedge, \tag{4.7}$$

where the vector fields $\overline{Z_i}$ etc act on forms over X_0 by acting on the coefficients where $d\overline{z_i}\wedge$ etc denotes the operator acting on forms on X, obtained by wedging with $d\overline{z_i}$. The adjoint operator will be denoted by $d\overline{z_i}^*$. Note that the expression for $\overline{\partial}$ is independent of the ordering of the operators, since the elements in the corresponding frame of $(0,1)-$forms are $\overline{\partial}-$closed. We denote by Δ_T and Δ_N the corresponding Laplace operators, i.e.

$$\Delta_T = \overline{\partial_T}\overline{\partial_T}^* + \overline{\partial_T}^*\overline{\partial_T}, \ \ \Delta_N = \overline{\partial_N}\overline{\partial_N}^* + \overline{\partial_N}^*\overline{\partial_N},$$

Recall that α is said to satisfy $\overline{\partial}-$Neumann boundary conditions if $\overline{\partial\rho'}^{*}$ applied to α and $\overline{\partial}\alpha$ vanishes on the boundary of X_0, or equivalently if α_N and $(\overline{\partial}\alpha)_N$ vanishes there.

Lemma 4.6. *Denote by* $\overline{Z_i}^{*}$ *and* $\overline{N}^{*}$ *the formal adjoint operators of the operators* $\overline{Z}_i$ *and* $\overline{N}$ *acting on* $\Omega^{0,*}(X_0)$*. Then*

$$\begin{aligned} \overline{Z_i}^{*} &= -e^{\phi_0}Z_ie^{-\phi_0} \\ \overline{N}^{*} &= -a^{1/2}Na^{-1/2} \end{aligned} \tag{4.8}$$

Moreover, if the form α *has relatively compact support in* X_0 *then for any form smooth form* β *in* X_0 *we have that* $(\overline{\partial_T}^{*}\alpha,\beta)=(\alpha,\overline{\partial_T}\beta)$ *and if furthermore* α *satisfies* $\overline{\partial}-$*Neumann boundary conditions, then* $(\overline{\partial_N}^{*}\alpha,\beta)=(\alpha,\overline{\partial_N}\beta)$ *(in terms of the formal adjoint operators).*

Proof. It is clearly enough to prove 4.8 for the action of the operators on smooth *functions* with compact support (i.e we write $\alpha=f$ and $\beta=g$, where f and g are smooth functions with compact support). To prove the first statement in 4.8 it is, using Leibniz rule, enough to show that

$$\int_X(\overline{Z_i}(f\overline{g}e^{-\phi_0})(\omega_0)_n=0$$

But this follows from Stokes theorem since the integrand can be written as a constant times the form

$$d(f\overline{g}e^{-\phi_0}a^{-1}(\bigwedge_{j\neq i}dz_j\wedge d\overline{z_j})\wedge dz_i\wedge dw\wedge d\overline{w}),$$

using that $\overline{Z_i}(a^{-1})=0$, since $\overline{Z_i}$ is tangential. Similarly, to prove the second statement in 4.8 it is, using Stokes theorem, enough to observe that

$$\int_X d(f\overline{g}e^{-\phi_0}a^{-1/2}(\partial\overline{\partial}\left|z\right|^2)^{n-1}\wedge dw)=0,$$

Indeed, we have that $\overline{N}:=-ia^{1/2}\frac{\partial}{\partial\overline{w}}$, so the statement now follows from Leibniz' rule. Finally, the last two statements follow from the arguments above, since the boundary integrals obtained from Stokes theorem vanish. □

Lemma 4.7. *The* $\overline{\partial}-$*Laplacian* Δ *acting on* $\Omega^{0,*}(X_0)$ *decomposes as*

$$\Delta=\Delta_T+\Delta_N.$$

Proof. Expanding with respect to the decomposition 4.7 we just have to show that the sum of the mixed terms

$$(\overline{\partial_N\partial_T}^{*}+\overline{\partial_T}^{*}\overline{\partial_N})+(\overline{\partial_N}^{*}\overline{\partial_T}+\overline{\partial_T\partial_N}^{*})$$

vanishes. Let us first show that the first term vanishes. Observe that the following anti-commutation relations hold:

$$d\overline{z_i}\wedge\overline{\partial\rho'}^{*}+\overline{\partial\rho'}^{*}d\overline{z_i}\wedge=0.$$

Indeed, this is equivalent to the corresponding forms being orthogonal. Using this and the expansion 4.7 we get that

$$\overline{(\partial_N \overline{\partial_T}^* + \overline{\partial_T}^* \partial_N)} = \sum_i [\overline{N}, \overline{Z}_i^*] \overline{\partial \rho'}^* d\overline{z_i} \wedge .$$

But this equals zero since the commutators $[\overline{N}, \overline{Z}_i^*]$ vanish, using the expressions in lemma 4.6. To see that the second terms vanishes one can go through the same argument again, now using that the commutators $[\overline{N}^*, \overline{Z}_i]$ vanish. □

We will call a sequence χ_i of non-negative functions on X_0 a *relative exhaustion sequence* if there is sequence of balls B_{R_i} centered at the origin and exhausting $\mathbb{C}^n$, such that χ_i is identically equal to 1 on $B_{R_i/2}$ and with support in B_{R_i}. Moreover, if the metric ω is such that the sequence χ_i can be chosen to make $|d\chi_i|$ uniformly bounded then (X_0, ω) is called *relatively complete.* The point is that when (X_0, ω) is relatively complete, one can integrate partially without getting boundary terms "at infinity". For a complete manifold this was shown in [17] and the extension to the relative case is straightforward.

Lemma 4.8. *Suppose that (X_0, ω) is relatively complete. Then there is a relative exhaustion sequence χ_i of X_0such that, if $\alpha$is a smooth form in $L^2(X_0)$, then*

$$\lim_i (\chi_i \Delta_T \alpha, \alpha) = (\overline{\partial_T}^* \alpha, \overline{\partial_T}^* \alpha) + (\overline{\partial_T} \alpha, \overline{\partial_T} \alpha).$$

Moreover, if α satisfies $\overline{\partial}-$Neumann boundary conditions on ∂X_0, then

$$\lim_i (\chi_i \Delta_N \alpha, \alpha) = (\overline{\partial_N}^* \alpha, \overline{\partial_N}^* \alpha) + (\overline{\partial_N} \alpha, \overline{\partial_N} \alpha). \tag{4.9}$$

Proof. Since (X_0, ω) is relatively complete, following section $1.1B$ in [17] it is enough to prove the statements for a form α with relatively compact support, with χ_i identically equal to 1 (this is called the Gaffney cutoff trick in [17]). Assuming this, the first statement then follows immediately from lemma 4.6. To prove the second statement we assume that α satisfies $\overline{\partial}-$Neumann boundary conditions, i.e. $\alpha_N = (\overline{\partial}\alpha)_N = 0$ on ∂X_0. According to lemma 4.6 the first term in the right hand side of 4.9 may be written as $(\overline{\partial_N \partial_N}^* \alpha, \alpha)$, since $\alpha_N = 0$ on ∂X_0 by assumption. To show that the second term may be written as $(\overline{\partial_N^* \partial_N} \alpha, \alpha)$ we just have to show that $(\overline{\partial_N} \alpha)_N = 0$ on ∂X_0. To this end, first observe that $\overline{\partial_N} \alpha = \overline{\partial_N} \alpha_T$ and $(\overline{\partial}\alpha)_N = \overline{\partial_T} \alpha_N + \overline{\partial_N} \alpha_T$. Now, by assumption $(\overline{\partial}\alpha)_N = 0$ on ∂X_0. Combing this with the previous two identities we deduce that $\overline{\partial_N} \alpha = -\overline{\partial_T} \alpha_N$ on ∂X_0. But $\alpha_N = 0$ on ∂X_0 and $\overline{\partial_T}$ is a tangential operator, so it follows that $\overline{\partial_T} \alpha_N = 0$ on ∂X_0. This proves that $(\overline{\partial_N} \alpha)_N = 0$ on ∂X_0. □

Finally, to finish the proof of proposition 4.1, first observe that the model metric ω_0 corresponding to $a(\rho_0) = (1 - \rho_0)^2$ is relatively complete. Now take an arbitrary form α in $\mathcal{H}^{0,q}(X_0, \phi_0)$. Then $(\chi_i \Delta \alpha, \alpha) = 0$ for each i. Hence, using lemma 4.7 together with lemma 4.8 we deduce, after letting i tend to infinity, that

$$0 = \left\|\overline{\partial_T}\alpha\right\|^2 + \left\|\overline{\partial_T}^*\alpha\right\| + \left\|\overline{\partial_N}\alpha\right\|^2 + \left\|\overline{\partial_N}^*\alpha\right\|^2. \tag{4.10}$$

In particular, $\overline{\partial_N}^*\alpha$ vanishes in X_0. If we write $\alpha_N = \overline{\partial\rho}' \wedge \sum_I g_I d\overline{z_I}$ this means that

$$\overline{N}^* g_I = 0$$

in X_0 for all I. Moreover, since α satisfies $\overline{\partial}-$Neumann boundary conditions, each function g_I vanishes on the boundary of X_0. It follows that $g_I = 0$ in all of X_0. Indeed, let $g_I' := a(\rho)^{-1/2}\overline{g_I}$ and consider the restriction of g_I' to the half planes in $\mathbb{C}$ obtained by freezing the z_i- variables. Then g_I' is holomorphic in the half plane, vanishing on the boundary. It is a classical fact that g_I' then actually vanishes identically. Moreover, 4.10 also gives that α is $\overline{\partial}-$ closed and coclosed. This finishes the proof of proposition 4.1.

5. Contributions from the three boundary regions

In this section we will estimate the integral of the Bergman function over the three different boundary regions. The contribution from the inner part of X was essentially computed in [4].

5.1. The first region.

Recall that the first region is the set where $\rho \geq -R/k)$. Fix a point σ in ∂X and take local holomorphic coordinates (z, w), where z is in $\mathbb{C}^{n-1}$ and $w = u + iv$. By an appropriate choice [8], we may assume that the coordinates are orthonormal at 0 and that

$$\rho(z,w) = \sum_{i=1}^{n-1} v + \mu_i |z_i|^2 + O(|(z,w)|^3) =: \rho_0(z,w) + O(|(z,w)|^3). \tag{5.1}$$

In a suitable local holomorphic trivialization of L close to the boundary point σ, the fiber metric may be written as

$$\phi(z) = \sum_{i,j=1}^{n-1} \lambda_{ij} z_i \overline{z_j} + O(|w|)O(|z|) + O(|w|^2) + O(|(z,w)|^3).$$

Denote by F_k the holomorphic scaling map

$$F_k(z,w) = (z/k^{1/2}, w/k),$$

so that

$$X_k = F_k(D_{\ln k}) \bigcap X$$

is a sequence of decreasing neighborhoods of the boundary point σ, where $D_{\ln k}$ denotes the polydisc of radius $\ln k$ in $\mathbb{C}^n$. Note that

$$F_k^{-1}(X_k) \to X_0,$$

in a certain sense, where X_0 is the model domain with defining function ρ_0. On $F_k^{-1}(X_k)$ we have the *scaled metrics* $F_k^* k\omega_k$ and $F_k^* k\phi$ that tend to the model metrics ω_0 and ϕ_0 on X_0, when k tends to infinity, where

$$\omega_0 = \frac{i}{2}\partial\overline{\partial}\left|z\right|^2 + a(\rho_0)^{-1}2i\partial\rho_0 \wedge \overline{\partial\rho_0} \text{ and } \phi_0(z) = \sum_{i,j=1}^{n-1} \lambda_{ij} z_i \overline{z_j}, \tag{5.2}$$

for a smooth function $a(\rho_0)$ that is positive on $]-\infty,0]$. The factor $a(\rho_0)$, and hence the model metric ω_0, really depends on the number R (used in the definition of the boundary regions). However, the dependence on R will play no role in the proofs, since R will be fixed when k tends to infinity. Recall that the space of model harmonic $(0,q)-$forms in $L^2(X_0,\omega_0,\phi_0)$ satisfying $\overline{\partial}-$Neumann boundary conditions is denoted by $\mathcal{H}^{0,q}(X_0,\phi_0)$.

