THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

On Integral Representation with Weights on Complex Manifolds

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Abstract

We present a method for finding weighted integral representation formulas for differential forms on a complex manifold X for which there exists a vector bundle $E \to X \times X$ of rank $\dim X$, and a holomorphic section η of E that defines the diagonal of $X \times X$.

The method is applied to Stein manifolds, where we look at some examples of the uses of weights. Most of our applications, however, are to compact manifolds, such as Grassmannians, where we find weights which allow representations of forms with values in any holomorphic line bundle as well as in the tautological vector bundle and its dual. As a consequence we obtain some vanishing theorems of the Bott-Borel-Weil type. We also relate the projection part of our formulas to the Bergman kernels associated to the line bundles. We treat the special case of complex projective space $\mathbb{P}^n$ in some detail, as well as applying the method to $\mathbb{P}^n \times \mathbb{P}^m$.

We also find new integral representations of solutions to division problems in $\mathbb{C}^n$ involving matrices of polynomials. We find estimates of the polynomial degree of the solutions by means of careful degree estimates of the so-called Hefer forms which are components of the representations.

Keywords: integral representation, Bochner-Martinelli formula, Grassmannians, complex projective space, residue currents, effective Nullstellensatz

AMS 2000 Subject Classification: 32A26; 32L20; 32M10; 32M05; 32Q99; 13P10

This thesis consists of an introduction and the following papers:

Paper I: Elin Götmark. Weighted integral formulas on manifolds. Arkiv för matematik, to appear.

Paper II: Elin Götmark, Henrik Seppänen, Håkan Samuelsson. Koppelman formulas on Grassmannians.

Paper III: Elin Götmark. Explicit solutions of division problems for matrices of polynomials.

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Contents

1. Historical overview of integral representation

To study a function, it is often useful to express it as a sum of simpler functions. A holomorphic function in one complex variable can always be expressed locally as a Taylor series, that is, a sum of monomials. We also have the Cauchy integral formula, which expresses a holomorphic function as a superposition of simple rational functions,

$$\phi(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\phi(\zeta) d\zeta}{\zeta - z},$$

if $z \in D$. Note that the kernel is holomorphic in z when $\zeta \neq z$, and that it works for any domain D. The Cauchy integral formula is an indispensable tool and is the basis of central theorems in single-variable complex analysis.

In several complex variables we can still easily expand a holomorphic function as a Taylor series, but things are more complicated with integral formulas. The challenge is, given a particular domain, to construct a kernel for this domain which is holomorphic in z when $\zeta \neq z$. However, this is not possible for all domains.

One obvious generalization of the Cauchy integral formula is the product formula, which is holomorphic and works for a polydisc. We find it by simply applying the Cauchy formula one variable at a time. The Cauchy-Weil kernel is a generalization of the product kernel to analytic polyhedra, which is used to approximate for example pseudoconvex domains. Fantappiè in 1943 represented a holomorphic function as a superposition of functions of the type $1/(1 + \zeta \cdot z)$. This kernel is also holomorphic, but only works in lineally convex domains, for example the ball. As a contrast, we have the Bochner-Martinelli kernel (discovered in 1938 by Martinelli [27] and independently in 1944 by Bochner [12]), which works for all domains but is not holomorphic anywhere if $n > 1$.

Proposition 1.1. *Let $z \in \mathbb{C}^n$ be fixed and let*

$$b(\zeta) = \frac{1}{2\pi i} \frac{\partial |\zeta - z|^2}{|\zeta - z|^2}. \tag{1}$$

The Bochner-Martinelli kernel is

$$u_{BM} = b \wedge (\bar{\partial} b)^{n-1} = \frac{1}{(2\pi i)^n} \frac{\sum_i (\bar{\zeta}_i - \bar{z}_i) d\zeta_i \wedge (\sum d\bar{\zeta}_i \wedge d\zeta_i)^{n-1}}{|\zeta - z|^{2n}} \tag{2}$$

and it satisfies $\bar{\partial} u = [z]$, where $[z]$ is the current of integration over $\{z\}$.

We can see the last equality in (2) by noting that if $\bar{\partial}$ falls on the denominator of s, we will get a factor $\partial|\zeta|^2 \wedge \partial|\zeta|^2 = 0$. A more general kernel, the Cauchy-Fantappiè-Leray (CFL) kernel, was discovered in 1959 by Leray [25], but in the name he honored Cauchy and Fantappiè as influential mathematicians in the field. Let $\delta_{\zeta - z}$ denote contraction with the vector field

$$2\pi i \sum_1^n (\zeta_i - z_i) \frac{\partial}{\partial \zeta_i}.$$

Proposition 1.2. *Let $s(\zeta, z)$ be a smooth $(1,0)$-form on ∂D which satisfies $\delta_{\zeta - z} s = 1$ if $z \in D$. The CFL kernel is $u_{CFL} = s \wedge (\bar{\partial} s)^{n-1}$, and we have*

$$\phi(z) = \int_{\partial D} \phi u_{CFL}.$$

This kernel includes the Bochner-Martinelli kernel as a special case, choosing $s = b$. However, the CFL kernel is more flexible, and can e. g. be used to obtain a holomorphic kernel for a convex domain D, choosing

$$s = \frac{\partial \rho}{\delta_{\zeta - z} \partial \rho},$$

where $D = \{\zeta : \rho(\zeta) < 0\}$ and $\partial \rho \neq 0$ on ∂D.

