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Hankel forms of higher weights and matrix-valued Bergman-type projections

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Abstract

Bilinear Hankel forms of higher weights on weighted Bergman spaces on the unit ball of $\mathbb{C}^d$ were introduced by Peetre. They form irreducible components in the tensor product of weighted Bergman spaces under the action of the Möbius group. Each Hankel form corresponds to a vector-valued holomorphic function, called the symbol of the form. In this thesis we characterize bounded, compact and Schatten-von Neumann $\mathcal{S}_p$ $(1 \leq p \leq \infty)$ Hankel forms in terms of the membership of the symbols in certain Besov spaces. We also present partial results for Hankel forms of higher weights on Hardy spaces.

The study of the Schatten-von Neumann properties is closely related to the boundedness of certain matrix-valued Bergman-type projections onto Hilbert spaces of vector-valued holomorphic functions. We establish some L^p-boundedness criteria for a general class of projections on bounded symmetric domains of type I. We prove also similar results for the orthogonal projections onto Hilbert spaces of nearly holomorphic functions on the unit ball of $\mathbb{C}^d$.

Keywords: Hankel forms, Schatten-von Neumann classes, Bergman spaces, Hardy spaces, Besov spaces, nearly holomorphic functions, Bergman projections, duality of Besov spaces, atomic decomposition, transvectants, unitary representations, Möbius group.

AMS 2000 Subject Classification: 32A25, 32A35, 32A36, 32A37, 47B32, 47B35.

This thesis consists of an introduction and the following four papers:

[i] *Schatten-von Neumann properties of bilinear Hankel forms of higher weights,* Math. Scand., 98 (2006), 283–319.

[ii] *Trace class criteria for bilinear Hankel forms of higher weights,* Proc. of AMS, to appear.

[iii] *Bilinear Hankel forms of higher weights on Hardy spaces*, preprint 2006:15, Chalmers University of Technology and Göteborg University, (2006).

[iv] *L^p-boundedness for orthogonal projections onto spaces of nearly holomorphic functions and of vector-valued holomorphic functions*, preprint 2006:16, Chalmers University of Technology and Göteborg University, (2006).

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HANKEL FORMS OF HIGHER WEIGHTS AND MATRIX-VALUED BERGMAN-TYPE PROJECTIONS

MARCUS SUNDHÄLL

INTRODUCTION

In the first three papers we study the Schatten-von Neumann properties for bilinear Hankel forms of higher weights on weighted Bergman spaces and Hardy spaces on the unit ball of $\mathbb{C}^d$. Closely related to the study of Schatten-von Neumann properties is the question of boundedness of matrix-valued Bergman-type projections, and this will be treated in paper [iv] along with the Bergman-type projections onto spaces of nearly holomorphic functions.

1. Hankel operators on Hardy spaces on the unit circle

The Hardy space $H^2(\mathbb{T})$ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ consists of functions f in $L^2(\mathbb{T})$ such that $\hat{f}(n) = 0$ for $n < 0$, where $\hat{f}(n)$ is the nth Fourier coefficient of f. It is a closed subspace of $L^2(\mathbb{T})$ so there exists an orthogonal projection P of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. The Szegö projection P is connected to the Hilbert transform H (defined on smooth functions) via

$$P = H + \frac{I}{2}\,, \tag{1}$$

where

$$(Hf)(z) = \lim_{\varepsilon \to 0} \int_{|w-z|>\varepsilon} \frac{f(w)}{1 - \bar{w}z}\, d\sigma(w)\,, \tag{2}$$

and $d\sigma$ is the normalized Lebesgue measure on $\mathbb{T}$; see equation (1.14) in [Pe5]. Let M_f denote the multiplication operator $g \to fg$. Then the *Hankel operator* mapping $H^2(\mathbb{T})$ into $H^2(\mathbb{T})^\perp$ is defined by

$$H_f = (I - P) M_{\bar{f}} P\,, \tag{3}$$

where $f \in L^\infty(\mathbb{T})$ is called the *symbol* of H_f, and we have the following identity on $L^2(\mathbb{T})$;

$$[M_f, P] = M_f P - P M_f = H_{\bar{f}} - H_f^*\,. \tag{4}$$

Also, given the Toeplitz operator $T_f = PM_fP$, then $T_f^* = PM_{\bar{f}}P$ so that

$$[T_f^*, T_f] = PM_f(I-P)^2M_{\bar{f}}P = H_{\bar{f}}^*H_{\bar{f}}\,, \tag{5}$$

on $L^2(\mathbb{T})$. Hence,

$$\operatorname{tr}\left([T_f^*, T_f]\right) = \operatorname{tr}\left(H_{\bar{f}}^*H_{\bar{f}}\right) = \|H_f\|_{\mathcal{S}_2}^2 \tag{6}$$

where $\|\cdot\|_{\mathcal{S}_2}$ is the Hilbert-Schmidt norm. Naturally, one would like to study the operator theoretic properties of H_f such as boundedness, compactness, finite rank, and Schatten-von Neumann properties, trying to understand the noncommutativity of M_f with P and T_f^* with T_f respectively.

The symbols of the Hankel operators studied in this thesis are holomorphic, and, under the orthogonal basis $\{z^n\}$ of the Hardy space, the Hankel operator with the symbol $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is given by the matrix (se [Pe5])

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{7}$$

Matrices of the form (7) were studied by Hankel around 1860, and later Kronecker characterized the Hankel matrices of finite rank. The Hardy space provides a natural setup for the study of the Hankel matrix.

2. Schatten-von Neumann Hankel operators on $H^2(\mathbb{T})$

To introduce linear operators of Schatten-von Neumann class, we define the singular numbers of $T : H_1 \to H_2$ as

$$s_n(T) = \inf\left\{||T-K|| : \ \operatorname{rank}(K) \le n\right\}, \tag{8}$$

where H_1 and H_2 are Hilbert spaces and $n \ge 0$. If T is compact, these singular numbers are equal to the eigenvalues of $|T| = (T^*T)^{1/2}$ (counted with multiplicities). We denote by $\mathcal{S}_p$ the ideal of operators for which $\{s_n(T)\}_{n\ge 0} \in l^p$, $0 < p \le \infty$; see [S], and remark that $\mathcal{S}_\infty$ is the class of bounded operators.

2.1. **Hankel operators on Hardy spaces of class $\mathcal{S}_p$.** An overview of the theory of Hankel operators on Hardy spaces can be found in the book [Pe5]; see also [Pe1], [Pe2], [Pe3], and [Pe4]. The boundedness property of the Hankel operators H_f has been settled in the middle of the last century, and it is characterized by the membership of the symbols in the BMOA space. A refinement of this statement is given in terms of the singular numbers, which was done by Krein et al., and it relates the eigenvalues of $|H_f| = (H_f H_f^*)^{1/2}$ to the approximation of f by rational functions (see Section 4.1 in [Pe5]). More precisely, for a given Hankel operator H_f one can find another Hankel operator H_g of rank at most n such that

$$s_n(H_f) = \|H_f - H_g\| \,. \tag{9}$$

By Kronecker's theorem, g is a rational function of degree at most n. Hence, by Theorem 4.1.2 in [Pe5], if $f \in L^\infty(\mathbb{T})$, then there exists a function $g \in L^\infty(\mathbb{T})$ such that $(I-P)g$ is a rational function, of degree at most n, with

$$s_n\left(H_f\right) = \|f - g\|_\infty \,. \tag{10}$$

Also, by Theorem 1.1.3 in [Pe5], if $f \in L^\infty(\mathbb{T})$, then

$$\|H_f\|_{\mathcal{S}_\infty} = \inf\left\{\|f-g\|_\infty : g \in H^\infty\right\} , \tag{11}$$

where H^∞ is the subspace of holomorphic functions in $L^\infty(\mathbb{T})$, Thus the Schatten-von Neumann $\mathcal{S}_p$ property becomes a very interesting problem in that it places the classical approximation theory in the setup of functional analysis and operator theory. In the particular case $\mathcal{S}_2$ one may define an isometry from the Dirichlet space into the Hilbert-Schmidt class $\mathcal{S}_2$, since, for holomorphic symbols f,

$$\operatorname{tr}\left(H_f^* H_f\right) = \|H_f\|_{\mathcal{S}_2}^2 = c\int_{\mathbb{D}} |f'(z)|^2\, dm(z)\,; \tag{12}$$

by, for instance, Theorem 3.1 in [AFP]. It was proved by Peller that a Hankel operator H_f is in $\mathcal{S}_p$ if and only if f is in a certain Besov space. The result has found many applications in approximation theory, and in the resolvent of the Halmos problem on similarity of polynomially bounded operators [Pe3], [Pe5], [AC].

In 1982, Rochberg presented analogous results for Hankel operators on the Hardy spaces $H^2(\mathbb{R})$ on the real line; see [R1]. Rochberg considered the Schatten-von Neumann classes $\mathcal{S}_p$, $p \geq 1$. Semmes extended this characterization to the case $0 < p < 1$ in [Se].

2.2. **Hankel forms on Hardy spaces.** Later, we will generalize Hankel forms to a larger class of bilinear forms on Hardy and Bergman spaces. Now, consider the bilinear form on $H^2(\mathbb{T})$;

$$H^2(\mathbb{T}) \times H^2(\mathbb{T}) \ni (g_1, g_2) \longmapsto \int_{\mathbb{T}} \overline{f(z)} g_1(z) g_2(z)\, d\sigma(z)\,. \tag{13}$$

Given the Hankel operator in (3) we may define a bilinear form on $H^2(\mathbb{T})$ by

$$H^2(\mathbb{T}) \times H^2(\mathbb{T}) \ni (g_1, g_2) \longmapsto \left\langle H_{\bar{f}}(g_1), \bar{g}_2 \right\rangle_{L^2(\mathbb{T})}\,,$$

which differ from (13) by a form of rank one.

3. Hankel forms and operators on weighted Bergman spaces on the unit disk

The study of Schatten-von Neumann class Hankel operators on weighted Bergman spaces were initiated by Arazy, Fisher, Janson, Peetre, Rochberg et al. (see e.g. [AFP], [JPR] and [R2]).

3.1. **Weighted Bergman spaces.** Let dm denote the Lebesgue measure on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, then the space of holomorphic functions in $L^2(dm)$ is a closed subspace $L^2_a(dm)$, called the Bergman space. We can extend this definition to the so-called *weighted Bergman spaces* $L^2_a(d\iota_\nu)$ where $d\iota_\nu(z) = c_\nu(1-|z|^2)^{\nu-2}dm(z)$ is defined for $\nu > 1$ and c_ν is a normalization constant.

3.2. **Big and small Hankel operators.** As in the case of the Hardy space, there exists an orthogonal projection P_ν from $L^2(d\iota_\nu)$ onto the space $L^2_a(d\iota_\nu)$, called the *Bergman projection*; see [HKZ]. We will also consider the space $\overline{L^2_a(d\iota_\nu)}$ of antiholomorphic functions in $L^2(d\iota_\nu)$, where $\bar{P}_\nu$ is the corresponding projection. Now we can define the big and the small Hankel operators. The *big Hankel operator*, $\tilde{H}_f$, is mapping $L^2_a(d\iota_\nu)$ into $L^2_a(d\iota_\nu)^\perp$ by $\tilde{H}_f(g) = (I-P)(\bar{f}g)$, and the *small Hankel operator*, H_f, is mapping $L^2_a(d\iota_\nu)$ into $\overline{L^2_a(d\iota_\nu)}$ by $H_f(g) = \bar{P}(\bar{f}g)$. In both cases, the symbol f is holomorphic. The *bilinear Hankel form* on a weighted Bergman space is defined by

$$H_f(g_1, g_2) = \int_{\mathbb{B}} g_1(z) g_2(z) \overline{f(z)}\, d\iota_{2\nu}(z)\,, \tag{14}$$

where the symbol f is holomorphic. Actually, the small Hankel operators correspond to the bilinear Hankel forms. Namely, making an appropriate choice of β, we can see that

$$H_f(g_1, g_2) = \left\langle g_1, \bar{P}_\beta(\bar{f} g_2) \right\rangle_\nu \, , \tag{15}$$

where $\langle \cdot, \cdot \rangle_\nu$ is the inner product of $L^2_a(d\iota_\nu)$.

More generally, with the form H_f one can associate the operator A_f defined by

$$H_f(g_1, g_2) = \langle g_1, A_f(g_2) \rangle_\nu \tag{16}$$

as in [JPR]. Notice that A_f is an antilinear operator on $L^2_a(d\iota_\nu)$. To get a linear operator one combines A_f with a conjugation, i.e., one instead considers the operator $\bar{A}_f : g \to \overline{A_f(g)}$. We say that the bilinear Hankel form H_f is of Schatten-von Neumann class $\mathcal{S}_p$, $0 < p < \infty$, if and only if $\bar{A}_f : L^2_a(d\iota_\nu) \to \overline{L^2(d\iota_\nu)}$ is of class $\mathcal{S}_p$.

Given a big Hankel operator, $\tilde{H}_f$, and a small Hankel operator, H_f, $H_f - \bar{P}_\nu \tilde{H}_f$ has rank (at most) one; see [J]. As a consequence, if the big Hankel operator is bounded, compact or of class $\mathcal{S}_p$, then so is the small Hankel operator. In 1985, Rochberg characterized the small Hankel operators on weighted Bergman spaces in terms of the membership of the symbols in certain Besov spaces, see [R2]. Janson later proved that these characterizations can be reduced to results by Peller and Semmes; see [J]. For the big Hankel operators, similar results was proved by Arazy, Fisher and Peetre in [AFP], Janson, Peetre and Rochberg in [JPR] and Janson in [J]. The study of Schatten-von Neumann properties for big Hankel operators is different from the one for small Hankel operators in one major way; for big Hankel operators we get a so-called cut-off, i.e., for p small enough the big Hankel operators are in the class $\mathcal{S}_p$ if and only if their symbols are constants (and thus the big Hankel operators are zero).

4. Möbius invariance of Hankel forms

There is a natural action, π_ν, on weighted Bergman and Hardy spaces of the automorphism group $G = Aut(\mathbb{D})$. Namely, if $g : \mathbb{D} \to \mathbb{D}$ is biholomorphic (g bijective, g and g^{-1} holomorphic), then we define

$$\left(\pi_\nu(g^{-1}) f\right)(w) = f(g(w)) \cdot g'(w)^{\nu/2} \, , \tag{17}$$

with the appropriate convention concerning the ambiguity of the definition of power (which we clarify in Remark 3.1 in paper [i]). Then

$\pi_\nu(g)$ acts unitarily on $L^2(d\iota_\nu)$ for $\nu > 1$ and $\pi_1(g)$ acts unitarily on $H^2(\mathbb{T})$. Also,

$$H_f\left(\pi_\nu(g)f_1, \pi_\nu(g)f_2\right) = H_{\pi_{2\nu}(g)f}(f_1, f_2)\,. \tag{18}$$

The question then arises of classifying all bilinear forms with such invariance. Now, consider the tensor product $L^2_a(d\iota_\nu)\otimes L^2_a(d\iota_\nu)$ viewed as the space of Hilbert-Schmidt ($\mathcal{S}_2$) bilinear forms on $L^2_a(d\iota_\nu)$ with kernels $F(z,w)$. The group G acts on the tensor product by

$$\left(\pi_\nu \otimes \pi_\nu\right)(g^{-1})F(z,w) \;=\; F(g(z), g(w)) \cdot g'(z)^{\nu/2} \cdot g'(w)^{\nu/2}\,. \tag{19}$$

Under this action, the tensor product decomposes as

$$L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu) \simeq \bigoplus_{s=0}^{\infty} L^2_a(d\iota_{2\nu+2s})\,, \tag{20}$$

see [JP]. The intertwining projection map, called the *transvectant*,

$$\mathcal{T}_s : L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu) \to L^2_a(d\iota_{2\nu+2s}) \tag{21}$$

is given, up to a multiplicative constant, by

$$\mathcal{T}_s(f_1 \otimes f_2)(z) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{f_1^k(z) f_2^{s-k}(z)}{(\nu)_k (\nu)_{s-k}}\,, \tag{22}$$

where $(\nu)_k = \nu(\nu+1)\cdots(\nu+k-1)$ is the Pochammer symbol. Actually, the bilinear forms corresponding to $s=0$ are the Hankel forms (small Hankel operators). One may then define the *higher order Hankel forms* (of weight s) as

$$H_f(f_1, f_2) = \langle \mathcal{T}_s(f_1, f_2), f\rangle_{2\nu+2s}\,. \tag{23}$$

Janson, Peetre and Zhang established the $\mathcal{S}_p$ criteria for bilinear Hankel forms of higher weights in [JP] and in [Z2] respectively. Rosengren generalized the results to the multilinear case in [Ro].

5. Hankel forms on the unit ball

Arazy, Fisher, Janson, Peetre, Rochberg and Wallstén (see [AFJP], [JPR] and [W]) studied the Hankel operators on weighted Bergman spaces on the unit ball of $\mathbb{C}^d$; see e.g. [JPR] where a theory of Hankel operators on weighted Bergman spaces on a general domain is established. Feldman and Rochberg [FeldR] and Zhang [Z1] studied Hankel operators on Hardy spaces on the unit ball, and a characterization for the small Hankel operators on Hardy spaces is given in [FeldR].

5.1. **Hankel forms on weighted Bergman spaces.** In [P1] Peetre defined the transvectant in several variables to get Hankel forms of higher weights in the case of the unit ball of $\mathbb{C}^d$. Later, Peetre generalized these invariant Hankel forms to certain symmetric domains; see [P2]. In 2004 Peng and Zhang [PZ] gave an irreducible decomposition of the tensor product of weighted Bergman spaces (Hardy spaces included) on bounded symmetric domains, classifying all the invariant Hankel forms. In this paper an exact formula similar to (22) was presented. Especially, in the case of the unit ball, the spaces of Hankel forms of higher weights form irreducible components in the tensor product of weighted Bergman spaces under the action of the Möbius group. The main objective of papers [i] and [ii] is the study of $\mathcal{S}_p$ criteria for bilinear Hankel forms of higher weights on a weighted Bergman space on the unit ball of $\mathbb{C}^d$. Combining the results in paper [i] with the results in paper [ii] we get a full characterization of $\mathcal{S}_p$ Hankel forms of higher weights on weighted Bergman spaces, $1 \leq p \leq \infty$. The main idea is to first establish the boundedness and trace class $\mathcal{S}_1$ criteria, since then we get the $\mathcal{S}_p$ class criteria for $1 < p < \infty$ by interpolation. To find the interpolation spaces we have to prove certain duality results on Besov spaces of vector-valued holomorphic functions; see also Section 6 below. The Hilbert and Banach spaces of vector-valued functions appearing in paper [i] and paper [ii] are closely related to the quotients of function modules studied by Ferguson and Rochberg in [FergR].

5.2. **Hankel forms on Hardy spaces.** In the same way as for the case of weighted Bergman spaces, Hankel forms of higher weights can be defined on Hardy spaces on the unit ball, which we focus on in paper [iii]. In the case of weight zero we give a full characterization for the $\mathcal{S}_p$ Hankel forms in terms of their membership for the symbols to be in certain Besov spaces, $1 \leq p < \infty$, results which can be deduced from [Z1]. Also, we find a necessary condition for the Hankel forms to be bounded, in terms of a certain Carleson measure property for the symbols. In the case of higher weights the problem becomes somewhat more complicated. We establish a sufficient condition for the Hankel forms to be of trace class $\mathcal{S}_1$. However, a boundedness criterion seems to be harder to find. The transvectant does not act on Hardy spaces in the same way as it does on weighted Bergman spaces, as we can see in Example 3.5 in paper [iii]. Generalizing the Hilbert-Schmidt criterion from paper [i] to include also the Hardy case, we get sufficient criteria for Hankel forms of higher weights on Hardy spaces to be of Schatten-von Neumann class $\mathcal{S}_p$, $1 \leq p \leq 2$.

6. Matrix-valued Bergman-type projections

When studying the Schatten-von Neumann properties of the Hankel forms in papers [i] and [ii], one needs to work with some types of matrix-valued Bergman projections. Basically, a matrix-valued Bergman type projection is given by a matrix-valued kernel consisting of the classical Bergman kernel multiplied with a tensor product of the inverse of the Bergman operator. More explicitly, in the case of the unit ball of $\mathbb{C}^d$, a matrix-valued Bergman-type kernel is given, for a certain nonzero constant c, by

$$K_{\nu,s}(z,w) = c(1-\langle z,w\rangle)^{-\nu} \odot^s B^t(z,w)^{-1} \tag{24}$$

where $\nu > d$, s nonnegative integer and where $\odot^s B^t(z,w)^{-1}$ is given by the following: Let $z \in \mathbb{B}$ and identify the tangent space $T_z(\mathbb{B})$ with $\mathbb{C}^d$. Then the *Bergman metric* (see [FK1]) at z is given by $\langle B(z,z)^{-1}u, v\rangle$ where $u, v \in \mathbb{C}^d$, and where the *Bergman operator* $B(z,w)$ acting on $\mathbb{C}^d$ (see [L], [Hua] or [HLZ] for the definition of the Bergman operator on the unit ball of $\mathbb{C}^d$ or, generally, on bounded symmetric domains) is given by

$$B(z,w) = (1-\langle z,w\rangle)(I - z\otimes w^*)\,,$$

where $z \otimes w^*$ is the rank one operator given by $(z\otimes w^*)(v) = \langle v,w\rangle z$ for $v \in \mathbb{C}^d$. Then $B^t(z,z)$ is the dual action on the dual space $(\mathbb{C}^d)'$ and $\odot^s B^t(z,z)$ is the induced action on $\odot^s(\mathbb{C}^d)'$.

In paper [i] we find L^p-boundedness criteria for these matrix-valued Bergman-type projections, and these results are generalized (in some weaker version) in paper [iv] to bounded symmetric domains of type I, i.e., to spaces of $m \times n$-matrices with the matrix norm less than 1. An essential tool used in these papers is the Forelli-Rudin type estimate [Ru], [FK2], [EZ].

The Hilbert spaces generated by the kernels given by (24) consist of holomorphic functions on the unit ball of $\mathbb{C}^d$ with values in symmetric tensor products of cotangent spaces. One might instead be interested in such Hilbert spaces when the functions take values in tensor products of tangent spaces. Such Bergman-type spaces are studied in paper

[iv] and are closely related to spaces of nearly holomorphic functions in the sense of Shimura; see [Sh1] and [Sh2]. More concretely, in the case of the unit ball of $\mathbb{C}^d$, the spaces of nearly holomorphic functions can be viewed as images of $L^2(d\mu_\alpha)$ under certain orthogonal projections [Z3], where $d\mu_\alpha(z) = (1-|z|^2)^\alpha \, dm(z)$, $\alpha > -1$. For certain non-negative integers l, the operators $\bar{D}^l$ are intertwining operators from spaces of nearly holomorphic functions onto vector-valued Bergman-type spaces, where $\bar{D} = B(z,z)\bar{\partial}$ is the invariant Cauchy-Riemann operator, and we have the following diagram (see [PZ] and [EP]);

$$\begin{array}{ccc} L^2(d\mu_\alpha) \cap C^\infty(\mathbb{B}) & \xrightarrow{\bar{D}^l} & L^2\left(\mathbb{B}, \odot^l \mathbb{C}^d, d\mu_\alpha\right) \cap C^\infty\left(\mathbb{B}, \odot^l \mathbb{C}^d\right) \\ \Big\downarrow {\scriptstyle P_l} & & \Big\downarrow {\scriptstyle P_{\nu,\,l}} \\ A^2_l(d\mu_\alpha) & \xrightarrow{\bar{D}^l} & L^2_a\left(\mathbb{B}, \odot^l \mathbb{C}^d, d\mu_\alpha\right) \end{array}$$

where P_l is the orthogonal projection from $L^2(d\mu_\alpha)$ onto the discrete part $A^2_l(d\mu_\alpha)$ of nearly holomorphic functions, $P_{\nu,\,l}$ ($\nu = \alpha + d + 1$) is the orthogonal projection from $L^2(\mathbb{B}, \odot^l \mathbb{C}^d, d\mu_\alpha)$ onto its holomorphic subspace and the L^2-norm (invariant under the action of the Möbius group) is given by

$$\|f\|_{l,\alpha,2} = \left(\int_{\mathbb{B}} \left\langle \odot^l B(z,z)^{-1} f(z), f(z) \right\rangle \, d\mu_\alpha(z) \right)^{1/2} .$$

In paper [iv] we find necessary and sufficient conditions for the orthogonal projections P_l to be L^p-bounded, namely if and only if

$$\frac{\alpha+1}{\alpha+1-l} < p < \frac{\alpha+1}{l}, \tag{25}$$

and we prove that also the Bergman-type projections $P_{\nu,\,l}$ are L^p-bounded if condition (25) is satisfied. (In both cases we restrict α to satisfy $\alpha > 2l - 1$.)

In the same way as for the case of the unit ball of $\mathbb{C}^d$, one can relate spaces of nearly holomorphic functions with vector-valued Bergman-type spaces, via powers of the Cauchy-Riemann operator, on bounded symmetric domains; see [Z4]. But here we do not yet have explicit formulas for the orthogonal projections onto the spaces of nearly holomorphic functions as we do have in the case of the unit ball of $\mathbb{C}^d$; see [Z3]. However, a sufficient criterion for the corresponding Bergman-type projections to be L^p-bounded can be found in paper [iv].

7. A small selection of open problems

The ball case. The transvectant behaves quite differently in the Hardy case compared to the Bergman case. Therefore some new techniques are needed to be able to prove boundedness and compactness properties for bilinear Hankel forms of higher weights on Hardy spaces. Naturally, one would also like to find the Schatten-von Neumann $\mathcal{S}_p$ properties for bilinear Hankel forms of higher weights on Hardy spaces, for the case $2 < p \leq \infty$.

Also, a Kronecker theorem for Hankel forms of higher weights on weighted Bergman spaces might be interesting to find; see [R3].

Bounded symmetric domains. Hankel forms of higher weights can be defined on bounded symmetric domains; see [P2] and [PZ]. The Hilbert-Schmidt Hankel forms have been characterized (see [PZ]) in terms of the membership for the symbols in certain Hilbert spaces of vector-valued holomorphic functions. To find their S_p characterization, we need further results about expansion of reproducing kernels and Besov space characterization of Bergman spaces. The Hilbert spaces of vector-valued functions are often of considerable interest in the theory of Hankel operators and forms and in the analysis of symmetric spaces; see [ØZ1] and [ØZ2].

Multilinear Hankel forms. Schatten-von Neumann properties of multilinear Hankel forms of higher weights on weighted Bergman spaces have been studied in [Ro] in the one dimensional case. Naturally one would like to generalize these results to the several variable case.

Nearly holomorphic functions. The spaces of nearly holomorphic functions studied in the paper [iv] can also be defined on bounded symmetric domains; see [Z4]. Here one would like to find a formula for the reproducing kernels for these spaces; see [FK1]. It would also be interesting to generalize the results of paper [iv] to general bounded symmetric domains.

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Paper I

SCHATTEN-VON NEUMANN PROPERTIES OF BILINEAR HANKEL FORMS OF HIGHER WEIGHTS

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ABSTRACT. Bilinear Hankel forms of higher weights on weighted Bergman spaces on the unit ball of $\mathbb{C}^d$ were introduced by Peetre. Each Hankel form corresponds to a vector-valued holomorphic function, called the symbol of the form. In this paper we characterize bounded, compact and Schatten-von Neumann $\mathcal{S}_p$ class ($2 \leq p < \infty$) Hankel forms in terms of the membership of the symbols in certain Besov spaces.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Introduction.** Hankel operators on the unit disk have been studied extensively and have found many applications, see [Pe1], [Zh] and [JPR]. One of the central problems is to study the characterization of their Schatten-von Neumann properties. We recall briefly the definition of Hankel operators on a Hardy space on the unit disc. Consider the Hardy space $H^2(T) \subset L^2(T)$ of holomorphic functions, where $T = \{z \in \mathbb{C} : |z| = 1\}$. Let $P : L^2(T) \to H^2(T)$ be the Szegö projection. The Hankel operator $\tilde{H}_f$ with holomorphic symbol f is defined by $\tilde{H}_f g = (I - P)(\bar{f}g)$, $g \in H^2(T)$. It can also be viewed (up to a rank one operator) as a bilinear form H_f on $H^2(T)$, namely

$$H_f(g_1, g_2) = \int_{\partial D} \overline{f(z)} g_1(z) g_2(z) \, d\sigma(z) \, .$$

Their Schatten-von Neumann properties were studied first by Peller, see [Pe2]. It is proved there that H_f is of Schatten-von Neumann class if and only if f is in a certain Besov space. The corresponding problem for Hankel forms on a Bergman space has been studied in [JPR] and [R2]. It was realized later that the Hilbert-Schmidt Hankel forms on a weighted Bergman space can be viewed as the first irreducible

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component in the irreducible decomposition of the tensor product of two copies of the Bergman spaces, and subsequently Janson and Peetre [JP] introduced the Hankel forms of higher weights on Bergman spaces on the unit disc; see also [Ro] where multilinear Hankel forms are studied.

A natural problem is to consider Hankel forms on the unit ball in $\mathbb{C}^d$. In [P1] Peetre introduced Hankel forms on the unit ball. As in the case of the unit disk, the spaces of Hankel forms of higher weights are explicit characterizations of irreducible components in the tensor product of Bergman spaces under the Möbius group, see [JP], [P1] and [PZ]. However their Schatten-von Neumann properties have not been studied so far. In this paper we will address this problem.

The Hilbert and Banach spaces of symbols appearing in this paper are closely related to the quotients of function modules studied in [FR], and the expansion of the reproducing kernels of some similar spaces have been studied in [HLZ]. It is interesting to consider those problems in our context.

The paper is arranged in the following manner. In Section 1 we introduce the Hankel forms and state the main results in the form of three theorems. Section 2 consists of preliminary results. Section 3 is devoted to certain Banach spaces of vector-valued holomorphic functions. Section 4 gives an equivalent description for certain Besov spaces. The proofs of Theorem 1.1(a) and Theorem 1.1(b) are given in Section 5 and Section 6 respectively. The proof of Theorem 1.2 is given in Section 6. In Section 7 we prove some L^p-boundedness properties of certain Bergman-type projections, which are used in Section 8 to prove Theorem 1.3.