Lemma 5.1. *For the component-wise uniform norms on $F_k^{-1}(X_k)$ we have that*

$$\begin{aligned} \left\|F_k^* k\rho - \rho_0\right\|_\infty &\to 0 \\ \left\|F_k^* k\omega_k - \omega_0\right\|_\infty &\to 0 \\ \left\|F_k^* k\phi - \phi_0\right\|_\infty &\to 0 \end{aligned}$$

and similarly for all derivatives.

Proof. The convergence for ρ and ϕ is straightforward (compare [4]) and the convergence for ω_k will be showed in section 5.4 once ω_k has been constructed. □

The Laplacian on $F_k^{-1}(X)$ taken with respect to the scaled metrics will be denoted by $\Delta^{(k)}$ and the corresponding formal adjoint of $\overline{\partial}$ will be denoted by $\overline{\partial}^{*(k)}$. The Laplacian on X_0 taken with respect to the model metrics will be denoted by Δ_0. Because of the convergence property of the metrics above it is not hard to check that

$$\Delta^{(k)} = \Delta_0 + \varepsilon_k \mathcal{D}_k, \tag{5.3}$$

where $\mathcal{D}_k$ is a second order partial differential operator with bounded variable coefficients on $F_k^{-1}(X_k)$ and ϵ_k is a sequence tending to zero with k. Next, given a $(0,q)-$form α_k on X_k with values in L^k, define the *scaled form* $\alpha^{(k)}$ on $F_k^{-1}(X_k)$ by

$$\alpha^{(k)} := F_k^* \alpha_k.$$

Then

$$F_k^* \left|\alpha_k\right|^2 = \left|\alpha^{(k)}\right|^2, \tag{5.4}$$

where the norm of α_k is the one induced by the metrics $k\omega_k$ and $k\phi$ and the norm of the scaled form $\alpha^{(k)}$ is taken with respect to the scaled metrics $F_k^* k\omega_k$ and $F_k^* k\phi$. This is a direct consequence of the definitions. Moreover, the next lemma gives the transformation of the Laplacian.

Lemma 5.2. *The following relation between the Laplacians holds:*

$$\Delta^{(k)}\alpha^{(k)} = (\Delta_k \alpha_k)^{(k)}. \tag{5.5}$$

Proof. Since the Laplacian is naturally defined with respect to any given metric it is invariant under pull-back, proving the lemma. □

In the following, all norms over $F_k^{-1}(X_k)$ will be taken with respect to the model metrics ω_0 and ϕ_0. The point is that these norms anyway coincide, asymptotically in k, with the norms defined with respect to the scaled metrics used above, by the following lemma.

Lemma 5.3. *We have that uniformly on $F_k^{-1}(X_k)$*

$$\begin{array}{ccc} F_k^*\left|\alpha_k\right| & \sim & \left|\alpha^{(k)}\right| \\ \left\|\alpha_k\right\|_{X_k} & \sim & \left\|\alpha^{(k)}\right\|_{F_k^{-1}(X_k)} \end{array}$$

Moreover,for any sequence θ_k^I of ω_k-orthonormal bases of direction forms in $\Lambda^{0,q}(X)_x$ at $F_k(x)$, there is a bases of ω_0-orthonormal direction forms at x, such that the following asymptotic equality holds, when k tends to infinity:

$$F_k^*\left|\alpha_{k,\theta_k^I}\right| \sim \left|\alpha_{\theta^I}^{(k)}\right|$$

for each index I.

Proof. The lemma follows immediately from 5.4 and the convergence of the metrics in the previous lemma. □

Now we can prove the following lemma that makes precise the statement that, in the large k limit, harmonic forms α_k are harmonic with respect to the model metrics and the model domain on a small scale close to the boundary of X.

Lemma 5.4. *Suppose that the boundary of X satisfies condition $Z(q)$ (see remark 1.2). For each k, suppose that $\alpha^{(k)}$ is a $\overline{\partial}-$closed smooth $(0,q)-$form on $F_k^{-1}(X_k)$ such that $\overline{\partial}^{*(k)}\alpha^{(k)}=0$ and that $\alpha^{(k)}$ satisfies $\overline{\partial}-$Neumann boundary conditions on $F_k^{-1}(\partial X)$. Identify $\alpha^{(k)}$ with a form in $L^2(\mathbb{C}^n)$ by extending with zero. Then there is a constant C_R independent of k such that*

$$\sup_{D_R\bigcap F_k^{-1}(X_k).}\left|\alpha^{(k)}\right|^2\leq C_R\left\|\alpha^{(k)}\right\|^2_{D_{2R}\bigcap F_k^{-1}(X_k)}$$

Moreover, if the sequence of norms $\left\|\alpha^{(k)}\right\|^2_{F_k^{-1}(X_k)}$ is bounded, then there is a subsequence of $\left\{\alpha^{(k)}\right\}$ which converges uniformly with all derivatives on any compactly included set in X_0 to a smooth form β, where β is in $\mathcal{H}^{0,q}(X_0)$. The convergence is uniform on $D_R\bigcap F_k^{-1}(X_k)$.

Proof. Fix a k and consider the intersection of the polydisc D_R of radius R with $F_k^{-1}(X_k)$. It is well-known that the Laplace operator $\Delta^{(k)}$ acting on $(0,q)-$ forms is sub-elliptic close to a point x in the boundary satisfying the condition $Z(q)$ (see [14]). In particular, sub-elliptic estimates give for any smooth form $\beta^{(k)}$ satisfying $\overline{\partial}-$Neumann boundary conditions on $F_k^{-1}(\partial X)$ that

$$\left\|\beta^{(k)}\right\|^2_{D_R,m-1}\leq C_{k,R}(\left\|\beta^{(k)}\right\|^2_{D_{2R}}+\left\|\Delta^{(k)}\beta^{(k)}\right\|^2_{D_{2R},m}), \tag{5.6}$$

where the subscript m indicates a Sobolev norm with m derivatives in L^2 and where the norms are taken over $F_k^{-1}(X)$ with respect to the scaled metrics. The $k-$dependence of the constants $C_{k,R}$ comes from the boundary $F_k^{-1}(\partial X)$ and the scaled metrics $F_k^* k\omega_k$ and $F_k^* k\phi$. However, thanks to the convergence of the metrics in lemma 5.1 one can check that the dependence is uniform in k. Hence, applying the subelliptic estimates 5.6 to $\alpha^{(k)}$ we get

$$\left\|\alpha^{(k)}\right\|^2_{D_R \bigcap F_k^{-1}(X),m} \leq C_R \left\|\alpha^{(k)}\right\|^2_{D_{2R} \bigcap F_k^{-1}(X)} \tag{5.7}$$

and the continuous injection $L^{2,l} \hookrightarrow C^0$, $l > n$, provided by the Sobolev embedding theorem, proves the first statement in the lemma. To prove the second statement assume that $\left\|\alpha^{(k)}\right\|^2_{F_k^{-1}(X)}$ is uniformly bounded in k. Take a sequence of sets K_n, compactly included in X_0, exhausting X_0 when n tends to infinity. Then the estimate 5.6 (applied to polydiscs of increasing radii) shows that

$$\left\|\alpha^{(k)}\right\|^2_{K_n,m} \leq C'_n \tag{5.8}$$

Since this holds for any $m \geq 1$, Rellich's compactness theorem yields, for each n, a subsequence of $\left\{\alpha^{(k)}\right\}$, which converges in all Sobolev spaces $L^{2,l}(K_n)$ for $l \geq 0$ for a fixed n. The compact embedding $L^{2,l} \hookrightarrow C^p$, $k > n + \frac{1}{2}p$, shows that the sequence converges in all $C^p(K_n)$. Choosing a diagonal sequence with respect to k and n, yields convergence on any compactly included set K. Finally, we will prove that the limit form β is in $\mathcal{H}(X_0)$. First observe that by weak compactness we may assume that the sequence $1_{X_0}\alpha^{(k)}$ tends to β weakly in $L^2(\mathbb{C}^n)$, where 1_{X_0} is the characteristic function of X_0 and β is extended by zero to all of $\mathbb{C}^n$. In particular, the form β is weakly $\overline{\partial}$-closed in X_0. To prove that β is in $\mathcal{H}(X_0)$ it will now be enough to show that

$$(\beta, \overline{\partial}\eta)_{X_0} = 0 \tag{5.9}$$

for any form η in X_0 that is smooth up to the boundary and with a relatively compact support in X_0. Indeed, it is well-known that β then is in the kernel of the Hilbert adjoint of the densely defined operator $\overline{\partial}$. Moreover, the regularity theory then shows that β is smooth up to the boundary, where it satisfies $\overline{\partial}-$Neumann boundary conditions (actually this is shown using sub-elliptic estimates as in 5.6) [19, 14]. To see that 5.9 holds, we write the left hand side, using the weak convergence of $1_{X_0}\alpha^{(k)}$, as

$$\lim_k (\alpha^{(k)}, \overline{\partial}\eta)_{X_0} = \lim_k (\alpha^{(k)}, \overline{\partial}\eta)_{X_0 \bigcap F_k^{-1}(X)}. \tag{5.10}$$

Extending η to a smooth form on some neighborhood of X_0 in $\mathbb{C}^n$ we may now write this as a scalar product, with respect to the scaled metrics, over $F_k^{-1}(X)$, thanks to the convergence in lemma 5.1 of the scaled metrics and the scaled defining function. Since, $\alpha^{(k)}$ is assumed to satisfy $\overline{\partial}-$Neumann-boundary conditions on $F_k^{-1}(\partial X)$ and be in the kernel of the formal adjoint of $\overline{\partial}$, taken with respect to the scaled metrics, this means that the right hand side of 5.10 vanishes. This proves 5.9 and finishes the proof of the lemma. □

The following proposition will give the boundary contribution to the holomorphic Morse inequalities in theorem 2.1.

Proposition 5.5. *Let*

$$I_R := \limsup_k \int_{-\rho<Rk^{-1}} B_X^{q,k}\omega_n$$

Then

$$\limsup_R I_R \leq (-1)^q(\frac{1}{2\pi})^n \int_{\partial X}\int_{T(q)_{\rho,x}} (\Theta + t\mathcal{L})_{n-1} \wedge \partial\rho \wedge dt$$

where $T(q)_{\rho,x} = \{t > 0 : \text{ index}(\Theta + t\mathcal{L}) = q \text{ along } T^{1,0}(\partial X)_x\}$.

