THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Branching laws for holomorphic representations and the quaternionic discrete series for $Sp(1,1)$

HENRIK SEPPÄNEN

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and Göteborg University
Göteborg, Sweden 2006

Branching laws for holomorphic representations
and the quaternionic discrete series for $Sp(1,1)$
Henrik Seppänen
ISBN 91-628-7041-6

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31 772 10 00

Printed in Göteborg, Sweden 2006

Branching laws for holomorphic representations and the discrete series for $Sp(1,1)$

Henrik Seppänen
Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and Göteborg University

Abstract

We study the branching laws when representations given by the analytic continuation of the scalar holomorphic discrete series for a semisimple Hermitian Lie group, G, are restricted to a symmetric subgroup. The symmetric subgroups we consider are the fixed point groups for the lifts to G of antiholomorphic involutions on the corresponding bounded symmetric domain G/K. We prove a general theorem stating that a multiplicity free direct integral decomposition always exists. Explicit decomposition theorems are given for some of the given representations for three of the four types of classical bounded symmetric domains. The methods that are used include explicit intertwining operators and the spectral decomposition of an associated Casimir operator.

We also consider the quaternionic discrete series for the group $Sp(1,1)$. A generalised Szegö map is used to compute highest weight vectors for K-types in a homogeneous vector bundle model.

Keywords: Lie groups, unitary representations, branching law, bounded symmetric domains, hypergeometric functions, quaternionic structure, discrete series representations, Szegö map

AMS 2000 Subject Classification: 22E45, 32M15, 33C45, 43A85

This thesis consists of an introduction and the following four papers:

[i] *Branching of some holomorphic representations of* $SO(2,n)$, Journal of Lie theory, to appear,

[ii] *Tube domains and restrictions of holomorphic representations*, Preprint,

[iii] *Branching laws for minimal holomorphic representations*, Preprint,

[iv] *Quaternionic discrete series for* $Sp(1,1)$, Preprint.

Acknowledgements

First of all, I would like to thank my advisor Professor Genkai Zhang for his guidance during my time as a PhD student, for being so generous with his time, and for always having the patience to scrutinise my arguments carefully. On numerous occasions his illuminating comments have helped me during the preparation of my papers.

I am also grateful to the research group in harmonic analysis and the HARP-network for having partially financed my travels and thus enabled me to participate in stimulating summer schools and conferences.

I thank my friends and fellow graduate students at the department of mathematics, especially Bettan, Bom, Elin, Feif, Greken, Hövvet, Jonas, Julf, Kenny, Looben, Musse, Segge, Sälen, Unge Herrn and Blojne.

I thank my family for encouragement and support. Especially, I want to thank Katarina for making my life so pleasant.

Henrik Seppänen
Göteborg, November 2006

BRANCHING LAWS FOR HOLOMORPHIC REPRESENTATIONS AND THE QUATERNIONIC DISCRETE SERIES FOR $Sp(1,1)$

HENRIK SEPPÄNEN

INTRODUCTION

In this introduction we give an overview of the framework in which the representations we study take place. Since three of the papers are directly related to bounded symmetric domains, also the major part of this introduction is concerned with these. We briefly describe their geometry and the trivialisations of certain line bundles defined over them. Also, this gives a background to some representations defined by continuation in a parameter describing these line bundles. After this general framework we consider the restrictions to some symmetric subgroups that define totally real submanifolds. In this context, we give brief introductions to Papers $I-III$ that treat the decompositions of restrictions to subgroups, or the *branching law*. Following this, we put a so-called multiplicity-free property of the decompositions in a natural geometric and complex analytical context by presenting some recent results by T. Kobayashi ([19], [20]) on representations related to holomorphic vector bundles and certain types of actions of Lie groups on complex manifolds.

The remaining part of this introduction treats the quaternionic discrete series. Here we present the results of Paper IV and relate it to previous work by Gross and Wallach ([9]). We end the introduction by posing some open problems. These are divided into branching laws and quaternionic representations.

0.1. **Bounded symmetric domains.** In the present thesis we are interested in representations realised on Hilbert spaces of holomorphic functions on *bounded symmetric domains.* These are bounded open domains, $\mathscr{D}$, in some $\mathbb{C}^n$ with the property that for every $z \in \mathcal{D}$ there exists an automorphism (i.e. biholomorphic mapping on $\mathscr{D}$), s_z, of period 2 having z as an isolated fixed point. A bounded domain, $\mathscr{D}$, is *circled* (with respect to 0) if $0 \in \mathscr{D}$ and $e^{i\theta} z \in \mathscr{D}$ for every $z \in \mathscr{D}$ and real θ. One can prove that every bounded symmetric domain in $\mathbb{C}^n$ is biholomorphically equivalent to a bounded symmetric and circled domain which is unique up to a linear isomorphism of $\mathbb{C}^n$.

Since each s_z preserves the Bergman metric, it follows that it the geodesic symmetry around z, and that the Bergman metric is complete. Hence any two points in $\mathscr{D}$ can be joined by a geodesic, and reflection in the midpoint of this geodesic interchanges the two points. In particular, it follows that the group of automorphisms of $\mathscr{D}$ acts transitively on $\mathscr{D}$, and hence we can write

$$\mathscr{D} \cong G/K, \tag{1}$$

i.e., $\mathscr{D}$ is biholomorphically equivalent to a homogeneous space G/K, where G is a Lie group and K is a closed subgroup. In fact, G can be chosen as the connected component containing the identity element in the group $\mathrm{Aut}(\mathscr{D})$ of automorphism of $\mathscr{D}$, and K is the isotropy subgroup of the origin in G, i.e.,

$$K := \{g \in G | g(0) = 0\}. \tag{2}$$

The Lie theoretic description of bounded symmetric domains allows a classification of them. The classification reduces to a classification of irreducible bounded symmetric domains, i.e., those which are not equivalent to Cartesian products of bounded symmetric domains. The result is that any irreducible bounded symmetric domains belongs to one of six classes. Firstly, there are the four classical domains

$$\begin{array}{rrcl} I_{n,m}: & \mathscr{D} & = & \{Z \in M_{nm}(\mathbb{C}) | I_m - Z^*Z > 0\} \\ II_n: & \mathscr{D} & = & \{Z \in M_{nn}(\mathbb{C}) | I_n - Z^*Z > 0, Z = Z^t\} \\ III_n: & \mathscr{D} & = & \{Z \in M_{nn}(\mathbb{C}) | I_n - Z^*Z > 0, Z = -Z^t\} \\ IV_n: & \mathscr{D} & = & \{z \in \mathbb{C}^n | 1 - 2|z|^2 + |(z,z)|^2 > 0, |z| < 1\}, \\ & & & ((z,z) = z_1^2 + \cdots + z_n^2) \end{array}$$

Secondly, there are two exceptional domains in dimensions 16 and 27, respectively. In this thesis we shall only be concerned with the classical domains. It should be pointed out that the list of classical domains is not a disjoint list of irreducible domains. For instance, there is an isomorphism $IV_2 \cong IV_1 \times IV_1$, showing that the domain IV_2 is not even irreducible. Moreover, there are some isomorphism between the irreducible ones. Except for IV_2, all the other classical domains are irreducible, and every irreducible classical bounded symmetric domain is isomorphic to one of those listed. A complete list of all existing isomorphisms between the classical domains listed above can be found in Loos ([21]).

0.1.1. *Harish-Chandra realisation.* We shall now briefly describe how the bounded symmetric domains are obtained when the starting point is the Lie theoretic one. For a thorough treatment, we refer to Helgason [10] and to Knapp [15].

Suppose that G is a simple, connected noncompact Lie group with finite centre, and that K is a maximal compact subgroup with non-discrete centre. Then there exists a Cartan involution $\theta : G \to G$, such that its differential at the identity (which we also denote θ) is an involution of the Lie algebra $\mathfrak{g}$, admitting a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{3}$$

into the ± 1 eigenspaces for θ, where the subspace $\mathfrak{k}$ equals the Lie algebra of K. It can be proved that the centre, $\mathfrak{c}$, of $\mathfrak{k}$ is one dimensional and that the centraliser in $\mathfrak{g}$ of $\mathfrak{c}$, $Z_{\mathfrak{g}}(\mathfrak{c})$, equals the centraliser in $\mathfrak{k}$ of $\mathfrak{c}$, $Z_{\mathfrak{k}}(\mathfrak{c})$. As a consequence, any maximal abelian subalgebra of $\mathfrak{k}$ is also a maximal abelian subalgebra of $\mathfrak{g}$. We now fix a choice $\mathfrak{t} \subset \mathfrak{k}$ of such a subalgebra. Let $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}$, and let $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^{\mathbb{C}}$ denote the complexifications of $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Then we have the direct sum decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}, \tag{4}$$

and $\mathfrak{k}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$ is a maximal abelian subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

There exists an element $Z_0 \in \mathfrak{c}$ such that we have a decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^{+} \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{-}, \tag{5}$$

where $\mathfrak{p}^{\pm}$ denotes the $\pm i$-eigenspace for $\mathrm{ad}(Z_0)$.

These subspace are abelian Lie subalgebras of $\mathfrak{p}^{\mathbb{C}}$. Moreover, the relations

$$[\mathfrak{k}^{\mathbb{C}}, \mathfrak{p}^{+}] \subseteq \mathfrak{p}^{+}, [\mathfrak{k}^{\mathbb{C}}, \mathfrak{p}^{-}] \subseteq \mathfrak{p}^{-}, [\mathfrak{p}^{+}, \mathfrak{p}^{-}] \subseteq \mathfrak{k}^{\mathbb{C}} \tag{6}$$

hold.

Let $G^{\mathbb{C}}$ be a simply connected Lie group [1] with Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and let P^+, $K^{\mathbb{C}}$, and P^- denote the connected subgroups with Lie algebras $\mathfrak{p}^+$, $\mathfrak{k}^{\mathbb{C}}$, and $\mathfrak{p}^-$ respectively. Then the exponential maps $\exp : \mathfrak{p}^{\pm} \to P^{\pm}$ are diffeomorphic homomorphisms of abelian groups.

On a group level, we have the following analogue of (5).

Theorem 1 (Harish-Chandra decomposition)**.** *The multiplication map*

$$P^+ \times K^{\mathbb{C}} \times P^- \to G^{\mathbb{C}}, (p, k, q) \mapsto pkq \tag{7}$$

is injective, holomorphic and regular with open image. Moreover, there exists a bounded open subset $\mathscr{D} \subset \mathfrak{p}^+$ *such that with* $\Omega := \exp \mathscr{D} \subset P^+$

$$G\Omega \subset \Omega K^{\mathbb{C}} P^-. \tag{8}$$

Moreover, G/K *has a complex structure such that the action* $g'K \overset{g}{\mapsto} gg'K$ *of* G *is holomorphic. In fact, the map* $g \mapsto \log(g)_+$*, where* $(\cdot)_+$ *denotes the* P^+*-component, descends to a* G*-equivariant diffeomorphism between* G/K *and* $\mathscr{D}$*, and* G *acts holomorphically on* $\mathscr{D}$ *by* $g(z) := \log(g \exp z)_+$.

The complex structure J_0 at the tangent space of the identity coset, $T_{eK}(G/K) \cong \mathfrak{p}$, is of the form

$$J_0 = \mathrm{ad}(Z_0)|_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}. \tag{9}$$

(cf. [15], Thm 7.117.).

For $g \in G, z \in \mathscr{D}$, we denote the $K^{\mathbb{C}}$-component of $g \exp z$. The map $J : G \times \mathscr{D} \to K^{\mathbb{C}}$ is called the *automorphic factor*. It satisfies the cocycle condition

$$J(g_1 g_2, z) = J(g_1, g_2 z) J(g_2, z). \tag{10}$$

The diffeomorphism $G/K \cong \mathscr{D}$ gives a global trivialisation

$$T(G/K) \cong \mathscr{D} \times \mathfrak{p}^+ \tag{11}$$

of the tangent bundle. For $g \in G$ and $z \in \mathscr{D}$, the differential $dg(z)$ is given by (cf. [31])

$$dg(z)v = \mathrm{Ad}(J(g, z))v. \tag{12}$$

[1]The simple connectivity is not really necessary here, rather it singles out a canonical choice of complexification since it gives the universal covering of any other choice. In the case that G is a linear algebraic group, $G^{\mathbb{C}}$ can also be naturally chosen as a linear algebraic group. This will be the case in the next example.

Example 2.

$$\begin{aligned} G &= SU(n,m) \\ &:= \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(n+m,\mathbb{C}) | g^* J g = J, \det g = 1 \right\}, \\ K &= S(U(n)\times U(m)) \\ &:= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} | A \in U(n), D \in U(m), \det(A)\det(D) = 1 \right\}, \end{aligned}$$

where the sizes of the blocks are determined by A being $n \times n$, and $J = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$. Moreover,

$$G^{\mathbb{C}} = SL(n+m,\mathbb{C}) \tag{13}$$

can be chosen as a linear algebraic group. On the level of Lie algebras we have

$$\begin{aligned} \mathfrak{g}^{\mathbb{C}} &= \mathfrak{sl}(n+m,\mathbb{C}) = \{ Z \in M_{n+m}(\mathbb{C}) | \text{Trace } Z = 0 \}, \\ \mathfrak{k}^{\mathbb{C}} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} | \text{Trace } A + \text{Trace } B = 0 \right\}, \\ Z_0 &= \frac{m}{m+n} \begin{pmatrix} iI_n & 0 \\ 0 & -i\frac{n}{m} I_m \end{pmatrix}, \\ \mathfrak{p}^+ &= \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} | Z \in M_{nm}(\mathbb{C}) \right\}, \\ \mathfrak{p}^- &= \left\{ \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} | Z \in M_{mn}(\mathbb{C}) \right\}. \end{aligned}$$

The $P^+K^{\mathbb{C}}P^-$-decomposition is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_n & BD^{-1} \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_n & 0 \\ D^{-1}C & I_m \end{pmatrix}.$$

So, for

$$\begin{aligned} z &= \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \in \mathscr{D}, \qquad \exp(z) = \begin{pmatrix} I_n & Z \\ 0 & I_m \end{pmatrix} \in \Omega, \\ g &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n,m), \\ g\exp(z) &= \begin{pmatrix} A & AZ+B \\ C & CZ+D \end{pmatrix}, \end{aligned} \tag{14}$$

and the logarithm of the P^+-component is
$\begin{pmatrix} 0 & (AZ+B)(CZ+D)^{-1} \\ 0 & 0 \end{pmatrix}$. The identification $z \leftrightarrow Z$, with z and Z as in (14) defines a biholomorphic equivalence between the domain $I_{n.m}$ and the subset $\mathscr{D} \subset \mathfrak{p}^+$ in the Harish-Chandra decomposition. In particular, it follows that the fractional linear action

$$Z \mapsto (AZ+B)(CZ+D)^{-1}, Z \in \mathscr{D}, g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n,m)$$

gives a description of the type $I_{n,m}$ domain as the homogeneous space $SU(n,m)/S(U(n) \times U(m))$. Note that as a special case we have the description $SU(1,1)/U(1)$ of the unit disc.

0.2. **Homogeneous vector bundles.** Most of the representations that we consider in this thesis are defined on spaces of sections of homogeneous vector bundles. We therefore give a brief introduction to this topic here. Examples will be given in the following section that treats weighted Bergman spaces.

Definition 3. Let G/K be a homogeneous space. We fix the point $0 := eK$ as a reference point.

A vector bundle $\mathcal{V} \to G/K$ is said to be homogeneous if G acts on it by bundle automorphisms.

This means that for each $x \in G/K$ there is a linear mapping

$$g_x : \mathcal{V}_x \to \mathcal{V}_{gx}. \tag{15}$$

In particular, the fibre $\mathcal{V}_0$ is the representation space for a representation of K.

Conversely, if τ is a representation of K on a finite dimensional vector space V, we define the equivalence relation

$$(g,v) \sim (gk^{-1}, \tau(k)v) \qquad \forall (g,v) \in G \times V, \forall k \in K \tag{16}$$

on the Cartesian product $G \times K$. We let $[(g,v)]$ denote the equivalence class of the pair $(g,v)]$, and and we let

$$G \times_K V := \{[(g,v)] | (g,v) \in G \times V\} \tag{17}$$

be the set of equivalence classes. Then $G \times_K V$ carries the structure of a smooth vector bundle over G/K when equipped with the projection

$$\begin{aligned} p : G \times_K V &\to G/K, \\ p([(g,v)]) &:= gK. \end{aligned}$$

The continuous sections of this bundle can be identified with the set

$$C(G,\tau) := \{f \in C(G,V) | f(gk^{-1}) = \tau(k)f(g)\}. \tag{18}$$

Indeed, if $f \in C(G,\tau)$, the mapping

$$s_f : G/K \to G \times_K V,$$
$$s_f(gK) := [(g, f(g))]$$

defines a continuous section. On the other hand, for a section s, we can write

$$s(gK) = [(g, f_s(g))] \tag{19}$$

for a unique function $f : G \to V$. This is well defined with respect to the equivalence relation (16) if and only if f_s satisfies the property (18). The mappings $s \mapsto f_s$ and $f \mapsto s_f$ are the inverses of each other, and this characterises the space of continuous sections.

0.3. **Weighted Bergman spaces.** We now let $\mathscr{D} \cong G/K$ be a bounded symmetric domain of complex dimension n. The differential action of K on the tangent space $T_{eK}(G/K)$ is equivalent to the representation $\mathrm{Ad}_{\mathfrak{p}}$ of K. Taking exterior powers of the dual version of this isomorphism, we obtain an isomorphism between the representations $\overset{n}{\wedge}\ d^*(eK)$ and δ, where $\overset{n}{\wedge}\ d^*(eK)$ is the representation on the top power in the exterior algebra $\wedge T^*_{eK}$, and the latter representation is defined as

$$\delta(k) := \det(\mathrm{Ad}^*_{\mathfrak{p}}(k)). \tag{20}$$

The smoothly induced representation $\mathrm{Ind}_K^G(\delta)$ is defined on the space of smooth sections in the homogeneous vector bundle $G \times_K \overset{n}{\wedge}\ (\mathfrak{p}^*)^{\mathbb{C}}$, which can be identified with the space of sections in the complex determinant bundle. We can also form the analogous construction, starting with the representation $S^m(\delta)$, i.e., with the mth symmetric tensor of δ on the corresponding vector space $S^m\left(\overset{n}{\wedge}\ (\mathfrak{p}^*)^{\mathbb{C}}\right)$. The homogeneous vector bundle $G \times_K \overset{n}{\wedge}\ (\mathfrak{p}^*)^{\mathbb{C}}$ is then isomorphic to the mth symmetric power of the determinant bundle. Since the isomorphism of vector bundles is G-equivariant, the induced action on sections is equivalent with the action as pullbacks on top-forms and their symmetric tensor powers.

The biholomorphic equivalence $G/K \cong \mathscr{D}$ realising the symmetric space G/K as a bounded open domain in $\mathfrak{p}^+$, together with the description of the group action in this model, can be used to define a global trivialisation of the aforementioned vector bundles. Indeed, for any holomorphic representation, σ, of $K^{\mathbb{C}}$ on a finite dimensional complex vector space, V^σ, we can define an isomorphism of vector bundles

$$\begin{gathered}\varphi_\sigma : G \times_K V^\sigma \to \mathscr{D} \times V^\sigma \\ \varphi_\sigma([(g,v)]) = (g(0), \sigma(J(g,0))v).\end{gathered} \tag{21}$$

An action of G as vector bundle autmorphisms of $\mathscr{D} \times V^\sigma$ is defined by the requirement that φ_σ be G-equivariant; namely

$$(g)_z(z,v) := (g(z), \sigma(J(g,z))v). \tag{22}$$

If $F : G \to V^\sigma$ is a K-equivariant function corresponding to a section of the bundle $G \times_K V^\sigma$, it defines a function $f : \mathscr{D} \to V^\sigma$ by

$$f(g \cdot 0) = \sigma(J(g,0))F(g). \tag{23}$$

By the K-equivariance of F, this is indeed well-defined. The action of G on the space of sections then translates to an action on V^σ-valued functions by

$$gf(z) := \sigma(J(g^{-1},z))^{-1} f(g^{-1}z). \tag{24}$$

In particular, when $\sigma = Ad(k)^*|_{\mathfrak{p}}$ we get a trivialisation of the cotangent bundle. Taking σ to be the nth exterior power of $Ad(k)^*|_{\mathfrak{p}}$, (21) gives a trivialisation of the determinant bundle. The action of G on sections now corresponds to the action

$$gf(z) := \det(dg^{-1}(z)) f(g^{-1}z) = J_{g^{-1}}(z) f(g^{-1}z) \tag{25}$$

on complex valued functions on $\mathscr{D}$. Analogously, the action of G on sections in the mth symmetric power of the determinant bundle corresponds to the action

$$gf(z) = J_{g^{-1}}(z)^m f(g^{-1}z) \tag{26}$$

on complex valued functions in the trivialised picture.

For a fixed K-invariant inner product, $\langle\, , \,\rangle$, on V^σ, we define an Hermitian metric on $\mathcal{V}^\sigma := G \times_K V^\sigma$ by

$$h_z(u,v) := \langle (g^{-1})_z u, (g^{-1})_z v \rangle_k, \qquad u, v \in \mathcal{V}^k_z, \tag{27}$$

where $z = gK$ and $(g^{-1})_z$ denotes the fibre map $\mathcal{V}^\sigma_z \to \mathcal{V}^\sigma_0 \cong V^\sigma$ associated with g^{-1}. For a fixed choice, ι, of G-invariant measure on

G/K we define $L^2(\mathrm{Ind}_K^G(\sigma))$ as the Hilbert space completion of the space

$$\left\{ s \in \Gamma(G/K, \mathcal{V}^\sigma) | \int_{G/K} h_z(s,s) d\iota(z) < \infty \right\}. \tag{28}$$

In view of (22), the norm on V^σ-valued functions is given by

$$(f,f) := \int_{\mathscr{D}} \|\sigma(J(g^{-1},z))f(z)\|^2 d\iota(z). \tag{29}$$

The holomorphic functions in $L^2(\mathrm{Ind}_K^G(\sigma))$ form a closed subspace

$$\mathcal{B}^2(\sigma) := \left\{ f \in \mathcal{O}(\mathscr{D}, V^\sigma) | \int_{\mathscr{D}} \|\sigma(J(g^{-1},z))f(z)\|^2 d\iota(z) < \infty \right\}.$$

Since G acts holomorphically on $\mathscr{D} \times V^\sigma$ and the Hermitian metric is G-invariant, it follows that $\mathcal{B}^2(\sigma)$ is a G-invariant subspace of $L^2(\mathrm{Ind}_K^G(\sigma))$ and that the action (26) defines a unitary representation of G. We temporarily denote this representation by $\mathcal{O}(\mathrm{Ind}_K^G(\sigma))$. When $\sigma = S^m(\delta)$ for some m, these representation spaces are the *weighted Bergman spaces*. For each $z \in \mathscr{D}$, the evaluation functional

$$ev_z : f \mapsto f(z) \in V^\sigma \cong \mathbb{C} \tag{30}$$

is a bounded linear functional on $\mathcal{B}^2(S^m(\delta))$. Hence, $\mathcal{B}^2(S^m(\delta))$ admits a reproducing kernel, i.e., a function

$$K : \mathscr{D} \times \mathscr{D} \to \mathbb{C}$$

which is holomorphic in the first argument, antiholomorphic in the second one, such that $K(\cdot, w) \in \mathcal{B}^2(S^m(\delta))$ and

$$f(z) = (f, K(\cdot, z)), \qquad f \in \mathcal{B}^2(S^m(\delta)), z \in \mathscr{D}. \tag{31}$$

It follows by a theorem by S. Kobayashi (cf. [17]) that all the representations
$\mathcal{O}(\mathrm{Ind}_K^G(S^m(\delta)))$ are irreducible.

0.4. **Analytic continuation of the scalar holomorphic discrete series.** The spaces $\mathcal{B}^2(S^m(\delta))$ from the previous section all have reproducing kernels of the form $h(z,w)^{-mc_0}$, where h is a polynomial, holomorphic in z and anti-holomorphic in w, and c_0 is a constant related to the Lie algebra $\mathfrak{g}$ (cf. [7]). The set of all positive real ν for which the kernel function

$$h(z,w)^{-\nu} \tag{32}$$

is positive definite is called the *Wallach set*. It is given by

$$\mathcal{W} = \{(a/2)j | 0 \leq j \leq r-1\} \cup ((a/2)(r-1), \infty), \tag{33}$$

where a is a constant related to the root system for $\mathfrak{g}^{\mathbb{C}}$, and r is the rank of $\mathfrak{g}^{\mathbb{C}}$. For $\nu \in \mathcal{W}$, $h(z,w)^{-\nu}$ is the reproducing kernel for a Hilbert space, $\mathscr{H}_\nu$, of holomorphic functions on $\mathscr{D}$, and one defines an irreducible (projective) representation, π_ν, of G by

$$\pi_\nu(f)(z) = (\det dg^{-1}(z))^{\nu/c_0} f(g^{-1}z). \tag{34}$$

These (projective) representations constitute the *analytic continuations of the scalar holomorphic discrete series.*

This was proved by Wallach (cf. [37]), who worked algebraically with highest weight modules for the universal enveloping algebra $U(\mathfrak{g}^{\mathbb{C}})$ of $\mathfrak{g}^{\mathbb{C}}$, and independently by Rossi and Vergne ([30]) who used more analytic techniques.

Rossi and Vergne were able to realise all the spaces corresponding to the discrete points in the Wallach set as L^2-spaces of functions defined on some boundary orbits on a convex symmetric cone [2]. The number r in fact coincides with the so called rank for the cone, and this number also counts the number of orbits on the boundary of the cone under the automorphism group of the cone. The smallest nonzero discrete point in the Wallach set corresponds to the *minimal representation.* By the Rossi-Vergne characterisation, it is realised on a space of functions on the set of elements of minimal rank on the boundary of some cone. This is the model we use for the minimal representation in paper II, where we construct an even more explicit realisation.

0.5. **The branching rule.** Assume that $(\pi, \mathcal{H})$ is an irreducible unitary representation of a Lie group G. If $H \subset G$ is a closed subgroup, the restriction, π_H, of π to H need not be irreducible. One might therefore be interested in the decomposition

$$\pi_H \cong \int_{\widehat{H}} \sigma d\mu(\sigma)$$

[2]Actually, this holds only for the *tube domains*. In general, each point in the orbit parametrises a Fock-space and one considers functions of two variables such that the Fock-norm in one variable (taken pointwise) is a square integrable function of the other variable.

of π_H into a *direct integral* of irreducible representations. Here $\widehat{H}$ denotes the unitary dual of H, and μ is some positive Borel measure on $\widehat{H}$. The decomposition of this restriction is called a *branching rule* for the pair (G, H). A famous classical example of this is the Clebsch-Gordan decomposition for the restriction of the tensor product of two irreducible $SU(2)$-representations (which is a representation of $SU(2) \times SU(2)$) to the diagonal subgroup. For an introduction to the general theory for compact connected Lie groups, we refer to [15]. When the groups G and H are noncompact, there is yet no general theory.

Since the work by Howe ([11]) and by Kashiwara-Vergne ([13]), the study of branching rules for singular and minimal representations on spaces of holomorphic functions on bounded symmetric domains has been an active area of research. In [12], Jakobsen and Vergne studied the restriction to the diagonal subgroup of two holomorphic representations. More recently, Peng and Zhang ([29]) studied the corresponding decomposition for the tensor product of arbitrary (projective) representations in the analytic continuation of the scalar holomorphic discrete series. Zhang also studied the restriction to the diagonal of a minimal representation in this family tensored with its own anti-linear dual ([40]).

In this thesis we are concerned with the following situation. Let $\mathscr{D} = G/K$ be a bounded symmetric domain. Let $\tau : \mathscr{D} \to \mathscr{D}$ be an anti-holomorphic involutive diffeomorphism which lifts to an involution of the group G by $g \mapsto \tau g \tau^{-1}$. The fixed point group

$$H := G^\tau = \{g \in G | \tau g \tau^{-1} = g\}$$

then acts transitively on the fixed point set

$$\mathscr{X} := \{z \in \mathscr{D} | \tau(z) = z\},$$

and if we define $L := H \cap K$, we have the identification $\mathscr{X} = H/L$. The manifold $\mathscr{X}$ is then a totally real submanifold of $\mathscr{D}$, and so the restriction of holomorphic function on $\mathscr{D}$ of $\mathscr{X}$ is injective. It is therefore of interest to restrict (projective) representations of G on spaces of holomorphic functions on $\mathscr{D}$ to the subgroup H and find the irreducible decomposition. This problem has been studied recently by Davidson, Ólafsson, and Zhang ([6]), Neretin ([22], [23], van Dijk and Pevzner ([35]), and Zhang ([39], [40], [41]). More specifically, in this thesis we restrict the (projective) representations π_ν from the previous section to symmetric subgroups and find explicit Plancherel measures for the direct integral decompositions. We now briefly describe the contents of the first three papers that are concerned with this branching problem.

Paper I: In this paper, $\mathscr{D} \subset \mathbb{C}^n$ is the bounded domain of type IV, also known as the *Lie ball*. The involution τ is the standard coordinatewise conjugation, and the corresponding real submanifold, $\mathscr{X}$, is the real n-dimensional unit ball. In terms of Lie groups, $G = SO(2,n), H = SO(1,n)$. We consider representations π_ν for arbitrary ν in the Wallach set and find the branching rule for the restrictions to H. An explicit intertwining unitary operator is defined using a power series expansion for the spherical eigenfunctions of the Casimir operator associated with $\pi_\nu|_H$. The restriction of the decomposing unitary operator to the subspace $\mathscr{H}_\nu^L$ of L-invariants is mapped onto an L^2 space with an orthonormal basis given by certain continuous dual Hahn polynomials. The restriction of the minimal representation is proven to be irreducible by realising it as a Hilbert space of functions on the unit sphere S^n with explicitly given inner product.

Also, in this paper we prove the following general decomposition theorem.

Theorem 4. *Let π be a unitary representation of the semisimple Lie group H on a Hilbert space, $\mathscr{H}$. Suppose further that L is a maximal compact subgroup and that the representation has a cyclic L-invariant vector. Then π can be decomposed as a multiplicity-free direct integral of irreducible representations,*

$$\pi \cong \int_\Lambda \pi_\lambda \, d\mu(\lambda), \tag{35}$$

where Λ is a subset of the set of positive definite spherical functions on H and for $\lambda \in \Lambda, \pi_\lambda$ is the corresponding unitary spherical representation.

The idea of the proof is to use representation theory for C^*-algebras.

An important point to be made here is that we obtain the multiplicity-freeness directly from the construction; we obtain a decomposition on the $L^1(H)$-level which is multiplicity-free. Hence also the derived decomposition for the group representation is multiplicity-free. We will return to this issue in the section “Recent advancements”.

Paper II: In this paper we treat the branching rules for the pairs $(Sp(n,\mathbb{R}), GL(n,\mathbb{R}))$ and $(SU(n,n), GL(n,\mathbb{C}))$ and for the respective minimal representations. The corresponding bounded symmetric domains are those of type II_n and $I_{n,n}$ respectively. In the first case, an antiholomorphic involution, τ, is furnished by conjugating with respect to the real form given by the set of real symmetric matrices. In the second case, we consider the conjugation with respect to the Hermitian $n \times n$-matrices. Both domains are tube domains, and the groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ are the autmorphism groups for the associated symmetric cones consisting of the positive definite real symmetric matrices and the positive definite Hermitian matrices respectively. We give explicit realisations for the models of the minimal representations as L^2-spaces on boundary orbits for the cones. Also, the spherical representations occurring in Theorem 4 are given explicit realisations admitting the construction of intertwining operators as (the "analytic continuations of") integral operators. Inversion formulas and Plancherel theorems are proved.

The proof of the surjectivity of the intertwining operators relies on uniqueness properties coming from the construction of the intertwining operator of Theorem 4.

Paper III: In this paper, we consider the minimal representation for the group $SU(n,m)$ $(n \geq m)$ and find the branching law for the pair $(SU(n,m), SO(n,m))$. We find it by considering the spectral decomposition for the associated Casimir operator. By the construction in Theorem 4, it suffices to consider the subspace of L-invariants, which is invariant under the Casimir operator. We prove that the Casimir operator acts on this subspace as a Jacobi operator and identify the matrix elements. Thereby, we are able to find unitary operator intertwining the Casimir operator with a multiplication operator on an L^2-space for which certain continuous dual Hahn polynomials furnish an orthonormal basis. The corresponding Plancherel measure is shown to have point masses precisely when $n - m > 2$. In this case, we construct a Hilbert space of (equivalence classes of) functions on the real Stiefel manifold given by the real part of the Shilov boundary of $\mathscr{D}$. This Hilbert space carries a natural unitary representation of $SO(n,m)$. We identify it with a parabolically induced representation and construct an intertwining operator that embeds it unitarily into the representation space for the minimal representation. Finally, we identify it with a certain point in the set of point masses for the Plancherel measure.

Also in this paper, the surjectivity of the intertwining operator relies on uniqueness properties in the proof of Theorem (4).

0.6. **Recent advancements on decomposition.** We recall the discussion of
multiplicity-freeness following Theorem 4.

On the other hand, all the representations defined as restrictions to symmetric subgroups that we have considered in this thesis can be seen to be multiplicity-free, given that one knows that a direct integral decomposition exists. This follows from a recent theorem by Kobayashi. The theorem assumes some geometric conditions on the action of a Lie group on a holomorphic vector bundle over a complex manifold. We shall now give some background to these conditions, state the theorem, and then see how it fits into the framework of the present thesis.

Definition 5. Let M be a connected complex manifold with a complex structure J, H a Lie group acting holomorphically on M. We say that the action of H is *visible* if there exists an H-invariant non-empty open subset $D \subseteq M$, and a totally real submanifold, $S \subseteq M$ such that
(i) $S \cap H \cdot z \neq \emptyset$, for every $z \in D$,
(ii) $J_z T_z(S) \subseteq T_z(H \cdot z)$, for every $z \in S$ (J-transversality).

A stronger condition on the group action is provided in the following definition.

Definition 6. The action of H is *strongly visible* if there exists an H-invariant open subset $D \subseteq M$, a submanifold, S, of D, and an anti-holomorphic diffeomorphism, σ, of D satisfying the three conditions
(i), $S \cap H \cdot z \neq \emptyset$, for every $z \in D$,
(ii) $\sigma|_S = \mathrm{id}$,
(iii) σ preserves each H-orbit in D.

Also, an important aspect is the compatibility of twisting the H-action by σ with the automorphism of H, i.e., if the action of H on D given by $z \overset{h}{\mapsto} \sigma(h\sigma^{-1}(x))$ can be realised by composing the given action by a group automorphism. More precisely, the definition is as follows.

Definition 7. The strongly visible action of H has a compatible automorphism if there exists an automorphism $\tilde{\sigma} \in \mathrm{Aut}(H)$ such that

$$\tilde{\sigma}(h)z = \sigma(h\sigma^{-1}(z)), z \in D.$$

Kobayashi's theorem on propagation of the multiplicity free property deals with unitary representations of a Lie group, H, that can be realised in the space of holomorphic sections of a vector bundle, i.e., that there is an H-equivariant continuous embedding $\mathcal{H} \to \mathcal{O}(M, \mathcal{V})$, where $\mathcal{H}$ denotes the representation space, and $\mathcal{V} \to M$ is a Hermitian holomorphic H-equivariant vector bundle. We are now ready to state the theorem.

Theorem 8 (T. Kobayashi)**.** *Let $\mathcal{V} \to M$ be a holomorphic, Hermitian vector bundle over the connected complex manifold M. Assume that the Lie group H acts by isometric automorphism of the bundle such that the following three conditions are satisfied:*

(i) The action on the base space is strongly visible and with a compatible automorphism of the group H (see Def. 7).
(ii) The representation of the isotropy group, H_z, of z on the fibre $\mathcal{V}_z$ is multiplicity-free for any $z \in S$.
We write its decomposition into irreducibles as

$$\mathcal{V}_z = \bigoplus_{i=1}^{n(z)} \mathcal{V}_z^{(i)}.$$

(iii) The diffeomorphism σ lifts to an anti-holomorphic endomorphism (which we also denote by σ) of $\mathcal{V}$ such that

$$\sigma_z(\mathcal{V}_z^{(i)}) = \mathcal{V}_z^{(i)}, 1 \leq i \leq n(z), z \in S. \tag{36}$$

Then, any unitary representation of H which is realised in $\mathcal{O}(M, \mathcal{V})$ is multiplicity-free.

The representations we are interested in decomposing in this thesis are all given as restrictions of representations of a semisimple group G to the fixed point group, G^τ, of an involution. The involution τ comes from an anti-holomorphic involutive diffeomorphism (which we also denote by τ) of the bounded symmetric domain, $\mathscr{D}$, by $\tau(g) = \tau g \tau^{-1}$. Kobayashi has proved that in this setting the action of $H = G^\tau$ on $\mathscr{D}$ is always strongly visible (cf. [20]). In fact, the submanifold S can be taken as $\exp \mathfrak{a} \cdot 0$, and the diffeomorphism σ can be constructed directly on the group level from τ by $\sigma := \tau\theta$, where θ is the Cartan involution of G. The diffeomorphism σ on $\mathscr{D}$ is then of the form $\sigma(gK) = \sigma(g)K$ and hence the compatibility condition (Def. 7) follows by definition. Finally, since our vector bundles are trivial, σ automatically lifts to an antiholomorphic endomorphism of $\mathscr{D} \times \mathbb{C}$ by setting $(z, v) \mapsto (\sigma(z), \overline{v})$.

0.7. **Quaternionic discrete series.** The representations on weighted Bergman spaces that we have discussed all have the common feature that they belong to the *discrete series* for G, namely, if $(\pi, \mathcal{H})$ denotes one of these representations, then the function

$$m_{u,v} : G \to \mathbb{C},$$
$$m_{u,v}(g) = \langle \pi(g)u, v\rangle$$

is in $L^2(G)$ for all $u, v \in \mathcal{H}$. In general, discrete series representations cannot be realised as spaces of holomorphic sections of holomorphic vector bundles. However, if G is a connected linear semi-simple group with a compact Cartan subgroup $T \subset G$, the quotient space G/T is a complex Kähler manifold. Any discrete series representation of G can be realised as the space $H^p(G/T, \mathcal{L})$ of square integrable harmonic $(0, p)$ forms with values in a holomorphic line bundle, $\mathcal{L}$ over G/T. This was proved by Schmid in a series of papers ([32], [33], [34]) ending with [34].

In [9], Gross and Wallach considered representations of simple Lie groups G with maximal compact subgroup K such that the associated symmetric space G/K has a G-equivariant quaternionic structure (cf. [38]). This amounts to the group K containing a normal subgroup isomorphic to $SU(2)$. In fact, there is an isomorphism $K \cong SU(2) \times M$ for a subgroup $M \subseteq K$, and by setting $L := U(1) \times M$, the associated homogeneous space G/L is fibred over G/K with fibres diffeomorphic to $P^1(\mathbb{C})$. The *quaternionic discrete series representations* are then realised on the sheaf cohomology groups $H^1(G/L, \mathcal{L})$, where $\mathcal{L} \to G/L$ is a holomorphic line bundle. In this model they are able to classify all the K-types occurring in each of the obtained discrete series representations. Moreover, they consider the continuation of the discrete series and characterise the unitarisability of the underlying $(\mathfrak{g}, K)$-modules.

Paper IV:

In this paper we work with another model for discrete series given by Schmid in his thesis ([32]) . If π is a quaternionic discrete series representation realised on the cohomology group $H^1(G/L,\mathcal{L})$, and τ is its minimal K-type, then the *Schmid D-operator* acts on the sections of the homogeneous vector bundle $G\times_K V_\tau \to G/K$ where V_τ is some vector space on which the K-type is unitarily realised. The Hilbert space $\ker D \cap L^2(G,\tau)$ then furnishes another realisation of the representation π. We consider the special case when $G = Sp(1,1)$. In this case the symmetric space G/K can be embedded into the bounded symmetric domain $SU(2,2)/S(U(2)\times U(2))$ consisting of complex 2×2-matrices of norm less than one. The restriction of the Harish-Chandra embedding to G/K then yields a global trivialisation of the vector bundle $G\times_K V_\tau$. In this model we compute the restrictions to the submanifold $A\cdot 0$ [3] of the highest weight vectors for the occurring K-types. This is carried out by using the Szegö map defined by Knapp and Wallach in [16] which exhibits any discrete series representation as a quotient of a nonunitary principal series representations. The K-types are determined on the level of the principal series representation, and then the Szegö map is applied to compute the above mentioned restrictions.

These functions turn out to be fibrewise highest weight vectors with a hypergeometric function as a coefficient. Similar functions have been studied by Castro and Grünbaum in [5]. As we already saw in the descriptions of papers I and III, hypergeometric functions occur frequently in representation theory. For an example outside the theory of Lie groups, we refer to [27], where they play a role in the context of Hecke algebras.

0.8. **Open problems.**

0.8.1. *Branching laws.* In this thesis we have only considered representations in the scalar holomorphic discrete series and its analytic continuation. The restriction to a symmetric subgroup also makes sense if we consider holomorphic sections of vector bundles over the symmetric space G/K, or, in the trivialisation, vector valued holomorphic functions. Related spaces of functions have been consider by other people. For instance, in [28], Pedon considers spaces of harmonic (p,q)-forms over the complex unit ball. In the context of restricting holomorphic

[3] A is associated with a particular Iwasawa decomposition $G = NAK$.

sections to a real form, there is an operator constructed by Ólafsson and Ørsted, the *generalised Segal-Bargmann transform*, that maps holomorphic square integrable sections in $G \times_K V$ H-equivariantly into a dense subspace of the Hilbert space $L^2(\mathrm{Ind}_L^H(\sigma_L))$ (cf.[25]). An account of this can also be found in Ólafssons lecture notes [24]. It is interesting in this context that Camporesi has proved a generalisation of Helgasons Plancherel formula for $L^2(H/L)$ to the space $L^2(\mathrm{Ind}_L^H(\sigma_L))$ for a general symmetric space H/L of the noncompact type (cf. [3], [4]). It would therefore be interesting to study the branching law for the pair (G, H) for some suitable vector valued holomorphic representations.

0.8.2. *Quaternionic discrete series.* In Paper *IV* we considered only discrete series. In general, Gross and Wallach in [9] gave a criterion for the existence of a continuation of the quaternionic discrete series. Also, they classified the unitarisability of the underlying $(\mathfrak{g}, K)$-modules. Their proof uses Vogan's method of unitarising the derived functor modules constructed by cohomological induction (cf. [36]).

It would be a challenging problem to realise these Hilbert spaces analytically with an explicit inner product. A related problem is to understand the reproducing kernels for the discrete series representations by some geometric and/or algebraic construction and to see whether they admit an analytic continuation in some parameter, as is the case for the holomorphic discrete series.

References

[1] Bertram, Wolfgang, "The geometry of Jordan and Lie structure", Lecture Notes in Mathematics, 1754, Springer-Verlag, Berlin, 2000

[2] Bertram, Wolfgang and Hilgert, Joachim, *Reproducing Kernels on Vector Bundles*, Lie theory and its applications in physics. II Proceedings of the 2nd International Workshop held at the Technical University of Clausthal, Clausthal, August 17–20, 1997, Doebner, H.-D. and Dobrev, V. K. and Hilgert, J., World Scientific Publishing Co. Inc., River Edge, NJ, 1998

[3] Camporesi, Roberto, *The Helgason Fourier transform for homogeneous vector bundles over Riemannian symmetric spaces*, Pacific J. Math., **179**, (1997),2, 263–300

[4] Camporesi, Roberto, *The spherical transform for homogeneous vector bundles over Riemannian symmetric spaces*, J. Lie Theory, **7**, (1997), 1, 29–60

[5] Castro, Mirta M. and Grünbaum, F. Alberto, *The algebra of differential operators associated to a family of matrix-valued orthogonal polynomials: five instructive examples*, Int. Math. Res. Not.,2006

[6] Davidson, Mark and Ólafsson, Gestur and Zhang, Genkai, *Laplace and Segal-Bargmann transforms on Hermitian symmetric spaces and orthogonal polynomials*, J. Funct. Anal., **204**, (2003), 1, 157–195

[7] Faraut, J. and Korányi, A.*Function spaces and reproducing kernels on bounded symmetric domains* J. Funct. Anal. **88**,(1990), 1, 64–89

[8] Faraut, J. and Korányi, A. "Analysis on symmetric cones" Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1994

[9] Gross, Benedict H. and Wallach, Nolan R., *On quaternionic discrete series representations, and their continuations*, J. Reine Angew. Math.,**481**, (1996), 73–123

[10] Helgason, Sigurdur, "Differential geometry, Lie groups, and symmetric spaces", Pure and Applied Mathematics, 80, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978

[11] Howe, Roger *Transcending classical invariant theory* J. Amer. Math. Soc., **2**, (1989, 3, 535–552

[12] Jakobsen, Hans Plesner and Vergne, Michèle *Restrictions and expansions of holomorphic representations* J. Funct. Anal., **34**, (1979), 1, 29–53

[13] Kashiwara, M. and Vergne, M. *On the Segal-Shale-Weil representations and harmonic polynomials* Invent. Math., **44**, (1978), 1, 1–47

[14] Knapp, Anthony W. "Representation theory of semisimple groups, an overview based on examples", Princeton University Press, Princeton, NJ, 2001

[15] Knapp, Anthony W., "Lie groups beyond an introduction", Progress in Mathematics, 140, Birkhäuser Boston Inc., Boston, MA, 2002

[16] Knapp, A. W. and Wallach, N. R., *Szegö kernels associated with discrete series,* Invent. Math., **34**, (1976), 3, 163–200

[17] Kobayashi, Shoshichi, *Irreducibility of certain unitary representations*, J. Math. Soc. Japan, **20**, (1968), 638–642

[18] Kobayashi, Toshiyuki, *Multiplicity-free representations and visible actions on complex manifolds,* Publ. Res. Inst. Math. Sci., Kyoto University. Research Institute for Mathematical Sciences. Publications, **41**, (2005), 3, 497–549

[19] Kobayashi, Toshiyuki, *Propagation of multiplicity-free property for holomorphic vector bundles,* Preprint.

[20] Kobayashi, Toshiyuki, *Visible actions on symmetric spaces,* Preprint.