1.2. **Notation.** Let H_1 and H_2 be Hilbert spaces and let $T : H_1 \to H_2$ be a linear operator. Define the singular numbers

$$s_n(T) = \inf\left\{||T - K|| : \text{ rank}(K) \leq n\right\},$$

$n \geq 0$. If T is compact, these singular numbers are equal to the eigenvalues of $|T| = (T^*T)^{1/2}$ (counted with multiplicities). We denote by $\mathcal{S}_p$ the ideal of operators for which $\{s_n(T)\}_{n\geq 0} \in l^p$, $0 < p \leq \infty$, see [S]. We remark that $\mathcal{S}_\infty$ is the class of bounded operators. (The compact operators correspond to c_0, not to l^∞.)

Let dm denote the Lebesgue measure on the unit ball $\mathbb{B} \subset \mathbb{C}^d$ and let $d\iota(z)$ be the measure $(1 - |z|^2)^{-d-1}dm(z)$. For $d < \nu < \infty$ let

$d\iota_\nu(z)$ be the measure $c_\nu(1-|z|^2)^\nu\, d\iota(z)$, where c_ν is chosen such that

$$\int_{\mathbb{B}} d\iota_\nu(z) = 1\,,$$

i.e., $c_\nu = \Gamma(\nu)/(\pi^d\Gamma(\nu-d))$. The closed subspace of all holomorphic functions in $L^2(d\iota_\nu)$ is denoted by $L_a^2(d\iota_\nu)$ and is called a weighted Bergman space. Note that the space $L_a^2(d\iota_\nu)$ has a reproducing kernel $K_z(w) = (1-\langle w,z\rangle)^{-\nu}$, that is,

$$f(z) = \langle f, K_z\rangle_\nu = \int_{\mathbb{B}} f(w)\overline{K_z(w)}\, d\iota_\nu(w)\,, \tag{1}$$

where $f \in L_a^2(d\iota_\nu)$ and $z \in \mathbb{B}$.

Denote by $B(z,w)$ the Bergman operator on $V = \mathbb{C}^d$ as in [L], namely

$$B(z,w) = (1-\langle z,w\rangle)(I - z\otimes w^*), \tag{2}$$

where $z\otimes w^*$ stands for the rank one operator given by $(z\otimes w^*)(v) = \langle v,w\rangle z$. Viewed as a matrix acting on column vectors it is

$$B(z,w) = (1-\langle z,w\rangle)(I - z\bar{w}^t), \tag{3}$$

where w^t is the transpose of w. $B(z,w)$ is holomorphic in z and antiholomorphic in w.

The Bergman metric at $z\in\mathbb{B}$, when we identify the tangent space with V, is $\langle B(z,z)^{-1}u,v\rangle$ for $u,v\in V$. We note that

$$B(z,w)^{-1} = (1-\langle z,w\rangle)^{-2}\big((1-\langle z,w\rangle)I + z\otimes w^*\big)\,. \tag{4}$$

Let $B^t(z,w)$ denote the dual of $B(z,w)$ acting on the dual space V' of V. When acting on a vector $v'\in V'$ it is

$$B^t(z,w)v' = (1-\langle z,w\rangle)v'(I - z\bar{w}^t)\,. \tag{5}$$

Actually we may identify $B^t(z,w)$ with $(1-\langle z,w\rangle)(I-\bar{w}z^t)$.

For a nonnegative integer s, let $\otimes^s V'$ be the tensor product of s factors V' and let $\otimes^0 V' = \mathbb{C}$. The space $\otimes^s V'$ is equipped with a natural Hermitian inner product induced by that of V', so that

$$\langle v_1\otimes\cdots\otimes v_s, w_1\otimes\cdots\otimes w_s\rangle = \prod_{j=1}^s \langle v_j, w_j\rangle$$

where $v_j, w_j \in V'$, $j = 1,\ldots,s$.

Let $\{u_1, \ldots, u_d\} \subset V'$. Denote by $u_1^{i_1} \odot u_2^{i_2} \odot \cdots \odot u_d^{i_d}$ the sum

$$\frac{i_1! \cdots i_d!}{s!} \sum_{\pi \in S} \pi\big(\underbrace{u_1 \otimes \cdots \otimes u_1}_{i_1 \text{ factors}} \otimes \cdots \otimes \underbrace{u_d \otimes \cdots \otimes u_d}_{i_d \text{ factors}} \big)$$

where $i_1 + \ldots + i_d = s$, $S = S_s/(S_{i_1} \times \cdots \times S_{i_d})$, S_s is the permutation group acting on the tensor by permutating the factors in the tensor and $S_{i_1}, \ldots, S_{i_d}$ are the subgroups permutating the first i_1, the second i_2, ... , the last i_d elements respectively.

Let $\{e_1, \ldots, e_d\}$ be a basis for V'. Denote by $\odot^s V'$ the subspace of symmetric tensors of length s

$$\Big\{ \sum_{i_1 + \cdots + i_d = s} v_i \, e_1^{i_1} \odot e_2^{i_2} \odot \cdots \odot e_d^{i_d} \ : \quad i = (i_1, \ldots, i_d) \in \mathbb{N}^d \ , \quad v_i \in \mathbb{C} \Big\}.$$

Also, denote by $\otimes^s B^t(z,z)$ the operator on $\otimes^s V'$ induced by the action of $B^t(z,z)$ on V', where $\otimes^0 B^t(z,z) = I$.

1.3. **Hankel forms and main results.** The transvectant $\mathcal{T}_s$ defined on $L_a^2(d\iota_\nu) \otimes L_a^2(d\iota_\nu)$ (introduced in [P1], see also [P2] and [PZ]) is given by

$$\mathcal{T}_s(f,g)(z) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\partial^k f(z) \odot \partial^{s-k} g(z)}{(\nu)_k (\nu)_{s-k}} \tag{6}$$

where

$$\partial^s f(z) = \sum_{j_1 \ldots j_s = 1}^{d} \partial_{j_1} \cdots \partial_{j_s} f(z) \, dz_{j_1} \otimes \cdots \otimes dz_{j_s} \in \odot^s V'$$

and $(\nu)_k = \nu(\nu+1) \cdots (\nu+k-1)$, $(\nu)_0 = 1$, is the Pochammer symbol.

The *Hankel bilinear form of weight* s, H_F^s, on $L_a^2(d\iota_\nu) \otimes L_a^2(d\iota_\nu)$ is defined by

$$H_F^s(f,g) = \int_{\mathbb{B}} \big\langle \otimes^s B^t(z,z) \mathcal{T}_s(f,g)(z), F(z) \big\rangle \, d\iota_{2\nu}(z) \tag{7}$$

where $F : \mathbb{B} \to \odot^s V'$ is holomorphic. We call F the symbol of the corresponding Hankel form. We remark that

$$H_F^0(f,g) = \int_{\mathbb{B}} f(z) g(z) \overline{F(z)} \, d\iota_{2\nu}(z) \, .$$

This is the classical Hankel form studied in [JPR].

With the form H_F^s one can associate the operator A_F^s defined by

$$H_F^s(f,g) = \langle f, A_F^s g\rangle_\nu$$

as in [JPR]. Notice that A_F^s is an anti-linear operator on $L_a^2(d\iota_\nu)$. To get a linear operator one combines A_F^s with a conjugation, i.e., one instead considers the operator $\overline{A}_F^s : g \to \overline{A_F^s g}$. We say that H_F^s is of Schatten-von Neumann class $\mathcal{S}_p$, for $0 < p < \infty$, if and only if $\overline{A}_F^s : L_a^2(d\iota_\nu) \to \overline{L^2(d\iota_\nu)}$ is of class $\mathcal{S}_p$.

Finally we present the main results, of this paper, in the form of three theorems where we let s be a nonnegative integer.

Theorem 1.1. *Let $F : \mathbb{B} \to \odot^s V'$ be a holomorphic function.*

(a) *H_F^s is bounded if and only if*

$$\sup_{z\in\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z)\right\rangle < +\infty\,,$$

(b) *H_F^s is compact if and only if*

$$\left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z)\right\rangle \to 0 \quad \text{as} \quad |z| \nearrow 1\,.$$

Theorem 1.2. *H_F^s is of Hilbert-Schmidt class $\mathcal{S}_2$ if and only if*

$$\int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z)\right\rangle \, d\iota(z) < +\infty\,.$$

Theorem 1.3. *H_F^s is of class $\mathcal{S}_p$, for $2 < p < \infty$, if and only if*

$$\int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z)\right\rangle^{p/2} \, d\iota(z) < +\infty\,.$$

2. Preliminaries

2.1. $G = Aut(\mathbb{B})$**: The automorphisms of** $\mathbb{B}$**.** We shall need some results on the group $G = Aut(\mathbb{B})$ of biholomorphic mappings of $\mathbb{B}$.

Let P_z be the orthogonal projection of $\mathbb{C}^d$ into $\mathbb{C}z$ and let $Q_z = I - P_z$. Put $s_z = (1-|z|^2)^{1/2}$ and define a linear fractional mapping φ_z on $\mathbb{B}$ by

$$\varphi_z(w) = \frac{z - P_z w - s_z Q_z w}{1 - \langle w, z\rangle}\,. \tag{8}$$

If $g \in G$ and $g(z) = 0$, then there is a unique unitary operator $U : \mathbb{C}^d \to \mathbb{C}^d$ such that

$$g = U\varphi_z\,.$$

Sometimes $g(z)$ will be written as gz. Define the complex Jacobian, J_g, by $J_g(w) = \det(g'(w))$. Then we have $J_g(w) = \det U \cdot J_{\varphi_z}(w)$. Lemma 2.1 gives the differential of the Möbius transformations. It can be proved by similar computations as in the proof of Theorem 2.2.2 in [Ru].

Lemma 2.1. *Let φ_z be the linear fractional mapping (8) on $\mathbb{B}$. Then*

$$\varphi_z'(w) = \frac{-s_z^2 P_z - s_z Q_z + s_z(\langle w, z\rangle - w \otimes z^*)}{(1 - \langle w, z\rangle)^2}.$$

By computating the determinant of $\varphi_z'(w)$ we get the next proposition. It is a refinement of Theorem 2.2.6 in [Ru], which we state as a corollary.

Proposition 2.2. *Let φ_z be the linear fractional mapping (8) on $\mathbb{B}$. Then*

$$J_{\varphi_z}(w) = (-1)^d \left(\frac{s_z}{1 - \langle w, z\rangle}\right)^{d+1}.$$

Corollary 2.3. *Let $g \in G$. Then the real Jacobian $J_{\mathbb{R},g}$ of g is*

$$J_{\mathbb{R},g}(w) = |J_g(w)|^2 = \left(\frac{1 - |z|^2}{|1 - \langle w, z\rangle|^2}\right)^{d+1}.$$

We need also the Forelli-Rudin estimate (see Proposition 1.4.10 in [Ru]).

Lemma 2.4. *Let $\gamma > \alpha > d$. Then*

$$\int_{\mathbb{B}} \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w\rangle|^\gamma}\, d\iota(w) \le C(1 - |z|^2)^{-(\gamma-\alpha)}.$$

2.2. **Some elementary properties of the Bergman operator.** Let $g \in G$. Combining Proposition IX.1.1 with Proposition IX.2.6 in [FK] we get

$$B(z, w)^{-1} = (dg(w))^* \, B(gz, gw)^{-1} dg(z).$$

This yields

$$B^t(gz, gw) = \left(dg(w)^t\right)^* B^t(z, w) dg(z)^t. \tag{9}$$

Now we consider another property of the Bergman operator. It holds that

$$B^t(z, z) = (1 - |z|^2) Q_{\bar z} + (1 - |z|^2)^2 P_{\bar z}. \tag{10}$$

Thus

$$(1-|z|^2)^2 I \leq B^t(z,z) \leq (1-|z|^2)I\,; \tag{11}$$

in particular $B^t(z,z)$ is a positive operator. Actually $\otimes^s B^t(z,z)$ is positive on $\otimes^s V'$. To prove this we need an elementary observation.

Lemma 2.5. *Let H_1 and H_2 be Hilbert spaces. Let A and B be positive operators on H_1 and H_2 respectively. Then the operator $A \otimes B$ is positive on the induced Hilbert space $H_1 \otimes H_2$.*

Remark 2.6. Since $B^t(z,z)$ is positive on V' we have now that $\otimes^s B^t(z,z)$ is positive for $s = 0, 1, 2, \cdots$.

2.3. **The norm of z^α in the Bergman space $L^2_a(d\iota_\nu)$.** Denote by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ the ordered d-tuples of nonnegative integers α_i and denote $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Then the polynomials $\{z^\alpha\}$ forms an orthogonal basis for $L^2_a(d\iota_\nu)$ and

$$||z^\alpha||^2_\nu = \int_{\mathbb{B}} |z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_d^{\alpha_d}|^2 \, d\iota_\nu(z) = \frac{\alpha_1! \alpha_2! \cdots \alpha_d!}{(\nu)_{|\alpha|}} \tag{12}$$

where $(\nu)_{|\alpha|} = \nu(\nu+1)\cdots(\nu+|\alpha|-1) = \Gamma(\nu+|\alpha|)/\Gamma(\nu)$, $(\nu)_0 = 1$, is the Pochammer symbol.

2.4. **Some remarks on boundedness, compactness and $\mathcal{S}_2$.** Consider the bilinear Hankel form H^s_F with symbol F. First observe that the operator norm of the corresponding operator $\overline{A}^s_F$ equals

$$||H^s_F|| = \sup_{||f||_\nu = ||g||_\nu = 1} |H^s_F(f,g)|\,.$$

If $\overline{A}^s_F$ is compact and $\{g_n\}_{n=1}^\infty \subset L^2_a(d\iota_\nu)$, with $||g_n||_\nu = 1$, $g_n \to 0$ weakly as $n \to \infty$, then there is a sequence $\{c_n\}_{n=1}^\infty$ of positive numbers such that

$$|H^s_F(f, g_n)| \leq c_n ||f||_\nu$$

for all n. Also $c_n \to 0$ as $n \to \infty$. On the other hand, if $\{A_n\}_{n=1}^\infty$ is a sequence of compact bilinear forms on $L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu)$ such that $A_n \to H^s_F$ in operator norm, then H^s_F is compact. Also H^s_F is of Hilbert-Schmidt class $\mathcal{S}_2$ if and only if

$$||H^s_F||^2_{\mathcal{S}_2} = \sum_{|\alpha|=0}^\infty \sum_{|\beta|=0}^\infty |H^s_F(e_\alpha, e_\beta)|^2 < \infty$$

where $e_\alpha = z^\alpha/||z^\alpha||_\nu$. In addition, if A is a bilinear form on $L_a^2(d\iota_\nu)\otimes L_a^2(d\iota_\nu)$ of Schatten-von Neumann class $\mathcal{S}_p$, $1 \le p < \infty$, then A is compact.

3. The Banach space $\mathcal{H}^p_{\nu,s}$

Let $L^p_{\nu,s}$, for $1 \le p < \infty$, be the space of measurable functions $S : \mathbb{B} \to \odot^s V'$ such that

$$\|S\|_{\nu,s,p} = \left(\int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)S(z), S(z)\right\rangle^{p/2} d\iota(z)\right)^{1/p} < \infty\,.$$

Then $L^p_{\nu,s}$ is a Banach space. The closed subspace of holomorphic functions in $L^p_{\nu,s}$ is denoted by $\mathcal{H}^p_{\nu,s}$.

3.1. **Transformation properties of H^s_F.** Define an action π_ν of G on $L_a^2(d\iota_\nu)$ by

$$\pi_\nu: \ g \in G\ ,\ f(w) \to f(g^{-1}w)\,(J_{g^{-1}}(w))^{\nu/(d+1)}\,. \tag{13}$$

Remark 3.1. Let $z \in \mathbb{B}$. Then $\Re(1-\langle w,z\rangle) \ge (1-|z|) > 0$ for all $w \in \mathbb{B}$ so that $(1-\langle w,z\rangle)^\alpha$ can be defined as a holomorphic function in w for any real α. Thus for any $g \in G$, writing $g = U\varphi_z$ where $U \in \mathcal{U}(d)$ and φ_z is the linear fractional mapping (8), we let, according to Proposition 2.2,

$$(J_{g^{-1}}(w))^{\nu/(d+1)} = \left((-1)^d(1-|z|^2)^{(d+1)/2}\det U\right)^{\nu/(d+1)} \cdot (1-\langle w,z\rangle)^{-\nu}$$

which then defines a holomorphic function in w.

Actually $\pi_\nu : g \to \pi_\nu(g)$ is a projective unitary representation on $L_a^2(d\iota_\nu)$, that is $||\pi_\nu(g)f||_\nu = ||f||_\nu$ and $\pi_\nu(g_1g_2) = C(g_1,g_2)\pi_\nu(g_1)\pi_\nu(g_2)$ for some constant $C(g_1,g_2)$. This yields the following equality of two operator norms

$$\|H^s_F(\pi_\nu(g)(\cdot), \pi_\nu(g)(\cdot))\| = ||H^s_F||. \tag{14}$$

Define an action $\pi_{\nu,s}$ on $\mathcal{H}^2_{\nu,s}$ by

(15)

$$\pi_{\nu,s}: \ g \in G\ ,\ S(z)\ \to\ \left(\otimes^s \left(dg^{-1}(z)\right)^t\right) S(g^{-1}z)\,(J_{g^{-1}}(z))^{2\nu/(d+1)}\,.$$

Then

$$H^s_F(\pi_\nu(g)f_1, \pi_\nu(g)f_2) = H^s_S(f_1, f_2) \tag{16}$$

where $S(z) = \pi_{\nu,s}(g^{-1})F(z)$, by Lemma 3.2 below. Define an action $\pi_\nu(\cdot) \otimes \pi_\nu(\cdot)$ on $L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu)$ by

$$\begin{aligned}(17)\quad \pi_\nu \otimes \pi_\nu :\ g \in G\ ,\ &(f_1(w_1), f_2(w_2)) \\ &\to f_1(g^{-1}w_1)f_2(g^{-1}w_2)\left(J_{g^{-1}}(w_1)\right)^{\nu/(d+1)}\left(J_{g^{-1}}(w_2)\right)^{\nu/(d+1)}.\end{aligned}$$

The following invariance property of the transvectant is proved in [P1], see also [PZ].

Lemma 3.2. *Let $\pi_{\nu,s}$ and $\pi_\nu(\cdot) \otimes \pi_\nu(\cdot)$ be the representations given by (15) and (17) respectively. Let $g \in G$. Then*

$$\mathcal{T}_s\left(\pi_\nu(g) \otimes \pi_\nu(g)\right)(f_1, f_2) = \pi_{\nu,s}(g)\mathcal{T}_s(f_1, f_2).$$

Remark 3.3. It follows from Theorem 4.1 that $\mathcal{T}_s$ takes values in $\mathcal{H}^2_{\nu,s}$. In fact, Theorem 4.1 shows that $\mathcal{T}_s : L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu) \to \mathcal{H}^2_{\nu,s}$ is a bounded bilinear form.

Remark 3.4. As a consequence of Lemma 3.2 we have (16), namely

$$\begin{aligned} & H^s_F\Big(\left(\pi_\nu(g) \otimes \pi_\nu(g)\right)(f_1, f_2)\Big) \\ = \ & \langle \mathcal{T}_s\left(\pi_\nu(g) \otimes \pi_\nu(g)\right)(f_1, f_2), F\rangle_{\nu,s,2} \\ = \ & \langle \pi_{\nu,s}(g)\mathcal{T}_s(f_1, f_2), F\rangle_{\nu,s,2} \\ = \ & \left\langle \mathcal{T}_s(f_1, f_2), \pi_{\nu,s}(g^{-1})F\right\rangle_{\nu,s,2}\end{aligned}$$

which gives the result if we observe that $S = \pi_{\nu,s}(g^{-1})F$.

3.2. **Reproducing kernel of the space $\mathcal{H}^2_{\nu,s}$.**

Lemma 3.5. *The reproducing kernel of $\mathcal{H}^2_{\nu,s}$ is, up to a nonzero constant,*

$$K_{\nu,s}(z, w) = (1 - \langle z, w\rangle)^{-2\nu} \otimes^s \left(B^t(z, w)\right)^{-1}.$$

Namely, for any $f \in \mathcal{H}^2_{\nu,s}$ and any $v \in \odot^s V'$ it holds that

$$\begin{aligned}\langle f(z), v\rangle &= c\,\langle f(\cdot), K_{\nu,s}(\cdot, z)v\rangle_{\nu,s,2} \\ &= c\int_{\mathbb{B}} \left\langle (1 - |w|^2)^{2\nu} \otimes^s B^t(w, w)f(w), K_{\nu,s}(w, z)v\right\rangle d\iota(w).\end{aligned}$$

Proof. For any $v \in \odot^s V'$ we prove that $f \to \langle f(z), v\rangle$ is a bounded functional on $\mathcal{H}^2_{\nu,s}$. It follows then by Riesz lemma that there exists a function $R(z,w) : \odot^s V' \to \odot^s V'$ such that $\langle f(z), v\rangle = \langle f, R(\cdot, z)v\rangle_{\nu,s,2}$. Let $f \in \mathcal{H}^2_{\nu,s}$ and let $z \in \mathbb{B}$. Since $z \to \|f(z)\|$ is subharmonic then

$$\|f(z)\| \le C_{d,r,\nu} \int_{z+r\mathbb{B}} \|f(w)\| \, d\iota_{2\nu}(w)$$

so by Jensen's inequality

$$\|f(z)\|^2 \le C'_{d,r,\nu} \int_{z+r\mathbb{B}} \|f(w)\|^2 \, d\iota_{2\nu}(w)$$

if $\overline{z + r\mathbb{B}} \subset \mathbb{B}$. On the other hand, there is a constant $d_r > 0$ such that $d_r I \le \otimes^s B^t(w,w)$ for all $w \in \overline{z + r\mathbb{B}}$. Hence

$$\|f(z)\|^2 \le D_{d,r,\nu} \int_{z+r\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(w,w) f(w), f(w) \right\rangle d\iota(w)$$

so that $f \to \langle f(z), v\rangle$ is bounded. Then the reproducing property at $z = 0$ reads as

$$\langle f(0), v\rangle = \langle f(\cdot), R(\cdot, 0)v\rangle_{\nu,s,2} \, .$$

On the other hand, the space of $\odot^s V'$-valued polynomials is dense in $\mathcal{H}^2_{\nu,s}$ and $\langle p(\cdot), v\rangle_{\nu,s,2} = 0$ for all homogeneous polynomials of degree ≥ 1. Thus if

$$f(z) = \sum_{m=0}^{\infty} f_m(z)$$

where f_m are homogeneous polynomials of degree m, then

$$\langle f(\cdot), v\rangle_{\nu,s,2} = \langle f_0(\cdot), v\rangle_{\nu,s,2} = \langle f(0), v\rangle_{\nu,s,2} = c'\langle f(0), v\rangle \, .$$

Therefore

$$\langle f(\cdot), R(\cdot, 0)v\rangle_{\nu,s,2} = \langle f(0), v\rangle = \frac{1}{c'} \langle f(\cdot), v\rangle_{\nu,s,2}$$

so that $R(\cdot, 0) = cI$ with $c \neq 0$. Next we prove that $R(z,w)$ transforms under G as follows

$$R(gz, gw) = \left(\otimes^s dg(z)^t\right)^{-1} R(z,w) \left(\otimes^s \left(dg(w)^t\right)^*\right)^{-1} \cdot \left(J_g(z)\right)^{-2\nu/(d+1)} \left(\overline{J_g(w)}\right)^{-2\nu/(d+1)} \tag{18}$$

where $g \in G$. Indeed, for all $F \in \mathcal{H}^2_{\nu,s}$

$$\langle F(z), v\rangle = \int_{\mathbb{B}} \left\langle (1-|w|^2)^{2\nu} \otimes^s B^t(w,w) F(w), R(w,z)v \right\rangle d\iota(w)$$

from which it follows that for all $f \in L^2_a(d\iota_\nu)$

$$\left\langle J_g(z)^{2\nu/(d+1)} \otimes^s dg(z)^t f(gz), v \right\rangle = \int_{\mathbb{B}} \langle H_s(w), R(w,z)v \rangle \, d\iota(w) \,, \tag{19}$$

where $H_s(w) = (1-|w|^2)^{2\nu} \otimes^s B^t(w,w) J_g(w)^{2\nu/(d+1)} \otimes^s dg(w)^t f(gw)$. On the other hand, it follows from (9) that

$$\begin{aligned}
&\left\langle f(gz), \left(\overline{J_g(z)}\right)^{2\nu/(d+1)} \otimes^s \left(dg(z)^t\right)^* v \right\rangle \\
&= \int_{\mathbb{B}} \Bigg\langle \otimes^s B^t(w,w) f(w), \\
&\qquad\qquad R(w,gz) \left(\overline{J_g(z)}\right)^{2\nu/(d+1)} \otimes^s \left(dg(z)^t\right)^* v \Bigg\rangle \frac{d\iota_{2\nu}(w)}{c_{2\nu}} \\
&= \int_{\mathbb{B}} \left\langle \otimes^s B^t(gw,gw) f(gw), R(gw,gz) \left(\overline{J_g(z)}\right)^{2\nu/(d+1)} \otimes^s \left(dg(z)^t\right)^* v \right\rangle \\
&\qquad\qquad \cdot \left| (J_g(w))^{2\nu/(d+1)} \right|^2 \frac{d\iota_{2\nu}(w)}{c_{2\nu}} \\
&= \int_{\mathbb{B}} \left\langle H_s(w), \otimes^s dg(w)^t R(gw,gz) \otimes^s \left(dg(z)^t\right)^* v \right\rangle \\
&\qquad\qquad \cdot (J_g(z))^{2\nu/(d+1)} \left(\overline{J_g(w)}\right)^{2\nu/(d+1)} d\iota(w) \,.
\end{aligned}$$

Comparing this with (19) we get (18). Now both $R(z,w)/c$ and $K_{\nu,s}(z,w)$ satisfy the same transformation rule (18) and are identity operator at $z=0$. Thus they are the same for all $z, w \in \mathbb{B}$. This completes the proof of the lemma. □

4. The Besov space $\mathcal{B}_{\nu,s}$

Let $s = 1,2,3,\dots$ and define

$$\begin{aligned}
\mathcal{B}_{\nu,s} = \Bigg\{ & f : \mathbb{B} \to \mathbb{C} \text{ holomorphic} \,, \\
& \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z) \partial^s f(z), \partial^s f(z) \right\rangle d\iota_\nu(z) < +\infty \Bigg\} .
\end{aligned}$$

The space $\mathcal{B}_{\nu,s}$ is called a Besov space. It is a Hilbert space, equipped with the inner product $\langle\cdot,\cdot\rangle_{\nu,s}$ given by

$$\begin{aligned}\langle f,g\rangle_{\nu,s} = f(0)\overline{g(0)} \quad &+ \quad \cdots + \left\langle \left(\partial^{(s-1)}f\right)(0), \left(\partial^{(s-1)}g\right)(0)\right\rangle + \\ &+ \quad \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z)\partial^s f(z), \partial^s g(z)\right\rangle \, d\iota_\nu(z)\,.\end{aligned}$$

Actually $\mathcal{B}_{\nu,s} = L^2_a(d\iota_\nu)$, namely they are equal as sets and their norms are equivalent, as is shown below.

Theorem 4.1. *There exist constants $C_{\nu,s}, D_{\nu,s} > 0$ such that*

$$C_{\nu,s}\cdot ||f||_\nu \leq ||f||_{\nu,s} \leq D_{\nu,s}\cdot ||f||_\nu$$

for all holomorphic $f : \mathbb{B} \to \mathbb{C}$.

We need first some elementary lemmas.

Lemma 4.2. *Let f_m and f_n be homogeneous holomorphic polynomials of degree m and n respectively, with $m \neq n$. Then $\langle f_m, f_n\rangle_{\nu,s} = 0$.*

Proof. Let $0 < \theta < 2\pi$. Then $e^{i\theta} \neq 1$. Since f_m is a homogeneous polynomial of degree m we have $f_m(e^{i\theta}z) = e^{im\theta}f_m(z)$. Given m and n with $m \neq n$, it is enough to prove that

$$\langle f_m, f_n\rangle_{\nu,s} = e^{i(m-n)\theta}\langle f_m, f_n\rangle_{\nu,s} \tag{20}$$

The case $s = 0$ follows directly from the homogeneity. Now consider the case $s = 1$. It is easy to see that $B^t(z,z) = B^t(e^{-i\theta}z, e^{-i\theta}z)$. By the chain rule and homogeneity it follows that

$$(\partial f_m)(e^{i\theta}w) = e^{-i\theta}\partial(f_m(e^{i\theta}\cdot))(w) = e^{i(m-1)\theta}(\partial f_m)(w)$$

so that the equation (20) holds for $s = 1$. The cases $s = 2, 3, \ldots$ now follow in the same way if we first notice that $(\partial^s f_m)(e^{i\theta}w) = e^{i(m-s)\theta}(\partial^s f_m)(w)$. This completes the proof. □

We recall now a result from Rudin (Theorem 12.2.8 in [Ru]). Consider the space $\mathcal{P}_m$ of all homogeneous holomorphic polynomials of degree m on $\mathbb{B}$ with the natural group action of the unitary group $\mathcal{U}(d)$:

$$(\pi_g f)(z) = f(g^{-1}z), \quad f \in \mathcal{P}_m, \quad g \in \mathcal{U}(d)\,.$$

Then $(\mathcal{P}_m, \pi_g)$ is a unitary irreducible representation of $\mathcal{U}(d)$. As a consequence of Schur's lemma (Theorem 1.10 in [BD]) we have the following lemma.

Lemma 4.3. *Let m be a nonnegative integer. Then there exists a positive constant $C_{\nu,s,m}$ such that*

$$||f_m||_{\nu,s} = C_{\nu,s,m} \cdot ||f_m||_\nu$$

for all $f_m \in \mathcal{P}_m$. .