Proof. We may assume that the boundary of X satisfies condition $Z(q)$ (compare remark 1.2). Using the expression 2.1 for the metric ω_k, the volume form $(\omega_k)_n$ may be written as $a_k(\rho)^{-1}(\omega_T)_{n-1} \wedge 2i\partial\rho \wedge d\rho$. Hence, I_R can be expressed as

$$\limsup_k \int_{\partial X} (\omega_T)_{n-1} \wedge 2i\partial\rho \int_{-R/k}^0 a_k(\rho)^{-1} B_X^k d\rho$$

Now, fix a point in the boundary of X and take local coordinates as in the beginning of the section. To make the argument cleaner we will first assume that the restriction of ρ to the ray close to the boundary where z and the real part of w vanish, coincides with v. Then, after a change of variables, the inner integral along the ray in the first region becomes

$$1/k \int_0^R a_k(v/k)^{-1} B_X^{q,k}(v/k)dv. \tag{5.11}$$

Moreover, by the scaling properties of the metrics $k\omega_k$ in lemma 5.1 we have the uniform convergence

$$ka_k(v/k) \to a(v).$$

on the segment $[0, R]$ (see also section 5.4). Thus,

$$I_R = \limsup_k \int_{\partial X} (\omega_T)_{n-1} \wedge 2i\partial\rho \int_0^R a(v)^{-1} B_X^{q,k}(v/k)dv.$$

Let us now show that

$$\begin{array}{lrcl} (i) & B_X^{q,k}(0, iv/k) & \lesssim & B_{X_0}^q(0, iv) \\ (ii) & B_X^{q,k}(0, iv/k) & \leq & C_R \end{array} \tag{5.12}$$

We first prove (i). According to the extremal property 3.4 it is enough to show that

$$S_{X,\theta_k}^{q,k}(0, iv/k) \lesssim S_{X_0,\theta}^q((0, iv) \tag{5.13}$$

for any sequence of direction forms θ_k at $(0, iv/k)$ as in lemma 5.3. Given this, the bound 5.12 is obtained after summing over the base elements θ_k. To prove 5.13 we have to estimate

$$|\alpha_{k,\theta_k}(0, iv/k|^2 ,$$

where α_k is a normalized harmonic form with values in L^k that is extremal at $(0, iv/k)$ in the direction θ_k. Moreover, it is clearly enough to estimate some subsequence of α_k. By lemma 5.3 it is equivalent to estimate

$$\left|\alpha_\theta^{(k)}(0, iv)\right|^2 ,$$

where the scaled form $\alpha^{(k)}$ is defined on $F_k^{-1}(X_k)$ and extended by zero to all of X_0. Note that, according to lemma 5.3 the norms of the sequence of scaled forms $\alpha^{(k)}$ are asymptotically less than one:

$$\left\|\alpha^{(k)}\right\|^2_{F_k^{-1}(X)} \sim \|\alpha_k\|^2_{X_k} \leq 1, \tag{5.14}$$

since the global norm of α_k is equal to one. Hence, by lemma 5.4 there is a subsequence $\alpha^{(k_j)}$ that converges uniformly to β with all derivatives on the segment $0 \leq v \leq R$ in X_0 and where the limit form β is in $\mathcal{H}^{0,q}(X_0, \phi_0)$ and its norm is less than one (by 5.14). This means that

$$|\alpha_{k,\theta}(0, iv/k)|^2 \sim \left|\alpha_\theta^{(k_j)}(0, iv)\right|^2 \sim |\beta_\theta(0, iv)|^2 ,$$

Since the limit form β is a contender for the model extremal function $S^q_{X_0,\theta}(0)$, this proves 5.13, and hence we obtain (i). To show (ii), just observe that lemma 5.4 says that there is a constant C_R such that there is a uniform estimate

$$\left|\alpha^{(k)}(0, iv)\right|^2 \leq C_R,$$

By the extremal characterization 3.4 of $B_X^{q,k}$ this proves (ii). Now using 5.21 and Fatou's lemma to interchange the limits, I_R may be estimated by

$$\int_{\partial X} (\int_0^\infty B^q_{X_0}(0, iv)(\omega_0)_n, \tag{5.15}$$

in terms of the model metric ω_0 on X_0. By theorem 4.5 this equals

$$(\frac{i}{2\pi})^n(-1)^q \int_{\partial X} \int_{T(q)} (\partial\overline{\partial}\phi_0 + t\partial\overline{\partial}\rho_0)_{n-1} \wedge \partial\rho \wedge dt.$$

This finishes the proof of the proposition under the simplifying assumption that the restriction of ρ to the ray introduced above, coincides with the restriction of v. In general this is only true up to terms of order $O\left|(z, w)\right|^3$, given the expression 5.1. To handle the general case one writes the integral 5.11 as

$$1/k \int_{I_k} a_k B_X^{q,k} dv, \tag{5.16}$$

where I_k is the inverse image under F_k of the ray. Clearly, I_k tends to the segment $[0, R]$ in X_0 obtained by keeping all variables except v equal to zero. Moreover, since the sequence $\alpha^{(k_j)}$ above converges uniformly with all derivatives on $D_R \bigcap F_k^{-1}(X_k)$ it forms an equicontinous family, so the same argument as above gives that 5.16 may be estimated by 5.15. This finishes the proof of the proposition. □

5.2. **The second and third region.** Let us first consider the second region, i.e. where $-1/k^{1/2} \leq \rho \leq -R/k$. Given a k, consider a fixed point $(0, iv) = (0, ik^{-s})$, where $1/2 \leq s < 1$. Any point in the second region may be written in this way. Let (z', w') be coordinates on the unit polydisc D. Define the following holomorphic map from the unit polydisc D to a neighborhood of the fixed point:

$$F_{k,s}(z', w') = (k^{-1/2}z', k^{-s} + \frac{1}{2}k^{-s}w')$$

so that

$$X_{k,s} := F_{k,s}(D),$$

is a neighborhood of the fixed point, staying away from the boundary of X. On D we will use the scaled metrics $F^*_{k,s}k\omega_k$ and $kF^*_{k,s}k\phi$ that have bounded derivatives and are comparable to flat metrics in the following sense:

$$\begin{array}{rcccl} C^{-1}\omega_E & \leq & F^*_{k,s}k\omega_k & \leq & C\omega_E \\ & & \left|F^*_{k,s}k\phi\right| & \leq & C \end{array} \tag{5.17}$$

where ω_E is the Euclidean metric. Note that the scaling property of ω_k is equivalent to

$$C^{-1}k\rho^2 \leq a_k(\rho) \leq Ck\rho^2. \tag{5.18}$$

These properties will be verified in section 5.4 once ω_k is defined. The Laplacian on D taken with respect to the scaled metrics will be denoted by $\Delta^{(k,s)}$ and the Laplacian on X_0 taken with respect to the model metrics will be denoted by Δ_0. Next, given a $(0,q)-$form α_k on X_k with values in L^k, define the scaled form $\alpha^{(k)}$ on D by

$$\alpha^{(k,s)} := F^*_{k,s}\alpha_k.$$

Using 5.17 one can see that the following equivalence of norms holds:

$$\begin{array}{rcccl} C^{-1}\left|\alpha^{(k,s)}\right|^2 & \leq & F^*_{k,s}\left|\alpha_k\right|^2 & \leq & C\left|\alpha^{(k,s)}\right|^2 \\ C^{-1}\left\|\alpha^{(k,s)}\right\|^2_D & \leq & \left\|\alpha_k\right\|^2_{X_k} & \leq & C\left\|\alpha^{(k,s)}\right\|^2_D \end{array} \tag{5.19}$$

In the following, all norms over $F^{-1}_{k,s}(X_{k,s})$ will be taken with respect to the Euclidean metric ω_E and the trivial fiber metric.

Proposition 5.6. *Let*

$$II_R := \limsup_k \int_{Rk^{-1}<-\rho<k^{-1/2}} B^{q,k}_X(\omega_k)_n$$

Then

$$\lim_R II_R = 0.$$

Proof. Fix a k and a point $(0, iv) = (0, ik^{-s})$, where s is in $[1/2, 1[$. From the scaling properties 5.18 of ω_k it follows that at the point $(0, ik^{-s})$

$$\omega_k^n \leq Ck^{(2s-1)}\omega^n.$$

Next, observe that

$$B_X^{q,k}(0, ik^{-s}) \leq C \tag{5.20}$$

Accepting this for the moment, it follows that

$$B^{q,k}(0, iv)\omega_k^n \leq Ck^{-1}v^{-2}\omega^n,$$

since we have assumed that $v = k^{-s}$. Hence, the integral in II_R may be estimated by

$$\int_{\partial X} Ck^{-1} \int_{-k^{-1/2}}^{-Rk^{-1}} v^{-2} \mathrm{d}v = Ck^{-1}(R^{-1}k - k^{1/2})$$

which tends to CR^{-1} when k tends to infinity. This proves that II_R tends to zero when R tends to infinity, which proves the proposition, given 5.20.

Finally, let us prove the claim 5.20. For a given k consider the point $(0, ik^{-s})$ as above. As in the proof of the previous proposition we have to prove the estimate

$$\left|\alpha_k(0, ik^{-s})\right|^2 \leq C, \tag{5.21}$$

where α_k is a normalized harmonic section with values in L^k that is extremal at $(0, ik^{-s})$. By the equivalence of norms 5.19, it is equivalent to prove

$$\left|\alpha^{(k,s)}(0)\right|^2 \leq C,$$

where the scaled form $\alpha^{(k,s)}$ is defined on D. Note that, according to 5.19

$$\left\|\alpha^{(k,s)}\right\|_D^2 \sim \|\alpha_k\|_{X_k}^2 \leq C, \tag{5.22}$$

since the global norm of α_k is equal to 1. Moreover, a simple modification of lemma 5.2 gives

$$\Delta^{(k,s)}\alpha^{(k,s)} = 0$$

on D. Since $\Delta^{(k,s)}$ is an elliptic operator on the polydisc D, inner elliptic estimates (i.e. Gårding's inequality) and the Sobolev embedding theorem can be used as in [4] to get

$$\left|\alpha^{(k,s)}(0)\right|^2 \leq C \left\|\alpha^{(k,s)}\right\|_D^2,$$

where the constant C is independent of k and s thanks to the equivalence 5.17 of the metrics. Using 5.22, we obtain the claim 5.21. □

Let us now consider the third region where $-\varepsilon < \rho < -k^{-1/2}$.

Proposition 5.7. *Let*

$$III_\varepsilon := \int_{k^{-1/2} < -\rho < \varepsilon} B_X^{q,k}(\omega_k)_n$$

Then

$$III_\varepsilon = O(\varepsilon).$$

Proof. We just have to observe that

$$k^{-n}B_X^{q,k} \leq C, \tag{5.23}$$

when $\rho < -k^{-1/2}$. This follows from inner elliptic estimates as in the proof of the previous proposition, now using $s = 1/2$ (compare [4]). □

5.3. **End of the proof of theorem 2.1 (the weak Morse inequalities).** First observe that

(5.24)

$$\int_{0<-\rho<\varepsilon} B_X^{q,k}(\omega_k)_n \lesssim (-1)^q(\frac{1}{2\pi})^n \int_{\partial X}\int_{T(q)} dt(\partial\overline{\partial}\phi + t\partial\overline{\partial}\rho)^{n-1}\wedge\partial\rho + o(\varepsilon).$$

Indeed, for a fixed R we may write the limit of integrals above as the sum $I_R + II_R + III_\varepsilon$. Letting R tend to infinity and using the previous three propositions we get the estimate above. Moreover, we have that

$$\int_{X_\varepsilon} B_X^{q,k} \lesssim \int_{X_\varepsilon} (\partial\overline{\partial}\phi)^n, \tag{5.25}$$

where X_ε denotes the set where $-\rho$ is larger than ϵ. This follows from the estimates

$$B_X^{q,k}\omega_n \lesssim (\frac{i}{2\pi})^n(-1)^q 1_{X(q)}(\partial\overline{\partial}\phi)_n \text{ and } B_X^{q,k} \leq \text{C in X}_\varepsilon,$$

proved in [4] (compare 5.23 for the uniform estimate). Finally, writing $\dim_{\mathbb{C}} \mathcal{H}^{0,q}(X, L^k)$ as the sum of the integrals in 5.24 and 5.25 and letting ϵ tend to zero, yields the dimension bound in theorem 2.1 for the space of harmonic forms. By the Hodge theorem we are then done.