Example 1. If D is the unit ball, we can take $\rho = |\zeta|^2 - 1$, which yields

$$u = s \wedge (\bar{\partial} s)^{n-1} = \frac{\partial|\zeta|^2 \wedge (\partial\bar{\partial}|\zeta|^2)^{n-1}}{(|\zeta|^2 - z \cdot \bar{\zeta})^n}.$$

More generally, we have the following definition.

Definition 1. For a given smooth domain D, a $(1,0)$-form $\sigma(\zeta, z)$ is called a holomorphic support function if it depends holomorphically on z in a neighborhood of $\bar{D}$ when $\zeta \in \partial D$, and $\delta_{\zeta - z} \sigma \neq 0$ when $z \in D$.

This has a geometric interpretation: let $\zeta \in \partial D$ be fixed. Then $\{z : \delta_{\zeta - z} \sigma(\zeta, z) = 0\}$ defines an analytic hypersurface that contains ζ and does not intersect D. We can set $s = \sigma / \delta_{\zeta - z} \sigma$ and use it in the construction of the CFL kernel. It is known that not all pseudoconvex domains admit a holomorphic support function.

In 1969 holomorphic support functions, and thus holomorphic representation kernels, for strictly pseudoconvex domains were found by Henkin [16] and independently by Ramirez [32].

Koppelman [24] rediscovered the CFL kernel in 1967, and shortly afterwards he introduced formulas to represent forms ϕ of degree (p, q) in some domain D. To do this, we need to regard z as a variable. If $\Delta = \{(\zeta, z) : \zeta = z\} \subset \mathbb{C}^n_\zeta \times \mathbb{C}^n_z$ is the diagonal, then to obtain a kernel for representing ϕ, we need to solve

$$\bar{\partial} K = [\Delta], \tag{3}$$

where $[\Delta]$ is the current of integration over Δ (cf. solving the equation $\bar{\partial} u = [z]$ to represent a holomorphic function). Note that $\bar{\partial}$ now acts on both ζ and z. The reason we need to solve (3) is that

$$\int_z \left(\int_\zeta \phi(\zeta) \wedge [\Delta] \right) \wedge \psi(z) = \int_{z,\zeta} \phi(\zeta) \wedge \psi(z) \wedge [\Delta] = \int_z \phi(z) \wedge \psi(z),$$

where ψ is an $(n-p, n-q)$ test form, so that

$$\int_\zeta \phi(\zeta) \wedge [\Delta] = \phi(z)$$

holds in the current sense. If $K(\zeta, z)$ solves (3), the so-called Koppelman formula will hold:

$$\phi(z) = \int_{\partial D} K \wedge \phi + \int_D K \wedge \bar{\partial}\phi + \bar{\partial}_z \int_D K \wedge \phi.$$

If $\bar{\partial}\phi = 0$ and the boundary integral vanishes, then we get a solution to the equation $\phi = \bar{\partial} u$ in D.

Example 2. Assume that D is a domain where we can find a holomorphic support function σ. Let

$$s = \frac{\sum \sigma_i (d\zeta_i - dz_i)}{\sum \sigma_i (\zeta_i - z_i)}.$$

We can extend s to be defined in D, as shown in e. g. [7], so that $K = s \wedge (\bar{\partial} s)^{n-1}$ solves (3). If ϕ is of bidegree (p, q) with $q > 1$, we have

$$\int_{\partial D} K \wedge \phi = 0.$$

This is because K cannot contain any $d\bar{z}_i$ differentials, since σ is holomorphic in z. So finding holomorphic support functions implies that we can solve the $\bar{\partial}$-equation.

In 1982, Andersson and Berndtsson [7] found a method of generating more flexible kernels, by using so-called weights (we will return to these later). Berndtsson further developed these in [10] and applied them to interpolation and division problems. To solve an interpolation problem is to extend holomorphically a given holomorphic function which originally is defined only on a subvariety, and to solve a division problem is to solve the equation $\phi = f \cdot p$, where ϕ is a given holomorphic function, and f is a given tuple of holomorphic functions. For both interpolation and division problems, explicit solutions can be constructed with integral formulas in many cases. We will return to division problems in Section 4.

Another application of integral representations is finding an elementary solution to the Levi problem, that is, to prove that a strictly pseudoconvex domain D is also a domain of holomorphy. This question was first posed by Levi in 1912, and answered affirmatively by Oka, Norguet and Bremermann independently in 1953. One can find a considerably simpler proof than theirs by using integral formulas: indeed, if $a \in \partial D$, one can explicitly construct a function f which is holomorphic in D and singular at a (see for example [33]).

2. A method of generating kernels for integral representation in $\mathbb{C}^n$

We will now discuss a method of generating kernels, presented originally in [1], which gives more general weighted formulas than [7]. This method is the inspiration for both the method used in Papers I and II, and the method of solving division problems used in Paper III.