Remark 4.4. Actually, this lemma is a special case of the result in exercise 1.16.7 in [BD].

Now we can prove the norm-equivalence of $\mathcal{B}_{\nu,s}$ and $L^2_a(d\iota_\nu)$.

Proof of Theorem 4.1. It is enough to prove the theorem for f with $f(0) = \cdots = \partial^{s-1} f(0) = 0$. Write $f = \sum_{m=0}^\infty f_m$ where $f_m \in \mathcal{P}_m$. By Lemma 4.2 we have that $\{f_m\}_{m=0}^\infty$ is an orthogonal set in both $L^2_a(d\iota_\nu)$ and $\mathcal{B}_{\nu,s}$. Also, by Lemma 4.3 we have $||f_m||_{\nu,s} = C_{\nu,s,m} \cdot ||f_m||_\nu$ where $C_{\nu,s,m}$ does not depended on f_m of degree m. We compute $C_{\nu,s,m}$ and prove that there exist positive constants $C_{\nu,s}$ and $D_{\nu,s}$ such that

$$C_{\nu,s} \le C_{\nu,s,m} \le D_{\nu,s} \tag{21}$$

for all m. We may assume that $m \ge s$. Take $f_m(z) = z_1^m$. We shall calculate

$$||f_m||^2_{\nu,s} = \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z) \partial^s f_m(z), \partial^s f_m(z) \right\rangle \, d\iota_\nu(z).$$

First observe that

$$\begin{aligned} & \left\langle \otimes^s B^t(z,z) \partial^s f_m(z), \partial^s f_m(z) \right\rangle \\ = & \left\langle \otimes^s B^t(z,z) \left(\partial_1^s z_1^m\right) \otimes^s dz_1, \left(\partial_1^s z_1^m\right) \otimes^s dz_1 \right\rangle \\ = & \left\langle B^t(z,z) (\partial_1^s z_1^m) dz_1, (\partial_1^s z_1^m) dz_1 \right\rangle \cdot \left\langle B^t(z,z) dz_1, dz_1 \right\rangle^{s-1} \\ = & \frac{\Gamma(m+1)^2}{\Gamma(m-s+1)^2} (1-|z|^2)^s (1-|z_1|^2)^s |z_1|^{2(m-s)}. \end{aligned}$$

We have

$$\begin{aligned} C_\nu & \int_{\mathbb{B}} |z_1|^{2(m-s)} (1-|z_1|^2)(1-|z|^2)^{\nu+s} \, d\iota(z) \\ = & \int_{|z_1|<1} |z_1|^{2(m-s)} (1-|z_1|^2)^s \\ & \cdot \left(\int_{|z'|<\sqrt{1-|z_1|^2}} (1-|z_1|^2 - |z'|^2)^{\nu+s-d-1} dm(z') \right) dm(z_1) \end{aligned}$$

and

$$\int_{|z'|<\sqrt{1-|z_1|^2}} (1-|z_1|^2-|z'|^2)^{\nu+s-d-1} dm(z') = C'_\nu \cdot (1-|z_1|^2)^{\nu+s-2}\,.$$

Since

$$\int_{|z_1|<1} |z_1|^{2(m-s)}(1-|z_1|^2)^s(1-|z_1|^2)^{s+\nu-2}\,dm(z_1)$$
$$= C''_\nu \cdot \frac{\Gamma(m-s+1)\Gamma(\nu+2s-1)}{\Gamma(m+s+\nu)}$$

we get

$$||f_m||^2_{\nu,s} = a_\nu \cdot \frac{\Gamma(m+1)^2\Gamma(\nu+2s-1)}{\Gamma(m-s+1)\Gamma(m+s+\nu)}\,.$$

On the other hand

$$||f_m||^2_\nu = \frac{\Gamma(m+1)\Gamma(\nu)}{\Gamma(m+\nu)}$$

so that

$$C^2_{\nu,s,m} = \frac{||f_m||^2_{\nu,s}}{||f_m||^2_\nu} = a_\nu \cdot \frac{\Gamma(m+1)\Gamma(\nu+2s-1)\Gamma(m+\nu)}{\Gamma(m-s+1)\Gamma(m+s+\nu)\Gamma(\nu)}\,.$$

For $m \geq s$ we have

$$\begin{aligned} \frac{\Gamma(m+1)\Gamma(m+\nu)}{\Gamma(m-s+1)\Gamma(m+s+\nu)} &= \frac{m(m-1)\cdots(m-s+1)}{(m+s+\nu-1)\cdots(m+\nu)} \\ &= \frac{\left(1-\frac{1}{m}\right)\cdots\left(1-\frac{s-1}{m}\right)}{\left(1+\frac{s+\nu-1}{m}\right)\cdots\left(1+\frac{\nu}{m}\right)} \end{aligned}$$

so that

$$b_{\nu,s} = \frac{\left(1-\frac{1}{s}\right)\cdots\left(1-\frac{s-1}{s}\right)}{\left(1+\frac{s+\nu-1}{s}\right)\cdots\left(1+\frac{\nu}{s}\right)} \leq \frac{\Gamma(m+1)\Gamma(m+\nu)}{\Gamma(m-s+1)\Gamma(m+s+\nu)} \leq 1\,.$$

So (21) follows by putting

$$C_{\nu,s} = \sqrt{\frac{a_\nu \cdot b_{\nu,s} \cdot \Gamma(\nu+2s-1)}{\Gamma(\nu)}}$$

and

$$D_{\nu,s} = \sqrt{\frac{a_\nu \cdot \Gamma(\nu+2s-1)}{\Gamma(\nu)}}\,.$$

□

5. Boundedness

5.1. **The Banach space $\mathcal{H}^\infty_{\nu,s}$.** Denote by $L^\infty_{\nu,s}$ the space of functions $F : \mathbb{B} \to \odot^s V'$ such that

$$\|F\|_{\nu,s,\infty} = \sup_{z\in B} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z)\right\rangle^{1/2} < \infty\,.$$

If we write $\|F\|_{\nu,s,\infty} = \sup_{z\in B} \|S(z)\|_W$ where

$$\|S(z)\|_W = \left\| \left((1-|z|^2)^{2\nu} \otimes^s B^t(z,z)\right)^{1/2} F(z)\right\|$$

and $W = \odot^s V'$, then $L^\infty_{\nu,s}$ is a Banach space since it is easy to see that, if $S_n : \mathbb{B} \to W$ satisfies

$$\sum_{n=1}^{\infty} \sup_{z\in B} \|S_n(z)\|_W < \infty$$

then there is a $S : \mathbb{B} \to W$ with $\sup_{z\in\mathbb{B}} \|S(z)\|_W < \infty$ such that

$$\sup_{z\in\mathbb{B}} \left\| S(z) - \sum_{n=1}^{N} S_n(z) \right\|_W \to 0 \quad \text{as} \quad N \to \infty\,.$$

The closed subspace of holomorphic functions in $L^\infty_{\nu,s}$ is denoted by $\mathcal{H}^\infty_{\nu,s}$.

5.2. **Proof of Theorem 1.1(a).**

Proof of sufficiency. The Hankel form in (7) can be written as a sum of certain integrals, we estimate each one, as follows,

$$\left|\int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)\partial^k f(z) \otimes \partial^{s-k} g(z), F(z)\right\rangle\, d\iota(z)\right| \le$$
$$||F||_{\nu,s,\infty} \cdot \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z)\partial^k f(z) \otimes \partial^{s-k} g(z), \partial^k f(z) \otimes \partial^{s-k} g(z)\right\rangle^{1/2} \frac{d\iota_\nu(z)}{c_\nu}$$

and

$$\left\langle \otimes^s B^t(z,z)\partial^k f(z) \otimes \partial^{s-k} g(z), \partial^k f(z) \otimes \partial^{s-k} g(z)\right\rangle =$$
$$\left\langle \otimes^k B^t(z,z)\partial^k f(z), \partial^k f(z)\right\rangle \cdot \left\langle \otimes^{s-k} B^t(z,z)\partial^{s-k} g(z), \partial^{s-k} g(z)\right\rangle$$

so that

$$\left|\int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)\partial^k f(z) \otimes \partial^{s-k} g(z), F(z) \right\rangle \, d\iota(z)\right|$$
$$\leq c'_\nu \cdot ||F||_{\nu,s,\infty} \cdot ||f||_{\nu,k} \cdot ||g||_{\nu,s-k} \leq C_{\nu,s} \cdot ||F||_{s,\infty} \cdot ||f||_\nu \cdot ||g||_\nu \,,$$

where the last inequality follows from Theorem 4.1. □

For notational convenience we denote

$$\langle u, v \rangle_z = \left\langle \otimes^s B^t(z,z) u, v \right\rangle$$

where $u, v \in \odot^s V'$, and it defines an inner product on $\odot^s V'$.

Proof of necessity. Let $v \in \odot^s V'$. By Lemma 3.5 we have

$$\langle F(0), v \rangle = c \int_{\mathbb{B}} \left\langle (1-|w|^2)^{2\nu} \otimes^s B^t(w,w) F(w), v \right\rangle \, d\iota(w) \,.$$

We may write

$$v = \sum_{|i|=s} v_i \, e_1^{i_1} \odot \cdots \odot e_d^{i_d}$$

where $i = (i_1, \ldots, i_d)$ and $v_i \in \mathbb{C}$. Take

$$f(w) = \sum_{|i|=s} w_1^{i_1} \cdots w_d^{i_s} \cdot v_i \quad \text{and} \quad g(w) = 1 \,.$$

Then $f, g \in L^2_a(d\iota_\nu)$. By (6),

$$\mathcal{T}_s(f,g)(w)$$
$$= \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\partial^k f(w) \odot \partial^{s-k} g(w)}{(\nu)_k (\nu)_{s-k}} = \binom{s}{0} \frac{\partial^s f(w) \odot g(w)}{(\nu)_s (\nu)_0}$$

where

$$\partial^s f(w) = \sum_{|i|=s} \partial^s \left(w_1^{i_1} \cdots w_d^{i_d} \right) \cdot v_i = \sum_{|i|=s} s! \cdot v_i \, e_1^{i_1} \odot \cdots \odot e_d^{i_d} = s! \, v$$

so that

$$\mathcal{T}_s(f,g)(w) = \frac{s!}{(\nu)_s} \, v \,.$$

Hence

$$|\langle F(0), v \rangle|^2 = c^2 (\nu)_s^2 \cdot \frac{1}{(s!)^2} \, |H_F^s(f,g)|^2 \tag{22}$$

so that

$$|\langle F(0), v\rangle|^2 \le C_{\nu,s}\|H^s_F\|^2\|f\|^2_\nu\|g\|^2_\nu \le C_{\nu,s}\|H^s_F\|^2\|v\|^2 . \tag{23}$$

Define

$$S(w) = (\pi_{\nu,s}(\varphi_z)F)(w) = \left(\otimes^s\varphi_z'(w)^t\right) F(\varphi_z(w))\,(J_{\varphi_z}(w))^{2\nu/(d+1)} .$$

Then $S : \mathbb{B} \to \odot^s V'$ is holomorphic. Also by equations (14) and (16)

$$\|H^s_S\| = \|H^s_F\| < \infty ,$$

so by (23) with F replaced by S

$$|\langle S(0), v\rangle|^2 \le C\|H^s_S\|^2\|v\|^2 = C\|H^s_F\|^2\|v\|^2 . \tag{24}$$

Now

$$S(0) = \left(\otimes^s\varphi_z'(0)^t\right) F(z)\,(J_{\varphi_z}(0))^{2\nu/(d+1)} .$$

Since $-\varphi_z'(0)^t = s_z^2P_{\bar z} + s_zQ_{\bar z} \ge 0$ then $\left(-\varphi_z'(0)^t\right)^2 = B^t(z,z)$ and by the uniqueness of positive square root $B^t(z,z)^{1/2} = -\varphi_z'(0)^t$. Thus

$$\begin{aligned}\left(\otimes^s B^t(z,z)\right)^{1/2} &= \otimes^s B^t(z,z)^{1/2}\\ &= (-1)^s\otimes^s\varphi_z'(0)^t .\end{aligned}$$

Hence

$$S(0) = \rho(1-|z|^2)^\nu\left(\otimes^s B^t(z,z)\right)^{1/2} F(z) ,$$

where $|\rho| = 1$, so that (24) becomes

$$\begin{aligned}&\left|\left\langle F(z), \left(\otimes^s B^t(z,z)\right)^{1/2} v\right\rangle\right|^2\\ &\qquad\le C\|H^s_F\|^2\left\|\left(\otimes^s B^t(z,z)\right)^{-1/2} v\right\|_z^2 (1-|z|^2)^{-2\nu} .\end{aligned}$$

Observe that

$$\left\langle F(z), \left(\otimes^s B^t(z,z)\right)^{1/2} v\right\rangle = \left\langle F(z), \left(\otimes^s B^t(z,z)\right)^{-1/2} v\right\rangle_z$$

so the result follows from Riesz lemma, for the inner product $\langle\cdot,\cdot\rangle_z$. $\square$

6. Compactness and Hilbert-Schmidt properties

6.1. **Compactness.** In this subsection we prove Theorem 1.1(b).

Remark 6.1. Let $\{e_1, \ldots, e_d\}$ be a basis for V'. Then we can write

$$F(z) = \sum_{i_1+\cdots+i_d=s} F_i(z)\, e_1^{i_1} \odot \cdots \odot e_d^{i_d}$$

where $i = (i_1, \ldots, i_d)$ and $F_i : \mathbb{B} \to \mathbb{C}$ are holomorphic. Also

$$F_i(z) = \sum_{m=0}^{\infty} p_m^{(i)}(z)$$

where $p_m^{(i)}$ are homogeneous holomorphic polynomials of degree m.

To prove the sufficiency of Theorem 1.1(b) we need the following result.

Lemma 6.2. *Let $F : \mathbb{B} \to \odot^s V'$ be holomorphic with the property*

$$\left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z) \right\rangle \to 0 \quad \textit{if} \quad |z| \nearrow 1\,.$$

Let $\varepsilon > 0$ be given. Then there exists a number r' with $0 < r' < 1$ and a natural number N such that

$$\|F - P_N\|_{\nu,s,\infty} < \varepsilon$$

where

$$P_N(z) = \sum_{|i|=s} \sum_{m=0}^{N} p_m^{(i)}(r'z)\, e_1^{i_1} \odot \cdots \odot e_d^{i_d}\,.$$

Remark 6.3. Remember that we have already defined

$$\|F\|_{\nu,s,\infty} = \sup_{z\in\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z) \right\rangle^{1/2}$$

for holomorphic $F : \mathbb{B} \to \odot^s V'$.

Remark 6.4. Let H_1 and H_2 be Hilbert spaces and let $A_1, B_1 : H_1 \to H_1$ and $A_2, B_2 : H_2 \to H_2$ be positive operators. Then

(25)
$$(A_1-B_1)\otimes(A_2+B_2)+(A_1+B_1)\otimes(A_2-B_2) = 2\left(A_1 \otimes A_2 - B_1 \otimes B_2\right)\,.$$

Thus it follows from (25) that

(26) $$A_1 \geq B_1 \quad , \quad A_2 \geq B_2 \quad \Longrightarrow \quad A_1 \otimes A_2 \geq B_1 \otimes B_2\,.$$

Proof of Lemma 6.2. Let $\varepsilon > 0$ be given. Then there exists $0 < r_0 < 1$ such that

$$\sup_{r_0<|z|<1} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z)\right\rangle < \frac{\varepsilon^2}{32}.$$

Define $F_r(z) = F(rz)$ where $0 < r < 1$. Since $P_{r\bar{z}} = P_{\bar{z}}$ then

$$B^t(rz,rz) = (1-r^2|z|^2)(I - r^2|z|^2 P_{r\bar{z}}) \geq B^t(z,z)$$

for all $0 < r < 1$. By (26) it then follows that

$$\otimes^s B^t(rz,rz) \geq \otimes^s B^t(z,z)$$

for all $0 < r < 1$. Hence,

$$\begin{aligned} &\left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F_r(z), F_r(z)\right\rangle \\ &\qquad\qquad \leq \left\langle (1-|rz|^2)^{2\nu} \otimes^s B^t(rz,rz)F(rz), F(rz)\right\rangle . \end{aligned}$$

Then it follows from the inequalities

$$\begin{aligned} &\left\langle \otimes^s B^t(z,z)\left(F(z) - F_r(z)\right), F(z) - F_r(z)\right\rangle \\ &\qquad \leq \left\langle \otimes^s B^t(z,z)F(z), F(z)\right\rangle + \left\langle \otimes^s B^t(z,z)F_r(z), F_r(z)\right\rangle \\ &\qquad\qquad + 2\left|\left\langle \otimes^s B^t(z,z)F(z), F_r(z)\right\rangle\right| \end{aligned}$$

and

$$\begin{aligned} &\left|\left\langle \otimes^s B^t(z,z)F(z), F_r(z)\right\rangle\right| \\ &\qquad \leq \left\langle \otimes^s B^t(z,z)F(z), F(z)\right\rangle^{1/2} \left\langle \otimes^s B^t(z,z)F_r(z), F_r(z)\right\rangle^{1/2} \end{aligned}$$

that, if $1 > r > r_1 = 2r_0/(1+r_0)$ and $R_0 = (1+r_0)/2$,

$$\sup_{R_0<|z|<1} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)\left(F(z) - F_r(z)\right), F(z) - F_r(z)\right\rangle < \frac{\varepsilon^2}{8},$$

since, if $r_1 < r < 1$,

$$\begin{aligned} &\sup_{R_0<|z|<1} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(rz), F(rz)\right\rangle \\ \leq\ &\sup_{R_0 r<|rz|<r} \left\langle (1-|rz|^2)^{2\nu} \otimes^s B^t(rz,rz)F(rz), F(rz)\right\rangle \\ \leq\ &\sup_{r_0<|rz|<1} \left\langle (1-|rz|^2)^{2\nu} \otimes^s B^t(rz,rz)F(rz), F(rz)\right\rangle < \frac{\varepsilon^2}{32}. \end{aligned}$$

As $F_r \to F$ uniformly, $r \to 1$, on every compact subset of $\mathbb{B}$, there is a number r_2 such that if $r_2 < r < 1$, then

$$\sup_{|z| \le R_0} \langle F(z) - F_r(z), F(z) - F_r(z) \rangle < \frac{\varepsilon^2}{8}.$$

Since $B^t(z,z) \le (1-|z|^2)I \le I$ then (26) yields $\otimes^s B^t(z,z) \le \otimes^s I$ so that if $r_2 < r < 1$, then

$$\sup_{|z| \le R_0} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) \left(F(z) - F_r(z)\right), F(z) - F_r(z) \right\rangle$$
$$\le \sup_{|z| \le R_0} \langle F(z) - F_r(z), F(z) - F_r(z) \rangle < \frac{\varepsilon^2}{8}.$$

Hence for $\max(r_1, r_2) < r < 1$ it holds that

$$\|F - F_r\|^2_{\nu,s,\infty}$$
$$\le \sup_{|z| \le R_0} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) \left(F(z) - F_r(z)\right), F(z) - F_r(z) \right\rangle +$$
$$\sup_{R_0 < |z| < 1} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) \left(F(z) - F_r(z)\right), F(z) - F_r(z) \right\rangle < \frac{\varepsilon^2}{4}.$$

Now, take r' such that $\max(r_1, r_2) < r' < 1$. Actually, then the sum $\sum_{|i|=s} \sum_{m=0}^{\infty} p_m^{(i)}(r'z) e_1^{i_1} \odot \cdots \odot e_d^{i_d}$ converges uniformly to $F_{r'}(z)$ on $\mathbb{B}$. Hence there exists a natural number N such that

$$\|F_{r'} - P_N\|^2_{\nu,s,\infty} \le \sup_{z \in \mathbb{B}} \langle F_{r'}(z) - P_N(z), F_{r'}(z) - P_N(z) \rangle < \frac{\varepsilon^2}{4}$$

where $P_N(z) = \sum_{|i|=s} \sum_{m=0}^{N} p_m^{(i)}(r'z)\, e_1^{i_1} \odot \cdots \odot e_d^{i_d}$. This yields

$$\|F - P_N\|_{\nu,s,\infty} \le \|F - F_{r'}\|_{\nu,s,\infty} + \|F_{r'} - P_N\|_{\nu,s,\infty} < \varepsilon$$

which completes the proof of the lemma. □

Now we can prove the sufficiency of Theorem 1.1(b).

Proof of sufficiency. Let $\varepsilon > 0$ be given. Then, by Lemma 6.2, there is a P_N such that $\|F - P_N\|_{\nu,s,\infty} < \varepsilon$. Then the bilinear Hankel form $H^s_{F-P_N} = H^s_F - H^s_{P_N}$ with $F - P_N$ is bounded. In fact, the operator norm $\|\cdot\|$ satisfies

$$\|H^s_F - H^s_{P_N}\| \le C\|F - P_N\|_{\nu,s,\infty} < C\varepsilon.$$

If we can prove that $H^s_{P_N}$ is compact then we are done. Actually we shall find that $H^s_{P_N}$ is of Hilbert-Schmidt class $\mathcal{S}_2$ and thus especially compact. By construction (see Lemma 6.2) P_N is a linear combination of terms $z^{\gamma'}e^{\gamma} = z^{\gamma'}e_1^{\gamma_1}\odot\cdots\odot e_d^{\gamma_d}$ so it is enough to prove that $H^s_{z^{\gamma'}e^{\gamma}} \in \mathcal{S}_2$. Consider

$$\begin{aligned} &H^s_{z^{\gamma'}e^{\gamma}}(z^\alpha, z^\beta) \\ &\quad = \int_{\mathbb{B}} \left\langle \otimes^s B^t(w,w)\mathcal{T}_s(z^\alpha, z^\beta)(w), w^{\gamma'}e_1^{\gamma_1}\odot\cdots\odot e_d^{\gamma_d}\right\rangle d\iota_{2\nu}(w) \,. \end{aligned}$$

First we observe that

$$\left\langle \otimes^s B^t(w,w)\mathcal{T}_s\left(z^\alpha, z^\beta\right)(w), w^{\gamma'}e_1^{\gamma_1}\odot\cdots\odot e_d^{\gamma_d}\right\rangle$$

is a linear combination of terms

(27)

$$\begin{aligned} \Big\langle &\otimes^s B^t(w,w)\left(\partial_1^{i_1}\cdots\partial_d^{i_d}\right)(w^\alpha)\left(\partial_1^{j_1}\cdots\partial_d^{j_d}\right)\left(w^\beta\right) u_1\otimes u_2\otimes\cdots\otimes u_s, \\ &\qquad\qquad\qquad\qquad w^{\gamma'}v_1\otimes v_2\otimes\cdots\otimes v_s\Big\rangle \end{aligned}$$

where $u_1\otimes\cdots\otimes u_s$ and $v_1\otimes\cdots\otimes v_s$ contains i_k+j_k copies and γ_k copies of e_k respectively. We may assume that $\alpha_k \ge i_k$ and $\beta_k \ge j_k$ for $k = 1,2,\ldots d$. Denote $i = (i_1,\ldots,i_d)$ and $j = (j_1,\ldots,j_d)$. Then the term (27) equals

$$C_{i,j}(1-|w|^2)^s w^{(\alpha+\beta)-(i+j)}\bar{w}^{\gamma'}\prod_{m=1}^{s}\left(\langle u_m, v_m\rangle - \langle u_m,\bar{w}\rangle\langle\bar{w}, v_m\rangle\right) \,.$$

But this term yields a nonzero integral only for those α and β with $|\alpha+\beta| \le |\gamma'| + s$. In fact, this proves that the form $H^s_{z^{\gamma'}e^{\gamma}}$ has finite rank. Thus

$$\|H^s_{z^{\gamma'}e^{\gamma}}\|^2_{\mathcal{S}_2} = \sum_{\alpha,\beta}\frac{\left|H^s_{z^{\gamma'}e^{\gamma}}\left(z^\alpha, z^\beta\right)\right|^2}{\|z^\alpha\|^2_\nu\|z^\beta\|^2_\nu}$$

with a finite sum. Hence $H^s_{z^{\gamma'}e^{\gamma}} \in \mathcal{S}_2$ so that $H^s_{P_N} \in \mathcal{S}_2$. □

Now we prove the necessity of Theorem 1.1(b).

Proof of the necessity. Let F be a symbol such that H^s_F is compact. Since $\odot^s V'$ is a finite dimensional Hilbert space we need only to prove that $\langle u_n, v\rangle \to 0$ as $n\to\infty$ where

$$u_n = \left((1-|z_n|^2)^{2\nu}\otimes^s B^t(z_n,z_n)\right)^{1/2} F(z_n)$$

and $|z_n| \nearrow 1$ as $n \to \infty$, for any $v \in \odot^s V'$. As in the proof of the necessity of Theorem 1.1(a) we write

$$v = \sum_{|i|=s} v_i \, e_1^{i_1} \odot e_2^{i_2} \odot \cdots \odot e_d^{i_d}$$

and let

$$f(w) = \sum_{|i|=s} w_1^{i_1} \cdots w_d^{i_s} \cdot v_i \quad \text{and} \quad g(w) = 1 \, .$$

So for any symbol S we have

$$|\langle S(0), v\rangle| = C_{\nu,s} |H_S^s(f,g)| \, ,$$

by the same arguments as for (22) in the proof of the necessity of Theorem 1.1(a). Let

$$S(w) = \pi_{\nu,s}\left(\left(\varphi_{z_n}\right) F\right)(w) \otimes^s \varphi'_{z_n}(w)^t F(\varphi_{z_n}(w)) \left(J_{\varphi_{z_n}}(w)\right)^{2\nu/(d+1)}$$

so that

$$S(0) = \otimes^s \varphi'_{z_n}(0)^t F(z_n) \left(J_{\varphi_{z_n}}(0)\right)^{2\nu/(d+1)} \, . \tag{28}$$

By Proposition 2.2,

$$J_{\varphi_{z_n}}(0) = (-1)^d (1 - |z_n|^2)^{(d+1)/2} \quad \text{and} \quad B^t(z_n, z_n)^{1/2} = -\varphi'_{z_n}(0)^t$$

so that

$$|\langle S(0), v\rangle| = |\langle u_n, v\rangle| \, . \tag{29}$$

On the other hand

$$H_S^s(f,g) = H_F^s\left(f \circ \varphi_{z_n} \cdot J_{\varphi_{z_n}}^{\nu/(d+1)}, k_{z_n}\right)$$

where

$$k_{z_n}(w) = (g \circ \varphi_{z_n})(w) \left(J_{\varphi_{z_n}}(w)\right)^{\nu/(d+1)} = \rho \cdot \frac{(1 - |z_n|^2)^{\nu/2}}{(1 - \langle w, z_n\rangle)^\nu} \, , \quad |\rho| = 1 \, ,$$

so that $k_{z_n}(w) \to 0$ weakly as $n \to \infty$ and $\|k_{z_n}\|_\nu = 1$. Since H_F^s is compact then there is a sequence $\{c_n\}_{n=0}^\infty$ of positive numbers such that $c_n \to 0$ and

$$|H_F^s(h, k_{z_n})| \leq c_n \|h\|_\nu$$

for all $h \in L_a^2(d\iota_\nu)$. Let $h = f \circ \varphi_{z_n} \cdot J_{\varphi_{z_n}}^{\nu/(d+1)} = \pi_\nu\left(\varphi_{z_n}\right) f$ which yields

$$\|h\|_\nu^2 = \|f\|_\nu^2 \, .$$

Then

$$|\langle u_n, v\rangle| \leq C_{\nu,s} c_n \|f\|_\nu \leq C'_{\nu,s} c_n \|v\|$$

so that $\langle u_n, v\rangle \to 0$ as $n \to \infty$, which, combined with the equalities (28) and (29), implies that

$$\big\langle (1-|z_n|^2)^{2\nu} \otimes^s B^t(z_n, z_n)F(z_n), F(z_n)\big\rangle \to 0 \quad \text{as } |z_n| \nearrow 1\,.$$

□

6.2. **Hilbert-Schmidt properties.** In this subsection we prove Theorem 1.2. Denote by $\mathcal{H}'_{\nu,s}$ the space of all holomorphic functions $F : \mathbb{B} \to \odot^s V'$ such that the corresponding bilinear Hankel form on $L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu)$

$$H^s_F(f,g) = \int_{\mathbb{B}} \big\langle \otimes^s B^t(z,z)\mathcal{T}_s(f,g)(z), F(z)\big\rangle \, d\iota_{2\nu}(z)$$

is of Hilbert-Schmidt class $\mathcal{S}_2$. By Lemma 6.5, it is a Hilbert space with an inner product $\langle F, S\rangle'_{\nu,s} = \langle H^s_F, H^s_S\rangle_{\mathcal{S}_2}$ where

$$\langle H^s_F, H^s_S\rangle_{\mathcal{S}_2} = \sum_{|\alpha|=0}^{\infty}\sum_{|\beta|=0}^{\infty} H^s_F(e_\alpha, e_\beta)\overline{H^s_S(e_\alpha, e_\beta)}$$

and $e_\alpha = z^\alpha/\|z^\alpha\|_\nu$.

Lemma 6.5. *The space $\mathcal{H}'_{\nu,s}$ is a Hilbert space.*

Proof. Let $\{F_n\}_{n=0}^\infty$ be a Cauchy sequence in $\mathcal{H}'_{\nu,s}$. Then $\{H^s_{F_n}\}_{n=0}^\infty$ is Cauchy in operator norm so that $\{F_n\}_{n=0}^\infty$ is Cauchy in $\|\cdot\|_{\nu,s,\infty}$. Then there is a $F \in \mathcal{H}^\infty_{\nu,s}$ such that $F_n \to F$ in $\|\cdot\|_{\nu,s,\infty}$. Thus $H^s_{F_n} \to H^s_F$ in operator norm. On the other hand, the space of all bilinear forms of Hilbert-Schmidt class $\mathcal{S}_2$ is a Hilbert space so that $H^s_{F_n} \to H \in \mathcal{S}_2$ in $\|\cdot\|_{\mathcal{S}_2}$. Then $H^s_{F_n} \to H$ in operator norm so that $H = H^s_F$. Thus $F \in \mathcal{H}'_{\nu,s}$ and $F_n \to F$ in $\mathcal{H}'_{\nu,s}$. □

We now shall see that $\mathcal{H}'_{\nu,s} = \mathcal{H}^2_{\nu,s}$, namely they are equal as sets and the norms are equivalent, as is shown below. Actually, Theorem 1.2 is a direct consequence of Theorem 6.6.