5.4. **The sequence of metrics ω_k.** In this section the metrics ω_k will be defined and their scaling properties, that were used above, will be verified. Recall that we have to define a sequence of smooth functions a_k such that the metrics

$$\omega_k = \omega_T + a_k(\rho)^{-1}2i\partial\rho\wedge\overline{\partial\rho},$$

have the scaling properties of lemma 5.1 in the first region and satisfy 5.17 in the second region. First observe that the tangential part ω_T clearly scales the right way, i.e. that $F_k^* k\omega_T$ tends to $\frac{i}{2}\partial\overline{\partial}\left|z\right|^2$. Indeed, since the coordinates (z, w) are orthonormal at 0 the forms ω_T and $\frac{i}{2}\partial\overline{\partial}\left|z\right|^2$ coincide at 0. Since $\frac{i}{2}\partial\overline{\partial}\left|z\right|^2$ is invariant under $F_k^* k$ the convergence then follows immediately. We now consider the normal part of ω_k and show how to define the functions a_k. Consider first the piecewise smooth functions $\widetilde{a_k}$ where $\widetilde{a_k}$ is defined as R^2/k in the first region, as $k\rho^2$, in the second region and as 1 in the third region and on the rest of X. Then it is not hard to check that $\widetilde{a_k}$ satisfies our demands, except at the two middle boundaries between the three regions. We will now construct a_k as a regularization of $\widetilde{a_k}$. To this end we write $\widetilde{a_k} = k\widetilde{b_k}^2$, where $\widetilde{b_k}$ is defined by

$$\begin{cases} Rk^{-1}, & -\rho \leq Rk^{-1} \\ -\rho, & Rk^{-1} \leq -\rho \leq k^{-1/2} \\ k^{-1/2}, & -\rho \leq k^{-1/2} \end{cases}$$

in the three regions. It will be enough to regularize the sequence of continuous piecewise linear functions $\widetilde{b}_k$. Decompose $\widetilde{b}_k$ as a sum of continuous piecewise linear functions

$$\widetilde{b}_k(-\rho) = \frac{R}{k}\widetilde{b}_{1,k}(-\frac{k}{R}\rho) + \frac{1}{k^{1/2}}\widetilde{b}_{2,k}(-k^{1/2}\rho),$$

where $\widetilde{b}_{1,k}$ is determined by linearly interpolating between

$$\widetilde{b}_{1,k}(0) = 1 \quad \widetilde{b}_{1,k}(1) = 1 \quad \widetilde{b}_{1,k}(k^{1/2}/R) = 0 \quad \widetilde{b}_{1,k}(\infty) = 0$$

and $\widetilde{b}_{2,k}$ is determined by

$$\widetilde{b}_{2,k}(0) = 0 \quad \widetilde{b}_{2,k}(Rk^{-1/2}) = 0 \quad \widetilde{b}_{2,k}(1) = 1 \quad \widetilde{b}_{2,k}(\infty) = 1$$

Now consider the function b_k obtained by replacing $\widetilde{b}_{1,k}$ and $\widetilde{b}_{2,k}$ with the continuous piecewise linear functions b_1 and b_2, where b_1 is determined by

$$b_1(0) = 1 \quad b_1(1/2) = 1 \quad b_1(1) = 0 \quad b_1(\infty) = 0,$$

and b_2 is determined by

$$b_2(0) = 0 \quad b_2(1) = 1 \quad b(\infty) = 1 \ .$$

Finally, we smooth the corners of the two functions b_1 and b_2. Let us now show that the sequence of regularized functions b_k scales in the right way. In the first region we have to prove that lemma 5.1 is valid, which is equivalent to showing that there is a function b_0 such that

$$kb_k(t/k) \to b_0 \tag{5.26}$$

with all derivatives, for t such that $0 \leq t \leq \text{Rln}k$. From the definition we have that

$$kb_k(t/k) = Rb_1(t/R) + t, \tag{5.27}$$

which is even independent of k, so 5.26 is trivial then. Next, consider the second region. To show that 5.17 holds we have to show that, for parameters s such that $1/2 \leq s < 1$, the $t-$dependent functions $k^s b_k(1/k^s + t/2k^s)$ (where $|t| \leq 1$) are uniformly bounded from above and below by positive constants and have uniformly bounded derivatives. First observe that in the second region the sequence of functions may be written as

$$k^{s-1/2}b_2(1/k^{1/2-s} + t/2k^{1/2-s}),$$

and it is not hard to see that it is bounded from above and below by positive constants, independently of s and k. Moreover, differentiating with respect to t shows that all derivatives are bounded, independently of s and k. All in all this means that we have constructed a sequence of metrics ω_k with the right scaling properties. In particular, 5.27 shows that the factor $a^{-1}(\rho_0)$ in the model metric ω_0 5.2 satisfies

$$C_R^{-1}(1-\rho_0)^{-2} \leq a^{-1}(\rho_0) \leq C_R(1-\rho_0)^{-2}$$

for some constant C_R depending on R.

Part 2. The strong Morse inequalities and sharp examples

6. The strong Morse inequalities

We will assume that the boundary of X satisfies condition $Z(q)$ (compare remark 1.2) and use the same notation as in section 1. Let μ_k be a sequence tending to zero. Denote by $\mathcal{H}^{0,q}_{\leq \mu_k}(X)$ the space spanned by the $(0,q)-$eigen forms of the Laplacian Δ, with eigenvalues bounded by μ_k. The forms are assumed to satisfy $\overline{\partial}-$Neumann boundary conditions and they will be called *low energy forms.* Since we have assumed that condition $Z(q)$ holds, this space is finite dimensional for each k [14]. Recall that the Laplacian is defined with respect to the metric $k\omega_k$, so that the eigen values corresponding to μ_k are multiplied with k if the metric ω_k is used instead.

We will first show that the weak holomorphic Morse inequalities are *equalities* for the space $\mathcal{H}^{0,q}_{\leq \mu_k}(X)$ of low energy forms. When X has no boundary this yields *strong* Morse inequalities for the truncated Euler characteristics of the Dolbeault complex with values in L^k. However, when X has a boundary one has to assume that the boundary of X has either concave or convexity properties to ensure that the corresponding cohomology groups are finite dimensional, in order to obtain strong Morse inequalities.

The Bergman form for the space $\mathcal{H}^{0,q}_{\leq \mu_k}(X)$ defined as in section 3 will be denoted by $\mathbb{B}^q_{\leq \mu_k}$. By $L^2_m(X)$ we will denote the Sobolev space with m derivatives in $L^2(X)$ and a subscript m on a norm will indicate the corresponding Sobolev norm. The essential part in proving that we now get equality in the weak Morse inequalities is to show that the estimate on the Bergman form 5.12 in the proof of proposition 5.5 becomes an asymptotic equality, when considering low energy forms. The rest of the argument is more or less as before.

Let us first prove the upper bound, i.e. that the low energy Bergman form $\mathbb{B}^q_{\leq \mu_k}$ is asymptotically bounded by the model harmonic Bergman form.

Proposition 6.1. *We have that*

$$B^q_{\leq \mu_k,\theta_k}(0,iv/k) \lesssim B_{X_0,\theta}(0,v)$$

and the sequence $B^q_{\leq \mu_k}(0,iv/k)$ is uniformly bounded in the first region.

Proof. Let α_k be a sequence of normalized forms, such that α_k is an extremal for the Hilbert space $\mathcal{H}^{0,q}_{\leq \mu_k}(X)$ at the point $(0,iv/k)$ in the direction θ_k. In the following all norms will be taken over $F_k^{-1}(X)$. Observe that by the invariance property in lemma 5.2 of the Laplacian, the scaled form $\alpha^{(k)}$ satisfies

$$\left\|(\Delta^{(k)})^p\alpha^{(k)}\right\|^2 \leq \mu_k^{2p} \to 0 \tag{6.1}$$

for all positive integers p. Let us now show that

$$\left\|\Delta^{(k)}\alpha^{(k)}\right\|_m^2 \to 0 \tag{6.2}$$

for all non-negative integers m. First observe that $(\Delta^{(k)})^p\alpha^{(k)}$ satisfies $\overline{\partial}-$Neumann boundary conditions for all p. Indeed, by definition all forms in the space $\mathcal{H}_{\leq\mu_k}(X_0)$ satisfy $\overline{\partial}-$Neumann boundary conditions and since Δ preserves this space, the forms $(\Delta)^p\alpha_k$ also satisfy $\overline{\partial}-$Neumann boundary conditions for all p. By the scaling of the Laplacian this means that the forms $(\Delta^{(k)})^p\alpha^{(k)}$ satisfy $\overline{\partial}-$Neumann boundary conditions with respect to the scaled metrics. Now applying the subelliptic estimates 5.6 to forms of the type $(\Delta^{(k)})^p\alpha^{(k)}$ one gets , using induction, that

$$\left\|\Delta^{(k)}\alpha^{(k)}\right\|_m^2 \leq C\sum_{j=1}^{m+1}\left\|(\Delta^{(k)})^j\alpha^{(k)}\right\|^2.$$

Combining this with 6.1 proves 6.2. Now the rest of the argument proceeds almost word for word as in the proof of the claim 5.12 in the proof of proposition 5.5. The point is that the limit form β will still be in $\mathcal{H}(X_0)$, thanks to 6.1. □

Let us now show how to get the corresponding reverse bound for $\mathbb{B}^{q,k}_{\leq\mu_k}$. We will have use for the following lemma.

Lemma 6.2. *Suppose that β is a normalized extremal form for $\mathcal{H}^{0,q}(X_0,\phi_0)$ at the point $(0,iv_0)$ in the direction θ. Then*

$$\left|\beta_\theta(0,iv_0)\right|^2 = \frac{1}{4\pi}\int_{T(q)} B_{t,\theta}(z,z)e^{v_0t}b(t)^{-1}dt. \tag{6.3}$$

with notation as in lemma 4.4. Moreover, β is in $L^2_m(X_0)$ for all m.

Proof. Let x_0 be the point $(0,iv_0)$ in X_0. Since β is extremal we have, according to section 3, that $\left|\beta_\theta(0,iv_0)\right|^2 = B_{X_0,\theta}(0,iv_0)$, which in turn gives 6.3 according to lemma 4.4. To prove that β is in $L^2_m(X_0)$ for all m, we write β as

$$\beta(z,w) = \int_{T(q)}\widehat{\beta}_t(z)e^{\frac{i}{2}wt}dt,$$

in terms of its Fourier transform as in section 4. Recall that we have assumed that condition $Z(q)$ holds on the boundary of X, so that $T(q)$ is finite. Using proposition 4.3 we can write

$$\left\|\frac{\partial^l}{\partial^l w}\partial^{I\overline{J}}\beta\right\|_{X_0}^2 = 4\pi\int_{T(q)}\left\|\partial^{I\overline{J}}\widehat{\beta}_t(z)\right\|_t^2 t^{2l}b(t)dt,$$

where $\partial^{I\overline{J}}$ denotes the complex partial derivatives taken with respect to $z_i$and $\overline{z_j}$ for i and j in the multi index set I and J, respectively. Since, by assumption, β is in $L^2(X_0)$ the integral converges for $l=0$ with I and J

empty. Now it is enough to show that $\widehat{\beta_t}$ is in $L^2_m(\mathbb{C}^n, t\psi+\phi)$ for all t and positive integers m. To this end we will use the following generalization of 3.3:

$$|\mathbb{K}_{x,\theta}(y)|^2 = |\beta(y)|^2 B_\theta(x) \tag{6.4}$$

if β is an extremal form at the point x, in the direction θ (compare [5]). By lemma 4.4 the Fourier transform of $\mathbb{K}_{x,\theta}$ evaluated at t is proportional to $\mathbb{K}_{z,\theta,t}$ where $\mathbb{K}_{z,\theta,t}$ is the Bergman kernel form for the space $\mathcal{H}^{0,q}(\mathbb{C}^{n-1}, t\psi+\phi)$ at the point z (and $x = (z, w)$) in the direction θ. In [4] it was essentially shown that $\mathbb{K}_{z,\theta,t}$ is in $L^2_m(\mathbb{C}^n, t\psi + \phi)$ (more precisely: the property was shown to hold for the corresponding extremal form). Hence, the same thing holds for $\widehat{\beta_t}$, according to 6.4, which finishes the proof of the lemma. □

Now we can construct a sequence α_k of approximate extremals for the space $\mathcal{H}_{\leq \mu_k k}(X)$ of low energy forms.