[21] Loos, O. "Bounded symmetric domains and Jordan pairs" Lecture notes, University of California, Irvine, 1977

[22] Neretin, Yu. A. *Beta integrals and finite orthogonal systems of Wilson polynomials* Mat. Sb., **193**, (2002), 7, 131–148

[23] Neretin, Yurii A. *Plancherel formula for Berezin deformation of L^2 on Riemannian symmetric space* J. Funct. Anal., **189**, (2002), 2, 336–408

[24] Ólafsson, Gestur, *Analytic continuation in representation theory and harmonic analysis*, Global analysis and harmonic analysis (Marseille-Luminy, 1999), Sémin. Congr., 4, 201–233, Soc. Math. France, Paris, 2000

[25] Ólafsson, Gestur and Ørsted, Bent, *Generalizations of the Bargmann transform*, Lie theory and its applications in physics (Clausthal, 1995), 3–14, World Sci. Publ., River Edge, NJ, 1996

[26] Ørsted, Bent and Vargas, Jorge, Restriction of square integrable representations: discrete spectrum, Duke Math. J., **123**, (2004),3, 609–633

[27] Opdam, Eric M., *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math., **175**, (1995), 1, 75–121

[28] Pedon, Emmanuel, *Harmonic analysis for differential forms on complex hyperbolic spaces*, J. Geom. Phys.,**32**,(1999),2, 102–130

[29] Peng, Lizhong and Zhang, Genkai, *Tensor products of holomorphic representations and bilinear differential operators*, J. Funct. Anal., **210**, 2004, 1, 171–192

[30] Rossi, H. and Vergne, M. *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group* Acta Math., **136**, (1976), 1-2, 1–59

[31] Satake, Ichirô, "Algebraic structures of symmetric domains", Kanô Memorial Lectures, 4, Iwanami Shoten, Tokyo, 1980

[32] Schmid, Wilfried, *Homogeneous complex manifolds and representations of semisimple Lie groups*, Representation theory and harmonic analysis on semisimple Lie groups, Math. Surveys Monogr., **31**, 223–286 Dissertation, University of California, Berkeley, CA, 1967

[33] Schmid, Wilfried, *Some properties of square-integrable representations of semisimple Lie groups*, Ann. of Math. (2), **102**,(1975),3, 535–564

[34] Schmid, Wilfried, L^2-cohomology and the discrete series, Ann. of Math. (2), **103**,(1976), 2, 375–394

[35] van Dijk, G. and Pevzner, M. *Berezin kernels of tube domains* J. Funct. Anal., **181**, (2001), 2, 189–208

[36] Vogan, Jr., David A., *Unitarizability of certain series of representations*, Ann. of Math. (2), **120**,(1984),1, 141–187

[37] Wallach, Nolan R. *The analytic continuation of the discrete series. I, II*, Trans. Amer. Math. Soc., **251**, (1979), 1–17, 19–37

[38] Wolf, Joseph A., *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech., **14**, 1965, 1033–1047

[39] Zhang, Genkai, *Berezin transform on real bounded symmetric domains*, Trans. Amer. Math. Soc., **353**, (2001), 9, 3769–3787 (electronic)

[40] Zhang, Genkai, *Tensor products of minimal holomorphic representations*, Represent. Theory, **5**, (2001), 164–190 (electronic)

[41] Zhang, Genkai, *Branching coefficients of holomorphic representations and Segal-Bargmann transform* J. Funct. Anal., **195**, (2002), 2, 306–349

Paper I

BRANCHING OF SOME HOLOMORPHIC REPRESENTATIONS OF SO(2,N)

HENRIK SEPPÄNEN

Abstract. In this paper we consider the analytic continuation of the weighted Bergman spaces on the Lie ball

$$\mathscr{D} = SO(2,n)/S(O(2) \times O(n))$$

and the corresponding holomorphic unitary (projective) representations of $SO(2,n)$ on these spaces. These representations are known to be irreducible. Our aim is to decompose them under the subgroup $SO(1,n)$ which acts as the isometry group of a totally real submanifold $\mathscr{X}$ of $\mathscr{D}$. We give a proof of a general decomposition theorem for certain unitary representations of semisimple Lie groups. In the particular case we are concerned with, we find an explicit formula for the Plancherel measure of the decomposition as the orthogonalising measure for certain hypergeometric polynomials. Moreover, we construct an explicit generalised Fourier transform that plays the role of the intertwining operator for the decomposition. We prove an inversion formula and a Plancherel formula for this transform. Finally we construct explicit realisations of the discrete part appearing in the decomposition and also for the minimal representation in this family.

Introduction

One of the main problems in the representation theory of Lie groups and harmonic analysis on Lie groups is to decompose some interesting representations of a Lie group G under a subgroup $H \subset G$. This decomposition is also called the *branching rule*. Among other things, this has led to the discovery of new interesting representations. An exposition of the general theory for compact connected Lie groups,

1991 *Mathematics Subject Classification.* 32M15, 22E46, 22E43, 43A90, 32A36.

Key words and phrases. Bounded symmetric domains, Lie groups, Lie algebras, unitary representations, spherical functions, hypergeometric functions, intertwining operator.

including the classical results for $U(n)$ and $SO(n)$ (by Weyl and Murnaghan respectively), can be found in [12].

Since the work by R. Howe [7] and M. Kashiwara and M. Vergne (cf [10]), it has turned out to be fruitful to study the branching of singular and minimal holomorphic representations of a Lie group acting on a function space of holomorphic functions on a bounded symmetric domain. In [9], Jakobsen and Vergne study the restriction of the tensor product of two holomorphic representations to the diagonal subgroup.

In this paper we will study the branching of the analytic continuation of the scalar holomorphic discrete series of $SO(2,n)$ under the subgroup $H = SO_0(1,n)$. The subgroup H here is realised as the isometry group of a totally real submanifold of the Lie ball $SO(2,n)/S(O(2) \times O(n))$. The branching for a general Lie group G of Hermitian type under a symmetric subgroup H has been studied recently by Neretin ([19], [18]), Zhang ([28],[30],[29]) and by van Dijk and Pevzner [25]. In [14], Kobayashi and Ørsted studied the branching for some minimal representations. The branching rule for regular parameter and for some minimal representations is now well understood. However, the problem of finding the branching rule for non-discrete, non-regular parameter is a difficult one, and there is still no complete theory for the general case.

We find the branching rule for arbitrary scalar parameter ν in the Wallach set of $SO(2,n)$. It turns out that for small parameters ν there appears a discrete part in the decomposition. We discover here an intertwining operator realising the corresponding representation. It should be mentioned that for large parameter (in this case $\nu > n-1$) the corresponding branching problem has been solved by Zhang in [28] for arbitrary bounded symmetric domains.

The paper is organised as follows. In Section 1 we describe the geometry of the Lie ball. In Section 2 we recall some facts about general bounded symmetric domains and Jordan triple systems. In Section 3 we establish some facts about the real part of the Lie ball. In Section 4 we consider a family of function spaces and corresponding unitary representations. Section 5 is devoted to branching theorems and to finding the Plancherel measure. In Sections 6 and 7 we find realisations of the representations corresponding to the discrete part in the decomposition and to the minimal point in the Wallach set respectively.

Acknowledgement. I would like to thank Prof. Genkai Zhang for his guidance and support during the preparation of this paper and for all the discussions we have had on related topics. It is also my pleasure to thank Prof. Andreas Juhl for having read an earlier version of this manuscript and for having provided valuable comments. Finally, I would like to thank the anonymous referee for valuable comments, especially for having brought to my attention several interesting papers on branching that I was not aware of.

1. The Lie ball as a symmetric space $SO_0(2,n)/SO(2) \times SO(n)$

In this paper we study representations on function spaces on the domain

$$\mathscr{D} = \{z \in \mathbb{C}^n | 1 - 2\langle z, z\rangle + |zz^t|^2 > 0, |z| < 1\}. \tag{1}$$

We will only be concerned with the case $n > 2$. (If $n = 1$ it is the unit disk, U, and if $n = 2$, $\mathscr{D} \cong U \times U$). In this section we describe $\mathscr{D}$ as the quotient of $SO_0(2,n)$ by $SO(2) \times SO(n))$ by studying a holomorphically equivalent model on which we have a natural group action induced by the linear action on a submanifold of a Grassmanian manifold. Consider $\mathbb{R}^{n+2} \cong \mathbb{R}^2 \oplus \mathbb{R}^n$ equipped with the non-degenerate bilinear form

$$(x|y) := x_1y_1 + x_2y_2 - x_3y_3 - \ldots - x_{n+2}y_{n+2},$$

where the coordinates are with respect to the standard basis $e_1, \cdots, e_{n+2}$. Let $SO(2,n)$ be the group of all linear transformations on $\mathbb{R}^{n+2}$ that preserve this form and have determinant 1, i.e.,

$$SO(2,n) = \{g \in GL(2+n,\mathbb{R}) | (gx|gy) = (x|y), x, y \in \mathbb{R}^{2+n}, \det g = 1\}$$

Let $\mathcal{G}^+_{(2,n)}$ denote the set of all two-dimensional subspaces of $\mathbb{R}^2 \oplus \mathbb{R}^n$ on which $(\cdot|\cdot)$ is positive definite. Clearly $\mathbb{R}^2 \oplus \{0\}$ is one of these subspaces. It will be the reference point in $\mathcal{G}^+_{(2,n)}$ and we will denote it by V_0. The group $SO(2,n)$ acts naturally on this set and the action is transitive. In fact, the connected component of the identity, $SO_0(2,n)$ acts transitively. We will let G denote this group.

We denote by K the stabilizer subroup of V_0, i.e.,

$$K = \{g \in G | g(V_0) = V_0\}. \tag{2}$$

Any element $g \in G$ can be identified with a $(2+n) \times (2+n)$- matrix of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3}$$

where A is a 2×2-matrix. With this identification, K clearly corresponds to the matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

where A and D are orthogonal 2×2- and $n \times n$-matrices with deerminant one respectively, i.e., $K \cong SO(2) \times SO(n)$. The space $\mathcal{G}^+_{(2,n)}$ can be realised as the unit ball in $M_{n2}(\mathbb{R})$ with the operator norm. Indeed, let $V \in \mathcal{G}^+_{(2,n)}$. If $v = v_1 + v_2 \in V$, then $v_1 = 0$ implies that $v_2 = 0$, i.e., the projection $v \mapsto v_1$ is an injective mapping. This means that there is a real $n \times 2$ matrix Z with $Z^t Z < I_2$, such that

$$V = \{(v \oplus Zv) | v \in \mathbb{R}^2\}. \tag{4}$$

Conversely, if $Z \in M_{n2}(\mathbb{R})$ satisfies $Z^tZ < I_2$, then (4) defines an element in $\mathcal{G}^+_{(2,n)}$.

Using (3) to identify g with a matrix and letting V correspond to the matrix Z, then clearly

$$\begin{aligned} gV &= \{(Av + BZv \oplus Cv + DZv) | v \in \mathbb{R}^2)\} \\ &= \{v \oplus (C + DZ)(A + BZ)^{-1}v) | v \in \mathbb{R}^2)\}. \end{aligned}$$

In other words, we have a G-action on the set

$$M = \{Z \in M_{n2}(\mathbb{R}) | Z^tZ < I_2\}$$

given by

$$Z \mapsto (C + DZ)(A + BZ)^{-1}.$$

This exhibits M as a symmetric space.

$$M \cong G/K.$$

Moreover, we identify the matrix $Z = (XY)$ with the vector $X + iY$ in $\mathbb{C}^n$ in order to obtain an almost complex structure on M. With respect to this almost complex structure, the action of G is in fact holomorphic. Moreover we have the following result by Hua (see [8]).

Theorem 1. *The mapping*

$$\mathcal{H} : z \mapsto Z = 2\left(\begin{pmatrix} zz^t + 1 & i(zz^t - 1) \\ \overline{zz}^t + 1 & -i(\overline{zz}^t - 1) \end{pmatrix}^{-1} \begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)^t,$$

where $zz^t = z_1^2 + \cdots + z_n^2$, *is a holomorphic diffeomorphism of the bounded domain*

$$\mathscr{D} = \{z \in \mathbb{C}^n | 1 - 2\langle z, z\rangle + |zz^t|^2 > 0, |z| < 1\}$$

onto M*.*

We will call this mapping the *Hua transform.* It allows us to describe $\mathscr{D}$ as a symmetric space

$$\mathscr{D} \cong M \cong G/K.$$

2. Bounded symmetric domains and Jordan pairs

In this section we review briefly some general theory on bounded symmetric domains and Jordan pairs. All proofs are omitted. For a more detailed account we refer to Loos ([15]) and to Faraut-Koranyi ([2]).

Let $\mathcal{D}$ be a bounded open domain in $\mathbb{C}^n$ and $\mathcal{H}^2(\mathcal{D})$ be the Hilbert space of all square integrable holomorphic functions on $\mathcal{D}$,

$$\mathcal{H}^2(\mathcal{D}) = \{f, f \text{ holomorphic on } \mathcal{D} \,|\, \int_{\mathcal{D}} |f(z)|^2 dm(z) < \infty\},$$

where m is the $2n$-dimensional Lebesgue measure. It is a closed subspace of $L^2(\mathcal{D})$. For every $w \in \mathcal{D}$, the evaluation functional $f \mapsto f(w)$ is continuous, hence $\mathcal{H}^2(\mathcal{D})$ has a reproducing kernel $K(z, w)$, holomorphic in z and antiholomorphic in w such that

$$f(w) = \int_{\mathcal{D}} f(z)\overline{K(z, w)}dm(z).$$

$K(z, w)$ is called the Bergman kernel. It has the transformation property

$$K(\varphi(z), \varphi(w)) = J_\varphi(z)^{-1} K(z, w) \overline{J_\varphi(w)}^{-1}, \tag{5}$$

for any biholomorphic mapping φ on $\mathcal{D}$ with complex Jacobian $J_\varphi(z) = \det d\varphi(z)$. Hereafter biholomorphic mappings will be referred to as automorphisms. The formula

$$h_z(u,v) = \partial_u \partial_{\overline{v}} \log K(z,z) \tag{6}$$

defines a Hermitian metric, called the Bergman metric. It is invariant under automorphisms and its real part is a Riemannian metric on $\mathcal{D}$.

A bounded domain $\mathcal{D}$ is called *symmetric* if, for each $z \in \mathcal{D}$ there is an involutive automorphism s_z with z as an isolated fixed point. Since the group of automorphisms, $Aut(\mathcal{D})$ preserves the Bergman metric, s_z coincides with the local geodesic symmetry around z. Hence $\mathcal{D}$ is a Hermitian symmetric space.

A domain $\mathcal{D}$ is called *circled* (with respect to 0) if $0 \in \mathcal{D}$ and $e^{it}z \in \mathcal{D}$ for every $z \in \mathcal{D}$ and real t.

Every bounded symmetric domain is holomorphically isomorphic with a bounded symmetric and circled domain. It is unique up to linear isomorphisms.

From now on $\mathcal{D}$ denotes a circled bounded symmetric domain. G is the identity component of $Aut(\mathcal{D})$, K is the isotropy group of 0 in G. The Lie algebra $\mathfrak{g}$ will be considered as a Lie algebra of holomorphic vector fields on $\mathcal{D}$, i.e., vector fields X on $\mathcal{D}$ such that Xf is holomorphic if f is. The symmetry $s, z \mapsto -z$ around the origin induces an invoulution on G by $g \mapsto sgs^{-1}$ and, by differentiating, an involution $Ad(s)$ of $\mathfrak{g}$. We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

into the ± 1-eigenspaces.

For every $v \in \mathbb{C}^n$, let ξ_v be the unique vector field in $\mathfrak{p}$ that takes the value v at the origin. Then

$$\xi_v(z) = v - Q(z)\overline{v} \tag{7}$$

where $Q(z) : \overline{V} \to V$ is a complex linear mapping and $Q : V \to \mathrm{Hom}(\overline{V}, V)$ is a homogeneous quadratic polynomial. Hence $Q(x,z) = Q(x+z) - Q(x) - Q(z) : \overline{V} \to V$ is bilinear and symmetric in x and z. For $x, y, z \in V$, we define

$$\{x\overline{y}z\} = D(x,\overline{y})z = Q(x,z)\overline{y} \tag{8}$$

Thus $\{x\overline{y}z\}$ is complex bilinear and symmetric in x and z and complex antilinear in y, and $D(x,\overline{y})$ is the endomorphism $z \mapsto \{x\overline{y}z\}$ of V.

The pair $(V, \{\ \})$ is called a *Jordan triple system.* This Jordan triple system is positive in the sense that if $v \in V, v \neq 0$ and $Q(v)\overline{v} = \lambda v$ for some $\lambda \in \mathbb{C}$, then λ is positive. We introduce the endomorphisms

$$B(x, y) = I - D(x, \overline{y}) + Q(x)\overline{Q}(\overline{y}) \tag{9}$$

of V for $x, y \in V$, where $\overline{Q}(\overline{y})x = \overline{Q(y)\bar{x}}$. We summarise some results in the following proposition.

Proposition 2. *a) The Lie algebra* $\mathfrak{g}$ *satisfies the relations*

$$[\xi_u, \xi_v] = D(u, \overline{v}) - D(v, \overline{u}) \tag{10}$$

$$[l, \xi_u] = \xi_{lu} \tag{11}$$

for $u, v \in V$ *and* $l \in \mathfrak{k}$
b) The Bergman kernel $k(x, y)$ *of* $\mathcal{D}$ *is*

$$m(\mathcal{D})^{-1} \det B(x, y)^{-1} \tag{12}$$

c) The Bergman metric at 0 *is*

$$h_0(u, v) = trD(u, \overline{v}), \tag{13}$$

and at an arbitrary point $z \in \mathcal{D}$

$$h_z(u, v) = h_0(B(z, z)^{-1}u, v) \tag{14}$$

d) The triple product $\{\ \}$ *is given by*

$$h_0(\{u\overline{v}w\}, y) = \partial_u \partial_{\overline{v}} \partial_x \partial_{\overline{y}} \log K(z, z)|_{z=0} \tag{15}$$

We define odd powers of an element $x \in V$ by

$$x^1 = x,\ x^3 = Q(x)\overline{x}, \cdots, x^{2n+1} = Q(x)\overline{x^{2n-1}}.$$

An element $x \in V$ is said to be *tripotent* if $x^3 = x$, i.e., if $\{x\overline{x}x\} = 2x$. Two tripotents c and e are called orthogonal if $D(c, \overline{e}) = 0$. In this case $D(c, \overline{c})$ and $D(e, \overline{e})$ commute and $e + c$ is a tripotent.

Every $x \in V$ can be written uniquely

$$x = \lambda_1 c_1 + \cdots + \lambda_n c_n,$$

where the c_i are pairwise orthogonal nonzero tripotents which are real linear combinations of odd powers of x, and the λ_i satisfy

$$0 < \lambda_1 < \cdots < \lambda_n.$$

This expression for x is called its *spectral decomposition* and the λ_i the eigenvalues of x. Moreover, the domain $\mathcal{D}$ can be realised as the unit ball in V with the spectral norm

$$\|x\| = \max|\lambda_i|,$$

where the λ_i are the eigenvalues of x, i.e.,

$$\mathcal{D} = \{x \in V | \|x\| < 1\}.$$

Let $f(t)$ be an odd complex valued function of the real variable t, defined for $|t| < \rho$. For every $x \in V$ with $|x| < \rho$ we define $f(x) \in V$ by

$$f(x) = f(\lambda_1)c_1 + \cdots + f(\lambda_n)c_n, \tag{16}$$

where $x = \lambda_1 c_1 + \cdots + \lambda_n c_n$ is the spectral resolution of x. This functional calculus is used in expressing the action on $\mathcal{D}$ of the elements $\exp \xi_v$ in G:

$$\exp \xi_v(z) = u + B(u,u)^{1/2} B(z,-u)^{-1}(z + Q(z)\overline{u}) \tag{17}$$

and

$$d(\exp \xi_v)(z) = B(u,u)^{1/2} B(z,-u)^{-1}, \tag{18}$$

where $u = \tanh v$, for $v \in \mathbb{C}^n$ and $z \in \mathcal{D}$.

3. The real part of the Lie ball

We consider the non-degenerate quadratic form

$$q(z) = z_1^2 + \cdots + z_n^2 \tag{19}$$

on $V = \mathbb{C}^n$. In the following we will often denote $q(z,w)$ by (z,w). Defining $Q(x)y = q(x,y)x - q(x)y$, where $q(x,y) = q(x+y) - q(x) - q(y)$, we get a Jordan triple system. The Lie ball $\mathscr{D} = \{z \in \mathbb{C}^n | 1 - 2\langle z,z\rangle + |zz^t|^2 > 0, |z| < 1\}$ is the open unit ball in this Jordan triple system. An easy computation shows the following identity.

$$D(x,\overline{y})z = 2(\sum_{k=1}^n x_k \overline{y_k})z + 2(\sum_{k=1}^n z_k \overline{y_k})x - 2(\sum_{k=1}^n x_k z_k)\overline{y}$$

Recalling that $B(x,y) = I - D(x,\overline{y}) + Q(x)\overline{Q}(\overline{y})$. The Bergman kernel of $\mathscr{D}$ is

$$K(z,w) = (1 - 2\langle z,w\rangle + (zz^t)\overline{(ww^t)})^{-n}. \tag{20}$$

We will hereafter denote it by $h(z,w)^{-n}$. Consider the real form $\mathbb{R}^n$ in $\mathbb{C}^n$. Observe that

$$\mathscr{X} := \mathscr{D} \bigcap \mathbb{R}^n$$

is the unit ball of $\mathbb{R}^n$. On $\mathscr{X}$ we have a simple expression for the Bergman metric:

$$B(x,x) = (1-|x|^2)^{-2} I, x \in \mathscr{X}. \tag{21}$$

The submanifold $\mathscr{X}$ is a totally real form of $\mathscr{D}$ in the sense that

$$T_x(\mathscr{X}) + iT_x(\mathscr{X}) = T_x(\mathscr{D}),\ T_x(\mathscr{X}) \bigcap iT_x(\mathscr{X}) = \{0\}$$

This implies that every holomorphic function on $\mathscr{D}$ that vanishes on $\mathscr{X}$ is identically zero. We define the subgroup H as the identity component of

$$\{h \in G | h(x) \in \mathscr{X} \text{ if} x \in \mathscr{X}\}$$

We will denote $H \bigcap K$ by L.

Using the fact that the real form $\mathbb{R}^n$ is a sub-triple system of $\mathbb{C}^n$, one can show that $\mathscr{X}$ is a totally geodesic submanifold of $\mathscr{D}$ (cf Loos [15]). Hence we can describe $\mathscr{X}$ as a symmetric space

$$\mathscr{X} \cong H/L.$$

We now study the image of $\mathscr{X}$ in the $M_{n2}(\mathbb{R})$- model of the Lie ball. For computational convenience, we now work with the transposes of these matrices. The defining equation of the Hua-transform can be written as

$$\frac{1}{2}\begin{pmatrix} zz^t+1 & i(zz^t-1) \\ \overline{zz}^t+1 & -i(\overline{zz}^t-1) \end{pmatrix} Z = \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \tag{22}$$

In the coordinates $(z_1, \ldots, z_n)$ of z, this identity takes the form

$$z_k = \frac{1}{2}(\ (zz^t+1)x_k + i(zz^t-1)y_k). \tag{23}$$

This gives

$$4zz^t = (zz^t)^2(X+iY)(X+iY)^t + 2(XX^t+YX^t)zz^t \tag{24}$$

$$+(X-iY)(X-iY)^t, \tag{25}$$

which is a quadratic equation in zz^t with unique solution

$$zz^t = \frac{2-(XX^t+YY^t)-2\sqrt{(1-XX^t)(1-YY^t)-(YX^t)^2}}{(X+iY)(X+iY)^t}. \tag{26}$$

From (23) we see that if z is real, then $y_k = 0$ for all k. On the other hand, if $Y = 0$, then (26) shows that zz^t is real and therefore z is real by (23). Hence the image of the real part $\mathscr{X} \subset \mathscr{D}$ under the Hua-transform is the set

$$\mathcal{H}(\mathscr{X}) = \{Z = (X\,0)\,|X \in M_{n1}(\mathbb{R}), |X| < 1\}, \tag{27}$$

since for an element $Z = (X\,0)$, the condition that $Z^t Z < I_2$ is clearly equivalent with $|X| < 1$.

Recall that the real n-dimensional unit ball can be described as a symmetric space $SO_0(1,n)/SO(n)$ by a procedure analogous to the one in the first section. One first considers all lines in $\mathbb{R}^{1+n}$ on which the quadratic form $x_1^2 - x_2^2 - \cdots - x_{n+1}^2$ is positive definite and identifies these lines with all real $n \times 1$-matrices with norm less than one. If we write elements $g \in SO(1,n)$ as matrices of the form

$$g = \begin{pmatrix} a & - & b & - \\ | & & & \\ c & & D & \\ | & & & \end{pmatrix}, \tag{28}$$

the action is given by

$$X \mapsto (c + DX)(a + bX)^{-1}. \tag{29}$$

The group $SO(1,n)$ can be embedded into $SO(2,n)$. Indeed, the equality

$$\begin{pmatrix} a & 0 & - & b & - \\ 0 & 1 & - & 0 & - \\ | & | & & & \\ c & 0 & & D & \\ | & | & & & \end{pmatrix} \begin{pmatrix} a' & 0 & - & b' & - \\ 0 & 1 & - & 0 & - \\ | & | & & & \\ c' & 0 & & D' & \\ | & | & & & \end{pmatrix}$$

$$= \begin{pmatrix} aa' + bc' & 0 & - & ab' + bD' & - \\ 0 & 1 & - & 0 & - \\ | & | & & & \\ ca' + Dc' & 0 & & cb' + DD' & \\ | & | & & & \end{pmatrix}$$

shows that we can define an injective homomorphism $\theta : SO(1,n) \to SO(2,n)$ by

$$\theta : \begin{pmatrix} a & - & b & - \\ | & & & \\ c & & D & \\ | & & & \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & - & b & - \\ 0 & 1 & - & 0 & - \\ | & | & & & \\ c & 0 & & D & \\ | & | & & & \end{pmatrix}.$$

This subgroup acts on $\mathcal{H}(\mathscr{X})$ as

$$(X\,0) \mapsto \big((c + DX)(a + bX)^{-1}\ 0\big)$$

and the action is transitive. Suppose now that $h \in SO(2,n)$ preserves $H(\mathscr{X})$. Let $p = h(0)$. We can choose a $g \in SO_0(1,n)$ such that $g(0) = p$ (here we identify g with $\theta(g)$). Then $g^{-1}h(0) = 0$ and hence we can write it in block form as

$$g^{-1}h = \begin{pmatrix} I_2 & 0 \\ 0 & D \end{pmatrix},$$

with $D \in SO(n)$. This is an element in $\theta(SO(1,n))$ and hence $h \in \theta(SO(1,n))$. We have now proved the following theorem.

Theorem 3. *The Hua transform $\mathcal{H} : \mathscr{D} \to M$ maps the real part $\mathscr{X}$ diffeomorphically onto*

(30) $$\mathcal{H}(\mathscr{X}) = \{Z = (X\,0)\,|X \in M_{n1}(\mathbb{R}), |X| < 1\}$$

by $x \mapsto \frac{2x}{1+|x|^2}$. Moreover, the induced group homomorphism $h \mapsto \mathcal{H}\,h\,\mathcal{H}^{-1}$ is an isomorphism between the groups H and $SO_0(1,n)$

Remark. The model $\mathcal{H}(\mathscr{X})$ of $SO_0(1,n)/SO(n)$ is the real part of the complex n-dimensional unit ball $SU(1,n)/SU(n)$ with fractional-linear group action. It is therefore equipped with a Riemannian metric given by the restriction of the Bergman metric of the complex unit ball. If $x \in \mathcal{H}(\mathscr{X}), x \neq 0$, we decompose $\mathbb{R}^n = \mathbb{R}x \oplus (\mathbb{R}x)^\perp$. We let $v = v_x + v_{x^\perp}$ be the corresponding decomposition of a tangent vector v at x. In this model, the Riemannian metric at x is (cf [21])

$$g_x(v,v) = \frac{|v_x|^2}{(1-|x|^2)^2} + \frac{|v_{x^\perp}|^2}{(1-|x|^2)}.$$

We recall from equation (21) that if $x \in X$, then the Riemannian metric at x is

$$h_x(v,v) = \frac{1}{2n}\frac{|v|^2}{(1-|x|^2)^2}.$$

The Hua transform thus induces an isometry (up to a constant) of the real n-dimensional unit ball equipped with two different Riemannian structures.

3.1. **Iwasawa decomposition of $\mathfrak{h}$.** The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ induces a decomposition $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{q}$. We let

$$\mathfrak{a} = \mathbb{R}\xi_e$$

be the one-dimensional subspace of $\mathfrak{q}$, where $e = e_1$ denotes the first standard basis vector and the corresponding vector field ξ_e is defined in (7).

Proposition 4. *The Lie algebra $\mathfrak{h}$ has rank one, and the roots with respect to the abelian subalgebra $\mathfrak{a}$ of $\mathfrak{q}$ are $\{\alpha, -\alpha\}$, where $\alpha(\xi_e) = 2$. The corresponding posive root space is*

$$\mathfrak{q}_\alpha = \{\xi_v + \frac{1}{2}(D(e,v) - D(v,e)) | v \in \mathbb{R}e_2 \oplus \cdots \oplus \mathbb{R}e_n\}$$

Proof. This is known in a general context, but we give here an elementary proof.

Take u and v in $\mathbb{R}^n$ and assume that $[\xi_u, \xi_v] = 0$. Then, for any $x \in \mathbb{R}^n$ we have

$$D(u,v)x = D(v,u)x.$$

A simple calculation shows that this amounts to

$$(u,x)v = (v,x)u,$$

which can only hold for all real x if $u = v$.

Thus $\mathfrak{a}$ is a maximal abelian subalgebra in $\mathfrak{q}$. The vector e is a maximal tripotent in the Jordan triple system corresponding to $\mathscr{D}$. Suppose that $[\xi_e, \xi_v + l] = \alpha(\xi_e)(\xi_v + l)$. Identifying the q- and l-components yields

$$D(e,v) - D(v,e) = \alpha(\xi_e)l \tag{31}$$

$$-\xi_{le} = \alpha(\xi_e)\xi_v \tag{32}$$

From (32) it follows that $le = -\alpha(\xi_e)v$ and, thus, applying both sides of (31) to e gives

$$D(e,v)e - D(v,e)e = -\alpha(\xi_e)^2 v,$$

i.e.,

$$D(e,e)v - D(e,v)e = \alpha(\xi_e)^2 v,$$

An easy computation gives

$$4v - 4(e,v)e = \alpha(\xi_e)^2 v.$$

Hence e is orthogonal to v and $\alpha(\xi_e)^2 = 4$. The rest follows immediately. □

We shall fix the positive root α. Elements in $\mathfrak{a}_{\mathbb{C}}^*$ are of the form $\lambda\alpha$ and will hereafter be identified with the complex numbers λ. In particular, the half sum of the positive roots (with multiplicities), ρ, will be identified with the number $(n-1)/2$.

3.2. **The Cayley transform.** The Cayley transform is a biholomorphic mapping from a bounded symmetric domain onto a *Siegel domain.* We describe it for the domain $\mathscr{D}$ and use it to express the spherical functions on $\mathscr{X}$ in terms of the spherical functions on the unbounded domain. We fix the maximal tripotent e. Then $\mathbb{C}^n$ equipped with the bilininear mapping

$$(z,w) \mapsto z \circ w = \frac{1}{2}\{zew\} \tag{33}$$

is a complex Jordan algebra. Observe that since e is a tripotent, it is a unity for this multiplication. The Cayley transform is the mapping $c : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$c(z) = (e+z) \circ (e-z)^{-1}, \tag{34}$$

where $(e-z)^{-1}$ denotes the inverse of $(e-z)$ with respect to the Jordan product.

Proposition 5. *The Cayley transform is given by the formula*

$$c(z) = \frac{1 - zz^t}{1 - 2z_1 + (zz^t)^2} e + \frac{2z'}{1 - 2z_1 + (zz^t)^2}, \tag{35}$$

for $z = (z_1, z') = z_1 e + z' \in \mathscr{D}$. *Moreover, it maps* $\mathscr{X}$ *onto the halfspace*

$$\{(x_1, \ldots, x_n\} \in \mathbb{R}^n | x_1 > 0\}.$$

Proof. We first find the inverse for an element x. Suppose therefore that $e = \frac{1}{2}\{xez\} = \frac{1}{2}D(x,e)z$, i.e.,

$$e = (x,e)z + (z,e)x - (x,z)e = x_1 z + z_1 x - (x,z)e$$

Identifying coordinates gives

$$\begin{aligned} 1 &= 2x_1 z_1 - (x,z) \\ 0 &= x_1 z' + z_1 x' \end{aligned}$$

These equations have the solution

$$\begin{aligned} z_1 &= x_1/(x,x) \\ z' &= -x'/(x,x). \end{aligned}$$

If we apply this to the expression $(e-z)^{-1}$ in the definition of c, we get

$$(e-z)^{-1} = \frac{1-z_1}{(1-z_1)^2 + (z',z')} e + \frac{z'}{(1-z_1)^2 + (z',z')}.$$

Now the formula (35) follows by an easy computation. Moreover, we observe that the inverse transform is given by

$$w \mapsto (w-e) \circ (w+e)^{-1} = -c(-w).$$

Hence both c and c^{-1} preserve $\mathbb{R}^n$ and therefore

$$c(\mathscr{X}) = c(\mathscr{D}) \bigcap \mathbb{R}^n.$$

We now determine $c(\mathscr{X})$.

From ([15]) we know that (since e is a maximal tripotent)

$$c(\mathscr{D}) = \{u + iv | u \in A^+, v \in A\}, \tag{36}$$

where A is the real Jordan algebra

$$\{z \in V | Q(e)\overline{z} = z\}$$

and A^+ is the positive cone $\{z \circ z | z \in A\}$ in A. By a simple computation we see that

$$A = \mathbb{R}e \oplus \mathbb{R}ie_2 \oplus \cdots \oplus \mathbb{R}ie_n.$$

Since we have the identities

$$\begin{aligned} z + Q(e)\overline{z} &= 2u, \\ z - Q(e)\overline{z} &= 2iv \end{aligned}$$

and

$$Q(e)\overline{z} = 2\overline{z_1} - \overline{z},$$

we get expressions for u and v:

$$2u = (z_1 + \overline{z_1}, z_2 - \overline{z_2}, \dots, z_n - \overline{z_n})$$

$$2iv = (z_1 - \overline{z_1}, z_2 + \overline{z_2}, \dots, z_n + \overline{z_n})$$

The condition that $x = u + iv$ be in the image of $\mathscr{X}$ thus implies that

$$u = (x_1, 0, \dots, 0),$$

$$iv = (0, x_2, \dots, x_n).$$

Moreover we require that

$$u = w \circ w = 2w_1 w - (w, w)e,$$

for some

$$w = c_1 e + c_2 i e_2 + \cdots + c_n i e_n.$$

This yields

$$(x_1, \dots, 0) = (c_1^2 + \cdots + c_n^2, ic_1c_2, \dots, ic_1c_n).$$

Hence

$$c_1^2 = x_1, c_2 = \cdots = c_n = 0,$$

and thus

$$u + iv = (c_1^2, x_2, \dots, x_n).$$

This proves the claim. □

Recall the expression for the spherical functions on a symmetric space of noncompact type (cf [6] Thm 4.3)

$$\varphi_\lambda(h) = \int_L e^{(i\lambda+\rho)A(lh)} dl,$$

where $A(lh)$ is the (logarithm) of the A part of lh in the Iwasawa decomposition $H = NAL$. The integrand in this formula is called the *Harish-Chandra e-function.* For the above Siegel domain it has the form $e_\lambda(w) = (w_1)^{i\lambda+\rho}$ (cf [24]). Hence we have the following corollary.

Corollary 6. *The spherical function φ_λ on $\mathscr{X} = H/L$ is*

$$\varphi_\lambda(x) = \int_{S^{n-1}} \left(\frac{1 - |x|^2}{1 - 2(x, \zeta) + xx^t} \right)^{i\lambda+\rho} d\sigma(\zeta). \tag{37}$$

where σ is the $O(n)$-invariant probability measure on S^{n-1}.

4. A FAMILY OF UNITARY REPRESENTATIONS OF G

4.1. **The function spaces $\mathscr{H}_\nu$.** The Bergman space $\mathcal{H}^2(\mathscr{D})$ has the reproducing kernel $h(z,w)^{-n}$. This means in particular that the function $h(z,w)^{-n}$ is positive definite in the sense that

$$\sum_{i,j=1}^{m} \alpha_i \overline{\alpha_j} h(z_i, z_j)^{-n} \geq 0,$$

for all $z_1, \dots, z_n \in \mathscr{D}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. It has been proved by Wallach ([26]) and Rossi-Vergne ([20]) that $h(z,w)^{-\nu}$ is positive definite precisely when ν in the set

$$\{0, (n-2)/2\} \bigcup ((n-2)/2, \infty)$$

This set will also be referred as the *Wallach set* (cf [3]). For ν in the Wallach set above, $h(z,w)^{-\nu}$ is the reproducing kernel of a Hilbert space of holomorphic functions on $\mathscr{D}$. We will call this space $\mathscr{H}_\nu$ and the reproducing kernel $K_\nu(z,w)$. The mapping $g \mapsto \pi_\nu(g)$, where

$$\pi_\nu(g) f(z) = J_{g^{-1}}(z)^{\frac{\nu}{n}} f(g^{-1}z)$$

defines a unitary projective representation of G on $\mathscr{H}_\nu$. Indeed, comparison with the Bergman kernel shows that $h(z,w)^{-\nu}$ transforms under automorphisms according to the rule

$$h(gz, gw)^{-\nu} = J_g(z)^{-\frac{\nu}{n}} h(z,w)^{-\nu} \overline{J_g(w)}^{-\frac{\nu}{n}}. \tag{38}$$

Recall that for functions f_1 and f_2 of the form

$$f_1(z) = \sum_{k=1}^{l} \alpha_k K_\nu(z, w_k), \;\; f_2(z) = \sum_{k=1}^{m} \beta_k K_\nu(z, w'_k),$$

the inner product is defined as

$$\langle f_1, f_2 \rangle_\nu = \sum_{i,j} \alpha_i \overline{\beta_j} K_\nu(w_i, w'_j) \tag{39}$$

Equation (38) implies that

$$K_\nu(g^{-1}z, w) = J_{g^{-1}}(z)^{-\frac{\nu}{n}} K_\nu(z, gw) \overline{J_{g^{-1}}(w)}^{-\frac{\nu}{n}}. \tag{40}$$

Hence we have the following two equalities

$$\pi_\nu(g)f_1(z) = \sum_{k=1}^{l} \alpha_k \overline{J_{g^{-1}}(w_k)}^{-\frac{\nu}{n}} K_\nu(z, gw_k)$$
$$\pi_\nu(g)f_2(z) = \sum_{k=1}^{m} \beta_k \overline{J_{g^{-1}}(w_k')}^{-\frac{\nu}{n}} K_\nu(z, gw_k').$$

The unitarity

$$\langle \pi_\nu(g)f_1, \pi_\nu(g)f_2\rangle_\nu = \langle f_1, f_2\rangle_\nu$$

now follows by an application of the transformation rule (38) in the definition (39). Since functions of the form above are dense in $\mathscr{H}_\nu$, it follows that each $\pi_\nu(g)$ is a unitary operator and it is easy to see that $g \mapsto \pi_\nu(g)$ is a projective homomorphism of groups. In fact, π_ν is an irreducible projective representation, cf [2].

4.2. **Fock-Fischer spaces.** It can be shown that for $\nu > (n-2)/2$ all holomorphic polynomials are in $\mathscr{H}_\nu$ and that polynomials of different homogeneous degree are orthogonal. In this context, the spaces $\mathscr{H}_\nu$ are closely linked with the *Fock-Fischer space*, $\mathscr{F}$, which we will now describe. The basis vector e_1 is a maximal tripotent which is decomposed into minimal tripotents as $e_1 = \frac{1}{2}(1, i, 0, \dots, 0) + \frac{1}{2}(1, -i, 0, \dots, 0)$. (We omit the easy computations.) In order to expand the reproducing kernel K_ν into a power series consistent with the treatment in [2], we need to introduce a new norm on $\mathbb{C}^n$ so that the minimal tripotents have norm 1, i.e., the Euclidean norm multiplied with $\sqrt{2}$. Then

$$\{f_1, \dots, f_n\} := \left\{\frac{1}{\sqrt{2}}e_1, \dots, \frac{1}{\sqrt{2}}e_n\right\}$$

is an orthonormal basis with respect to this new norm. We write points $z \in \mathscr{D}$ as $z = w_1 f_1 + \dots + w_n f_n$. For polynomials $p(w) = \sum_\alpha a_\alpha w^\alpha$, we define

$$p^*(w) = \sum_\alpha \overline{a_\alpha} w^\alpha.$$

The Fock-Fischer inner product is now defined as

$$\langle p, q\rangle_{\mathscr{F}} = p(\partial)(q^*)|_{w=0},$$

where $p(\partial)$ is the differential operator $\sum_\alpha a_\alpha \frac{\partial^\alpha}{\partial w^\alpha}$, for p as above. The Fock-Fischer space, $\mathscr{F}$, is the completion of the space of polynomials. It is easy to see that polynomials of different homogeneous degree are orthogonal in $\mathscr{F}$. Moreover, the representation of $SO(n)$ on $\mathcal{P}^m$, the polynomials of homogeneous degree m, can be decomposed into irreducible subspaces as

$$\mathcal{P}^m = \bigoplus_{m-2k\geq 0} E_{m-2k} \otimes \mathbb{C}(ww^t)^k, \tag{41}$$

where E_i are the spherical harmonic polynomials of degree i (cf [23]). This is a special case of the general Hua-Schmid decomposition (cf [2]).The following relation holds between the Fock-Fischer norm and the $\mathscr{H}_\nu$-norm on the space $E_{m-2k} \otimes \mathbb{C}(ww^t)^k$ (cf [2]).

$$\|p\|^2_\nu = \frac{\|p\|^2_{\mathscr{F}}}{(\nu)_{m-k}(\nu - \frac{n-2}{2})_k}, \tag{42}$$

for $p \in E_{m-2k} \otimes \mathbb{C}(ww^t)^k$. We have the following decomposition of $\mathscr{H}_\nu$ under K:

Proposition 7. *(Faraut-Korànyi,* [2]*) a) If $\nu > \frac{n-2}{2}$, then*

$$\mathscr{H}_\nu|_K = \bigoplus \sum_{m-2k\geq 0} E_{m-2k} \otimes \mathbb{C}(zz^t)^k, \tag{43}$$

where E_{m-2k} is the space of spherical harmonic polynomials of degree $m-2k$. Moreover, we have the following expansion of the kernel function:

$$h(z,w)^{-\nu} = \sum_{m-2k\geq 0} (\nu)_{m-k} \left(\nu - \frac{n-2}{2}\right)_k K_{(m-k,k)}(z,w), \tag{44}$$

where $K_{(m-k,k)}$ is the reproducing kernel for the subspace $E_{m-2k} \otimes \mathbb{C}(zz^t)^k$ with the Fock-Fischer norm. The series converges in norm and uniformly on compact sets of $\mathscr{D} \times \mathscr{D}$.
b) If $\nu = \frac{n-2}{2}$, then

$$\mathscr{H}_\nu|_K = \bigoplus \sum_m E_m \tag{45}$$

We will later need the norm of $(zz^t)^k$ in $\mathscr{H}_\nu$.

Proposition 8.

$$\|(zz^t)^k\|_\nu^2 = \frac{k!\left(\frac{n}{2}\right)_k}{(\nu)_k\left(\nu - \frac{n-2}{2}\right)_k} \tag{46}$$

Proof. A straightforward computation shows that

$$(\frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2})(z_1^2 + \cdots + z_n^2)^k = (2^2 k(k-1) + n2k)(z_1^2 + \cdots + z_n^2)^{k-1}$$

Proceeding inductively, we obtain

$$\begin{aligned}\left(\frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}\right)^k (z_1^2 + \cdots + z_n^2)^k &= \prod_{j=1}^k 2j(2(j-1)+n) \\ &= 4^k k! \left(\frac{n}{2}\right)_k\end{aligned}$$

The Fock-Fischer norm is computed in the w-coordinates $w_i = \sqrt{2} z_i$, so

$$(zz^t)^k = 2^{-k}(ww^t)^k$$

and

$$\left(\frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}\right)^k = 2^{-k}\left(\frac{\partial^2}{\partial w_1^2} + \cdots + \frac{\partial^2}{\partial w_n^2}\right)^k.$$

Hence

$$\|(zz^t)^k\|_{\mathscr{F}}^2 = k!\left(\frac{n}{2}\right)_k$$

and an application of Prop. 7 gives the result. □

5. Branching of π_ν under the subgroup H

5.1. **A decomposition theorem.** Recall the irreducible (projective) representations π_ν from the previous section. Our main objective is to decompose these into irreducible representations under the subgroup H. The fact that $\mathscr{X}$ is a totally real form is reflected in the restrictions of the representations π_ν to H.

Proposition 9. *The constant function* 1 *is in* $\mathscr{H}_\nu$ *and is an* L*-invariant cyclic vector for the representation* $\pi_\nu : H \to \mathscr{U}(\mathscr{H}_\nu)$.

Proof. First note that

$$\begin{aligned} K_\nu(z, h0) &= J_h(h^{-1}z)^{-\nu/n} K_\nu(h^{-1}z, 0)\overline{J_h(0)^{-\nu/n}} \\ &= \overline{J_h(0)^{-\nu/n}} J_{h^{-1}}(z)^{\nu/n} K_\nu(h^{-1}z, 0) \\ &= \overline{J_h(0)^{-\nu/n}} \pi_\nu(h)1(z) \end{aligned}$$

Suppose now that the function $f \in \mathscr{H}_\nu$ is orthogonal to the linear span of the elements $\pi_\nu(h)1, h \in H$. By the above identity we have

$$\begin{aligned} f(h0) &= \langle f, K_\nu(\cdot, h0)\rangle_\nu \\ &= 0. \end{aligned}$$

Since H acts transitively on $\mathscr{X}$, f is zero on $\mathscr{X}$. Hence it is identically zero. ☐

We want decompose the representation of H into a *direct integral* of irreducible representations. For the definition of a direct integral over a measurable field of Hilbert spaces we refer to Naimark ([17]). The following general decomposition theorem is stated in several references (e.g. [19]), but the author has not been able to find a proof of it in the literature. A proof for abelian groups can be found in [17]. The proof we present below is based on the Gelfand-Naimark representation theory for C^*-algebras.

Theorem 10. *Let π be a unitary representation of the semisimple Lie group H on a Hilbert space, $\mathscr{H}$. Suppose further that L is a maximal compact subgroup and that the representation has a cyclic L-invariant vector. Then π can be decomposed as a multiplicity-free direct integral of irreducible representations,*

$$\pi \cong \int_\Lambda \pi_\lambda \, d\mu(\lambda), \tag{47}$$

where Λ is a subset of the set of positive definite spherical functions on H and for $\lambda \in \Lambda, \pi_\lambda$ is the corresponding unitary spherical representation.

Proof. We consider the Banach space $L^1(H)$. This is a Banach $*$-algebra with multiplication defined as the convolution

$$(f * g)(x) = \int_H f(y)g(y^{-1}x)dy$$

and involution defined by

$$f^*(x) = \overline{f(x^{-1})}.$$

Recall that the representation π extends to a representation of the Banach algebra $L^1(H)$ by

$$f \mapsto \int_H f(x)\pi(x)dx.$$

We will also denote this mapping of $L^1(H)$ into $\mathscr{B}(\mathscr{H})$ (the set of bounded linear operators on $\mathscr{H}$) by π. This representation will also be cyclic as the following argument shows. Denote by ξ the L-invariant cyclic unit vector for H. Vectors of the form

$$\pi(f_\epsilon)(\pi(h_1)\xi + \cdots + \pi(h_n)\xi),$$

where $\{f_\epsilon\}$ is an approximate identity on H, will then be dense in $\mathscr{H}$. Moreover the identity

$$\pi(f)(\pi(h_1)\xi + \cdots + \pi(h_n)\xi) = \pi((R_{h_1^{-1}} + \cdots + R_{h_n^{-1}})f)\xi,$$

holds for $f \in L^1(H)$ and $h_1, \ldots, h_n \in H$. (Here $R_h f$ denotes the right-translation of the argument of f; $f \mapsto f(\cdot\, h)$. We similarly define $L_h f$.) Hence vectors of the form $\pi(f)\xi$, where $f \in L^1(H)$, form a dense subset in $\mathscr{H}$.