Theorem 6.6. *There is a constant $C_{\nu,s} > 0$ such that*

$$\|F\|'_{\nu,s} = C_{\nu,s}\|F\|_{\nu,s,2}$$

for all holomorphic $F : \mathbb{B} \to \odot^s V'$.

To prove Theorem 6.6 we need some lemmas.

Lemma 6.7. *Let $\{e_1, \dots, e_d\}$ be an orthonormal basis for V'. Then the spaces $\mathcal{H}'_{\nu,s}$ and $\mathcal{H}^2_{\nu,s}$ contains the element $e_1^s = e_1 \otimes \cdots \otimes e_1$.*

Proof. Clearly $e_1^s \in \mathcal{H}^2_{\nu,s}$. The fact that $e_1^s \in \mathcal{H}'_{\nu,s}$ follows from (27), letting $\gamma' = 0$ and $\gamma_j = s \cdot \delta_{1j}$ for $j = 1, \ldots, d$. □

Lemma 6.8. *The action $\pi_{\nu,s}$, defined in (15), is unitary on both $\mathcal{H}'_{\nu,s}$ and $\mathcal{H}^2_{\nu,s}$.*

Proof. Clearly, $\pi_{\nu,s}$ is unitary on $\mathcal{H}^2_{\nu,s}$. That $\pi_{\nu,s}$ is also unitary on $\mathcal{H}'_{\nu,s}$ follows from Lemma 3.2 and the fact that π_ν, defined in (13), is unitary on $L^2_a(d\iota_\nu)$. □

Lemma 6.9. *The space $\mathcal{H}^2_{\nu,s}$ is irreducible with respect to the action $\pi_{\nu,s}$, defined in (15).*

Proof. Let $\mathcal{H}_0$ be an invariant closed subspace of $\mathcal{H}^2_{\nu.s}$ under the action $\pi_{\nu,s}(g)$, $g \in G$, and assume that $h \in \mathcal{H}_0$ for some $h \neq 0$. We may assume, by replacing h by an action of $\pi_{\nu,s}(g)$ on h if necessary, that $h(0) \neq 0$. We need to prove

$$f \in \mathcal{H}^2_{\nu,s} \quad , \quad f \perp \mathcal{H}_0 \quad \Longrightarrow \quad f = 0\,. \tag{30}$$

Take such an $f \in \mathcal{H}^2_{\nu,s}$. Since $e^{-i\theta} : z \to e^{-i\theta} z$ is in G and

$$\mathcal{H}_0 \ni \left(\pi_{\nu,s}(e^{-i\theta})h\right)(z) = \left(e^{i\theta d}\right)^{2\nu/(d+1)} \cdot e^{i\theta s} \cdot h(e^{i\theta} z)\,,$$

then $h(e^{i\theta} z) \in \mathcal{H}_0$. Hence, by the mean value property,

$$h(0) = \int_0^{2\pi} h(e^{i\theta} z)\, d\theta \in \mathcal{H}_0\,.$$

Then we have found a nonzero element in $\odot^s V'$ which is also contained in $\mathcal{H}_0$. Then $v \in \mathcal{H}_0$ for any $v \in \odot^s V'$ (by Theorem 12.2.8 in [Ru]). Then $[\pi_{\nu,s}(\varphi_w)v](z) = c \cdot K(z,w)v$ is in $\mathcal{H}_0$, for any $v \in \odot^s V'$, where $K(\cdot, w)$ is the reproducing kernel for $\mathcal{H}^2_{\nu,s}$ and c is a nonzero constant. Hence

$$f \perp K(\cdot\,, w)v$$

so that

$$f(w) = 0 \quad \text{for all} \quad w \in \mathbb{B}$$

by the reproducing property. This proves (30). □

Now we can prove Theorem 6.6.

Proof of Theorem 6.6. As a consequence of Theorem VI.23 in [RS] we can make the following identification of the space $\mathcal{S}_2(L^2_a(d\iota_\nu), L^2_a(d\iota_\nu))$

of Hilbert-Schmidt bilinear forms on $L^2_a(d\iota_\nu)$ with the tensor product, that is,

$$\mathcal{S}_2\left(L^2_a(d\iota_\nu), L^2_a(d\iota_\nu)\right) = L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu).$$

Moreover $L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu)$ can be decomposed into irreducible subspaces $\tilde{\mathcal{H}}_{\nu,s}$ of Hankel forms of weight s with an intertwining operator $T : \mathcal{H}^2_{\nu,s} \to \tilde{\mathcal{H}}_{\nu,s}$, (see [PZ]). Also, H^s_F defined in (7) is a Hankel form of weight s and by Lemma 6.7 there is a nonzero element in $\mathcal{H}^2_{\nu,s}$ which yields a nonzero element in $\mathcal{H}'_{\nu,s}$. Thus

$$\mathcal{H}'_{\nu,s} = \mathcal{H}^2_{\nu,s}$$

whose norms are the same up to a constant, by Corollary 8.13 in [K]. □

7. Matrix-valued Bergman-type projections

To prove Theorem 1.3 we need certain interpolation results for the spaces $\mathcal{H}^p_{\nu,s}$, which will then be derived from certain L^p-boundedness properties of some matrix-valued Bergman projections. The results in this section might be of independent interests. We refer to Zhu [Zh] for the study of boundedness property of scalar Bergman projections.

We start with a technical lemma.

Lemma 7.1. *Let s be a positive integer. Then*

$$\left\|\otimes^s \left(B^t(w,w)^{1/2}\left(B^t(w,z)\right)^{-1} B^t(z,z)^{1/2}\right) v\right\| \leq C_s \cdot \frac{(1-|w|^2)^{s/2}(1-|z|^2)^{s/2}}{|1-\langle w,z\rangle|^s}\|v\|$$

for all $w, z \in \mathbb{B}$ and $v \in \otimes^s V'$.

Proof. First we shall prove the lemma for $s = 1$ by using the following identities (see (10) and (4)):

$$B^t(z,z)^{1/2} = s_z(s_z P_{\bar{z}} + Q_{\bar{z}}) \quad \text{where} \quad s_z = (1-|z|^2)^{1/2}$$

and

$$B^t(w,z)^{-1} = (1-\langle w,z\rangle)^{-2}\Big((1-\langle w,z\rangle)I + \bar{z}\otimes\bar{w}^*\Big).$$

Note that

$$\begin{aligned}B^t(w,w)^{1/2}&B^t(w,z)^{-1}B^t(z,z)^{1/2}\\ &= s_w s_z(1-\langle w,z\rangle)^{-1}(s_wP_{\bar w}+Q_{\bar w})(s_zP_{\bar z}+Q_{\bar z})+\\ &\qquad s_w s_z(1-\langle w,z\rangle)^{-2}(s_wP_{\bar w}+Q_{\bar w})(\bar z\otimes \bar w^*)(s_zP_{\bar z}+Q_{\bar z})\,.\end{aligned}$$

Thus, by the inequality

$$\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle w,z\rangle|^2}\le 1 \quad \text{for all} \quad z,w\in\mathbb{B},$$

it is enough to show that

$$\|(s_wP_{\bar w}+Q_{\bar w})(\bar z\otimes \bar w^*)(s_zP_{\bar z}+Q_{\bar z})\|\le C|1-\langle w,z\rangle|\,. \tag{31}$$

To this end, we may assume $|z|\ge 1/2$ and $|w|\ge 1/2$. Expand the product $(\cdot)(\cdot)(\cdot)$ as a sum of four terms. First we note that

$$\|s_wP_{\bar w}(\bar z\otimes \bar w^*)s_zP_{\bar z}\|\le s_w s_z\le |1-\langle w,z\rangle|\,. \tag{32}$$

There are three parts left to consider. Let $v\in V'$. The first part to estimate is

$$s_wP_{\bar w}(\bar z\otimes \bar w^*)Q_{\bar z}v = s_w\left\langle v,\bar w-\frac{\langle z,w\rangle}{|z|^2}\bar z\right\rangle\frac{\langle z,w\rangle}{|w|^2}\bar w\,.$$

We use Cauchy-Schwarz' inequality. Note that

$$\left\|\bar w-\frac{\langle z,w\rangle}{|z|^2}\bar z\right\|^2=\frac{|w|^2|z|^2-|\langle z,w\rangle|^2}{|z|^2}\le 8|1-\langle z,w\rangle|\,.$$

Thus the inequalities $1-|w|^2\le 2|1-\langle z,w\rangle|$ and $|w|\ge 1/2$ yield the estimation

$$\|s_wP_{\bar w}(\bar z\otimes \bar w^*)Q_{\bar z}v\|\le 16|1-\langle z,w\rangle|\|v\| \tag{33}$$

Since

$$Q_{\bar w}(\bar z\otimes \bar w^*)s_zP_{\bar z}=(s_zP_{\bar z}(\bar w\otimes \bar z^*)Q_{\bar w})^*$$

we have an estimation of the second part

$$\|Q_{\bar w}(\bar z\otimes \bar w^*)s_zP_{\bar z}v\|\le 16|1-\langle z,w\rangle|\|v\|\,. \tag{34}$$

Finally consider

$$Q_{\bar w}(\bar z\otimes \bar w^*)Q_{\bar z}v=\left\langle v,\bar w-\frac{\langle z,w\rangle}{|z|^2}\bar z\right\rangle\left(\bar z-\frac{\langle w,z\rangle}{|w|^2}\bar w\right)\,.$$

The same estimates as above yield

$$\|Q_{\bar w}(\bar z\otimes \bar w^*)Q_{\bar z}v\|\le 8|1-\langle z,w\rangle|\|v\|\,. \tag{35}$$

Thus the four estimations (32), (33), (34) and (35) yields (31). We have proved the lemma for $s = 1$. Now, consider the case where $s = 2, 3, \ldots$ and let

$$A_{w,z} = B^t(w,w)^{1/2} \left(B^t(w,z)\right)^{-1} B^t(z,z)^{1/2}$$

and

$$t_{w,z} = \frac{s_z s_w}{|1 - \langle z, w\rangle|}.$$

We have proved that

$$A^*_{w,z} A_{w,z} \le C^2 t^2_{w,z} I$$

so that

$$(\otimes^s A_{w,z})^* \otimes^s A_{w,z} = \otimes^s \left(A^*_{w,z} A_{w,z}\right) \le C^{2s} t^{2s}_{w,z} \otimes^s I$$

which proves the lemma. □

Theorem 7.2. *Let $\alpha > d$ and let $P_{\nu,s} : L^2_{\nu,s} \to \mathcal{H}^2_{\nu,s}$ be the orthogonal projection operator. If* $\max\{(\alpha - d)/(2\nu + s/2 - d), 1\} < p < \infty$, *then*

$$\int_{\mathbb{B}} \left\|\otimes^s B^t(z,z)^{1/2} P_{\nu,s} f(z)\right\|^p (1 - |z|^2)^\alpha \, d\iota(z) \le C \int_{\mathbb{B}} \left\|\otimes^s B^t(w,w)^{1/2} f(w)\right\|^p (1 - |w|^2)^\alpha \, d\iota(w).$$

Remark 7.3. By Lemma 3.5 the orthogonal projection operator $P_{\nu,s}$, such that for any $f \in L^2_{\nu,s}$ and any $v \in \odot^s V'$ we have that

$$\langle P_{\nu,s} f(z), v\rangle = c \int_{\mathbb{B}} \left\langle (1 - |w|^2)^{2\nu} \otimes^s B^t(w,w) f(w), K_{\nu,s}(w,z) v\right\rangle \, d\iota(w) \tag{36}$$

where

$$K_{\nu,s}(w,z) = \otimes^s \left(B^t(w,z)\right)^{-1} (1 - \langle w, z\rangle)^{-2\nu},$$

is well-defined.

Proof of Theorem 7.2. The formula (36) can be rewritten as

$$P_{\nu,s} f(z) = c \int_{\mathbb{B}} K_{\nu,s}(w,z)^* \otimes^s B^t(w,w) f(w) (1 - |w|^2)^{2\nu} \, d\iota(w).$$

Now let

$$T(z,w) = \frac{(1 - |z|^2)^{s/2} (1 - |w|^2)^{2\nu + s/2 - \alpha}}{|1 - \langle z, w\rangle|^{2\nu + s}}.$$

By the equality $K_{\nu,s}(w,z)^* = K_{\nu,s}(z,w)$ and Lemma 7.1 it follows that

$$\begin{aligned}&\left\|\otimes^s B^t(z,z)^{1/2} P_{\nu,s} f(z)\right\| \\ &\qquad \le C \int_{\mathbb{B}} T(z,w) \left\|\otimes^s B^t(w,w)^{1/2} f(w)\right\| (1-|w|^2)^\alpha \, d\iota(w)\,.\end{aligned}$$

We claim that there exists a real number t such that

$$\int_{\mathbb{B}} T(z,w)(1-|w|^2)^{qt}(1-|w|^2)^\alpha \, d\iota(w) \le M(1-|z|^2)^{qt} \tag{37}$$

and

$$\int_{\mathbb{B}} T(z,w)(1-|z|^2)^{pt}(1-|z|^2)^\alpha \, d\iota(z) \le M(1-|w|^2)^{pt} \tag{38}$$

holds for some constant M, where q is given by $1 = 1/p + 1/q$. Accepting temporarily the claim, using Hölder's inequality and (37),

$$\begin{aligned}&\left\|\otimes^s B^t(z,z)^{1/2} P_{\nu,s} f(z)\right\| \\ \le\ & C\left(\int_{\mathbb{B}} T(z,w)(1-|w|^2)^{qt}(1-|w|^2)^\alpha \, d\iota(w)\right)^{1/q} \times \\ &\left(\int_{\mathbb{B}} T(z,w)(1-|w|^2)^{-pt} \left\|\otimes^s B^t(w,w)^{1/2} f(w)\right\|^p (1-|w|^2)^\alpha \, d\iota(w)\right)^{1/p} \\ \le\ & CM^{1/q}(1-|z|^2)^t \times \\ &\left(\int_{\mathbb{B}} T(z,w)(1-|w|^2)^{-pt} \left\|\otimes^s B^t(w,w)^{1/2} f(w)\right\|^p (1-|w|^2)^\alpha \, d\iota(w)\right)^{1/p}.\end{aligned}$$

Thus, by Fubini-Tonelli's theorem and (38), we have that

$$\begin{aligned}&\int_{\mathbb{B}} \left\|\otimes^s B^t(z,z)^{1/2} P_{\nu,s} f(z)\right\|^p (1-|z|^2)^\alpha \, d\iota(z) \\ \le\ & C^p M^{p/q} \int_{\mathbb{B}} (1-|z|^2)^{pt} \left(\int_{\mathbb{B}} T(z,w)(1-|w|^2)^{-pt} \cdot \right. \\ &\left. \left\|\otimes^s B^t(w,w)^{1/2} f(w)\right\|^p (1-|w|^2)^\alpha \, d\iota(w)\right)(1-|z|^2)^\alpha \, d\iota(z)\end{aligned}$$

$$
\begin{aligned}
&= C^p M^{p/q} \int_{\mathbb{B}} (1-|w|^2)^{-pt} \left\| \otimes^s B^t(w,w)^{1/2} f(w) \right\|^p (1-|w|^2)^\alpha \cdot \\
&\qquad\qquad \left(\int_{\mathbb{B}} (1-|z|^2)^{pt} T(z,w) (1-|z|^2)^\alpha \, d\iota(z) \right) d\iota(w) \\
&\le C^p M^p \int_{\mathbb{B}} \left\| \otimes^s B^t(w,w)^{1/2} f(w) \right\|^p (1-|w|^2)^\alpha \, d\iota(z) \,,
\end{aligned}
$$

namely our theorem.

Now we go back to (37) and (38) which, by Lemma 2.4, holds if

$$
\frac{d-2\nu-s/2}{q} < t < \frac{s}{2q} \tag{39}
$$

and

$$
\frac{d-s/2-\alpha}{p} < t < \frac{2\nu+s/2-\alpha}{p} \tag{40}
$$

respectively. Actually, by simple computations,

$$
\left(\frac{d-2\nu-s/2}{q}, \frac{s}{2q} \right) \bigcap \left(\frac{d-s/2-\alpha}{p}, \frac{2\nu+s/2-\alpha}{p} \right) \neq \emptyset
$$

if $\max\{(\alpha-d)/(2\nu+s/2-d), 1\} < p < \infty$. □

Corollary 7.4. *If $1 < p < \infty$, then*

$$
P_{\nu,s} L^p_{\nu,s} = \mathcal{H}^p_{\nu,s} \,,
$$

namely $P_{\nu,s} : L^p_{\nu,s} \to \mathcal{H}^p_{\nu,s}$ is bounded.

8. Application of the boundedness of $P_{\nu,s}$

8.1. **Some interpolation results.** In this subsection we use the complex interpolation method of Banach spaces to prove Theorem 8.2, which we will use to prove Theorem 1.3 in subsection 8.2.

The spaces $\mathcal{A}_1 = L^2_{\nu,s} + L^\infty_{\nu,s}$ and $\mathcal{A}_2 = \mathcal{H}^2_{\nu,s} + \mathcal{H}^\infty_{\nu,s}$ are Banach spaces with the norms

$$
\|F\|_{\mathcal{A}_i} = \inf \left\{ \|F_2\|_{\nu,s,2} + \|F_\infty\|_{\nu,s,\infty} \,:\, F = F_2 + F_\infty \in \mathcal{A}_i \right\},
$$

$i = 1, 2$, respectively, by Lemma 2.3.1 in [BL]. Denote by $\mathcal{F}_i = \mathcal{F}(\mathcal{A}_i)$, $i = 1, 2$, the space of all functions with values in $\mathcal{A}_i$, which are bounded and continuous on the strip

$$
S = \{z \in \mathbb{C} : 0 \le \Re z \le 1\}
$$

and holomorphic on the open strip

$$S_0 = \{z \in \mathbb{C} : 0 < \Re z < 1\}$$

and moreover, the functions $t \to f(j+it)$ are continuous functions from the real line such that $f(it) \in L^2_{\nu,s}$ (resp. $\mathcal{H}^2_{\nu,s}$) and $f(1+it) \in L^\infty_{\nu,s}$ (resp. $\mathcal{H}^\infty_{\nu,s}$), which tends to zero as $|t| \to \infty$. Then $\mathcal{F}_i$, $i = 1, 2$, are Banach spaces with the same norm

$$\|f\|_{\mathcal{F}} = \max\left(\sup \|f(it)\|_{\nu,s,2}\, , \, \sup \|f(1+it)\|_{\nu,s,\infty} \right),$$

by Lemma 4.1.1 in [BL]. Now let $0 < \theta < 1$ and denote by $\left(L^2_{\nu,s}, L^\infty_{\nu,s}\right)_{[\theta]}$ and $\left(\mathcal{H}^2_{\nu,s}, \mathcal{H}^\infty_{\nu,s}\right)_{[\theta]}$ the space of all $S \in \mathcal{A}_i$ such that

$$\|S\|_{i,[\theta]} = \inf\left\{ \|f\|_{\mathcal{F}} \, : \, f(\theta) = S\, ,\, f \in \mathcal{F}_i \right\} < \infty\, ,$$

$i = 1, 2$, respectively.

Lemma 8.1. *If* $2 < p < \infty$, *then*

$$P_{\nu,s}\left(L^2_{\nu,s}, L^\infty_{\nu,s}\right)_{[1-2/p]} = \left(\mathcal{H}^2_{\nu,s}, \mathcal{H}^\infty_{\nu,s}\right)_{[1-2/p]}\, ,$$

namely $P_{\nu,s} : \left(L^2_{\nu,s}, L^\infty_{\nu,s}\right)_{[1-2/p]} \to \left(\mathcal{H}^2_{\nu,s}, \mathcal{H}^\infty_{\nu,s}\right)_{[1-2/p]}$ *is bounded.*

Proof. As a direct consequence of Lemma 7.1 we have that $P_{\nu,s} : L^\infty_{\nu,s} \to \mathcal{H}^\infty_{\nu,s}$ is bounded. Indeed, for any $f \in L^\infty_{\nu,s}$,

$$\begin{aligned}
& \left\|\otimes^s B^t(z,z)^{1/2} P_{\nu,s} f(z)\right\| \\
\le\;& C(1-|z|^2)^{s/2} \int_{\mathbb{B}} \left\|\otimes^s B^t(w,w)^{1/2} f(w)\right\| \cdot \frac{(1-|w|^2)^{2\nu+s/2}}{|1-\langle w,z\rangle|^{2\nu+s}}\, d\iota(w) \\
\le\;& C(1-|z|^2)^{s/2} \|f\|_{\nu,s,\infty} \int_{\mathbb{B}} \frac{(1-|w|^2)^{\nu+s/2}}{|1-\langle w,z\rangle|^{2\nu+s}}\, d\iota(w) \\
\le\;& C' \|f\|_{\nu,s,\infty} (1-|z|^2)^{-\nu}
\end{aligned}$$

where the last inequality follows from Lemma 2.4. Hence, the result follows from Riesz-Thorin's interpolation theorem. □

If we claim that

$$\left(L^2_{\nu,s}, L^\infty_{\nu,s}\right)_{[1-2/p]} = L^p_{\nu,s} \quad , \quad 2 < p < \infty\, , \tag{41}$$

then we have the following theorem.

Theorem 8.2. *If* $2 < p < \infty$, *then*

$$\mathcal{H}^p_{\nu,s} = \left(\mathcal{H}^2_{\nu,s}, \mathcal{H}^\infty_{\nu,s}\right)_{[1-2/p]} .$$

Proof. If $2 < p < \infty$, then by the identity (41) we have that

$$L^p_{\nu,s} = \left(L^2_{\nu,s}, L^\infty_{\nu,s}\right)_{[1-2/p]} .$$

Thus, by Corollary 7.4 and Lemma 8.1, if $2 < p < \infty$ then

$$\mathcal{H}^p_{\nu,s} = P_{\nu,s} L^p_{\nu,s} = P_{\nu,s}\left(L^2_{\nu,s}, L^\infty_{\nu,s}\right)_{[1-2/p]} = \left(\mathcal{H}^2_{\nu,s}, \mathcal{H}^\infty_{\nu,s}\right)_{[1-2/p]} .$$

□

The identity (41) can be proved by slightly modifying Theorem 5.1.1 in [BL] using

$$(42)\quad \|F\|_{\nu,s,p} = \sup\left\{ \left| \int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), S(z) \right\rangle d\iota(z) \right| : \right.$$
$$\left. S \text{ bounded with compact support }, \ \|S\|_{\nu,s,q} = 1 \right\}$$

where $1/p + 1/q = 1$. Indeed, to prove (42) let $F : \mathbb{B} \to \odot^s V'$ be measurable. Then

$$H = \left((1-|\cdot|^2)^{2\nu} \otimes^s B^t(\cdot,\cdot)\right)^{1/2} F : \mathbb{B} \to \odot^s V'$$

is measurable and we may write $H = (H_1, \ldots, H_N)$, where $\dim(\odot^s V') = N$. For $1 \le j \le N$ we can find bounded functions b^j_n with compact support in $\mathbb{B}$ such that $|b^j_n| \nearrow |H_j|$. Let

$$s^j_n = |b^j_n| \cdot e^{iArgH_j} .$$

Then s^j_n are bounded with compact support and

$$H_j \cdot \overline{s^j_n} = |H_j| \cdot |b^j_n| .$$

Let $s_n = (s^1_1, \ldots, s^N_n)$ and put

$$t_n(z) = \left((1-|z|^2)^{2\nu} \otimes^s B^t(z,z)\right)^{-1/2} s_n(z) .$$

Then $t_n : \mathbb{B} \to \odot^s V'$ is measurable and

$$\begin{aligned}
(43) \qquad & \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) t_n(z), t_n(z) \right\rangle \\
&= \sum_{j=1}^{N} s_n^j(z) \cdot \overline{s_n^j(z)} = \sum_{j=1}^{N} |b_n^j(z)| \cdot |b_n^j(z)| \\
&\leq \sum_{j=1}^{N} |H_j(z)| \cdot |b_n^j(z)| = \sum_{j=1}^{N} H_j(z) \cdot \overline{s_n^j(z)} \\
&= \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), t_n(z) \right\rangle .
\end{aligned}$$

Now, let

$$S_n(z) = \frac{\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) t_n(z), t_n(z) \rangle^{(q-2)/2} \cdot t_n(z)}{\|t_n\|_{\nu,s,q}^{q-1}} .$$

Then $S_n : \mathbb{B} \to \odot^s V'$ is measurable, $\|S_n\|_{\nu,s,q} = 1$ and

$$\int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) S_n(z), t_n(z) \right\rangle \, d\iota(z) = \|t_n\|_{\nu,s,p}$$

so by (43)

$$\begin{aligned}
\|F\|_{\nu,s,p} &\leq \underline{\lim} \, \|t_n\|_{\nu,s,p} \\
&= \underline{\lim} \int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) S_n(z), t_n(z) \right\rangle \, d\iota(z) \\
&\leq \underline{\lim} \int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) S_n(z), F(z) \right\rangle \, d\iota(z) \\
&\leq M_p(F)
\end{aligned}$$

where

$$M_p(F) = \sup \left\{ \left| \int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), S(z) \right\rangle \, d\iota(z) \right| : \right. \\
\left. S \text{ bounded with compact support }, \ \|S\|_{\nu,s,q} = 1 \right\} .$$

On the other hand

$$\left| \int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), S(z) \right\rangle d\iota(z) \right| \leq \|F\|_{\nu,s,p} \cdot \|S\|_{\nu,s,q}$$

which proves (42). The rest is almost the same as in [BL] loc. cit., only replacing the usual absolute value $|g(z)|$ of scalar functions $g(z)$

by the norm $\|S(z)\|_z = \| \left((1-|z|^2)^{2\nu} \otimes^s B^t(z,z)\right)^{1/2} S(z)\|$ of vector-valued functions $S(z)$, also $E(z) = \langle f(z), g(z)\rangle$ by

$$H(z) = \int_{\mathbb{B}} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), \overline{S(z)} \right\rangle d\iota(z) .$$

8.2. **Schatten-von Neumann properties.** In this subsection we prove Theorem 1.3.

Proof of sufficiency of Theorem 1.3. By Theorem 1.2 and Theorem 1.1 the operator $F \to H_F^s$ is bounded from $\mathcal{H}^2_{\nu,s}$ into $\mathcal{S}_2$ and from $\mathcal{H}^\infty_{\nu,s}$ into $\mathcal{S}_\infty$ respectively. Then it follows from Theorem 8.2 and Riesz-Thorin interpolation theorem that $F \to H_F^s$ is bounded from $\mathcal{H}^p_{\nu,s}$ into $(\mathcal{S}_2, \mathcal{S}_\infty)_{[1-2/p]}$ if $2 < p < \infty$. By Theorem 2.10 in [S] we have that

$$\mathcal{S}_p = (\mathcal{S}_2, \mathcal{S}_\infty)_{[1-2/p]} ,$$

so that the operator $F \to H_F^s$ is bounded from $\mathcal{H}^p_{\nu,s}$ into $\mathcal{S}_p$ if p satisfies $2 < p < \infty$. □

The necessity of Theorem 1.3 is a direct consequence of Lemma 8.6 below. This Lemma states some boundedness properties for an operator $\tilde{\mathcal{T}}_s$ closely related to the transvectant defined in (6) viewed as an operator from bilinear forms to vector-valued holomorphic functions, see also [FR] and [PZ]. We need to construct $\tilde{\mathcal{T}}_s$. Let $A \in \mathcal{S}_\infty\left(L^2_a(d\iota_\nu), \overline{L^2_a(d\iota_\nu)}\right)$. Then there is a conjugate linear operator $T : L^2_a(d\iota_\nu) \to \overline{L^2_a(d\iota_\nu)}$ such that

$$A(K_z, K_w) = \langle K_z, TK_w\rangle_\nu = \langle K_w, T^* K_z\rangle_\nu = \overline{T^* K_z(w)} .$$

Also,

$$\begin{aligned} A(f,g) = \langle f, Tg\rangle_\nu &= \int_{\mathbb{B}} f(z)\overline{(Tg)(z)}\, d\iota_\nu(z) \\ &= \int_{\mathbb{B}} f(z)\overline{\langle Tg, K_z\rangle_\nu}\, d\iota_\nu(z) \\ &= \int_{\mathbb{B}} f(z)\overline{\langle T^* K_z, g\rangle_\nu}\, d\iota_\nu(z) \\ &= \int_{\mathbb{B}}\int_{\mathbb{B}} \overline{T^* K_z(w)} f(z) g(w)\, d\iota_\nu(z) d\iota_\nu(w) . \end{aligned}$$

Define $G(z,w) := G_A(z,w) = \overline{A(K_z, K_w)}$. Then $G(z,w)$ is holomorphic in z and in w and

$$A(f,g) = \int_{\mathbb{B}}\int_{\mathbb{B}} \overline{G(z,w)} f(z) g(w)\, d\iota_\nu(z) d\iota_\nu(w)\,.$$

Now, define

$$\tilde{\mathcal{T}}_s(A)(z) = (\mathcal{T}_s G)(z,z) \tag{44}$$

where

$$(\mathcal{T}_s G)(z,w) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\partial_z^k \odot \partial_w^{s-k} G(z,w)}{(\nu)_k (\nu)_{s-k}}\,.$$

Remark 8.3. If $G(z,w) = f(z)g(w)$ where $f, g \in L_a^2(d\iota_\nu)$ then we have that $\mathcal{T}_s(G)(z,z) = \mathcal{T}_s(f,g)(z)$ where $\mathcal{T}_s(f,g)(z)$ is the transvectant defined in (6).