For motivation, we begin with the one-dimensional case where we want to represent a holomorphic function ϕ. Let z be a fixed point. It is clear that if $u(\zeta, z)$ is a solution to

$$\bar{\partial} u = [z],$$

where $[z]$ is the Dirac measure at z considered as a $(1,1)$-current, then one would get an integral representation

$$\int_{\partial D_\zeta} \phi(\zeta) u(\zeta, z) = \int_{D_\zeta} \phi(\zeta) \bar{\partial} u(\zeta, z) = \phi(z),$$

by Stokes' theorem. But we can also note that the kernel of Cauchy's integral formula

$$u = \frac{1}{2\pi i} \frac{d\zeta}{\zeta - z}$$

satisfies the equation

$$\delta_{\zeta - z} u = 1,$$

where $\delta_{\zeta - z}$ denotes contraction with the vector field $2\pi i(\zeta - z)(\partial/\partial\zeta)$. We now define the operator

$$\nabla_{\zeta - z} = \delta_{\zeta - z} - \bar{\partial},$$

which will be of central importance. Note that $\nabla_{\zeta - z}^2 = 0$. By combining the two previous equations, we get

$$\nabla_{\zeta - z} u = 1 - [z]. \tag{4}$$

To generalize this to several complex variables, we let $\delta_{\zeta-z}$ denote contraction with the vector field

$$2\pi i \sum(\zeta_j - z_j)\frac{\partial}{\partial \zeta_j},$$

and look for solutions to (4) in $\mathbb{C}^n$. Note that the right hand side now contains one function and one (n,n)-current. Considering the actions of $\nabla_{\zeta-z}$, then, we must have $u = u_{1,0} + u_{2,1} + \ldots + u_{n,n-1}$, where $u_{k,k-1}$ has bidegree $(k, k-1)$. Equation (4) can then be written as a system of equations

$$(5) \quad \delta_{\zeta-z}u_{1,0} = 1, \qquad \delta_{\zeta-z}u_{2,1} - \bar\partial u_{1,0} = 0 \qquad \ldots \qquad \bar\partial u_{n,n-1} = [z].$$

We can construct a solution to (4) which has the Bochner-Martinelli kernel as its top degree term:

Proposition 2.1. *If*

$$u = \frac{b}{\nabla_{\zeta-z}b} := \sum b \wedge (\bar\partial b)^{k-1},$$

where b is defined by (1), then u solves (4).

Proof. We start by showing that u solves (4) outside $\{\zeta = z\}$. In fact, one can justify the calculation

$$\nabla_{\zeta-z}u = \nabla_{\zeta-z}\frac{b}{\nabla_{\zeta-z}b} = \frac{\nabla_{\zeta-z}b}{\nabla_{\zeta-z}b} - \frac{b}{(\nabla_{\zeta-z}b)^2} \wedge \nabla^2_{\zeta-z}b = 1,$$

where the last equality follows because $\nabla^2_{\zeta-z} = 0$. Alternatively, we can look at the equations in (5) separately: we have $\delta_{\zeta-z}\bar\partial b = -\bar\partial\delta_{\zeta-z}b = \bar\partial 1 = 0$, and using this, we get

$$\delta_{\zeta-z}u_{k+1} - \bar\partial u_k = \delta_{\zeta-z}(b \wedge (\bar\partial b)^k) - \bar\partial(b \wedge (\bar\partial b)^{k-1}) = (\bar\partial b)^k - (\bar\partial b)^k = 0.$$

Obviously also $\delta_{\zeta-z}b = 1$.

What happens when $\zeta = z$? Proposition 1.1 tells us that $\bar\partial u_{n,n-1} = [z]$ in the current sense. To prove the other equations in (5), we have to use, among other things, that all terms of u are locally integrable. □

One can find other solutions to (4) by using the CFL kernel (see Proposition 1.2), so that $u_{k,k-1} = s \wedge (\bar\partial s)^{k-1}$. See Proposition 2.2 in Paper I for a more general statement on how to find other solutions.

From the last equation and Stokes' theorem, it is clear that we will have

$$(6) \qquad \phi(z) = \int_{\partial D} \phi u_{n,n-1},$$

but the purpose of the other terms in u is less clear. Why not just solve $\bar\partial u_{n,n-1} = [z]$ and ignore the other $u_{k,k-1}$? It turns out that to get more flexible kernels, we need all the terms in u. To this end, we define weights.

Definition 2. A weight with respect to the fixed point z is a smooth form $g(\zeta, z) = g_{0,0} + \cdots + g_{n,n}$ such that $\nabla_{\zeta - z} g = 0$ and $g_{0,0}(z,z) = 1$.

We can find new kernels by combining u and g:

Proposition 2.2. *If g is a weight with respect to z, and u solves (4), then*

$$\phi(z) = \int_{\partial D} \phi (u \wedge g)_{n,n} + \int_D \phi g_{n,n}$$

if $\phi \in \mathcal{O}(\overline{D})$ and $z \in D$.

Proof. We have

$$\nabla_{\zeta - z}(u \wedge g) = \nabla_{\zeta - z} u \wedge g = (1 - [z]) \wedge g = g - [z].$$

Then $\bar\partial (u \wedge g)_{n,n} = g_{n,n} - [z]$, and the proposition follows. □

Here are some examples of weights:

Example 3. If $q = q_{1,0} + \cdots + q_{n,n-1}$, then $g = 1 + \nabla_{\zeta - z} q$ is a weight.

Example 4. If g_1 and g_2 are weights, then $g_1 \wedge g_2$ is also a weight.