The function Φ defined as

$$\Phi : \pi(f) \mapsto \langle \pi(f)\xi, \xi\rangle \tag{48}$$

extends to a state on the C^*-algebra $\mathscr{C}$ generated by $\pi(L^1(H))$ and the identity operator. It is a well-known fact from the theory of C^*-algebras that the norm-decreasing positive functionals form a convex and weak*-compact set (cf [16]). For a C^*-algebra with identity, the extreme points of this set are the pure states. Therefore, Φ can be expressed as

$$\Phi = \int_X \varphi_x d\mu, \tag{49}$$

where X is the set of pure states and μ is a regular Borel measure on X (cf [22], Thm. 3.28). We recall the Gelfand-Naimark-Segal construction of a cyclic representation of a C^*-algebra associated with a given state (cf [16]). In this duality, the irreducible representations correspond to the pure states. So each φ_x in (49) parametrises an irreducible representation of $\pi(L^1(H))$ on some Hilbert space H_x with a $\pi(L^1(H))$-cyclic unit vector ξ_x.

Herafter we will, by an abuse of notation, write $\Phi(f)$ for $\Phi(\pi(f))$ and correspondingly for the functionals φ_x.

We define a unitary operator $T : \mathscr{H} \to \int_X H_x d\mu$ that intertwines the actions of $\mathscr{C}$ by

$$T : \pi(f)\xi \mapsto \{\pi_x(f)\xi_x\}, f \in L^1(H). \tag{50}$$

To see that this is well-defined, suppose that $\pi(f)\xi = 0$. Then we have

$$\langle \pi(f)\xi, \pi(f)\xi\rangle = \langle \pi(f^* * f)\xi, \xi\rangle = 0 \tag{51}$$

i.e.,

$$\Phi(f^* * f) = 0 \tag{52}$$

By (49) we have

$$\Phi(f^* * f) = \int_H \langle \pi_x(f^* * f)\xi_x, \xi_x\rangle_x d\mu = 0. \tag{53}$$

Therefore $\pi_x(f)\xi_x = 0$ for almost every x and hence T is well defined on a dense set of vectors. Note that (53) also shows that T is isometric on this set and it therefore extends to an isometry of $\mathscr{H}$ into $\int_X H_x d\mu$.

Consider now the subalgebra, $L^1(H)^\#$, consisting of all L^1-functions that are left- and right L-invariant, i.e.,

$$L_l f = R_l f = f,$$

for all l in L. This is a commutative Banach *-algebra (cf [6], Ch. IV). We know that $\varphi_x \circ \pi : L^1(H)^\# \to \mathbb{C}$ is a homomorphism of algebras and is therefore of the form (cf [6], Ch. IV)

$$\varphi_x(f) = \int_H f(h)\phi_x(h)dh, f \in L^1(H)^\#, \tag{54}$$

where ϕ_x is a bounded spherical function. In fact, this formula holds for all L^1-functions on H, as the following argument shows.

Since ξ is L-invariant, the identity

$$\pi(f)\xi = \pi(R_l f)\xi$$

holds for all L^1-functions f and $l \in L$. Applying T to both sides of this equality (and using the fact that both $L^1(H)$ and L are separable), we see that

$$\pi_x(f)\xi_x = \pi_x(R_l f)\xi_x \tag{55}$$

holds for all $f \in L^1(H)$ and $l \in L$ outside some set of measure zero with respect to μ. We now choose an approximation of the identity $\{\eta_\epsilon\}$ on H, and by replacing it with $\{\int_L \eta_\epsilon(l \cdot l^{-1})dl\}$ if necessary, we may assume that it is invariant under the conjugate action of L.

Consider now Φ_ϵ defined by

$$\varphi_\epsilon(f) = \langle \pi(f)\pi(\eta_\epsilon)\xi, \pi(\eta_\epsilon)\xi \rangle.$$

We define the functionals $\varphi_{x,\epsilon}$ analogously for all $x \in X$. Clearly $\Phi_\epsilon(f) \to \Phi(f)$ as $\epsilon \to 0$ and therefore

$$\lim_{\epsilon \to 0} \varphi_{x,\epsilon}(f) = \varphi_x(f)$$

holds for all L^1-functions f outside some set of measure zero with respect to μ. (Again we use the separability of $L^1(H)$.) Using the L-conjugacy invariance of η_ϵ and (55), a simple calculation shows that

$$\varphi_{x,\epsilon}(f) = \varphi_{x,\epsilon}(f^{\#}),$$

where

$$f(h) = \int_L \int_L f(l_1 h l_2) dl_1 dl_2,$$

and by letting ϵ tend to zero we get

$$\varphi_x(f) = \varphi_x(f^{\#})$$

for almost every x. Hence

$$\varphi_x(f) = \int_H f(h)\phi_x(h)dh,$$

for $f \in L^1(H)$.

Since φ_x also preserves the involution $*$, it is a positive linear functional, i.e.,

$$\int_H f(h)\phi_x(h)dh \geq 0, \tag{56}$$

for every $f \in L^1(H)$, such that $f = g * g^*$, for some $g \in L^1(H)$. The proof of the following lemma can be found in [4], p.85.

Lemma 11. *Suppose that φ is a bounded spherical function such that $\int_H f(h)\varphi(h)dh \geq 0$ for all $f \in L^1(H)$ of the form $f = g * g^*$ for some $g \in L^1(H)$. Then φ is positive definite.*

Since every positive definite spherical function defines an irreducible, unitary, spherical representation of H, it also gives rise to a representation $L^1(H)$. Its restriction to the subspace of L-invariant vectors, E_x will be $L^1(H)^{\#}$-invariant and one-dimensional (cf [6], Ch. IV). If the state φ_x corresponds to the spherical function ϕ_x, we denote by (π_x, H_x) both the representations of H and of $L^1(H)$ that it induces. Corresponding to this cyclic representation of $L^1(H)$ with cyclic unit vector ϕ_x, we have that the state $f \mapsto \langle \pi_x(f)\phi_x, \phi_x\rangle_x$ is

$$\begin{aligned}
\langle \pi_x(f)\phi_x, \phi_x\rangle_x &= \int_H f(h)\langle \pi_x(h)\phi_x, \phi_x\rangle_x dh \\
&= \int_H f(h)\langle L_h\phi_x, \phi_x\rangle_x dh \\
&= \int_H f(h)\phi_x(h^{-1}) dh \\
&= \int_H f(h)\overline{\phi_x(h)} dh.
\end{aligned}$$

Therefore this representation of $L^1(H)$ is unitarily equivalent to the one given by the Gelfand-Naimark-Segal correspondence, i.e., we can regard the representation as coming from a representation of the group H.

The operator T clearly intertwines the group representations π and $\int_X \pi_x d\mu$. The only thing that remains is to prove that T is surjective.

Suppose that $c = \{c_x\}$ is orthogonal to $T(\pi(L^1(H))$, i.e.,

$$\int_X \langle \pi_x(f)\xi_x, c_x\rangle_x d\mu = 0.$$

We observe that the restriction of T to the space $\mathscr{H}^L$ of L-invariant vectors intertwines the representations of $\pi(L^1(H)^{\#})$ on $\mathscr{H}^L$ and $\int_x E_x d\mu$. The mapping

$$\pi(f) \mapsto (x \mapsto \varphi_x(f))$$

is the Gelfand transform that realises the commutative C^*-algebra generated by $\pi(L^1(H)^{\#})$ and the identity operator as the algebra, $C(X)$, of continuous functions on X. Continuous functions of the form $\Psi(x) = \varphi_x(f^{\Psi})$, where $f^{\Psi} \in L^1(H)^{\#}$ are dense in $C(X)$. For such Ψ we have

$$\begin{aligned}\int_X \langle \pi_x(f)\xi_x, c_x\rangle_x \Psi(x) d\mu &= \int_X \langle \pi_x(f * f^{\Psi})\xi_x, c_x\rangle_x d\mu \\ &= 0\end{aligned}$$

From this we can conclude that (using once more the separability of $L^1(H)^{\#}$) for all x outside a set of μ-measure zero, the equality

$$\langle \pi_x(f)\xi_x, c_x\rangle_x = 0$$

holds for all $f \in L^1(H)^{\#}$. Since the vectors ξ_x are $L^1(H)^{\#}$-cyclic, we can conclude that $c = 0$ and this finishes the proof. □

Remark. The measure μ in the above theorem is called the *Plancherel measure* for the representation π.

5.2. **Extension and expansion of the spherical functions.** Consider the mapping $R : \mathscr{H}_\nu \to C^\infty(\mathscr{X})$ defined by

$$(Rf)(x) = h(x,x)^{\nu/2} f(x), x \in \mathscr{X}$$

(see [28]). When $\nu > n-1$, R is in fact an H-intertwining operator onto a dense subspace of $L^2(\mathscr{X}, d\iota)$ (where $d\iota$ is the H-invariant measure on $\mathscr{X}$) and the principal series representation gives the desired decomposition of π_ν into irreducible spherical representations. This is a heuristic motivation for studying the functions $R^{-1}\varphi_\lambda$, where φ_λ is a spherical function on $\mathscr{X}$.

Theorem 12. *Let $\nu > (n-2)/2$. The function $R^{-1}\varphi_\lambda(z)$ is holomorphic on $\mathscr{D}$ and has the power series expansion*

$$R^{-1}\varphi_\lambda(z) = \sum_k p_k(\lambda) e_k(z),$$

where $e_k(z)$ is the normalisation of the function $z \mapsto (zz^t)^k$ in the $\mathscr{H}_\nu$-norm, and the coefficients $p_k(\lambda)$ are polynomials of degree $2k$ of λ and satisfy the orthogonality relation a) If $\nu \geq \frac{n-1}{2}$, then

$$\begin{aligned}&\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(\frac{1}{2}+i\lambda)\Gamma(\frac{n-1}{2}+i\lambda)\Gamma(\nu-\frac{n-1}{2}+i\lambda)}{\Gamma(2i\lambda)}\right|^2 p_{\nu,k}(\lambda)\overline{p_{\nu,l}(\lambda)} d\lambda \\ = \;& \Gamma\left(\frac{n}{2}\right)\Gamma\left(\nu-\frac{n-2}{2}\right)\Gamma(\nu)\delta_{kl}.\end{aligned}$$

b) If $\nu < \frac{n-1}{2}$, then

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(\frac{1}{2}+i\lambda)\Gamma(\frac{n-1}{2}+i\lambda)\Gamma(\nu-\frac{n-1}{2}+i\lambda)}{\Gamma(2i\lambda)}\right|^2 p_{\nu,k}(\lambda)\overline{p_{\nu,l}(\lambda)}d\lambda \\
+ &\frac{\Gamma(\nu)\Gamma(\nu-\frac{n-2}{2})\Gamma(n-1-\nu)\Gamma(\frac{n}{2}-\nu)}{\Gamma(n-1-2\nu)} \\
&\times p_{\nu,k}\left(i\left(\nu-\frac{n-1}{2}\right)\right)\overline{p_{\nu,l}\left(i\left(\nu-\frac{n-1}{2}\right)\right)} \\
= &\Gamma\left(\frac{n}{2}\right)\Gamma\left(\nu-\frac{n-2}{2}\right)\Gamma(\nu)\delta_{kl}.
\end{aligned}$$

Proof. Recall the root space decomposition for $\mathfrak{h}$. Let $\langle\, ,\, \rangle$ denote the inner product on $\mathfrak{a}_{\mathbb{C}}$ that is dual to the restriction of the Killing form to $\mathfrak{a}$. Let α_0 denote $\alpha/\langle\alpha,\alpha\rangle$.
In this setting the spherical function φ_λ is determined by the formula (cf [6], Ch. IV, exercise 8)

$$\varphi_\lambda(\exp(t\xi_e)0) = {}_2F_1(a',b',c';-\sinh(\alpha(t\xi_e))^2), \tag{57}$$

where

$$\begin{aligned}
a' &= \frac{1}{2}\left(\frac{1}{2}m_\alpha+m_{2\alpha}+\langle i\lambda,\alpha_0\rangle\right) = \frac{1}{2}\left(\frac{n-1}{2}+i\lambda\right), \\
b' &= \frac{1}{2}\left(\frac{1}{2}m_\alpha+m_{2\alpha}-\langle i\lambda,\alpha_0\rangle\right) = \frac{1}{2}\left(\frac{n-1}{2}-i\lambda\right), \\
c' &= \frac{1}{2}\left(\frac{1}{2}m_\alpha+m_{2\alpha}+1\right) = \frac{1}{2}\left(\frac{n+1}{2}\right).
\end{aligned}$$

Letting $x = \exp(t\xi_e)0 = \tanh t$, (57) takes the form

$$\varphi_\lambda(x) = {}_2F_1(a',b',c';\frac{xx^t}{1-xx^t}) \tag{58}$$

By Euler's formula (cf [5]) we have

$$\varphi_\lambda(x) = {}_2F_1(a',b',c';\frac{xx^t}{1-xx^t}) = (1-xx^t)^{a'}\,{}_2F_1(a',c'-b',c';xx^t)$$

For the function $R^{-1}\varphi_\lambda$ we thus get the expression

$$R^{-1}\varphi_\lambda(z) = (1-zz^t)^{-\nu+a'}\,{}_2F_1(a',c'-b',c;zz^t) \tag{59}$$

Expanding (59) into a power series yields

$$(60)\qquad R^{-1}\varphi_\lambda(z)=\sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{(\nu-a')_{m-l}(a')_l(c'-b')_l}{(m-l)!l!(c')_l}(zz^t)^m,$$

noticing that $|zz^t|<1$ for $z\in\mathscr{D}$. Next, we use the following simple identities:

$$\begin{aligned}(\nu-a')_{m-l} &= \frac{(\nu-a')_m}{(\nu-a'+(m-l))_l}=\frac{(\nu-a')_m}{(-1)^l(-(\nu-a'+m-1))_l}\\ (m-l)! &= \frac{m!}{(m-l+1)_l}.\end{aligned}$$

Substitution of these in (60) yields

$$(61)\qquad \begin{aligned}&R^{-1}\varphi_\lambda(z)\\ &=\sum_{m=0}^{\infty}\frac{(\nu-a')_m}{m!}\sum_{l=0}^{m}\frac{(a')_l(c'-b')_l(-m)_l}{(c')_l(-(\nu-a'+m-1))_l}(zz^t)^m.\end{aligned}$$

The inner sum in (61) can be recognised as a hypergeometric function, i.e., we have

$$\sum_{l=0}^{m}\frac{(a')_l(c'-b')_l(-m)_l}{(c')_l(-(\nu-a'+m-1))_l}={}_3F_2(a',c'-b',-m;c',-(\nu-a'+m-1);1).$$

Now we use Thomae's transformation rule (cf [5]) for the function ${}_3F_2$:

$$\begin{aligned}&{}_3F_2(a',c'-b',-m;c',-(\nu-a'+m-1);1)\\ &=\frac{(-(\nu-a'+m-1)-(c'-b'))_m}{(-(\nu-a'+m-1))_m}\\ &\times\ {}_3F_2(c'-a',c'-b',-m;1+(c'-b')+(\nu-a'+m-1)-m;1)\end{aligned}$$

We finally obtain the following expression:

$$R^{-1}\varphi_\lambda(z)=\sum_{k=0}^{\infty}c_{n,\nu,k}(\lambda)(zz^t)^k,$$

where

$$c_{n,\nu,k}(\lambda)=\frac{(\nu-\frac{n-2}{2})_k}{k!}{}_3F_2(-k,\frac{1+i\lambda}{2},\frac{1-i\lambda}{2};\frac{n}{2},\nu-\frac{n-2}{2};1)$$

Recall the continuous dual Hahn polynomials (cf [27])

$$(62)\qquad \begin{aligned}S_k(x^2;a,b,c)&=(a+b)_k(a+c)_k\\ &\times{}_3F_2(-k,a+ix,a-ix;a+b,a+c;1)\end{aligned}$$

We can thus write

$$\begin{aligned} R^{-1}\varphi_\lambda(z) &= \sum_{k=0}^{\infty} \frac{(\nu-\frac{n-2}{2})_k}{(\frac{n}{2})_k(\nu-\frac{n-2}{2})_k k!} S_k\left((\frac{\lambda}{2})^2; \frac{1}{2}, \frac{n-1}{2}, \nu-\frac{n-2}{2}\right)(zz^t)^k \\ &= \sum_{k=0}^{\infty} p_{\nu,k}(\lambda)\frac{(zz^t)^k}{\|(zz^t)^k\|_\nu}. \end{aligned}$$

For the orthogonality relation in the claim, we refer to [27]. □

5.3. **Principal and complementary series representations.** In this section we let μ ($=\mu_\nu$) be the finite measure on the real line that orthogonalises the coefficients $p_k(\lambda)$ in (57). Let Λ_ν be its support. As we saw above, μ can, depending on the value of ν, either be absolutely continuous with respect to Lebesgue measure or have a point mass at $\lambda = i(\nu-(n-1)/2)$, i.e., we either have

$$\Lambda_\nu = (0,\infty)\bigcup\{i(\nu-(n-1)/2)\},\ \nu \in ((n-2)/2, (n-1)/2)$$

or

$$\Lambda_\nu = (0,\infty),\ \nu \geq (n-1)/2.$$

We will now construct explicit realisations for the spherical representations π_λ corresponding to the points $\lambda \in \Lambda_\nu$ on Hilbert spaces H_λ. For λ in the continuous part in Λ, the underlying space H_λ will be $L^2(S^{n-1})$ and for the discrete point $i(\nu-(n-1)/2)$, H_λ will be a Sobolev space.

We will hereafter suppress the index ν and simply denote the support of μ by Λ.

Lemma 13. *If $g \in H$, then g transforms the surface measure, σ, on S^{n-1} as*

$$d\sigma(g\zeta) = J_g(\zeta)^{\frac{n-1}{n}} d\sigma(\eta).$$

Proof. Clearly it suffices to prove the statement for automorphisms of the form

$$g = \exp \xi_v, v \in \mathbb{R}^n.$$

Moreover we can assume that $\zeta = e_1$, since any $\zeta \in S^{n-1}$ can be written as le_1, where $l \in L$, and

$$\begin{aligned} \exp\xi_v(le_1) = (\exp\xi_v l)(e_1) = (ll^{-1}\exp\xi_v l)(e_1) = (l\sigma_{l^{-1}}(\exp\ \xi_v))(e_1) \\ = l\exp\ (Ad(l^{-1})\xi_v)(e_1) = l\exp\xi_{\,l^{-1}v}(e_1). \end{aligned}$$

Consider now the tangent space of $\mathbb{R}^n$ at e_1. We have an orthogonal decomposition

$$T_{e_1}(\mathbb{R}^n) = T_{e_1}(S^{n-1}) \oplus \mathbb{R}e_1 .$$

At ge_1 we have the corresponding decomposition

$$T_{ge_1}(\mathbb{R}^n) = T_{ge_1}(S^{n-1}) \oplus \mathbb{R}ge_1 .$$

Since H preserves S^{n-1},

$$dg(e_1)\, T_{e_1}(S^{n-1}) = T_{ge_1}(S^{n-1}),$$

and by completing e_1 and ge_1 to orthonormal bases for their respective tangent spaces, $dg(e_1)$ corresponds to a matrix of the form

$$\begin{pmatrix} c & & 0 & \\ | & * & * & * \\ v & * & * & * \\ | & * & * & * \end{pmatrix}$$

Hence

$$J_g(e_1) = cJ_{g|_{S^{n-1}}}(e_1), \tag{63}$$

where

$$c = (dg(e_1)e_1, ge_1). \tag{64}$$

We next determine this constant c.

We have

$$c = (dg(e_1)e_1, ge_1) = \lim_{r\to 1}(dg(re_1)re_1, gre_1).$$

For fixed $r < 1$ we have

$$\begin{aligned} \exp \xi_v(re_1) &= u + B(u,u)^{1/2}B(re_1,-u)^{-1}(re_1 + Q(re_1)u) \\ &= u + dg(re_1)(re_1 + Q(re_1)u), \end{aligned}$$

and

$$J_g(re_1) = \left(\frac{h(re_1,-u)}{h(u,u)^{1/2}}\right)^{-n}, \tag{65}$$

where $u = \tanh v$. Since $Q(re_1)u = 2(u, re_1)re_1 - u$, we get

(66) $(dg(re_1)re_1, g(re_1))$

$$= (1 + 2(u, re_1))|dg(re_1)re_1|^2 + (dg(re_1)re_1, u - dg(re_1)u)$$

For any $z \in \mathscr{D} \bigcap \mathbb{R}^n$ and $v, w \in \mathbb{R}^n$, the identity

$$(dg(z)v, w) = \frac{h(gz, gz)}{h(z, z)}(v, dg(z)^{-1}w) \tag{67}$$

can be established using the transformation properties of the function h and the operator B. Applying (67) in the cases $z = re_1$, $v = re_1$, and $w = dg(re_1)re_1$ and
$w = u - dg(re_1)u$, repectively, yields

$$(dg(re_1)re_1, dg(re_1)re_1) = \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)}r^2 \tag{68}$$

and

$$\begin{aligned}(dg(re_1)re_1, u - dg(re_1)u) \\ = \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)}(re_1, dg(re_1)^{-1}u - u).\end{aligned} \tag{69}$$

The expressions above and an elementary computation shows that (66) can be written as

$$\begin{aligned}(dg(re_1)re_1, g(re_1)) \\ = \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)}\, r^2 \,\frac{1 + 2(u, re_1) + |u|^2}{1 - |u|^2}\end{aligned} \tag{70}$$

By the transformation rule for the Bergman kernel

$$h(g(re_1), g(re_1)) = |J_g(re_1)|^{2/n} h(re_1, re_1).$$

So,

$$\begin{aligned} c &= \lim_{r \to 1} |J_g(re_1)|^{2/n} r^2 \,\frac{1 + 2(u, re_1) + |u|^2}{1 - |u|^2} \\ &= |J_g(e_1)|^{2/n} \frac{h(e_1, -u)}{h(u, u)^{1/2}}. \end{aligned}$$

Comparing with the expression (65), we have determined the constant

$$c = J_g(e_1)^{1/n},$$

and this finishes the proof. □

For λ in the continuous part of Λ, the corresponding representation is a principal series representation described by the following proposition. (We will hereafter follow Helgason and in this context denote S^{n-1} by B. The measure σ will be denoted by db.)

Proposition 14. *For any real number λ, the map $h \mapsto \tau_\lambda(h)$, where*

$$\tau_\lambda(h)f(b) = J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}} f(h^{-1}b)$$

defines a unitary representation of H on $L^2(B)$.

Proof. We have

$$\begin{aligned}\int_B |J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}}|^2 |f(h^{-1}b)|^2 db &= \int_B J_{h^{-1}}(hb)^{\frac{2\rho}{n}} |f(b)|^2 d(hb) \\ &= \int_B J_h(b)^{-\frac{2\rho}{n}} |f(b)|^2 J_h(b)^{\frac{n-1}{n}} db \\ &= \int_B |f(b)|^2 db,\end{aligned}$$

where the last equality follows by lemma 13. □

It is well known that the representations τ_λ above are unitarily equivalent to the canonical spherical representations associated with the corresponding functionals λ on $\mathfrak{a}_\mathbb{C}$ (cf [11], ch. 7).

In order to realise the representation τ_λ for $\lambda = i(\nu - (n-1)/2)$, we consider the following Hilbert spaces.

Definition 15. For $\frac{n-2}{2n} \leq \alpha < \frac{n-1}{2n}$, let $\mathscr{C}_\alpha$ be the Hilbert space completion of the C^∞-functions on S^{n-1} with respect to the norm

$$\|f\|_{\mathscr{C}_\alpha} = \int_{S^{n-1}} \int_{S^{n-1}} f(\zeta)\overline{f(\eta)} K(\zeta,\eta)^\alpha d\sigma(\zeta) d\sigma(\eta)$$

Using the action of H on S^{n-1}, we can define a unitary representation of H on $\mathscr{C}_\alpha$ of the form

$$\sigma_\alpha : f \mapsto J_{h^{-1}}(\cdot)^\beta f(h^{-1}\cdot), h \in H,$$

where $\beta = -\alpha + (n-1)/n$. The unitarity follows from

$$\begin{aligned}&\int_{S^{n-1}} \int_{S^{n-1}} J_{h^{-1}}(\zeta)^\beta f(h^{-1}\zeta) \overline{J_{h^{-1}}(\eta)^\beta f(h^{-1}\eta)} K(\zeta,\eta)^\alpha d\sigma(\zeta) d\sigma(\eta) \\ &= \int_{S^{n-1}} \int_{S^{n-1}} J_h(\zeta)^{-\beta} f(\zeta) \overline{J_h(\eta)^{-\beta} f(\eta)} K(h\zeta, h\eta)^\alpha J_h(\zeta)^{\frac{n-1}{n}} J_h(\eta)^{\frac{n-1}{n}} d\sigma(\zeta) d\sigma(\eta) \\ &= \int_{S^{n-1}} \int_{S^{n-1}} J_h(\zeta)^{-\beta-\alpha+\frac{n-1}{n}} J_h(\eta)^{-\beta-\alpha+\frac{n-1}{n}} f(\zeta)\overline{f(\eta)} K(\zeta,\eta)^\alpha d\sigma(\zeta) d\sigma(\eta).\end{aligned}$$

In fact, this representation is irreducible (cf [1]). We denote this representation by σ_α. One can prove that for $\alpha = \nu/n$ and $\lambda = \nu - (n-1)/2$, σ_α and τ_λ are unitarily equivalent.

Recall the expression in Cor. 6 for the spherical functions. In this setting we write it as

$$\varphi_\lambda(x) = \int_B e_{\lambda,b}(x)db,$$

where

$$e_{\lambda,b}(x) = \left(\frac{h(x,x)^{1/2}}{h(x,b)}\right)^{i\lambda+\rho}$$

by Cor. 6. For fixed $z \in \mathscr{D}$ and $\lambda \in \Lambda$, $R^{-1}e_{\lambda,b}(z)$ is a function in $L^2(B)$. Moreover, $\pi_\nu(H)$ makes sense as a group of mappings on $\mathcal{O}(\mathscr{D})$, the set of holomorphic functions on $\mathscr{D}$. We have a relationship between these representations.

Lemma 16. *For every $g \in H$ and $\lambda \in \Lambda$,*

$$\pi_\nu(g)\tau_\lambda(g)R^{-1}e_{\lambda,b}(z) = R^{-1}e_{\lambda,b}(z). \tag{71}$$

Correspondingly, for $X \in \mathfrak{h}$, we have the relation

$$\pi_\nu(X)R^{-1}e_{\lambda,b}(z) = -\tau_\lambda(X)R^{-1}e_{\lambda,b}(z). \tag{72}$$

The proof is straightforward by applying the transformation rules for the function $h(z,w)$.

5.4. **The Fourier-Helgason transform.** The purpose of this section is to construct an H-intertwining unitary operator between the Hilbert spaces $\mathscr{H}_\nu$ and $\int_\Lambda H_\lambda d\mu$.

Any holomorphic function, f, on $\mathscr{D}$ has a power series expansion

$$f(z) = \sum_\alpha f_\alpha z^\alpha, \tag{73}$$

where $f_\alpha = \frac{\partial^\alpha f}{\alpha!\partial z^\alpha}(0)$. We can collect the powers of equal homogeneous degree together and write

$$f(z) = \sum_k f_k(z), \tag{74}$$

where f_k is of homogeneous degree k. We now consider the mapping

$$(\cdot,\cdot)_\nu : \mathcal{P} \times \mathcal{O}(\mathscr{D}) \to \mathbb{C}$$

defined as

$$(f,g)_\nu = \sum_k \langle f, g_k\rangle_\nu. \tag{75}$$

Observe that the definition makes sense since every polynomial is orthogonal to all but finitely many g_k.

Definition 17. If f is a polynomial in $\mathscr{H}_\nu$, its generalised Fourier-Helgason transform is the function $\tilde{f}$ on $\Lambda \times B$ defined by

$$\tilde{f}(\lambda, b) = (f, R^{-1}e_{\lambda,b})_\nu \tag{76}$$

Proposition 18. *(i) If the polynomial f is in $\mathscr{H}_\nu^L$, then $\tilde{f}$ is L-invariant and*

$$\|f\|_\nu^2 = \int_\Lambda \|\tilde{f}\|_\lambda^2 d\mu,$$

where $\|\cdot\|_\lambda$ is the norm on H_λ, and the Fourier-Helgason transform extends to an isometry from $\mathscr{H}_\nu^L$ onto $L^2(\Lambda, d\mu)$.
(ii) The inversion formula for L-invariant polynomials

$$f(z) = \int_\Lambda \tilde{f}(\lambda) R^{-1}\varphi_\lambda(z) d\mu(\lambda) \tag{77}$$

holds. Moreover, the above formula holds for arbitrary L-invariant functions, when restricted to the submanifold $\mathscr{X}$.

Proof. Writing

$$R^{-1}e_{\lambda,b} = \sum_\alpha c_\alpha(\lambda, b) z^\alpha = \sum_k e_{\lambda,b,k}$$

and

$$R^{-1}\varphi_\lambda(z) = \sum_\alpha c_\alpha(\lambda) z^\alpha = \sum_k p_k(\lambda) e_k(z),$$

we see that the coefficients and polynomials of homogeneous degree k are related by

$$c_\alpha(\lambda) = \int_B c_\alpha(\lambda, b) db \tag{78}$$

and

$$p_k(\lambda) e_k(z) = \int_B e_{\lambda,b,k}(z) db \tag{79}$$

respectively. Therefore we have

$$\begin{aligned}\tilde{f}(\lambda, b) &= \sum_k \langle f, e_{\lambda,b,k}\rangle_\nu \\ &= \sum_k \langle \int_L \pi_\nu(l) f dl, e_{\lambda,b,k}\rangle_\nu \\ &= \sum_k \langle f, \int_L \pi_\nu(l^{-1}) e_{\lambda,b,k} dl\rangle_\nu \\ &= \sum_k \langle f, \int_L \pi_\lambda(l) e_{\lambda,b,k} dl\rangle_\nu \\ &= (f, R^{-1}\varphi_\lambda)_\nu.\end{aligned}$$

This proves the L-invariance. Moreover, we have

$$(f, R^{-1}\varphi_\lambda)_\nu = \sum_k \overline{p_k(\lambda)}\langle f, e_k\rangle_\nu.$$

Hence

$$\int_\Lambda \|\tilde{f}\|_\lambda^2 d\mu = \sum_k |\langle f, e_k\rangle_\nu|^2 = \|f\|_\nu^2.$$

This proves the first part of the claim.

To prove the inversion formula, we now let f be an L-invariant polynomial and x be a point in $\mathscr{D}\bigcap\mathbb{R}^n$. Since we have an estimate of the form

$$|R^{-1}\varphi_\lambda(x)| \leq (1-|x|^2)^{-\frac{\nu}{2}} C(x), \tag{80}$$

where C is some function of x, independently of λ, the integral

$$\int_\Lambda \tilde{f}(\lambda) R^{-1}\varphi_\lambda(x) d\mu(\lambda)$$

makes sense for real x. We then have

$$\begin{aligned}\int_\Lambda \tilde{f}(\lambda) R^{-1}\varphi_\lambda(x) d\mu(\lambda) &= \sum_k \int_\Lambda \langle f, e_k\rangle_\nu \overline{p_k(\lambda)} R^{-1}\varphi_\lambda(x) d\mu(\lambda) \\ &= \sum_k \langle f, e_k\rangle_\nu \int_\Lambda \sum_j \overline{p_k(\lambda)} p_j(\lambda) e_j(x) d\mu(\lambda) \\ &= f(x).\end{aligned}$$

Now let $f \in \mathscr{H}_\nu^L$ be arbitrary. We choose a sequence of polynomials $f_n \in \mathscr{H}_\nu^L$ such that

$$f = \lim f_n.$$

Since the evaluation functionals are continuous, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_\Lambda \tilde{f}_n(\lambda) R^{-1} \varphi_\lambda(x) d\mu(\lambda)$$

for every real point x. By Jensen's inequality and (80)

$$\left| \int_\Lambda (\tilde{f}(\lambda) - \tilde{f}_n(\lambda)) R^{-1} \varphi_\lambda(x) d\mu(\lambda) \right|^2$$
$$\leq \mu(\Lambda) \int_\Lambda |\tilde{f}(\lambda) - \tilde{f}_n(\lambda)|^2 C(x)(1-|x|^2)^{-\nu} d\mu(\lambda).$$

Hence

$$f(x) = \int_\Lambda \tilde{f}(\lambda) R^{-1} \varphi_\lambda(x) d\mu(\lambda).$$

Thus the inversion formula holds for real points, x. To see that the formula holds for arbitrary points when f is a polynomial, we note that both the left hand- and the right hand side of the formula define holomorphic functions on $\mathscr{D}$. Since they agree on the totally real form $\mathscr{X}$, they are equal. □

Theorem 19 (The Plancherel Theorem)**.** *For $\nu > (n-2)/2$, the Fourier-Helgason transform is a unitary isomorphism from the H-modules $\mathscr{H}_\nu$ onto the H-module $\int_\Lambda H_\lambda d\mu$, i.e.,*

$$(\pi_\nu(h)f)\tilde{}(\lambda, b) = \tau_\lambda(h)\widetilde{f}(\lambda, b),$$

for $h \in H$, and

$$\|f\|_\nu^2 = \int_\Lambda \|\tilde{f}\|_\lambda^2 d\mu.$$

Proof. We divide the proof into three steps:
(i) We prove that the Fourier-Helgason transform intertwines the action of the Lie algebra of H.
(ii) We use (i) to prove that the norm is preserved.
(iii) We conclude that the group actions are intertwined from (i) and (ii).

We will see that these properties actually imply that the Fourier-Helgason transform is surjective.

Consider now the corresponding representations of the Lie algebra $\mathfrak{h}$. These will also be denoted by π_ν and τ_λ respectively. Moreover they extend naturally to representations of the universal enveloping algebra, $\mathfrak{U}(\mathfrak{h})$, of $\mathfrak{h}$.
Let $X \in \mathfrak{h}$. If f is a polynomial in $\mathscr{H}_\nu$, then differentiation of the mapping

$$t \mapsto J_{\exp tX}(z)^{\nu/n} f((\exp tX)z)$$

at $t = 0$ shows that $\pi_\nu(X)f$ is also a polynomial, and

$$\begin{aligned}
\widetilde{\pi_\nu(X)f}(\lambda, b) &= \sum_k \langle \pi_\nu(X)f, e_{\lambda,b,k}\rangle_\nu \\
&= \sum_k \langle f, -\pi_\nu(X)e_{\lambda,b,k}\rangle_\nu \\
&= \sum_k \langle f, \tau_\lambda(X)e_{\lambda,b,k}\rangle_\nu \\
&= (f, \tau_\lambda(X)R^{-1}e_{\lambda,b})_\nu \\
&= \tau_\lambda(X)(f, R^{-1}e_{\lambda,b})_\nu,
\end{aligned}$$

which proves (i).

To prove the second step, we recall that the adjoint representation of L on $\mathfrak{h}$ extends to an action on $\mathfrak{U}(\mathfrak{h})$ as homomorphisms of an associative algebra. The L-invariant elements in $\mathfrak{U}(\mathfrak{h})$ form a subalgebra, $\mathfrak{U}(\mathfrak{h})^L$. We let p denote the projection

$$X \mapsto \int_L Ad(l)X dl$$

of $\mathfrak{U}(\mathfrak{h})$ onto $\mathfrak{U}(\mathfrak{h})^L$. This action of L connects the representations of H and $\mathfrak{U}(\mathfrak{h})$ according to the following identity:

$$\pi_\nu(l)\pi_\nu(X)\pi_\nu(l^{-1}) = \pi_\nu(Ad(l)X),$$

for $l \in L$ and $X \in \mathfrak{U}(\mathfrak{h})$.

Since the vector $1 \in \mathscr{H}_\nu$ is cyclic for the representation of H, it is also cyclic for the representation of $\mathfrak{U}(\mathfrak{h})$. Hence it suffices to prove that the norm is preserved for elements of the form $\pi_\nu(X)1$, where $X \in \mathfrak{U}(\mathfrak{h})$. In the following equalities, we temporarily let τ denote the direct integral of the representations τ_λ, and analogously we let $\langle , \rangle$

denote the direct integral of the corresponding inner products.

$$\begin{aligned}\langle \pi_\nu(X)1, \pi_\nu(X)1\rangle_\nu &= \langle \pi_\nu(X)^*\pi_\nu(X)1, 1\rangle_\nu \\ &= \langle -\pi_\nu(X^2)1, 1\rangle_\nu\end{aligned}$$

Since the vector 1 is L-invariant, the last expression equals $\langle -\pi_\nu(p(X^2))1, 1\rangle_\nu$, and by proposition (18), we have

$$\langle -\pi_\nu(p(X^2))1, 1\rangle_\nu = \langle -\pi_\nu(\widetilde{p(X^2)})1, \tilde{1}\rangle.$$

By (i), the expression on the right-hand side equals $\langle -\tau(p(X^2))\tilde{1}, \tilde{1}\rangle$, and since $\tilde{1}$ is L-invariant, we have

$$\langle -\tau(p(X^2))\tilde{1}, \tilde{1}\rangle = \langle -\tau(X^2)\tilde{1}, \tilde{1}\rangle.$$

Thus (ii) is proved.

To prove (iii), we recall the following equalities (on the respective dense spaces of analytic vectors):

$$\begin{aligned}\pi_\nu(\exp(X)) &= e^{\pi_\nu(X)} \\ \tau_\lambda(\exp(X)) &= e^{\tau_\lambda(X)}.\end{aligned}$$

From this and the facts that H is connected and that the Fourier-Helgason transform is bounded operator, we immediately see that (iii) holds.

To see that the operator is surjective, note that by (ii) and (iii)

$$\langle \pi_\nu(f)1, 1\rangle_\nu = \int_\Lambda \langle \tau_\lambda \tilde{1}(\lambda, \cdot), \tilde{1}(\lambda, \cdot)\rangle_\lambda d\mu,$$

for $f \in L^1(H)^\#$, i.e., we can write the positive functional

$$f \mapsto \langle \pi_\nu(f)1, 1\rangle_\nu$$

as an integral of pure states with respect to some measure. By uniqueness, it is the measure in Theorem 10. Since the Fourier-Helgason transform intertwines the group action, it is the intertwining operator constructed in Theorem 10. Thus it is surjective. □

Theorem 20 (The Inversion Formula). *If f is a polynomial in $\mathscr{H}_\nu$, then*

$$f(z) = \int_\Lambda \int_B \widetilde{f}(\lambda, b) R^{-1} e_{\lambda,b}(z)\, db\, d\mu(\lambda). \tag{81}$$

Proof. Take $h \in H$. Define

$$f_1(z) = \int_L \pi_\nu(l)\pi_\nu(h)f(z)dl$$

This is a radial function, and we have that

$$f_1(0) = J_{h^{-1}}(0)^{\frac{\nu}{n}} f(h^{-1}0). \tag{82}$$

Prop. 18 gives

$$f_1(0) = \int_\Lambda \tilde{f}_1(\lambda) R^{-1}\varphi_\lambda(z) d\mu(\lambda). \tag{83}$$

Moreover

$$\begin{aligned} \widetilde{f_1}(\lambda) = (f_1, R^{-1}\varphi_\lambda)_\nu &= (\int_L \pi_\nu(l)\pi_\nu(h)f dl, R^{-1}\varphi_\lambda)_\nu \\ &= (\pi_\nu(h)f, R^{-1}\varphi_\lambda)_\nu. \end{aligned} \tag{84}$$

By Thm. 19 we have

$$\begin{aligned} (\pi_\nu(h)f, R^{-1}\varphi_\lambda)_\nu &= (f, \pi_\nu(h^{-1})R^{-1}\varphi_\lambda)_\nu \\ &= (f, \int_B \pi_\nu(h^{-1})R^{-1}e_{\lambda,b}\, db)_\nu \\ &= (f, \int_B \pi_\nu(h^{-1})R^{-1}e_{\lambda,b}\, db)_\nu \\ &= (f, \int_B \tau_\lambda(h)R^{-1}e_{\lambda,b}\, db)_\nu \\ &= (f, \int_B J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}} R^{-1}e_{\lambda,h^{-1}b}\, db)_\nu. \end{aligned}$$

The integrand above has a power series expansion where the coefficients are functions of b. If we integrate, we obtain a holomorphic functions for which the coefficients in the power series expansion are obtained by integrating the aforementioned coefficients over B. Hence we can proceed as follows.

$$\begin{aligned}
(f, \int_B J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}} R^{-1} e_{\lambda,h^{-1}b}\, db)_\nu &= \int_B J_{h^{-1}}(b)^{\frac{-i\lambda+\rho}{n}} (f, R^{-1}e_{\lambda,h^{-1}b})_\nu db \\
&= \int_B J_{h^{-1}}(b)^{\frac{-i\lambda+\rho}{n}} \widetilde{f}(\lambda, h^{-1}b) db \\
&= \int_B J_{h^{-1}}(hb)^{\frac{-i\lambda+\rho}{n}} \widetilde{f}(\lambda, b) J_h(b)^{\frac{n-1}{n}} db \\
(85) \qquad &= \int_B \widetilde{f}(\lambda, b) J_h(b)^{\frac{i\lambda+\rho}{n}} db.
\end{aligned}$$

It is easy to see that

$$(86) \qquad J_h(b)^{\frac{i\lambda+\rho}{n}} = J_{h^{-1}}(0)^{\frac{\nu}{n}} R^{-1} e_{\lambda,b}(h^{-1}0),$$

and so combining (82), (83) and (85) finally yields

$$(87) \qquad f(h^{-1}0) = \int_\Lambda \int_B \widetilde{f}(\lambda, b) R^{-1} e_{\lambda,b}(h^{-1}0) db\, d\mu.$$

Thus the inversion formula holds for real points, hence for all points by the same argument as in the proof of Prop. 18. □

6. Realisation of the discrete part of the decomposition

We recall the earlier defined complementary series representations. The following theorem states that $\sigma_{\nu/n}$ is the representation corresponding to the singular point in the decomposition theorem.

Theorem 21. *The operator T_ν defined by the formula*

$$(T_\nu f)(z) = \int_{S^{n-1}} f(\zeta) K_\nu(z, \zeta) d\sigma(\zeta)$$

is a unitary H-intertwining operator from $\mathscr{C}_{\nu/n}$ onto an irreducible H-submodule of $\mathscr{H}_\nu$.

Proof. First of all we note that T_ν maps functions in $\mathscr{C}_{\nu/n}$ to holomorphic functions on $\mathscr{D}$ and thus π_ν has a meaning on the range of T_ν.

We start by showing that T_ν is formally intertwining. We have

$$\begin{aligned} T_\nu(\sigma_{\nu/n})f(z) &= \int_{S^{n-1}} J_{h^{-1}}(\zeta)^{-\nu/n+\frac{n-1}{n}} f(h^{-1}\zeta)K_\nu(z,\zeta)d\sigma(\zeta) \\ &= \int_{S^{n-1}} J_h(\zeta)^{\nu/n-\frac{n-1}{n}} f(\zeta)K_\nu(z,h\zeta)J_h(\zeta)^{\frac{n-1}{n}}d\sigma(\zeta) \\ &= \int_{S^{n-1}} J_h(\zeta)^{\nu/n} f(\zeta)K_\nu(h^{-1}z,\zeta)J_h(h^{-1}z)^{-\frac{\nu}{n}}J_h(\zeta)^{-\frac{\nu}{n}}d\sigma(\zeta) \\ &= J_{h^{-1}}(z)^{\frac{\nu}{n}}\int_{S^{n-1}} f(\zeta)K_\nu(h^{-1}z,\zeta)d\sigma(\zeta), \end{aligned}$$

i.e.,

$$T_\nu\sigma_{\nu/n} = \pi_\nu T_\nu.$$

The next step is to prove that the constant function 1 is mapped into $\mathscr{H}_\nu$ and that its norm is preserved. Note that for $\alpha = \nu/n$, $K(z,\zeta)^\alpha = K_\nu(z,\zeta)$, and by Prop. 7 we have an expansion

$$K_\nu(\zeta,e_1) = \sum_{m-2k\geq 0} c_{m,k}(\nu)K_{(m-k,k)}(\zeta,e_1),$$

where the coefficients $c_{m,k}(\nu)$ are given explicitly. Now, since $K_\nu(\zeta,e_1)$ is $SO(n-1)$-invariant and the action of $SO(n-1)$ is linear, each $K_{(m,k)}(\zeta,e_1)$ must also be $SO(n-1)$-invariant. Hence, $K_{(m,k)}(\zeta,e_1)$ can be assumed to be $\phi_{m-2k}(\zeta)(\zeta\zeta^t)^k$, where ϕ_{m-2k} is the unique element in E_{m-2k} that assumes the value 1 in e_1. Therefore

$$\begin{aligned} &\int_{S^{n-1}} K(\zeta,\eta)^\alpha d\sigma(\zeta) && (88)\\ &= \int_L K_\nu(\zeta,le_1)dl \int_L K_\nu(l^{-1}\zeta,e_1)dl \\ &= \sum_{m-2k\geq 0} c_{m,k}(\nu)\int_L (l^{-1}\zeta(l^{-1}\zeta)^t)^k\phi_{m-2k}(l^{-1}\zeta)dl && (89)\\ &= \sum_{m-2k\geq 0} c_{m,k}(\nu)(\zeta\zeta^t)^k\int_L \phi_{m-2k}(l^{-1}\zeta)dl && (90) \end{aligned}$$

Since $SO(n)$ acts irreducibly on E_{m-2k} and the function $\int_L \phi_{m-2k}(l^{-1}z)dl$ is an $SO(n)$-invariant element in E_{m-2k} it must be identically zero unless $m-2k=0$. Since

$$\|1\|^2_{\mathscr{C}_{\frac{\nu}{n}}} = \int_{S^{n-1}}\int_{S^{n-1}} K_\nu(\zeta,\eta)d\sigma(\zeta)d\sigma(\eta),$$

the computation above implies that

$$\begin{aligned}
\|1\|^2_{\mathscr{C}_{\frac{\nu}{n}}} &= \int_{S^{n-1}} \sum_{k=0}^{\infty} c_{2k,k}(\nu)(\zeta\zeta^t)^k d\sigma(\zeta) \\
&= \sum_{k=0}^{\infty} c_{2k,k}(\nu) = \sum_{k=0}^{\infty} \frac{(\nu)_k(\nu-\frac{n-2}{2})_k}{\|(zz^t)^k\|^2_{\mathscr{F}}} \\
&= \sum_{k=0}^{\infty} \frac{(\nu)_k(\nu-\frac{n-2}{2})_k}{k!(\frac{n}{2})_k} \qquad (91)
\end{aligned}$$

On the other hand, the equalities (88)-(90) also show that

$$\begin{aligned}
T_\nu 1(z) &= \sum_{k=0}^{\infty} \frac{(\nu)_k(\nu-\frac{n-2}{2})_k}{k!(\frac{n}{2})_k}(zz^t)^k \\
&= \sum_{k=0}^{\infty} \frac{((\nu)_k(\nu-\frac{n-2}{2})_k)^{1/2}}{(k!(\frac{n}{2})_k)^{1/2}} \frac{(zz^t)^k}{\|(zz^t)^k\|_\nu}. \qquad (92)
\end{aligned}$$

If we compare (91) and (92), we see that $T_\nu 1 \in \mathscr{H}_\nu$ and that $\|1\|_{\mathscr{C}_{\nu/n}} = \|1\|_\nu$. Recall that

$$\mathscr{C}_{\nu/n} = \overline{\bigoplus_m E_m(S^{n-1})}$$

and that the representation of $\mathfrak{h}$ on the algebraic sum $\bigoplus_m E_m(S^{n-1})$ is irreducible. Hence

$$\bigoplus_m E_m(S^{n-1}) = \mathrm{Span}_{\mathbb{C}}\{\sigma_{\nu/n}(X_1)\ldots\sigma_{\nu/n}(X_k)1 | X_i \in \mathfrak{h}, 1 \le i \le k\}$$

Since T_ν interwines the representations of $\mathfrak{h}$, we have that π_ν is an irreducible representation of $\mathfrak{h}$ on the space $T_\nu(\bigoplus_m E_m(S^{n-1})) \subseteq \mathscr{H}_\nu$. By Schur's lemma ([13], ch.4)

$$\langle T_\nu f, T_\nu g\rangle_\nu = c\langle f, g\rangle_{\mathscr{C}_{\nu/n}},$$

for some real constant c. Putting, f and g equal to the constant function 1 and applying, we see that $c = 1$. Therefore, T_ν extends to a unitary operator

$$T_\nu : \mathscr{C}_{\nu/n} \to \overline{T_\nu(\bigoplus_m E_m(S^{n-1}))}$$

and we have proved the theorem. □

7. Realisation of the minimal representation $\pi_{(n-2)/2}$

In this section we show that the representation $\pi_{(n-2)/2}$ of H is irreducible by realising it as a complementary series representation.