Lemma 8.4. *Let $\tilde{\mathcal{T}}_s$ be defined on $\mathcal{S}_\infty$ as in (44). Then $\tilde{\mathcal{T}}_s : \mathcal{S}_\infty \to \mathcal{H}_{\nu,s}^\infty$ is bounded.*

Proof. Let $A \in \mathcal{S}_\infty$. Let $G(z,w) = G_A(z,w)$. First we note that $(\mathcal{T}_s G)(z,w)$ is a linear combination of terms

$$\partial_z^k \partial_w^{s-k} G(z,w) = \sum_{|I|=k, |J|=s-k} \partial_z^I \partial_w^J \overline{A(K_z, K_w)}\, dz_I \otimes dw_J$$

where $i_1, \cdots, i_k \in \{1, \ldots, d\}$, $I = (i_1, \ldots, i_k)$, $dz_I = dz_{i_1} \otimes \cdots \otimes dz_{i_k}$ and $\partial_z^I = \partial_{i_1}\partial_{i_2} \cdots \partial_{i_k}$. By the identity

$$\partial_{i_1} \cdots \partial_{i_k} \partial_{j_1} \cdots \partial_{j_{s-k}} \overline{A(K_z, K_w)} = \overline{A(E_z, E_w)}$$

where

$$\begin{aligned} E_z(\zeta) &= (\nu)_k e_I(\zeta)(1 - \langle \zeta, z\rangle)^{-\nu-k}, & e_I(\zeta) &= \zeta_{i_1} \cdots \zeta_{i_k}, \\ E_w(\zeta) &= (\nu)_{s-k} e_J(\zeta)(1 - \langle \zeta, w\rangle)^{-\nu+s-k}, & e_J(\zeta) &= \zeta_{j_1} \cdots \zeta_{j_{s-k}}\,, \end{aligned}$$

it follows that

$$\left(\partial_z^k \partial_w^{s-k} G\right)(0,0) = (\nu)_k (\nu)_{s-k} \sum_{|I|=k, |J|=s-k} \overline{A(e_I, e_J)}\, dz_I \otimes dw_J\,.$$

Since A is bounded then

$$\left\| \left(\partial_z^k \partial_w^{s-k} G\right)(0,0) \right\| \le C\|A\|\,. \tag{45}$$

Let $z \in \mathbb{B}$ and define a bilinear form A_z on $L_a^2(d\iota_\nu)$ such that

$$A_z(f,g) = A\left(\pi_\nu(\varphi_z) f, \pi_\nu(\varphi_z) g\right)\,,$$

where φ_z is the linear fractional mapping (8) and π_ν is the action (13). Then it holds that $\|A_z\| = \|A\|$ and by the same transformation property as in Lemma 3.2, see also [PZ], it follows that $\tilde{\mathcal{T}}_s(A_z) = \pi_{\nu,s}(\varphi_z)\tilde{\mathcal{T}}_s(A)$. Hence, replacing A by A_z in (45) yields

$$\left\|\tilde{\mathcal{T}}_s(A_z)(0)\right\| \le C\|A_z\| = C\|A\|$$

and

$$\tilde{\mathcal{T}}_s(A_z)(0) = \left(\pi_{\nu,s}(\varphi_z)\tilde{\mathcal{T}}_s(A)\right)(0) = \otimes^s \varphi_z'(0)^t \tilde{\mathcal{T}}_s(A)(z) J_{\varphi_z}(0)^{2\nu/(d+1)}$$

so that

$$\left\|\otimes^s B^t(z,z)^{1/2}\tilde{\mathcal{T}}_s(A)(z)(1-|z|^2)^\nu\right\| \le C\|A\|\,.$$

This proves the lemma. □

Lemma 8.5. *Let $\tilde{\mathcal{T}}_s$ be defined on $\mathcal{S}_2$ as in (44). Then $\tilde{\mathcal{T}}_s : \mathcal{S}_2 \to \mathcal{H}^2_{\nu,s}$ is bounded.*

Proof. By Theorem 6.6 it follows that $\sigma : \mathcal{H}^2_{\nu,s} \to \mathcal{S}_2$, $\sigma(F) = H_F^s$ defines an isometry. Thus $\sigma^* : \mathcal{S}_2 \to \mathcal{H}^2_{\nu,s}$ is a partial isometry and therefore bounded. We claim that $\sigma^* = \tilde{\mathcal{T}}_s$, which actually follows by an identification. Indeed let A be a bilinear form of finite rank. We shall prove that $\sigma^*(A) = \tilde{\mathcal{T}}_s(A)$, which gives the general case. Let H_F^s be a Hilbert-Schmidt Hankel form. Then

$$\langle H_F^s, A\rangle_{\mathcal{S}_2} = \sum_{i,j=1}^N H_F^s(e_i,e_j)\overline{A(e_i,e_j)}$$

where $\{e_i\}_{i=1}^N$ is an orthonormal set in $\mathcal{H}^2_{\nu,s}$. Since

$$H_F^s(e_i,e_j) = \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z)\mathcal{T}_s(e_i,e_j)(z), F(z)\right\rangle \, d\iota_{2\nu}(z)$$

then

$$\sum_{i,j=1}^N H_F^s(e_i,e_j)\overline{A(e_i,e_j)}$$

$$= \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z) \sum_{i,j=1}^N \mathcal{T}_s(e_i,e_j)(z)\overline{A(e_i,e_j)}, F(z)\right\rangle \, d\iota_{2\nu}(z)\,.$$

On the other hand

$$\langle H_F^s, A\rangle_{\mathcal{S}_2} = \langle \sigma(F), A\rangle_{\mathcal{S}_2} = \langle \sigma^*(A), F\rangle_{\nu,s,2}\,.$$

Thus, it remains to prove that

$$\tilde{\mathcal{T}}_s(A)(z) = \sum_{i,j=1}^{N} \mathcal{T}_s(e_i,e_j)(z)\overline{A(e_i,e_j)}\,. \tag{46}$$

Since $A(f,g)=0$ if f or g is in $\operatorname{span}\{e_1,\ldots,e_N\}^\perp$ and since $\{\bar{e}_i\otimes\bar{e}_j\}$ is an orthonormal set in $\mathcal{S}_2$, where $\bar{e}_i\otimes\bar{e}_j(f,g)=\langle f,e_i\rangle_\nu\langle g,e_j\rangle_\nu$, then

$$A = \sum_{i,j=1}^{N} \langle A, \bar{e}_i\otimes\bar{e}_j\rangle_{\mathcal{S}_2}\,\bar{e}_i\otimes\bar{e}_j = \sum_{i,j=1}^{N} A(e_i,e_j)\,\bar{e}_i\otimes\bar{e}_j\,.$$

Hence

$$G(z,w) = \overline{A(K_z,K_w)} = \sum_{i,j=1}^{N} \overline{A(e_i,e_j)}\,e_i(z)e_j(w)$$

so that

$$\tilde{\mathcal{T}}_s(A)(z) = (\mathcal{T}_s G)\,(z,z) = \sum_{i,j=1}^{N} \overline{A(e_i,e_j)}\mathcal{T}_s(e_i,e_j)(z)$$

which proves (46). □

Lemma 8.6. *Let $\tilde{\mathcal{T}}_s$ be defined on $\mathcal{S}_p$ as in (44), $2<p<\infty$. Then $\tilde{\mathcal{T}}_s:\mathcal{S}_p\to\mathcal{H}^p_{\nu,s}$ is bounded and $\tilde{\mathcal{T}}_s(H_F^s)=F$ if $H_F^s\in\mathcal{S}_p$.*

Proof. It follows from Lemma 8.4, Lemma 8.5 and Riesz-Thorin's interpolation theorem that $\tilde{\mathcal{T}}_s:\mathcal{S}_p\to\mathcal{H}^p_{\nu,s}$ is bounded for $2<p<\infty$. Also, $\tilde{\mathcal{T}}_s(H_F^s)=F$ if $H_F^s\in\mathcal{S}_2$. Now define $F_r(z)=F(rz)$ for $0<r<1$. Then $H^s_{F_r}\in\mathcal{S}_2$ so that $\tilde{\mathcal{T}}_s(H^s_{F_r})=F_r$. Since H_F^s is compact then $F_r\to F$ in $\mathcal{H}^\infty_{\nu,s}$, by the necessity of Theorem 1.1(b) and the proof of Lemma 6.2. On one hand $F_r\to F$ pointwise. On the other hand, by Theorem 1.1(a) and Lemma 8.4, it follows that $\tilde{\mathcal{T}}_s(H^s_{F_r})\to\tilde{\mathcal{T}}_s(H^s_F)$. Thus $\tilde{\mathcal{T}}_s(H^s_F)=F$ if $H^s_F\in\mathcal{S}_p$. □

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Paper II

TRACE CLASS CRITERIA FOR BILINEAR HANKEL FORMS OF HIGHER WEIGHTS

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Abstract. In this paper we give a complete characterization of higher weight Hankel forms, on the unit ball of $\mathbb{C}^d$, of Schatten-von Neumann class $\mathcal{S}_p$, $1 \leq p \leq \infty$. For this purpose we give an atomic decomposition for certain Besov-type spaces. The main result is then obtained by combining the decomposition and our earlier results [Su].

1. Introduction

Hankel operators on the unit disk have been studied extensively, see [Pe2] for a systematic treatment. One of the main topics is to study Schatten-von Neumann properties of Hankel operators, see [Pe1] and [Pe2]. In [JP] Janson and Peetre introduced Hankel forms of higher weights on the unit disk. Their Schatten-von Neumann properties were studied in [Ro] and [Z].

In [P1] Peetre introduced Hankel forms of higher weights on the unit ball of $\mathbb{C}^d$. Their Schatten-von Neumann, $\mathcal{S}_p$, properties were studied in [Su] for $2 \leq p \leq \infty$. See also [FR] for a different approach.

The results for $2 \leq p \leq \infty$ in [Su] were proved by using interpolation between $\mathcal{S}_2$ and $\mathcal{S}_\infty$ (bounded operators) and boundedness of certain matrix-valued Bergman-type projections, but the case of $1 \leq p < 2$ was left open there.

In this paper we extend the results in [Su] to $1 \leq p \leq \infty$. For this purpose we study the atomic decomposition for some Besov spaces of vector-valued holomorphic functions, see Section 4, which then gives the $\mathcal{S}_1$ properties. Our results follow by interpolation and we get a full characterization for $1 \leq p \leq \infty$. Some of the proofs in this paper are

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based on techniques used in [Su] and will therefore be given briefly. The reader is referred to that article for more details.

The paper is organized as follows. In Section 2 we recall briefly some notation and we prove Theorem 2.1, generalizing the result for $p = 2$ in [Su]. Section 3 is devoted to duality relations for the spaces of symbols. In Section 4 we give an atomic decomposition for a certain space of symbols, which will be used in Section 5 to prove the $\mathcal{S}_1$ criterion.

2. Preliminaries

2.1. **The Banach space $\mathcal{H}^p_{\nu,s}$ for $1 \leq p \leq \infty$.** Let dm denote the Lebesgue measure on the unit ball $\mathbb{B} \subset \mathbb{C}^d$ and let $d\iota(z)$ be the measure $(1-|z|^2)^{-d-1}dm(z)$. For $d < \nu < \infty$ let $d\iota_\nu(z) = c_\nu(1-|z|^2)^\nu \, d\iota(z)$, where c_ν is chosen such that

$$\int_{\mathbb{B}} d\iota_\nu(z) = 1\,.$$

The closed subspace of all holomorphic functions in $L^2(d\iota_\nu)$ is denoted by $L^2_a(d\iota_\nu)$ and is called a weighted Bergman space. Note that the space $L^2_a(d\iota_\nu)$ has a reproducing kernel $K_z(w) = (1-\langle w,z\rangle)^{-\nu}$, that is,

$$f(z) = \langle f, K_z\rangle_\nu = \int_{\mathbb{B}} f(w)\overline{K_z(w)}\, d\iota_\nu(w)\,,\ \ f \in L^2_a(d\iota_\nu)\,,\ \ z \in \mathbb{B}\,. \tag{1}$$

Denote by $B(z,w)$ the Bergman operator on $V = \mathbb{C}^d$ as in [L], namely

$$B(z,w) = (1-\langle z,w\rangle)(I - z\otimes w^*), \tag{2}$$

where $z \otimes w^*$ stands for the rank one operator given by $(z\otimes w^*)(v) = \langle v, w\rangle z$.

The Bergman metric at $z \in \mathbb{B}$, when we identify the tangent space with V, is $\langle B(z,z)^{-1}u, v\rangle$ for $u, v \in V$. We note that

$$B(z,w)^{-1} = (1-\langle z,w\rangle)^{-2}\big((1-\langle z,w\rangle)I + z\otimes w^*\big)\,. \tag{3}$$

Let $B^t(z,w)$ denote the dual of $B(z,w)$ acting on the dual space V' of V. When acting on a vector $v' \in V'$ it is

$$B^t(z,w)v' = (1-\langle z,w\rangle)v'(I - z\bar{w}^t)\,. \tag{4}$$

For a nonnegative integer s, let $\otimes^s V'$ be the tensor product of s copies of V' and let $\otimes^0 V' = \mathbb{C}$. The space $\otimes^s V'$ is equipped with a natural Hermitian inner product induced by that of V'. Denote by $\odot^s V'$ the subspace of symmetric tensors of length s and denote by $\otimes^s B^t(z,z)$ the operator on $\otimes^s V'$ induced by the action of $B^t(z,z)$ on V', where $\otimes^0 B^t(z,z) = I$. Recall, generally, that if A acts on V', $\otimes^s A$ acts on $\otimes^s V'$ by

$$(\otimes^s A)(u_1 \otimes u_2 \otimes \cdots \otimes u_s) = (Au_1) \otimes (Au_2) \otimes \cdots \otimes (Au_s) .$$

For example, in the case $s = 2$ the operator $\otimes^2 B^t(z,z)$ becomes

$$(1 - |z|^2)^2 (I \otimes I - I \otimes A_z - A_z \otimes I + A_z \otimes A_z) ,$$

where $A_z = \bar{z} \otimes \bar{z}^*$. Let $L^p_{\nu,s} = L^p_\nu(\mathbb{B}, \odot^s V')$ be the space of functions $G : \mathbb{B} \to \odot^s V'$ such that

$$\|G\|_{\nu,s,p} = \left(\int_{\mathbb{B}} \left\langle (1 - |z|^2)^{2\nu} \otimes^s B^t(z,z) G(z), G(z) \right\rangle^{p/2} d\iota(z) \right)^{1/p} < \infty ,$$

where $1 \le p < \infty$, and let $L^\infty_{\nu,s}$ be the space of functions $G : \mathbb{B} \to \odot^s V'$ such that

$$\|G\|_{\nu,s,\infty} = \sup_{z \in \mathbb{B}} \left\langle (1 - |z|^2)^{2\nu} \otimes^s B^t(z,z) G(z), G(z) \right\rangle^{1/2} < \infty .$$

Let $\mathcal{H}^p_{\nu,s}$ be the closed subspace of all holomorphic functions in $L^p_{\nu,s}$, $1 \le p \le \infty$.

Also, we need the group G of biholomorphic mappings of $\mathbb{B}$. Let P_z be the orthogonal projection of $\mathbb{C}^d$ onto $\mathbb{C}z$ and let $Q_z = I - P_z$. Put $s_z = (1 - |z|^2)^{1/2}$ and define a linear fractional mapping φ_z on $\mathbb{B}$ by (see [Ru])

$$\varphi_z(w) = \frac{z - P_z w - s_z Q_z w}{1 - \langle w, z \rangle} . \tag{5}$$

If $g \in G$ and $g(z) = 0$, then there is a unique unitary operator $U : \mathbb{C}^d \to \mathbb{C}^d$ such that

$$g = U \varphi_z .$$

Define the complex Jacobian J_g by $J_g(w) = \det(g'(w))$. Now, let $z_0 \in \mathbb{B}$. Then by arguments in Remark 3.1 in [Su] it follows that there is a constant c with $|c| = 1$ such that

$$J_{\varphi_{z_0}}(w)^{2\nu/(d+1)} = c \cdot \frac{(1 - |z_0|^2)^\nu}{(1 - \langle w, z_0 \rangle)^{2\nu}} . \tag{6}$$

The next theorem gives the reproducing properties for $\mathcal{H}^p_{\nu,s}$.

Theorem 2.1. *Let $1 \leq p \leq \infty$. There is a nonzero constant c such that, for any $G \in \mathcal{H}^p_{\nu,s}$ and any $v \in \odot^s V'$,*

$$\langle G(z), v \rangle = c \int_{\mathbb{B}} \left\langle \otimes^s B^t(w,w) G(w), K_{\nu,s}(w,z) v \right\rangle (1-|w|^2)^{2\nu} \, d\iota(w) \,,$$

where

$$K_{\nu,s}(w,z) = (1-\langle w,z \rangle)^{-2\nu} \otimes^s B^t(w,z)^{-1} \,.$$

The proof of this theorem is given at the end of this subsection.

Remark 2.2. Consider $\mathcal{H}^2_{\nu,s} \subset L^2_{\nu,s}$. According to Lemma 3.5 in [Su] the orthogonal projection operator $P_{\nu,s}$ of $L^2_{\nu,s}$ onto $\mathcal{H}^2_{\nu,s}$, is given by

$$P_{\nu,s} G(z) \; = \; c \int_{\mathbb{B}} (1-|w|^2)^{2\nu} K_{\nu,s}(z,w) \otimes^s B^t(w,w) G(w) \, d\iota(w) \,. \tag{7}$$

Namely, for any $G \in L^2_{\nu,s}$ and any $v \in \odot^s V'$ it follows that

$$\langle P_{\nu,s} G(z), v \rangle = c \int_{\mathbb{B}} \left\langle \otimes^s B^t(w,w) G(w), K_{\nu,s}(w,z) v \right\rangle (1-|w|^2)^{2\nu} \, d\iota(w) \,.$$

The orthogonal projection operator has the following boundedness property.

Proposition 2.3. *If $1 \leq p < \infty$, then $P_{\nu,s} : L^p_{\nu,s} \to \mathcal{H}^p_{\nu,s}$ is bounded.*

Proof. The case $1 < p < \infty$ is just Corollary 7.4 in [Su]. Now, consider the case $p = 1$. Let $F \in L^1_{\nu,s}$. Then it follows from Theorem 2.1 above and Lemma 7.1 in [Su] that

$$\begin{aligned} &\left\| \otimes^s B^t(z,z)^{1/2} P_{\nu,s} F(z) \right\| \\ &\qquad \leq C_s \int_{\mathbb{B}} T(z,w) \left\| \otimes^s B^t(w,w)^{1/2} F(w) \right\| (1-|w|^2)^{2\nu} \, d\iota(w) \,, \end{aligned}$$

where

$$T(z,w) = \frac{(1-|z|^2)^{s/2}(1-|w|^2)^{s/2}}{|1-\langle z,w \rangle|^{2\nu+s}} \,.$$

Thus, by Fubini-Tonelli's theorem and Proposition 1.4.10 in [Ru] it follows that

$$\begin{aligned}
&\|P_{\nu,s}F\|_{\nu,s,1} \\
&\leq C_s \int_{\mathbb{B}} \left\|\otimes^s B^t(w,w)^{1/2}F(w)\right\| (1-|w|^2)^{2\nu} \\
&\qquad\qquad \cdot \left(\int_{\mathbb{B}} T(z,w)(1-|z|^2)^{\nu}\, d\iota(z)\right) d\iota(w) \\
&\leq C_s' \int_{\mathbb{B}} \left\|\otimes^s B^t(w,w)^{1/2}F(w)\right\| (1-|w|^2)^{\nu}\, d\iota(w) = C_s'\|F\|_{\nu,s,1}\,.
\end{aligned}$$

□

Note that it is proved in [Su], using the complex interpolation method of Banach spaces, that $\mathcal{H}^p_{\nu,s} = (\mathcal{H}^2_{\nu,s}, \mathcal{H}^\infty_{\nu,s})_{[1-2/p]}$ if $2 < p < \infty$; see Theorem 8.2 in [Su]. However, Proposition 2.3 allows us to use the same proof as in [Su] to get the following result.

Corollary 2.4. *If* $1 < p < \infty$*, then*

$$\mathcal{H}^p_{\nu,s} = \left(\mathcal{H}^1_{\nu,s}, \mathcal{H}^\infty_{\nu,s}\right)_{[1-1/p]}\,.$$

Now we go back to Theorem 2.1. First we need a proposition and a lemma.

Proposition 2.5. *Let* s *be a nonnegative integer and let* $\nu > d$, $2\nu > \alpha > d$. *Then there is a constant* $C_s > 0$ *such that*

$$(1-|z|^2)^{2\nu-\alpha}\left\|K_{\nu,s}(\cdot,z)\otimes^s B^t(z,z)^{1/2}v\right\|_{\alpha,s,1} \leq C_s\|v\|$$

for all $z \in \mathbb{B}$ *and all* $v \in \odot^s V'$.

Proof. Let $v \in \odot^s V'$. It follows from Lemma 7.1 in [Su] and Proposition 1.4.10 in [Ru] that

$$\begin{aligned}
&\|K_{\nu,s}(\cdot,z)\otimes^s B^t(z,z)^{1/2}v\|_{\alpha,s,1} \\
&= \int_{\mathbb{B}} \left\|\otimes^s \left(B^t(w,w)^{1/2}B^t(w,z)^{-1}B^t(z,z)^{1/2}\right)v\right\| \frac{(1-|w|^2)^{\alpha}}{|1-\langle w,z\rangle|^{2\nu}}\, d\iota(w) \\
&\leq C_s\|v\| \int_{\mathbb{B}} \frac{(1-|z|^2)^{s/2}(1-|w|^2)^{\alpha+s/2}}{|1-\langle w,z\rangle|^{2\nu+s}}\, d\iota(w) \leq C_s'(1-|z|^2)^{\alpha-2\nu}\|v\|\,.
\end{aligned}$$

□

Lemma 2.6. *Let $z \in \mathbb{B}$. Then there is a constant $C_s > 0$ such that, for any $v \in \odot^s V'$ and any $1 \leq p \leq \infty$, it follows that*

$$\left\|(1-|z|^2)^\nu K_{\nu,s}(\cdot,z) \otimes^s B^t(z,z)^{1/2} v\right\|_{\nu,s,p} \leq C_s \|v\| .$$

Proof. Let $T_z = (1-|z|^2)^\nu K_{\nu,s}(\cdot,z) \otimes^s B^t(z,z)^{1/2}$. By Proposition 2.5 and by Lemma 7.1 in [Su] it follows that $\|T_z v\|_{\nu,s,1} \leq C_s\|v\|$ and $\|T_z v\|_{\nu,s,\infty} \leq C'_s\|v\|$ respectively, for all $v \in \odot^s V'$. Thus the result follows from Riesz-Thorin's interpolation theorem. □

Now we can prove Theorem 2.1.

Proof of Theorem 2.1. Let $G \in \mathcal{H}^p_{\nu,s}$, $1 \leq p \leq \infty$. Then it follows from Lemma 2.6 that, for all $v \in \odot^s V'$,

$$\int_{\mathbb{B}} \left|\left\langle \otimes^s B^t(w,w) G(w), K_{\nu,s}(w,z)v\right\rangle\right| \, d\iota_{2\nu}(w) \\ \leq \|G\|_{\nu,s,p} \|K_{\nu,s}(\cdot,z)v\|_{\nu,s,q} < \infty .$$

In particular, if $z=0$, then

$$\int_{\mathbb{B}} \left|\left\langle \otimes^s B^t(w,w) G(w), v\right\rangle\right| (1-|w|^2)^{2\nu} \, d\iota(w) < \infty .$$

By the mean-value property for holomorphic functions and rotation invariance for integration,

$$\int_{\mathbb{B}} \left\langle (1-|w|^2)^{2\nu} \otimes^s B^t(w,w) G(w), v\right\rangle d\iota(w) = c'\langle G(0), v\rangle ,$$

where $c' \neq 0$ only depends on d, ν and s. Hence, there exists a nonzero constant c such that, for all $G \in \mathcal{H}^p_{\nu,s}$ and all $v \in \odot^s V'$,

$$\langle G(0), v\rangle = c\langle G, v\rangle_{\nu,s,2} , \tag{8}$$

where $\langle \cdot,\cdot\rangle_{\nu,s,2}$ is the $\mathcal{H}^2_{\nu,s}$-pairing. Now, define an isometry $\pi_{\nu,s}$ on $\mathcal{H}^2_{\nu,s}$ by

$$\pi_{\nu,s} : \; g \in G \, , \; S(z) \quad \to \quad \left(\otimes^s \left(dg^{-1}(z)\right)^t\right) S(g^{-1}z) \left(J_{g^{-1}}(z)\right)^{2\nu/(d+1)} ,$$

as in [Su]. Let $z_0 \in \mathbb{B}$. For notational convenience we prove the reproducing property only for $s=1$; the case for general s is identically the same. On the one hand,

$$\langle (\pi_{\nu,1}(\varphi_{z_0})G)(0), v\rangle = \left\langle G(z_0), \overline{J_{\varphi_{z_0}}(0)}^{2\nu/(d+1)} \left(\varphi'_{z_0}(0)^t\right)^* v\right\rangle . \tag{9}$$

By equation (6), $(1-|z_0|^2)^{(d+1)/2} < |J_{\varphi_{z_0}}(w)| < (1-|z_0|^2)^{-d-1}$ on $\mathbb{B}$, so $\pi_{\nu,1}(\varphi_{z_0})G \in \mathcal{H}^p_{\nu,1}$. However, using equation (8) above for $\pi_{\nu,1}(\varphi_{z_0})G$ and the transformation properties

$$B\left(\varphi_{z_0}(w), \varphi_{z_0}(z)\right) = \varphi'_{z_0}(w)B(w,z)\left(\varphi'_{z_0}(z)\right)^*$$

(see Equation (9) in [Su]) and

$$\begin{aligned} K_{\nu,1}\left(\varphi_{z_0}(w), \varphi_{z_0}(z)\right) &= J_{\varphi_{z_0}}(w)^{-2\nu/(d+1)} \cdot \overline{J_{\varphi_{z_0}}(z)}^{-2\nu/(d+1)} \times \\ &\qquad \left(\varphi'_{z_0}(w)^t\right)^{-1} K_{\nu,1}(w,z)\left(\left(\varphi'_{z_0}(z)^t\right)^*\right)^{-1} \end{aligned}$$

(see equation (9) in [Su] and Theorem 2.2.5 in [Ru]), the left-hand side in equation (9) above is

$$\langle G(z_0), u\rangle = c\langle G, K_{\nu,s}(\cdot, z_0)u\rangle_{\nu,s,2},$$

where $u = \overline{J_{\varphi_{z_0}}(0)}^{2\nu/(d+1)} \left(\varphi'_{z_0}(0)^t\right)^* v$. Since v is arbitrary, then so is $u \in \odot^s V'$, which proves the theorem. □

2.2. **Hankel forms of higher weights.** Let H_1 and H_2 be Hilbert spaces and let $T : H_1 \to H_2$ be a linear operator. Define the singular numbers $s_n(T) = \inf\{||T-K|| : \operatorname{rank}(K) \le n\}$, $n \ge 0$. If T is compact, these singular numbers are equal to the eigenvalues of $|T| = (T^*T)^{1/2}$ (counted with multiplicities). We denote by $\mathcal{S}_p$ the ideal of operators for which $\{s_n(T)\}_{n\ge 0} \in l^p$, $0 < p \le \infty$; see [S].

The transvectant $\mathcal{T}_s$ on $L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu)$ (introduced in [P1]; see also [P2], [PZ] and [Su]) is defined by

$$\mathcal{T}_s(f,g)(z) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\partial^k f(z) \odot \partial^{s-k} g(z)}{(\nu)_k (\nu)_{s-k}}, \tag{10}$$

where

$$\partial^s f(z) = \sum_{j_1 \dots j_s = 1}^{d} \partial_{j_1} \cdots \partial_{j_s} f(z)\, dz_{j_1} \otimes \cdots \otimes dz_{j_s} \in \odot^s V'$$

and $(\nu)_k = \nu(\nu+1)\cdots(\nu+k-1)$, $(\nu)_0 = 1$, is the Pochammer symbol.

Lemma 2.7. *There is a constant $C_s > 0$ such that*

$$\|\mathcal{T}_s(f,g)\|_{\nu,s,1} \le C_s \|f\|_\nu \|g\|_\nu$$

for all $f, g \in L^2_a(d\iota_\nu)$.

First we need a lemma, which actually is a consequence of Theorem 4.1 in [Su], but we give an independent and easier proof.

Lemma 2.8. *There is a constant $C_{\nu,s} > 0$ such that*

$$\int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z)\partial^s f(z), \partial^s f(z)\right\rangle (1-|z|^2)^\nu \, d\iota(z) \le C_{\nu,s}\|f\|_\nu$$

for all $f \in L^2_a(d\iota_\nu)$.