Lemma 6.3. *For any point $x_{0,k} = (0, iv_0/k)$ and direction form θ in the first region there is a sequence $\{\alpha_k\}$ and direction forms θ_k such that α_k is in $\Omega^{0,q}(X, L^k)$ and*

$$\begin{array}{lrcl} (i) & |\alpha_{k,\theta_k}(0, iv_0/k)|^2 & \sim & B_{X_0,\theta}(0, iv_0) \\ (ii) & \|\alpha_k\|^2_X & \sim & 1 \\ (iii) & \left\|(\overline{\partial} + \overline{\partial}^{*(k)})\alpha^{(k)}\right\|^2_m & \sim & 0 \\ (iv) & (\Delta\alpha_k, \alpha_k)_X & \leq & \delta_k \|\alpha_k\|^2 \end{array}$$

where δ_k is a sequence, independent of $x_{0,k}$, tending to zero, when k tends to infinity. Moreover, α_k satisfies $\overline{\partial}-$Neumann boundary conditions on $\partial X \bigcap F_k^{-1}(D)$, where D is a polydisc in $\mathbb{C}^n$ centered at 0.

Proof. Consider a sequence of points $x_{0,k}$ that can be written as $(0, iv_0/k)$ in local coordinates as in section 5.1. Let us first construct a form α_k with the properties (i) to (iv). It is defined by

$$\alpha_k := (F_k^{-1})^*(\chi_k\beta)$$

where $\chi_k(z, w) = \chi(z/\ln k, w/\ln k)$ for χ a smooth function in $\mathbb{C}^n$ that is equal to one on the polydisc D of radius one centered at 0, vanishing outside the polydisc of radius two and where β is the extremal form at the point $(0, v_0)$ in the direction θ from the previous lemma. The definition of α_k is made so that

$$\alpha^{(k)} = \chi_k\beta$$

We have used the fact that the form β extends naturally as a smooth form to the domain X_δ with defining function $\rho_0 - \delta$, to make sure that α_k is defined on all of X. The extension is obtained by writing β in terms of its Fourier transform with respect to t as in the proof of the previous lemma:

$$\beta(z, w) = \int_{T(q)} \widehat{\beta_t}(z) e^{-\frac{i}{2}wt} dt,$$

In fact, the right hand side is defined for all w since we have assumed that condition $Z(q)$ holds on the boundary so that $T(q)$ is finite. Note that the L_m^2-norm of β over X_δ tends to the L_m^2-norm of β over X_0 when δ tends to zero, as can be seen from the analog of proposition 4.3 on the domain X_δ. Indeed, the dependence on δ only appears in the definition of $b(t)$, where the upper integration limit is shifted to δ. Now the statements (i) and (ii) follow from the corresponding statements in the previous lemma. To see that (iii) holds, first observe that

$$\overline{\partial}^{*(k)} = \overline{\partial}^{*0} + \varepsilon_k \mathcal{D},$$

where $\mathcal{D}$ is a first order differential operator with bounded coefficients on the ball $B_{lnk}(0)$ and ε_k is a sequence tending to zero. Indeed, this is a simple modification of the statement 5.3. Moreover, by construction $(\overline{\partial} + \overline{\partial}^{*0})\beta = 0$. Hence, Leibniz rule gives

$$\left\| (\overline{\partial} + \overline{\partial}^{*(k)})\alpha^{(k)} \right\| \leq \delta_k \left\| \beta \right\|_1 + \left\| \left| d\chi_k \right| \beta \right\|$$

The first term tends to zero since β is in $L_1^2(X_0)$ and the second terms tends to zero, since it can be dominated by the "tail" of a convergent integral. The estimates for $m \geq 1$ are proved in a similar way (compare [4]). Finally, to prove (iv) observe that by the scaling property 5.5 for the Laplacian

$$(\Delta \alpha_k, \alpha_k)_X = \left\| (\overline{\partial} + \overline{\partial}^{*})\alpha_k \right\|_X^2 = \left\| (\overline{\partial} + \overline{\partial}^{*(k)})\alpha^{(k)} \right\|.$$

By (ii), the norm of $\left\| \alpha_k \right\|_X^2$ tends to one and the norm in the right hand side above can be estimated as above. To see that δ_k can be taken to be independent of the point $x_{0,k}$ in the first region, one just observes that the constants in the estimates depend continuously on the eigenvalues of the curvature forms (compare [4]). Finally, consider a polydisc D in $\mathbb{C}^n$ with small radius. We will perturb α_k slightly so that it satisfies $\overline{\partial}-$Neumann boundary conditions on $\partial X \bigcap F_k^{-1}(D)$ while preserving the properties (i) to (iv). Recall that a form $\overline{\partial}-$closed form η_k satisfies $\overline{\partial}-$Neumann boundary conditions on ∂X if

$$\overline{\partial \rho}^{*} \eta_k = 0, \tag{6.5}$$

where $\overline{\partial \rho}^{*}$ is the fiber-wise adjoint of the operator obtained by wedging with the form $\overline{\partial \rho}$, and where the adjoint is taken with respect to the metric ω_k on X. Equivalently,

$$\overline{\partial k \rho^{(k)}}^{*} \eta^{(k)} = 0,$$

where the adjoint is taken with respect to the scaled metrics. By construction we have that $\alpha^{(k)}$ is $\overline{\partial}-$closed on $F_k^{-1}(D)$ and

$$\overline{\partial \rho_0}^{*0} \alpha^{(k)} = 0, \tag{6.6}$$

where now the adjoint is taken with respect to the model metrics. Let

$$u^{(k)} := -\overline{\partial}(k\rho^{(k)} \overline{\partial k \rho^{(k)}}^{*} \alpha^{(k)})$$

and let $\widetilde{\alpha}^{(k)} := \alpha^{(k)} + \chi u^{(k)}$, where χ is the cut-off function defined above. Then, using that ρ vanishes on ∂X, we get that the $\overline{\partial}-$closed form $\widetilde{\alpha}^{(k)}$ satisfies the scaled $\overline{\partial}-$Neumann boundary conditions, i.e. the relation 6.5 on ∂X. Moreover, using that $k\rho^{(k)}$ converges to ρ with all derivatives on a fixed polydisc centered at 0 (lemma 5.1) and 6.6 one can check that $u^{(k)}$ tends to zero with all derivatives in X_δ. Finally, since $\chi u^{(k)}$ is supported on a bounded set in X_δ and converges to zero with all derivatives it is not hard to see that $\widetilde{\alpha_k}$ also satisfies the properties (i) to (iv), where $\widetilde{\alpha_k} := F_k^{-1*}(\widetilde{\alpha^{(k)}})$ □

By projecting the sequence α_k of approximate extremals, from the previous lemma, on the space of low energy forms we will now obtain the following lower bound on $\mathbb{B}^q_{\leq \mu_k}$.

Proposition 6.4. *There is a sequence μ_k tending to zero such that*

$$\liminf_k B^q_{\leq \mu_k, \theta_k}(0, iv/k) \geq B_{X_0}(0, v)_\theta.$$

Proof. The proof is a simple modification of the proof of proposition 5.3 in [4]. Let $\{\alpha_k\}$ be the sequence that the previous lemma provides and decompose it with respect to the orthogonal decomposition $\Omega^{0,q}(X, L^k) = \mathcal{H}^q_{\leq \mu_k}(X, L^k) \oplus \mathcal{H}^q_{> \mu_k}(X, L^k)$, induced by the spectral decomposition of the subelliptic operator Δ [14]:

$$\alpha_k = \alpha_{1,k} + \alpha_{2,k}$$

First, we prove that

$$\lim_k \left| \alpha_2^{(k)}(0, iv) \right|^2 = 0. \tag{6.7}$$

Since $\alpha_2^{(k)} = \alpha^{(k)} - \alpha_1^{(k)}$ the form $\alpha_2^{(k)}$ satisfies $\overline{\partial}-$Neumann boundary conditions on the intersection of the polydisc D with $F_k^{-1}(\partial X)$, using lemma 6.3. Subelliptic estimates as in the proof of lemma 6.1 then show that

$$\left| \alpha_2^{(k)}(0) \right|^2 \leq C(\left(\left\| \alpha_2^{(k)} \right\|^2_{D \bigcap F_k^{-1}(X)} + \left\| (\Delta^{(k)}) \alpha_2^{(k)} \right\|^2_{D \bigcap F_k^{-1}(\partial X), m} \right) \tag{6.8}$$

for some large integer m. To see that the first term in the right hand side tends to zero, we first estimate $\left\| \alpha_2^{(k)} \right\|^2_{D \bigcap F_k^{-1}(X)}$ with $\|\alpha_{k,2}\|^2_X$ using the norm localization in lemma 5.3. Next, by the spectral decomposition of Δ_k :

$$\|\alpha_{2,k}\|^2_X \leq \frac{1}{\mu_k} \langle \Delta_k \alpha_{2,k}, \alpha_{2,k} \rangle_X \leq \frac{1}{\mu_k} \langle \Delta_k \alpha_k, \alpha_k \rangle_X \leq \frac{\delta_k}{\mu_k} \|\alpha_k\|^2_X ,$$

using property (iv) in the previous lemma in the last step. By property (ii) in the same lemma $\|\alpha_k\|^2_X$ is asymptotically 1, which shows that the first term in 6.8 tends to zero if the sequence μ_k is chosen as $\delta_k^{1/2}$, for example. To see that the second term tends to zero as well, we estimate

$$\left\| (\Delta^{(k)}) \alpha_2^{(k)} \right\|_m \leq \left\| (\Delta^{(k)}) \alpha^{(k)} \right\|_m + \left\| (\Delta^{(k)}) \alpha_1^{(k)} \right\|_m .$$

The first term in the right hand side tends to zero by (iii) in the previous lemma and so does the second term, using 6.2 (that holds for any scaled sequence of forms in $\mathcal{H}^q_{\leq\mu_k}(X, L^k)$). This finishes the proof of the claim 6.7. Finally,

$$\liminf_k B^q_{\leq\mu_k}(0, iv/k)_{\theta_k} \geq |\alpha_{1,k}(0)|^2_{\theta_k}$$

and

$$\liminf_k |\alpha_{k,1,\theta_k}(0)|^2 \geq B_{X_0,\theta}(0, v) + 0,$$

when k tends to infinity, using 6.7 and (i) in the previous lemma. □

Now we can prove that the Morse inequalities are essentially equalities for the space $\mathcal{H}^{0,q}_{\leq\mu_k}(X, L^k)$ of low-energy forms. But first recall that $\mathcal{H}^{0,q}_{\leq\mu_k}(X, L^k)$ depend on a large parameter R, since the metrics ω_k depend on R.

Theorem 6.5. *Suppose that X is is a compact complex manifold with boundary satisfying condition $Z(q)$. Then there is a sequence μ_k tending to zero such that the limit of $k^{-n}\dim\mathcal{H}^{0,q}_{\leq\mu_k}(X, L^k)$ when k tends to infinity is equal to*

$$(-1)^q(\frac{1}{2\pi})^n(\int_{X(q)}\Theta_n + \int_{\partial X}\int_{T(q)_{\rho,x}}(\Theta + t\mathcal{L})_{n-1}\wedge\partial\rho\wedge dt) + \epsilon_R$$

where the sequence ϵ_R tends to zero when R tends to infinity.