We recall the space $\mathscr{C}_{\nu/n}$ from the previous section and the corresponding operator T_ν.

Theorem 22. *$T_{(n-2)/2}$ is a unitary H-intertwining operator from $\mathscr{C}_{(n-2)/2n}$ onto $\mathscr{H}_{(n-2)/2}$.*

Proof. Recall that

$$\mathscr{C}_{(n-2)/n} = \overline{\bigoplus_m E_m(S^{n-1})} \tag{93}$$

and that the sum is a decomposition into $SO(n)$-irreducible subspaces. If we let $\mathcal{P}_{(n-2)/n}$ denote the set of all finite sums in (93), $\sigma_{(n-2)/n}$ defines a representation of $\mathfrak{l}$ on $\mathcal{P}_{(n-2)/n}$. The polynomial $(\zeta_1 + i\zeta_2)^m$ is a highest weight vector in E_m for this representation. Moreover, the power series expansion of $K_{(n-2)/n}$ shows that $T_{(n-2)/2}$ is a polynomial in E_m. Since $T_{(n-2)/2}$ intertwines the $\mathfrak{l}$-actions, $T_{(n-2)/2}((\zeta_1 + i\zeta_2)^m)$ is a highest weight vector space for $\pi_{(n-2)/2}(\mathfrak{l})$, i.e.,

$$(T_{(n-2)/2}(\zeta_1 + i\zeta_2)^m)(z) = C_m(z + iz)^m, \tag{94}$$

for some constant C_m. We now determine C_m. Choose $z = w\frac{1}{2}(1, -i, 0, \ldots, 0)$, where w is a complex number with $|w| < 1$. In this case $zz^t = 0, (z + iz)^m = w^m$. We now compute $(T_{(n-2)/2}((\zeta_1 + i\zeta_2)^m)(z)$.

$$\begin{aligned}
&\int_{S^{n-1}} K_{(n-2)/n}(z, \zeta)(\zeta_1 + i\zeta_2)^m \\
&\quad = \int_{S^{n-1}} (1 - w(\zeta_1 - i\zeta_2))^{-(n-2)/n}(\zeta_1 + i\zeta_2)^m d\sigma(\zeta)
\end{aligned}$$

This integral only depends on the first two coordinates and can hence be converted to an integral over the unit disk, U (cf [21] Prop 1.4.4).

$$\begin{aligned}
&\int_{S^{n-1}} (1 - w(\zeta_1 - i\zeta_2))^{-(n-2)/n}(\zeta_1 + i\zeta_2)^m d\sigma(\zeta) \\
&\quad = \frac{\Gamma\left(\frac{n-2}{2}\right)}{\pi\Gamma\left(\frac{n}{2}\right)} \int_U (1 - w\overline{\zeta})^{-(n-2)/n}\zeta^m(1 - |\zeta|^2)^{(n-4)/2} dm(\zeta).
\end{aligned}$$

We have the power series expansion

$$(1-w\overline{\zeta})^{-(n-2)/2}=\sum_{k=0}^{\infty}\left(\frac{n-2}{2}\right)_k(z\overline{\zeta})^k$$

Recall that $(1-w\overline{\zeta})^{-n/2}$ is the reproducing kernel for the weighted Bergman space $\mathscr{H}_{n/2}(U)$, defined as

$$\mathscr{H}_{n/2}(U)=\{f\in\mathcal{O}(U)\mid \frac{\Gamma\left(\frac{n}{2}\right)}{\pi\Gamma\left(\frac{n-2}{2}\right)}\int_U|f(\zeta)|^2(1-|\zeta|^2)^{(n-4)/2}dm(\zeta)<\infty\},$$

Polynomials of different degree are orthogonal in $\mathscr{H}_{n/2}(U)$ and hence we have

$$\begin{aligned}
&\int_U(1-w\overline{\zeta})^{-(n-2)/n}\zeta^m(1-|\zeta|^2)^{(n-4)/2}dm(\zeta)\\
&= \int_U\sum_{k=0}^{\infty}\left(\frac{n-2}{2}\right)_k(z\overline{\zeta})^k\zeta^m(1-|\zeta|^2)^{(n-4)/2}dm(\zeta)\\
&= \int_U\sum_{k=0}^{\infty}\left(\frac{n-2}{2}\right)\left(\frac{n}{2}\right)_m(z\overline{\zeta})^m\zeta^m(1-|\zeta|^2)^{(n-4)/2}dm(\zeta)\\
&= \pi w^m,
\end{aligned}$$

where the last equality follows from the reproducing property in $\mathscr{H}_{n/2}(U)$. Summing up, we have

$$(T_{(n-2)/2}(\zeta_1+i\zeta_2)^m)(z)=\frac{n-2}{2\pi^2}(z_1+iz_2)^m \tag{95}$$

From this and the intertwining of the $\mathfrak{l}$-action, it follows that

$$T_{(n-2)/2}\left(\bigoplus_m E_m(S^{n-1})\right)\subseteq\bigoplus_m E_m \tag{96}$$

To compute the norm of $T_{(n-2)/2}(p)$ where $p\in E_k(S^{n-1})$, we first fix $r<1$ and consider the polynomial $T_{(n-2)/2}(p(rz))$. By definition

$$\begin{aligned}
T_{(n-2)/2}(p)(rz) &= \int_{S^{n-1}}K_\nu(rz,\zeta)p(\zeta)d\sigma(\zeta)\\
&= \int_{S^{n-1}}K_\nu(z,r\zeta)p(\zeta)d\sigma(\zeta).
\end{aligned} \tag{97}$$

The norm is given by

$$\|T_{(n-2)/2}(p)(r\,\cdot)\|_\nu^2 = \int_{S^{n-1}}\int_{S^{n-1}} p(\zeta)\overline{p(\eta)}K_\nu(r\zeta, r\eta)d\sigma(\zeta)d\sigma(\eta).$$

Finally, we let $r \to 1$ and obtain

$$\|T_{(n-2)/2}(p)\|_\nu^2 = \int_{S^{n-1}}\int_{S^{n-1}} p(\zeta)\overline{p(\eta)}K_\nu(\zeta, \eta)d\sigma(\zeta)d\sigma(\eta).$$

From this and the orthogonality of the spaces E_k, it follows that $T_{(n-2)/2}$ maps $(\bigoplus_m E_m(S^{n-1})$ isometrically onto $(\bigoplus_m E_m)$. Hence it extends to a unitary operator from $\mathscr{C}_{(n-2)/2n}$ onto $\mathscr{H}_{(n-2)/2}$. □

References

1. Cowling, Michael *Harmonic analysis on some nilpotent Lie groups (with application to the representation theory of some semisimple Lie groups)* Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, (1982), 81–123, Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983
2. Faraut, J. and Korányi, A.*Function spaces and reproducing kernels on bounded symmetric domains* J. Funct. Anal. **88**,(1990), 1, 64–89
3. Faraut, J. and Korányi, A. "Analysis on symmetric cones" Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1994
4. Folland, Gerald B. "A course in abstract harmonic analysis" Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995
5. Gasper, George and Rahman, Mizan "Basic hypergeometric series" Encyclopedia of Mathematics and its Applications, 35, Cambridge University Press, Cambridge, 1990
6. Helgason, Sigurdur "Groups and geometric analysis" Mathematical Surveys and Monographs,83, American Mathematical Society,Providence, RI, 2000
7. Howe, Roger *Transcending classical invariant theory* J. Amer. Math. Soc., **2**, (1989, 3, 535–552
8. Hua, Loo-keng *On the theory of Fuchsian functions of several variables* Ann. of Math. (2), **47**, (1946), 167–191
9. Jakobsen, Hans Plesner and Vergne, Michèle *Restrictions and expansions of holomorphic representations* J. Funct. Anal., **34**, (1979), 1, 29–53
10. Kashiwara, M. and Vergne, M. *On the Segal-Shale-Weil representations and harmonic polynomials* Invent. Math., **44**, (1978), 1, 1–47
11. Knapp, Anthony W. "Representation theory of semisimple groups , an overview based on examples" Princeton University Press, Princeton, NJ, 2001
12. Knapp, Anthony W. "Lie groups beyond an introduction" Progress in Mathematics, 140, Birkhäuser Boston Inc., Boston, MA, 2002
13. Knapp, Anthony W. and Vogan, Jr., David A. "Cohomological induction and unitary representations" Princeton Mathematical Series, 45, Princeton University Press, Princeton, NJ, 1995

14. Kobayashi, Toshiyuki and Ørsted, Bent *Analysis on the minimal representation of $O(p,q)$, II.* Branching laws Adv. Math., **180**, (2003), 2, 513–550
15. Loos, O. "Bounded symmetric domains and Jordan pairs" Lecture notes, University of California, Irvine, 1977
16. Murphy, Gerard J. "C^*-algebras and operator theory" Academic Press Inc.,Boston, MA, 1990
17. Naĭmark, M. A. "Normed rings" P. Noordhoff N. V., Groningen, 1964
18. Neretin, Yu. A. *Beta integrals and finite orthogonal systems of Wilson polynomials* Mat. Sb., **193**, (2002), 7, 131–148
19. Neretin, Yurii A. *Plancherel formula for Berezin deformation of L^2 on Riemannian symmetric space* J. Funct. Anal., **189**, (2002), 2, 336–408
20. Rossi, H. and Vergne, M. *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group* Acta Math., **136**, (1976), 1-2, 1–59
21. Rudin, Walter "Function theory in the unit ball of $\mathbf{C}^n$" Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 241, Springer-Verlag, New York, 1980
22. Rudin, Walter "Functional analysis" International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991
23. Stein, Elias M. and Weiss, Guido "Introduction to Fourier analysis on Euclidean spaces",Princeton Mathematical Series, No. 32 Princeton University Press, Princeton, N.J., 1971
24. Unterberger, A. and Upmeier, H. *The Berezin transform and invariant differential operators* Comm. Math. Phys., **164**, (1994), 3, 563–597
25. van Dijk, G. and Pevzner, M. *Berezin kernels of tube domains* J. Funct. Anal., **181**, (2001), 2, 189–208
26. Wallach, Nolan R. *The analytic continuation of the discrete series. I, II* Trans. Amer. Math. Soc., **251**, (1979), 1–17, 19–37
27. Wilson, James A. *Some hypergeometric orthogonal polynomials* SIAM J. Math. Anal., **11**, (1980), 4, 690–701
28. Zhang, Genkai *Berezin transform on real bounded symmetric domains* Trans. Amer. Math. Soc., **353**, (2001), 9, 3769–3787 (electronic)
29. Zhang, Genkai *Tensor products of minimal holomorphic representations* Represent. Theory, **5**, (2001), 164–190 (electronic)
30. Zhang, Genkai *Branching coefficients of holomorphic representations and Segal-Bargmann transform* J. Funct. Anal., **195**, (2002), 2, 306–349

Department of Mathematics, Chalmers University of Technology and Göteborg University , Göteborg, Sweden
E-mail address: `henriks@math.chalmers.se`

Paper II

TUBE DOMAINS AND RESTRICTIONS OF MINIMAL REPRESENTATIONS

HENRIK SEPPÄNEN

Abstract. In this paper we study the restrictions of the minimal representation in the analytic continuation of the scalar holomorphic discrete series from $Sp(n,\mathbb{R})$ to $GL(n,\mathbb{R})$, and from $SU(n,n)$ to $GL(n,\mathbb{C})$ respectively. We work with the realisations of the representation spaces as L^2-spaces on the boundary orbits of rank one of the corresponding cones, and give explicit integral operators that play the role of the intertwining operators for the decomposition. We prove inversion formulas for dense subspaces and use them to prove the Plancherel theorem for the respective decomposition. The Plancherel measure turns out to be absolutely continuous with respect to the Lebesgue measure in both cases.

1. Introduction

The unitary representations obtained by continuation of the scalar holomorphic discrete series of a hermitian Lie group, G, were classified by Wallach in [10], and independently by Rossi and Vergne ([5]). The classification amounts to membership in the *Wallach set* for the linear functionals on the compact Cartan subalgebra that extend the family of weights parametrising the weighted Bergman spaces on the symmetric space G/K.

These unitary representations can all be realised on Hilbert spaces of holomorphic functions on the corresponding bounded symmetric domain $\mathscr{D} \cong G/K$. However, in this model the unitary structure cannot be described in a uniform way even though the corresponding reproducing kernels can. In any case, the restriction to any totally real submanifold defines an injective mapping. Therefore it is natural to consider an antiholomorphic involution $\tau : \mathscr{D} \to \mathscr{D}$ that lifts to an

1991 *Mathematics Subject Classification.* 22E45, 32M15, 33C45, 43A85.

Key words and phrases. Lie groups, unitary representations, branching law, real bounded symmetric domains.

involutive automorphism (which we also denote by τ) of the group G. Letting $H = G^\tau$ denote the fixed point group, and $L = K \cap H$, the space $\mathscr{X} := H/L$ is a totally real submanifold. The decomposition of the restriction to H of the unitary representations obtained by analytic continuation, or more generally, the restriction of holomorphic representations to symmetric subgroups, has lately been an area of intensive research. Among those who have studied this problem we find, for example, Davidson, Ólafsson, and Zhang ([1]), van Dijk and Pevzner ([9]), Zhang ([11], [12], [13]), and the author ([7], [8]). However, there does not yet seem to be any uniform way of dealing with this problem. For regular parameter, the Segal-Bargmann transform provides a unitary equivalence between the restriction to the group H and the left regular representation of H on the space $L^2(H/L)$ and in this case the decomposition is determined explicitly by the Helgason Fourier transform for the symmetric space H/L. Otherwise, the results obtained so far depend on particular features of the special cases. In [12], Zhang decomposes the restriction to the diagonal subgroup of the tensor product of a minimal representation and its dual by finding the spectral decomposition for the Casimir operator. The same method is used in [8], where the author determines the restriction of the minimal representation for $SU(n,m)$ to the subgroup $SO(n,m)$. It should be noted that this approach identifies the representations occurring in the decomposition and determines the Plancherel measure explicitly, but it does not provide an intertwining operator. In [7], the author determines the restriction from $SO(2,n)$ to $SO(1,n)$ for general parameter in the Wallach set and gives an intertwining operator. This was possible thanks to an explicit power series expansion for the spherical functions on the group $SO(1,n)$.

In this paper we consider the minimal representations for the groups $Sp(n,\mathbb{R})$ and $SU(n,n)$ and restrict to the automorphism groups for the cones associated with the respective tube domains. We use the model in [5] that realises the representations as L^2-spaces on the orbits of rank one elements on the boundaries of the respective cones. The intertwining operators are given explicitly as integral transforms [1]. The two cases we deal with are identical in principle. However, the proofs are rather technical when it comes to parameters, so we

[1] or, rather, as analytic continuations of operators defined as integral transforms

chose to avoid a uniform presentation to increase the readability. Instead we present two parallel cases where the solutions follow the same guideline.

The paper is organised as follows. Section 2 contains preliminaries for the two cases separately. In section 3 we describe the constituents in the decomposition for the restriction from $Sp(n,\mathbb{R})$, construct an intertwining operator and prove the Plancherel theorem. Section 4 is the analogue of section 3 for the group $SU(n,n)$.

Acknowledgement: The author is grateful to his advisor Professor Genkai Zhang for illuminating discussions on the topic of this paper.

2. Preliminaries

2.1. **Type II_n.** Let V be the real vector space of symmetric $n\times n$-matrices. The complexification, $V^{\mathbb{C}} = V \oplus iV$, of V consists of all complex symmetric $n\times n$ matrices. Consider the bounded symmetric domain

$$\mathscr{D} = \{Z \in V^{\mathbb{C}} | I - Z^*Z > 0\}. \tag{1}$$

The group

$$G = Sp\,(n,\mathbb{R}) = \left\{ g \in SU(n,n) | g^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\}$$

acts transitively on $\mathscr{D}$ by

$$Z \mapsto (AZ+B)(CZ+D)^{-1}, \tag{2}$$

where

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3}$$

consists of the $n\times n$ blocks A, B, C and D. The isotropy group of 0 is

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} | A \in U(n) \right\}, \tag{4}$$

and hence

$$\mathscr{D} \cong G/K. \tag{5}$$

Let

$$\Omega = \{X \in V | X > 0\}.$$

Then Ω is a symmetric cone in V with automorphism group $GL(n, \mathbb{R})$ acting as

$$X \overset{g}{\mapsto} gXg^t.$$

In fact

$$\Omega \cong GL(n, \mathbb{R})/O(n).$$

Moreover, the boundary of Ω is partitioned into n orbits under $GL(n, \mathbb{R})$,

$$\partial\Omega = \cup_{i=1}^n \Omega^{(i)},$$

where $\Omega^{(i)}$ is the set of positive semidefinite matrices of rank i. Each orbit carries a *quasi-invariant* measure, μ_i, transforming in the fashion

$$g^*\mu_i = |\det g|^i \mu_i$$

under the action of $GL(n, \mathbb{R})$.

The Cayley transform

$$c(Z) = (I - Z)(I + Z)^{-1}$$

maps $\mathscr{D}$ biholomorphically onto the tube domain

$$T_\Omega := \{Z = U + iV \in V^{\mathbb{C}} | U \in \Omega\}.$$

Let τ denote the conjugation with respect to V, i.e.,

$$\tau(u + iv) = u - iv.$$

The set, $\mathscr{X}$ of fixed points of τ in $\mathscr{D}$, $V \cap \mathscr{D}$ is a totally real and totally geodesic real submanifold of $\mathscr{D}$, and the Cayley transform restricts to a diffeomorphism

$$\mathscr{X} \cong \Omega.$$

In particular, $\mathscr{X}$ is a homogeneous space

$$\mathscr{X} \cong H/L,$$

where $H \cong GL(n, \mathbb{R})$ and $L \cong O(n)$. Consider now the isomorphism

$$\Psi : GL(n, \mathbb{R}) \to \mathbb{R}^* \times SL(n, \mathbb{R})$$

given by

$$g \overset{\Psi}{\mapsto} (\det(g), \det(g)^{-1/n} g)$$

with inverse Ψ^{-1} given by

$$(\lambda, h) \overset{\Psi^{-1}}{\mapsto} \lambda^{1/n} h.$$

The differential of Ψ at the identity element gives an isomorphism of Lie algebras

$$\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R} \oplus \mathfrak{sl}(n, \mathbb{R}).$$

We will denote the $SL(n, \mathbb{R})$-factor in H by H', and correspondingly we let $\mathfrak{h}'$ denote $\mathfrak{sl}(n, \mathbb{R})$.

The minimal representation in the analytic continuation of the scalar holomorphic discrete series of G can be defined as a Hilbert space of functions, $\mathscr{H}_{1/2}$, on T_Ω. This space has the reproducing kernel

$$K_{1/2}(z, w) := \det(z - w^*)^{-1/2}. \tag{6}$$

By (an analytic continuation of) a Laplace transform, it is unitarily and G-equivariantly equivalent to the the Hilbert space $L^2(\Omega^{(1)}, \mu_1)$ (cf. [5]). Another proof of this can be found in [2]. We will now give an even more explicit model for this representation space. Consider therefore the mapping

$$\eta : \mathbb{R}^n \setminus \{0\} \to \Omega^{(1)},$$

defined by

$$\eta(x) = xx^t.$$

Here we identify $\mathbb{R}^n$ with the space of all $n \times 1$ real matrices. It is straigthforward to check that η is surjective and that

$$\eta(x) = \eta(y) \Leftrightarrow x = \pm y,$$

and hence we have a bijection

$$\Omega^{(1)} \cong (\mathbb{R}^n \setminus \{0\}) / \pm 1,$$

where the right hand side denotes the set of orbits under the linear action of the two-element group generated by the endomorphism $-I$. Moreover, the action of $GL(n, \mathbb{R})$ is covered by the linear action on $\mathbb{R}^n \setminus \{0\}$ so that we have the following commuting diagram.

$$\begin{array}{ccc} \mathbb{R}^n \setminus \{0\} & \xrightarrow{x \mapsto gx} & \mathbb{R}^n \setminus \{0\} \\ \eta \big\downarrow & & \big\downarrow \eta \\ \Omega^{(1)} & \xrightarrow{X \mapsto gXg^t} & \Omega^{(1)} \end{array}$$

The measure μ_1 is the pushforward under η of the Lebesgue measure on $\mathbb{R}^n \setminus \{0\}$. Hence, the minimal representation can be realised in the

Hilbert space of even square-integrable functions on $\mathbb{R}^n \setminus \{0\}$. In this picture we have the formula

$$f \overset{h}{\mapsto} \det h\,(f \circ h^t) \tag{7}$$

for the group action on functions. Obviously this Hilbert space contains L-invariant functions, i.e., the representation is spherical. Therefore, by [7], there exists a direct integral decomposition

$$L^2((\mathbb{R}^n \setminus \{0\})/\pm 1) \cong \int_\Lambda \mathcal{H}_\lambda d\nu,$$

where Λ is some parameter set, the $\mathcal{H}_\lambda$ are canonical representation spaces for irreducible spherical unitary representations of H', and ν is some positive measure on Λ, called the *Plancherel measure* for the minimal representation.

2.2. **Type I_{nn}.** Let V be the real vector space of Hermitian $n \times n$-matrices. The complexification, $V^{\mathbb{C}} = V \oplus iV$, of V consists of all complex $n \times n$ matrices. Consider the bounded symmetric domain

$$\mathscr{D} = \{Z \in V^{\mathbb{C}} | I - Z^* Z > 0\}.$$

The group

$$G = SU(n, n)$$

acts on $\mathscr{D}$ by

$$Z \mapsto (AZ + B)(CZ + D)^{-1},$$

where

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

consists of the $n \times n$ blocks A, B, C and D. We have the description

$$\mathscr{D} \cong G/K.$$

of $\mathscr{D}$ as a homogeneous space, where

$$K = S(U(n) \times U(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} | A, D \in U(n), \det(A)\det(D) = 1 \right\}.$$

The symmetric cone

$$\Omega = \{X \in V | X > 0\}$$

in V has automorphism group $GL(n, \mathbb{C})$ acting as

$$X \overset{g}{\mapsto} gXg^*,$$

and

$$\Omega \cong GL(n, \mathbb{C})/U(n).$$

The boundary of Ω partitions into $GL(n, \mathbb{C})$ orbits as

$$\partial\Omega = \cup_{i=1}^{n} \Omega^{(i)},$$

where $\Omega^{(i)}$ is the set of positive semidefinite matrices of rank i. Each orbit carries a *quasi-invariant* measure, μ_i, transforming in the fashion

$$g^* \mu_i = |\det g|^{2i} \mu_i$$

under the action of $GL(n, \mathbb{C})$. The Cayley transform

$$c(Z) = (I - Z)(I + Z)^{-1}$$

maps $\mathscr{D}$ biholomorphically onto the tube domain

$$T_\Omega := \{Z = U + iV \in V^{\mathbb{C}} | U \in \Omega\}.$$

Let τ denote the conjugation with respect to V, i.e.,

$$\tau(u + iv) = u - iv.$$

The set, $\mathscr{X}$ of fixed points of τ in $\mathscr{D}$, $V \cap \mathscr{D}$ is a totally real submanifold of $\mathscr{D}$ and the Cayley transform restricts to a diffeomorphism

$$\mathscr{X} \cong \Omega.$$

In particular, $\mathscr{X}$ is a homogeneous space

$$\mathscr{X} \cong H/L,$$

where $H \cong GL(n, \mathbb{C})$ and $L \cong U(n)$.

The minimal representation in the analytic continuation of the scalar holomorphic discrete series of G is a Hilbert space, $\mathscr{H}_1$, of functions on T_Ω with reproducing kernel

$$K_1(z, w) := \det(z - w^*)^{-1}.$$

Another realisation is furnished by the Hilbert space $L^2(\Omega^{(1)}, \mu_1)$ (cf. [5], [2]).

To give an explicit realisation, we now consider the mapping

$$\eta : \mathbb{C}^n \setminus \{0\} \to \Omega^{(1)}$$

defined by

$$\eta(z) = zz^*.$$

Here we identify $\mathbb{C}^n$ with the space of all $n \times 1$ complex matrices. It is straigthforward to check that η is surjective and that

$$\eta(z) = \eta(w) \Leftrightarrow z = e^{i\theta} w,$$

for some real θ, and hence we have a bijection

$$\Omega^{(1)} \cong \mathbb{C}^n / U(1).$$

The action of $GL(n, \mathbb{C})$ is covered by the linear action on $\mathbb{C}^n$ so that we have the following commuting diagram.

$$\begin{array}{ccc} \mathbb{C}^n \setminus \{0\} & \xrightarrow{z \mapsto gz} & \mathbb{C}^n \setminus \{0\} \\ \eta \big\downarrow & & \big\downarrow \eta \\ \Omega^{(1)} & \xrightarrow{Z \mapsto gZg^*} & \Omega^{(1)} \end{array}$$

The measure μ_1 is the pushforward under η of the Lebesgue measure on $\mathbb{C}^n \setminus \{0\}$. Hence, the minimal representation can be realised in the Hilbert space of $U(1)$-invariant square-integrable functions on $\mathbb{C}^n$. In this picture, we have the formula

$$f \stackrel{h}{\mapsto} \det{}_{\mathbb{R}} h^* \, (f \circ h^*), \tag{8}$$

for the group action on functions, where the subscript on the determinant means the determinant of h as an $\mathbb{R}$-linear operator on $\mathbb{R}^{2n}$.

3. The branching rule: type II_n

3.1. **Some parabolically induced representations.** In the following, we will consider some parabolically induced representations of $H' = SL(n, \mathbb{R})$.

Let $\mathfrak{a}_0 = \mathbb{R}e$, where

$$e = \begin{pmatrix} n-1 & 0 \\ 0 & -I_{n-1} \end{pmatrix},$$

where I_{n-1} denotes the identity matrix of size $(n-1) \times (n-1)$. The maximal parabolic subalgebra, $\mathfrak{q}_o$, determined by $\mathfrak{a}_0$ has a decomposition

$$\mathfrak{q}_o = \overline{\mathfrak{n}_0} \oplus \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0,$$

where

$$\begin{aligned}
\mathfrak{n}_\circ &= \left\{ \begin{pmatrix} 0 & x_1 & \cdots & x_{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} | x_1, \dots, x_{n-1} \in \mathbb{R} \right\}, \\
\mathfrak{m}_\circ &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} | M \in \mathfrak{sl}(n-1, \mathbb{R}) \right\}, \\
\overline{\mathfrak{n}_\circ} &= \left\{ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ x_{n-1} & 0 & \cdots & 0 \end{pmatrix} | x_1, \dots, x_{n-1} \in \mathbb{R} \right\}.
\end{aligned}$$

Here the subspace $\mathfrak{m}_0$ is defined by the property

$$Z_{\mathfrak{h}'}(\mathfrak{a}_0) = \mathfrak{a}_0 \oplus \mathfrak{m}_0,$$

and

$$\begin{aligned}
\mathfrak{n}_0 &= \{X \in \mathfrak{h}' | \quad [H, X] = \alpha(H)X, \quad \forall H \in \mathfrak{a}_0\}, \\
\overline{\mathfrak{n}_0} &= \{X \in \mathfrak{h}' | \quad [H, X] = -\alpha(H)X, \quad \forall H \in \mathfrak{a}_0\}
\end{aligned}$$

are the generalised root spaces, where the root $\alpha \in {\mathfrak{a}_\circ}^*$ is determined by

$$\alpha(e) = n.$$

We let ρ_0 denote the half sum of the positive roots counted with multiplicity, i.e.,

$$\rho_0 = \frac{n-1}{2}\alpha.$$

On the group level we have the corresponding decomposition

$$Q_0 = M_0 A_0 N_0,$$

where

$$
\begin{aligned}
A_0 &= \left\{ \begin{pmatrix} e^s & 0 \\ 0 & qI_{n-1} \end{pmatrix} | s, q \in \mathbb{R}, e^s q^{n-1} = 1 \right\} \\
M_0 &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} | M \in GL(n-1, \mathbb{R}) \right\}, \\
N_0 &= \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_{n-1} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} | x_1, \ldots, x_{n-1} \in \mathbb{R} \right\}
\end{aligned}
$$

Consider now the representation $1 \otimes \exp i\lambda \otimes 1$ of the group

$$Q_0 = M_0 A_0 N_0.$$

The induced representation

$$\tau_\lambda := \mathrm{Ind}_{Q_0}^{H'}(1 \otimes \exp(i\lambda + \rho_0) \otimes 1) \tag{9}$$

has a noncompact realisation in the Hilbert space $L^2(\overline{N_0}, d\overline{n})$ (cf. [4]). We have

$$\pi_\lambda(h) f(\overline{n}) = e^{-(i\lambda+\rho_0)(\log a_0(h^{-1}\overline{n}))} f(\overline{n_0}(h^{-1}\overline{n})), \tag{10}$$

The decomposition $\mathfrak{h}' = \overline{\mathfrak{n}_o} \oplus \mathfrak{m}_o \oplus \mathfrak{a}_o \oplus n_0$ gives a corresponding decomposition

$$H' \doteq \overline{N_0} M_0 A_0 N_0,$$

by which we mean that the equality holds outside a set of strictly lower dimension. The factorisation of a group element with respect to this decomposition is not unique, but the A_0-component is. For $h \in H'$, we let $a_0(h)$ denote this component. The mapping $\exp : \mathfrak{a}_0 \to A_0$ is a diffeomorphic homomorphism of abelian groups. We let $\log : A_0 \to \mathfrak{a}_0$ denote its inverse. The representation τ_λ is then given by

$$\tau_\lambda(h) f(\overline{n}) = e^{-(i\lambda+\rho_0)(\log a_0(h^{-1}\overline{n}))} f(\overline{n_0}(h^{-1}\overline{n})). \tag{11}$$

For arbitrary $h \in H', \overline{n} \in \overline{N_0}$, a factorisation of $h\overline{n}$ can be given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & I_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{c+dx}{a+bx} & I_{n-1} \end{pmatrix} \begin{pmatrix} \frac{a+bx}{|a+bx|} & 0 \\ 0 & |a+bx|^{1/n-1}(d - \left(\frac{c+dx}{a+bx}\right) b) \end{pmatrix}$$
$$\times \begin{pmatrix} |a+bx| & 0 \\ 0 & |a+bx|^{-1/n-1} I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{b^t}{a+bx} \\ 0 & I_{n-1} \end{pmatrix}.$$

In view of this, and identifying $L^2(\overline{N_0}, d\overline{n})$ with $L^2(\mathbb{R}^{n-1}, dx)$, we obtain the following explicit formula for the induced representation:

$$(\tau_\lambda(h)f)(x) = |ax+b|^{-(i\lambda+n/2)} f\left(\frac{c+dx}{a+bx}\right),$$

where $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

3.2. **An intertwining operator.** Let m_λ be the one-dimensional representation

$$m_\lambda(c)z := |c|^{i\lambda+\rho_0} z$$

of $\mathbb{R}^*$. Recalling the isomorphism

$$\Psi : GL(n,\mathbb{R}) \to \mathbb{R}^* \times SL(n,\mathbb{R})$$

from the previous section, we can now form the representation

$$\pi_\lambda := m_\lambda \otimes \tau_\lambda \tag{12}$$

of $GL(n,\mathbb{R})$. We let $\mathcal{H}_\lambda$ denote the associated representation space.

We now consider an operator, T, mapping a $C_0^\infty((\mathbb{R}^n \setminus \{0\})/\pm 1)$-function, f, to a function $Tf : \mathbb{C} \times \mathbb{R}^{n-1} \to \mathbb{R}$ which is meromorphic in the first variable. The function

$$Tf(z,\eta) := \int_{\mathbb{R}^n} f(x) |\langle x, (1,\eta)\rangle|^{-(iz+n/2)} dx \tag{13}$$

is well defined as a function of z and η when $\mathrm{Im}z - n/2 > -1$. For such z, a change of variables, followed by an integration by parts, yields

$$
\begin{aligned}
&\int_{\mathbb{R}^n} f(x)|\langle x,(1,\eta)\rangle|^{-(iz+n/2)}dx \\
= \; & \frac{1}{(1+|\eta|^2)^{iz+n/2}(-(iz+n/2)+1)} \\
&\times \int_{\mathbb{R}^{n-1}}\int_{y_1<0} \frac{\partial(f\circ g)(y)}{\partial y_1}|y_1|^{-(iz+n/2)+1}dy_1dy_2\dots dy_n \\
&- \frac{1}{(1+|\eta|^2)^{iz+n/2}(-(iz+n/2)+1)} \\
&\times \int_{\mathbb{R}^{n-1}}\int_{y_1>0} \frac{\partial(f\circ g)(y)}{\partial y_1}|y_1|^{-(iz+n/2)+1}dy_1dy_2\dots dy_n,
\end{aligned}
$$

where g is some orthogonal transformation such that

$$(1,\eta) = g(|(1,\eta)|e_1).$$

By repeated integration by parts we get the identity

$$
\begin{aligned}
&Tf(z,\eta) \\
= \; & \frac{1}{(1+|\eta|^2)^{iz+n/2}\Pi_{j=1}^k(-(iz+n/2)+j)} \\
&\times \int_{\mathbb{R}^{n-1}}\int_{y_1<0} \frac{\partial^k(f\circ g)(y)}{\partial y_1^k}|y_1|^{-(iz+n/2)+k}dy_1dy_2\dots dy_n \\
&+(-1)^k \frac{1}{(1+|\eta|^2)^{iz+n/2}\Pi_{j=1}^k(-(iz+n/2)+j)} \\
&\times \int_{\mathbb{R}^{n-1}}\int_{y_1>0} \frac{\partial^k(f\circ g)(y)}{\partial y_1^k}|y_1|^{-(iz+n/2)+k}dy_1dy_2\dots dy_n.
\end{aligned}
\tag{14}
$$

Therefore, $Tf(z,\eta)$ can be continued to a meromorphic function with poles at $z = i(n/2 - j)$, for $j = 1, 2, \dots$. In particular, $Tf(\lambda,\eta)$ is a well defined real analytic function for real λ (except possibly for $\lambda = 0$). We write $T_\lambda f(\eta)$ for $Tf(\lambda,\eta)$.

Proposition 1. *For $f \in C_0^\infty((\mathbb{R}^n\setminus\{0\})/\pm 1)$ and $\lambda \in \mathbb{R}$, the functions $T_\lambda f$ is in $L^2(\mathbb{R}^{n-1})$.*

Proof. For $\lambda \in \mathbb{R}$, choose the natural number $k > n/2$ in (14). The function

$$\frac{\partial^k}{\partial y_1^k}(f \circ h), \qquad h \in O(n) \tag{15}$$

constitute a uniformly bounded family in the supremum-norm. Hence, we have an estimate

$$|T_\lambda f(\eta)| \leq C(\lambda)(1 + |\eta|^2)^{-(i\lambda + n/2)}, \tag{16}$$

and this proves the claim. □

In what follows, we will state and prove properties for the functions $T_\lambda f$ for arbitrary real λ although the proofs will use the defining integral (13) which makes sense only when $\mathrm{Im} z > n/2 - 1$. The idea is then that both sides in the stated equalities are meromorphic functions, so by the uniqueness theorem for meromorphic functions it suffices to perform the calculations when the defining integral makes sense. All integral equalities should therefore be thought of as analytic continuations of the corresponding equalities when the integrals are convergent.

Proposition 2. *The operator*

$$T_\lambda : C_0^\infty(\mathbb{R}^n/\{\pm 1\}) \to \mathcal{H}_\lambda,$$

given by

$$T_\lambda f(\eta) = \int_{\mathbb{R}^n} f(x)|\langle x, (1, \eta)\rangle|^{-(i\lambda + n/2)} dx \tag{17}$$

is H-equivariant.

Proof. Take $g \in H$ and write $g = \zeta h$, where ζ is a diagonal matrix and h has determinant 1. Moreover we write $h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\begin{aligned} & T_\lambda(gf)(\eta) \\ = & \int_{\mathbb{R}^n} f(g^t x) |\langle (x_1, x'), (1, \eta) \rangle|^{-(i\lambda + n/2)} \det g dx \\ = & \int_{\mathbb{R}^n} f(x) |\langle (x_1, x'), g^{-1}(1, \eta) \rangle|^{-(i\lambda + n/2)} dx \\ = & |\zeta|^{i\lambda + n/2} \int_{\mathbb{R}^n} f(x) |(a + b\eta) x_1 + \langle x', c + d\eta \rangle|^{-(i\lambda + n/2)} dx \\ = & |\zeta|^{i\lambda + n/2} \int_{\mathbb{R}^n} f(x) |a + b\eta|^{-(i\lambda + n/2)} \\ & \qquad \times |\langle (x_1, x'), (1, (c + d\eta)(a + b\eta)^{-1}) \rangle|^{-(i\lambda + n/2)} dx \\ = & \pi_\lambda(g) T_\lambda f(\eta). \end{aligned}$$

□

If f is L-invariant, then $T_\lambda f$ is an L-invariant function in the representation space $\mathcal{H}_\lambda$. By the Cartan-Helgason theorem ([3]), the subspace of L-invariants is at most one dimensional. In fact, it is spanned by the function $\eta \mapsto (1 + |\eta|^2)^{-(i\lambda + n/2)/2}$. Thus, we can define a function $\tilde{f}$ by

$$T_\lambda f(\eta) = \tilde{f}(\lambda)(1 + |\eta|^2)^{-(i\lambda + n/2)/2}. \tag{18}$$

The plan is now to prove an inversion formula and a Plancherel theorem. The following lemma will be very useful in the sequel.

Lemma 3. *Let $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})^L$. Then the function $\tilde{f}$ can be written in the form*

$$\tilde{f}(\lambda) = 2\pi^{n/2} \frac{\Gamma(-(2i\lambda + (n-2))/2)}{\Gamma(1/2)\Gamma((-2i\lambda + n)/4)} \mathscr{M}(r \mapsto r^{n/2} f(re_1))(\lambda),$$

where $\mathscr{M}$ is the Mellin transform.

Proof. We start by observing that, since f has compact support outside the origin, the Mellin transform above admits an entire extension by the Paley-Wiener theorem. It thus suffices to prove the statement for $\lambda \in i(n/2 - 1, \infty)$ by the uniqueness of an analytic continuation.

By the L-invariance of f and of the Lebesgue measure, we have

$$\begin{aligned}(19)\qquad &\int_{\mathbb{R}^n} f(x)|\langle x,(1,\eta)\rangle|^{-(i\lambda+n/2)}dx.\\ &=\int_{\mathbb{R}^n} f(x)\int_{SO(n)}|\langle gx,(1,\eta)\rangle|^{-(i\lambda+n/2)}dgdx.\end{aligned}$$

Consider now the function

$$(20)\qquad R(x,y):=\int_{SO(n)}|\langle gx,y\rangle|^{-(i\lambda+n/2)}dg,\qquad x,y\in\mathbb{R}^n.$$

It is $SO(n)$-invariant in each variable separately, and it is homogeneous of degree $-(i\lambda+n/2)$. Hence,

$$(21)\quad R(x,y)=|x|^{-(i\lambda+n/2)}|y|^{-(i\lambda+n/2)}\int_{SO(n)}|\langle ge_1,e_1\rangle|^{-(i\lambda+n/2)}dg.$$

The integral on the right hand side can be expressed as an integral over the sphere S^{n-1}. Indeed, the fibration

$$(22)\qquad p:SO(n)\to S^{n-1}, p(g)=ge_1$$

defines a measure σ on S^{n-1} as the pushforward of the normalised Haar measure on $SO(n)$, i.e., σ is defined as an $SO(n)$-invariant linear functional on $C(S^{n-1})$ by the equation

$$(23)\quad \int_{S^{n-1}} f(\xi)d\sigma(\xi):=\int_{SO(n)} f(p(g))dg,\qquad f\in C(S^{n-1}).$$

By choosing f as a constant function in the above equality, we see that σ is the normalised surface measure on S^{n-1}. Applying (23) to the equality (21), we get

$$(24)\quad R(x,y)=|x|^{-(i\lambda+n/2)}|y|^{-(i\lambda+n/2)}\int_{S^{n-1}}|\zeta_1|^{-(i\lambda+n/2)}d\sigma(\zeta).$$

The last integrand depends only on one variable, and hence we can apply the “Functions of fewer variables”-theorem (cf. [6]) and replace the integral by an integral over the unit interval on the real line. This yields

$$\begin{aligned}&\int_{S^{n-1}}|\zeta_1|^{-(i\lambda+n/2)}d\sigma(\zeta)\\ &\frac{2\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)}\int_0^1(1-t^2)^{\frac{n-3}{2}}t^{-(i\lambda+n/2)}dt.\end{aligned}$$

By performing the change of variables $s = 1 - t^2$, we obtain

$$\begin{aligned} & \frac{2\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \int_0^1 (1-t^2)^{\frac{n-3}{2}} t^{-(i\lambda+n/2)} dt \\ = \; & \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \int_0^1 s^{\frac{n-1}{2}-1} (1-s)^{-\frac{2i\lambda+(n-2)}{4}-1} ds \\ = \; & \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \beta\left(\frac{n-1}{2}, -\frac{2i\lambda+(n-2)}{4}\right) \\ = \; & \frac{\Gamma(n/2)\Gamma(-(2i\lambda+(n-2))/2)}{\Gamma(1/2)\Gamma((-2i\lambda+n)/4)}, \end{aligned}$$

and hence

$$\int_{S^{n-1}} |\zeta_1|^{-(i\lambda+n/2)} d\sigma(\zeta) = \frac{\Gamma(n/2)\Gamma(-(2i\lambda+(n-2))/2)}{\Gamma(1/2)\Gamma((-2i\lambda+n)/4)}. \tag{25}$$

Inserting (24) (with $y = (1, \eta)$) and (25) into (19) gives

$$\begin{aligned} & \int_{\mathbb{R}^n} f(x) |\langle x, (1,\eta)\rangle|^{-(i\lambda+n/2)} dx \\ = \; & (1+|\eta|^2)^{-(i\lambda+n/2)/2} \frac{\Gamma(n/2)\Gamma(-(2i\lambda+(n-2))/2)}{\Gamma(1/2)\Gamma((-2i\lambda+n)/4)} \\ & \times \int_{\mathbb{R}^n} f(x)|x|^{-(i\lambda+n/2)} dx. \end{aligned}$$

Finally, we use polar coordinates to compute the integral on the right hand side. Then

$$\int_{\mathbb{R}^n} f(x)|x|^{-(i\lambda+n/2)} dx = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty r^{n/2} f(re_1) r^{-i\lambda} \frac{dr}{r},$$

and hence the lemma is proved. □

Theorem 4 (Inversion formula)**.** *If* $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})^L$, *then*

$$f(re_1) = \frac{\Gamma(1/2)}{2\pi^{n/2}} \int_{\mathbb{R}} \tilde{f}(\lambda) r^{i\lambda-n/2} \frac{\Gamma((-2i\lambda+n)/4)}{\Gamma((-(2i\lambda+(n-2))/2)} d\lambda.$$

Proof. By the previous lemma, we can write

$$\mathscr{M}(r \mapsto r^{n/2} f(re_1))(\lambda) := \tilde{f}(\lambda) b(\lambda). \tag{26}$$

By the assumptions on f, the inverse Mellin transform is defined for the left hand side since it is in L^1, and the inversion formula for the

Mellin transform yields

$$r^{n/2}f(re_1) = \int_{\mathbb{R}} \tilde{f}(\lambda)b(\lambda)r^{i\lambda-1}d\lambda, \tag{27}$$

i.e.,

$$f(re_1) = \frac{\Gamma(1/2)}{2\pi^{n/2}} \int_{\mathbb{R}} \tilde{f}(\lambda)r^{i\lambda-n/2}\frac{\Gamma((-2i\lambda+n)/4)}{\Gamma((-(2i\lambda+(n-2))/2)}d\lambda.$$

□

Remark. Note that this is a somewhat peculiar looking "Inversion formula". It does not express the function f as a weighted superposition of some canonical functions with respect to the Plancherel measure for the given representation. This will become clear by the next theorem. The reason that we prove it is rather because it serves as a means for proving the Plancherel theorem.