Proof. First,

$$\partial^s f(z) = c_\nu(\nu)_s \int_{\mathbb{B}} \frac{f(w)\otimes^s \bar{w}}{(1-\langle z,w\rangle)^{\nu+s}} \cdot (1-|w|^2)^\nu \, d\iota(w)\,,$$

so that

$$\begin{aligned} &\left\|\otimes^s B^t(z,z)^{1/2}\partial^s f(z)\right\| \\ &\qquad \le C_{\nu,s}\int_{\mathbb{B}} \frac{|f(w)|\cdot \left\|B^t(z,z)^{1/2}\bar{w}\right\|^s}{|1-\langle z,w\rangle|^{\nu+s}} \cdot (1-|w|^2)^\nu \, d\iota(w)\,. \end{aligned}$$

We can estimate

$$\begin{aligned} \left\|B^t(z,z)^{1/2}\bar{w}\right\| &= s_z\left(\|s_z P_{\bar{z}}\bar{w}\|^2 + \|Q_{\bar{z}}\bar{w}\|^2\right)^{1/2} \\ &= s_z\left(|w|^2 - |\langle z,w\rangle|^2\right)^{1/2} \\ &\le \sqrt{2}\cdot s_z|1-\langle z,w\rangle|^{1/2}\,. \end{aligned}$$

Hence,

$$\left\|\otimes^s B^t(z,z)^{1/2}\partial^s f(z)\right\| \le C'_{\nu,s}\int_{\mathbb{B}} T(z,w)|f(w)|(1-|w|^2)^\nu \, d\iota(w)\,,$$

where

$$T(z,w) = \frac{(1-|z|^2)^{s/2}}{|1-\langle z,w\rangle|^{\nu+s/2}}\,.$$

Now, the result follows by exactly the same arguments as in the proof of Theorem 7.2 in [Su] (where we let $t = -(\nu-d)/4$). □

Proof of Lemma 2.7. The transvectant is a linear combination of terms $\partial^k f(z)\otimes\partial^{s-k}g(z)$ so we need only to estimate $\|\partial^k f(z)\otimes\partial^{s-k}g(z)\|_{\nu,s,1}$ for $0\le k\le s$. First we observe that

$$\begin{aligned} &\left\|\otimes^s B^t(z,z)^{1/2}\partial^k f(z)\otimes\partial^{s-k}g(z)\right\| \\ &\qquad = \left\|\otimes^k B^t(z,z)^{1/2}\partial^k f(z)\right\|\cdot\left\|\otimes^{s-k}B^t(z,z)^{1/2}\partial^{s-k}g(z)\right\|\,. \end{aligned}$$

Thus by Hölder's inequality and Lemma 2.8 it follows that

$$\int_{\mathbb{B}} \left\| \otimes^s B^t(z,z)^{1/2} \partial^k f(z) \otimes \partial^{s-k} g(z) \right\| (1-|z|^2)^\nu \, d\iota(z) \\ \leq C \|f\|_{\nu,k} \|g\|_{\nu,s-k} \leq C_s \|f\|_\nu \|g\|_\nu \, .$$

□

The Hankel bilinear form H_F^s on $L_a^2(d\iota_\nu) \otimes L_a^2(d\iota_\nu)$ is defined by

$$H_F^s(f,g) = \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z) \mathcal{T}_s(f,g)(z), F(z) \right\rangle \, d\iota_{2\nu}(z) \tag{11}$$

where $F : \mathbb{B} \to \odot^s V'$ is holomorphic. We call F the symbol of the corresponding Hankel form. We remark that

$$H_F^0(f,g) = \int_{\mathbb{B}} f(z) g(z) \overline{F(z)} \, d\iota_{2\nu}(z) \, .$$

This is the classical Hankel form studied in [JPR].

With the form H_F^s one can associate the operator A_F^s defined by

$$H_F^s(f,g) = \langle f, A_F^s g \rangle_\nu$$

as in [JPR]. Notice that A_F^s is an anti-linear operator on $L_a^2(d\iota_\nu)$. To get a linear operator one combines A_F^s with a conjugation, i.e., one instead considers the operator $\overline{A}_F^s : g \to \overline{A_F^s g}$. We say that H_F^s is of Schatten-von Neumann class $\mathcal{S}_p$, for $0 < p < \infty$, if and only if $\overline{A}_F^s : L_a^2(d\iota_\nu) \to \overline{L^2(d\iota_\nu)}$ is of class $\mathcal{S}_p$.

3. Duality of $\mathcal{H}_{\nu,s}^p$

In this section we determine the dual space $(\mathcal{H}_{\nu,s}^p)^*$ of $\mathcal{H}_{\nu,s}^p$, where $1 \leq p < \infty$.

Lemma 3.1. *Let $1 \leq p < \infty$. If $\Phi \in \left(L_{\nu,s}^p\right)^*$, then there is a function $G \in L_{\nu,s}^q$ such that*

$$\Phi(F) = \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z) F(z), G(z) \right\rangle (1-|z|^2)^{2\nu} \, d\iota(z)$$

and $\|\Phi\| = \|G\|_{\nu,s,q}$ where $1/q + 1/p = 1$.

Proof. Define $A(z) = (1-|z|^2)^\nu \otimes^s B^t(z,z)^{1/2}$ and $(M_A F)(z) = A(z)F(z)$. Then M_A is an isometry from $L^p_{\nu,s}$ onto $L^p = \{F : \mathbb{B} \to V \ : \ \|F\|_p < \infty\}$ where

$$\|F\|_p = \left(\int_{\mathbb{B}} \|F(z)\|^p \, d\iota(z)\right)^{1/p} .$$

Consider $\Theta = \Phi M_A^{-1}$. Then Θ is a bounded linear functional on L^p and $\Theta(AF) = \Phi(F)$. Then we can find a function $H \in L^q$ such that

$$\Phi(F) = \int_{\mathbb{B}} \langle (AF)(z), H(z)\rangle \, d\iota(z)$$

with $\|\Theta\| = \|H\|_q$. Let $G = M_A^{-1}H$. Then $G \in L^q_{\nu,s}$ and

$$\Phi(F) = \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z)F(z), G(z)\right\rangle (1-|z|^2)^{2\nu} \, d\iota(z) .$$

Also $\|\Phi\| = \|G\|_{\nu,s,q}$. □

Theorem 3.2. *For $1 \le p < \infty$ we have $(\mathcal{H}^p_{\nu,s})^* = \mathcal{H}^q_{\nu,s}$, under the integral pairing*

$$\langle F, G\rangle_{\nu,s,2} = \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z)F(z), G(z)\right\rangle (1-|z|^2)^{2\nu} \, d\iota(z) ,$$

$F \in \mathcal{H}^p_{\nu,s}$, $G \in \mathcal{H}^q_{\nu,s}$, where $1/p + 1/q = 1$. Namely, for any bounded linear functional $\Phi : \mathcal{H}^p_{\nu,s} \to \mathbb{C}$ there is a function $G \in \mathcal{H}^q_{\nu,s}$ such that $\Phi(F) = \langle F, G\rangle_{\nu,s,2}$ for all $F \in \mathcal{H}^p_{\nu,s}$ with

$$C\|G\|_{\nu,s,q} \le \|\Phi\| \le \|G\|_{\nu,s,q} .$$

Proof. By Hölder's inequality, every function $G \in \mathcal{H}^q_{\nu,s}$ defines a bounded linear functional Φ on $\mathcal{H}^p_{\nu,s}$ under the above integral pairing with $\|\Phi\| \le \|G\|_{\nu,s,q}$.

Conversely, let $\Phi \in \left(\mathcal{H}^p_{\nu,s}\right)^*$. By the Hahn-Banach theorem we can extend Φ to a bounded linear functional $\tilde{\Phi}$ on $L^p_{\nu,s}$ such that $\Phi(F) = \tilde{\Phi}(F)$ for all $F \in \mathcal{H}^p_{\nu,s}$ with $\|\Phi\| = \|\tilde{\Phi}\|$. By Lemma 3.1 there is a function $H \in L^q_{\nu,s}$ such that

$$\tilde{\Phi}(F) = \int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z)F(z), H(z)\right\rangle (1-|z|^2)^{2\nu} \, d\iota(z) \tag{12}$$

for all $F \in L^p_{\nu,s}$, with $\|\tilde{\Phi}\| = \|H\|_{\nu,s,q}$. However, Theorem 2.1 implies that, for any $F \in \mathcal{H}^p_{\nu,s}$,

$$F(z) = (P_{\nu,s}F)(z) = c \int_{\mathbb{B}} (1-|w|^2)^{2\nu} K_{\nu,s}(w,z)^* \otimes^s B^t(w,w) F(w)\, d\iota(w)\,.$$

Substituting this into formula (12) and using Fubini-Tonelli's theorem we get that

$$\Phi(F) = \tilde{\Phi}(F) = \int_{\mathbb{B}} \left\langle \otimes^s B^t(w,w)F(w), (P_{\nu,s}H)(w) \right\rangle (1-|w|^2)^{2\nu}\, d\iota(w)\,.$$

Let $G = P_{\nu,s}H$. By Proposition 2.3, $\|P_{\nu,s}H\|_{\nu,s,q} \leq C'\|H\|_{\nu,s,q}$. Then $G \in \mathcal{H}^q_{\nu,s}$, $\Phi(F) = \langle F, G\rangle_{\nu,s,2}$ for all $F \in \mathcal{H}^p_{\nu,s}$ and $C\|G\|_{\nu,s,q} \leq \|\Phi\|$. □

4. Atomic decomposition of $\mathcal{H}^1_{\nu,s}$

Following [JPR], we denote by $l^1(\mathbb{B}, \odot^s V')$ the space of all functions $a : \mathbb{B} \to \odot^s V'$, with support in $\{z_j\}_{j=1}^{\infty} \subset \mathbb{B}$, such that

$$\|a\|_{l^1} = \sum_{j=1}^{\infty} \|a(z_j)\| < \infty\,.$$

Also, denote by $l^\infty(\mathbb{B}, \odot^s V')$ the space of all functions $a : \mathbb{B} \to \odot^s V'$ such that

$$\|a\|_{l^\infty} = \sup_{z\in\mathbb{B}} \|a(z)\| < \infty\,.$$

Then it is elementary that

$$l^\infty(\mathbb{B}, \odot^s V') = \left(l^1(\mathbb{B}, \odot^s V')\right)^*\,, \tag{13}$$

under the pairing

$$\langle a, b\rangle' = \sum_{j=1}^{\infty} \langle a(z_j), b(z_j)\rangle$$

where $a \in l^1(\mathbb{B}, \odot^s V')$ with support $\{z_j\}_{j=1}^{\infty} \subset \mathbb{B}$ and $b \in l^\infty(\mathbb{B}, \odot^s V')$. Namely, for any bounded linear functional $\Phi : l^1(\mathbb{B}, \odot^s V') \to \mathbb{C}$ there is a function b in $l^\infty(\mathbb{B}, \odot^s V')$ such that $\Phi(a) = \langle a, b\rangle'$ for all $a \in l^1(\mathbb{B}, \odot^s V')$ with $\|\Phi\| = \|b\|_{l^\infty}$.

Theorem 4.1. *It follows that $F \in \mathcal{H}^1_{\nu,s}$ if and only if there is a sequence $\{z_j\}_{j=1}^{\infty} \subset \mathbb{B}$ and a sequence $\{a_j\}_{j=1}^{\infty} \in l^1(\mathbb{B}, \odot^s V')$ such that*

$$F(w) = \sum_{j=1}^{\infty} (1-|z_j|^2)^{\nu} K_{\nu,s}(w, z_j) \otimes^s B^t(z_j, z_j)^{1/2} a_j\,.$$

Proof. By Proposition 2.5, for any $v \in \odot^s V'$ and any $z \in \mathbb{B}$,

$$\left\|K_{\nu,s}(\cdot,z)\otimes^s B^t(z,z)^{1/2}v\right\|_{\nu,s,1} \leq C_s(1-|z|^2)^{-\nu}\|v\|.$$

Thus, the operator $T : l^1(\mathbb{B},\odot^s V') \to \mathcal{H}^1_{\nu,s}$ defined by

$$(Ta)(w) = \sum_{j=1}^{\infty}(1-|z_j|^2)^{\nu}K_{\nu,s}(w,z_j)\otimes^s B^t(z_j,z_j)^{1/2}a_j$$

is bounded, where $a_j = a(z_j)$ and the support of a is $\{z_j\}_{j=1}^{\infty}$. We need to prove that T is onto. Consider $T^* : \left(\mathcal{H}^1_{\nu,s}\right)^* \to \left(l^1(\mathbb{B},\odot^s V')\right)^*$, $T^*(\Phi)(a) = \Phi(Ta)$, which is bounded, where $\Phi \in \left(\mathcal{H}^1_{\nu,s}\right)^*$ and $a \in l^1(\mathbb{B},\odot^s V')$. By Theorem 3.2, for any $\Phi \in \left(\mathcal{H}^1_{\nu,s}\right)^*$ there is a $G \in \mathcal{H}^{\infty}_{\nu,s}$ such that $\Phi(F) = \langle F,G\rangle_{\nu,s,2}$ for all $F \in \mathcal{H}^1_{\nu,s}$ with $C\|G\|_{\nu,s,\infty} \leq \|\Phi\| \leq \|G\|_{\nu,s,\infty}$. Now, let $a \in l^1(\mathbb{B},\odot^s V')$ with support $\{z_j\}_{j=1}^{\infty} \subset \mathbb{B}$. By the reproducing property in Theorem 2.1 it follows that

$$\begin{aligned} T^*(\Phi)(a) &= \Phi(Ta) \\ &= \langle Ta, G\rangle_{\nu,s,2} \\ &= c\sum_{j=1}^{\infty}\langle a_j,(1-|z_j|^2)^{\nu}\otimes B^t(z_j,z_j)G(z_j)\rangle. \end{aligned}$$

Hence, by (13) and Theorem 3.2 it follows that

$$\begin{aligned} &\frac{1}{c}\cdot\|T^*\Phi\|_{(l^1)^*} \\ &\quad= \sup_{z\in\mathbb{B}}\left\|(1-|z|^2)^{\nu}\otimes^s B^t(z,z)G(z)\right\| = \|G\|_{\nu,s,\infty} \geq \|\Phi\|. \end{aligned} \tag{14}$$

On the one hand, (14) yields that $\ker T^* = \{0\}$ and consequently the range of T is dense in $\mathcal{H}^1_{\nu,s}$. On the other hand, (14) yields that the range of T^* is closed and so is the range of T by the Closed Range Theorem. ☐

5. Trace class $\mathcal{S}_1$

We consider now the trace class property of H^s_F in (11).

Theorem 5.1. *The Hankel form H^s_F is of trace class $\mathcal{S}_1$ if and only if $F \in \mathcal{H}^1_{\nu,s}$.*

Combining the results in [Su] we have now a complete characterization of the Schatten-von Neumann class Hankel forms.

Theorem 5.2. *The Hankel form H_F^s is of Schatten-von Neumann class $\mathcal{S}_p$ if and only if $F \in \mathcal{H}^p_{\nu,s}$, $1 \le p \le \infty$.*

Proof of Theorem 5.2. It follows from Lemma 5.5 below and Theorem 1.1(a) in [Su] that the operator $\Gamma : F \to H_F^s$ is bounded from $\mathcal{H}^1_{\nu,s}$ into $\mathcal{S}_1$ and from $\mathcal{H}^\infty_{\nu,s}$ into $\mathcal{S}_\infty$, respectively. Since $\mathcal{S}_p = (\mathcal{S}_1, \mathcal{S}_\infty)_{[1-1/p]}$ if $1 < p < \infty$, then it follows by Riesz-Thorin's interpolation theorem and Corollary 2.4 that Γ is bounded from $\mathcal{H}^p_{\nu,s}$ into $\mathcal{S}_p$ if $1 < p < \infty$.

On the other hand, it follows from Lemma 5.6 below and Theorem 1.1(a) in [Su] that $\tilde{\mathcal{T}}_s$, defined in (16), is bounded from $\mathcal{S}_1$ into $\mathcal{H}^1_{\nu,s}$ and from $\mathcal{S}_\infty$ into $\mathcal{H}^\infty_{\nu,s}$, respectively. Again, by interpolation $\tilde{\mathcal{T}}_s$ is bounded from $\mathcal{S}_p$ into $\mathcal{H}^p_{\nu,s}$ if $1 < p < \infty$. Also, if $H_F^s \in \mathcal{S}_p$ for $1 \le p < \infty$, then $\tilde{\mathcal{T}}_s(H_F^s) = F$, which follows by the same arguments as in the proof of Lemma 8.6 in [Su]. ☐

The proof of Theorem 5.1 will be divided into a few lemmas. We will first show in Lemma 5.3 that every H_F^s is of trace class $\mathcal{S}_1$ if F is in $\mathcal{H}^1_{\nu,s}$ and then in Lemma 5.4 that $\mathcal{H}^1_{\nu,s}$ can be continuously embedded into $\mathcal{H}^\infty_{\nu,s}$. Using these results we prove, in Lemma 5.5, that $F \to H_F^s$ is bounded from $\mathcal{H}^1_{\nu,s}$ into $\mathcal{S}_1$. Finally, in Lemma 5.6 we find a bounded mapping $\tilde{\mathcal{T}}_s$ from the trace class $\mathcal{S}_1$ into $\mathcal{H}^1_{\nu,s}$ such that $\tilde{\mathcal{T}}_s(H_F^s) = F$.

Lemma 5.3. *If $F \in \mathcal{H}^1_{\nu,s}$, then $H_F^s \in \mathcal{S}_1$.*

Proof. Let $F \in \mathcal{H}^1_{\nu,s}$. By Theorem 4.1, $F = \sum_{j=1}^\infty F_j$ where

$$F_j(w) = (1 - |z_j|^2)^\nu K_{\nu,s}(w, z_j) \otimes^s B^t(z_j, z_j)^{1/2} a_j$$

for some $\{z_j\}_{j=1}^\infty \subset \mathbb{B}$ and some $\{a_j\}_{j=1}^\infty \in l^1(\mathbb{B}, \odot^s V')$. We claim that

$$\text{rank}\, H^s_{F_j} \le M_s \qquad \text{for all } j = 1, 2, 3, \dots, \tag{15}$$

where M_s depends only on s and d. Accepting temporarily the claim and using Theorem 1.1(a) in [Su] we get that

$$\begin{aligned} \|H_F^s\|_{\mathcal{S}_1} \le \sum_{j=1}^\infty \|H^s_{F_j}\|_{\mathcal{S}_1} &\le M_s \sum_{j=1}^\infty \|H^s_{F_j}\|_{\mathcal{S}_\infty} \\ &\le M_s' \sum_{j=1}^\infty \|F_j\|_{\nu,s,\infty} \le M_s'' \sum_{j=1}^\infty \|a_j\| \,. \end{aligned}$$

Now we go back to claim (15). By Lemma 2.7, $\mathcal{T}_s(f,g) \in \mathcal{H}^1_{\nu,s}$ for all $f, g \in L^2_a(d\iota_\nu)$. Thus, by the reproducing property in Theorem 2.1,

$$H^s_{F_j}(f,g) = c\left\langle \mathcal{T}_s(f,g)(z_j), (1-|z_j|^2)^\nu \otimes^s B^t(z_j,z_j)^{1/2} a_j \right\rangle .$$

Fix $z_0 \in \mathbb{B}$. Then $\mathcal{T}_s(f,g)(z_0)$ is a sum of finitely many rank one forms where the number M_s of summands depends only on s and d. To see this, we consider $f(z_0) = \langle f, K_{z_0}\rangle_\nu$. Since

$$\partial^k f(z_0) \otimes \partial^{s-k} g(z_0) = \langle f, \overline{\partial^k} K_{z_0}\rangle_\nu \otimes \langle g, \overline{\partial^{s-k}} K_{z_0}\rangle_\nu ,$$

then $(f,g) \to \partial^{s-k} f(z_0) \otimes \partial^k g(z_0)$ is a rank one form. Thus, the bilinear form $(f,g) \to \mathcal{T}_s(f,g)(z_0)$ has rank at most M_s and so has H_{F_j}. □

Lemma 5.4. *The operator* $\mathcal{I} : \mathcal{H}^1_{\nu,s} \to \mathcal{H}^\infty_{\nu,s}$, $\mathcal{I}(F) = F$, *is bounded.*

Proof. First, let $F \in \mathcal{H}^1_{\nu,s}$. Then $H^s_F \in \mathcal{S}_1$ by Lemma 5.3. Hence $H^s_F \in \mathcal{S}_\infty$, so by Theorem 1.1(a) in [Su] it follows that $F \in \mathcal{H}^\infty_{\nu,s}$. Thus $\mathcal{I}$ is well-defined.

Now, assume that $F_n \to F$ in $\mathcal{H}^1_{\nu,s}$ and that $\mathcal{I}(F_n) \to G$ in $\mathcal{H}^\infty_{\nu,s}$. We shall prove that $\mathcal{I}(F) = G$. On the one hand, since $F_n \to F$ in $\mathcal{H}^1_{\nu,s}$, then there is a subsequence $\mathcal{I}(F_{n_j})$ converging pointwise to $\mathcal{I}(F)$. On the other hand, since $\mathcal{I}(F_n) \to G$ in $\mathcal{H}^\infty_{\nu,s}$, then $\mathcal{I}(F_{n_j}) \to G$ pointwise. Thus $\mathcal{I}(F) = G$ and the operator $\mathcal{I}$ is bounded by the Closed Graph Theorem. □

Lemma 5.5. *The operator* $\Gamma : \mathcal{H}^1_{\nu,s} \to \mathcal{S}_1$, $\Gamma(F) = H^s_F$, *is bounded.*

Proof. The operator Γ is well defined by Lemma 5.3. We use the Closed Graph Theorem. Assume that $F_n \to F$ in $\mathcal{H}^1_{\nu,s}$ and that $\Gamma(F_n) \to B$ in $\mathcal{S}_1$. We shall prove that $H^s_F = B$. On one hand, by Theorem 1.1(a) in [Su] and Lemma 5.4 it follows that

$$\|H^s_{F_n - F}\|_{\mathcal{S}_\infty} \leq C\|F_n - F\|_{\nu,s,\infty} \leq C'\|F_n - F\|_{\nu,s,1}$$

so that $H^s_{F_n} \to H^s_F$ in $\mathcal{S}_\infty$. On the other hand,

$$\|\Gamma(F_n) - B\|_{\mathcal{S}_\infty} \leq \|\Gamma(F_n) - B\|_{\mathcal{S}_1}$$

so that $H^s_{F_n} \to B$ in $\mathcal{S}_\infty$. Thus $H^s_F = B$ so that Γ has the closed graph property. Hence, Γ is bounded. □

We recall the transvectant $\tilde{\mathcal{T}}_s : \mathcal{S}_\infty\left(L^2_a(d\iota_\nu), L^2_a(d\iota_\nu)\right) \to \mathcal{A}_s(\mathbb{B} \times \mathbb{B})$ defined in [Su] (see also [FR] and [PZ]), where $\mathcal{A}_s(\mathbb{B}\times\mathbb{B})$ consists of all holomorphic functions $G : \mathbb{B} \times \mathbb{B} \to \odot^s V'$. We recall further that the transvectant $\mathcal{T}_s$ in (10) can be defined for any holomorphic function $G(z,w)$ on $\mathbb{B}\times\mathbb{B}$, namely

$$(\mathcal{T}_s G)(z,w) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\partial_z^k \odot \partial_w^{s-k} G(z,w)}{(\nu)_k (\nu)_{s-k}} .$$

For bounded bilinear forms A on $L_a^2(d\iota_\nu)$, we define

$$\tilde{\mathcal{T}}_s(A)(z) = (\mathcal{T}_s G)(z,z) , \tag{16}$$

where $G(z,w) = \overline{A(K_z, K_w)}$.

Lemma 5.6. *The operator $\tilde{\mathcal{T}}_s : \mathcal{S}_1 \to \mathcal{H}^1_{\nu,s}$ defined in (16) is bounded. Also, $\tilde{\mathcal{T}}_s(H_F^s) = F$ if $H_F^s \in \mathcal{S}_1$.*

Proof. First, let $B \in \mathcal{S}_1$ be of rank one. Then there exists $\phi, \varphi \in L_a^2(d\iota_\nu)$ such that

$$B(f,g) = \langle f, \phi\rangle_\nu \langle g, \varphi\rangle_\nu$$

for all $f, g \in L_a^2(d\iota_\nu)$. Then $\|B\|_{\mathcal{S}_1} = \|\phi\|_\nu \|\varphi\|_\nu$ and $\tilde{\mathcal{T}}_s(B)(z) = \mathcal{T}_s(\phi,\varphi)(z)$, so by Lemma 2.7 it follows that

$$\|\tilde{\mathcal{T}}_s(B)\|_{\nu,s,1} \le C_s \|\phi\|_\nu \|\varphi\|_\nu \le C_s \|B\|_{\mathcal{S}_1} . \tag{17}$$

In general, if $B \in \mathcal{S}_1$ we can write $B = \sum_{n=1}^\infty B_n$, $\operatorname{rank} B_n = 1$ such that

$$\|B^N\|_{\mathcal{S}_1} = \sum_{n=1}^{N} \|B_n\|_{\mathcal{S}_1} \to \|B\|_{\mathcal{S}_1} \text{ , as } N \to \infty ,$$

where $B^N = \sum_{n=1}^N B_n$. By (17) the sequence $\{\tilde{\mathcal{T}}_s(B^N)\}_{n=1}^\infty$ is Cauchy and hence converges to some G in $\mathcal{H}^1_{\nu,s}$. Now, since $B^N \to B$ in $\mathcal{S}_\infty$ it follows by Lemma 8.4 in [Su] that $\tilde{\mathcal{T}}_s(B^N) \to \tilde{\mathcal{T}}_s(B)$ in $\mathcal{H}^\infty_{\nu,s}$. Hence $\tilde{\mathcal{T}}_s(B) = G$ so that (17) holds for any $B \in \mathcal{S}_1$.

Also, if $H_F^s \in \mathcal{S}_1$, then $\tilde{\mathcal{T}}_s(H_F^s) = F$. (As in the proof of Theorem 5.2 we refer to the proof of Lemma 8.6 in [Su].) □

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Paper III

BILINEAR HANKEL FORMS OF HIGHER WEIGHTS ON HARDY SPACES

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Abstract. In this paper we study bilinear Hankel forms of higher weights on Hardy spaces in several dimensions (see [Su1] and [Su2] for Hankel forms of higher weights on weighted Bergman spaces). For the case of weight zero we get a full characterization of $\mathcal{S}_p$ class Hankel forms, $1 \leq p < \infty$, in terms of the membership for the symbols to be in certain Besov spaces. Also, in this case, if a Hankel form is bounded, then the symbol satisfies a certain Carleson measure criterion. For the case of higher weights, we find sufficient criteria for Hankel forms to be in class $\mathcal{S}_p$, $1 \leq p \leq 2$.

1. Introduction

Schatten-von Neumann class Hankel forms of higher weights on Bergman spaces are characterized in [Su1] and [Su2]. In the same way, as for the case of Bergman spaces, Hankel forms of higher weights on a Hardy space are explicit characterizations of irreducible components in the tensor product of Hardy spaces under the Möbius group, see [PZ].

In this paper we use the same notations as in [Su1] and [Su2]. Now, let $\partial\mathbb{B}$ be the boundary of the unit ball $\mathbb{B}$ of $\mathbb{C}^d$. We denote by H_F^s the *bilinear Hankel forms of weight s* on the Hardy space $H^2(\partial\mathbb{B})$ if

$$H_F^s(f,g) = \int_{\mathbb{B}} \left\langle \otimes^m B^t(z,z)\mathcal{T}_s(f,g)(z), F(z) \right\rangle (1-|z|^2)^{d-1}\, dm(z)\,, \tag{1}$$

where $\mathcal{T}_s$ is the transvectant given by

$$\mathcal{T}_s(f,g)(z) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\partial^k f(z) \odot \partial^{s-k} g(z)}{(d)_k (d)_{s-k}}\,,$$

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and $(d)_k = d(d+1)\cdots(d+k-1)$ is the Pochammer symbol. The tensor-valued holomorphic function F is called the *symbol* corresponding to the Hankel form H_F^s. In fact, this is the limiting case $\nu = d$ of (7) in [Su1].

In Section 2 we establish the Schatten-von Neumann class criteria for bilinear Hankel forms of weight zero. In this case we get a full characterization of $\mathcal{S}_p$ class Hankel forms, $1 \leq p < \infty$, in terms of the membership for the symbols in certain Besov spaces. Also a sufficient criterion for boundedness, in terms of Carleson measures, is presented there. The main theorems in Section 2 are Theorem 2.5 and Theorem 2.16. In section 3 we study the case of higher weight. Here a new difficulty appears. The transvectant does not behave in the same way as for the case of Bergman spaces, see Example 3.5. Therefore we cannot generalize the techniques used in [Su1] to find boundedness and compactness criteria, but we establish sufficient criteria for Hankel forms of nonzero weight to be of class $\mathcal{S}_p$, $1 \leq p \leq 2$; see Theorem 3.9.

Notation. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on a vector space X, then we write $\|x\|_1 \simeq \|x\|_2$, $x \in X$. Also, for two functions f and g we write $f \lesssim g$ if there is a constant $C > 0$, independent of the variables in questions, such that $Cf(z) \leq g(z)$.

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2. Hankel forms of weight zero

To find the Schatten-von Neumann class Hankel forms of weight zero on Hardy spaces we shall rewrite H_F^0 in terms of the small Hankel operators studied in [Z]. The problem then boils down to finding the relationship between the corresponding symbols.

The Hankel form, H_G, in [Z] is given by

$$H_G(f,g) = \int_{\partial\mathbb{B}} \overline{G(w)} f(w) g(w)\, d\sigma(w)\,, \tag{2}$$

where $d\sigma$ is the normalized Lebesgue measure on $\partial\mathbb{B}$. By the reproducing property on the Hardy space $H^2(\partial\mathbb{B})$ we have the following relationship between F and G.

Lemma 2.1. *Let H_F^0 be given by (1) and H_G by (2). Then $H_F^0 = H_G$, if and only if*

$$(R+d)_d G(w) = c_d (d)_d F(w)\,, \tag{3}$$

where c_d is a normalization constant for dm on $\mathbb{B}$.