Example 5. If $G(\lambda)$ is a holomorphic function of one complex variable such that $G(0) = 1$, and g is a weight, then it is possible to define a new weight $G(g)$, since there is a functional calculus for forms of even degree. If $g = 1 + \nabla_{\zeta - z} q$ with q a $(1,0)$-form, then

$$G(g) = \sum_0^n G^{(k)}(\delta_{\zeta - z} q)(-\bar\partial q)^k / k!.$$

Example 6. Let χ be a cut-off function that is 1 in a neighborhood of z and has compact support in D. If u solves (4), it is easy to see that $g = \chi - \bar\partial \chi \wedge u$ is a weight with compact support in D. Note that this gets rid of the boundary integral in Proposition 2.2, and we get the representation

$$\phi(z) = \int \bar\partial \chi \wedge \phi u_{n,n-1},$$

which can be seen as a "smoothed-out" version of the unweighted representation formula (6).

Weights are useful for example if our function ϕ behaves badly close to ∂D. We can then use a weight to compensate for the behavior of ϕ:

Example 7. Let ϕ be a holomorphic function in the unit ball B, and suppose that ϕ grows polynomially at the boundary, that is,

$$|\phi(\zeta)|^2 \leq C(1 - |\zeta|^2)^{-\rho}$$

close to ∂B for some ρ. If we just use the representation (6), the integral may not converge. Let

$$g = \left(1 + \nabla_{\zeta - z} \frac{\partial |\zeta|^2}{2\pi i (1 - |\zeta|^2)}\right)^{-r} = \left(\frac{1 - \bar\zeta \cdot z}{1 - |\zeta|^2} - \bar\partial \frac{\partial |\zeta|^2}{2\pi i (1 - |\zeta|^2)}\right)^{-r}.$$

The form g is a weight according to Example 5. Note that we have

$$g_{k,k} = \frac{1}{(2\pi i)^k}\binom{-r}{k}\left(\frac{1-|\zeta|^2}{1-\bar\zeta\cdot z}\right)^{r+k}\left(\bar\partial\frac{\partial|\zeta|^2}{|\zeta|^2-1}\right)^k,$$

so that $g_{k,k}$ vanishes on the boundary if r is large enough. We can use the representation

$$\phi(z) = \int_{\partial B}\phi(u\wedge g)_{n,n} + \int_B \phi g_{n,n}.$$

If r is large enough, g will compensate for the growth of ϕ close to the boundary, so that the first integral vanishes and the second one is convergent. This results in the formula

$$\phi(z) = \int_B \phi g_{n,n} = C\int_B \phi(\zeta)\left(\frac{1-|\zeta|^2}{1-\bar\zeta\cdot z}\right)^{r+n}\left(\bar\partial\frac{\partial|\zeta|^2}{1-\bar\zeta\cdot z}\right)^n.$$

We can construct an analogous weight on strictly pseudoconvex domains.

Example 8. Let $D=\{\rho<0\}$ be a strictly pseudoconvex domain with a strictly plurisubharmonic defining function ρ. To construct weights for D, we find a smooth form $h(\zeta,z)$ on $\bar D$, holomorphic in z, such that

$$2\text{Re}\,\delta_{\zeta-z}h(\zeta,z)\ge\rho(\zeta)-\rho(z)+\delta|\zeta-z|^2,$$

and then set $v(\zeta,z)=\delta_{\zeta-z}h(\zeta,z)-\rho(\zeta)$. In fact, $v(\zeta,z)$ will then be an approximation of the polarization of ρ. For a weight, we can take

$$g=(1-\nabla_{\zeta-z}(h/\rho))^{-r}=\left(-\frac{v}{\rho}+\bar\partial\frac{h}{\rho}\right)^{-r},$$

which vanishes on ∂D if $r>1$. We then have the representation

$$\phi(z)=\int_D\phi g_{n,n}=\int_D\phi(-\rho/v)^{r+n}(\bar\partial(h/\rho))^n.$$

If ϕ is a (p,q)-form instead of a function, the kernels discussed in this section so far will not work. As discussed before, we must instead solve (3). In Paper I, we give a more detailed presentation of how to find kernels for Koppelman formulas in $\mathbb{C}^n$, based on [1]. We will sketch the ideas here.

Consider the subbundle $E^*=\text{Span}\{d\zeta_1-dz_1,\dots,d\zeta_n-dz_n\}$ of the cotangent bundle $T^*_{1,0}(\mathbb{C}^n_\zeta\times\mathbb{C}^n_z)$. Let E be its dual bundle, and let

$$\eta = 2\pi i\sum_1^n(\zeta_j-z_j)e_j, \tag{7}$$

where $\{e_j\}$ is the dual basis to $\{d\zeta_j-dz_j\}$. Contraction with η will then act on E^*. We can now define our operator $\nabla_\eta=\delta_\eta-\bar\partial$, and find a solution to

$$\nabla_\eta u = 1-[\Delta], \tag{8}$$

since $u_{n,n-1}$ will then solve (3). In fact, if we replace $d\zeta_i$ in (1) with $d\zeta_i - dz_i$ and let

$$u = \frac{b}{\nabla_\eta b} := \sum b \wedge (\bar\partial b)^{k-1}, \tag{9}$$

then u will solve (8).