Theorem 5 (Plancherel theorem). *For all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})^L$ we have*

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}} |\tilde{f}(\lambda)|^2 \left| \frac{\Gamma(1/2)}{2\pi^{n/2}} \frac{\Gamma((-2i\lambda+n)/4)}{\Gamma((-(2i\lambda+(n-2))/2)} \right|^2 d\lambda.$$

Proof. We introduce some temporary notation and write the inversion formula in the simplified form

$$f(re_1) = \int_{\mathbb{R}} \tilde{f}(\lambda)r^{i\lambda-n/2}\phi(\lambda)d\lambda.$$

By the inversion formula we then have

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^2 dx &= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}} \overline{\tilde{f}(\lambda)}|x|^{-i\lambda-n/2}\overline{\phi(\lambda)}d\lambda dx \\
&= \int_{\mathbb{R}} \overline{\tilde{f}(\lambda)} \int_{\mathbb{R}^n} f(x)|x|^{-i\lambda-n/2}dx\overline{\phi(\lambda)}d\lambda.
\end{aligned}$$

By the proof of Lemma 3, the inner integral can be seen to be equal to $\tilde{f}(\lambda)\phi(\lambda)$, and hence

$$\int_{\mathbb{R}} \overline{\tilde{f}(\lambda)} \int_{\mathbb{R}^n} f(x)|x|^{-i\lambda-n/2}dx\overline{\phi(\lambda)}d\lambda = \int_{\mathbb{R}} |\tilde{f}(\lambda)|^2|\phi(\lambda)|^2 d\lambda,$$

and this concludes the proof. □

Theorem 6. *The operator T extends to a unitary H-intertwining operator*

$$U : L^2((\mathbb{R}^n \setminus \{0\})/\pm 1) \to \int_{\mathbb{R}} \mathcal{H}_\lambda d\mu(\lambda), \tag{28}$$

where μ is the measure determined by the identity

$$\int_{\mathbb{R}} f(\lambda) d\mu(\lambda) := \int_{\mathbb{R}} f(\lambda) \left| \frac{\Gamma(1/2)}{2\pi^{n/2}} \frac{\Gamma((-2i\lambda + n)/4)}{\Gamma((-(2i\lambda + (n-2))/2)} \right|^2 d\lambda.$$

Proof. By Prop. 2 and Thm. 5, there exists a unique H-intertwining extension $U : L^2((\mathbb{R}^n \setminus \{0\})/\pm 1) \to \int_{\mathbb{R}} \mathcal{H}_\lambda d\mu(\lambda)$ of T. The only thing that remains to prove is the surjectivity of U.

This follows immediately from the proof of Theorem 9 in [7]. Indeed, by the H-equivariance of the operator U the action of the commutative Banach algebra $L^1(H)^{\#}$ of left and right L-invariant L^1-functions on H is intertwined. On each subspace $\mathcal{H}_\lambda^L$, a function $f \in L^1(H)^{\#}$ acts as a scalar operator, $\hat{f}(\lambda)$. If we let v_λ denote the canonical L-invariant vector associated with the spherical representation on $\mathcal{H}_\lambda$, and $\omega \in L^2((\mathbb{R}^n \setminus \{0\})/\pm 1)$ denote an L-invariant vector in the minimal K-type, then the positive functional Φ on $L^1(H)^{\#}$ given by $\Phi(f) = \langle \pi(f)\omega, \omega \rangle$ can be written as the integral

$$\Phi(f) = \int_{\mathbb{R}} \phi_\lambda(f) d\mu(\lambda), \tag{29}$$

where ϕ_λ is the multiplicative functional $f \mapsto \langle f(\lambda) v_\lambda, v_\lambda \rangle_\lambda$. The surjectivity now follows from the proof of Theorem 9 in [7] by uniqueness of such an integral decomposition of Φ. □

4. The branching rule: type I_{nn}

We consider the diffeomorphism

$$\Psi : GL(n, \mathbb{C}) \to \mathbb{C}^* \times SL(n, \mathbb{C})$$

given by

$$g \overset{\Psi}{\mapsto} (\det(g), \det(g)^{-1/n} g),$$

where we have chosen the branch of the nth-root multifunction determined by the root of unity with the least argument (i.e. in polar coordinates $(re^{i\theta})^{1/n} := r^{1/n} e^{i\theta/n}$). The mapping Ψ has inverse

$$\Psi^{-1} : (\lambda, h) \mapsto \lambda^{1/n} h.$$

In this case, however, Ψ is not a group homomorphism since the chosen branch of the multifunction is not multiplicative. Instead Ψ is multiplicative up to scalar multiples of modulus one. We shall see later that we can still use this diffeomorphism to construct representations of $GL(n,\mathbb{C})$ from representations of $SL(n,\mathbb{C})$ and $\mathbb{C}^*$ respectively.

4.1. **Some parabolically induced representations of $SL(n,\mathbb{C})$.** Let $\mathfrak{a}_0 = \mathbb{R}e$, where

$$e = \begin{pmatrix} n-1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}.$$

Consider the maximal parabolic subalgebra, $\mathfrak{q}_\circ$, determined by $\mathfrak{a}_0$, with decomposition

$$\mathfrak{q}_\circ = \overline{\mathfrak{n}_0} \oplus \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0,$$

where

$$\begin{aligned}
\mathfrak{n}_\circ &= \left\{ \begin{pmatrix} 0 & z_1 & \cdots & z_{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} | z_1, \ldots, z_{n-1} \in \mathbb{C} \right\}, \\
\mathfrak{m}_\circ &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} | M \in \mathfrak{sl}(n-1,\mathbb{C}) \right\}, \\
\overline{\mathfrak{n}_\circ} &= \left\{ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ z_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ z_{n-1} & 0 & \cdots & 0 \end{pmatrix} | z_1, \ldots, z_{n-1} \in \mathbb{C} \right\}.
\end{aligned}$$

Here the subspace $\mathfrak{m}_0$ is defined by the property

$$Z_{\mathfrak{h}'}(\mathfrak{a}_0) = \mathfrak{a}_0 \oplus \mathfrak{m}_0,$$

and

$$\begin{aligned}
\mathfrak{n}_0 &= \{X \in \mathfrak{h}' | \, [H,X] = \alpha(H)X, \qquad \forall H \in \mathfrak{a}_0\}, \\
\overline{\mathfrak{n}_0} &= \{X \in \mathfrak{h}' | \, [H,X] = -\alpha(H)X, \qquad \forall H \in \mathfrak{a}_0\}
\end{aligned}$$

are the generalised root spaces, where the root $\alpha \in \mathfrak{a}_\circ{}^*$ is determined by

$$\alpha(e) = n.$$

We let ρ_0 denote the half sum of the positive roots counted with multiplicity, i.e.,

$$\rho_0 = (n-1)\alpha.$$

On the group level we have the corresponding decomposition

$$Q_0 = M_0 A_0 N_0,$$

where

$$\begin{aligned} A_0 &= \left\{ \begin{pmatrix} e^s & 0 \\ 0 & qI_{n-1} \end{pmatrix} | s, q \in \mathbb{R}, e^s q^{n-1} = 1 \right\} \\ M_0 &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} | M \in GL(n-1, \mathbb{C}) \right\}, \\ N_0 &= \left\{ \begin{pmatrix} 1 & z_1 & z_2 & \cdots & z_{n-1} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} | z_1, \ldots, z_{n-1} \in \mathbb{C} \right\} \end{aligned}$$

Consider now the representation $1 \otimes \exp i\lambda \otimes 1$ of the group

$$Q_0 = M_0 A_0 N_0.$$

We realise the induced representation

$$\tau_\lambda := \operatorname{Ind}_{Q_0}^{H'}(1 \otimes \exp(i\lambda + \rho_0) \otimes 1) \tag{30}$$

in the Hilbert space $L^2(\overline{N_0}, d\overline{n})$.

The decomposition on the group level

$$H' \doteq \overline{N_0} M_0 A_0 N_0,$$

gives that for $h \in H', \overline{n} \in \overline{N_0}$, $h\overline{n}$ can be factorised as

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & I_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{c+dz}{a+bz} & I_{n-1} \end{pmatrix} \begin{pmatrix} \frac{a+bz}{|a+bz|} & 0 \\ 0 & |a+bx|^{1/n-1}(d - \left(\frac{c+dz}{a+bz}\right) b) \end{pmatrix} \\ &\times \begin{pmatrix} |a+bz| & 0 \\ 0 & |a+bz|^{-1/n-1} I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{b^t}{a+bz} \\ 0 & I_{n-1} \end{pmatrix}. \end{aligned}$$

Hence, by identifying $L^2(\overline{N_0}, d\overline{n})$ with $L^2(\mathbb{C}^{n-1}, dm(z))$, where $dm(z)$ is the Lebesgue measure on $\mathbb{C}^n$, we obtain the following formula for the action of H' on functions in the representation space:

$$\tau_\lambda(h)f(z) = |az+b|^{-(i\lambda+n)} f\left(\frac{c+dz}{a+bz}\right),$$

where $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

4.2. **An intertwining operator.** Recalling the diffeomorphism

$$\Psi : GL(n,\mathbb{C}) \to \mathbb{C}^* \times SL(n,\mathbb{C})$$

from the previous section, we can now form the representation

$$m_\lambda \otimes \mathrm{Ind}_{Q_0}^{H'}(1 \otimes \exp(i\lambda + \rho_0) \otimes 1),$$

where $m_\lambda(c) = |c|^{i\lambda+\rho}$, of $\mathbb{C}^* \times SL(n,\mathbb{C})$. This will in fact give a representation of $GL(n,\mathbb{C})$. Indeed, suppose that we have $g_1, g_2 \in GL(n,\mathbb{C})$. We can write

$$g_1 = \lambda_1^{1/n} h_1,$$

and

$$g_2 = \lambda_2^{1/n} h_2,$$

with $\lambda_1, \lambda_2 \in \mathbb{C}^*$ and $h_1, h_2 \in SL(n,\mathbb{C})$. Then

$$g_1 g_2 = (\lambda_1\lambda_2)^{1/n}\xi(\lambda_1,\lambda_2)h_1h_2,$$

where $|\xi(\lambda_1,\lambda_2)| = 1$ and hence the mapping

$$\pi_\lambda : g \mapsto m_\lambda \otimes \mathrm{Ind}_{Q_0}^{H'}(1 \otimes \exp(i\lambda + \rho_0) \otimes 1) \circ \Psi(g) \tag{31}$$

defines a unitary representation of $GL(n,\mathbb{C})$. We let $\mathcal{H}_\lambda$ denote the corresponding representation space.

For $f \in C_0^\infty(\mathbb{C}^n)$, we define the function $Tf : \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{C}$ by

$$Tf(\lambda,\eta) := \int_{\mathbb{C}^n} f(z)|\langle z,(1,\eta)\rangle|^{-(i\lambda+n)} dm(z),$$

where the right hand side is to be interpreted using analytic continuation in the variable λ as in the previous section (eq.(13) and the following discussion). We have the following analog of Prop. 1.

Proposition 7. *For $f \in C_0^\infty((\mathbb{C}^n \setminus \{0\})/\pm 1)$ and $\lambda \in \mathbb{R}$, the functions $T_\lambda f$ is in $L^2(\mathbb{C}^{n-1})$.*

The proof is the same as that of Prop. 1.

All the following integral equalities where the variable λ occurs are to be thought of as analytic continuations of the corresponding equalities involving convergent integrals. We write $T_\lambda f$ for the function $\eta \mapsto Tf(\lambda, \eta)$.

Proposition 8. *The operator*

$$T_\lambda : C_0^\infty((\mathbb{C}^n \setminus \{0\})/U(1)) \to \mathcal{H}_\lambda \tag{32}$$

is H-equivariant.

Proof. Take $g \in H$ and write $g = \zeta h$, where ζ is a diagonal matrix and h has determinant 1. Moreover we write $h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\begin{aligned}
T_\lambda(gf)(\eta) &= \int_{\mathbb{C}^n} f(g^* z)|\langle (z_1, z'), (1, \eta)\rangle|^{-(i\lambda+n)}|\det g^*|^2 dm(z) \\
&= \int_{\mathbb{C}^n} f(x)|\langle (z_1, z'), g^{-1}(1, \eta)\rangle|^{-(i\lambda+n)} dm(z) \\
&= |\zeta|^{i\lambda+n} \int_{\mathbb{C}^n} f(x)|(a + b\eta)z_1 + \langle z', c + d\eta\rangle|^{-(i\lambda+n)} dm(z) \\
&= |\zeta|^{i\lambda+n} \int_{\mathbb{C}^n} f(x)|a + b\eta|^{-(i\lambda+n)} \\
&\qquad \times |\langle (z_1, z'), (1, (c + d\eta)(a + b\eta)^{-1})\rangle|^{-(i\lambda+n/2)} dm(z) \\
&= \pi_\lambda(g) T_\lambda(f)(\eta).
\end{aligned}$$

□

For an L-invariant function f, the function $\eta \mapsto Tf(\lambda, \eta)$ is L-invariant by the above proposition. The Cartan-Helgason theorem ([3]) therefore allows us to define the function $\tilde{f}$ by

$$T(\lambda, \eta) = \tilde{f}(\lambda)(1 + |\eta|^2)^{-(i\lambda+n)/2}.$$

Lemma 9. *Let $f \in C_0^\infty(\mathbb{C}^n \setminus \{0\})^L$. Then the function $\tilde{f}$ can be written in the form*

$$\tilde{f}(\lambda) = 4\pi^n \frac{\Gamma\left(-\frac{i\lambda+(n-2)}{2}\right)}{\Gamma\left(-\frac{i\lambda+(n-2)}{2} + n - 1\right)} \mathscr{M}(r \mapsto r^n f(re_1))(\lambda), \tag{33}$$

where $\mathscr{M}$ is the Mellin transform.

Proof. The proof is almost identical to that of Lemma 3. We assume that λ is purely imaginary with big enough imaginary part. Then

$$\begin{aligned}(34)\qquad &\int_{\mathbb{C}^n} f(z)|\langle z,(1,\eta)\rangle|^{-(i\lambda+n)}dm(z)\\ &=\int_{\mathbb{C}^n} f(z)\int_{SU(n)}|\langle gz,(1,\eta)\rangle|^{-(i\lambda+n)}dgdm(z).\end{aligned}$$

The inner integral can be written as

$$\begin{aligned}(35)\qquad &\int_{\mathbb{C}^n} f(z)|\langle z,(1,\eta)\rangle|^{-(i\lambda+n)}dm(z)\\ &=\int_{\mathbb{C}^n} f(z)\int_{SU(n)}|\langle gz,(1,\eta)\rangle|^{-(i\lambda+n)}dg\\ &=|z|^{-(i\lambda+n)}(1+|\eta|^2)^{-(i\lambda+n)/2}\int_{S^{2n-1}}|\zeta_1|^{-(i\lambda+n)}d\sigma(\zeta).\end{aligned}$$

The integrand on the right hand side depends only on one variable, and hence we can apply [6], Prop. 1.4.4. This yields

$$\begin{aligned}(36)\qquad &\int_{S^{2n-1}}|\zeta_1|^{-(i\lambda+n)}d\sigma(\zeta)\\ &=\frac{n-1}{\pi}\int_U(1-|z|^2)^{n-2}|z|^{-(i\lambda+n)}dm(z),\end{aligned}$$

where U is the unit disc in $\mathbb{C}$. The last integral can be written as

$$\begin{aligned}(37)\qquad &\frac{n-1}{\pi}\int_U(1-|z|^2)^{n-2}|z|^{-(i\lambda+n)}dm(z)\\ =&\ 2\pi(n-1)\int_0^1(1-t)^{(n-1)-1}t^{-(i\lambda+(n-2))/2-1}dt\\ :=&\ 2\pi(n-1)\beta(n-1,-(i\lambda+(n-2))/2)\\ =&\ 2\pi\frac{\Gamma\left(-\frac{i\lambda+(n-2)}{2}\right)}{\Gamma\left(-\frac{i\lambda+(n-2)}{2}+n-1\right)}.\end{aligned}$$

Using (35), (36), and (37), the identity (34) can be rewritten in the form

$$\begin{aligned}&\int_{\mathbb{C}^n} f(z)|\langle z,(1,\eta)\rangle|^{-(i\lambda+n)}dm(z)\\ =\ & 4\pi^n(1+|\eta|^2)^{-(i\lambda+n)/2}\frac{\Gamma\left(-\frac{i\lambda+(n-2)}{2}\right)}{\Gamma\left(-\frac{i\lambda+(n-2)}{2}+n-1\right)}\\ &\times\int_{\mathbb{C}^n} f(z)|z|^{-(i\lambda+n)}dm(z).\end{aligned}$$

Using polar coordinates, the integral on the right is given by

$$\int_{\mathbb{C}^n} f(z)|z|^{-(i\lambda+n)}dm(z)=\frac{2\pi^n}{\Gamma(n)}\int_0^\infty r^n f(re_1)r^{-i\lambda}\frac{dr}{r},$$

and this proves the statement. □

Since f has compact support outside the origin, the right hand side admits an extension to an entire function by the Paley-Wiener theorem.

Theorem 10 (Inversion formula). *If* $f\in C_0^\infty(\mathbb{C}^n\setminus\{0\})^L$, *then*

$$f(re_1)=\frac{1}{4\pi^n}\int_{\mathbb{R}}\tilde{f}(\lambda)r^{i\lambda-n}\left(-\frac{i\lambda+n-2}{2}\right)_{n-1}d\lambda,$$

where $(\cdot)_k$ *denotes the* Pochhammer symbol *defined as*

$$\begin{aligned}(t)_0&=1,\\ (t)_k&=t(t+1)\cdots(t+k-1),\ k\in\mathbb{N}^+.\end{aligned}$$

Proof. The identity

$$(x)_k=\frac{\Gamma(x+k)}{\Gamma(x)}$$

shows that the statement of Lemma 9 can be written in the form

$$\mathscr{M}(r\mapsto r^n f(re_1))=\frac{1}{4\pi^n}\left(-\frac{i\lambda+n-2}{2}\right)_{n-1}\tilde{f}(\lambda) \tag{38}$$

The inversion formula for the Mellin transform then yields the identity

$$f(re_1)=\frac{1}{4\pi^n}\int_{\mathbb{R}}\left(-\frac{i\lambda+n-2}{2}\right)_{n-1}\tilde{f}(\lambda)r^{i\lambda-n}d\lambda.$$

□

Theorem 11 (Plancherel theorem). *For all $f \in C_0^\infty(\mathbb{C}^n \setminus \{0\})^L$ we have*

$$\int_{\mathbb{C}^n} |f(z)|^2 dm(z) = \left(\frac{1}{4\pi^n}\right)^2 \int_{\mathbb{R}} |\tilde{f}(\lambda)|^2 \left| \left(-\frac{i\lambda + n - 2}{2}\right)_{n-1} \right|^2 d\lambda.$$

Proof. For simplicity, we write the inversion formula in the form

$$f(re_1) = \int_{\mathbb{R}} \tilde{f}(\lambda) r^{i\lambda - n} \phi(\lambda) d\lambda.$$

By the inversion formula we then have

$$\begin{aligned} \int_{\mathbb{C}^n} |f(z)|^2 dm(z) &= \int_{\mathbb{C}^n} f(z) \int_{\mathbb{R}} \overline{\tilde{f}(\lambda)} |z|^{-i\lambda - n} \overline{\phi(\lambda)} dm(z) \\ &= \int_{\mathbb{R}} \overline{\tilde{f}(\lambda)} \int_{\mathbb{C}^n} f(z) |z|^{-i\lambda - n} dm(z) \overline{\phi(\lambda)} d\lambda. \end{aligned}$$

By the proof of Lemma 9, the inner integral can be seen to be equal to $\tilde{f}(\lambda)\phi(\lambda)$, and hence

$$\int_{\mathbb{R}} \overline{\tilde{f}(\lambda)} \int_{\mathbb{C}^n} f(z) |z|^{-i\lambda - n} dm(z) \overline{\phi(\lambda)} d\lambda = \int_{\mathbb{R}} |\tilde{f}(\lambda)|^2 |\phi(\lambda)|^2 d\lambda,$$

and this concludes the proof. □

By the same argument that we used to prove Theorem 6, we have the following branching law.

Theorem 12. *The operator T extends to a unitary H-intertwining operator*

$$U : L^2((\mathbb{C}^n \setminus \{0\})/U(1)) \to \int_{\mathbb{R}} \mathcal{H}_\lambda d\mu(\lambda), \tag{39}$$

where μ is the measure determined by the identity

$$\int_{\mathbb{R}} f(\lambda) d\mu(\lambda) := \left(\frac{1}{4\pi^n}\right)^2 \int_{\mathbb{R}} f(\lambda) \left| \left(-\frac{i\lambda + n - 2}{2}\right)_{n-1} \right|^2 d\lambda.$$

Remark. The ideas in this paper could probably be extended to the case of the type III_n bounded symmetric domain consisting of complex antisymmetric $n \times n$ matrices by realising the corresponding minimal representation as the Hilbert space
$L^2((\mathbb{H}^n \setminus \{0\})/Sp(1))$ and proceeding in an analogous way.

References

1. Davidson, Mark and Ólafsson, Gestur and Zhang, Genkai, *Laplace and Segal-Bargmann transforms on Hermitian symmetric spaces and orthogonal polynomials*, J. Funct. Anal., **204**, (2003), 1, 157–195
2. Faraut, J. and Korányi, A. "Analysis on symmetric cones", Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1994
3. Helgason, Sigurdur "Groups and geometric analysis", Mathematical Surveys and Monographs,83, American Mathematical Society,Providence, RI, 2000
4. Knapp, Anthony W. "Representation theory of semisimple groups , an overview based on examples" Princeton University Press, Princeton, NJ, 2001
5. Rossi, H. and Vergne, M. *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group*, Acta Math., **136**, (1976), 1-2, 1–59
6. Rudin, Walter "Function theory in the unit ball of $\mathbf{C}^n$", Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 241, Springer-Verlag, New York, 1980
7. Seppänen, H. *Branching of some holomorphic representations of $SO(2,n)$*, Journal of Lie theory (to appear)
8. Seppänen, H. *Branching laws for minimal holomorphic representations,* Preprint
9. van Dijk, G. and Pevzner, M. *Berezin kernels of tube domains*, J. Funct. Anal., **181**, (2001), 2, 189–208
10. Wallach, Nolan R. *The analytic continuation of the discrete series. I, II*, Trans. Amer. Math. Soc., **251**, (1979), 1–17, 19–37
11. Zhang, Genkai *Berezin transform on real bounded symmetric domains*, Trans. Amer. Math. Soc., **353**, (2001), 9, 3769–3787 (electronic)
12. Zhang, Genkai *Tensor products of minimal holomorphic representations*, Represent. Theory, **5**, (2001), 164–190 (electronic)
13. Zhang, Genkai *Branching coefficients of holomorphic representations and Segal-Bargmann transform*, J. Funct. Anal., **195**, (2002), 2, 306–349

Department of Mathematics, Chalmers University of Technology and Göteborg University, Göteborg, Sweden

E-mail address: henriks@math.chalmers.se

Paper III

BRANCHING LAWS FOR MINIMAL HOLOMORPHIC REPRESENTATIONS

HENRIK SEPPÄNEN

ABSTRACT. In this paper we study the branching law for the restriction from $SU(n,m)$ to $SO(n,m)$ of the minimal representation in the analytic continuation of the scalar holomorphic discrete series. We identify the the group decomposition with the spectral decomposition of the action of the Casimir operator on the subspace of $S(O(n) \times O(m))$-invariants. The Plancherel measure of the decomposition defines an L^2-space of functions, for which certain continuous dual Hahn polynomials furnish an orthonormal basis. It turns out that the measure has point masses precisely when $n-m > 2$. Under these conditions we construct an irreducible representation of $SO(n,m)$, identify it with a parabolically induced representation, and construct a unitary embedding into the representation space for the minimal representation of $SU(n,m)$.

1. INTRODUCTION

One of the most important problems in harmonic analysis and in representation theory is that of decomposing group representations into irreducible ones. When the given representation arises as the restriction of an irreducible representation of a bigger group, the decomposition is referred to as a *branching law*. One of the most famous examples of this is the Clebsch-Gordan decomposition for the restriction of the tensor product of two irreducible $SU(2)$-representations (which is a representation of $SU(2) \times SU(2)$) to the diagonal subgroup. For an introduction to the general theory for compact connected Lie groups, we refer to [11].

Since the work by Howe ([7]) and by Kashiwara-Vergne ([9]), the study of branching rules for singular and minimal representations on

1991 *Mathematics Subject Classification.* 22E45, 32M15, 33C45, 43A85.

Key words and phrases. Unitary representations, Lie groups, branching law, bounded symmetric domains, real bounded symmetric domains.

spaces of holomorphic functions on bounded symmetric domains has been an active area of research. In [8], Jakobsen and Vergne studied the restriction to the diagonal subgroup of two holomorphic representations. More recently, Peng and Zhang ([22]) studied the corresponding decomposition for the tensor product of arbitrary (projective) representations in the analytic continuation of the scalar holomorphic discrete series. Zhang also studied the restriction to the diagonal of a minimal representation in this family tensored with its own anti-linear dual ([33]).

The restriction of the representations given by the analytic continuation of the scalar holomorphic discrete series to symmetric subgroups (fixed point groups for involutions) has been studied recently by Neretin ([18], [17]), Davidson, Ólafsson, and Zhang ([2]), Zhang ([32], [34]), van Dijk and Pevzner ([30]) and by the author ([28]).

All the above mentioned decompositions have the common feature that they are multiplicity free. This general result follows from a recent theorem by Kobayashi ([13]), where some geometric conditions are given for the action of a Lie group as isometric automorphisms of a Hermitian holomorphic vector bundle over a connected complex manifold to guarantee the multiplicity-freeness in the decomposition of any Hilbert space of holomorphic sections of the bundle. The action of a symmetric subgroup on the trivial line bundle over a bounded symmetric domain then satisfies these conditions (cf. [14]).

In this paper we study the branching rule for the restriction from $G := SU(n,m)$ to $H := SO(n,m)$ of the minimal representation in the analytic continuation of the scalar holomorphic discrete series. We consider the the subspace of $L := S(O(n) \times O(m))$-invariants and study the spectral decomposition for the action of Casimir element of the Lie algebra of H. The diagonalisation gives a unitary isomorphism between the subspace of L-invariants and an L^2-space with a Hilbert basis given by certain continuous dual Hahn polynomials. The main theorem is Theorem 9, where the decomposition on the group level is identified with this spectral decomposition. The Plancherel measure turns out to have point masses precisely when $n-m>2$. The second half of the paper is devoted to the realisation of the representation associated with one of these points and the unitary embedding into the representation space for the minimal representation. The main theorem of the second half is Theorem 21.

The paper is organised as follows. In Section 2 we begin with some preliminaries on the structure of the Lie algebra $\mathfrak{g}$, the group action, and the minimal representation. In Section 3 we construct an orthonormal basis for the subspace of L-invariants. In Section 4 we compute the action of the Casimir elements on the L-invariants and find its diagonalisation. We also state the branching theorem. In Section 5 we construct an irreducible representation of the group H (for $n-m>2$, i.e., when point masses occur in the Plancherel measure), identify it with a parabolically induced representation, and finally we construct a unitary embedding that realises one of the discrete points in the spectrum.

Acknowledgement: The author would like to thank his advisor Professor Genkai Zhang for support and for many valuable suggestions during the preparation of this paper.

2. Preliminaries

Let $\mathscr{D}$ be the bounded symmetric domain of type $I_{mn} (n \geq m)$, i.e.,

$$\mathscr{D} := \{z \in M_{nm}(\mathbb{C}) | I_n - zz^* > 0\}. \tag{1}$$

Here $M_{nm}(\mathbb{C})$ denotes the complex vector space of $n \times m$ matrices. We let G be the group $SU(n,m)$, i.e., the group of all complex $(n+m) \times (n+m)$ matrices of determinant one preserving the sesquilinear form $\langle \cdot, \cdot \rangle_{n,m}$ on $\mathbb{C}^{n+m}$ given by

$$\langle u, v\rangle_{n,m} = u_1\overline{v}_1 + \cdots + u_n\overline{v}_n - u_{n+1}\overline{v}_{n+1} - \cdots - u_{n+m}\overline{v}_{n+m}. \tag{2}$$

The group G acts holomorphically on $\mathscr{D}$ by

$$g(z) = (Az+B)(Cz+D)^{-1}, \tag{3}$$

if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a block matrix determined by the size of A being $n \times n$. The isotropy group of the origin is

$$\begin{aligned} K &:= S(U(n) \times U(m)) \\ &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} | A \in U(n), D \in U(m), \det(A)\det(D) = 1 \right\}, \end{aligned}$$

and hence

$$\mathscr{D} \cong G/K. \tag{4}$$

2.1. **Harish-Chandra decomposition.** Let θ denote the Cartan involution $g \mapsto (g^*)^{-1}$ on G. We use the same letter to denote its differential $\theta : \mathfrak{g} \to \mathfrak{g}$ at the identity. Here, we have identified $T_e(G)$ with $\mathfrak{g}$. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{5}$$

be the decomposition into the ± 1 eigenspaces of θ respectively. In terms of matrices,

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} | A^* = -A, D^* = -D, \mathrm{tr}(A) + \mathrm{tr}(D) = 0 \right\}, \tag{6}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \right\}, \tag{7}$$

where the size A is $n \times n$.

The Lie algebra $\mathfrak{g}$ has a compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$, where

$$\mathfrak{t} = \left\{ \begin{pmatrix} is_1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & is_n & 0 & \cdots & 0 \\ 0 & \cdots & 0 & it_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & it_m \end{pmatrix} | \begin{matrix} s_i, t_j \in \mathbb{R} \\ \sum_i s_i + \sum_j t_j = 0 \end{matrix} \right\}. \tag{8}$$

Its complexification, $\mathfrak{t}^{\mathbb{C}}$ (the set of complex diagonal traceless matrices), is a Cartan subalgebra of the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n+m, \mathbb{C})$, where

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}. \tag{9}$$

We let E_{ij} denote the matrix with 1 at the entry corresponding to the ith row and the jth column and zeros elsewhere. By E_{ij}^* we mean the dual linear functional, i.e., $E_{ij}^*(z) = z_{ij}$ for $z \in M_{nm}(\mathbb{C})$. Moreover, we define an ordered basis $\{F_j\}$ for $\mathfrak{t}^{\mathbb{C}}$ by

$$\begin{aligned} & F_j := E_{jj}^* - E_{j+1\,j+1}^*, \; j = 1, \dots, n+m-1, \\ & F_1 \le \cdots \le F_{n+m-1}. \end{aligned} \tag{10}$$

The root system, $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ is given by

$$\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) = \{E_{ii}^* - E_{jj}^* | 1 \le i, j \le n+m, i \ne j\}. \tag{11}$$

We denote the root $E_{ii}^* - E_{jj}^*$ by α_{ij}. We define a system of positive roots, Δ^+, by the ordering (10). Then

$$\Delta^+ = \{\alpha_{ij} | j > i\}, \tag{12}$$

and we let Δ^- denote the complement so that $\Delta = \Delta^+ \cup \Delta^-$. For a root, α, we let $\mathfrak{g}^\alpha$ stand for the corresponding root space. Then $\mathfrak{g}^{\alpha_{ij}} = \mathbb{C}E_{ij}$. For a root space, $\mathfrak{g}^\alpha$, we either have $\mathfrak{g}^\alpha \subset \mathfrak{k}^{\mathbb{C}}$ or $\mathfrak{g}^\alpha \subset \mathfrak{p}^{\mathbb{C}}$. In the first case, we call the corresponding root compact, and in the second case we call it non-compact. We denote the sets of compact and non-compact roots by $\Delta_{\mathfrak{k}}$ and $\Delta_{\mathfrak{p}}$ respectively. Finally, we let $\Delta_{\mathfrak{p}}^+$ and $\Delta_{\mathfrak{p}}^-$ denote the set of non-compact positive roots and the set of non-compact negative roots respectively. We set

$$\mathfrak{p}^+ = \sum_{\alpha \in \Delta_{\mathfrak{p}}^+} \mathfrak{g}^\alpha, \tag{13}$$

$$\mathfrak{p}^- = \sum_{\alpha \in \Delta_{\mathfrak{p}}^-} \mathfrak{g}^\alpha. \tag{14}$$

These subspace are abelian Lie subalgebras of $\mathfrak{p}^{\mathbb{C}}$. Moreover, the relations

$$[\mathfrak{k}^{\mathbb{C}}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+, [\mathfrak{k}^{\mathbb{C}}, \mathfrak{p}^-] \subseteq \mathfrak{p}^-, [\mathfrak{p}^+, \mathfrak{p}^-] \subseteq \mathfrak{k}^{\mathbb{C}} \tag{15}$$

hold. We let $K^{\mathbb{C}}, P^+$, and P^- denote the connected Lie subgroups of the complexification of G, $G^{\mathbb{C}}$, with Lie algebras $\mathfrak{k}^{\mathbb{C}}, \mathfrak{p}^+$, and $\mathfrak{p}^-$ respectively. The exponential mapping $\exp : \mathfrak{p}^{\pm} \to P^{\pm}$ is a diffeomorphic isomorphism of abelian groups. As subspaces of the Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n+m)$ we have the matrix realisations

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} | z \in M_{nm}(\mathbb{C}) \right\}, \tag{16}$$

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} | z \in M_{mn}(\mathbb{C}) \right\}. \tag{17}$$

The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ can be decomposed as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-. \tag{18}$$

On a group level, the multiplication map

$$P^+ \times K^{\mathbb{C}} \times P^- \to G^{\mathbb{C}}, (p, k, q) \mapsto pkq \tag{19}$$

is injective, holomorphic and regular with open image containing GP^+. In fact, identifying the domain $\mathscr{D}$ with the subset $\left\{\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} | z \in \mathscr{D}\right\} \subset \mathfrak{p}^+$ and letting

$$\Omega := \exp \mathscr{D} = \left\{\begin{pmatrix} I_n & z \\ 0 & I_m \end{pmatrix} | z \in \mathscr{D}\right\},$$

there is an inclusion

$$GP^+ \subset \Omega K^{\mathbb{C}} P^-. \tag{20}$$

For $g \in G$, we let $(g)_+, (g)_0$, and $(g)_-$ denote its $P^+, K^{\mathbb{C}}$, and P^- factors respectively. The action of g on $\mathscr{D}$ defined by

$$g(z) = \log((g \exp z)_+) \tag{21}$$

then coincides with the action (3). In fact, for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the Harish-Chandra factorisation is given by

$$\begin{aligned} &\begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} I_n & BD^{-1} \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_n & 0 \\ D^{-1}C & I_m \end{pmatrix}. \end{aligned} \tag{22}$$

For g as above, and $\exp z = \begin{pmatrix} I_n & z \\ 0 & I_m \end{pmatrix}$,

$$g \exp z = \begin{pmatrix} A & Az + B \\ C & Cz + D \end{pmatrix}, \tag{23}$$

and hence

$$(g \exp z)_+ = \begin{pmatrix} I_n & (Az + B)(Cz + D)^{-1} \\ 0 & I_m \end{pmatrix} \tag{24}$$

by (22).

We also use the Harish-Chandra decomposition to describe the differentials $dg(z)$ for group elements g at points z. We identify all tangent spaces $T_z(\mathscr{D})$ with $\mathfrak{p}^+ (\cong M_{nm}(\mathbb{C}))$. Then $dg(z) : \mathfrak{p}^+ \to \mathfrak{p}^+$ is given by the mapping

$$dg(z) = \mathrm{Ad}((g \exp z)_0)|_{\mathfrak{p}^+} \tag{25}$$

(cf. [26]). In the explicit terms given by (22), this mapping is given by

$$dg(z)Y = (A - (Az + B)(Cz + D)^{-1}C)YD^{-1}, \quad Y \in M_{nm}(\mathbb{C}).$$

2.2. **Strongly orthogonal roots.** We recall that two roots, α and β, are *strongly orthogonal* if neither $\alpha + \beta$, nor $\alpha - \beta$ is a root. We define a maximal set of strongly orthogonal noncompact roots, Γ, inductively by choosing γ_{k+1} as the smallest noncompact root strongly orthogonal to each of the members $\{\gamma_1, \dots, \gamma_k\}$ already chosen. When the ordering of the roots in given as in (10), we get

$$\Gamma = \{\gamma_1, \dots, \gamma_m\},\ \gamma_j = E^*_{jj} - E^*_{j+n\,j+n}. \tag{26}$$

We now let E_{γ_j} denote the elementary matrix that spans the root space $\mathfrak{g}^{\gamma_j}$. Then the real vector space

$$\mathfrak{a} := \sum_{j=1}^{n} \mathbb{R}(E_{\gamma_j} - \theta E_{\gamma_j}) \tag{27}$$

is a maximal abelian subspace of $\mathfrak{p}$. We set

$$E_j := E_{\gamma_j} - \theta E_{\gamma_j}. \tag{28}$$

2.3. **Shilov boundary.** Let $\mathcal{O}(\mathscr{D})$ denote the set of holomorphic functions on $\mathscr{D}$, and let $\mathcal{O}(\overline{\mathscr{D}})$ denote the subset consisting of those which have continuous extensions to the boundary. The Shilov boundary of $\mathscr{D}$ is the set

$$\mathcal{S} = \{z \in V | I_m - z^*z = 0\}.$$

It has the property that

$$\sup_{z \in \overline{\mathscr{D}}} |f(z)| = \sup_{z \in \mathcal{S}} |f(z)|, f \in \mathcal{O}(\overline{\mathscr{D}}), \tag{29}$$

and it is minimal with respect to this property, i.e., no proper subset of $\mathcal{S}$ has the property. The set $\mathcal{S}$ can also be described as the set of all rank m partial isometries from $\mathbb{C}^m$ to $\mathbb{C}^n$. The group $K = U(n) \times U(m)$ acts transitively on $\mathcal{S}$ by

$$(g, h)(z) = gzh^{-1}.$$

To find the isotropy group of the fixed element $z_0 := \begin{pmatrix} I_m \\ 0 \end{pmatrix}$, let $(g,h) \in U(n) \times U(m)$ and write g in the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is of size $m \times m$. Then

$$gz_0h^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} h^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} Ah^{-1} \\ Ch^{-1} \end{pmatrix}.$$

So, the equality $gz_0h^{-1} = z_0$ holds if and only if $A = h$ and $C = 0$. Since g is unitary, the last condition implies that also $B = 0$ and hence the isotropy group is

$$K_0 := (U(n) \times U(m))_{z_0} = \left\{ (g,h) \in U(n) \times U(m) | g = \begin{pmatrix} h & 0 \\ 0 & D \end{pmatrix} \right\}.$$

Thus we have the description

$$\mathcal{S} = K/K_0 = (U(n) \times U(m))/U(n-m) \times U(m)$$

of the Shilov boundary as a homogeneous space.

In the sequel, we will often be concerned with the submanifold $\mathcal{S}_\Delta$ of $\mathcal{S}$, where

$$\mathcal{S}_\Delta := \left\{ z_{\underline{\xi}} := \begin{pmatrix} \xi_1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & \xi_m \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathcal{S} | \xi_1, \ldots, \xi_m \in S^1 \right\}. \tag{30}$$

Also, we let $\operatorname{diag}(\underline{\xi})$ denote the $m \times m$-matrix $\begin{pmatrix} \xi_1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & \xi_m \end{pmatrix}$. The identity

$$z_{\underline{\xi}} = \begin{pmatrix} \operatorname{diag}(\underline{\xi}) \\ 0 \end{pmatrix} = \begin{pmatrix} \operatorname{diag}(\underline{\xi}) & 0 \\ 0 & I_{n-m} \end{pmatrix} \begin{pmatrix} I_m \\ 0 \end{pmatrix} \tag{31}$$

identifies the matrices in the submanifold $\mathcal{S}_\Delta$ with certain cosets in K/K_0.

2.4. **The real form $\mathscr{X}$.** Consider the mapping $\tau : \mathscr{D} \to \mathscr{D}$ defined by

$$\tau(z) = \overline{z}, \tag{32}$$

where the conjugation is entrywise. It is an antiholomorphic involutive diffeomorphism of $\mathscr{D}$. We let $\mathscr{X}$ denote the set of fixed points of τ, i.e.,

$$\mathscr{X} = \{z \in \mathscr{D} | \tau(z) = z\}. \tag{33}$$

Moreover, τ defines an involution, which we also denote by τ, of G given by

$$\tau(g) = \tau g \tau^{-1}. \tag{34}$$

We let H denote the set of fixed points, i.e.,

$$H = G^{\tau} = \{g \in G | \tau(g) = g\}. \tag{35}$$

Clearly, $H = SO(n,m)$, i.e., the elements in G with real entries. The group H acts transitively on $\mathscr{X}$, and the isotropy group of 0 in H is $L := H \cap K$. Hence

$$\mathscr{X} \cong H/L. \tag{36}$$

2.5. **Minimal representation $\mathscr{H}_1$.** We recall that the Bergman kernel of $\mathscr{D}$ is given by

$$K(z,w) = \det(I_n - zw^*)^{-(n+m)}. \tag{37}$$

It has the transformation property

$$K(gz,gw) = J_g(z)^{-1} K(z,w) \overline{J_g(w)}^{-1}, \tag{38}$$

where $J_g(z)$ denotes the complex Jacobian of g at z. We let $h(z,w)$ denote the function

$$h(z,w) = \det(I_n - zw^*). \tag{39}$$

Then, for real ν, the kernel

$$h(\cdot,\cdot)^{-\nu} \tag{40}$$

is positive definite if and only if ν belongs to the Wallach set, $\mathcal{W}$. Here,

$$\mathcal{W} = \{0, 1, \dots, m-1\} \bigcup (m-1, \infty) \tag{41}$$

(cf. [3]). The kernel $h(\cdot,\cdot)^{-\nu}$ satisfies the transformation rule

$$h(gz,gw)^{-\nu} = J_g(z)^{-\frac{\nu}{n+m}} h(z,w)^{-\nu} \overline{J_g(w)}^{-\frac{\nu}{n+m}}. \tag{42}$$

For $\nu \in \mathcal{W}$, we denote the Hilbert space defined by the kernel $h(\cdot,\cdot)^{-\nu}$ by $\mathscr{H}_\nu$. A projective representation, π_ν, of G is defined on $\mathscr{H}_\nu$ by

$$\pi_\nu(g)f(z) = J_{g^{-1}}(z)^{\frac{\nu}{n+m}} f(g^{-1}z). \tag{43}$$

We will be concerned with the so called minimal representation, i.e, with the representation π_1 on the space $\mathscr{H}_1$.

3. The L-invariants

For any $\nu \in \mathcal{W}$, let

$$\mathscr{H}_\nu = \bigoplus_{\underline{k}:=-(k_1\gamma_1+\cdots+k_m\gamma_m)} \mathcal{P}^{\underline{k}}$$

be the decomposition into K-types. Here $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ is the maximal strongly orthogonal set in $\Delta_{\mathfrak{p}}^+$ with ordering $\gamma_1 < \cdots < \gamma_m$ defined in the previous section, and

$$k_1 \geq \cdots \geq k_m, k_i \in \mathbb{N}, \tag{44}$$

and $\mathcal{P}^{\underline{k}}$ is a representation space for the K-representation of highest weight that is realised inside the space of homogeneous polynomials of degree $|\underline{k}| = k_1 + \cdots + k_m$ on $\mathfrak{p}^+$. When $\nu = 1$, the weights occurring in this sum are all of the form

$$\underline{k} = -k\gamma_1 \tag{45}$$

(cf. [3]). Taking L-invariants, we have

$$\mathscr{H}_1^L = \bigoplus_{\underline{k}} (\mathcal{P}^{\underline{k}})^L.$$

The data (K, L, τ) defines a Riemannian symmetric pair, and hence $(V^{\underline{k}})^L$ is at most one dimensional by the Cartan-Helgason theorem (cf. [6], Ch. IV, Lemma 3.6.).

We recall the compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$ in (8). We let $\tilde{\mathfrak{t}}$ denote the Cartan subalgebra of $\mathfrak{u}(n) \oplus \mathfrak{u}(m)$ consisting of all diagonal imaginary matrices, i.e., matrices of the form (8) but without the requirement that the trace be zero. Then we have an orthogonal decomposition

$$\tilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathfrak{t}^\perp \tag{46}$$

given by the Killing form.

Any linear functional $l \in \mathfrak{t}^*$ extends uniquely to a functional on $\tilde{\mathfrak{t}}$ which annihilates the orthogonal complement $\tilde{\mathfrak{t}}^{\perp}$. We will denote these extensions by the same letter l. Therefore, any dominant integral weight on $\mathfrak{t}$ parametrises an irreducible representation of $\mathfrak{u}(n) \oplus \mathfrak{u}(m)$ in which $\tilde{\mathfrak{t}}^{\perp}$ acts trivially. When $\lambda = \underline{k} = k\gamma_1$ is a K-type occurring in $\mathscr{H}_1$, we denote the underlying representation space for $\mathfrak{u}(n) \oplus \mathfrak{u}(m)$ by V^{λ}. Moreover, the Cartan subalgebra $\tilde{\mathfrak{t}}$ is the sum

$$\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$$

of the corresponding subalgebras of $\mathfrak{u}(n)$ and $\mathfrak{u}(m)$ respectively. The restrictions of λ to $\mathfrak{t}_1$ and $\mathfrak{t}_2$ respectively define integral weights, hence they parametrise irreducible representations of the Lie algebras $\mathfrak{u}(n)$ and $\mathfrak{u}(m)$ respectively. We denote the corresponding representation spaces by V_n^{λ} and V_m^{λ}. In what follows, λ will always denote the extension to $\mathfrak{u}(n) \oplus \mathfrak{u}(m)$ of a weight of the form $\underline{k}$ in (45). We will use the explicit realisations

$$V_n^{\lambda} = \overset{k}{\odot}\, \mathbb{C}^n, \tag{47}$$

where the right hand side denotes the symmetric tensor product defined as a quotient of the k-fold tensor product of $\mathbb{C}^n$. In the following, for a multiindex $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, we let

$$|\alpha| := \alpha_1 + \cdots + \alpha_n, \tag{48}$$

$$\alpha! := \alpha_1! \cdots \alpha_n!. \tag{49}$$

For any choice of orthonormal basis $\{e_1, \ldots, e_n\}$ for $\mathbb{C}^n$, the set

$$\{e^{\alpha} := e_1^{\alpha_1} \cdots e_n^{\alpha_n} \,|\, |\alpha| = k\} \tag{50}$$

furnishes a basis for $\overset{k}{\odot}\, \mathbb{C}^n$. We fix an K-invariant inner product, $\|\cdot\|_{\mathcal{F}}$, [1] on $\overset{k}{\odot}\, \mathbb{C}^n$ by the normalisation

$$\|e_1^k\|_{\mathcal{F}}^2 = k!. \tag{51}$$

Observe that we have suppressed both the indices k and n here. For n fixed, the norm in fact equals the restriction of the norm defined on all polynomial functions on $\mathbb{C}^n$ (we use the natural identification $e^{\alpha} \leftrightarrow z^{\alpha}$ of symmetric tensor power with polynomial functions)

$$\langle p, q \rangle_k := p(\partial)(q^*)(0), \tag{52}$$

[1] This is often called the *Fock-Fischer* inner product (cf. [3]).

where $p(\partial)$ is the differential operator defined by substituting $\frac{\partial}{\partial e_j}$ for e_j in p, and for $q = \sum_\alpha a_\alpha z^\alpha$, q^* is defined as

$$(\sum_\alpha a_\alpha z^\alpha)^* := \sum_\alpha \overline{a_\alpha} z^\alpha. \tag{53}$$

The suppressing of the index n will not cause any confusion in what follows. Finally, on the dual space V_m^λ we have the corresponding basis

$$\{(e^*)^\alpha := (e_1^*)^{\alpha_1} \cdots (e_n^*)^{\alpha_n} | |\alpha| = k\}, \tag{54}$$

where $\{e_1^*, \dots, e_n^*\}$ is the dual basis to $\{e_1, \dots, e_n\}$ with respect to the standard inner product on $\mathbb{C}^n$. We also let $\|\cdot\|_{\mathcal{F}}$ denote the K-invariant norm on V_m^λ normalised by

$$\|(e_1^*)^k\|_{\mathcal{F}}^2 = k!. \tag{55}$$

Lemma 1. *For any choice of orthonormal basis $\{e_1, \dots, e_m\}$ for $\mathbb{C}^m$ and extension $\{e_1, \dots, e_m, e_{m+1}, \dots, e_n\}$ to an orthonormal basis for $\mathbb{C}^n$, the vector*

$$\iota_\lambda := \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| = k}} f_\alpha \otimes f_\alpha^* \in V_n^\lambda \otimes (V_m^\lambda)^*,$$

where $f_\alpha = \frac{e^\alpha}{(\alpha!)^{1/2}}$ and $f_\alpha^* = \frac{(e*)^\alpha}{(\alpha!)^{1/2}}$, is K_0-invariant.