Proof. The proof is completely analogous to the proof of theorem 2.1. In the first region one just replaces the claim 5.12 in the proof of proposition 5.5 by the asymptotic equality for $B^q_{\leq\mu_k}(0, iv/k)_{\theta_k}$ obtained by combing the propositions 6.1 and 6.4. Moreover, a simple modification of the proof of proposition 6.1 shows that there is no contribution from the integrals over the second and third region, when R tends to infinity, as before. Finally, the convergence on the inner part of X was shown in ([4]). □

Recall that the Dolbeault cohomology group $H^{0,q}(X, L^k)$ is isomorphic to the space of harmonic forms, which is a subspace of $\mathcal{H}^{0,q}_{\leq\mu_k}(X, L^k)$. Hence, the previous theorem is stronger than the weak Morse inequalities for the dimensions $h^q(L^k)$ of $H^{0,q}(X, L^k)$ (theorem 2.1). When X has no boundary Demailly showed that, by combining a version of theorem 6.5 with some homological algebra, one gets strong Morse inequalities for the Dolbeault cohomology groups. These are inequalities for an alternating sum of all $h^i(L^k)$ when the degree i varies between 0 and a fixed degree q [9]. In fact, a variation of the homological algebra argument yields inequalities for alternating sums when the degree i varies between a fixed degree q and the complex dimension n of X (the two versions are related by Serre duality). However, when X has a boundary one has to impose certain curvature conditions on ∂X to obtain strong Morse inequalities from theorem 6.5 . Indeed, to apply the theorem one has to assume that ∂X satisfies condition $Z(i)$ for all degrees i in the corresponding range. In particular the corresponding dimensions will then be finite dimensional so that the alternating sum makes sense. Now, to state the strong holomorphic Morse inequalities for a manifold with boundary, recall that the boundary of a compact complex manifold is called $q-$convex if the Levi form $\mathcal{L}$ has at least $n-q$ positive eigenvalues along $T^{1,0}(\partial X)$ and it is called $q-$concave if the Levi form has at least $n-q$ negative eigenvalues along $T^{1,0}(\partial X)$ (i.e ∂X is $q-$convex "from the inside" precisely when it is $q-$concave "from the outside"). We will denote by $X(\geq q)$ the union of all sets $X(i)$ with $i \geq q$ and $T(\geq q)_{\rho,x}$ is defined similarly. The sets $X(\leq q)$ and $T(\leq q)_{\rho,x}$ are defined by putting $i \leq q$ in the previous definitions. Finally, we set

$$I_{\geq q} := (\frac{1}{2\pi})^n (\int_{X(\geq q)} \Theta_n + \int_{\partial X} \int_{T(\geq q)_{\rho,x}} (\Theta + t\mathcal{L})_{n-1} \wedge \partial\rho \wedge dt)$$

and define $I_{\leq n-1-q}$ similarly.

Theorem 6.6. *Suppose that X is an $n-$dimensional compact manifold with boundary. If the boundary is strongly $q-$convex, then*

$$k^{-n} \sum_{i=q}^{n} (-1)^{q-j} h^j(L^k) \leq I_{\geq q} k^n + o(k^n).$$

If X has strongly $q-$concave boundary, then

$$\sum_{i=0}^{n-1-q} (-1)^{q-j} h^j(L^k) \leq I_{\leq n-1-q} k^n + o(k^n).$$

Proof. First note that if ∂X is $q-$convex, then ∂X satisfies condition $Z(i)$ for i such that $n-q \leq i \leq n$. Similarly, if ∂X is $q-$concave, then ∂X satisfies condition $Z(i)$ for i such that $0 \leq i \leq n-q-1$. The proof then follows from theorem 6.5 and the homological algebra argument in [9],[10]. See also [7] and [21]. □

6.1. **Strong Morse inequalities on open manifolds.** One can also define $q-$convexity and $q-$concavity on open manifolds following Andreotti and Grauert [2]. First, one says that a function ρ is $q-$convex if $i\partial\overline{\partial}\rho$ has at least $n-q+1$ positive eigen values. Next, an open manifold Y is said to be $q-$convex if it has an exhaustion function ρ that is $q-$convex outside some compact subset K of Y. The point is that the regular sublevel sets of ρ are then $q-$convex considered as compact manifolds with boundary. The extra positive eigen value occurring in the definition of $q-$convexity for an open manifold is needed to make sure that $i\partial\overline{\partial}\rho$ still has at least $n-q$ positive eigen values along a regular level surface of ρ. Finally, an open manifold Y is said to be $q-$concave if it has an exhaustion function ρ such that $-\rho$ is $q-$convex outside some compact subset K of Y.

Now, by remark 1.3 theorem 6.6 extends to any $q-$convex open manifold Y with a line bundle L if one uses the usual Dolbeault cohomology $H^{0,*}(Y, L^k)$ (or equivalently the sheaf cohomology $H^*(Y, \mathcal{O}(L^k))$ and the curvature integrals are taken over a regular level surface of ρ in the complement of the compact set K. However, for a $q-$concave open manifold

Y one only gets the corresponding result if $n-q-1$ is replaced with $n-q-2$. Indeed, by remark 1.3 one has to make sure that condition $Z(i+1)$ holds for the highest degree i occurring in the alternating sum. In this form the $q-$convex case and $q-$concave case was obtained by Bouche [7] and Marinescu [21], respectively, under the assumption that the curvature of the line bundle L is adapted to the curvature of the boundary of X in a certain way. Comparing with theorem 6.6 their assumptions imply that the boundary integral vanishes. There is also a very recent preprint [22] of Marinescu where strong Morse inequalities on a $q-$concave manifold with an arbitrary line bundle L are obtained. However, the corresponding boundary term is not as precise as the one in theorem 6.6 and in section 7 we will show that theorem 6.6 is sharp.

Note that since the curvature integrals are taken over *any* regular level surface of ρ in Y one expects that $I(\geq q)$ and $I(\leq n-1-q)$ are independent of the level surface. This is indeed the case (see remark 7.4).

6.2. **Application to the volume of semi-positive line bundles.** Now assume that X is a strongly pseudoconcave manifold X with a semi-positive line bundle L (i.e the Levi form $\mathcal{L}$ is negative along $T^{1,0}(\partial X)$ and the curvature form of L is semi-positive in X). The case of pseudoconcave surfaces has been recently studied in [18][12], by different methods. When the dimension of X is at least three, the strong Morse inequalities give a lower bound on the dimension of the space of holomorphic sections with values in L^k. Namely, $h^0(L^k)$ is asymptotically bounded from below by

$$(6.9)\quad (\frac{1}{2\pi})^n(\int_{X(0)}\Theta_n+\int_{\partial X}\int_{T(\leq 1)}(\Theta+t\mathcal{L})_{n-1}\wedge\partial\rho\wedge dt)k^n+h^1(L^k)+o(k^n)$$

In particular, if the curvatures are such that the coefficient in front of k^n is positive, then the dimension of $H^0(X,L^k)$ grows as k^n. In other words, the line bundle L is *big* then. For example, this happens when the curvature forms are conformally equivalent along the complex tangential directions, i.e. if there is a function f on ∂X such that

$$\mathcal{L}=-f\Theta \tag{6.10}$$

when restricted to $T^{1,0}(\partial X)\otimes T^{0,1}(\partial X)$. In fact, by multiplying the original ρ by f^{-1} we may and will assume that $f=1$. The lower bound 6.9 combined with the upper bound from the weak Morse inequalities (theorem 2.1) then gives the following corollary.

Corollary 6.7. *Suppose that X is a strongly pseudoconcave manifold X of dimension $n\geq 3$ with a semi-positive line bundle L. Then if the curvature forms are conformally equivalent at the boundary*

$$h^0(L^k)=k^n(\frac{1}{2\pi})^n(\int_X\Theta_n+\frac{1}{n}\int_{\partial X}(i\partial\overline{\partial}\rho)_{n-1}\wedge i\partial\rho)+o(k^n).$$

When L is positive, the conformal equivalence in the previous corollary says that the symplectic structure on X determined by L is compatible with the contact structure of ∂X determined by the complex structure, in a strong sense (compare [12]) and the conclusion of the corollary may be expressed by the formula

$$\text{Vol}(L) = \text{Vol}(X) + \frac{1}{n}\text{Vol}(\partial X) \tag{6.11}$$

in terms of the symplectic and contact volume of X and ∂X, respectively (where the volume of a line bundle L is defined as the lim sup of $(2\pi)^n k^{-n} h^0(L^k)$ [20]). The factor $\frac{1}{n}$ in the formula is related to the fact that if $(X_+, d\alpha)$ is a $2n-$dimensional real symplectic manifold with boundary, such that α is a contact form for ∂X_+, then, by Stokes theorem, the contact volume of ∂X_+ divided by n is equal to the symplectic volume of X_+. In fact, this is how we will show that 6.11 is compatible with hole filling in section 7.1.

7. Sharp examples and hole filling

In this section we will show that the leading constant in the Morse inequalities 2.1 is sharp. When X is a compact manifold without boundary, this is well-known. Indeed, let X be the $n-$dimensional flat complex torus $\mathbb{C}^n/\mathbb{Z}^n + i\mathbb{Z}^n$ and consider the hermitian holomorphic line bundle L_λ over T^n determined by the constant curvature form

$$\Theta = \sum_{i=1}^{n} \frac{i}{2}\lambda_i dz_i \wedge \overline{dz_i},$$

where λ_i are given non-zero integers [16]. Then one can show (see the remark at the end of the section) that

$$B^q(x) \equiv \frac{1}{\pi^n} 1_{X(q)} \left|\det_\omega \Theta\right|, \tag{7.1}$$

where $1_{X(q)}$ is identically equal to one if exactly q of the eigenvalues λ_i are negative and equal to zero otherwise. This shows that the leading constant in the Morse inequalities on a compact manifold is sharp.

Let us now return to the case of a manifold with boundary. We let X be the manifold obtained as the total space of the unit disc bundle in the dual of the line bundle L_μ (where L_μ is defined as above) over the torus T^{n-1}, where μ_i are $n-1$ given non-zero integers. Next, we define a hermitian holomorphic line bundle over X. Denote by π the natural projection from X onto the torus T^{n-1}. Then the pulled back line bundle $\pi^* L_\lambda$ is a line bundle over X. The construction is summarized by the following commuting diagram

$$\begin{array}{ccccc} \pi^* L_\lambda & & & & L_\lambda \\ \downarrow & & & & \downarrow \\ X & \hookrightarrow & L_\mu^* & \rightarrow & T^{n-1} \end{array}$$

Let h be the positive real-valued function on X, defined as the restriction to X of the squared fiber norm on L_μ^*. Then $\rho := \ln h$ is a defining function for X close to the boundary and we define a hermitian metric ω on X by

$$\omega = \frac{i}{2}\partial\overline{\partial}\left|z\right|^2 + \frac{i}{2} h^{-1}\partial h \wedge \overline{\partial h}$$

extended smoothly to the base T^{n-1} of X. The following local description of the situation is useful. The part of X that lies over a fundamental domain of T^{n-1} can be represented in local holomorphic coordinates (z,w), where w is the fiber coordinate, as the set of all (z,w) such that

$$h(z,w) = |w|^2 \exp(+\sum_{i=1}^{n-1} \mu_i |z_i|^2) \leq 1$$

and the fiber metric ϕ for the line bundle $\pi^* L_\lambda$ over X may be written as

$$\phi(z,w) = \sum_{i=1}^{n-1} \lambda_i |z_i|^2 .$$

The proof of the following proposition is very similar to the proof of theorem 4.5, but instead of Fourier transforms we will use Fourier series, since the $\mathbb{R}-$ symmetry is replaced by an S^1- symmetry (the model domain X_0 in section 4 is the universal cover of X defined above).