Proof. Let $f, g \in H^2(\partial\mathbb{B})$. Using the reproducing property of fg and Fubini-Tonelli's theorem,

$$\int_{\mathbb{B}} f(z)g(z)\overline{F(z)}(1-|z|^2)^{d-1}\,dm(z) \\ = \int_{\partial\mathbb{B}} f(w)g(w)\int_{\mathbb{B}} \frac{\overline{F(z)}(1-|z|^2)^{d-1}}{(1-\langle z,w\rangle)^d}\,dm(z)\,d\sigma(w)\,.$$

Hence $H_F^0 = H_G$ if and only if

$$G(w) = \int_{\mathbb{B}} \frac{F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^d}\,dm(z)\,. \tag{4}$$

Apply the radial differentiation R,

$$\begin{aligned} &(RG)(w) \\ = &\sum_{i=1}^{d} w_i \frac{\partial G}{\partial w_i}(w) = d\int_{\mathbb{B}} \frac{\langle w,z\rangle F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^{d+1}}\,dm(z) \\ = &-d\int_{\mathbb{B}} \frac{F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^d}\,dm(z) + d\int_{\mathbb{B}} \frac{F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^{d+1}}\,dm(z)\,, \end{aligned}$$

so that

$$(R+d)G(w) = d\int_{\mathbb{B}} \frac{F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^{d+1}}\,dm(z)\,.$$

Repeating this procedure,

$$(R+d)_d G(w) = (d)_d \int_{\mathbb{B}} \frac{F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^{2d}}\,dm(z)\,.$$

Hence, by the reproducing property on the Bergman space $L_a^2(dm)$, equation (4) can be reformulated into

$$(R+d)_d G(w) = c_d (d)_d F(w)\,, \tag{5}$$

where c_d is a normalization constant for dm on $\mathbb{B}$. On the other hand, if $F(w) = (R+d)_d G(w)/c_d(d)_d$ then equation (4) holds by symmetry of $(R+a)$, $a>0$, w.r.t. the inner product

$$\int_{\mathbb{B}} h_1(z)\overline{h_2(z)}(1-|z|^2)^{d-1}\,dm(z)\,.$$

□

Remark 2.2. For convenience we denote $D = R+1$. Then D is symmetric w.r.t. the inner product

$$\langle h_1, h_2\rangle_{\alpha'} = \int_{\mathbb{B}} h_1(z)\overline{h_2(z)}(1-|z|^2)^{\alpha'}\,dm(z)$$

where $\alpha' > -1$ and $h_1, h_2 : \mathbb{B} \to \mathbb{C}$ are holomorphic.

2.1. **Schatten-von Neumann class $\mathcal{S}_p$ Hankel forms.** In this subsection we present sufficient and necessary conditions for Hankel forms of weight zero to be in Schatten-von Neumann class $\mathcal{S}_p$, $1 \le p < \infty$ (see Theorem 2.5) and the following lemmas are useful in the proof this theorem.

Lemma 2.3. *Let $a_1, \cdots a_k, b_1, \cdots, b_k > 0$, $\alpha > -1$ and $1 < p < \infty$. Then*

$$\|(R+a_k)\cdots(R+a_1)f\|_{\alpha,p} \simeq \|(R+b_k)\cdots(R+b_1)f\|_{\alpha,p}$$

for all holomorphic $f : \mathbb{B} \to \mathbb{C}$, where $\|f\|_{\alpha,p} = \|f\|_{L^p((1-|z|^2)^\alpha dm(z))}$.

Proof. This result follows using the same arguments as in the proof of Theorem 5.3 in [BB]. □

Lemma 2.4. *If $\alpha > -1$ and $1 < p < \infty$, then*

$$\left\|\big((R+\alpha+d+1)f\big)(\cdot)(1-|\cdot|^2)\right\|_{\alpha,p} \simeq \|f\|_{\alpha,p}\ ,$$

for all holomorphic $f : \mathbb{B} \to \mathbb{C}$.

Proof. If $\beta > 0$, then

$$\frac{1}{\beta}\left((R+\beta)\left(\frac{1}{(1-\langle\cdot,w\rangle)^\beta}\right)\right)(z) = \frac{1}{(1-\langle z,w\rangle)^{\beta+1}}\,,$$

and hence the result follows by using the same arguments as in the proof of Theorem 2.19 in [Zhu1]. □

Theorem 2.5. *The Hankel form H_F^0 is of Schatten-von Neumann class $\mathcal{S}_p$, for $1 < p < \infty$, if and only*

$$\|F(\cdot)(1-|\cdot|^2)^d\|_{L^p(d\iota)} = \left(\int_{\mathbb{B}} \left|F(z)(1-|z|^2)^d\right|^p\,d\iota(z)\right)^{1/p} < \infty\,.$$

Also, H_F^0 is of trace class $\mathcal{S}_1$ if and only if $DF \in L^1(dm)$.

Remark 2.6. The measure $d\iota(z) = (1 - |z|^2)^{-(d+1)}\, dm(z)$ is a Möbius invariant measure on $\mathbb{B}$.

To prove the theorem we use Theorem 1 in [Z] (see also Theorem C in [FeldR]) given below.

Theorem 2.7. *Let $\alpha > -1$ and $1 \le p < \infty$. Then the Hankel form H_G, defined by (2), is of Schatten-von Neumann class $\mathcal{S}_p$ if and only if*

$$\sum_{|\alpha|=d+1} \left\| (\partial^\alpha G(\cdot))\,(1 - |\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} < \infty.$$

Proof of Theorem 2.5. We shall make use of the fact that $H_F^0 = H_G$ if and only if F and G satisfies equation (3), which follows by Lemma 2.1. Then $DF(z) = c_d D(R+d)_d G(z)$. In view of Theorem 2.7, it is enough to prove that, for $1 < p < \infty$,

$$\left\| F(\cdot)(1 - |\cdot|^2)^d \right\|_{L^p(d\iota)} \simeq \left\| (DF(\cdot))\,(1 - |\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} \tag{6}$$

and that, for $1 \le p < \infty$,

$$\begin{aligned} &\left\| \left(D^{d+1} G(\cdot)\right)(1 - |\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} \\ &\quad \simeq \sum_{|\alpha|=d+1} \left\| (\partial^\alpha G(\cdot))\,(1 - |\cdot|^2)^{d+1} \right\|_{L^P(d\iota)} + \sum_{|\alpha|\le d} \left| (\partial^\alpha G)(0) \right|, \end{aligned} \tag{7}$$

since then it will follow by Lemma 2.3 that

$$\begin{aligned} &\left\| F(\cdot)(1 - |\cdot|^2)^d \right\|_{L^p(d\iota)} \\ &\quad \simeq \left\| (DF(\cdot))\,(1 - |\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} \\ &\quad \simeq \left\| \left(D^{d+1} G(\cdot)\right)(1 - |\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} \\ &\quad \simeq \sum_{|\alpha|=d+1} \left\| (\partial^\alpha G(\cdot))\,(1 - |\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} + \sum_{|\alpha|\le d} \left| (\partial^\alpha G)(0) \right|. \end{aligned}$$

Actually, (6) is a direct consequence of Lemma 2.3 and Lemma 2.4, and (7) is a consequence of Theorem 5.3 in [BB]. □

2.2. **Bounded Hankel forms.** In this subsection we present a necessary condition for Hankel forms of weight zero to be bounded; see Theorem 2.16. First we need some preliminaries, which basically can be found in [Zhu1]. We also remark that equivalence in Lemma 2.12 holds in the one dimensional case, due to Corollary 15 in [Zhu2].

Definition 2.8 (See [Zhu1])**.** Let $\zeta \in \partial\mathbb{B}$ and $r > 0$ and let

$$Q_r(\zeta) = \{z \in \mathbb{B} : d(z, \zeta) < r\}$$

where $d(z,\zeta) = |1 - \langle z, \zeta\rangle|^{1/2}$ is the non-isotropic metric on $\partial\mathbb{B}$. A positive Borel measure μ in $\mathbb{B}$ is called a *Carleson* measure if there exists a constant $C > 0$ such that

$$\mu(Q_r(\zeta)) \leq Cr^{2d},$$

for all $\zeta \in \partial\mathbb{B}$ and $r > 0$.

Lemma 2.9 (Theorem 5.4 in [Zhu1])**.** *A positive Borel measure μ in $\mathbb{B}$ is Carleson if and only if*

$$\sup_{z\in\mathbb{B}} \int_{\mathbb{B}} P(z,w)\, d\mu(w) < \infty,$$

where

$$P(z,w) = \frac{(1-|z|^2)^d}{|1-\langle z,w\rangle|^{2d}}; \quad z, w \in \mathbb{B}.$$

Lemma 2.10 (Theorem 5.9 in [Zhu1])**.** *A positive Borel measure μ in $\mathbb{B}$ is Carleson if and only if there exists a constant $C > 0$ such that*

$$\int_{\mathbb{B}} |f(z)|^2\, d\mu(z) \leq C\|f\|^2_{H^2(\partial\mathbb{B})},$$

for all $f \in H^2(\partial\mathbb{B})$.

Lemma 2.11 (Theorem 50 in [ZhZh])**.** *Let μ be a positive Borel measure in $\mathbb{B}$. Then the following conditions are equivalent*

(a) *There is a constant $C > 0$ such that*

$$\int_{\mathbb{B}} |(Rf)(z)|^2 d\mu(z) \leq C\|f\|^2_{H^2(\partial\mathbb{B})},$$

for all $f \in H^2(\partial\mathbb{B})$.

(b) *There is a constant $C > 0$ such that*

$$\mu(Q_r(\zeta)) \leq Cr^{2(d+2)},$$

for all $\zeta \in \partial\mathbb{B}$ and $r > 0$.

Lemma 2.12. *Let $\alpha > -1$. For any holomorphic function $g : \mathbb{B} \to \mathbb{C}$, if $d\mu_1(z) = |g(z)|^2(1-|z|^2)^\alpha \, dm(z)$ is a Carleson measure then so is $d\mu_2(z) = |Rg(z)|^2(1-|z|^2)^{\alpha+2} \, dm(z)$.*

Proof. If $d\mu_1$ is a Carleson measure, then there is a constant $C > 0$ such that

$$\int_{Q_r(\zeta)} (1-|z|^2)^2 \, d\mu_1(z) \leq 4r^4 \int_{Q_r(\zeta)} d\mu_1(z) \leq 4Cr^{2(d+2)} ,$$

for all $\zeta \in \mathbb{B}$ and $r > 0$, so that $(1-|z|^2)^2 d\mu_1(z)$ satisfies the condition (b) in Lemma 2.11. Hence there is a constant $C_1 > 0$ such that

$$\int_{\mathbb{B}} |(Rf)(z)|^2(1-|z|^2)^2 \, d\mu_1(z) \leq C_1 \|f\|_{H^2(\partial\mathbb{B})} , \tag{8}$$

for all $f \in H^2(\partial\mathbb{B})$. By Theorem 2.16 in [Zhu1] (used on fg, assuming, without loss of generality, that $f(0) = 0$) and by the inequality (8),

$$\begin{aligned} &\left(\int_{\mathbb{B}} |f(z)|^2 \, d\mu_2(z)\right)^{1/2} \\ &\leq \left(\int_{\mathbb{B}} |(R(fg))(z)|^2(1-|z|^2)^{\alpha+2} \, dm(z)\right)^{1/2} + \\ &\qquad\qquad \left(\int_{\mathbb{B}} |(Rf)(z)|^2(1-|z|^2)^2 \, d\mu_1(z)\right)^{1/2} \\ &\leq C_2 \left(\int_{\mathbb{B}} |f(z)|^2 \, d\mu_1(z)\right)^{1/2} + C_1\|f\|_{H^2(\partial\mathbb{B})} \leq C_3\|f\|_{H^2(\partial\mathbb{B})} , \end{aligned}$$

for all $f \in H^2(\partial\mathbb{B})$, so that $d\mu_2$ is Carleson by Lemma 2.10. □

Definition 2.13 (See [Zhu1])**.** Let BMOA denote the space of functions $f \in H^2(\partial\mathbb{B})$ such that

$$\|f\|_{BMO}^2 = |f(0)|^2 + \sup_{Q(\zeta,r)} \frac{1}{Q(\zeta,r)} \int_{Q(\zeta,r)} |f(\xi) - f_{Q(\zeta,r)}|^2 \, d\sigma(\xi) < \infty ,$$

where, for any $\zeta \in \partial\mathbb{B}$ and $r > 0$,

$$Q(\zeta,r) = \left\{\xi \in \partial\mathbb{B} : |1 - \langle\zeta,\xi\rangle|^{1/2} < r\right\} ,$$

and

$$f_{Q(\zeta,r)} = \frac{1}{Q(\zeta,r)} \int_{Q(\zeta,r)} f(\xi) \, d\sigma(\xi) .$$

Lemma 2.14 (Theorem 5.3 in [Zhu1]). *A function $f \in H^2(\partial\mathbb{B})$ belongs to BMOA if and only if*

$$\sup_{z\in\mathbb{B}} \int_{\partial\mathbb{B}} |f(\varphi_z(\zeta)) - f(z)|^2 \, d\sigma(\zeta) < \infty ,$$

where φ_z is the linear fractional map given by (8) in [Su1].

Lemma 2.15. *If f is in BMOA, then $|f(z)|^2 \, dm(z)$ is a Carleson measure on $\mathbb{B}$.*

Proof. If f is in BMOA, then there is a constant $C > 0$ such that

$$\begin{aligned} &\sup_{z\in\mathbb{B}} \int_{\mathbb{B}} P(z,w)|f(w) - f(0)|^2 \, dm(w) \\ \leq \;& C \cdot \sup_{z\in\mathbb{B}} \int_{\partial\mathbb{B}} P(z,\zeta)|f(\zeta) - f(0)|^2 \, d\sigma(\zeta) \\ = \;& C \cdot \sup_{z\in\mathbb{B}} \int_{\partial\mathbb{B}} |f(\varphi_z(\zeta)) - f(z)|^2 \, d\sigma(\zeta) < \infty , \end{aligned}$$

by Lemma 2.14. Then $|f(w) - f(0)|^2 \, dm(w)$ is Carleson by Lemma 2.9, so that $|f(w)|^2 \, dm(w)$ is Carleson. ☐

Theorem 2.16. *If the Hankel form H_F^0 is bounded, then*

$$|F(z)|^2(1 - |z|^2)^{2d-1} \, dm(z)$$

is a Carleson measure on $\mathbb{B}$.

Proof. The classical Hankel form (small Hankel operator) H_G on the Hardy space $H^2(\partial\mathbb{B})$, as in [Z], is bounded if and only if $G \in BMOA$ and by Theorem 5.14 in [Zhu1],

$$G \in BMOA \quad \Longleftrightarrow \quad |(RG)(z)|^2(1 - |z|^2) \, dm(z) \text{ is Carleson} .$$

Now, $H_F^0 = H_G$ if and only if the equation (3) holds. Hence, if H_F^0 is bounded, then $|(RG)(z)|^2(1 - |z|^2) \, dm(z)$ is a Carleson measure and, since G is in BMOA, then $|((R+d)G)(z)|^2(1 - |z|^2) \, dm(z)$ is a Carleson measure by Lemma 2.15. Using Lemma 2.12,

$$|(R(R + d)G)(z)|^2(1 - |z|^2)^{2\cdot 2-1} \quad \text{is Carleson,}$$

and hence $|((R + d + 1)(R + d)G)(z)|^2(1 - |z|^2)^{2\cdot 2-1} \, dm(z)$ is Carleson. Repeating this procedure we get that

$$|((R + d)_d G)(z)|^2(1 - |z|^2)^{2d-1} \, dm(z) \quad \text{is Carleson.}$$

By equation (3), the proof is complete. ☐

3. The case $s = 1, 2, 3, \cdots$

In this section we study the class $\mathcal{S}_p$ properties, $1 \le p \le 2$, for the case $s \ge 1$. Denote by $\mathcal{H}^p_{d,s}$ the space of holomorphic functions $F : \mathbb{B} \to \odot^s V'$ such that $\|F\|_{d,s,p} < \infty$ where

$$\|F\|_{d,s,p} = \left(\int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z) F(z), F(z) \right\rangle^{p/2} (1 - |z|^2)^{pd} \frac{dm(z)}{(1 - |z|^2)^{d+1}} \right)^{1/p} .$$

This space is a well-defined Banach space if $1 \le p < \infty$. Also, denote by $\mathcal{H}^\infty_{d,s}$ the space of holomorphic functions such that

$$\|F\|_{d,s,\infty} = \sup_{z \in \mathbb{B}} \left\| (1 - |z|^2)^d \otimes^s B^t(z,z)^{1/2} F(z) \right\| < \infty .$$

3.1. **Results about $\mathcal{H}^p_{\nu,s}$ for $\nu \ge d$.** In [Su1] and in [Su2] there are several results about $\mathcal{H}^p_{\nu,s}$ for $s = 0, 1, 2, \cdots$ and $\nu > d$. Now, if we consider $s = 1, 2, \cdots$, i.e., $s \ne 0$, then we can use the same arguments as in [Su1] and [Su2] to generalize results about $\mathcal{H}^p_{\nu,s}$ for $\nu > d$ to $\nu \ge d$, where $1 \le p \le \infty$. Hence, the results below will be stated without proofs. The reader is referred to [Su1] and [Su2] for more details.

Lemma 3.1. *Let $\nu \ge d$ and let s be a positive integer. Then the reproducing kernel of $\mathcal{H}^2_{\nu,s}$ is, up to a nonzero constant c, given by*

$$K_{\nu,s}(w,z) = (1 - \langle w, z \rangle)^{-2\nu} \otimes^s B^t(w,z)^{-1} .$$

Namely, for any $v \in \odot^s V'$ and any $F \in \mathcal{H}^2_{\nu,s}$,

$$\begin{aligned} \langle F(z), v \rangle &= c \langle F, K_{\nu,s}(\cdot, z) v \rangle_{\nu,s,2} \\ &= c \int_{\mathbb{B}} \left\langle \otimes^s B^t(w,w) F(w), K_{\nu,s}(w,z) v \right\rangle (1 - |w|^2)^{2\nu} \, d\iota(w) . \end{aligned}$$

Let $\mathcal{H}'_{\nu,s}$ be the space of holomorphic functions $F : \mathbb{B} \to \odot^s V'$ such that the corresponding bilinear Hankel form on $H^2(\partial\mathbb{B}) \otimes H^2(\partial\mathbb{B})$, defined by (1), is of Hilbert-Schmidt class $\mathcal{S}_2$. The norm on $\mathcal{H}'_{\nu,s}$ is given by $\|F\|'_{\nu,s} = \|H^s_F\|_{\mathcal{S}_2}$.

Theorem 3.2. *Let $\nu \ge d$ and let s be a nonnegative integer. Then there is a constant $C_{\nu,s} > 0$ such that*

$$\|F\|'_{\nu,s} = C_{\nu,s} \|F\|_{\nu,s,2}$$

Theorem 3.3. *Let $\nu \geq d$ and let s be a positive integer. If $1 < p < 2$, then*

$$\mathcal{H}^p_{\nu,s} = (\mathcal{H}^1_{\nu,s}, \mathcal{H}^2_{\nu,s})_{[2(1-1/p)]} .$$

Theorem 3.4. *Let $\nu \geq d$ and let s be a positive integer. Then $F \in \mathcal{H}^1_{d,s}$ if and only if there is a function $a \in l^1(\mathbb{B}, \odot^s V')$ with support in $\{z_j\}_{j=1}^{\infty} \subset \mathbb{B}$, $a_j = a(z_j)$, with $\sum_{j=0}^{\infty} \|a_j\| < \infty$ such that*

$$F(w) = \sum_{j=1}^{\infty} (1 - |z_j|^2)^d K_{d,s}(w, z_j) \otimes^s B^t(z_j, z_j)^{1/2} a_j . \tag{9}$$

3.2. **The transvectant.** If we were able to prove that $\mathcal{T}_s(f,g) \in \mathcal{H}^1_{d,s}$, for positive s, that is generalize the analogous result for the case of Bergman spaces (see Lemma 2.7 in [Su2]), then boundedness properties and compactness properties would follow in the same way as for the case of Bergman spaces; see [Su1]. But, unfortunately, we can find $f, g \in H^2(\partial\mathbb{B})$ such that $\|\mathcal{T}_s(f,g)\|_{d,s,1} = \infty$.

Example 3.5. This example is based on the proof of Theorem II in [Ru]. First consider the case when $s = 1$ and $d = 1$. Let

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{2^k} \quad \text{and} \quad g(z) = 1 .$$

Then $f, g \in H^2(\partial\mathbb{D})$ and since the series $f(z)$ is lacunary then

$$\|\mathcal{T}_1(f,g)\|_{1,1,1} = \int_{\mathbb{D}} |f'(z)| \, dm(z) = \infty .$$

This is a consequence of a result about lacunary series by Zygmund; see [Ru]. Namely, if $n_{k+1}/n_k > \lambda$ for some $\lambda > 1$, and if $h(z) = \sum_{k=0}^{\infty} c_k z^{n_k}$ satisfies

$$\int_0^1 |h'(re^{i\theta})| \, dr < \infty$$

for some θ, then $\sum_{k=0}^{\infty} |c_k| < \infty$.

In the general case, $d \geq 1$ and $s = 1, 2, \cdots$, we just change f into

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z_1^{2^k} ,$$

and still let $g(z) = 1$. Then

$$\begin{aligned} \|\mathcal{T}_s(f,g)\|_{d,s,1} &= \int_{\mathbb{B}} (1-|z_1|^2)^{s/2}(1-|z|^2)^{s/2-1} \left| \frac{\partial^s f}{\partial z_1^s}(z) \right| \, dm(z) \\ &\geq \int_{\mathbb{B}} (1-|z|^2)^{s-1} \left| \frac{\partial^s f}{\partial z_1^s}(z) \right| \, dm(z) \, . \end{aligned}$$

By Theorem 2.17 in [Zhu1] there is a constant $C > 0$ such that

$$\int_{\mathbb{B}} (1-|z|^2)^{s-1} \left| \frac{\partial^s f}{\partial z_1^s}(z) \right| \, dm(z) \geq C \int_{\mathbb{B}} \left| \frac{\partial f}{\partial z_1}(z) \right| \, dm(z)$$

and the right hand side of the inequality above is infinite, as we can see in the initial case ($s = 1$, $d = 1$).

But, what we can prove is the following Lemma.

Lemma 3.6. *Let s be a nonnegative integer and let $\varepsilon > 0$. Then there is a constant $C_\varepsilon > 0$ such that*

$$\int_{\mathbb{B}} \left\| \otimes^s B^t(z,z)^{1/2} \mathcal{T}_s(f,g)(z) \right\| (1-|z|^2)^{\varepsilon-1} \, dm(z) \leq C_\varepsilon \cdot \|f\|_{H^2} \cdot \|g\|_{H^2} \, .$$

Proof. It follows by exactly the same arguments as in the proof of Theorem 4.1 in [Su1] that, for $k \neq 0$,

$$\left(\int_{\mathbb{B}} \left\langle \otimes^k B^t(z,z) \partial^k f(z), \partial^k f(z) \right\rangle \frac{dm(z)}{(1-|z|^2)} \right)^{1/2} \leq C_{d,k} \cdot \|f\|_{H^2} \, . \tag{10}$$

This yields the result, since $\mathcal{T}_s(f,g)$ is a linear combination of terms $\partial^k f(z) \otimes \partial^{s-k} g(z)$ and by Hölder's inequality

$$\left\| \otimes^s B^t(\cdot,\cdot)^{1/2} \partial^k f(\cdot) \otimes \partial^{s-k} g(\cdot) \right\|_{L^1((1-|z|^2)^{-1} dm)} \leq C_{d,s} \cdot \|f\|_{H^2} \cdot \|g\|_{H^2}$$

if $k \neq 0$ and, for $k = 0$,

$$\begin{aligned} & \int_{\mathbb{B}} \left\| \otimes^s B^t(z,z)^{1/2} g(z) \partial^s f(z) \right\| (1-|z|^2)^{\varepsilon-1} \, dm(z) \\ \leq\ & C_{d,s} \cdot \|f\|_{H^2} \cdot \left(\int_{\mathbb{B}} |g(z)|^2 (1-|z|^2)^{\varepsilon-1} \, dm(z) \right)^{1/2} \\ \leq\ & C_\varepsilon \cdot \|f\|_{H^2} \cdot \|g\|_{H^2} \, . \end{aligned}$$

□

3.3. Class $\mathcal{S}_p$ Hankel forms for $1 \le p \le 2$.

Theorem 3.7. *Let s be a positive integer. If $F \in \mathcal{H}^1_{d,s}$, then the corresponding Hankel form H^s_F is of class $\mathcal{S}_1$.*

Proof. Let $F \in \mathcal{H}^1_{d,s}$. Then, by Theorem 3.4 we can write

$$F(w) = \sum_{j=1}^{\infty} F_j(w)$$

where $F_j(w) = (1 - |z_j|^2)^d K_{d,s}(w, z_j) \otimes^s B^t(z_j, z_j)^{1/2} a_j$. As a consequence of Lemma 7.1 in [Su1],

$$\|F_j\|' = \sup_{w \in \mathbb{B}} \| \otimes^s B^t(w, w)^{1/2} F_j(w)\| \le 2^d (1 - |z_j|^2)^{-d} \|a_j\| \,. \tag{11}$$

By Lemma 3.6 and (11) it then follows that

$$\begin{aligned} &\int_{\mathbb{B}} \left|\left\langle \otimes^s B^t(w,w) \mathcal{T}_s(f,g)(w), F_j(w) \right\rangle\right| (1 - |w|^2)^{d-1} \, dm(w) \\ \le\ & \|F_j(w)\|' \cdot \int_{\mathbb{B}} \left\| \otimes^s B^t(w,w)^{1/2} \mathcal{T}_s(f,g)(w) \right\| (1 - |w|^2)^{d-1} \, dm(w) \\ \le\ & C_d \cdot (1 - |z_j|^2)^{-d} \cdot \|a_j\| \cdot \|f\|_{H^2} \cdot \|g\|_{H^2} < \infty \,. \end{aligned}$$

Hence, by the reproducing property,

$$H_{F_j}(f, g) = c \left\langle \mathcal{T}_s(f,g)(z_j), (1 - |z_j|^2)^d \otimes^s B^t(z, z)^{1/2} a_j \right\rangle \,.$$

The bilinear form $(f, g) \to \mathcal{T}_s(f, g)(z_j)$ is a sum of finitely many rank one forms where the number of summands M_s only depends on s. We see this by writing $f(z_j) = c\langle f, K_{z_j} \rangle_{H^2}$, where $K_{z_j}(w) = (1 - \langle w, z_j \rangle)^{-d}$, so that

$$\partial^{s-k} f(z_j) \otimes \partial^k g(z_j) = c^2 \langle f, \overline{\partial^{s-k}} K_{z_j} \rangle_{H^2} \otimes \langle g, \overline{\partial^k} K_{z_j} \rangle_{H^2} \,.$$

Hence

$$\|H^s_{F_j}\|_{\mathcal{S}_1} \le \sqrt{M_s} \cdot \|H_{F_j}\|_{\mathcal{S}_2}$$

for all $j = 1, 2, \cdots$ so by Theorem 3.2 it follows that

$$\|H^s_F\|_{\mathcal{S}_1} \le \sum_{j=1}^{\infty} \|H^s_{F_j}\|_{\mathcal{S}_1} \le \sqrt{M_s} \cdot \sum_{j=1}^{\infty} \|H^s_{F_j}\|_{\mathcal{S}_2} = C \cdot \sum_{j=1}^{\infty} \|F_j\|_{\mathcal{H}^2_{d,s}}$$

and

$$\|F_j\|^2_{\mathcal{H}^2_{d,s}} = c' \cdot \|a_j\|^2$$

by the reproducing property. Thus

$$\|H_F\|_{\mathcal{S}_1} \leq C' \cdot \sum_{j=1}^{\infty} \|a_j\| < \infty\,.$$

This completes the proof. □

Corollary 3.8. *The map* $\Gamma : \mathcal{H}^1_{d,s} \to \mathcal{S}_2$, $\Gamma(F) = H^s_F$, *is bounded.*

Proof. This follows immediately from the last inequality in the proof of the theorem above and from the fact that $\|F\|_{d,s,1}$ is equivalent to

$$\|F\|_{\inf} = \inf\left\{\sum_{j=1}^{\infty} \|a_j\| : \{a_j\}_{j=1}^{\infty} \text{ defines } F \text{ by } (9)\right\}. \tag{12}$$

We need to prove that $\|F\|_{d,s,1}$ is equivalent to $\|F\|_{\inf}$. If we let $\mathcal{B}$ be the Banach space of holomorphic $F : \mathbb{B} \to \odot^s V'$ such that $\|F\|_{\inf} < \infty$, then the bijection $I : \mathcal{H}^1_{d,s} \to \mathcal{B}$, $F \mapsto F$, is bounded. Hence, by the Open Mapping Theorem $I : \mathcal{B} \to \mathcal{H}^1_{d,s}$ is also bounded, and thus we get equivalent norms. □

Theorem 3.9. *Let s be a positive integer and let $1 \leq p \leq 2$. Then* $\Gamma : \mathcal{H}^p_{d,s} \to \mathcal{S}_p$, $\Gamma(F) = H^s_F$, *is bounded.*

Proof. By Corollary 3.8 and by Theorem 3.2 it follows that $\Gamma : \mathcal{H}^i_{d,s} \to \mathcal{S}_i$ is bounded for $i = 1, 2$ respectively. Then the theorem follows by interpolation and Theorem 3.3. □

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Paper IV

L^p-BOUNDEDNESS FOR ORTHOGONAL PROJECTIONS ONTO SPACES OF NEARLY HOLOMORPHIC FUNCTIONS AND OF VECTOR-VALUED HOLOMORPHIC FUNCTIONS

MARCUS SUNDHÄLL

ABSTRACT. In this paper we establish L^p-boundedness criteria for orthogonal projections from $L^2(d\mu_\alpha)$ onto the discrete parts in the irreducible decomposition of $L^2(d\mu_\alpha)$ under the action of the Möbius group, where $d\mu_\alpha(z) = (1 - |z|^2)^\alpha \, dm(z)$, $\alpha > -1$, and dm is the Lebesgue measure on the unit ball, $\mathbb{B}$, of $\mathbb{C}^d$. These spaces can be realized as kernels of the power $\bar{D}^{l+1}$ of the invariant Cauchy-Riemann operator $\bar{D} = B(z,z)\bar{\partial}$ (where $B(z,z)^{-1}$ is the Bergman metric) and are therefore spaces of nearly holomorphic functions in the sense of Shimura. The operators $\bar{D}^l$ are intertwining operators from these spaces of nearly holomorphic functions into certain vector-valued Bergman-type spaces of holomorphic functions in $\mathbb{B}$. The orthogonal projections onto these spaces are given by matrix-valued Bergman-type kernels, and we study their L^p-boundedness properties for bounded symmetric domains of type I.