Definition 3. A weight is a smooth form $g = g_{0,0} + \cdots + g_{n,n}$, where $g_{k,k}$ takes values in $\Lambda^k E^* \wedge \Lambda^k T^*_{0,1}$, such that $\nabla_\eta g = 0$ and $g_{0,0}(z,z) = 1$.

If we set $K = (u \wedge g)_{n,n-1}$ and $P = g_{n,n}$, then they satisfy

$$\bar\partial K = [\Delta] - P.$$

Note the analogy with Proposition 2.2. Using K and P, we get the weighted Koppelman formula

$$\phi(z) = \int_{\partial D} K \wedge \phi + \int_D K \wedge \bar\partial\phi + \bar\partial_z \int_D K \wedge \phi + \int_D P \wedge \phi \tag{10}$$

for (p,q)-forms ϕ.

Example 9. We can adapt Example 8 to (p,q)-forms by letting $h(\zeta,z)$ be a smooth form in E^* such that

$$2\mathrm{Re}\, \delta_\eta h(\zeta,z) \geq \rho(\zeta) - \rho(z) + \delta|\zeta - z|^2$$

holds, and then set $v(\zeta,z) = \delta_\eta h(\zeta,z) + \rho(\zeta)$. We can then proceed analogously and obtain weighted formulas for (p,q)-forms on strictly pseudoconvex domains.

3. Approaches to solving the $\bar\partial$-equation

The equation $\bar\partial u = f$, where f is a (p,q)-form such that $\bar\partial f = 0$, is of central importance in complex analysis. We will discuss different approaches to solving this problem. It is always possible to solve the $\bar\partial$-equation locally - this is the well-known Dolbeault's lemma. The question then becomes how to solve it globally in a given domain or manifold, and obtaining estimates of various norms on u.

In the 1950's, Kodaira [22] proved vanishing theorems for positive line bundles on compact manifolds. A vanishing theorem gives conditions under which cohomology groups are trivial, which means exactly that all $\bar\partial$-closed forms are $\bar\partial$-exact. He used the isomorphism between cohomology groups and harmonic forms, and what is now called the Bochner-Kodaira-Nakano identity for (p,q)-forms with values in a vector bundle. In the 1960's, Kohn and Morrey solved the $\bar\partial$-Neumann problem, which gives a solution to the $\bar\partial$-problem on strictly pseudoconvex domains. However, there is a problem with regularity on the boundary. Hörmander [19] used the Bochner-Kodaira-Nakano identity in $\mathbb{C}^n$ to prove existence theorems for the $\bar\partial$-equation with weighted L^2-estimates in pseudoconvex domains in $\mathbb{C}^n$, and he was able to use the weights to resolve the problems at the boundary. For an historical overview, see [21]. The L^2-method of solving the $\bar{\partial}$-equation is still an active field of research, both on manifolds and on domains in $\mathbb{C}^n$ with boundary.

During the 1970's, integral formulas were first used by Henkin to solve $\bar{\partial}$-equations, and the techniques were developed further by for example Ovrelid, Kerzman, Grauert and Lieb. This strategy gave explicit solutions to the $\bar{\partial}$-equations, whose norms could then be estimated. For example, Henkin used Koppelman formulas together with his and Ramirez' result from 1969 to solve the $\bar{\partial}$-equation in strictly pseudoconvex domains. This paved the way for the Henkin-Skoda theorem ([17], [34]), which provided improved L^1-estimates on the boundary for solutions of the $\bar{\partial}$-equation. This is the first time weights were used, albeit in an ad hoc way.

We can exemplify the approach in one complex variable. If $f \in L^1(D)$ and we want to solve $\bar{\partial} u = f$ in the unit disc D, the ordinary Cauchy formula gives the solution

$$u = \frac{1}{\pi} \int_D \frac{f(\zeta)}{\zeta - z}.$$

Unfortunately, u is not in $L^1(\partial D)$. But for the weighted solution

$$\tilde{u} = \frac{1}{\pi} \int_D \frac{1 - |\zeta|^2}{1 - \zeta \cdot \bar{z}} \cdot \frac{f(\zeta)}{\zeta - z},$$

we have the estimate

$$\int_{\partial D} |\tilde{u}| \leq C \int_D |f|.$$

If we are in a strictly pseudoconvex domain in $\mathbb{C}^n$, we can use the weight in Example 9.

The advent of a more systematic approach to weights in the 1980's, following [7], allowed the integral formula approach to be used on new classes of functions, as in the following example:

Example 10. Let ϕ be a $\bar{\partial}$-closed $(0,1)$-form on $\mathbb{C}^n$ which grows slower than e^u, where u is a convex function. Let $u_j = \partial u / \partial \zeta_j$. We can set

$$g = \exp[-\nabla_\eta (2 \sum u_j(\zeta)(d\zeta_j - dz_j))],$$

and it is easy to see that g is a weight in the sense of Definition 3. If we take $D = B(0, R)$ in the weighted Koppelman formula, and let $R \to \infty$, one can prove that the first integral will vanish and the others will be convergent, using the growth condition on ϕ and the fact that u is convex. Since g contains no $d\bar{z}_i$, the last integral will vanish, and we get an explicit solution to the $\bar{\partial}$-problem.