Theorem 7.1. *Let $J(q)$ be the set of all integers j such that the form $\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho$ has exactly q negative eigenvalues. Then*

$$B_X^q = (\frac{1}{2\pi})^n \sum_{j \in J(q)} det_\omega(i(\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho))\frac{1}{2}(j+1)h^j. \tag{7.2}$$

In particular, the dimension of $H^{0,q}(X, \pi^* L_\lambda)$ is given by

$$(\frac{i}{2\pi})^n \int_{\partial X} \sum_{j \in J(q)} (\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho)_{n-1} \wedge \partial\rho \tag{7.3}$$

and the limit of the dimensions of $H^{0,q}(X, (\pi^* L_\lambda)^k)$ divided by k^n is

$$(\frac{i}{2\pi})^n \int_{\partial X} \int_{T_{x,\rho}(q)} (\partial\overline{\partial}\phi + t\partial\overline{\partial}\rho)_{n-1} \wedge \partial\rho \wedge dt \tag{7.4}$$

Proof. First note that if α_j is a form on T^{n-1} with values in $L_\mu^j \otimes L_\lambda$, then

$$\alpha(z,w) := \sum_{j \geq 0} \alpha_j(z) w^j$$

defines a global form on X with values in $\pi^* L_\lambda$. The proof of proposition 4.1 can be adapted to the present situation to show that any form α in $\mathcal{H}^{0,q}(X,\phi)$ is of this form with α_j in $\mathcal{H}^{0,q}(T^{n-1}, L_\mu^j \otimes L_\lambda)$. Actually, since X is a fiber bundle over T^{n-1} with *compact* fibers one can also give a somewhat simpler proof. For example, to show that α is tangential one solves the $\overline{\partial}-$ equation along the fibers of closed discs in order to replace the normal part α_N with an exact form. Then using the assumption the α is coclosed one shows that the exact form must vanish. The details are omitted. We now have the following analog of proposition 4.3 for any α in $\mathcal{H}^{0,q}(X,\phi)$

$$(\alpha,\alpha) = 2\pi \sum_j (\alpha_j, \alpha_j) b_j, \;\; b_j = \int_0^1 (r^2)^j r dr = 1/2(j+1)^{-1} \tag{7.5}$$

in terms of the induced norms. To see this, one proceed as in the proof of proposition 4.3, now using the Taylor expansion of α. Writing $\psi(z) = \sum_{i=1}^{n-1} \mu_i |z_i|^2$ and restricting z to the fundamental region of T^{n-1} we get that (α, α) is given by

$$\int_{|w|^2 < e^{-\psi(z)}} \left| \sum_j \alpha_j(z) w^j \right|^2 e^{-\phi(z)} (\frac{i}{2} \partial\overline{\partial} |z|^2)_{n-1} e^{\psi(z)} r dr d\theta.$$

Now using Parceval's formula for Fourier series in the integration over θ this can be written as

$$2\pi \sum_j \int_z |\widehat{\alpha_j}(z)|^2 e^{-\phi(z)} (\frac{i}{2} \partial\overline{\partial} |z|^2)_{n-1} e^{\psi(z)} \int_0^{e^{-\psi(z)/2}} (r^2)^j r dr.$$

Finally, the change of variables $r' = e^{\psi(z)/2} r$ in the integral over r gives a factor $e^{-j\psi(z)}$ and the upper integration limit becomes 1. This proves 7.5.

As in the proof of theorem 4.5 we infer that B_X may be expanded as

$$B_X(z, w) = \frac{1}{2\pi} \sum_j B_j(z) h^j b_j^{-1},$$

where B_j is the Bergman function of the space $\mathcal{H}^{0,q}(T^{n-1}, L_\mu^j \otimes L_\lambda)$. According to 7.1, we have that

$$B_j(z) \equiv (\frac{1}{2\pi})^{n-1} \delta_{j,J(q)} \left| \det_\omega (i(\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho)) \right|,$$

where $J(q) = \{j : \text{index}(\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho) = q\}$and where the sequence $\delta_{j,J(q)}$ is equal to 1 if $j \in J(q)$ and zero otherwise. Thus, 7.2 is obtained. Integrating 7.2 over X gives

$$\int_X B_X \omega_n = \frac{1}{2\pi} \int_{T^{n-1}} \sum_j B_j (\frac{i}{2} \partial\overline{\partial} |z|^2)_{n-1} \int_0^{2\pi} d\theta \int_0^1 (r^2)^j r dr b_j^{-1}.$$

The integral over the radial coordinate r is cancelled by b_j^{-1} and we may write the resulting integral as

$$\frac{1}{2\pi} \int_{\partial X} \sum_j B_j \omega_{n-1} \wedge i\partial\rho = (\frac{i}{2\pi})^n \int_{\partial X} \sum_{j \in J(q)} (\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho)_{n-1} \wedge \partial\rho$$

Hence, 7.3 is obtained. Finally, applying the formula 7.3 to the line bundle $(\pi^* L_\lambda)^k = \pi^*(L_\lambda^k)$ shows, since the curvature form of $\pi^*(L_\lambda^k)$ is equal to $k\partial\overline{\partial}\phi$, that

$$k^{-n} \int_X B_X^{q,k} \omega_n = (\frac{i}{2\pi})^n \int_{\partial X} \sum_j (\partial\overline{\partial}\phi + \frac{j}{k} \partial\overline{\partial}\rho)^{n-1} \frac{1}{k} \wedge \partial\rho,$$

where the sum is over all integers j such that $\partial\overline{\partial}\phi + \frac{j}{k}\partial\overline{\partial}\rho$ has exactly q negative eigenvalues. Observe that the sum is a Riemann sum and when k tends to infinity we obtain 7.4. $\square$

Note that since the line bundle $\pi^* L_\lambda$ over X is flat in the fiber direction the integral over X in 1.2 vanishes. Hence, the theorem above shows that the holomorphic Morse inequalities are sharp. The most interesting case covered by the theorem above is when the line bundle $\pi^* L_\lambda$ (simply denoted by L) over X is semi-positive, and positive along the tangential directions, and X is strongly pseudoconcave. This happens precisely when all λ_i are positive and all μ_i are negative. Then, for $n \geq 3$, the theorem above shows that the dimension of $H^{0,1}(X, L^k)$ grows as k^n unless the curvature of L is a multiple of the Levi curvature of the boundary, i.e. unless λ and μ are parallel as vectors. This is in contrast to the case of a manifold without boundary, where the corresponding growth is of the order $o(k^n)$ for a semi-positive line bundle. Note that the bundle L above always admits a metric of *positive* curvature. Indeed, the fiber metric $\phi + \epsilon h$ on L can be seen to have positive curvature, if the positive number ϵ is taken sufficiently small. However, if λ and μ are not parallel as vectors, there is no metric of positive curvature which is conformally equivalent to the Levi curvature at the boundary. This follows from the weak holomorphic Morse inequalities, theorem 2.1, since the growth of the dimensions of $H^{0,1}(X, L^k)$ would be of the order $o(k^n)$ then.

Remark 7.2. To get examples of open manifold Y as described in remark 1.3 one may take the total space of the line bundle L_μ^* over T^{n-1}, as defined in the beginning of the section. Then ρ is an exhaustion function, exhausting L_μ^* by disc bundles. Furthermore, to get examples of manifolds with boundary X where the index of the Levi curvature form is non-constant one may take X to be an annulus bundle in L_μ^*. Such a manifold is neither $q-$convex or $q-$concave for any q. Theorem 7.1 extends to such manifolds X if one uses Laurent expansions of sections instead of Taylor expansions. A concrete example is given by the hyper plane bundle $O(1)$ over $\mathbb{P}^{n-1}$. Then the corresponding annulus bundle is biholomorphic to a spherical shell in $\mathbb{C}^n$, i.e. all z in $\mathbb{C}^n$ such that $r \leq |z| \leq r'$ for some given numbers r and r'. It has one pseudoconvex and one pseudoconcave boundary component.

Finally, a remark about the proof of formula 7.1.

Remark 7.3. To prove formula 7.1 one can for example reduce the problem to holomorphic sections, i.e. when $q = 0$ (compare [4]). One could also use symmetry to first show that the Bergman kernel is constant and then compute the dimension of $H^q(T^n, L_\lambda)$ by standard methods. To compute the dimension one writes the line bundle L_λ as $L_\lambda = \pi_1^* L^{\lambda_1} \otimes \pi_2^* L^{\lambda_2} \otimes ... \otimes \pi_n^* L^{\lambda_n}$, using projections on the factors of T^n, where L is the classical line bundle over the elliptic curve $T^1 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$, such that $H^0(\mathbb{C}/\mathbb{Z} + i\mathbb{Z}, L)$ is generated by the Riemann theta function [16]. Now, using Kunneth's theorem one gets that $H^q(T^n, L_\lambda)$ is isomorphic to the direct sum of all tensor products of the form

$$H^1(T^1, L^{\lambda_{i_1}}) \otimes \cdots \otimes H^1(T^1, L^{\lambda_{i_q}}) \otimes H^0(T^1, L^{\lambda_{i_{q+1}}}) \otimes \cdots \otimes H^0(T^1, L^{\lambda_{i_n}}).$$

Observe that this product vanishes unless the index $I = (i_1, ..., i_n)$ is such that the first q indices are negative while the others are positive. Indeed, first observe that if m is a positive integer, the dimension of $H^0(T^1, L^{-m})$ vanishes, since L^{-m} is a negative line bundle. Next, by Serre duality $H^1(T^1, L^m) \approxeq H^0(T^1, L^{-m})$, since the canonical line bundle on T^1 is trivial. So the dimension of $H^1(T^1, L^m)$ vanishes as well. In particular, the dimension of $H^q(T^n, L_\lambda)$ vanishes unless exactly q of the numbers λ_i are negative, i.e. unless the index of the curvature of L_λ is equal to q. Finally, if the index is equal to q, then, using that $H^0(T^1, L)$ is one-dimensional, combined with Serre duality and Kunneth's formula again, one gets that the dimension of $H^0(T^1, L^{-1})$ is equal to the absolute value of the product of all eigenvalues λ_i. This proves 7.1.

7.1. **Relation to hole filling and contact geometry.** Consider a compact strongly pseudoconcave manifold X with a semi-positive line bundle L. We will say that the pair (X, L) may be *filled* if there is a compact complex manifold $\widetilde{X}$, without boundary, with a semi-positive line bundle $\widetilde{L}$ such that there is a holomorphic line bundle injection of L into $\widetilde{L}$. [2] The simplest situation is as follows. Start with a compact complex manifold $\widetilde{X}$ with a positive line bundle $\widetilde{L}$ (by the Kodaira embedding theorem $\widetilde{X}$ is then automatically a projective variety [16]). We then obtain a pseudoconcave manifold X by making a small hole in $\widetilde{X}$ in the following way. Consider a small neighborhood of a fixed point x in $\widetilde{X}$, holomorphically equivalent to a ball in $\mathbb{C}^n$, where $\widetilde{L}$ is holomorphically trivial and let ϕ be the local fiber metric. We may assume that $\phi(x) = 0$ and that ϕ is non-negative close to x. Then for a sufficiently small ϵ the set where ϕ is strictly less than ϵ is a strongly pseudoconvex domain of $\widetilde{X}$ and its complement is then a strongly pseudoconcave manifold that we take to be our manifold X. We let L be the restriction of $\widetilde{L}$ to X. A defining function of the boundary of X can be obtained as $\rho = -\phi$. Now, since $\widetilde{L}$ is a positive line bundle it is well-known that

$$\lim_k k^{-n} \dim_{\mathbb{C}} H^0(\widetilde{X}, \widetilde{L}^k) = (\frac{i}{2\pi})^n \frac{1}{n!} \int_{\widetilde{X}} (\partial\overline{\partial}\widetilde{\phi})^n.$$

In fact, this holds for any semi-positive line bundle, as can be seen by combining Demailly's holomorphic Morse inequalities 1.1 with the Riemann-Roch theorem (this was first proved by different methods in [26]).