Proof. We recall the identification of the isotropic subgroup of the fixed element z_0 with $U(n-m) \times U(m)$. From this it is clear that it suffices to prove that the vector $\iota_\lambda \in V_m^\lambda \otimes (V_m^\lambda)^* \subset V_n^\lambda \otimes (V_m^\lambda)^*$ is invariant under the restriction of the representation of $U(m) \times U(m)$ to the diagonal subgroup.

The vector space $V_m^\lambda \otimes (V_m^\lambda)^*$ is naturally isomorphic to $\mathrm{End}(V_m^\lambda)$, the isomorphism being given by $(u \otimes v^*)(y) = v^*(y)u$. Then, if $y \in V_m^\lambda$ is the linear combination $y = \sum_\beta c_\beta f_\beta$,

$$\sum_\alpha f_\alpha \otimes f_\alpha^*(y) = \sum_{\alpha,\beta} c_\beta \langle f_\beta, f_\alpha \rangle f_\beta = y;$$

i.e., ι_λ corresponds to the identity operator. Moreover, for the action of $\mathfrak{u}(m)$ on the tensor product $V_m^\lambda \otimes (V_m^\lambda)^*$, we have

$$\begin{aligned} X(u \otimes v^*)(y) &= (Xu \otimes v^*)(y) + (u \otimes Xv^*)(y) \\ &= v(y)Xu + (Xv^*)(y)u \\ &= v(y)Xu + \langle y, Xv\rangle u \\ &= v(y)Xu - \langle Xy, v\rangle u \\ &= [X, u \otimes v^*](y), \end{aligned}$$

where $X \in \mathfrak{u}(m), u \in V_m^\lambda, v^* \in (V_m^\lambda)^*$, i.e., the action as derivations of the tensor product corresponds to the commutator action on the endomorphisms. In particular, $X\iota_\lambda = 0$ for all X in $\mathfrak{u}(m)$. This proves the lemma. □

Since the vectors in the representation space V^λ are holomorphic polynomials, they are determined by their restrictions to the Shilov boundary $\mathcal{S}$.

In the sequel, we use the Fock inner product to define an antilinear identification of V_m^λ with $(V_m^\lambda)^*$ by

$$v \mapsto v^*, \quad v^*(w) = \langle w, v\rangle_{\mathcal{F}}, \qquad w \in V_m^\lambda.$$

We let $\langle \cdot, \cdot \rangle$ denote the inner product on the tensor product $V_n^\lambda \otimes (V_m^\lambda)^*$ induced by the Fock inner products on the factors.

Proposition 2. *The operator $T_\lambda : V_n^\lambda \otimes (V_m^\lambda)^* \to V^\lambda$ defined by*

$$T_\lambda(u \otimes v^*)(z) = \langle (g, h)\iota_\lambda, u \otimes v^*\rangle,$$

where $z = (g, h)K_0 \in \mathcal{S}$, is a $\mathbb{C}$-antilinear isomorphism of $U(n) \times U(m)$-representations.

Proof. We first observe that the left hand side is well defined as a function of z by the invariance of ι_λ.

The root system $\Delta(\mathfrak{u}(n) \oplus \mathfrak{u}(m), \mathfrak{t})$ is the union of the root systems $\Delta(\mathfrak{u}(n), \mathfrak{t}_1)$ and $\Delta(\mathfrak{u}(m), \mathfrak{t}_2)$. Fix choices of positive roots $\Delta^+(\mathfrak{u}(n), \mathfrak{t}_1)$, and $\Delta^+(\mathfrak{u}(m), \mathfrak{t}_2)$ respectively. We define a system of positive roots in $\Delta(\mathfrak{u}(n) \oplus \mathfrak{u}(m), \mathfrak{t})$ by

$$\Delta^+(\mathfrak{u}(n) \oplus \mathfrak{u}(m), \mathfrak{t}) := \Delta^+(\mathfrak{u}(n), \mathfrak{t}_1) \cup \Delta^+(\mathfrak{u}(m), \mathfrak{t}_2).$$

Let $u_\lambda \in V_n^\lambda$ be a lowest weight-vector, and $v_\lambda \in V_m^\lambda$ be a highest weight-vector. Then $u_\lambda \otimes v_\lambda^*$ is a lowest weight-vector in $V_n^\lambda \otimes (V_m^\lambda)^*$.

For $H=(H_1,H_2)\in\mathfrak{t}_1\oplus\mathfrak{t}_2$ we have

$$\begin{aligned}
& \frac{d}{dt}(T_\lambda(u_\lambda\otimes v_\lambda^*))(\exp tH\cdot z)_{t=0}\\
= & \frac{d}{dt}\langle(\exp tH_1 g,\exp tH_2 h)\iota_\lambda, u_\lambda\otimes v_\lambda^*\rangle_{t=0}\\
= & \frac{d}{dt}\left(\langle(g,h)\iota_\lambda,(\exp -tH_1,\exp -tH_2)(u_\lambda\otimes v_\lambda^*)\rangle\right)_{t=0}\\
= & \langle(g,h)\iota_\lambda,\lambda(-H_1)u_\lambda\otimes v_\lambda^*\rangle\\
& +\langle(g,h)\iota_\lambda,u_\lambda\otimes\lambda(-H_2)v_\lambda^*\rangle\\
= & \lambda(H)T_\lambda(u_\lambda\otimes v_\lambda^*)(z).
\end{aligned}$$

Thus $T_\lambda(u_\lambda\otimes v_\lambda^*)$ is a vector of weight λ.

Any root vector in $\mathfrak{u}(n)\oplus\mathfrak{u}(m)$ lies in either of the components. Take therefore a positive root vector $E+iF\in\mathfrak{u}(n)^{\mathbb{C}}$. Then

$$\begin{aligned}
& (E+iF,0)(T_\lambda(u_\lambda\otimes v_\lambda^*))(z)\\
= & \frac{d}{dt}\left(\langle(\exp tEg,h)\iota_\lambda,u_\lambda\otimes v_\lambda^*\rangle\right)_{t=0}\\
& +i\frac{d}{dt}\left(\langle(\exp tFg,h)\iota_\lambda,u_\lambda\otimes v_\lambda^*\rangle\right)_{t=0}\\
= & \langle(g,h)\iota_\lambda,(-(E-iF)u_\lambda\otimes v_\lambda^*)\rangle\\
= & 0,
\end{aligned}$$

since $E-iF$ is a negative root vector. Similarly one shows that the positive root vectors in $\mathfrak{u}(m)$ annihilate $T_\lambda(u_\lambda\otimes v_\lambda^*)$. The function $T_\lambda(u_\lambda\otimes v_\lambda^*)$ on the Shilov boundary naturally extends to a holomorphic polynomial on $\mathscr{D}$ which belongs to $\mathscr{H}_1$. Hence $T_\lambda(u_\lambda\otimes v_\lambda^*)$ can be written as finite sum of highest weight-vectors from the K-types of $\mathscr{H}_1$. But it is a vector of weight λ, and so by the multiplicity-freeness of the K-type decomposition, $T_\lambda(u_\lambda\otimes v_\lambda^*)$ is a highest weight-vector in V^λ. □

Lemma 3. *The space $(V^\lambda)^L$ is nonzero if and only if $\lambda=-2k\gamma_1$ for $k\in\mathbb{N}$. In this case, it is one-dimensional with a basis vector ψ_k, where*

$$\psi_k(z_{\underline{\xi}}):=\sum_{\substack{\beta\in\mathbb{N}^m\\|\beta|=k}}\binom{k}{\beta}^2(2\beta!)\xi^{2\beta}, \tag{56}$$

where $z_{\underline{\xi}}$ is the matrix defined in (30).

Proof. We use the isomorphism from the proposition above. Then the first statement is obvious, since for any $\lambda = -j\gamma_1$, the representation space V_n^λ is isomorphic to the space of all polynomials of homogeneous degree j on $\mathbb{C}^n$, and the corresponding statement holds for V_m^λ. Assume therefore that $\lambda = -2k\gamma_1$.

Clearly, the vector $(e_1^2 + \cdots + e_n^2)^k \otimes ((e_1^*)^2 + \cdots + (e_m^*)^2)^k$ is an L-invariant vector in $V_n^\lambda \otimes (V_m^\lambda)^*$. We compute its image under T_λ when restricted to the matrices in $\mathcal{S}_\Delta$.

$$\begin{aligned} T_\lambda((e_1^2 &+ \cdots + e_n^2)^k \otimes ((e_1^*)^2 + \cdots + (e_m^*)^2)^k)(z_{\underline{\xi}}) \\ &= \langle (g_{\underline{\xi}}, I_m)\iota_\lambda, (e_1^2 + \cdots + e_n^2)^k \otimes ((e_1^*)^2 + \cdots + (e_m^*)^2)^k \rangle \\ &= \langle \sum_\alpha \xi^\alpha f_\alpha \otimes f_\alpha^*, (e_1^2 + \cdots + e_n^2)^k \otimes ((e_1^*)^2 + \cdots + (e_m^*)^2)^k \rangle \\ &= \sum_\alpha \xi^\alpha \langle f_\alpha, (e_1^2 + \cdots + e_n^2)^k \rangle \langle f_\alpha^*, ((e_1^*)^2 + \cdots + (e_m^*)^2)^k \rangle. \end{aligned}$$

Since the symmetric tensor $(e_1^2 + \cdots + e_n^2)^k$ has the monomial expansion

$$(e_1^2 + \cdots + e_n^2)^k = \sum_{|\beta|=k} \binom{k}{\beta} e^{2\beta},$$

we get the equality

$$T_\lambda((e_1^2 + \cdots + e_n^2)^k \otimes ((e_1^*)^2 + \cdots + (e_m^*)^2)^k)(z_{\underline{\xi}}) = \sum_{|\beta|=k} \binom{k}{\beta}^2 (2\beta!)\xi^{2\beta}.$$

□

Theorem 4. *The polynomials φ_k of degree $2k$, for $k \in \mathbb{N}$, given by*

$$\varphi_k(z_{\underline{\xi}}) = \frac{1}{4^k k! \left(\frac{m}{2}\right)_k^{1/2} \left(\frac{n}{2}\right)_k^{1/2}} \sum_{|\beta|=k} \binom{k}{\beta}^2 (2\beta!)\xi^{2\beta}$$

constitute an orthonormal basis for the subspace, $\mathscr{H}_1^L$, of L-invariants.

Proof. The only thing that is left to prove is the normalisation part of the statement, i.e., we need to compute the norms of the polynomials ψ_k.

Using the antilinear isomorphism T_λ, we can introduce an inner product

$$\langle \cdot, \cdot \rangle_\lambda' := \overline{\langle T_\lambda^{-1} \cdot, T_\lambda^{-1} \cdot \rangle},$$

where the the right hand side denotes the conjugate of the inner product on the tensor product induced by the Fock inner products on the factors, on V^λ. By Schur's lemma, the equality

$$\|\cdot\|_{\mathcal{F}} = C_\lambda \|\cdot\|_\lambda'$$

holds on V^λ for some complex constant C_λ. To compute this constant, we compare the norms of the lowest weight-vector $u_\lambda \otimes v_\lambda^*$ and the highest weigh-vector $T_\lambda(u_\lambda \otimes v_\lambda^*)$ in their respective representation spaces. Let $\{e_1, \dots, e_m\}$ and $\{e_1, \dots, e_n\}$ denote the standard orthonormal bases for $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Then $u_\lambda \otimes v_\lambda^* = e_1^{2k} \otimes (e_1^{2k})^*$, and

$$\|e_1^{2k} \otimes (e_1^{2k})^*\| = (2k)!.$$

Moreover, the normalised lowest weight-vector $\frac{e_1^{2k} \otimes (e_1^*)^{2k}}{(2k)!}$ maps to

$$T_\lambda\left(\frac{e_1^{2k} \otimes (e_1^*)^{2k}}{(2k)!}\right),$$

where

$$\begin{aligned} T_\lambda\left(\frac{e_1^{2k} \otimes (e_1^*)^{2k}}{(2k)!}\right)(z_{\underline{\xi}}) &= \xi_1^{2k} \\ &= p_{11}(z_{\underline{\xi}}), \end{aligned}$$

where p_{11} is the highest weight vector given by $p_{11}(z) = z_{11}^{2k}$. Since $\|p_{11}\|_{\mathcal{F}} = \sqrt{(2k)!}$, we see that $C_\lambda = \sqrt{(2k)!}$.

The norm of $(e_1^2+\cdots+e_n^2)^k \otimes (((e_1^*)^2+\cdots+(e_m^*)^2)$ is straightforward to compute. In fact,

$$\|(e_1^2+\cdots+e_n^2)^k\|_{\mathcal{F}}^2 \, \|((e_1^*)^2+\cdots+(e_m^*)^2)^k\|_{\mathcal{F}}^2 = (k!)^2 \left(\frac{m}{2}\right)_k \left(\frac{n}{2}\right)_k.$$

Finally, we have the equality

$$\|\cdot\|_1^2 = \frac{1}{(2k)!}\|\cdot\|_{\mathcal{F}}^2 \tag{57}$$

(cf. [3]) relating the $\mathscr{H}_1$-norm to the Fock-Fischer norm on the K-type $\underline{2k} = -2k\gamma_1$, and this ends the proof. □

4. The action of the Casimir element on the L-invariants

We consider the representation of the universal enveloping algebra $U(\mathfrak{h}^{\mathbb{C}})$ defined for all $X \in \mathfrak{h}$ by

$$f \mapsto \frac{d}{dt}\pi_1(\exp tX)f|_{t=0}, \tag{58}$$

for f in the dense subspace, $\mathscr{H}_1^\infty$, of analytic vectors, and extended to a homomorphism $U(\mathfrak{h}^{\mathbb{C}}) \to \mathrm{End}(\mathscr{H}_1^\infty)$. We will denote this representation too by π_1. We recall that the Casimir element, $\mathcal{C} \in U(\mathfrak{h}^{\mathbb{C}})$ is given by

$$\mathcal{C} = X_1^2 + \cdots + X_p^2 - Y_1^2 - \cdots - Y_q^2, \tag{59}$$

where $\{X_i, i = 1, \ldots, \dim\mathfrak{q}\}$ and $\{Y_i, i = 1, \ldots, \dim\mathfrak{l}\}$ are any orthogonal bases for $\mathfrak{q}$ and $\mathfrak{l}$ respectively with respect to the Killing form, $B(\cdot,\cdot)$, on $\mathfrak{h}$ such that

$$\begin{aligned} B(X_i, X_i) &= 1, \qquad i = 1, \ldots, \dim\mathfrak{q}, \\ B(Y_i, Y_i) &= -1, \qquad i = 1, \ldots, \dim\mathfrak{l}. \end{aligned}$$

Consider now the left regular representation, l, of H on $C^\infty(H/L)$, i.e., $l(h)f(x) = f(h^{-1}x)$. We define an operator $R_1 : \mathscr{H}_1 \to C^\infty(H/L)$ by

$$R_1 f(x) := h(x,x)^{-1/2} f(x). \tag{60}$$

This is the *generalised Segal-Bargmann transform* due to Ólafsson and Ørsted (cf. [21]). A nice introduction to this transform in a more general context can also be found in Ólafssons overview paper [20]. The following lemma is an immediate consequence of the transformation rule (42).

Lemma 5. *The operator $R_1 : \mathscr{H}_1 \to C^\infty(H/L)$ is H-equivariant.*

Moreover, the Casimir element acts on $C^\infty(H/L)$ as the Laplace-Beltrami operator, $\mathcal{L}$, for the symmetric space H/L. We recall the "polar coordinate map" (cf. [5], Ch.IX)

$$\begin{aligned} \phi : L/M \times A^+ &\to (H/L)', \\ (lM, a) &\mapsto laL \end{aligned} \tag{61}$$

Here $(H/L)' := H'/L$, where H' is the set of regular elements in H, and $A^+ = \exp\mathfrak{a}^+$, where

$$\mathfrak{a}^+ = \{t_1E_1 + \cdots + t_mE_m | t_i \geq 0, i = 1, \ldots, m\}. \tag{62}$$

The map ϕ is a diffeomorphism onto an open dense set in H/L. Hence, any $f \in C^\infty(H/L)^L$ is uniquely determined by its restriction to the submanifold $A^+ \cdot 0 = \psi(\{eM\} \times A^+)$. In fact, the restriction mapping $f \mapsto f|_{A^+ \cdot 0}$ defines an isomorphism between the spaces $C^\infty(H/L)^L$ and $C^\infty(A^+ \cdot 0)^{N_L(\mathfrak{a})/Z_L(\mathfrak{a})}$. The space $C^\infty(H/L)^L$ is invariant under the

Laplace-Beltrami operator. Recall that the radial part of the Laplace-Beltrami operator is a differential operator, $\Delta\mathcal{L}$, on the submanifold $A^+ \cdot 0$ with the property that the diagram

$$\begin{array}{ccc} C^\infty(H/L) & \xrightarrow{\mathcal{L}} & C^\infty(H/L) \\ \downarrow & & \downarrow \\ C^\infty(A^+ \cdot 0) & \xrightarrow{\Delta\mathcal{L}} & C^\infty(A^+ \cdot 0) \end{array},$$

where the vertical arrows denote the restriction map, commutes.

Moreover, the functions in $\mathscr{H}_1$ are determined by their restrictions to the real submanifold H/L, and the L-invariant functions are determined by their restrictions to $A^+ \cdot 0$. By Lemma 5 and the above discussion, we have the following commuting diagram.

$$\begin{array}{ccc} \mathscr{H}_1^L & \xrightarrow{\pi_1(C)} & \mathscr{H}_1^L \\ \downarrow & & \downarrow \\ C^\infty(A^+ \cdot 0) & \xrightarrow{R_1^{-1}\Delta\mathcal{L}R_1} & C^\infty(A^+ \cdot 0) \end{array},$$

where, again, the vertical arrows denote the restriction maps.

In what follows, we will compute the action of the operator $R_1^{-1}\Delta\mathcal{L}R_1$ on the subspace $\mathscr{H}_1^L$.

The radial part of the Laplace-Beltrami operator of H/L is given by (cf.[6], Ch. II, Prop. 3.9)

$$\begin{aligned} 4\Delta\mathcal{L} = &\sum_{j=1}^{m} \frac{\partial^2}{\partial t_j^2} + \sum_{m \geq i \geq j \geq 1} \coth(t_i \pm t_j)(\frac{\partial}{\partial t_i} \pm \frac{\partial}{\partial t_j}) \\ &+(n-m)\sum_{j=1}^{m} \coth t_j \frac{\partial}{\partial t_j}. \end{aligned}$$

The coordinates t_i are related to the Euclidean coordinates x_i by $x_i = \tanh t_i$, i.e.,

$$A^+ \cdot 0 = \{(x_1, \dots, x_m) | 0 \leq x_1 \leq x_2 \leq \dots \leq x_m < 1\} \tag{63}$$

In the coordinates x_i, the operator $4R_1^{-1}\Delta\mathcal{L}R_1 := 4\mathcal{L}^1$ has the expression

$$\begin{aligned}
4\mathcal{L}^1 &= \sum_{i=1}^m \left(-(1-x_i^2) - x_i^2 - 2x_i(1-x_i^2)\frac{\partial}{\partial x_i} + (1-x_i^2)^2 \frac{\partial^2}{\partial x_i^2} \right) \\
&\quad + \sum_{i=1}^m \left(2x_i^2 - 2x_i(1-x_i^2)\frac{\partial}{\partial x_i} \right) \\
&\quad + (n-m)\sum_{i=1}^m \left(-1 - x_i\frac{\partial}{\partial x_i} + \frac{1}{x_i}\frac{\partial}{\partial x_i} \right) \\
&\quad + 2\sum_{m\geq i>j\geq 1} \left(-1 + \frac{(1-x_i^2)(1-x_j^2)}{x_i^2 - x_j^2} \left(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j} \right) \right).
\end{aligned}$$

The following lemma is proved by a straightforward calculation. A proof for a similar decomposition can be found in [33].

Lemma 6. *The operator $4R_1^{-1}\Delta\mathcal{L}R_1$ can be written as a sum of three operators, $\mathcal{L}_-, \mathcal{L}_0$ and $\mathcal{L}_+$ that lower, keep and, respectively, raise the degrees of the polynomials ψ_k. In fact,*

$$\begin{aligned}
\mathcal{L}_- &= \sum_{i=1}^m \left(\frac{\partial^2}{\partial x_i^2} + \frac{n-m}{x_i}\frac{\partial}{\partial x_i} \right) + 2\sum_{m\geq i>j\geq 1} \frac{1}{x_i^2 - x_j^2}\left(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j} \right), \\
\mathcal{L}_0 &= -mn + \sum_{i=1}^m \left((-4-(n-m))x_i\frac{\partial}{\partial x_i} - 2x_i^2\frac{\partial^2}{\partial x_i^2} \right) \\
&\quad -2\sum_{m\geq i>j\geq 1} \frac{x_i^2 + x_j^2}{x_i^2 - x_j^2}\left(x_j\frac{\partial}{\partial x_j} - x_i\frac{\partial}{\partial x_i} \right), \\
\mathcal{L}_+ &= \sum_{i=1}^m \left(2x_i^2 + 4x_i^3\frac{\partial}{\partial x_i} + x_i^4\frac{\partial^2}{\partial x_i^2} \right) \\
&\quad +2\sum_{m\geq i>j\geq 1} \frac{x_i^2 x_j^2}{x_i^2 - x_j^2}\left(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j} \right).
\end{aligned}$$

Proposition 7. *The operator $\mathcal{L}^1$ acts on the (unnormalised) orthogonal system $\{\psi_k\}$ as the Jacobi operator*

$$\mathcal{L}^1\psi_k = A_k\psi_{k-1} + B_k\psi_k + C_k\psi_{k+1},$$

where

$$
\begin{aligned}
A_k &= 4k^4+(4(m-2)+2(n-m))k^3 \\
&\quad +((m^2-4m+4)+(n-m)(m-2))k^2, \\
B_k &= -2k^2-\frac{n+m}{2}k-\frac{mn}{4}, \\
C_k &= \frac{1}{4}.
\end{aligned} \tag{64}
$$

Proof. It follows from the above lemma that the operator is a Jacobi operator. In order to identify the coefficients A_k, B_k, and C_k, we evaluate the polynomials at points $(x_1,0):=(x_1,0,\ldots,0)$. Then we have

$$
\begin{aligned}
\mathcal{L}^+\psi_k((x_1,0)) &= \left(2x_1^2+4x_1^3\frac{\partial_1}{\partial x_1}+x_1^4\frac{\partial_1^2}{\partial x_1^2}\right)\psi_k((x_1,0)) \\
&= (2+8k+2k(2k-1))(2k)!x_1^{2k+2} \\
&= \frac{4k^2+6k+2}{(2k+2)(2k+1)}\psi_{k+1}((x_1,0)) \\
&= \psi_{k+1}((x_1,0)),
\end{aligned}
$$

whence $C_k=\frac{1}{4}$.

We now investigate the action of the operators $\frac{x_i\frac{\partial}{\partial x_i}-x_j\frac{\partial}{\partial x_j}}{x_i^2-x_j^2}$ that occur in $\mathcal{L}_-$ and in $\mathcal{L}_0$. For i and j fixed, we write the symmetric polynomial ψ_k as a sum (suppressing here the indices k,i and j in order to increase readability)

$$
\psi_k=\sum_{c\geq d\geq 0} p_{c,d}(x)(x_i^{2c}x_j^{2d}+x_i^{2d}x_j^{2c}),
$$

where the $p_{c,d}$ are symmetric polynomials in the variables other than x_i and x_j. The operator then acts on the second factor of each term, and

$$
\begin{aligned}
&\frac{x_i\frac{\partial}{\partial x_i}-x_j\frac{\partial}{\partial x_j}}{x_i^2-x_j^2}(x_i^{2c}x_j^{2d}+x_i^{2d}x_j^{2c}) \\
&=2(c-d)(x_ix_j)^{2d}(x_i^{2(c-d-1)}+\cdots+x_j^{2(c-d-1)}).
\end{aligned}
$$

Evaluating the right hand side at $(x_1, 0)$ (whence $x_i = 0$) yields zero unless $d = 0$, in which case we get $2cx_j^{2(c-1)}$. Therefore,

$$\frac{1}{x_i^2 - x_j^2}\left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right)(\psi_k)((x_1,0)) = \sum_{c=0}^{k} p_{c,0}((x_1,0))(2cx_j^{2(c-1)})((x_1,0)).$$

We now consider two separate cases.

(1) If $j = 1$, then evaluating the polynomial $p_{c,0}$ at a point $(x_1, 0)$ yields zero unless it is a constant polynomial, i.e., unless $c = k$. In this case, $p_{k,0} = (2k)!$.

(2) If $j \neq 1$, then evaluating $p_{c,0} 2cx_j^{2(c-1)}$ at $(x_1, 0)$ gives zero unless $c = 1$, in which case we get the value

$$2p_{1,0}(x_1,0) = 2\left(\frac{k!}{(k-1)!}\right)^2 (2(k-1))!2!x_1^{2k-2} = 4k^2(2(k-1))!x_1^{2k-2}.$$

Hence, we have

$$\sum_{m \geq i > j \geq 1} \frac{1}{x_i^2 - x_j^2}\left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right)(\psi_k)((x_1,0)) = (m-1)2k(2k)!x_1^{2k-2} + \binom{m-1}{2} 4k^2(2(k-1))!x_1^{2k-2}.$$

From this, we conclude that

$$\begin{aligned}
&\mathcal{L}_{-}\psi_k((x_1,0)) \\
= \;& 2\left((m-1)2k(2k)!x_1^{2k-2} + \binom{m-1}{2} 4k^2(2(k-1))!x_1^{2k-2}\right) x_1^{2k-2} \\
&+ \left(2k(2k-1)(2k)! + (m-1)4k^2(2(k-1))!\right) x_1^{2k-2} \\
&+ \left((n-m)2k(2k)! + 4(m-1)(n-m)k^2(2(k-1))!\right) x_1^{2k-2} \\
&+ (4(m^2-4m+4) + 4(n-m)(m-2))(2(k-1))!x_1^{2k-2},
\end{aligned}$$

and hence

$$\begin{aligned}
A_k &= 4k^4 + (4(m-2) + 2(n-m))k^3 \\
&\quad + ((m^2-4m+4) + (n-m)(m-2))k^2.
\end{aligned}$$

Similarly, we see that

$$\begin{aligned}\mathcal{L}_0\psi_k((x_1,0)) &= (-mn+(-(n-m)-4)2k-4k(2k-1))(2k)!x_1^{2k}\\ &\quad -2(m-1)2k(2k)!x_1^{2k}\\ &= (-8k^2+(-4(m-1)-2(n-m)-4)k-mn)\psi_k((x_1,0)),\end{aligned}$$

and hence the value of B_k. □

Theorem 8. *The Hilbert space $\mathscr{H}_1^L$ is isometrically isomorphic to the Hilbert space $L^2(\Sigma,\mu)$, where*

$$\Sigma=(0,\infty)\cup\{i(\frac{1}{2}-\frac{n-m}{4}+k)|k\in\mathbb{N},\frac{1}{2}-\frac{n-m}{4}+k<0\},$$

and μ is the measure defined by

$$\int_\Sigma f d\mu=\frac{1}{2\pi}\int_0^\infty\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 f(x)dx+ \tag{65}$$
$$\frac{\Gamma(a+c)\Gamma(c+b)\Gamma(b-c)\Gamma(a-c)}{\Gamma(-2c)}\times\sum_{\substack{j\in\mathbb{N}\\ c+j<0}}\frac{(2c)_j(c+1)_j(c+b)_j(c+a)_j}{(c)_j(c-b+1)_j(c-a+1)_j}(-1)^j$$
$$\times f(-(c+j)^2),$$

where the constants $a,b,$ and c are given by

$$\begin{aligned}a &= \frac{m-1}{2}+\frac{n-m}{4},\\ b &= \frac{1}{2}+\frac{n-m}{4},\\ c &= \frac{1}{2}-\frac{n-m}{4}.\end{aligned} \tag{66}$$

Under the isomorphism, the operator $\mathcal{L}^1$ corresponds to the multiplication operator $f\mapsto-(a^2+x^2)f$.

Proof. We recall the continuous dual Hahn polynomials, $S_k(x^2;a,b,c)$, (cf. [16]) defined by

$$\frac{S_k(x^2;a,b,c)}{(a+b)_k(a+c)_k}={}_3F_2\left(\begin{matrix}-k,a+ix,a-ix\\ a+b,a+c\end{matrix}|1\right). \tag{67}$$

Here, $(\cdot)_k$ denotes the *Pochhammer symbol* defined as

$$\begin{aligned}(t)_0 &= 1,\\ (t)_k &= t(t+1)\cdots(t+k-1),\ k\in\mathbb{N}^+.\end{aligned}$$

Suppressing the parameters and denoting the left hand side above by $\tilde{S}_k(x^2)$, these polynomials satisfy the recurrence relation

$$-(a^2+x^2)\tilde{S}_k(x^2) = A'_k\tilde{S}_{k-1}(x^2) + B'_k\tilde{S}_k(x^2) + C'_k\tilde{S}_{k+1}(x^2), \tag{68}$$

where the recursion constants A'_k, B'_k, and C'_k are given by

$$A'_k = k(k+b+c-1), \tag{69}$$

$$C'_k = (k+a+b)(k+a+c), \tag{70}$$

$$B'_k = -(A'_k + C'_k). \tag{71}$$

Under a renormalisation of the form

$$S_k(x^2,a,b,c) \mapsto \alpha_k S_k(x^2,a,b,c) := S_k(x^2,a,b,c)^\alpha,$$

where α_k is some sequence of complex numbers, the corresponding polynomials $\tilde{S^\alpha}_k$ will also satisfy a recurrence relation of the type in (68), with constants, $A^\alpha_k, B^\alpha_k, C^\alpha_k$, given by

$$A^\alpha_k = \frac{\alpha_k}{\alpha_{k-1}}A'_k, \tag{72}$$

$$B^\alpha_k = B'_k, \tag{73}$$

$$C^\alpha_k = \frac{\alpha_k}{\alpha_{k+1}}C'_k. \tag{74}$$

From this we can see that the product $A'_{k+1}C'_k = A^\alpha_{k+1}C^\alpha_k$ is invariant.

Consider now the continuous dual Hahn polynomials with $S_k(x^2;a,b,c)$, with the parameters a,b,c from (66). These polynomials satisfy the orthogonality relation (cf. [16])

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 S_k(x^2;a,b,c)S_l(x^2;a,b,c)dx\\
&+\frac{\Gamma(a+c)\Gamma(c+b)\Gamma(b-c)\Gamma(a-c)}{\Gamma(-2c)}\\
&\times\sum_{\substack{j\in\mathbb{N}\\ c+j<0}}\frac{(2c)_j(c+1)_j(c+b)_j(c+a)_j}{(c)_j(c-b+1)_j(c-a+1)_j}(-1)^j\\
&\times S_k(-(c+j)^2;a,b,c)S_l(-(c+j)^2;a,b,c)\\
&=\Gamma(k+a+b)\Gamma(k+a+c)\Gamma(k+b+c)k!\delta_{kl}.
\end{aligned} \tag{75}$$

By a straightforward computation one sees that the corresponding constants A'_k, B'_k, and C'_k are related to the Jacobi constants A_k, B_k, and C_k in (65) by

$$\begin{aligned} A_{k+1}C_k &= A'_{k+1}C'_k, \\ B_k &= B'_k. \end{aligned}$$

We can thus use (74) to define a sequence α_k recursively in such a way that the resulting polynomials $\tilde{S}^\alpha_k$ satisfy the recurrence relation

$$(76) \quad -(a^2+x^2)\tilde{S}^\alpha_k(x^2) = A_k\tilde{S}^\alpha_{k-1}(x^2) + B_k\tilde{S}^\alpha_k(x^2) + C_k\tilde{S}^\alpha_{k+1}(x^2)$$

with the same Jacobi constants as the operator $4\mathcal{L}^1$. More precisely, we set

$$\alpha_0 := \left(\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\right)^{-1/2}, \tag{77}$$

$$\alpha_{k+1} := \frac{1}{4}\left(k+\frac{m}{2}\right)\left(k+\frac{n}{2}\right)^{-1}\alpha_k. \tag{78}$$

Then $\alpha_k = \left(\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})\right)^{-1/2} 4^k \left(\frac{m}{2}\right)_k \left(\frac{n}{2}\right)_k$, and hence, by (75), we have

$$\|\tilde{S}^\alpha_k\|^2_{L^2} = 4^{2k}(k!)^2\left(\frac{m}{2}\right)_k\left(\frac{n}{2}\right)_k \tag{79}$$

$$= \|\psi_k\|^2_1. \tag{80}$$

Therefore, the operator $T_0 : \mathscr{H}^L_1 \to L^2(\Sigma, d\mu)$ defined by

$$T_0\psi_k = \tilde{S}^\alpha_k \tag{81}$$

is a unitary operator which diagonalises the restriction of the operator $\mathcal{L}^1$ to $\mathscr{H}^L_1$. □

Theorem 9. *For each $x \in \Sigma$, there exists a Hilbert space $\mathscr{H}_x$ and an irreducible unitary spherical representation, π_x, of H on $\mathscr{H}_x$ such that*

(1) *If $v_x \in \mathscr{H}_x$ is the canonical spherical vector, then there is an isometric embedding of Hilbert spaces $L^2(\Sigma,\mu) \subset \int_\Sigma \mathscr{H}_x d\mu(x)$ given by*

$$f \mapsto s_f,$$

where $s_f(x) := f(x)v_x$.

(2) *The operator T_0 extends uniquely to an H-intertwining unitary operator*

$$T : (\pi_1, \mathscr{H}_1) \to \left(\int_\Sigma \pi_x d\mu(x), \int_\Sigma \mathscr{H}_x d\mu(x)\right). \tag{82}$$

Proof. The Banach algebra $L^1(H)$ equipped with convolution as multiplication carries the structure of a Banach *-algebra when the involution * is defined as $f^*(h) = \overline{f(h^{-1})}$. The representation π_1 of H induces a representation of $L^1(H)$ by

$$\pi(f) = \int_H f(h)\pi_1(h)dh. \tag{83}$$

If $L^1(H)^\#$ denotes the subalgebra of left and right L-invariant L^1-functions, the closed C^*-algebra generated by $\pi_1(L^1(H)^\#)$ and the identity operator is a commutative C^*-algebra. Moreover, the Casimir operator $\pi_1(\mathcal{C})$ commutes with all the operators $\pi_1(f)$ for $f \in L^1(H)^\#$. Hence, (by [1],Vol. I, Thm 1, p. 77), the diagonalisation of the Casimir operator yields a simultaneous diagonalisation of the whole commutative algebra $\pi_1(L^1(H)^\#)$.

For $f \in L^1(H)^\#$, we let the function $\tilde{f} : \Sigma \to \mathbb{C}$ be the multiplier corresponding to the operator $T\pi_1(f)T^{-1} : L^2(\Sigma, \mu) \to L^2(\Sigma, \mu)$. For each $x \in \Sigma$, we let λ_x denote the multiplicative functional

$$\lambda_x(f) := \tilde{f}(x), \tag{84}$$

which clearly is bounded almost everywhere with respect to μ. The equality

$$\langle \pi_1(f)\varphi_0, \varphi_0\rangle_1 = \int_\Sigma \lambda_x(f)d\mu(x)$$

holds for $f \in L^1(H)^\#$, i.e., the positive functional

$$\Phi_0(f) := \langle \pi_1(f)\varphi_0, \varphi_0\rangle_1, f \in L^1(H)^\# \tag{85}$$

is expressed as an integral of characters.

By [28] (Thm. 10) there exists a direct integral decomposition into unitary spherical irreducible representations of the form (82), and it expresses the functional Φ_0 as an integral of characters against the corresponding measure. This measure is supported on the characters given by positive definite spherical functions. By [25] (Thm. 11.32), such an integral expression for bounded positive functionals is unique, and hence every character λ_x can be expressed by a positive definite spherical function ϕ_x as

$$\lambda_x(f) = \int_H f(h)\phi_x(h)dh.$$

The rest now follows from the proof of Thm. 10 in [28].

□

5. A subrepresentation of $\pi_1|_H$

Recall that the boundary $\partial\mathscr{D}$ is the disjoint union of m G-orbits. More specifically, for $j = 1, \dots, m$, let e_j denote the $n \times m$ matrix with 1 at position (j,j) and all other entries zero. Then

$$\partial\mathscr{D} = \bigcup_{r=1}^{m} G(e_1 + \cdots + e_r)$$

and the inclusion

$$\overline{G(e_1 + \cdots + e_{r+1})} \subseteq G(e_1 + \cdots + e_r)$$

holds for $r = 1, \dots, m-1$. The Shilov boundary is the G-orbit of the rank m partial isometry $e_1 + \cdots + e_m$. It is also the K-orbit of this element. We consider now the "real part", Y, of the Shilov boundary, i.e.,

$$Y := \mathcal{S} \cap M_{nm}(\mathbb{R}). \tag{86}$$

Then Y is the homogeneous space H/P_0, where P_0 is the maximal parabolic subgroup defined by the one dimensional subalgebra

$$\mathfrak{a}_0 = \mathbb{R}(E_1 + \cdots + E_m)$$

of $\mathfrak{a}$ (cf. (28)). We let $P_0 = M_0A_0N_0$ be the Langlands decomposition. Then Y can also be described as a homogeneous space $Y = L/L \cap M_0$. Consider the one dimensional representation with character

$$l \mapsto |\det \mathrm{Ad}^{-1}_{\mathfrak{l}/\mathfrak{l}\cap\mathfrak{m}_0}(l)| \tag{87}$$

of $L \cap M_0$. The induced representation $\mathrm{Ind}^{L}_{L\cap M_0}(|\det \mathrm{Ad}^{-1}_{\mathfrak{l}/\mathfrak{l}\cap\mathfrak{m}_0}|)$ is realised on the space of sections of the density bundle of $Y = L/L \cap M_0$. The representation (87) is in fact trivial, and this allows us to define an L-invariant section, ω, by

$$\omega(l(L \cap M_0)) := l_{e(L\cap M_0)}\omega_0, \tag{88}$$

where $\omega_0 \neq 0 \in \mathcal{D}(T_{e(L\cap M_0)})$ is arbitrary, where $\mathcal{D}(T_{e(L\cap M_0)})$ denotes the vector space of densities on $T_{e(L\cap M_0)}$. The section ω then corresponds to a constant function $F_\omega : L \to \mathbb{C}$. In the usual way, we will sometimes identify ω with the measure it defines by integration against continuous functions. We then use measure theoretic notation and write $\int_Y \varphi\, d\omega$ for $\int_Y \varphi\, \omega$. Moreover, we choose ω_0 in (88) so that this measure is normalised.

Using the identification $\mathfrak{l}/\mathfrak{l} \cap \mathfrak{m}_0 \simeq \mathfrak{h}/\mathfrak{p}_0$, the representation (87) extends to the representation δ_0 of $\mathfrak{p}_0$ given by

$$\delta_0(m_0 a_0 n_0) = |\det(\mathrm{Ad}_{\mathfrak{h}/\mathfrak{p}_0}(m_0 a_0 n_0)^{-1}|. \tag{89}$$

Clearly, $\delta_0(m_0 a_0 n_0) = e^{2\rho_0(\log a_0)}$, where ρ_0 denotes the half sum of the restricted roots. The action of H as pullbacks (actually, the inverse mapping composed with pullback) on densities is equivalent to the left action defined by the representation $\mathrm{Ind}_{P_0}^H(\delta_0)$. For the extension of the function F_ω to a P_0 equivariant function $H \to \mathbb{C}$ (which we still denote by F_ω), we then have

$$F_\omega(k_0 m_0 a_0 n_0) = e^{-2\rho_0(\log a_0)} F_\omega(k_0) = e^{-2\rho_0(\log a_0)} F_\omega(e). \tag{90}$$

From this, it follows that

$$h^*\omega(l(L \cap M_0) = e^{-2\rho_0(\log A_0(hl)} \omega(l(L \cap M_0)). \tag{91}$$

The action of H on Y can either be described on the coset space H/P_0 in terms of the Langlands decomposition for P_0, or in terms of the geometric action on the boundary of $\mathscr{D}$ defined by the Harish-Chandra decomposition. The next proposition expresses the transformation of ω under H in terms of the latter description.

Lemma 10. *The density ω transforms under the action of H as*

$$h^*\omega(v) = J_h(v)^{\left(\frac{n-1}{n+m}\right)} \omega(v). \tag{92}$$

Proof. The idea of the proof is to use the (non-unique) factorisation $H = LM_0A_0N_0$ of H. We prove that the group N_0 fixes the reference point $e_1 + \cdots + e_m$ and acts with Jacobian equal to one on the tangent space at $e_1 + \cdots + e_m$, and the group elements in M_0 have Jacobian equal to one at $e_1 + \cdots + e_m$. By the chain rule for differentiation, it then suffices to prove the statement for all group elements in A_0.

In the Langlands decomposition $\mathfrak{p}_{\min} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ for the minimal parabolic subgroup, the subalgebra $\mathfrak{n}$ is generated by the restricted root spaces

$$\begin{aligned}
\bigoplus_{m\geq j>i\geq 1} \mathfrak{h}_{E_j^*+E_i^*} &= \left\{ X_q = \begin{pmatrix} -q & 0 & q \\ 0 & 0 & 0 \\ -q & 0 & q \end{pmatrix} | q^t = -q \right\}, \\
\bigoplus_{m\geq j>i\geq 1} \mathfrak{h}_{E_j^*-E_i^*} &= \left\{ X_u = \begin{pmatrix} u^t - u & 0 & u + u^t \\ 0 & 0 & 0 \\ u + u^t & 0 & u^t - u \end{pmatrix} | u \text{ is upper triang.} \right\}, \\
\bigoplus_{j=1}^{m} \mathfrak{h}_{E_j^*} &= \left\{ X_z = \begin{pmatrix} 0 & z^t & 0 \\ -z & 0 & z \\ 0 & z^t & 0 \end{pmatrix} \right\},
\end{aligned}$$

where the matrices are written in blocks in such a way that the block-rows are of height $m, n-m$, and m respectively, and the block-columns are of width $m, n-m$, and m respectively.

In the Langlands decomposition $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$, the centraliser, $\mathfrak{m}_0$ of $\mathfrak{a}_0$ is the direct sum

$$\mathfrak{m}_0 = \mathfrak{m} \oplus \bigoplus_{m\geq j>i\geq 1} \mathfrak{h}_{E_j^*-E_i^*},$$

and

$$\mathfrak{n}_0 = \bigoplus_{m\geq j>i\geq 1} \mathfrak{h}_{E_j^*+E_i^*} \oplus \bigoplus_{j=1}^{m} \mathfrak{h}_{E_j^*}. \tag{93}$$

The matrices X_q and X_z commute, so in order to prove that the elements in N_0 have Jacobian equal to one at $e_1 + \cdots + e_m$, it suffices to consider elements of the form

$$\begin{aligned}
\exp X_q &= \begin{pmatrix} 1-q & 0 & q \\ 0 & 1 & 0 \\ -q & 0 & 1+q \end{pmatrix}, \\
\exp X_z &= \begin{pmatrix} 1 - \frac{z^t z}{2} & z^t & \frac{z^t z}{2} \\ -z & 1 & z \\ -\frac{z^t z}{2} & z^t & 1 + \frac{z^t z}{2} \end{pmatrix}
\end{aligned}$$

separately.

We have

$$\begin{aligned}\exp X_q \exp(e_1+\cdots+e_m) &= \begin{pmatrix} 1-q & 0 & q \\ 0 & 1 & 0 \\ -q & 0 & 1+q \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-q & 0 & 1 \\ 0 & 1 & 0 \\ -q & 0 & 1 \end{pmatrix}.\end{aligned}$$

If we write this matrix in the block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then the $K^{\mathbb{C}}$-component in the Harish-Chandra decomposition is given by

$$\begin{pmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{pmatrix} = I_{n+m},$$

and hence

(94) $$J_{\exp X_q}(e_1+\cdots+e_m) = 1.$$

Next, we consider the action of $\exp X_z$. We have

$$\exp X_z \exp(e_1+\cdots+e_m) = \begin{pmatrix} 1-\frac{z^t z}{2} & z^t & 1 \\ -z & 1 & 0 \\ -\frac{z^t z}{2} & z^t & 1 \end{pmatrix}.$$

Here, the $K^{\mathbb{C}}$-component is given by

$$K^{\mathbb{C}}(\exp X_z \exp(e_1+\cdots+e_m)) = \begin{pmatrix} 1 & 0 & 0 \\ -z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The complex differential of $\exp X_z$ at $e_1+\cdots+e_m$ is then the linear mapping

(95) $$d\exp X_z(e_1+\cdots+e_m)Y = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} Y,$$

where we have identified the tangent spaces with $\mathfrak{p}^+ = M_{nm}(\mathbb{C})$. Clearly, the determinant of this mapping is

(96) $$\det \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}^m = 1.$$

Consider now the subgroup M_0. Its Lie algebra $\mathfrak{m}_0$ is reductive with Cartan involution given by the restriction of θ and the corresponding decomposition is

$$\mathfrak{m}_0 = \mathfrak{m}_0 \cap \mathfrak{l} \oplus \mathfrak{m}_0 \cap \mathfrak{q}.$$

The abelian subalgebra $\mathfrak{a}$ is included in $\mathfrak{m}_0 \cap \mathfrak{q}$, and therefore (cf. [10], Prop. 7.29)

$$\mathfrak{m}_0 \cap \mathfrak{q} = \bigcup_{l \in M_0 \cap L} \mathrm{Ad}(l)\mathfrak{a}. \tag{97}$$

We now investigate the Jacobians of arbitrary group elements in A. For $H = t_1 E_1 + \cdots + t_m E_m$,

$$\exp H = \begin{pmatrix} \Delta(\underline{\cosh t}) & 0 & \Delta(\underline{\sinh t}) \\ 0 & 1 & 0 \\ \Delta(\underline{\sinh t}) & 0 & \Delta(\underline{\cosh t}) \end{pmatrix},$$

where $\Delta(\underline{\cosh t})$ denotes the $m \times m$ diagonal matrix with entries $\cosh t_1, \ldots, \cosh t_m$, and the other blocks are analogously defined. Then

$$\begin{aligned} &\exp(t_1 E_1 + \cdots + t_m E_m) \exp(e_1 + \cdots + e_m) \\ &= \begin{pmatrix} \Delta(\underline{\cosh t}) & 0 & \Delta(\underline{\cosh t + \sinh t}) \\ 0 & 1 & 0 \\ \Delta(\underline{\sinh t}) & 0 & \Delta(\underline{\cosh t + \sinh t}) \end{pmatrix} \end{aligned}$$

The $K^{\mathbb{C}}$-component is

$$\begin{aligned} &K^{\mathbb{C}}(\exp(t_1 E_1 + \cdots + t_m E_m) \exp(e_1 + \cdots + e_m)) \\ &= \begin{pmatrix} \Delta(\underline{e^{-t}}) & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \Delta(\underline{e^{t}}) \end{pmatrix}, \end{aligned}$$

so the differential $d(\exp(t_1 E_1 + \cdots + t_m E_m))(e_1 + \cdots + e_m)$ is the mapping

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \mapsto \begin{pmatrix} \Delta(\underline{e^{-2t}}) Y_1 \\ Y_2 \Delta(\underline{e^{-t}}) \end{pmatrix},$$

where Y_1 is the upper $m \times m$ block of the $n \times m$ matrix in the tangent space. Counting the multiplicities of the eigenvalues e^{-t_j}, we see that

$$J_{\exp(t_1 E_1 + \cdots + t_m E_m)}(e_1 + \cdots + e_m) = e^{-(n+m)\sum_{j=1}^{m} t_j}. \tag{98}$$

If we write $\mathfrak{a}$ as the orthogonal sum $\mathfrak{a} = \mathfrak{a}_0 \oplus (\mathfrak{a}_0)^{\perp}$ (with respect to the Killing form), then $(\mathfrak{a}_0)^{\perp}$ consists of those $t_1 E_1 + \cdots + t_m E_m$ in $\mathfrak{a}$

for which $\sum_{j=1}^{m} t_j = 0$. From the identities (94), (96), (97), and (98) we can thus conclude that

$$J_h(e_1 + \cdots + e_m) = J_{A_0(h)}(e_1 + \cdots + e_m). \tag{99}$$

On the other hand, by (93),

$$\begin{aligned} &2\rho_0(t(E_1 + \cdots + E_m)))) \\ &= 2\frac{m(m-1)}{2}t + m(n-m)t = m(n-1)t, \end{aligned} \tag{100}$$

so

$$\begin{aligned} &e^{-2\rho_0(t(E_1+\cdots+E_m))} \\ &= \left(J_{\exp(t(E_1+\cdots+E_m))}(e_1 + \cdots + e_m)\right)^{\frac{n-1}{n+m}}. \end{aligned} \tag{101}$$

□

In what follows, we will define a Hilbert space of functions on the manifold Y. Hilbert spaces of a similar kind were also considered by Neretin and Olshanski in [19]. One difference is that their spaces were not defined using a limit procedure (see the next definition below).