1. INTRODUCTION

Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^d$ with the Lebesgue measure dm. Consider the weighted L^2-space $L^2(d\mu_\alpha)$, where $d\mu_\alpha(z) = (1-|z|^2)^\alpha dm(z)$, $\alpha > -1$. The Möbius group of biholomorphic mappings of $\mathbb{B}$ acts on $L^2(d\mu_\alpha)$ as unitary (projective) representations. A weighted Plancherel formula was established by Peetre, Peng and Zhang in [PPZ] and Zhang [Z1], giving an explicit decomposition of the representation. There are continuous and disctrete parts in the decomposition. The discrete parts can be viewed as images of $L^2(d\mu_\alpha)$ under certain orthogonal projections. These spaces can be realized (see [Z2]) as the kernels of powers, $\bar{D}^{m+1}$, of the invariant Cauchy-Riemann operator

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$\bar{D} = B(z,z)\bar{\partial}$, where $B(z,z)^{-1}$ is the Bergman metric) and are therefore spaces of nearly holomorphic functions in the sense of Shimura (see [Sh1] and [Sh2]). Actually, for a certain k, the operators $\bar{D}^l$, $l = 0, 1, \ldots, k$, are intertwining operators from the spaces of nearly holomorphic functions onto certain Bergman spaces of vector-valued holomorphic functions on $\mathbb{B}$ (see [PZ] and [EP]). We have the following diagram;

$$\begin{array}{ccc} L^2(d\mu_\alpha) \cap C^\infty(\mathbb{B}) & \xrightarrow{\bar{D}^l} & L^2\left(\mathbb{B}, \odot^l \mathbb{C}^d, d\mu_\alpha\right) \cap C^\infty\left(\mathbb{B}, \odot^l \mathbb{C}^d\right) \\ \Big\downarrow P_l & & \Big\downarrow P_{\nu,l} \\ A_l^2(d\mu_\alpha) & \xrightarrow{\bar{D}^l} & L_a^2\left(\mathbb{B}, \odot^l \mathbb{C}^d, d\mu_\alpha\right) \end{array}$$

where P_l is the orthogonal projection from $L^2(d\mu_\alpha)$ onto the discrete part $A_l^2(d\mu_\alpha)$ of nearly holomorphic functions, $P_{\nu,l}$ ($\nu = \alpha + d + 1$) is the orthogonal projection from $L^2(\mathbb{B}, \odot^l \mathbb{C}^d, d\mu_\alpha)$ onto its holomorphic subspace and the L^2-norm (invariant under the action of the Möbius group) is given by

$$\|f\|_{l,\alpha,2} = \left(\int_{\mathbb{B}} \left\langle \otimes^l B(z,z)^{-1} f(z), f(z) \right\rangle \, d\mu_\alpha(z) \right)^{1/2},$$

where $\otimes^l B(z,z)^{-1}$ is the action on $\otimes^l \mathbb{C}^d$ induced by the action of $B(z,z)^{-1}$ on $\mathbb{C}^d$. This can be generalized into the setting of bounded symmetric domains [Z2].

The main objective of this paper is to establish the L^p-boundedness criteria for the orthogonal projections P_l onto the spaces of nearly holomorphic functions (Section 2) and also for the related Bergman-type projections $P_{\nu,l}$ onto the Bergman spaces of vector-valued holomorphic functions (Section 3) for the unit ball of $\mathbb{C}^d$; the problem makes also sense for general bounded symmetric domains, and we study the Bergman-type projections, $P_{\nu,l}$, for bounded symmetric domains of type I.

More concretely, if $\alpha > 2l - 1$ then, on one hand, Theorem 2.1 states that P_l is L^p-bounded if and only if

$$\frac{\alpha+1}{\alpha+1-1} < p < \frac{\alpha+1}{l}. \tag{1}$$

On the other hand, Theorem 3.6 states that $P_{\nu,l}$ is L^p-bounded if condition (1) is satisfied.

To find L^p-boundedness criteria for P_l we use concrete formulas, which can be found in [Z1]. In the more general setting of bounded symmetric domains we do not yet have such formulas. However, the generalizations of $P_{\nu,l}$ to bounded symmetric domains of type I is studied in Section 3. In this section, a sufficient conditions for these projections to be L^p-bounded is presented. Actually, this is a weak generalization of the corresponding result for the case of the unit ball of $\mathbb{C}^d$, weaker since the Forelli-Rudin type estimate is different in the general case (see [FK2] and [EZ]). The Bergman-type projections mentioned above are closely related to vector-valued Bergman-type projections studied in [Su1]. A weak generalization of the L^p-boundedness criteria for these projections is presented in Section 3.

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2. The projection operators onto nearly holomorphic functions

2.1. **The action of the Möbius group.** Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^d$, and let $G = Aut(\mathbb{B})$ be the group of holomorphic bijections on $\mathbb{B}$ with holomorphic inverse. An element $g \in G$, $g(z) = 0$, can be decomposed as $g = U\varphi_z$ where $U : \mathbb{C}^d \to \mathbb{C}^d$ is a unitary map and φ_z is a linear fractional map, taking 0 to z, see [Ru]. The complex Jacobian $J_{\varphi_z}(w)$ is given in [Su1] by

$$J_{\varphi_z}(w) = (-1)^d \frac{(1-|z|^2)^{(d+1)/2}}{(1-\langle w, z\rangle)^{d+1}},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{C}^d$. Since G acts transitively on $\mathbb{B}$ (see [Ru]) we get $J_g(w)$ for any $g \in G$ in this way. Hence, we can define an action, π_ν, of G on $L^2(d\mu_\alpha)$ by

$$(\pi_\nu(g)f)(w) = f(g^{-1}(w)) \cdot J_{g^{-1}}(w)^{\nu/(d+1)}, \tag{2}$$

where $\nu = \alpha + d + 1$, and where we use the same convention as in [Su1] concerning the ambiguity of the definition of power. Then π_ν is a unitary projective representation of G.

2.3. **An L^p-boundedness criterion.** The orthogonal projection operators from $L^2(d\mu_\alpha)$ onto the discrete parts of the irreducible decomposition under the action (2) of $L^2(d\mu_\alpha)$ are given explicitly in [Z1] by P_l, for $l = 0, 1, \cdots, k = [(\alpha+1)/2]$ (α is not an odd integer), where

$$P_l f(z) = \langle f, K_l(\cdot, z)\rangle_\alpha = \int_{\mathbb{B}} f(w) K_l(z,w)\, d\mu_\alpha(w) \tag{3}$$

and

$$K_l(z,w) = c_l \frac{1}{(1-\langle z,w\rangle)^{\alpha+d+1}} \times \sum_{i=0}^{l} \frac{(-l)_i (l-\alpha-1)_i (-1)^i}{(d)_i i!} \left(1 - \frac{|1-\langle z,w\rangle|^2}{(1-|z|^2)(1-|w|^2)}\right)^i,$$

where c_l is a normalization constant and $(d)_n = d(d+1)\cdots(d+n-1)$ is the Pochammer symbol. In the next theorem we present necessary and sufficient conditions on $1 < p < \infty$ to make the projection operators P_l bounded on $L^p(d\mu_\alpha)$.

Theorem 2.1. *If $l \in \{0, 1, 2, \cdots, k\}$, $k = [(\alpha+1)/2]$ (α is not an odd integer), then the orthogonal projection operator P_l, defined in (3), is bounded on $L^p(d\mu_\alpha)$ if and only if*

$$\frac{\alpha+1}{\alpha+1-l} < p < \frac{\alpha+1}{l}$$

when $l \neq 0$, and $1 < p < \infty$ when $l = 0$.

Proof. The case $l = 0$ is classical (see for instance Theorem 2.11 in [Zhu]). Assume now $l \neq 0$. We can write the reproducing kernel K_l as

$$K_l(z,w) = h_{l-1}(z,w) + c_l T_l(z,w)$$

where

$$T_l(z,w) = \frac{(1-|z|^2)^{-l}(1-|w|^2)^{-l}(1-\langle w,z\rangle)^l}{(1-\langle z,w\rangle)^{\alpha-l+d+1}}.$$

First we observe that there is a constant $C > 0$ such that

$$|K_l(z,w)| \le C \cdot \frac{(1-|z|^2)^{-l}(1-|w|^2)^{-l}}{|1-\langle z,w\rangle|^{\alpha+1+d-2l}} = C \cdot T_l(z,w)\,.$$

Hence,

$$|P_l f(z)| \leq C \int_{\mathbb{B}} T_l(z,w)|f(w)|\, d\mu_\alpha(w)\,.$$

We claim that there are real numbers $M > 0$ and t such that the inequalities

$$\int_{\mathbb{B}} T_l(z,w)(1-|z|^2)^{pt}\, d\mu_\alpha(z) \leq M(1-|w|^2)^{pt} \tag{4}$$

and

$$\int_{\mathbb{B}} T_l(z,w)(1-|w|^2)^{qt}\, d\mu_\alpha(w) \leq M(1-|z|^2)^{qt} \tag{5}$$

hold for q with $1/q + 1/p = 1$. If the claim is true then P_l is bounded on $L^p(d\mu_\alpha)$, by Schur's test (see [HKZ]). By the same arguments as in the proof of Theorem 7.2 in [Su1] it follows that the claim is true if

$$\frac{\alpha+1}{\alpha+1-l} < p < \frac{\alpha+1}{l}\,.$$

Now we consider the cases when $1 < p \leq (\alpha+1)/(\alpha+1-l)$ or $(\alpha+1)/l \leq p < \infty$. Actually, for duality reasons we need only to consider the case when $(\alpha+1)/l \leq p < \infty$. Let $\varepsilon > 0$ and define χ_ε to be the characteristic function on $\mathbb{B}_\varepsilon = \{z \in \mathbb{C}^d : |z| < \varepsilon\}$. If a is a positive real number, then

$$(1-\langle z,w\rangle)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!}\langle z,w\rangle^k\,.$$

By binomial expansion and orthogonality,

$$\begin{aligned}&\int_{\mathbb{B}_\varepsilon} \frac{(1-\langle w,z\rangle)^j}{(1-\langle z,w\rangle)^{\alpha-j+d+1}}\,(1-|w|^2)^{\alpha-j}dm(w)\\ &= \sum_{i=0}^{j} \binom{j}{i} \frac{(\alpha-j+d+1)_i}{i!}(-1)^i \int_{\mathbb{B}_\varepsilon} |\langle z,w\rangle|^{2i}\,(1-|w|^2)^{\alpha-j}dm(w)\end{aligned}$$

for all $j = 0,1,2,\cdots,l$. Clearly we can find a constant D_l such that

$$|\langle \chi_\varepsilon, h_{l-1}(\cdot,z)\rangle_\alpha| \leq D_l(1-|z|^2)^{-l+1}\,. \tag{6}$$

Also,

$$(7)\quad |\langle \chi_\varepsilon, T_l(\cdot, z)\rangle_\alpha| \geq$$

$$(1-|z|^2)^{-l}\int_{\mathbb{B}_\varepsilon}(1-|w|^2)^{\alpha-l}dm(w)\left(1-\sum_{i=1}^{l}\binom{l}{i}\frac{(\alpha-l+d+1)_i}{i!}\,\varepsilon^{2i}\right).$$

Thus by (6) and (7), if we choose ε to be small enough and $K<1$ large enough, there is a positive constant C_l such that

$$|\langle \chi_\varepsilon, K_l(\cdot, z)\rangle_\alpha| \geq |\langle \chi_\varepsilon, c_l T_l(\cdot, z)\rangle_\alpha| - |\langle \chi_\varepsilon, h_{l-1}(\cdot, z)\rangle_\alpha| \geq C_l(1-|z|^2)^{-l}$$

if $K<|z|<1$. Hence,

$$(8)\int_{\mathbb{B}}|\langle \chi_\varepsilon, K_l(\cdot, z)\rangle_\alpha|^p\, d\mu_\alpha(z) \geq C_l^p\int_{K<|z|<1}(1-|z|^2)^{\alpha-pl}dm(z)$$

and the integral on the right side of the inequality (8) is infinite if $p\geq(\alpha+1)/l$. □

3. Bergman spaces of vector-valued holomorphic functions

3.1. **Bounded symmetric domains of type I.** Let $\mathcal{D}$ be a type I bounded symmetric domain, i.e., $\mathcal{D}=\{Z\in M_{m,n}(\mathbb{C}) : ZZ^*<I_m\}$ and let $dm(Z)$ be the Lebesgue measure on $\mathcal{D}$. By Theorem 4.3.1 in [H], the Bergman kernel is given by

$$(9)\qquad k(Z,W)=c\cdot h(Z,W)^{-(m+n)},$$

where c is a certain nonzero constant, and where

$$h(Z,W)=\det(I-ZW^*).$$

If $g:\mathcal{D}\to\mathcal{D}$ is biholomorphic, then, by Theorem 2.10 in [FK1],

$$(10)\qquad k(Z,W)=\det(dg(Z))\cdot k(g(Z),g(W))\cdot\overline{\det(dg(W))}$$

where $dg(Z):T_Z(\mathcal{D})\to T_{g(Z)}(\mathcal{D})$ is the differential map.

The Bergman operator defined for $Z,W\in\mathcal{D}$ is given in [L] by

$$B(Z,W)X=(I-ZW^*)X(I-W^*Z),$$

for matrices $X\in M_{m,m}(\mathbb{C})$. By Lemma 2.11 in [L],

$$(11)\qquad B(g(Z),g(W))=dg(Z)B(Z,W)dg(W)^*.$$

3.2. **Values in tensor products of a tangent space.** Consider the measure $d\mu_\alpha(Z) = h(Z,Z)^\alpha dm(Z)$, for $\alpha > 2s-1$, and the corresponding L^2-space $L^2(\mathcal{D}, \odot^s V, d\mu_\alpha)$ where $V = M_{m,m}(\mathbb{C})$ (so that we can identify a tangent space on $\mathcal{D}$ with V) and $\odot^s V$ is the induced symmetric tensor product for s copies of V where s is a nonnegative integer. The functions in $L^2(\mathcal{D}, \odot^s V, d\mu_\alpha)$ are tensor-valued and the L^2-norm is given by

$$\|f\|_{s,\alpha,2} = \left(\int_{\mathcal{D}} \left\langle \otimes^s B(Z,Z)^{-1} f(Z), f(Z)\right\rangle \, d\mu_\alpha(Z)\right)^{1/2},$$

where $\langle X, Y\rangle = \mathrm{tr}(XY^*)$. The reproducing kernel is up to a constant

$$K_{\nu,s}(Z,W) = h(Z,W)^{-\nu} \otimes^s B(Z,W) \tag{12}$$

where $\nu = \alpha + m + n$. This can be proved by using the transformation properties (10) and (11) of $h(Z,W)$ and $B(Z,W)$ respectively (see e.g. [Su1] for the case of the unit ball and [ØZ] for similar results).

Lemma 3.1. *Let s be a nonnegative integer. Then there is a constant $C_s > 0$ such that*

$$\left\|\otimes^s \left(B(Z,Z)^{-1/2} B(Z,W) B(W,W)^{-1/2}\right) X\right\| \le C_s \frac{|h(Z,W)|^{2s}}{h(Z,Z)^s h(W,W)^s} \|X\|,$$

for all $X \in M_{m,m}(\mathbb{C})^s$.

Proof. The case $s = 0$ is trivial, so first we prove the case $s = 1$. If $X \in M_{m,m}(\mathbb{C})$, then

$$\begin{aligned} B(Z,Z)^{-1/2} X &= (I - ZZ^*)^{-1/2} X (I - Z^*Z)^{-1/2} \\ &= h(Z,Z)^{-1} \left(\mathrm{adj}(I - ZZ^*)\right)^{1/2} X \left(\mathrm{adj}(I - Z^*Z)\right)^{1/2}, \end{aligned}$$

where $\mathrm{adj}(A)$ is the adjoint of the matrix A. Thus, if $W = 0$, then

$$\|B(Z,Z)^{-1/2} X\| \le C \cdot h(Z,Z)^{-1} \cdot \|X\|.$$

Now, let g be a biholomorphic map on $\mathcal{D}$ such that $g(0) = W$ and $g^{-1} = g$. On one hand,

$$\|B(g^{-1}(Z), g^{-1}(Z))^{-1/2} X\| \le C \cdot h(g^{-1}(Z), g^{-1}(Z))^{-1} \|X\|$$

On the other hand, by (10),

$$h(g^{-1}(Z), g^{-1}(Z)) = h(g(Z), g(Z)) = \frac{h(W,W) h(Z,Z)}{|h(Z,W)|^2}.$$

Hence

$$\|B(g^{-1}(Z), g^{-1}(Z))^{-1/2}X\| \leq C \cdot \frac{|h(Z,W)|^2}{h(Z,Z)h(W,W)} \cdot \|X\| \,. \tag{13}$$

Let $Y = dg(0)^* B(W,W)^{-1/2}X$. Then we can replace X by Y in the inequality (13). Also $\|Y\| = \|X\|$, so if we let $Z_0 = g^{-1}(Z)$ then by (11),

$$\begin{aligned} & \left\|B(Z,Z)^{-1/2}B(Z,W)B(W,W)^{-1/2}X\right\|^2 \\ = & \left\|B(Z,Z)^{-1/2}dg(Z_0)Y\right\|^2 \\ = & \operatorname{tr}\left(B(Z,Z)^{-1/2}dg(Z_0)YY^*dg(Z_0)^*B(Z,Z)^{-1/2}\right) \\ = & \operatorname{tr}\left(dg(Z_0)^*B(Z,Z)^{-1}dg(Z_0)YY^*\right) \\ = & \operatorname{tr}\left(B(Z_0,Z_0)^{-1}YY^*\right) \\ \leq & C^2 \cdot \left(\frac{|h(Z,W)|^2}{h(Z,Z)h(W,W)}\right)^2 \cdot \|X\|^2 \,. \end{aligned}$$

Hence, the lemma is proved for the case $s = 1$. Now, consider the case where $s = 2, 3, \ldots$ and let

$$A_{Z,W} = B(Z,Z)^{-1/2}B(Z,W)B(W,W)^{-1/2}$$

and

$$t_{Z,W} = \frac{|h(Z,W)|^2}{h(Z,Z)h(W,W)} \,.$$

We have proved that

$$A^*_{Z,W}A_{Z,W} \leq C^2 t^2_{Z,W} I$$

so that

$$(\otimes^s A_{Z,W})^* \otimes^s A_{Z,W} = \otimes^s \left(A^*_{Z,W}A_{Z,W}\right) \leq C^{2s} t^{2s}_{Z,W} \otimes^s I$$

which proves the lemma. □

As a special case we get the following lemma.

Lemma 3.2. *If $\mathcal{D} = \mathbb{B}$, then for any nonnegative integer s, there is a constant $C_s > 0$ such that*

$$\left\|\otimes^s \left(B(z,z)^{-1/2}B(z,w)B(w,w)^{-1/2}\right) x\right\| \leq C_s \frac{|1 - \langle z,w\rangle|^{2s}}{(1-|z|^2)^s(1-|w|^2)^s}\|x\| \,,$$

for all $x \in \otimes^s V$.

As a special case of Theorem 4.1 in [FK2] we have the following lemma.

Lemma 3.3. *Let $\beta - 1 > \alpha > -1$. Then there is a constant $C > 0$ such that*
$$\int_{\mathcal{D}} \frac{h(Z,Z)^{\alpha}}{|h(Z,W)|^{\beta+m+n}}\, dm(Z) \leq C \cdot h(W,W)^{-(\beta-\alpha)}\,.$$

Remark 3.4. There is an orthogonal projection $P_{\nu,s}$, from the Hilbert space $L^2(\mathcal{D}, \odot^s V, d\mu_\alpha)$ into its holomorphic subspace, such that for any $f \in L^2(\mathcal{D}, \odot^s V, d\mu_\alpha)$ and any $X \in \odot^s V$ we have that

$$\begin{aligned} &\langle P_{\nu,s} f(Z), X\rangle \\ &\qquad = c\int_{\mathcal{D}} \big\langle \otimes^s B(W,W)^{-1} f(W), K_{\nu,s}(W,Z)X \big\rangle \, d\mu_\alpha(W)\,. \end{aligned} \tag{14}$$

Theorem 3.5. *Let $\alpha > 2s$ and let $P_{\nu,s}$ be the orthogonal projection operator, where $\nu = \alpha + m + n$. If*
$$\frac{\alpha+2}{\alpha+1-s} < p < \frac{\alpha+2}{s+1}\,,$$
then $P_{\nu,s}$ is bounded on $L^p(\mathcal{D}, \odot^s V, d\mu_\alpha)$.

Proof. The formula (14) can be rewritten as
$$P_{\nu,s} f(Z) = c\int_{\mathcal{D}} K_{\nu,s}(W,Z)^* \otimes^s B(W,W)^{-1} f(W)\, d\mu_\alpha(W)\,.$$
Let
$$T(Z,W) = \frac{h(Z,Z)^{-s} h(W,W)^{-s}}{|h(Z,W)|^{\nu-2s}}\,.$$
By the equality $K_{\nu,s}(W,Z)^* = K_{\nu,s}(Z,W)$ and Lemma 3.1 it follows that
$$\begin{aligned} &\big\|\otimes^s B(Z,Z)^{-1/2} P_{\nu,s} f(Z)\big\| \\ &\qquad \leq C\int_{\mathcal{D}} T(Z,W)\, \big\|\otimes^s B(W,W)^{-1/2} f(W)\big\|\, d\mu_\alpha(W)\,. \end{aligned}$$
Now by Lemma 3.3, using the same techniques as in the proof of Theorem 7.2 in [Su1], it follows that there exists a real number t and a constant $M > 0$ such that
$$\int_{\mathcal{D}} T(Z,W) h(Z,Z)^{pt+\alpha}\, dm(Z) \leq M \cdot h(W,W)^{pt}$$

and

$$\int_{\mathcal{D}} T(Z,W)h(W,W)^{qt+\alpha}\, dm(W) \leq M \cdot h(Z,Z)^{qt}$$

where $1/p+1/q=1$. Namely, there exists such t if p and α satisfies the condition

$$\frac{\alpha+2}{\alpha+1-s} < p < \frac{\alpha+2}{s+1}\,.$$

So, with this condition for p and α it follows by Schur's test that

$$\int_{\mathcal{D}} \left\|\otimes^s B(Z,Z)^{-1/2} P_{\nu,s} f(Z)\right\|^p \, d\mu_\alpha(Z) \leq C \int_{\mathcal{D}} \left\|\otimes^s B(W,W)^{-1/2} f(W)\right\|^p \, d\mu_\alpha(W)\,.$$

□

Theorem 3.6. *Let $\alpha > 2s-1$ and let $P_{\nu,s}$ be the orthogonal projection where $\nu = \alpha+d+1$, i.e. $\mathcal{D} = \mathbb{B}$. If $s \neq 0$ and*

$$\frac{\alpha+1}{\alpha+1-s} < p < \frac{\alpha+1}{s}\,,$$

then $P_{\nu,s}$ is bounded on $L^p(\mathbb{B}, \odot^s V, d\mu_\alpha)$. If $s=0$, then $P_{\nu,s}$ is bounded on $L^p(\mathbb{B}, \odot^s V, d\mu_\alpha)$ for any $1 < p < \infty$.

Proof. The case $s=0$ is classical (see for instance Theorem 2.11 in [Zhu]). Assume now $s \neq 0$. By similar arguments as in the proof of Theorem 3.5, using Lemma 3.2, we get that

$$\left\|\otimes^s B(z,z)^{-1/2} P_{\nu,s} f(z)\right\| \leq C \int_{\mathbb{B}} T(z,w) \left\|\otimes^s B(w,w)^{-1/2} f(w)\right\| \, d\mu_\alpha(w)$$

where

$$T(z,w) = \frac{(1-|z|^2)^{-s}(1-|w|^2)^{-s}}{|1-\langle z,w\rangle|^{\nu-2s}}\,.$$

Again, following the proof of Theorem 3.5, using Proposition 1.4.10 in [Ru] instead of Lemma 3.3, we get the desired result. □

3.3. **Values in tensor products of a cotangent space.** Once we have studied the L^p-boundedness for Bergman-type projections onto Bergman spaces with functions with values in symmetric tensor products of a tangent space, it is natural to do so even for the case of cotangent spaces. These Bergman-type projections are closely related to the Bergman-type projections studied in [Su1] and [Su2].

Let $\mathcal{D}$ be the type I bounded symmetric domain given in the previous subsection. Most notation are the same as in the previous subsection, only $\alpha > -1$ and $L^2(\mathcal{D}, \odot^s V, d\mu_\alpha)$ is replaced by $L^2(\mathcal{D}, \odot^s V', d\mu_\alpha)$ with norm

$$\|f\|'_{s,\alpha,2} = \left(\int_{\mathbb{B}} \langle \otimes^s B(Z,Z)' f(Z), f(Z) \rangle \, d\mu_\alpha(Z) \right)^{1/2},$$

where $B(Z,Z)'$ is the dual action of $B(Z,Z)$ acting on the dual space V'. Also $B(Z,Z)'$ may be identified with $B^t(Z,Z)$ where

$$B^t(Z,W)X = (I - ZW^*)^t X (I - W^*Z)^t,$$

for matrices $X \in M_{m,m}(\mathbb{C})$ and where t is the transpose of a matrix. The reproducing kernel for $L^2(\mathcal{D}, \odot^s V', d\mu_\alpha)$ is given, up to a nonzero constant, by

$$K'_{\nu,s}(Z,W) = h(Z,W)^{-\nu} \otimes^s B^t(Z,W)^{-1},$$

where again $\nu = \alpha + m + n$. The orthogonal projection, $P'_{\nu,s}$, in question from $L^2(\mathcal{D}, \odot^s V', d\mu_\alpha)$ onto its holomorphic subspace, is defined in the following way. For any $f \in L^2(\mathcal{D}, \odot^s V', d\mu_\alpha)$ and any $X \in \odot^s V'$ we have that

$$\begin{aligned} (15)\quad & \left\langle P'_{\nu,s} f(Z), X \right\rangle \\ & \qquad = c' \int_{\mathcal{D}} \left\langle \otimes^s B^t(W,W) f(W), K'_{\nu,s}(W,Z) X \right\rangle \, d\mu_\alpha(W). \end{aligned}$$

Hence, if we can find a result similar to Lemma 3.1 then we can use the same arguments as in the proof of Theorem 3.5 to find criteria for the projections $P'_{\nu,s}$ to be bounded on $L^p(\mathcal{D}, \odot^s V', d\mu_\alpha)$.

Lemma 3.7. *Let s be a nonnegative integer. Then*

$$\left\| \otimes^s \left(B^t(Z,Z)^{1/2} B^t(Z,W)^{-1} B^t(W,W)^{1/2} \right) X \right\| \le \|X\|,$$

for all $X \in M_{m,m}(\mathbb{C})^s$.

Proof. By the definition of the Bergman operator it follows that

$$\left\|B^t(Z,Z)^{1/2}X\right\| \leq \|X\| \tag{16}$$

Actually, if $\mathcal{D}$ is not the unit ball of $\mathbb{C}^d$ then we can find $Z \in \mathcal{D}$ such that $B^t(Z,Z)X = X$ for all $X \in M_{m,m}(\mathbb{C})$ and therefore (16) is actually the best estimate we can get in the general case. Now, given $W \in \mathcal{D}$, choose g as in the proof of Lemma 3.1. Then

$$\left\|B^t(Z_0,Z_0)^{1/2}X\right\| \leq \|X\|,$$

if $g(Z_0) = Z$. Since

$$\left(dg(Z_0)^t\right)^{-1} : T_{Z_0}(\mathcal{D})' \to T_Z(\mathcal{D})'$$

is an isometry then

$$\left\|B^t(Z,Z)^{1/2}\left(dg(Z_0)^t\right)^{-1}X\right\| \leq \|X\|.$$

Hence

$$\begin{aligned}&\left\|B^t(Z,Z)^{1/2}B^t(Z,W)^{-1}B^t(W,W)^{1/2}X\right\|\\ &= \left\|B^t(Z,Z)^{1/2}\left(dg(Z_0)^t\right)^{-1}Y\right\| \leq \|Y\|,\end{aligned}$$

where $Y = ((dg(0)^*)^t)^{-1}B^t(W,W)^{1/2}X$. Also, $\|Y\| = \|X\|$ which follows in the same way as in the proof of Lemma 3.1. Thus, the lemma is proved for the case when $s = 1$ and the proof of the general case is done in exactly the same way as in the proof of Lemma 3.1. □

As we could see in the proof of the lemma above we need to treat the particular case $\mathcal{D} = \mathbb{B}$ separately. The following lemma can be proved by using the same techniques as in the proof of Lemma 3.1 and in Lemma 3.7. However, the same result can also be found in [Su1].

Lemma 3.8 (Lemma 7.1 in [Su1])**.** *If $\mathcal{D} = \mathbb{B}$, then for any nonnegative integer s, there is a constant $C_s > 0$ such that*

$$\begin{aligned}&\left\|\otimes^s\left(B^t(z,z)^{1/2}B^t(z,w)^{-1}B^t(w,w)^{1/2}\right)x\right\|\\ &\qquad\leq C_s\frac{(1-|z|^2)^{s/2}(1-|w|^2)^{s/2}}{|1-\langle z,w\rangle|^s}\|x\|,\end{aligned}$$

for all $x \in \otimes^s V'$.

Now we can get the desired boundedness condition. This result is a weaker generalization of Theorem 7.2 in [Su1].

Theorem 3.9. *Let $\alpha > 0$ and let $P'_{\nu,s}$ be the orthogonal projection operator, where $\nu = \alpha + m + n$. If*

$$\frac{\alpha+2}{\alpha+1} < p < \alpha + 2\,,$$

then $P'_{\nu,s}$ is bounded on $L^p(\mathcal{D}, \odot^s V', d\mu_\alpha)$.

Proof. The result follows by exactly the same arguments as we used to prove Theorem 3.5. □

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