On the other hand we have by Harthog's phenomena (assuming that $n \geq 2$), that $H^0(\widetilde{X}, \widetilde{L}^k)$ is isomorphic to $H^0(X, L^k)$. So decomposing the integral above with respect to

$$\widetilde{X} = X \bigsqcup X^c \tag{7.6}$$

[2]By a theorem of Rossi [23], the pair (X, L) may always be filled if L is trivial close to the boundary and the dimension of X is at least 3.

and using Stokes theorem gives that

(7.7)

$$\lim_k k^{-n}\dim_{\mathbb{C}} H^0(X,L^k)=(\frac{i}{2\pi})^n(\frac{1}{n!}\int_X(\partial\overline{\partial}\phi)_n-\frac{1}{n!}\int_{\partial X}(\partial\overline{\partial}\phi)^{n-1}\wedge\partial\phi)$$

Let us now compare the boundary integral above with the curvature integral in the holomorphic Morse inequalities 1.2. Since $\rho=-\phi$ this integral equals

$$-\frac{1}{(n-1)!}\int_{\partial X\times[0,1]}((1-t)\partial\overline{\partial}\phi)^{n-1}\wedge\partial\phi\wedge dt,$$

which coincides with the boundary integral in 7.7 since $\int_o^1(1-t)^{n-1}dt=1/n$. This shows that the holomorphic Morse inequalities, theorem 6.5 are sharp for the line bundle L over X. To show that the Morse inequalities are sharp as soon as a pair (X,L) may be filled by a Stein manifold it is useful to reformulate the boundary term in 1.2 in terms of the contact geometry of the boundary ∂X.

Let us first recall some basic notions of contact geometry ([3]). The distribution $T^{1,0}(\partial X)$ can be obtained as $\ker(-i\partial\rho)$ and since, by assumption, the restriction of $d(-i\partial\rho)$ is non-degenerate it defines a so called *contact* distribution and ∂X is hence called a *contact manifold.* By duality $T^{1,0}(\partial X)$ determines a real line bundle in the real cotangent bundle $T^*(\partial X)$ that can be globally trivialized by the form $-i\partial\rho$. Denote by X_+ the associated fiber bundle over ∂X of "positive" rays and denote by α the tautological one form on $T^*(\partial X)$, so that $d\alpha$ is the standard symplectic form on $T^*(\partial X)$. The pair $(X_+,d\alpha)$ is called the *symplectification* of the contact manifold ∂X in the literature [3]. More concretely,

$$X_+=\{t(-i\partial\rho_x):\,x\in\partial X,t\geq 0\},$$

i.e. $X_+$is isomorphic to $\partial X\times[0,\infty[$ and $\alpha=-it\partial\rho$ so that $d\alpha=i(t\partial\overline{\partial}\rho+\partial\rho\wedge dt)$. The boundary integral in 1.2 may now be compactly written as

$$\int_{X_+(q)}(\Theta+d\alpha)_n,$$

where $X_+(q)$ denotes the part of X_+ where the pushdown of $d\alpha$ to ∂X has exactly q negative eigenvalues along the contact distribution $T^{1,0}(\partial X)$.

Let us now assume that X is strongly pseudoconcave and that (X,L) is filled by $(\widetilde{X},L)$ (abusing notation slightly). We will also assume that the strongly pseudoconvex manifold Y, in $\widetilde{X}$, obtained as the closure of the complement of X in $\widetilde{X}$, has a defining function that we write as $-\rho$ which is plurisubharmonic on Y. We may assume that the set of critical points of $-\rho$ on Y is finite and to simplify the notation in the argument we assume that there is exactly one critical point x_0 in Y and we assume that $\rho(x_0)=1$ (the general argument is the same). For a regular value c of ρ we let $X_+(0)_c$ be the subset of the symplectification of $\rho^{-1}(c)$ defined as above, thinking of $\rho^{-1}(c)$ as a strictly pseudoconcave boundary. Now consider the following manifold with boundary:

$$\mathcal{X}_\varepsilon(0) = \bigcup_{c\in[0,1-\varepsilon]} X_+(0)_c.$$

More concretely, $\mathcal{X}_\varepsilon(0)$ can be identified with a subset of the positive closed cone in $T^*(Y,\mathbb{C})$ determined by $\partial\rho$:

$$\{t(-i\partial\rho_x) : x \in Y, t \geq 0\}.$$

Hence, $\mathcal{X}_\varepsilon(0)$ is a fiber bundle over a subset of Y and when ϵ tends to zero, the base of $\mathcal{X}_\varepsilon(0)$ tends to Y. Note that the fibers of $\mathcal{X}_\varepsilon(0)$ are a finite number of intervals and the induced function t on $\mathcal{X}_\varepsilon$ is uniformly bounded with respect to ε (i.e. the "height" of the fiber is uniformly bounded). Indeed, we have assumed that $i\partial\overline{\partial}\rho$ is strictly negative. This forces $\Theta + ti\partial\overline{\partial}\rho$ to be negative on all of Y for all t larger then some fixed number t_0. In particular such a t is not in $T_x(0)$ for any x in Y, i.e. not in any fiber of $\mathcal{X}_\varepsilon(0)$. Now observe that the form $\Theta + d\alpha$ on $X_+(0)$ extends to a closed form in $\mathcal{X}_\varepsilon(0)$ and the restriction of the form to Y coincides with Θ. Let us now integrate the form $(\Theta + d\alpha)_n$ over the boundary of $\mathcal{X}_\varepsilon(0)$. The boundary can be written as

$$\partial(\mathcal{X}_\varepsilon(0)) = X_+(0) \bigcup \left(\bigcup_{c\in]0,1-\varepsilon[} \partial(X_+(0)_c) \right) \bigcup X_+(0)_\varepsilon.$$

Since the form is closed, the integral over $\partial(\mathcal{X}_\varepsilon(0))$ vanishes according to Stokes theorem, giving

$$0 = \int_{X_+(0)} (\Theta + d\alpha)_n - \int_Y \Theta_n + 0 + O(\varepsilon)$$

where the zero contribution comes from the fact that the form $(\Theta + d\alpha)_n$ vanishes along $\left(\bigcup_{c\in]0,1-\varepsilon[} \partial(X_+(0)_c)\right) - Y$. The term $O(\varepsilon)$ comes from the integral (of a uniformly bounded function) over the "cylinder" $X_+(0)_\varepsilon$ around the point x_0. Finally, by letting ϵ tend to zero we see that the Morse inequalities for L over X are sharp in this situation as well.

Remark 7.4. The preceding argument also shows that if ρ is a function on an open manifold Y with regular values c and c' (where c is less than c'), then

$$\int_{X_+(i)_c} (\Theta + d\alpha)_n = \int_{\rho^{-1}]c,c']} \Theta_n + \int_{X_+(i)_{c'}} (\Theta + d\alpha)_n$$

for all i such that $i \geq q$, if ρ is $q-$convex on $\rho^{-1}]c, c']$. In other words, the right hand sides in the weak Morse inequalities for $\rho^{-1}(\leq c)$ and $\rho^{-1}(\leq c')$ coincide. The analogous statement also holds in the $q-$concave case.

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References

[1] Andreotti, A: Theoremes de dependence algebrique sur les espaces complexes pseudoconcaves. Bull. Soc. Math. France, 91, 1963, 1–38.

[2] Andreotti, A, Grauert, H: Theoremes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France, 90, 1962, 193–259.

[3] Arnold, V.I: Symplectic Geometry. In Dynamical systems IV, 1–138 Encyclopaedia Math. Sci, 4, Springer, Berlin, 2001

[4] Berman, R: Bergman kernels and local holomorphic Morse inequalities. Math Z., Vol 248, Nr 2 (2004), 325–344 (arXiv.org/abs/math.CV/0211235)

[5] Berman, R: Super Toeplitz operators on holomorphic line bundles (arXiv.org/abs/math.CV/0406032)

[6] Berndtsson, Bo: Bergman kernels related to hermitian line bundles over compact comlex manifolds. Explorations in complex and Riemannian geometry, 1–17, Contemp. Math, 332, Amer. Math. Soc, Providence, RI, 2003.

[7] Bouche, T: Inegalite de Morse pour la d''-cohomologie sur une variete noncompacte. Ann. Sci. Ecole Norm. Sup. 22, 1989, 501-513

[8] Chern, S.S; Moser.J.K: Real hypersurfaces in complex manifolds, Acta Math. 133 (1974), 219–271

[9] Demailly, J-P: Champs magnetiques et inegalite de Morse pour la d''-cohomologie., Ann Inst Fourier, 355 (1985,185-229)

[10] Demailly, J-P: Holomorphic Morse inequalities. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 93-114

[11] Demailly, J-P: Introduction à la théorie de Hodge. In "Transcendental methods in algebraic geometry. Lectures given at the 3rd C.I.M.E. Session held in Cetraro, July 4–12, 1994." Lecture Notes in Mathematics, 1646. Springer-Verlag, 1996.

[12] Eliashberg, Y: A few remarks about symplectic filling. Geometry and topology, Vol 8 (2004) Nr 6, 277—293

[13] Epstein, C: Geometric bounds on the relative index. J. Inst. Math. Jussieu 1 (2002), no 3, 441–465.

[14] Folland, G.B, Kohn J.J:The Neumann problem for the Cauchy-Riemann complex. Annals of Math. Studies 75, Princeton University Press, 1972.

[15] Getzler, E: An analogue of Demailly's inequality for strictly pseudoconvex CR manifolds. J. Differential Geom. 29 (1989), no. 2, 231–244

[16] Griffiths, P; Harris, J: Principles of algebraic geometry. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[17] Gromov, M: Kähler hyperbolicity and L^2-Hodge theory. J. Differential Geom. 33 (1991), no. 1, 263–292.

[18] Henkin, G, Epstein, C: Stability of embeddings for prseudoconcave surfaces and their boundaries. Acta Math. 185 (200), no 2, 161–237

[19] Hörmander, L: L^2 estimates and existence theorems for the $\overline{\partial}-$operator. Acta Math. 113 1965 89–152.

[20] Lazarsfeld, Robert: Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. A series of modern surveys in Mathematics, 48. Springer-Verlag, Berlin, 2004

[21] Marinescu, G: Asymptotic Morse inequalities for Pseudoconcave manifolds. Ann. Scuola. Norm. Sup. Pisa CL Sci (4) 23 (1996), no 1, 27–55

[22] Marinescu, G: Existence of holomorphic sections and perturbation of positive line bundles over $q-$concave manifolds. (arXiv.org/abs/math.CV/0402041)

[23] Rossi, H: Attaching analytic spaces to an analytic space along a pseudoconcave boundary. Proc. Conf. Complex. Manifolds (Minneapolis), Springer-Verlag, New York 1965, 242–256

[24] Rudin, W: Real and complex analysis. McGraw-Hill Book Company, international edition 1987.

[25] Siu, Y. T: Some recent results in complex manifold theory related to vanishing theorems for the semipositive case. Workshop Bonn 1984 (Bonn, 1984), 169–192, Lecture Notes in Math., 1111, Springer, Berlin, 1985.
[26] Siu, Y.T: A vanishing theorem for semipositive line bundles over non-Kähler manifolds. J. Differential Geom. 19 (1984), no. 2, 431–452.
[27] Wells, R. O., Jr.: Differential analysis on complex manifolds. Graduate Texts in Mathematics, 65. Springer-Verlag, New York-Berlin, 1980.
[28] Witten, E: Supersymmetry and Morse theory. J. Differential Geom. 17 (1982), no. 4, 661–692.

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