We begin by introducing some notation. For a continuous function, f, on Y and $r \in (0,1)$, we define the function $F_r : Y \to \mathbb{C}$ by

$$F_r(u) := \int_Y f(v) \det(I_n - ruv^t)^{-1} d\omega(v). \tag{102}$$

We construct the Hilbert space by requiring that the following space of functions be dense.

Definition 11. Let $\mathscr{C}_0$ denote the set of all continuous functions $f : Y \to \mathbb{C}$ such that the limit function

$$F(u) := \lim_{r \to 1} F_r(u)$$

exists in the supremum norm.

On $\mathscr{C}_0$ we define a sesquilinear form $\langle \ , \ \rangle_{\mathscr{C}_0}$ by

$$\langle f, g \rangle_{\mathscr{C}_0} = \int_Y f(u)\overline{G(u)} d\omega(u). \tag{103}$$

By the Dominated Convergence Theorem, we have

$$\begin{aligned} &\int_Y f(u)\overline{G(u)} d\omega(u) \\ &= \lim_{r \to 1} \int_Y f(u) \int_Y \overline{g(v)} \det(I_n - ruv^t)^{-1} d\omega(v) d\omega(u), \end{aligned} \tag{104}$$

and hence the form $\langle \quad , \quad \rangle_{\mathscr{C}_0}$ is positive semidefinite. Let $\mathcal{N}$ denote the space of functions of norm zero, i.e.,

$$\mathcal{N} = \{f \in \mathscr{C}_0 | \langle f, f\rangle_{\mathscr{C}_0} = 0\}. \tag{105}$$

Then the quotient space $\mathscr{C}_0/\mathcal{N}$ together with the induced sesquilinear form, $\widetilde{\langle \quad , \quad \rangle}_{\mathscr{C}_0}$, is a pre-Hilbert space. We define $\mathscr{C}$ to be the Hilbert space completion of $\mathscr{C}_0$ with respect to $\widetilde{\langle \quad , \quad \rangle}_{\mathscr{C}_0}$. We denote the inner product on $\mathscr{C}$ by $\langle \quad , \quad \rangle_{\mathscr{C}}$.

Proposition 12. *The action τ of H on $\mathscr{C}_0$ given by*

$$\tau(h)f(\eta) := J_{h^{-1}}(\eta)^{\beta} f(h^{-1}\eta), \tag{106}$$

where $\beta = \frac{n-2}{n+m}$, descends to a unitary representation of H on $\mathscr{C}$.

Proof. It suffices to prove that the dense subspace $\mathscr{C}_0/\mathcal{N}$ of $\mathscr{C}$ is H-invariant and that the action is unitary on $\mathscr{C}_0/\mathcal{N}$. For this, it clearly suffices to prove that the space $\mathscr{C}_0$ is H-invariant, and that H preserves the sesquilinear form $\langle \quad , \quad \rangle_{\mathscr{C}_0}$, since then the subspace $\mathcal{N}$ is also H-invariant.

Consider first the mapping $f \mapsto F$ in Definition 11. We write K_1 for the reproducing kernel. For $h \in H$, we then have

$$\begin{aligned}\int_Y \tau(h)f(v)K_1(ru,v)d\omega(v) &= \int_Y J_h(h^{-1}v)^{-\beta} f(h^{-1})K_1(ru_1,v)d\omega(v)\\ &= \int_Y J_h(v')^{-\beta+\frac{n-1}{n+m}} f(v')K_1(ru_1,hv')d\omega(v'),\end{aligned}$$

by the transformation property for the measure ω. By the transformation rule for the reproducing kernel K_1, we have

$$\begin{aligned}&\int_Y J_h(v')^{-\beta+\frac{n-1}{n+m}} f(v')K_1(ru_1,hv')d\omega(v')\\ &= \int_Y J_h(h^{-1}ru)^{-\frac{1}{n+m}} f(v')K_1(h^{-1}ru,v')d\omega(v').\end{aligned}$$

Therefore,

$$\lim_{r\to 1}\int_Y \tau(h)f(v)K_1(ru,v)d\omega(v) = J_h(h^{-1}u)^{-\frac{1}{n+m}} F(h^{-1}u),$$

where the convergence is uniform in u, so $\mathscr{C}_0$ is H-invariant.

Next, take $f, g \in \mathscr{C}_0$. Then, $\langle \tau(h)f, \tau(h)g\rangle_{\mathscr{C}_0}$ is given by

$$\begin{aligned}\langle \tau(h)f, \tau(h)g\rangle_{\mathscr{C}_0} &= \int_Y J_h(h^{-1}u)f(h^{-1}u)J_h(h^{-1}u)^{-\frac{1}{n+m}}\overline{G(h^{-1}u)}d\omega(u) \\ &= \int_Y f(v')\overline{G(v')}d\omega(v') \\ &= \langle f, g\rangle_{\mathscr{C}_0},\end{aligned}$$

where the second equality follows from the transformation property of ω. □

The next proposition gives a sufficient condition for the Hilbert space $\mathscr{C}$ to be nonzero.

Proposition 13. *The (equivalence class modulo $\mathcal{N}$ of the) constant function* 1 *belongs to the pre-Hilbert space $\mathscr{C}_0/\mathcal{N}$ if and only if $n-m >$* 2.

Proof. Recall that the reproducing kernel has a series expansion

$$\det(I_n - zw^*)^{-1} = \sum_{k=0}^{\infty} k! K_k(z, w),$$

where $K_k(z, w)$ is the reproducing kernel with respect to the Fock-Fischer norm for the K-type indexed by k. The functions

$$z \mapsto \int_Y K_{2k}(z, v)d\omega$$

are then L-invariant vectors in the K-type $2k$ and hence differ from the L-invariants ψ_k by some constants depending on k. We determine these by computing the integrals for a suitable choice of z.

Before we begin with the computations, consider the fibration

$$p : Y \to S^{n-1}, p(v) = v(e_1).$$

For $u \in S^{n-1}$, the fibre $p^{-1}(u)$ can be identified with the set of all rank $m-1$ partial isometries from $\mathbb{R}^m$ to $(\mathbb{R}u)^{\perp}$. Moreover, p is equivariant with respect to the actions of $O(n)$ on Y and S^{n-1}. Hence the equality

$$\int_{S^{n-1}} f d\sigma = \int_Y f \circ p \, d\omega, \tag{107}$$

where σ denotes the normalised rotation invariant measure on S^{n-1}, holds for all $f \in C(S^{n-1})$.

Choose now $z = \lambda e_1$, where $0 < \lambda < 1$. Since zv^t is a matrix of rank one, $\det(I_n - zv^t)^{-1} = (1 - \mathrm{tr}(zv^t))^{-1}$. Hence

$$\int_Y (1 - \mathrm{tr}(zv^t))^{-1} d\omega = \int_Y (1 - \lambda v_{11})^{-1} d\omega = \int_Y (1 - \lambda p(v)_1)^{-1} d\omega.$$

By (107), we have

$$\int_Y (1 - \lambda p(v)_1)^{-1} d\omega = \int_{S^{n-1}} (1 - \lambda u_1)^{-1} d\sigma(u).$$

Moreover,

$$\int_{S^{n-1}} (1 - \lambda u_1)^{-1} d\sigma(u) = \sum_{j=0}^{\infty} \lambda^j \int_{S^{n-1}} u_1^j d\sigma(u).$$

The integrands on the right hand side depend only on the first coordinate, and hence the integrals can be written as integrals over the open interval $(-1, 1)$ in $\mathbb{R}$ (cf. [24] 1.4.4.). In fact,

$$\int_{S^{n-1}} u_1^j d\sigma(u) = \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \int_{-1}^{1} (1 - x^2)^{(n-2)/2-1} x^j dx.$$

This integral is zero for odd j, and for $j = 2k$, we have

$$\int_{-1}^{1} (1 - x^2)^{(n-2)/2-1} x^j dx = B\left(\frac{2k+1}{2}, \frac{n-1}{2}\right) := \frac{\Gamma(\frac{2k+1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{2k+n}{2})}.$$

Therefore,

$$\int_{S^{n-1}} (1 - \lambda u_1)^{-1} d\sigma(u) = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{2k+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{2k+n}{2})} \lambda^{2k}.$$

From this, it follows that for an arbitrary $z \in \mathscr{D}$, we have the expansion

$$\begin{aligned} &\int_Y \det(I_n - zv^t)^{-1} d\omega(v) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{2k+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{2k+n}{2})\Gamma(2k+1)} \psi_k(z). \end{aligned} \tag{108}$$

Since the functions ψ_k are L-invariant, they are constant on the set $\{ru | u \in Y, 0 < r < 1\}$. This value equals

$$\psi_k(ru) = r^{2k}\psi_k(u) = r^{2k} 4^k k! \left(\frac{m}{2}\right)_k. \tag{109}$$

Suppose now that $|r-r'|<\epsilon$. By (108) and (109),

$$\int_Y K_1(ru,v)d\omega(v) - \int_Y K_1(r'u,v)d\omega(v)$$
$$= \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{2k+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{2k+n}{2})\Gamma(2k+1)} 4^k k! \left(\frac{m}{2}\right)_k (r^{2k} - (r')^{2k}),$$

and hence we have the estimate

$$(110) \quad \left| \int_Y K_1(ru,v)d\omega(v) - \int_Y K_1(r'u,v)d\omega(v) \right| \leq$$
$$\epsilon \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{2k+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{2k+n}{2})\Gamma(2k+1)} 4^k k! \left(\frac{m}{2}\right)_k .$$

Applying Sterling's formula to the kth term on the right hand side yields

$$(111) \qquad \frac{\Gamma(\frac{n}{2})\Gamma(\frac{2k+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{2k+n}{2})\Gamma(2k+1)} 4^k k! \left(\frac{m}{2}\right)_k = O(k^{-\frac{n-m}{2}}).$$

Hence, the sum in (110) converges if and only if $n-m>2$. In this case, the corresponding net $\left\{\int_Y K_1(r\cdot,v)d\omega\right\}_r$ is Cauchy in the supremum norm, and hence converges uniformly. □

Lemma 14. *Consider the representation τ in (106). On the space of continuous functions on Y, it is equivalent to the representation $Ind_P^H(1 \otimes (i\lambda+\rho) \otimes 1)$, where P is the minimal parabolic subgroup defined by the maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, and $\lambda \in (\mathfrak{a}^{\mathbb{C}})^*$ is defined as*

$$(112) \qquad -(i\lambda+\rho)|_{\mathfrak{a}_0} = -\frac{2(n-2)}{n-1}\rho_0,$$
$$(113) \qquad -(i\lambda+\rho)|_{\mathfrak{a}_0^{\perp}} = 0.$$

In fact, when the continuous functions on Y are identified with right $L\cap M_0$-invariant functions on L, we can extend them to functions on H in such a way that the two representations are equal in this setting.

Proof. By (101), we can rewrite the action of H in (106) as

$$(114) \qquad \tau(h)f(x) = e^{-\frac{2(n-2)}{n-1}\rho_0(\log A_0(g^{-1}x))} f(\kappa(g^{-1}x)),$$

where

$$g^{-1}x = \kappa(g^{-1}x)m_0(g^{-1}x)A_0(g^{-1}x)n_0(g^{-1}x) \in LM_0A_0N_0.$$

We now let $\lambda \in (\mathfrak{a}^{\mathbb{C}})^*$ be defined by the requirements (112) and (113).

By (113), $-(i\lambda + \rho)$ has to annihilate all the restricted root spaces $\mathfrak{h}_{E_j^* - E_i^*}$, and hence be of the form $c(E_1^* + \cdots + E_m^*)$ for some constant c. By (100) it follows that $c = -m(n-2)$.

Consider now the parabolically induced representation

$$\mathrm{Ind}_P^H(1 \otimes \exp(i\lambda + \rho) \otimes 1)$$

acting on on continuous functions on H. By definition, this representation is defined on the space of continuous functions $f : H \to \mathbb{C}$ having the P-equivariant property

$$f(xman) = e^{-i(\lambda+\rho)(\log a)} f(x). \tag{115}$$

The action of H is given by

$$f \overset{h}{\mapsto} e^{-(i\lambda+\rho)(A(h^{-1}x))} f(\kappa(h^{-1}x)). \tag{116}$$

On the other hand, the restriction of the representation τ to the space of continuous functions on Y coincides with the H-action defined by the parabolically induced representation $\mathrm{Ind}_{P_0}^H(\exp)$. Since $P \subset P_0$, and

$$e^{-(i\lambda+\rho)(\log A(x))} = e^{-(i\lambda+\rho)(\log A_0(x))}, \tag{117}$$

it follows that

$$\tau(h)f(x) = e^{-(i\lambda+\rho)(\log A(h^{-1}x))} f(\kappa(h^{-1}x), \tag{118}$$

where f is the extension of a continuous function on Y to a P_0-equivariant function on H. This finishes the proof. □

Proposition 15. *The operator $T : \mathscr{C}_0 \to \mathcal{O}(\mathscr{D})$ defined by*

$$Tf(z) = \int_Y f(v) \det(I_n - zv^t)^{-1} d\omega(v)$$

is H-equivariant.

Proof. We have

$$\begin{aligned} T(\tau(h)f)(z) &= \int_Y J_h(h^{-1}v)^\beta f(h^{-1}v) K_1(z,v) d\omega \\ &= \int_Y J_h(s)^{\beta + \frac{n-1}{n+m}} f(s) K_1(z, hs) d\omega \\ &= J_h(h^{-1}z)^{-\frac{1}{n+m}} \int_Y J_h(s)^{\beta + \frac{n-1}{n+m} - \frac{1}{n+m}} f(s) K_1(h^{-1}z, s) d\omega \\ &= \pi_1(h)(Tf)(z). \end{aligned}$$

□

Corollary 16. *The function $T1$ is a joint eigenfunction for all operators $\pi_1(Z)$, $Z \in Z(U(\mathfrak{h}^{\mathbb{C}}))$. In particular, it is an eigenfunction for the Casimir operator, $\pi_1(\mathcal{C})$, with eigenvalue $-\frac{m(n-2)}{4}$.*

Proof. By Lemma 14, we can identify the extension of constant function 1 on Y to a function on H with the *Harish-Chandra e-function* $e_\lambda : H \to \mathbb{C}$ given by

$$e_\lambda(h) = e^{-i(\lambda+\rho)(\log A(h))}. \tag{119}$$

Moreover, the representation $\mathrm{Ind}_P^H(1 \otimes \exp(i\lambda+\rho) \otimes 1)$ has infinitesimal character $i\lambda+\rho$ (cf. [10], Ch. VIII). The value of the Casimir element is $-(i\lambda+\rho)(\mathcal{C}) = -(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle) = -\frac{m(n-2)}{4}$ (cf. [11], Ch. V). □

Proposition 17. *The function $(T1)(z) = \int_Y \det(I_n - zv^t)^{-1} d\omega(v)$ belongs to $\mathscr{H}_1$.*

Proof. We rewrite the series expansion in (108) using the orthonormal basis $\{\varphi_k\}$, i.e.,

$$\int_Y K_1(z,v)d\omega(v) = \sum_{k=0}^{\infty} \alpha_k \varphi_k(z), \tag{120}$$

where $\alpha_k = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{2k+1}{2})4^k k! \left(\frac{n}{2}\right)_{2k}^{1/2} \left(\frac{m}{2}\right)_{2k}^{1/2}}{\Gamma(\frac{1}{2})\Gamma(\frac{2k+n}{2})\Gamma(2k+1)}$. By Sterling's formula

$$\alpha_k^2 = O(k^{-(n-m)/2}), \tag{121}$$

and hence the series $\sum_k \alpha_k^2$ converges if and only if $n - m > 2$. □

The operator T maps the H-span (the set of all finite linear combinations $c_1\tau(h_1)1 + \cdots + c_N\tau(h_1)1, h_i \in H, c_i \in \mathbb{C}$) of the function 1 into $\mathscr{H}_1$. We introduce the temporary notation $H \cdot 1$ to denote this subspace. Moreover, we let $\mathcal{N}_{H \cdot 1} := \mathcal{N} \cap H \cdot 1$.

Proposition 18. *The equality*

$$\langle Tf, Tf \rangle_1 = \langle f, f \rangle_{\mathscr{C}_0} \tag{122}$$

holds for $f \in H \cdot 1$.

Proof. For $f \in H \cdot 1$ and $r \in (0,1)$, consider the function $Tf(r\cdot)$. We have

$$Tf(rz) = \int_Y f(v)K_1(rz,v)d\omega(v) \tag{123}$$

$$= \int_Y f(v)K_1(z,rv)d\omega(v). \tag{124}$$

The square of the $\mathscr{H}_1$-norm is then given by

$$\|Tp(r\cdot)\|_1^2 = \int_Y \int_Y f(\zeta)\overline{f(\eta)}K_1(r\zeta, r\eta)d\omega(\zeta)d\omega(\eta).$$

These norms are uniformly bounded in r, and hence there is a convergent sequence $\{Tf(r_k\cdot)\}_k$ with respect to the $\mathscr{H}_1$-norm. Since point evaluation functionals are continuous, we also have pointwise convergence, and hence this limit function is Tf. Therefore,

$$\begin{aligned} \|Tf\|_1^2 &= \lim_{k\to\infty} \|Tf(r_k\cdot)\|_1^2 \\ &= \lim_{r\to 1} \int_Y \int_Y f(\zeta)\overline{f(\eta)}K_1(r^2\zeta, \eta)d\omega(\zeta)d\omega(\eta) \\ &= \|f\|_{\mathscr{C}_0}^2. \end{aligned}$$

□

We let T_1 denote the restriction of the operator T to the subspace $H \cdot 1$. Then, we have the following corollary.

Corollary 19. *For the operator $T_1 : H \cdot 1 \to \mathscr{H}_1$,*

$$\ker T_1 = \mathcal{N}_{H\cdot 1}. \tag{125}$$

The operator T_1 then descends to an operator $U_1 : H \cdot 1/\mathcal{N}_{H\cdot 1} \to \mathscr{H}_1$. Now let $\mathcal{H}$ denote the Hilbert space completion of the space $H \cdot 1/\mathcal{N}_{H\cdot 1}$. We keep the letter τ to denote the representation of H of this space (in reality, the representation we mean is derived from τ by first restricting, then descending to a quotient, and, finally, by extending uniquely to a Hilbert space completion).

Proposition 20. *The representation τ of H on $\mathcal{H}$ is irreducible.*

Proof. The representation τ is H-cyclic with a spherical (L-invariant) vector. Hence, there exists a unitary, H-equivariant direct integral decomposition

$$S : \mathcal{H} \to \int_\Lambda \mathcal{H}_\lambda d\mu(\lambda), \tag{126}$$

where Λ is a subset of the bounded spherical functions (or rather, the functionals on $\mathfrak{a}$ that parametrise them), μ is some measure on Λ, and $\mathcal{H}_\lambda$ is the canonical spherical unitary representation corresponding to the spherical function ϕ_λ. For each λ, we let v_λ denote the canonical spherical vector in $\mathcal{H}_\lambda$.

Suppose now that τ is not irreducible, i.e., the set Λ is not a singleton set. Then, we can choose two disjoint open subsets Ω_1, Ω_2 of Λ. We define vectors s_1 and s_2 in the Hilbert space $\int_\Lambda H_\lambda d\mu$ by

$$s_1(\lambda) = \begin{cases} v_\lambda, & \text{if } \lambda \in \Omega_1 \\ 0_\lambda, & \text{otherwise} \end{cases},$$

$$s_2(\lambda) = \begin{cases} v_\lambda, & \text{if } \lambda \in \Omega_2 \\ 0_\lambda, & \text{otherwise} \end{cases}.$$

The vectors $S^{-1}s_1$ and $S^{-1}s_2$ are then linearly independent spherical vectors in $\mathcal{H}$. But, clearly, the only spherical vectors in $\mathcal{H}$ are the (cosets modulo $\mathcal{N}_{H\cdot 1}$ of the) constant functions; a contradiction. □

We are now ready to state a subrepresentation theorem. The proof follows from Prop. 18, the above corollary, and Cor. 16.

Theorem 21. *The operator U_1 can be extended to an isometric H-intertwining operator*

$$U : \mathcal{H} \to \mathscr{H}_1. \tag{127}$$

Its image is isomorphic to the spherical unitary representation corresponding to the discrete point $\left\{i\left(\frac{1}{2} - \frac{n-m}{4}\right)\right\}$ in the spectral decomposition for the Casimir operator $\pi_1(\mathcal{C})$.

References

1. Akhiezer, N. I. and Glazman, I. M., "Theory of linear operators in Hilbert space", Translated from the Russian and with a preface by Merlynd Nestell, Reprint of the 1961 and 1963 translations, Two volumes bound as one, Dover Publications Inc., New York, 1993
2. Davidson, Mark and Ólafsson, Gestur and Zhang, Genkai, *Laplace and Segal-Bargmann transforms on Hermitian symmetric spaces and orthogonal polynomials*, J. Funct. Anal., **204**, (2003), 1, 157–195
3. Faraut, J. and Korányi, A.*Function spaces and reproducing kernels on bounded symmetric domains* J. Funct. Anal. **88**,(1990), 1, 64–89
4. Faraut, J. and Korányi, A. "Analysis on symmetric cones" Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1994

5. Helgason, Sigurdur "Differential geometry, Lie groups, and symmetric spaces", Pure and Applied Mathematics, 80, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978
6. Helgason, Sigurdur "Groups and geometric analysis" Mathematical Surveys and Monographs,83, American Mathematical Society,Providence, RI, 2000
7. Howe, Roger *Transcending classical invariant theory* J. Amer. Math. Soc., **2**, (1989, 3, 535–552
8. Jakobsen, Hans Plesner and Vergne, Michèle *Restrictions and expansions of holomorphic representations* J. Funct. Anal., **34**, (1979), 1, 29–53
9. Kashiwara, M. and Vergne, M. *On the Segal-Shale-Weil representations and harmonic polynomials* Invent. Math., **44**, (1978), 1, 1–47
10. Knapp, Anthony W. "Representation theory of semisimple groups , an overview based on examples" Princeton University Press, Princeton, NJ, 2001
11. Knapp, Anthony W. "Lie groups beyond an introduction" Progress in Mathematics, 140, Birkhäuser Boston Inc., Boston, MA, 2002
12. Kobayashi, Toshiyuki, *Multiplicity-free representations and visible actions on complex manifolds,* Publ. Res. Inst. Math. Sci., Kyoto University. Research Institute for Mathematical Sciences. Publications, **41**, (2005), 3, 497–549
13. Kobayashi, Toshiyuki, *Propagation of multiplicity-free property for holomorphic vector bundles,* Preprint.
14. Kobayashi, Toshiyuki, *Visible actions on symmetric spaces,* Preprint.
15. Kobayashi, Toshiyuki and Ørsted, Bent *Analysis on the minimal representation of $O(p,q)$, II.* Branching laws Adv. Math., **180**, (2003), 2, 513–550
16. Koekoek, R., Swarttouw, R.F., *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue,* Report 98-17, Techical University Delft, Delft, 1998
17. Neretin, Yu. A. *Beta integrals and finite orthogonal systems of Wilson polynomials* Mat. Sb., **193**, (2002), 7, 131–148
18. Neretin, Yurii A. *Plancherel formula for Berezin deformation of L^2 on Riemannian symmetric space* J. Funct. Anal., **189**, (2002), 2, 336–408
19. Neretin, Yu. A. and Olshanski, G. I., *Boundary values of holomorphic functions, singular unitary representations of the groups* $\mathrm{O}(p,q)$ *and their limits as* $q \to \infty$, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI),**223**, (1995), Teor. Predstav. Din. Sistemy, Kombin. i Algoritm. Metody. I, 9–91
20. Ólafsson, Gestur, *Analytic continuation in representation theory and harmonic analysis,* Global analysis and harmonic analysis (Marseille-Luminy, 1999), Sémin. Congr., 4, 201–233, Soc. Math. France, Paris, 2000
21. Ólafsson, Gestur and Ørsted, Bent, *Generalizations of the Bargmann transform,* Lie theory and its applications in physics (Clausthal, 1995), 3–14, World Sci. Publ., River Edge, NJ, 1996
22. Peng, Lizhong and Zhang, Genkai, *Tensor products of holomorphic representations and bilinear differential operators,* J. Funct. Anal., **210**, 2004, 1, 171–192
23. Rossi, H. and Vergne, M. *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group* Acta Math., **136**, (1976), 1-2, 1–59

24. Rudin, Walter "Function theory in the unit ball of $\mathbf{C}^n$" Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 241, Springer-Verlag, New York, 1980
25. Rudin, Walter "Functional analysis" International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991
26. Satake, Ichirô, "Algebraic structures of symmetric domains", Kanô Memorial Lectures, 4, Iwanami Shoten, Tokyo, 1980
27. Schmid, Wilfried, *Homogeneous complex manifolds and representations of semisimple Lie groups*, Representation theory and harmonic analysis on semisimple Lie groups, Math. Surveys Monogr., **31**, 223–286 Dissertation, University of California, Berkeley, CA, 1967
28. Seppänen, H., *Branching of some holomorphic representations of $SO(2,n)$*, Journal of Lie theory (to appear)
29. Seppänen, H. *Tube domains and restrictions of minimal representations*, Preprint
30. van Dijk, G. and Pevzner, M. *Berezin kernels of tube domains* J. Funct. Anal., **181**, (2001), 2, 189–208
31. Wallach, Nolan R. *The analytic continuation of the discrete series. I, II* Trans. Amer. Math. Soc., **251**, (1979), 1–17, 19–37
32. Zhang, Genkai, *Berezin transform on real bounded symmetric domains* Trans. Amer. Math. Soc., **353**, (2001), 9, 3769–3787 (electronic)
33. Zhang, Genkai, *Tensor products of minimal holomorphic representations* Represent. Theory, **5**, (2001), 164–190 (electronic)
34. Zhang, Genkai, *Branching coefficients of holomorphic representations and Segal-Bargmann transform* J. Funct. Anal., **195**, (2002), 2, 306–349

Department of Mathematics, Chalmers University of Technology and Göteborg University, Göteborg, Sweden

E-mail address: `henriks@math.chalmers.se`

Paper IV

QUATERNIONIC DISCRETE SERIES FOR $Sp(1,1)$

HENRIK SEPPÄNEN

ABSTRACT. In this paper we study the analytic realisation of the discrete series representations for the group $G = Sp(1,1)$ as a subspace of the space of square integrable sections in a homogeneous vector bundle over the symmetric space $G/K := Sp(1,1)/(Sp(1) \times Sp(1))$. We use the Szegö map to give expressions for the restrictions of the K-types occurring in the representation spaces to the submanifold AK/K.

1. INTRODUCTION

In [2], Gross and Wallach considered representations of simple Lie groups G with maximal compact subgroup K such that the associated symmetric space G/K has a G-equivariant quaternionic structure (cf. [11]). This amounts to the group K containing a normal subgroup isomorphic to $SU(2)$. In fact, there is an isomorphism $K \cong SU(2) \times M$ for a subgroup $M \subseteq K$, and by setting $L := U(1) \times M$, the associated homogeneous space G/L is fibred over G/K with fibres diffeomorphic to $P^1(\mathbb{C})$. The quaternionic discrete series representations are then realised on the Dolbeault cohomology groups $H^1(G/L, \mathcal{L})$, where $\mathcal{L} \to G/L$ is a holomorphic line bundle. In this model they are able to classify all the K-types occurring in each of the obtained discrete series representations. Moreover, they consider the continuation of the discrete series and characterise the unitarisability of the underlying $(\mathfrak{g}, K)$-modules.

In this paper we consider another model of the quaternionic discrete series. If π is a quaternionic discrete series representation realised on the cohomology group $H^1(G/L, \mathcal{L})$, and τ is its minimal K-type, then the *Schmid D-operator* acts on the sections of the homogeneous vector bundle $G \times_K V_\tau \to G/K$ where V_τ is some vector space on which the K-type is unitarily realised. The Hilbert space $\ker D \cap L^2(G, \tau)$

Key words and phrases. Lie groups, discrete series representation, quaternionic symmetric space, Szegö map.

then furnishes another realisation of the representation π. We consider the special case when $G = Sp(1,1)$. In this case the symmetric space G/K can be embedded into the bounded symmetric domain $SU(2,2)/S(U(2)\times U(2))$ consisting of complex 2×2-matrices of norm less than one. The restriction of the Harish-Chandra embedding to G/K then yields a global trivialisation of the vector bundle $G\times_K V_\tau$. In this model we compute the restrictions to the submanifold $A\cdot 0$ [1] of the highest weight vectors for the occurring K-types. These functions turn out to be fibrewise highest weight vectors with a hypergeometric function as a coefficient. Similar functions have been studied by Castro and Grünbaum in [1]. Hypergeometric functions occur frequently in representation theory, not only for Lie groups. For example, in [8], they play a role in the context of Hecke algebras.

We compute the K-types by using the Szegö map defined by Knapp and Wallach in [6] which exhibits any discrete series representation as a quotient of a nonunitary principal series representations. The K-types are determined on the level of the principal series representation, and then the Szegö map is applied to compute the above mentioned restrictions.

The paper is organised as follows. In section 2 we explicitly state some results from the structure theory of the Lie group $Sp(1,1)$ that will be needed. In section 3 we describe the models for the discrete series in the general context of induced representations, and also give an explicit global trivialisation. Section 4 describes the Szegö map by Knapp and Wallach, and we also compute K-types on the level of a nonunitary principal series representation. In section 5 we compute the images of the K-types under the Szegö map and trivialise them to yield vector valued functions. The main theorem of this paper is Theorem 8 of this section.

2. Preliminaries

2.1. **The quaternion algebra.** The quaternion algebra, $\mathbb{H}$, is a four-dimensional associative algebra over $\mathbb{R}$ with generators i, j, k satisfying

[1] A is associated with a particular Iwasawa decomposition $G = NAK$.

the relations

$$\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ ij = k, jk = i, ki &= j \text{ and} \\ ji = -ij, ik = -ki, kj &= -jk. \end{aligned}$$

Moreover, $\mathbb{H}$ is equipped with an involution, $*$, given by

$$(a+bi+cj+dk)^* = a - bi - cj - dk, a, b, c, d \in \mathbb{R}.$$

The Euclidean norm on the vector space $\mathbb{R}^4 \simeq \mathbb{H}$ can be expressed in terms of this involution by

$$|(a,b,c,d)|^2 = a^2+b^2+c^2+d^2 = (a+bi+cj+dk)^*(a+bi+cj+dk).$$

It follows immediately that the quaternions of norm one, $Sp(1)$, form a group. The algebra $\mathbb{H}$ can be embedded as a subalgebra of the algebra, $M_2(\mathbb{C})$, of 2×2 complex matrices by

$$\iota : \mathbb{H} \to M_2(\mathbb{C}), \tag{1}$$

where

$$\iota(a+bi+cj+dk) = \begin{pmatrix} a+bi & c+di \\ -(c-di) & a-bi \end{pmatrix}. \tag{2}$$

In particular, the generators $1, i, j, k$ are embedded as

$$\iota(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \iota(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \iota(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \iota(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The embedding ι also satisfies the relation

$$\iota((a+bi+cj+dk)^*) = \begin{pmatrix} a-bi & -c-di \\ -(-c+di) & a+bi \end{pmatrix} = \begin{pmatrix} a+bi & c+di \\ -(c-di) & a-bi \end{pmatrix}^*,$$

so ι is a homomorphism of involutive algebras. We observe that, letting $z = a+bi, w = c+di$,

$$\iota(\mathbb{H}) = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C} \right\}$$

and moreover, we have the identity

$$a^2+b^2+c^2+d^2 = |z|^2+|w|^2 = \det \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}.$$

In particular,

$$Sp(1) \simeq \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \mid |z|^2 + |w|^2 = 1 \right\} = SU(2).$$

2.2. **The group Sp(1,1).** The real vector space $\mathbb{H}^2 \cong \mathbb{R}^4$ is also equipped with the structure of an $\mathbb{H}$-module by

$$(\alpha, (h_1, h_2)) \mapsto (\alpha h_1, \alpha h_2), \alpha, h_1, h_2 \in \mathbb{H}. \tag{3}$$

If we identify $\mathbb{H}^2$ with the set of 2×1 matrices over $\mathbb{H}$, there is a natural $\mathbb{H}$-linear action of the matrix group $GL(2, \mathbb{H})$ on $\mathbb{H}^2$ given by

$$\begin{pmatrix} h_1 & h_2 \end{pmatrix} \mapsto \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4}$$

Consider the real vector space $\mathbb{H}^2$ equipped with the nondegenerate indefinite Hermitian form

$$\langle \cdot, \cdot \rangle_{1,1} : ((h_1, h_2), (h_1', h_2')) \mapsto h_1(h_1')^* - h_2(h_2')^*. \tag{5}$$

Recall that the group $Sp(1,1)$ is defined as

$$Sp(1,1) := \{g \in GL(2, \mathbb{H}) | \langle gh, gh' \rangle_{1,1} = \langle h, h' \rangle_{1,1}\}, \tag{6}$$

where $h := (h_1, h_2), h' := (h_1', h_2')$. The condition that the form $\langle , \rangle_{1,1}$ be preserved can be reformulated as

$$g^* J g = J, \tag{7}$$

where $g^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* := \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$, and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The embedding (1) induces an embedding (which we also denote by the same symbol)

$$\iota : M_2(\mathbb{H}) \to M_4(\mathbb{C}) \tag{8}$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \iota(a) & \iota(b) \\ \iota(c) & \iota(d) \end{pmatrix}. \tag{9}$$

This embedding is a homomorphism of algebras with involution. Applying it to the identity (7) reveals that the image of $Sp(1,1)$ is a subgroup of the group

$$\begin{aligned} SU(2,2) &= \{g \in M_4(\mathbb{C}) | g^* \tilde{J} g = \tilde{J}, \det g = 1\}, \\ \tilde{J} &:= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \end{aligned} \tag{10}$$

2.3. The symmetric space $B_1(\mathbb{H}) = Sp(1,1)/(Sp(1)\times Sp(1))$. Let $B_1(\mathbb{H})$ denote the unit ball

$$B_1(\mathbb{H}) := \{h \in \mathbb{H} | \, |h| < 1\} \tag{11}$$

in $\mathbb{H}$. The group $G := Sp(1,1)$ acts transitively on $B_1(\mathbb{H})$ by the fractional linear action

$$g(h) := (ah+b)(ch+d)^{-1}, \tag{12}$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, h \in B_1(\mathbb{H}).$$

The isotropic subgroup for the origin is

$$K := G_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G \right\} \cong Sp(1)\times Sp(1), \tag{13}$$

and hence we have the description

$$B_1(\mathbb{H}) \cong G/K \tag{14}$$

of $B_1(\mathbb{H})$ as a homogeneous space. Moreover, from eq. (7) it follows immediately that the group G is invariant under the the Cartan involution

$$\theta(g) := (g^*)^{-1} \tag{15}$$

and hence the space G/K is equipped with the family of reflections $\{\sigma_{gK}\}_{gK\in G/K}$ given by

$$\sigma_{gK}(xK) := g\theta(g^{-1}x)K \tag{16}$$

which furnish G/K with the structure of a Riemannian symmetric space of the noncompact type. In particular, for any $h \in B_1(\mathbb{H})$, there is a unique geodesic joining 0 and h. We let φ_h denote the reflection in the midpoint, m_h, of this geodesic. The isometry $\varphi_h \in G$ is uniquely characterised by the properties

$$\varphi_h(m_h) = h, \tag{17}$$
$$d\varphi_h(m_h) = -Id_{T_{m_h}(B_1(\mathbb{H}))}. \tag{18}$$

We let $Sp(1)_1$ and $Sp(1)_2$ denote the "upper" and "lower" subgroups of K given by

$$\begin{aligned} Sp(1)_1 &= \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in K \right\}, \\ Sp(1)_2 &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in K \right\}. \end{aligned}$$

For $k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in K$, we will write $k = (a,d) := (k_1, k_2)$. The group $Sp(1)_1 \cong SU(2)$ is then a normal subgroup of K. We will write

$$\pi : K \to K/Sp(1)_1 \cong SU(2) \tag{19}$$

for the natural projection onto the quotient group.

The group K acts on the tangent space $T_0(B_1(\mathbb{H}))$ by the differentials at 0 of the actions on $B_1(\mathbb{H})$. By the restriction to the subgroup $Sp(1)_1$ we have a representation of $SU(2)$ on $T_0(B_1(\mathbb{H}))$. We can define an $SU(2)$-representation, μ_h, on the tangent space $T_h(B_1(\mathbb{H}))$ for any h by the formula

$$\begin{aligned} \mu_h(l)v := d\varphi_h(0) \circ dl(0) \circ d\varphi_h^{-1}(h)v, \\ v \in T_h(B_1(\mathbb{H})), l \in Sp(1)_1. \end{aligned} \tag{20}$$

The family $\{\mu_h\}_{h \in B_1(\mathbb{H})}$ of $SU(2)$-representations amounts to an action of $SU(2)$ as gauge transformations of the tangent bundle $T(B_1(\mathbb{H}))$. It is, however, not invariant under the action of G as automorphisms of the bundle. Indeed, if we define,for $h \in B_1(\mathbb{H}), g \in G$,

$$\kappa_{g,h} := \varphi_{g(h)}^{-1} g \varphi_h \in K, \tag{21}$$

then

$$\mu_{g(h)}(l) dg(h) v = dg(h) \mu_h(\kappa_{g,h}^{-1} l \kappa_{g,h}) v, \tag{22}$$

where the element $\kappa_{g,h}^{-1} l \kappa_{g,h}$ belongs to the subgroup $Sp(1)_1$ since it is normalised by K. Hence the principal fibre bundle over $B_1(\mathbb{H})$ defined by the family $\{\mu_h\}_{h \in B_1(\mathbb{H})}$ is G-equivariant, though not elementwise. This shows that the symmetric space has a quaternionic structure and is a quaternionic symmetric space in the sense defined by Wolf (cf. [11]).

2.4. **Harish-Chandra realisation.** We consider again the embedding ι defined in eq. (8). If we set

$$\begin{aligned} G' &= SU(2,2), \\ K' &= S(U(2)\times U(2)) \\ &:= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in SU(2,2) | A \in U(2), D \in U(2), \det(A)\det(D) = 1 \right\}, \end{aligned}$$

ι induces an embedding of pairs $(G,K) \hookrightarrow (G',K')$ and hence descends to an embedding

$$G/K \hookrightarrow G'/K' \tag{23}$$

of the corresponding symmetric spaces. We will write $SU(2)_1$ and $SU(2)_2$ for the images $\iota(Sp(1)_1)$ and $\iota(Sp(1)_2)$ respectively.

The Hermitian symmetric space G'/K' is by the Harish-Chandra realisation holomorphically, and G-equivariantly, equivalent to the bounded symmetric domain of type I

$$G'/K' \cong \mathscr{D} := \{Z \in M_2(\mathbb{C}) | I_2 - Z^*Z > 0\}. \tag{24}$$

The action of G' on $\mathscr{D}$ is given by

$$g(Z) = (AZ+B)(CZ+D)^{-1}, \tag{25}$$

if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a block matrix with blocks of size 2×2. The symmetric space G/K is thus embedded into $\mathscr{D}$ as the subset

$$\mathcal{D} := \left\{ Z \in M_2(\mathbb{C}) | I_2 - Z^*Z > 0, Z = \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} z, w \in \mathbb{C} \right\},$$

and the action is given by

$$\iota(g)(\iota(h)) = (\iota(a)\iota(h) + \iota(b))(\iota(c)\iota(h) + \iota(d))^{-1} = \iota(g(h)), \tag{26}$$

where $g(h)$ is the action defined in (12).

For any $Z \in \mathscr{D}$, the tangent space $T_Z(\mathscr{D})$ is identified with the complex vector space $M_2(\mathbb{C})$ and the differentials at 0 of the K' actions are given by

$$dk(0)Z = AZD^{-1}, \qquad Z \in M_2(\mathbb{C}), k = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in K'. \tag{27}$$

2.5. **Cartan subalgebra and root system.** Recall the Cartan involution θ on G (15). Its differential at the identity determines a decomposition of $\mathfrak{g}$ into the ± 1-eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ respectively,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{28}$$

where

$$\begin{aligned} \mathfrak{k} &= \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} | X, Y \in \mathbb{H}, X^* = -X, Y^* = -Y, \mathrm{tr}X + \mathrm{tr}Y = 0 \right\}, \\ \mathfrak{p} &= \left\{ \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} | X \in \mathbb{H} \right\}. \end{aligned}$$

Let $\mathfrak{t} \subset \mathfrak{k}$ denote the subalgebra (realised as complex matrices)

$$\mathfrak{t} = \left\{ \begin{pmatrix} si & 0 & 0 & 0 \\ 0 & -si & 0 & 0 \\ 0 & 0 & ti & 0 \\ 0 & 0 & 0 & -ti \end{pmatrix} | s, t \in \mathbb{R} \right\}. \tag{29}$$

It has a basis $\{H_1, H_2\}$, where

$$H_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{30}$$

$$H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{31}$$

Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$, and $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^{\mathbb{C}}$ denote the complexifications of $\mathfrak{k}$ and $\mathfrak{p}$ respectively. The Cartan decomposition induces the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}. \tag{32}$$

The complexification $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$ is a compact Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let Δ denote the set of roots, and for $\alpha \in \Delta$, we let $\mathfrak{g}^{\alpha}$ denote the corresponding root space. Then, for each $\alpha \in \Delta$ either the inclusion $\mathfrak{g}^{\alpha} \subseteq \mathfrak{k}^{\mathbb{C}}$ or the inclusion $\mathfrak{g}^{\alpha} \subseteq \mathfrak{p}^{\mathbb{C}}$ holds. In the first case, we call the root compact, and in the second case we call it non-compact. Let $\Delta_{\mathfrak{k}}$, and $\Delta_{\mathfrak{p}}$ denote the set of compact roots and the set of non-compact roots respectively. We order the roots by letting the ordered basis $\{-\sqrt{-1}H_1^*, -\sqrt{-1}H_2^*\}$ for the real vector space $\sqrt{-1}\mathfrak{t}^*$ define a lexicographic ordering. We let $\Delta_{\mathfrak{k}}^+$ denote the set of positive compact roots, and we let $\Delta_{\mathfrak{p}}^+$ denote the set of positive non-compact roots.

The roots are given by

$$\Delta_{\mathfrak{k}} = \{\pm 2\sqrt{-1}H_1^*, \pm 2\sqrt{-1}H_2^*\}, \tag{33}$$

$$\Delta_{\mathfrak{p}} = \{\pm\sqrt{-1}(H_1^* + H_2^*), \pm\sqrt{-1}(H_1^* - H_2^*)\}. \tag{34}$$

In terms of quaternionic matrices, the corresponding root spaces are

$$\mathfrak{g}_{\pm 2\sqrt{-1}H_1^*} = \mathbb{C}\begin{pmatrix} j \mp \sqrt{-1}k & 0 \\ 0 & 0 \end{pmatrix}, \tag{35}$$

$$\mathfrak{g}_{\pm 2\sqrt{-1}H_2^*} = \mathbb{C}\begin{pmatrix} 0 & 0 \\ 0 & j \mp \sqrt{-1}k \end{pmatrix}, \tag{36}$$

and

$$\mathfrak{g}_{\pm\sqrt{-1}(H_1^*+H_2^*)} = \mathbb{C}\left(\begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \mp \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}\right), \tag{37}$$

$$\mathfrak{g}_{\pm\sqrt{-1}(H_1^*-H_2^*)} = \mathbb{C}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mp \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right) \tag{38}$$

respectively.