On manifolds, the integral formula approach is not as well developed as the L^2 methods. Stein manifolds have been treated mainly in [18] and [14], and complex projective space in [30] and [9]. As for more general approaches, there is [8], where integral formulas are found for manifolds X such that there exists a vector bundle $E \to X \times X$ of rank $\dim(X)$, and a holomorphic section η of E that defines the diagonal of $X \times X$. This approach is part of the inspiration for Paper I.

4. Approaches to solving division problems

Let $f = (f_1, \dots, f_m)$ be a tuple of holomorphic functions. If ϕ is a given holomorphic function, we can ask whether it is possible to write

$$\phi = f \cdot \psi := f_1\psi_1 + \cdots f_m\psi_m$$

where ψ is holomorphic, or equivalently, if ϕ lies in the ideal (f) generated by $f_1, \dots, f_m$. This is called the membership problem. Obviously a necessary condition is that ϕ vanishes on the variety $Z = \{z : f_1 = \cdots = f_m = 0\}$. Once we find a solution ψ, we are usually interested in its properties, such as estimates of norms on ψ.

One method of solving division problems is to use the Koszul complex, which was first done in this context by Hörmander [20]. The idea is to begin by finding a current solution, and then modify it by solving a succession of $\bar\partial$-equations, to obtain a holomorphic solution. In this method, we view $f = \sum f_i e_i^*$ as a section of a trivial vector bundle $E^* \to \mathbb{C}^n$ with holomorphic frame $\{e_i^*\}$, with dual bundle E. Regard the bundle $\Lambda[T^*(\mathbb{C}^n) \oplus E \oplus E^*]$, where we can for example take wedge products of differential forms and sections of E. The Koszul complex of f is

$$0 \longrightarrow \Lambda^m E \xrightarrow{\delta_f} \cdots \xrightarrow{\delta_f} \Lambda^2 E \xrightarrow{\delta_f} E \xrightarrow{\delta_f} \mathbb{C} \to 0,$$

where δ_f is contraction with f. We can now reformulate our division problem, and look for a holomorphic solution to $\delta_f \psi = \phi$. Obviously, outside Z we can choose $u_1 = \sum \bar f_i e_i / |f|^2$ as a smooth solution. Both δ_f and $\bar\partial$ extend to currents taking values in sections of $\Lambda[T^*(\mathbb{C}^n) \oplus E \oplus E^*]$, note also that they anticommute. Since $\delta_f \bar\partial u_1 = -\bar\partial \delta_f u_1 = 0$ and the Koszul complex is pointwise exact outside Z, we can solve $\delta_f u_2 = -\bar\partial u_1$. In the same way, we successively solve the equations $\delta_f u_k = -\bar\partial u_{k-1}$ for $k \geq 2$. We now assume that the u_k can be extended as currents U_k over Z such that these equations still hold. In that case, we can solve the following system of equations:

$$\begin{aligned} \bar\partial w_\mu &= U_\mu \\ \bar\partial w_{\mu-1} &= U_{\mu-1} + \delta_f w_\mu \\ &\vdots \\ \bar\partial w_2 &= U_2 + \delta_f w_3. \end{aligned} \tag{11}$$

and use $\psi = U_1 + \delta_f w_2$ as our holomorphic solution. This method relies on being able to solve $\bar\partial$-equations, as in the previous section. So if one wants to estimate the solutions, one has to rely on a succession of estimates of solutions of the $\bar\partial$-equation.

There is also a method due to Skoda [35], which is based on Hörmander's L^2-methods combined with complex geometry. We will sketch the idea behind this method. As above, we view ψ as a section of E, and look for a solution to $\delta_f \psi = \phi$. Obviously ϕu_1 is a smooth solution outside Z, so if we find v satisfying $\delta_f v = 0$ and $\bar\partial v = \phi \bar\partial u_1$, then $\psi = \phi u_1 - v$ is a holomorphic solution. This amounts to solving $\bar\partial v = \phi \bar\partial u_1$ in the subbundle $\operatorname{Ker} \delta_f \subset E$, since $\delta_f(\phi \bar\partial u_1) = 0$. Loosely speaking, the problem is that if E is a trivial bundle to begin with, then generally $\operatorname{Ker} \delta_f$ has negative curvature, and one cannot solve the $\bar\partial$-equation. The remedy is to make sure that the curvature of E is positive enough, by modifying the metric on E with a factor $e^{-c \log |f|}$, for a large enough c.

One form of Skoda's result is the following: suppose ϕ satisfies

$$I_1 = \int_\Omega |\phi|^2 |f|^{-2\alpha q - 2} e^{-\varphi} < \infty$$

where $\alpha > 1$, $\Omega \in \mathbb{C}^n$ is a plurisubharmonic domain, and φ is plurisubharmonic in Ω. We then have $\phi = f \cdot \psi$ where

$$\int_\Omega |\psi|^2 |f|^{-2\alpha q} e^{-\varphi} \le \alpha/(\alpha - 1) I_1.$$

One can also use weighted integral formulas to solve division problems. The idea is to find a weight that includes the factor $f(z)$. Here is an example from [10], reformulated to fit our formalism.