According to the lexicographic ordering on $\sqrt{-1}\mathfrak{t}^*$ determined by the ordered basis $\{-\sqrt{-1}H_1^*, -\sqrt{-1}H_2^*\}$, the positive noncompact roots are

$$\alpha_1 = -\sqrt{-1}H_1^* + \sqrt{-1}H_2^*, \tag{39}$$

$$\alpha_2 = -\sqrt{-1}H_1^* - \sqrt{-1}H_2^*, \tag{40}$$

and $\alpha_1 < \alpha_2$. Moreover, $\alpha_1 + \alpha_2 = -2\sqrt{-1}H_1^*$, i.e., the sum is a root. Hence $\{\alpha_1\}$ is a maximal sequence of strongly orthogonal positive noncompact roots. We let $B(\cdot,\cdot)$ denote the Killing form on $\mathfrak{g}$. We use it to identify $\mathfrak{g}^*$ with $\mathfrak{g}^*$ according to

$$\alpha(X) := B(X, H_\alpha), \alpha \in \mathfrak{g}^*, X \in \mathfrak{g}^*. \tag{41}$$

Via this identification, the Killing form induces a bilinear form on $\mathfrak{g}^*$ by

$$\langle \alpha, \beta \rangle := B(H_\alpha, H_\beta). \tag{42}$$

For $\alpha \in \Delta$, we select a root vector $E_\alpha \in \mathfrak{g}_\alpha$ in such a way that

$$B(E_\alpha, E_{-\alpha}) = \frac{2}{\langle \alpha, \alpha \rangle}. \tag{43}$$

2.6. **Iwasawa decomposition.** Consider the maximal abelian subspace

$$\mathfrak{a} := \mathbb{R}(E_{\alpha_1} + E_{-\alpha_1}) = \left\{ \begin{pmatrix} 0 & tI_2 \\ tI_2 & 0 \end{pmatrix} \mid t \in \mathbb{R} \right\} \tag{44}$$

of $\mathfrak{p}$. The Iwasawa decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is given by

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

The corresponding global decomposition is

$$G = NAK,$$

where, written as quaternionic matrices

$$\begin{aligned} N &= \left\{ \begin{pmatrix} 1+q & -q \\ q & 1-q \end{pmatrix} \mid q \in \mathbb{H}, q^* = -q \right\}, \\ A &= \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}. \end{aligned}$$

Remark. One can just as well use an Iwasawa decomposition $G = KAN$, the correpondence between these two decompositions being $(nak)^{-1} = k^{-1}a^{-1}n^{-1}$. In the sequel will see that it is sometimes convenient use this other decomposition as a means for finding the components in our decomposition.

In the sequel we will need the explicit formulas for the NAK-factorisation

$$g = n(g)a(g)\kappa(g) \tag{45}$$

of an element $g \in Sp(1,1)$.

Lemma 1. *For* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *and* $\log a(g) = t(E_{\alpha_1} + E_{-\alpha_1})$, e^t *and* $\kappa(g)$ *are given by*

$$\begin{aligned} e^t &= \frac{(1 - |bd^{-1}|^2)^{1/2}}{|1 - bd^{-1}|}, \\ \kappa(g) &= e^t \begin{pmatrix} a-c & 0 \\ 0 & d-b \end{pmatrix}. \end{aligned}$$

Proof. The proof is by straightforward computation. We prove only the second statement.

The identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1+q & -q \\ q & 1-q \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$$

is equivalent to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$= \begin{pmatrix} (\cosh t + q(\cosh t - \sinh t))u_1 & (\sinh t + q(\sinh t - \cosh t))u_2 \\ (\sinh t + q(\cosh t - \sinh t))u_1 & (\cosh t + q(\sinh t - \cosh t))u_2 \end{pmatrix}.$$

Hence

$$\begin{aligned} a - c &= e^{-t}u_1 \\ d - b &= e^{-t}u_2. \end{aligned}$$

□

3. Quaternionic discrete series representations

3.1. **Generalities.** The Cartan decomposition (28) decomposes $\mathfrak{g}$ into two invariant subspaces for the adjoint action of K. Moreover, we have the isomorphism of K-representations

(46) $$\mathrm{Ad}_{\mathfrak{g}/\mathfrak{k}} \cong \mathrm{Ad}_{\mathfrak{p}}.$$

We extend $\mathrm{Ad}_{\mathfrak{p}}$ to a complex linear representation of K on the space $(\mathfrak{g}/\mathfrak{k})^{\mathbb{C}} \cong \mathfrak{p}^{\mathbb{C}}$.

Consider now the surjective mapping

(47) $$p : G \to G/K, p(g) = gK.$$

The differential at the origin

(48) $$dp(e) : \mathfrak{g} \to T_{eK}(G/K)$$

intertwines the adjoint action of K on $\mathfrak{g}$ with the differential action on the tangent space $T_{eK}(G/K)$. The kernel of $dp(e)$ is $\mathfrak{k}$, and as K-representations we thus have the isomorphism

(49) $$\mathrm{Ad}_{\mathfrak{p}}^* \cong (dK(o)^{\mathbb{C}})^*,$$

where the right hand side denotes the complex linear dual to the representation given by the complexified actions of the tangent maps at the origin. Using the quotient mapping induced by (48) and the realisation of the differential action of K at the tangent space $T_0(\mathscr{D})$, we obtain the formula

$$\mathrm{Ad}_{\mathfrak{p}}^*(k)Z = (A^{-1})^t Z D^t. \tag{50}$$

Here $Z \in M_2(\mathbb{C}) \cong T_0^*(\mathscr{D}) \cong (T_0(\mathcal{D})^{\mathbb{C}})^*$, and $k = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in K \cong SU(2) \times SU(2)$. The restriction, $\mathrm{Ad}_{\mathfrak{p}}^*|_{SU(2)_1}$, to the subgroup $SU(2)_1$ is then given by

$$\mathrm{Ad}_{\mathfrak{p}}^*|_{SU(2)_1}(k)Z = (A^{-1})^t Z. \tag{51}$$

If we let $\{E_{ij}\}$, $i, j = 1, 2$ denote the standard basis for the complex vector space (i.e., E_{ij} has 1 at the position on the ith row and jth column and zeros elsewhere), then clearly the subspace

$$V := \mathbb{C}E_{11} \oplus \mathbb{C}E_{21} \cong \mathbb{C}^2 \tag{52}$$

spanned by the basis elements in the first column is $SU(2)_1$-invariant. Likewise, the subspace spanned by the basis elements of the second column is invariant. We now let τ denote the representation given by restricting the K-representation $\mathrm{Ad}_{\mathfrak{p}}^*|_{SU(2)_1}$ to the subspace V, and let τ_k denote the kth symmetric tensor power of the representation τ. Then clearly, the natural identification of τ_k with a representation of $SU(2)$ is equivalent to the standard representation of $SU(2)$ on the space of polynomial functions $p(z, w)$ on $\mathbb{C}^2$ of homogeneous degree k, i.e., we have

$$\tau_k(l_1, l_2)p(z, w) := p(l_1^{-1}(z, w)) := p(az + \overline{b}w, -bz + \overline{a}w), \tag{53}$$

where $l_1^{-1} = \begin{pmatrix} a & \overline{b} \\ -b & \overline{a} \end{pmatrix} \in SU(2)$. We let V_{τ_k} denote the representation space for τ_k. The (smoothly) induced representation $\mathrm{Ind}_K^G(\tau_k)$ is then defined on the space

$$C^\infty(G, \tau_k) := \{f \in C^\infty(G, V_{\tau_k}) | f(gl^{-1}) = \tau_k(l)f(g)\ \forall g \in G\, \forall l \in K\},$$

i.e., on the space of smooth sections on the G-homogeneous vector bundle

$$\mathcal{V}^k \to G/K := G \times_K V_{\tau_k} \to G/K. \tag{54}$$

We fix the K-invariant inner product on $\langle , \rangle_k$ on V_{τ_k} given by

$$\langle p, q \rangle_k := p(\partial)(q^*)(0), \tag{55}$$

where $p(\partial)$ is the differential operator defined by substituting $\frac{\partial}{\partial z}$ for z, and $\frac{\partial}{\partial w}$ for w in the polynomial function $p(z,w)$, and

$$(\sum_{j=1}^{k} a_j z^j w^{k-j})^* := \sum_{j=1}^{k} \overline{a_j} z^j w^{k-j}).$$

We use this inner product to define an Hermitian metric on $\mathcal{V}^k$ by

$$h_Z(u,v) := \langle (g^{-1})_Z u, (g^{-1})_Z v\rangle_k, \qquad u,v \in \mathcal{V}^k_Z, \tag{56}$$

where $Z = gK$ and $(g^{-1})_Z$ denotes the fibre map $\mathcal{V}^k_Z \to \mathcal{V}^k_0 \cong V_k$ associated with g^{-1}. For a fixed choice, ι, of G-invariant measure on G/K we define $L^2(\mathrm{Ind}_K^G(\tau_k))$ as the Hilbert space completion of the space

$$\left\{ s \in \Gamma(G/K, \mathcal{V}^k) | \int_{G/K} h_Z(s,s) d\iota(Z) < \infty \right\}. \tag{57}$$

The tensor product representation $\tau_k \otimes \mathrm{Ad}(K)|_{\mathfrak{p}^{\mathbb{C}}}$ decomposes into K-types according to

$$\tau_k \otimes \mathrm{Ad}(K)|_{\mathfrak{p}^{\mathbb{C}}} = \sum_{\beta \in \Delta_{\mathfrak{p}}} m_\beta \pi_{\beta - k\sqrt{-1}H_1^*}, \tag{58}$$

where $m_\beta \in \{0,1\}$, and $\pi_{\beta - k\sqrt{-1}H_1^*}$ is the irreducible representation of K with highest weight $\beta - k\sqrt{-1}H_1^*$. Let τ_k^- be the subrepresentation of the tensor product given by

$$\tau_k^- = \sum_{\beta \in \Delta_{\mathfrak{p}}^-} m_\beta \pi_{\beta - k\sqrt{-1}H_1^*}, \tag{59}$$

and let V_k^- be the subspace of $V_{\tau_k} \otimes \mathfrak{p}^{\mathbb{C}}$ on which τ_k^- operates. Let $P : V_{\tau_k} \otimes \mathfrak{p}^{\mathbb{C}} \to V_k^-$ be the orthogonal projection. Define the space $C^\infty(G, \tau_k^-)$ in analogy with (54). We recall that the *Schmid D operator* is a differential operator mapping the space $C^\infty(G,\tau_k)$ into $C^\infty(G,\tau_k^-)$ and is defined as

$$Df(g) = \sum_i P(X_i f(g) \otimes X_i), \tag{60}$$

where $\{X_i\}$ is any orthonormal basis for $\mathfrak{p}^{\mathbb{C}}$, and $X_i f$ denotes left invariant differentiation, i.e.,

$$\begin{aligned} Xf(g) &:= \frac{d}{dt} f(g\exp(tX))|_{t=0}, X \in \mathfrak{p}, \\ Zf(g) &:= Xf(g) + iY(g), Z = X + iY \in \mathfrak{p}^{\mathbb{C}}. \end{aligned}$$

The subspace $\ker D \cap L^2(\mathrm{Ind}_K^G(\tau_k))$ is then invariant under the left action of G and defines an irreducible representation of G belonging to the quaternionic discrete series. We let $\mathcal{H}_k$ denote this representation space. By [2], it belongs to the discrete series for $k \geq 1$.

Remark. The model we use to describe the Hilbert space $\mathcal{H}_k$ can be used to realise any discrete series representation by induction from K to G of the minimal K-type for any pair (G, K) where G is semisimple and K is maximal compact (cf. [5]). By [2], for $k \geq 1$, τ_k occurs as a minimal K-type for some discrete series representation of $G = Sp(1,1)$.

3.2. **Global trivialisation.** Let us for a while view the representation space $\mathcal{H}_k$ as a space of sections of the vector bundle $\mathcal{V}^k \to G/K$. We recall the diffeomorphism $G/K \cong \mathcal{D}$ given by $gK \mapsto g \cdot 0$. This lifts to a global trivialisation, Φ, of the bundle $\mathcal{V}^k \to G/K$ given by

$$\Phi : G \times_K V^{\tau_k} \to \mathcal{D} \times V_{\tau_k},\ \Phi([(g, v)]) := (g \cdot 0, \tau_k(J(g, 0))v), \tag{61}$$

where $J(g, Z)$ denotes the $K^{\mathbb{C}}$-component of $g \exp Z$ - the *automorphic factor of g at Z* (cf. [9]).

If $F : G \to V^{\tau_k}$ is a function in $C^\infty(G, \tau_k)$, its trivialised counterpart is the function $f : \mathcal{D} \to V_{\tau_k}$ given by

$$f(g \cdot 0) := \tau_k(J(g, 0))F(g). \tag{62}$$

In the trivialised picture, the group G acts on functions on $\mathcal{D}$ by

$$gf(Z) := \tau_k(J(g^{-1}, Z))^{-1} f(g^{-1}Z). \tag{63}$$

More explicitly, if $g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (considered as a matrix in $SU(2,2)$), then

$$J(g^{-1}Z) = \begin{pmatrix} A - (AZ+B)(CZ+D)^{-1}C & 0 \\ 0 & D \end{pmatrix} \in SL(4, \mathbb{C}),$$

and

$$gf(Z) = \overset{k}{\odot} (A - (AZ+B)(CZ+D)^{-1}C) f((AZ+B)(CZ+D)^{-1}).$$

The action of $SU(2)$ on the vector space V_{τ_k} is here naturally extended to an action of $SL(2,\mathbb{C})$ by the formula (53).

In the trivialised picture, the norm (57) can also be described explicitly.

Proposition 2. *Let $k \geq 1$. In the realisation of the Hilbert space $\mathcal{H}_k$ as a space of V_{τ_k}-valued functions on $\mathcal{D}$, the norm* (57) *is given by*

$$\|f\|_k := \int_{B_1(\mathbb{H})} (1-|q|^2)^k \langle f(q), f(q)\rangle_k (1-|q|^2)^{-4} dm(q). \tag{64}$$

Proof. For $Z = gK$, a fibre map $(g^{-1})_Z : V_{\tau_k} \to V_{\tau_k}$ is given by

$$(g^{-1})_Z v = \tau_k(J(g,0))^{-1} v. \tag{65}$$

If $g = \begin{pmatrix} \cosh tI_2 & \sinh tI_2 \\ \sinh tI_2 & \cosh tI_2 \end{pmatrix}$, the automorphic factor $J(g,0)$ is given by (cf.[5])

$$\begin{aligned} &J(g,0) \\ &= \begin{pmatrix} \cosh t^{-1} I_2 & 0 \\ 0 & \cosh tI_2 \end{pmatrix} = \begin{pmatrix} (1-\tanh^2 t)^{1/2} I_2 & 0 \\ 0 & \cosh tI_2 \end{pmatrix}. \end{aligned} \tag{66}$$

A general point $Z \in \mathcal{D}$ can be described as $Z = kgK$ for g as above. The cocycle condition

$$J(kg,0) = J(k,g0)J(g,0) \tag{67}$$

then implies that

$$J(kg,0) = \begin{pmatrix} k_1(1-\tanh^2 t)^{1/2} I_2 & 0 \\ 0 & k_2 \cosh tI_2 \end{pmatrix}, \tag{68}$$

if $k = (k_1,k_2) \in SU(2) \times SU(2)$. Hence, for $k = 1$

$$\begin{aligned} h_Z(u,v) &= \operatorname{tr}\left((k_1(1-\tanh^2 t)^{1/2} I_2)^t u ((k_1(1-\tanh^2 t)^{1/2} I_2)^t v)^*\right) \\ &= \operatorname{tr}\left((I_2 - ZZ^*)^t uv^*\right). \end{aligned}$$

For arbitrary k, we have

$$h_Z(u,v) = \operatorname{tr}\left(\overset{k}{\odot} (I_2 - ZZ^*)^t uv^*\right). \tag{69}$$

By analogous considerations, it follows that the invariant measure is given by

$$d\iota(Z) = \det(I_2 - Z^*Z)^{-2} dm(Z),$$

where $dm(Z)$ denotes the Lebesgue measure. Hence, we obtain the formula

$$\int_{\mathcal{D}} \langle \overset{k}{\odot} (I_2 - ZZ^*)^t f(Z), f(Z)\rangle_k \det(I_2 - Z^*Z)^{-2} dm(Z) \tag{70}$$

for the norm (57). In quaternionic notation, this translates into the statement of the proposition. □

4. Principal series representations and the Szegö map

In this section we will consider a realisation of the discrete series representation $L^2(\mathrm{Ind}K^G(\tau_k))$ as a quotient of a certain nonunitary principal series representation. We first state the theorem, and then we investigate how the given principal series representation decomposes into K-types. From now on we fix the number k and simply write τ for τ_k.

Recall the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and consider the parabolic subgroup

$$P = MAN$$

of G, where

$$M = Z_K(A) = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \in SU(2) \right\},$$

and A and N are the ones that occur in the Iwasawa decomposition. Let σ be the restriction of the representation τ to the subgroup M. Then, clearly, the subspace defined by the M-span of the τ-highest weight-vector equals V_τ and the representation σ is also irreducible. We will hereafter denote this representation space by V^σ. Recall the identification of V^σ with a space of homogeneous polynomials. We thus adopt a somewhat abusive notation and write z^σ for the highest weight-vector. Let $\nu \in \mathfrak{a}^*$ be a real-valued linear functional and consider the representation

$$\sigma \otimes \exp(\nu) \otimes 1 \tag{71}$$

of P. The induced representation $\mathrm{Ind}_P^G(\sigma \otimes \exp(\nu) \otimes 1)$ is defined on the set of continuous functions $f : G \to V_\sigma$ having the P-equivariant property

$$f(gman) = e^{-\nu(\log a)} \sigma(m)^{-1} f(g). \tag{72}$$

The action of G on this space is given by left translation,

$$\mathrm{Ind}_P^G(\sigma \otimes \exp(\nu) \otimes 1)(f)(x) := L_{g^{-1}} f(x) = f(g^{-1}x).$$

Consider now the smoothly induced representation $\mathrm{Ind}_M^K(\sigma)$ which operates on the space, $C^\infty(K,\sigma)$, of all smooth functions $f : K \to V_\sigma$ having the M-equivariance property

$$f(km) = \sigma(m)^{-1} f(k) \tag{73}$$

with K-action given by left translation. The Iwasawa decomposition $G = KAN$ shows that, a fortiori, $G = KMAN$ (although this factorisation is not unique). Given a linear functional $\nu \in \mathfrak{a}^*$, we can therefore extend any such function on K to a function on G by setting

$$f(kman) = e^{-\nu(\log a)} \sigma(m)^{-1} f(k), \text{for } g = kman.$$

The equivariance property (73) of f guarantees that this is indeed well-defined even though the factorisation of g is not. The extended function f has the P-equivariance property (72). In fact, this extension procedure defines a bijection between the representation spaces of the representations $\mathrm{Ind}_M^K(\sigma)$ and $\mathrm{Ind}_P^G(\sigma \otimes \exp(\nu) \otimes 1)$. There is a natural pre-Hilbert space structure on this representation space given by

$$\|f\|^2 = \int_K \|f(k)\|_\sigma^2 dk,$$

where $\|\cdot\|_\sigma$ denotes the inner product on V_σ and dk is the Haar measure on K. The completion of the space of M-equivariant smooth functions $K \to V_\sigma$ with respect to this sesquilinear form can be identified with the space of all square-integrable V_σ- valued functions having the property (73). We will denote the K-representation on this space by $L^2(\mathrm{Ind}_M^K(\sigma))$. By the extension procedure using ν described above, this completion can be extended to the space of all P-equivariant V_σ-valued functions on G such that the restriction to K is square-integrable.

We now state the theorem by Knapp and Wallach.

Theorem 3 ([6], Thm. 6.1). *The* Szegö mapping *with parameters* τ *and* ν *given by*

$$S(f)(x) := \int_K e^{\nu \log a(lx)} \tau(\kappa(lx)^{-1}) f(l^{-1}) dl \tag{74}$$

carries the the space $C^\infty(K,\sigma)$ *into* $C^\infty(G,\tau)\cap\ker D$*, provided that* ν *and* τ *are related by the formula*

$$\nu(E_{\alpha_1}+E_{-\alpha_1})=\frac{2\langle -k\sqrt{-1}H_1^*+n_1\alpha_1,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle}, \tag{75}$$

where

$$n_1=|\{\gamma\in\Delta_{\mathfrak{p}}^+|\alpha(\gamma)=\alpha_1 \text{ and } \alpha_1+\gamma\in\Delta\}|.$$

In this case $n_1=1$, since the root α_2 is the only one satisfying the above condition. Moreover, an easy calculation gives that

$$E_{\alpha_1}=\frac{1}{2}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}+\begin{pmatrix}0&i\\-i&0\end{pmatrix}\right). \tag{76}$$

Hence, the condition (75) takes the form

$$\nu\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)=k+2. \tag{77}$$

Hereafter, we will make the identification

$$\nu=k+2 \tag{78}$$

of the functional with a natural number. We now proceed with a more detailed study of the representation $L^2(Ind_M^K(\sigma))$.

Lemma 4. *The representation* $L^2(Ind_M^K(\sigma))$ *is* K*-equivalent to* $L^2(K/M)\otimes V_\sigma$*.*

Proof. Let f be a continuous function from K to V_σ having the property of M-equivariance

$$f(km)=\sigma(m)^{-1}f(k), k\in K, m\in M.$$

Then the function

$$\tilde{f}(k):=\tau(k)f(k)$$

is clearly right M-invariant and hence we can define the function $F: K/M\to V_\sigma$ by

$$F(kM)=\tilde{f}(k).$$

This is obviously well-defined. By choosing a basis $\{e_j\}$ for V_σ, we can write

$$F(kM)=\Sigma_j F_j(kM)e_j$$

for some complex-valued functions F_j. We now define a mapping

$$T: L^2(Ind_M^K(\sigma))\to L^2(K/M)\otimes V_\sigma$$

by

$$Tf := \Sigma_j F_j \otimes e_j.$$

To see that this mapping is a bijection, note that any vector in the Hilbert space $L^2(K/M) \otimes V_\sigma$ can be uniquely expressed in the form $\Sigma_j G_j \otimes e_j$. We can thus define a mapping

$$S : L^2(K/M) \otimes V_\sigma \to L^2(Ind_M^K(\sigma))$$

by

$$S(\Sigma_j G_j \otimes e_j)(k) := \tau(k)^{-1}\Sigma_j g_j(kM)e_j$$

and it is easy to see that S is the inverse of T.

It remains now only to prove the K-equivariance. Pick therefore any element
$\Sigma_j G_j \otimes e_j$ from the Hilbert space on the right hand side. We have

$$k(\Sigma_j G_j \otimes e_j) = \Sigma_j G_j \circ L_{k^{-1}} \otimes \sigma(k)e_j.$$

If we denote the matrix coefficients of $\sigma(k)$ with respect to the basis $\{e_j\}$ by $\sigma(k)_{ij}$, we have

$$\sigma(k)e_j = \Sigma_i \sigma(k)_{ij} e_i$$

and hence

$$\Sigma_j G_j \circ L_{k^{-1}} \otimes \sigma(k)e_j = \Sigma_{i,j} G_j \circ L_{k^{-1}} \otimes \sigma(k)_{ij} e_i.$$

Applying S to the above expression yields

$$\begin{aligned} S(\Sigma_{i,j} G_j \circ L_{k^{-1}} \otimes \sigma(k)_{ij} e_i)(k') &= \sigma(k')^{-1}\Sigma_{i,j} G_j(k^{-1}k'M)\sigma(k)_{ij} e_i \\ &= \sigma(k')^{-1}\sigma(k)\Sigma_j G_j(k^{-1}k'M)e_j \\ &= S(\Sigma_j G_j \otimes e_j) \circ L_{k^{-1}}(k'). \end{aligned}$$

□

We shall now examine the left action of K on the $L^2(K/M)$-factor in the tensor product more closely. In particular, we are interested in a certain K-invariant subspace defined by a subclass of the K-types occurring in $L^2(K/M)$. We recall the identification of the K-representation τ_j with a standard representation of $SU(2)$. We therefore let τ_j also denote the corresponding $SU(2)$-representation, and we let V_j denote the associated vector space of polynomials. Any irreducible representation of $K = SU(2) \times SU(2)$ is isomorphic to a tensor product of irreducible $SU(2)$-representations, i.e., it is realised on a space

$$V_j^* \otimes V_i, \tag{79}$$

for some $i, j \in \mathbb{N}$. With the fixed ordering of the roots, the polynomial function $(z, w) \mapsto z^j$ is a highest weight vector in V_j, and the polynomial function $(z, w) \mapsto w^j$ is a lowest weight vector. We will use the abusive notation where they are denoted by z^j and w^j respectively.

Proposition 5. *The algebraic sum*

$$W := \bigoplus_{j \in \mathbb{N}} V_j^* \otimes V_{\sigma+j} \tag{80}$$

of K-types is a subspace of $L^2(Ind_M^K(\sigma)$. The highest weight vector for the K-type $V_j^* \otimes V_{\sigma+j}$ is given by the function

$$f_j(k) := \langle \tau_j \circ \pi(k) z^j, w^j \rangle_j \, \tau(k)^{-1} z^\sigma. \tag{81}$$

Proof. For

$$k = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, k' = \begin{pmatrix} u_1' & 0 \\ 0 & u_2' \end{pmatrix} \in K,$$

we have

$$\begin{aligned} k'kM = \begin{pmatrix} u_1'u_1 & 0 \\ 0 & u_2'u_2 \end{pmatrix} M &= \begin{pmatrix} u_1'u_1u_2^{-1}(u_2')^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u_2'u_2 & 0 \\ 0 & u_2'u_2 \end{pmatrix} M \\ &= \begin{pmatrix} u_1'u_1u_2^{-1}(u_2')^{-1} & 0 \\ 0 & I \end{pmatrix} M \end{aligned}$$

and this shows that the left action of $K = SU(2) \times SU(2)$ on $L^2(SU(2))$-functions is equivalent to the action $L_{g^{-1}} \otimes R_h$:

$$(L_{g^{-1}} \otimes R_h)(g, h) f(l) := f(g^{-1} l h)$$

Then, by the Peter-Weyl Theorem, $L^2(K/M)$ decomposes into K-types according to

$$L^2(K/M) \simeq \bigoplus_{j \in \widehat{SU(2)}} (V_j \otimes V_j^*). \tag{82}$$

Tensoring with V_σ gives the sequence of K-isomorphisms

$$\begin{aligned} L^2(K/M)\otimes V_\sigma &\simeq \bigoplus_{j\in\widehat{SU(2)}} (V_j\otimes V_j^*)\otimes V_\sigma \\ &\simeq \bigoplus_{j\in\widehat{SU(2)}} (V_j^*\otimes V_j)\otimes V_\sigma \\ &\simeq \bigoplus_{j\in\widehat{SU(2)}} V_j^*\otimes (V_j\otimes V_\sigma). \end{aligned}$$

Moreover, each term $(V_j\otimes V_\sigma)$ has a *Clebsch-Gordan*-decomposition

$$(V_j\otimes V_\sigma)\simeq (V_{\sigma+j}\oplus\cdots)$$

and therefore each term $V_j\otimes V_{\sigma+j}$ will constitute a K-type in $L^2(K/M)\otimes V_\sigma$. Such a K-type has a highest weight-vector $(w^j)^*\otimes z^{\sigma+j}$. Using first the embedding into $V_j^*\otimes(V_j\otimes V_\sigma)$ and then the K-isomorphism given by Lemma 4, we see that highest weight-vector corresponds to the M-equivariant function

$$f_j(k) := \langle \tau_j\circ\pi(k)z^j, w^j\rangle_j\,\sigma(k)^{-1}z^\sigma. \tag{83}$$

□

5. Realisation of K-types

By [2], the only K-types occurring in the quaternionic discrete series for $Sp(1,1)$ are the ones that form the subspace W in Proposition 5. In this section we compute their realisations as V_τ-valued functions on $B_1(\mathbb{H})$ when restricted to the submanifold

$$A\cdot 0 = \{t\in\mathbb{H} \mid -1<t<1\} \tag{84}$$

of $B_1(\mathbb{H})$. For $s\in\mathbb{R}$, we let

$$a_s = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \in Sp(1,1). \tag{85}$$

Then $a_s\cdot 0=\tanh s\in A\cdot 0$. We start by computing the Szegö images of the f_j when restricted to points a_s.

Each of the standard $SU(2)$-representations, V_N, can be naturally extended to a representation of $GL(2,\mathbb{C})$ by

$$p\mapsto p\circ g^{-1}, \qquad p\in V_n, g\in GL(2,\mathbb{C}). \tag{86}$$

This action of $GL(2,\mathbb{C})$ will occur frequently in the sequel.

Lemma 6. *The Szegö transform of the highest weight-vector f_j is given by*

$$\begin{aligned} Sf_j(a_s) &= (\cosh s)^{-\nu} \\ &\times \int_{SU(2)} (\det(1 - l\tanh s))^{-(\nu+\sigma)/2} \langle \tau_j(l^{-1}) z^j, w^j \rangle_j \, \sigma(1 - l\tanh s) z^\sigma dl \end{aligned}$$

when restricted to the A-component in the decomposition $G = NAK$.

Proof. Take

$$\begin{aligned} k &= \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \text{ and} \\ x &= \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}. \end{aligned}$$

Then

$$kx = \begin{pmatrix} u_1 \cosh s & u_1 \sinh s \\ u_2 \sinh s & u_2 \cosh s \end{pmatrix},$$

and Lemma 1 gives that

$$\begin{aligned} e^{\nu(\log H(kx))} &= \left(\frac{1 - |u_1 u_2^{-1} \tanh s|^2}{|1 - u_1 u_2^{-1} \tanh s|^2} \right)^{\nu/2}, \\ \kappa(kx) &= \left(\frac{1 - |u_1 u_2^{-1} \tanh s|^2}{|1 - u_1 u_2^{-1} \tanh s|^2} \right)^{1/2} \\ &\times \begin{pmatrix} u_1 \cosh s - u_2 \sinh s & 0 \\ 0 & u_2 \cosh s - u_1 \sinh s \end{pmatrix}. \end{aligned}$$

Hence

$$\tau(\kappa(lx))^{-1} = \left(\frac{1 - |u_1 u_2^{-1} \tanh s|^2}{|1 - u_1 u_2^{-1} \tanh s|^2} \right)^{-\sigma} \sigma(u_1 \cosh s - u_2 \sinh s)^{-1}$$

and we get

$$\begin{aligned}Sf_j(a_s) &= \int_K \left(\frac{1-|u_1u_2^{-1}\tanh s|^2}{|1-u_1u_2^{-1}\tanh s|^2}\right)^{(\nu-\sigma)/2} \langle \tau_j \circ \pi(k^{-1})z^j, w^j\rangle_j \\ &\qquad \times\sigma(u_1\cosh s - u_2\sinh s)^{-1}\sigma(u_1)z^\sigma dk \\ &= \int_K \left(\frac{1-|\tanh s|^2}{|1-u_1u_2^{-1}\tanh s|^2}\right)^{(\nu-\sigma)/2} \langle \tau_j \circ \pi(k^{-1})z^j, w^j\rangle_j \\ &\qquad \times\sigma(\cosh s - u_1^{-1}u_2\sinh s)^{-1}z^\sigma dk \\ &= (\cosh s)^{-\nu}\int_K |1-u_1u_2^{-1}\tanh s|^{\sigma-\nu}\langle \tau_j \circ \pi(k^{-1})z^j, w^j\rangle_j \\ &\qquad \times\sigma(1-u_1^{-1}u_2\tanh s)^{-1}z^\sigma dk.\end{aligned}$$

Using the identities

$$(1-u_1^{-1}u_2\tanh s)^{-1} = \frac{1-u_2^{-1}u_1\tanh s}{|1-u_2^{-1}u_1\tanh s|^2}$$

and

$$|1-u_1u_2^{-1}\tanh s| = |u_1^{-1}(1-u_1u_2^{-1}\tanh s)u_1| = |1-u_2^{-1}u_1\tanh s|$$

in the above equality yields

$$\begin{aligned}&Sf_j(a_s) = (\cosh s)^{-\nu} \\ &\times \int_K |1-u_2^{-1}u_1\tanh s|^{-(\sigma+\nu)}\langle \tau_j \circ \pi(k^{-1})z^j, w^j\rangle_j \\ &\qquad \times\sigma(1-u_2^{-1}u_1\tanh s)z^\sigma dk.\end{aligned}$$

We observe that the integrand is right M-invariant. In fact,

$$\begin{aligned}&(\cosh s)^{-\nu}\int_K |1-u_1u_2^{-1}\tanh s|^{-(\sigma+\nu)}\langle \tau_j \circ \pi(k^{-1})z^j, w^j\rangle_j \\ &\qquad \times\sigma(1-u_2^{-1}u_1\tanh s)z^\sigma dk \\ &= (\cosh s)^{-\nu}\int_K |1-\pi(k^{-1})^{-1}\tanh s|^{-(\sigma+\nu)}\langle \tau_j \circ \pi(k^{-1})z^j, w^j\rangle_j \\ &\qquad \times\sigma(1-\pi(k^{-1})^{-1}\tanh s)z^\sigma dk.\end{aligned}$$

Therefore, it can be written as an integral over the coset space $K/M \simeq SU(2)$, i.e.,

$$Sf_j(a_s) = (\cosh s)^{-\nu}$$
$$\times \int_{SU(2)} |1 - l^{-1}\tanh s|^{-(\sigma+\nu)} \langle \tau_j(l) z^j, w^j\rangle_j \, \sigma(1 - l^{-1}\tanh s) z^\sigma dl.$$

Making the change of variables $l \mapsto l^{-1}$, and using the invariance of the Haar measure on $SU(2)$ under this map, yields

$$Sf_j(a_s) = (\cosh s)^{-\nu}$$
$$\times \int_{SU(2)} |1 - l\tanh s|^{-(\sigma+\nu)} \langle \tau_j(l^{-1}) z^j, w^j\rangle_j \, \sigma(1 - l\tanh s) z^\sigma dl,$$

and this finishes the proof. □

5.1. **Highest weight-vectors for K-types.** The polynomial functions p_{l_1,l_2} defined by

$$p_{l_1,l_2}(z,w) := z^{l_1} w^{l_2},$$

for which $l_1 + l_2 = N$ form a basis for V_N. Occasionally we will however use the somewhat ambiguous notation $z^{l_1} w^{l_2}$ when there is no risk for misinterpretation

We write $\zeta = \tanh s$ and consider the action of $(1 - \zeta l)$ on the basis vector p_{l_1,l_2}. We have

$$(87) \qquad (1 - \zeta l) p_{l_1,l_2}(z,w)$$
$$= \left((1-\zeta l)^{-1} \begin{pmatrix} z \\ w \end{pmatrix} \right)_1^{l_1} \left((1-\zeta l)^{-1} \begin{pmatrix} z \\ w \end{pmatrix} \right)_2^{l_2},$$

where the subscripts denote the projection functions

$$(z,w)_1 = z, \qquad (z,w)_2 = w$$

onto the first and second coordinate respectively.

The Binomial theorem gives the following expression for the first factor above:

$$\begin{aligned}
&\left((1-\zeta l)^{-1}\begin{pmatrix} z \\ w \end{pmatrix}\right)_1^{l_1} \\
&= \det(1-\zeta l)^{-l_1}\sum_{j_1=0}^{l_1}\begin{pmatrix} l_1 \\ j_1 \end{pmatrix} z^{j_1}(-\zeta)^{l_1-j_1}\left(l^{-1}\begin{pmatrix} z \\ w \end{pmatrix}\right)_1^{l_1-j_1} \\
&= \det(1-\zeta l)^{-l_1}\sum_{j_1=0}^{l_1} z^{j_1}\begin{pmatrix} l_1 \\ j_1 \end{pmatrix}(-\zeta)^{l_1-j_1} l p_{l_1-j_1,0}(z,w),
\end{aligned}$$

and the second factor has a similar expression. Substituting these into (87) yields the double sum

$$\begin{aligned}
(88)\ &(1-\zeta l)p_{l_1,l_2}(z,w) = \det(1-\zeta l)^{-\sigma} \\
&\times \sum_{j_1=0}^{l_1}\sum_{j_2=0}^{l_2}\begin{pmatrix} l_1 \\ j_1 \end{pmatrix}\begin{pmatrix} l_2 \\ j_2 \end{pmatrix}(-\zeta)^{(l_1+l_2-j_1-j_2)} z^{j_1} w^{j_2} l p_{l_1-j_1,l_2-j_2}(z,w).
\end{aligned}$$

Denote the normalisation of the basis vector $p_{r,s}$ by $e_{r,s}$. Then

$$p_{r,s} = (r!s!)^{1/2} e_{r,s}$$

and the term $lp_{l_1-j_1,l_2-j_2}$ in (89) can be written as the sum

$$\begin{aligned}
(89) \qquad & lp_{l_1-j_1,l_2-j_2} = (r!s!)^{1/2} \\
& \times \sum_{r+s=l_1+l_2-j_1-j_2} M(l;\, l_1-j_1, l_2-j_2;\, r,s) e_{r,s}.
\end{aligned}$$

In what follows, we will use an expression for the first factor in the integrand in Lemma 6 as a series of $SU(2)$-characters. The following result can be found in [4].

Lemma 7. *The function* $l \mapsto (\det(1-l\tanh s)^{-\lambda}$ *has the character expansion*

$$\begin{aligned}
&(\det(1-l\tanh s)^{-\lambda} \\
&= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(\lambda-1)_{i+j+1}}{(i+j+1)!}\frac{(\lambda-1)_i}{i!}(j+1)(\tanh s)^{2i+j}\chi_j \\
&= \sum_{j=0}^{\infty}\frac{(\lambda-1)_{j+1}}{j!}(\tanh s)^j\ {}_2F_1(\lambda+j,\lambda-1;j+2;\tanh^2 s)\chi_j.
\end{aligned}$$

Proposition 8. *The function Sf_j has the following expression when restricted to A.*

$$Sf_j(a_s) = (1-\tanh^2 s)^{\frac{\sigma+2}{2}} \sum_{i=0}^{\sigma} \binom{\sigma}{i} (-1)^{j+i} \frac{(2\sigma)_{j+i+1}\, j!}{(j+i+1)!} (\tanh s)^{j+2i} \times {}_2F_1(2\sigma+1+j+i, 2\sigma; j+i+2; \tanh^2 s) z^{\sigma}.$$

Proof. In the defining integral

$$Sf_j(a_s) = (\cosh s)^{-\nu} \int_{SU(2)} (\det(1-\zeta l))^{-(\sigma+1)} \langle \tau_j(l^{-1}) z^j, w^j \rangle_j \sigma(1-\zeta l) z^{\sigma} dl$$

we already have a character expansion for the factor $(\det(1-\zeta l))^{-\lambda}$. Therefore, it suffices to determine the expansion of

$$\langle \tau_j(l^{-1}) z^j, w^j \rangle_j \sigma(1-\zeta l) z^{\sigma}$$

into matrix coefficients. As a special case of (89) we have

$$\sigma(1-\zeta l) p_{\sigma,0} = \det(1-\zeta l)^{-\sigma} \sum_{i=0}^{\sigma} \binom{\sigma}{i} (-\zeta)^{\sigma-i} z^i l p_{\sigma-i,0},$$

and the special case of (90) is

$$l p_{\sigma-i,0} = ((\sigma-i)!)^{1/2} \sum_{r=0}^{\sigma-i} \langle \tau_{\sigma-i}(l) e_{\sigma-i,0}, e_{r,\sigma-i-r} \rangle_{\sigma-i} e_{r,\sigma-i-r}.$$

To sum up, we have

$$\sigma(1-\zeta l) p_{\sigma,0}(z,w) = \det(1-\zeta l)^{-\sigma} \times \sum_{i=0}^{\sigma} \binom{\sigma}{i} (-\zeta)^{\sigma-i} z^i ((\sigma-i)!)^{1/2} \sum_{r=0}^{\sigma-i} \langle \tau_{\sigma-i}(l) e_{\sigma-i,0}, e_{r,\sigma-i-r} \rangle_{\sigma-i} \times e_{r,\sigma-i-r}(z,w).$$

The integrand is thus a linear combination of terms of the form

$$\langle \tau_j(l^{-1}) z^j, w^j \rangle_j \langle \tau_{\sigma-i}(l) e_{\sigma-i,0}, e_{r,\sigma-i-r} \rangle_{\sigma-i}. \tag{90}$$

It is easy to see (using the formula (53)) that the identity

$$\langle \tau_j(l^{-1}) z^j, w^j \rangle_j = (-1)^j \langle \tau_j(l) z^j, w^j \rangle_j \tag{91}$$

holds, and using this identity in (90), the resulting terms are matrix coefficients for the tensor product representation $\tau_j \otimes \tau_{\sigma-i}$. In fact,

$$(92)\quad \langle \tau_j(l)z^j, w^j\rangle_j \langle \tau_{\sigma-i}(l)e_{\sigma-i,0}, e_{r,\sigma-i-r}\rangle_{\sigma-i}$$
$$= \left(\frac{1}{(\sigma-i)!r!(\sigma-i-r)!}\right)^{1/2} \langle (\tau_j\otimes\tau_{\sigma-i})(l)(z^j\otimes z^{\sigma-i}), w^j\otimes z^r w^{\sigma-i-r}\rangle_{j\,\otimes\,(\sigma-i)}.$$

We recall the Clebsch-Gordan decomposition for the tensor product $V_j \otimes V_{\sigma-i}$. There is an isometric diag$(SU(2)\times SU(2))$-intertwining operator

$$\phi : V_j \otimes V_{\sigma-i} \to V_{j+\sigma-i} \oplus \cdots$$

which is of the form

$$\phi = \phi_{j+\sigma-i} \oplus \cdots \oplus \phi_{(\pm(j-(\sigma-i))}$$

where each term is an intertwining partial isometry and the sum is orthogonal, i.e., the terms have mutually orthogonal kernels. The vector $z^j \otimes z^{\sigma-i}$ is a weight vector of weight $-(j+\sigma-i)\sqrt{-1}H_1^*$ and hence it maps to a highest weight vector in the summand $V_{j+\sigma-i}$. Therefore only the term corresponding to this summand in the orthogonal expansion of the inner product (92) is nonzero. To be more precise, we use the isometry ϕ to write the matrix coefficient (92) as the sum

$$(93)\quad \langle (\phi(\tau_j\otimes\tau_{\sigma-i})(l)(z^j\otimes z^{\sigma-i}), \phi(w^j\otimes z^r w^{\sigma-i-r})\rangle$$
$$= \sum_s \langle \phi_{j+\sigma-i-2s}(\tau_j\otimes\tau_{\sigma-i})(l)(z^j\otimes z^{\sigma-i}), \phi_{j+\sigma-i-2s}(w^j\otimes z^r w^{\sigma-i-r})\rangle_{j+\sigma-i-2s}.$$

Since

$$\phi_{j+\sigma-i}((\tau_j\otimes\tau_{\sigma-i})(l)(z^j\otimes z^{\sigma-i})) = \left(\frac{j!(\sigma-i)!}{(j+\sigma-i)!}\right)^{1/2} \tau_{j+\sigma-i}(l)z^{j+\sigma-i},$$

the sum (93) is equal to its first term

$$\frac{j!(\sigma-i)!}{(j+\sigma-i)!}\langle \tau_{j+\sigma-i}(l)z^{j+\sigma-i}, z^r w^{j+\sigma-i-r}\rangle_{j+\sigma-i}.$$

Moreover, since we are integrating against characters, only the term corresponding to $r = j+\sigma-i$ will contribute. Hence we have the equality

(94)

$$Sf_j(a_s) = (1-\tanh^2 s)^{\frac{\sigma+2}{2}} \sum_{i=0}^{\sigma} \binom{\sigma}{i} (-\zeta)^{\sigma-i}(-1)^j \frac{j!(\sigma-i)!}{(j+\sigma-i)!} z^\sigma$$
$$\times \int_{SU(2)} (\det(1-\zeta l))^{-(2\sigma+1)} \langle \tau_{j+\sigma-i}(l) z^{j+\sigma-i}, z^{j+\sigma-i}\rangle_{j+\sigma-i} dl.$$

So, by using the character expansion (90) and the Schur orthogonality relations for matrix coefficients, we get the following expression for the above integral with the index i fixed.

$$\begin{aligned} &\int_{SU(2)} (\det(1-\zeta l))^{-(2\sigma+1)} \langle \tau_{j+\sigma-i}(l) z^{j+\sigma-i}, z^{j+\sigma-i}\rangle_{j+\sigma-i} dl \\ = \; & \frac{(2\sigma)_{j+\sigma-i+1}}{(j+\sigma-i)!} (\tanh s)^{j+\sigma-i} \\ & \times 2F_1(2\sigma+1+j+\sigma-i, 2\sigma; j+\sigma-i+2; \tanh^2 s) \\ & \times \frac{(j+\sigma-i)!}{j+\sigma-i+1}. \end{aligned}$$

So, substitution of this into the sum (94) and reversing the order of summation yields

$$Sf_j(a_s) = (1-\tanh^2 s)^{\frac{\sigma+2}{2}} \sum_{i=0}^{\sigma} \binom{\sigma}{i} (-1)^{j+i} \frac{(2\sigma)_{j+i+1}\, j!}{(j+i+1)!} (\tanh s)^{j+2i}$$
$$\times_2 F_1(2\sigma+1+j+i, 2\sigma; j+i+2; \tanh^2 s) z^\sigma.$$

□

We now return to the language of section 3, so that σ corresponds to the natural number k. We can now state the main theorem on the K-types.

Theorem 9. *For $k \geq 1$, the highest weight vector for the K-type $V_j^* \otimes V_{k+j}$ is the function $F_j : \mathcal{D} \to V_\tau$, whose restriction to $A \cdot 0$ is given by*

$$F_j(t) = (1-t^2) \sum_{i=0}^{k} \binom{\sigma}{i} (-1)^{j+i} \frac{(2k)_{j+i+1}\, j!}{(j+i+1)!} t^{j+2i}$$
$$\times_2 F_1(2k+1+j+i, 2k; j+i+2; t^2) z^k.$$

Proof. Letting $t = \tanh s = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \cdot 0$, and applying the trivialisation mapping (61), together with (66), to the functions Sf_j in Proposition 8 immediately gives the result. □

References

1. Castro, Mirta M. and Grünbaum, F. Alberto, *The algebra of differential operators associated to a family of matrix-valued orthogonal polynomials: five instructive examples*, Int. Math. Res. Not.,2006
2. Gross, Benedict H. and Wallach, Nolan R., *On quaternionic discrete series representations, and their continuations*, J. Reine Angew. Math., **481**, (1996), 73–123
3. Helgason, Sigurdur "Differential geometry, Lie groups, and symmetric spaces", Pure and Applied Mathematics, 80, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978
4. Hua, L. K., "Harmonic analysis of functions of several complex variables in the classical domain", Translations of Mathematical Monographs, 6, American Mathematical Society, Providence, R.I., 1979
5. Knapp, Anthony W. "Representation theory of semisimple groups , an overview based on examples" Princeton University Press, Princeton, NJ, 2001
6. Knapp, A. W. and Wallach, N. R., *Szegö kernels associated with discrete series,* Invent. Math., **34**, (1976), 3, 163–200
7. Loos, O. "Bounded symmetric domains and Jordan pairs" Lecture notes, University of California, Irvine, 1977
8. Opdam, Eric M., *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math., **175**, (1995), 1, 75–121
9. Satake, Ichirô, "Algebraic structures of symmetric domains", Kanô Memorial Lectures, 4, Iwanami Shoten, Tokyo, 1980
10. Schmid, Wilfried, *Homogeneous complex manifolds and representations of semisimple Lie groups*, Representation theory and harmonic analysis on semisimple Lie groups, Math. Surveys Monogr., **31**, 223–286 Dissertation, University of California, Berkeley, CA, 1967
11. Wolf, Joseph A., *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech., **14**, 1965, 1033–1047

Department of Mathematics, Chalmers University of Technology and Göteborg University, Göteborg, Sweden
E-mail address: henriks@math.chalmers.se