Example 11. Berndtsson's division formula.
Let $f = (f_1, ..., f_m)$ be a tuple of holomorphic functions in a strictly pseudoconvex domain $D \subset \mathbb{C}^n$, with no common zeroes, and let ϕ be another holomorphic function in D. We will construct a holomorphic $\psi = (\psi_1, \dots, \psi_m)$ by means of integral formulas, such that $f \cdot \psi = \phi$. Let $\sigma = (\bar f_1/|f|^2, \dots, \bar f_m/|f|^2)$ and $\mu = \min(n+1, m)$.

Let h be a tuple of holomorphic $(1,0)$-forms $h(\zeta, z) = (h_1, \dots, h_m)$, so-called Hefer forms, such that $\delta_{\zeta - z} h_j = f_j(\zeta) - f_j(z)$. We can find such forms by writing

$$f(\zeta) - f(z) = \int_{\partial D_w} (u(w,\zeta) - u(w,z)) f(w),$$

where u is constructed by means of a holomorphic support function, and then finding Hefer forms forms for u.

Consider

$$g_1 = (1 - \nabla_{\zeta - z}(h \cdot \sigma))^\mu = (f(z) \cdot \sigma + h \cdot \bar\partial \sigma)^\mu,$$

which is a weight depending holomorphically on z. By Proposition 2.2 we have

$$\phi(z) = \int_{\partial D} \phi (u \wedge g_1)_{n,n} + \int_D \phi (g_1)_{n,n}. \tag{12}$$

Now, note that we have $(h \cdot \bar\partial\sigma)^\mu = 0$. In fact, it is clear that every term in $(\bar\partial\sigma)^{n+1}$ vanishes for degree reasons, and since $f \cdot \sigma = 1$, we have $f \cdot \bar\partial\sigma = 0$, which says that the $\bar\partial\sigma_i$ are linearly dependent, and so $\bar\partial\sigma_1 \wedge \ldots \wedge \bar\partial\sigma_m = 0$ and $(h \cdot \bar\partial\sigma)^m = 0$. This implies that $f(z)$ will be a factor in g_1, so that we can write $g_1 = f(z) \cdot A(\zeta, z)$, and by means of this we can define our ψ.

If we have no control over the boundary behavior of f, we can use the use the weight in Example 6 to eliminate the boundary integral and ensure convergence. If f has polynomial growth at the boundary, we can use a weight of the type in Example 7.

We can also look at a special case, where $D = \mathbb{C}^n$ and ϕ and the f_i's are polynomials. We can then use a weight

$$g_2 = \left(1 - \nabla \frac{\bar\zeta \cdot d\zeta}{1 + |\zeta|}\right)^r = \left(\frac{1 + z \cdot \bar\zeta}{1 + |\zeta|^2} + i\bar\partial \frac{\bar\zeta \cdot d\zeta}{1 + |\zeta|^2}\right)^r,$$

which will compensate for the polynomial growth of g_1 if r is large enough, so that the integral over $\mathbb{C}^n$ will be convergent.

Example 12. There is an alternative weight due to Andersson (originally in [2], also see [5]) which can be used to solve division problems. Andersson's weight is more amenable to generalizations.

Let f, σ, h and μ be as in the previous example, and let ϕ be a holomorphic function in $\mathbb{C}^n$. As in the section on the Koszul complex, we will view f as a section $\sum f_i e_i^*$ of a trivial bundle E^*, $\sigma = \sum \bar f_i e_i / |f|^2$ as taking values in E, and the Hefer form h as taking values in E^*. We define

$$g_1 = f(z) \cdot \sum_0^{\mu-1} \delta_h^k (\sigma \wedge (\bar\partial\sigma)^k).$$

It is not hard to show that g_1 is a weight, and obviously g_1 contains a factor $f(z)$. We then get a representation (12). As when we use Berndtsson's weight, we can use a second weight, which will depend on the domain D, to eliminate the boundary integral or compensate for the behavior of ϕ.

5. Papers I and II

The purpose of Papers I and II is to present a method for finding integral representation formulas on certain manifolds, and to apply the method to Grassmannians, complex projective space and Stein manifolds.

The idea is to generalize the method for obtaining kernels for representing (p, q)-forms on $\mathbb{C}^n$ that we sketched in the end of Section 2. Note that we cannot in general hope to solve the equation $\bar\partial K = [\Delta]$ on a manifold, since $[\Delta]$ may represent a non-trivial cohomology class. Instead, we aim to solve

$$\bar\partial K = [\Delta] - P, \tag{13}$$

where P is a smooth form. This will give us a Koppelman formula as in (10).

Let X be a complex manifold of dimension n. To generalize the method on $\mathbb{C}^n$, we need to find a replacement for the vector field η, so we assume that there exists a vector bundle $E \to X_\zeta \times X_z$ of rank n, and a holomorphic section η of E that defines the diagonal Δ of $X \times X$. The construction with E and η is originally from Berndtsson [8], and we get the same formulas as he does. However, combining his construction with the method involving the operator ∇_η, inspired by [1], will allow us to use weights easily.

In analogy with the $\mathbb{C}^n$ case, we define our operator

$$\nabla_\eta = \delta_\eta - \bar{\partial},$$

and we note that ∇_η is a superconnection in the sense of Quillen [31]. We will find a u such that

$$\nabla_\eta u = 1 - R, \tag{14}$$

where R is a $(0,n)$-current with values in $\Lambda^n E^*$ and support on Δ (cf. [6]).

We exemplify with the $\mathbb{C}^n$ case. The section η is defined by (7), and to find u we would set

$$b = \frac{1}{2\pi i} \frac{\sum (\zeta_i - z_i) e_i^*}{|\zeta - z|^2},$$

and then let $u = \sum b \wedge (\bar{\partial} b)^k$. Comparing this to (9), it is clear that u satisfies (14), and that if we replace the e_j^*'s in R with $d\zeta_j - dz_j$'s, we will get precisely $[\Delta]$. Of course, in this special case, the e_j^*'s already are equal to these differentials. In the general case we will instead need to find a form A taking values in $\Lambda^n E$ such that $R \cdot A = [\Delta]$ if we use the natural contraction, which together with (14) will give K and P which solve (13). We will not go into the details of this here, but A will actually involve the supercurvature of the operator ∇_η. Moreover, it turns out that $P = c_n(E)$, that is, the n:th Chern form of E.

We can also use weights in a similar way as before, and we then get representations of (p,q)-forms taking values in a vector bundle $H \to X$. A weight g for H is defined as a section $g = g_{0,0} + \cdots + g_{n,n}$, where $g_{k,k}$ takes values in

$$\operatorname{Hom}(H_\zeta, H_z) \otimes \left[\Lambda^k E^* \wedge \Lambda^k T^*_{0,1}(X \times X)\right], \tag{15}$$

such that $\nabla_\eta g = 0$ and $g_{0,0}(z,z) = \mathrm{Id}$. Note the similarities to Definition 3. This will yield a current K and a smooth form P taking values in $\operatorname{Hom}(H_\zeta, H_z)$, and by means of these we get a weighted Koppelman formula for (p,q)-forms taking values in H.

In Paper II we apply our method to Grassmannians. The Grassmannian $X = Gr(k,N)$ is defined as the set of k-dimensional subspaces of

$\mathbb{C}^N$, which one can give a natural structure as an $k(N-k)$-dimensional manifold.

We need to find a vector bundle E and a section η with the right properties. To this end, we first define the tautological vector bundle H, which is a rank k sub-bundle of the trivial rank N bundle $\mathbb{C}^N \to X$, such that the fiber of H above $p \in X$ is the k-plane in $\mathbb{C}^N$ corresponding to the point p. We can also define the quotient bundle $F := \mathbb{C}^N/H$, which is a holomorphic vector bundle of rank $N-k$. We let $\pi_z : X_\zeta \times X_z \to X_z$ be the natural projection, and likewise for π_ζ. Finally, we define

$$E = \pi_z^* F \otimes \pi_\zeta^* H,$$

or in simpler notation, $F_z \otimes H_\zeta^*$. We can also view E as $\mathrm{Hom}\,(H_\zeta, F_z)$. Note that the rank of E is $k(N-k)$.

To define a mapping $\eta : H_\zeta \to F_z$ defining the diagonal, we start with a vector $v \in H_\zeta$, which we can identify with a vector $\tilde{v} \in \mathbb{C}^N$. We then let $\eta(v)$ be the projection of $\tilde{v}$ onto $F_z = \mathbb{C}^N/H_z$. It is clear that $\eta(v)$ vanishes if and only if $H_z = H_\zeta$ and thus $z = \zeta$. Hence, η is a global section of E and vanishes precisely on Δ. In fact, with appropriately chosen coordinates and frames, η has the simple form $\eta = \zeta - z$.

We can also find weights for the bundles H, H^*, $L := \det H$, and L^k for all k. For each bundle, this gives us K and P with which we can get weighted Koppelman formulas. In fact, using representation theory one can prove that all holomorphic line bundles over X are given as some power of L. We can also identify the form P which is associated with L^k with the Bergman kernel for L^{-k}.

A special case of the Grassmannian is the complex projective space $\mathbb{P}^n$, and integral representation on $\mathbb{P}^n$ is treated in Paper I. We define $\mathbb{P}^n$ as the set of complex lines through the origin in $\mathbb{C}^{n+1}$, so that $\mathbb{P}^n = Gr(1, n+1)$. In the $\mathbb{P}^n$ case, H will be a line bundle and thus $H = L$.

Remark 1. There are some differences between the construction on $\mathbb{P}^n$ in Paper I and the construction on Grassmannians in Paper II. The first difference is that in Paper I, everything in $\mathbb{P}^n$ is expressed in homogeneous coordinates, while in Paper II, the section η and the weights have both a coordinate-free description and an expression in local coordinates. A difference in notation is that in Paper II we have the line bundle $L = \det H$, but this bundle is equal to L^{-1} in Paper I.

The weight

$$\alpha = \alpha_0 + \alpha_1 = \frac{z \cdot \bar{\zeta}}{|\zeta|^2} + 2\pi i \bar{\partial} \frac{\bar{\zeta} \cdot e^*}{|\zeta|^2}$$

from Paper I seemingly takes values in the wrong bundle. In the terminology of Paper I, it is clear that α_0 takes values in $L_z^1 \otimes L_\zeta^{-1} = \mathrm{Hom}\,(L_\zeta^1, L_z^1)$, as it should, cf. (15), α_1 ought to take values in $L_z^1 \otimes L_\zeta^{-1} \otimes (E^* \wedge T_{0,1}^*)$. As it is, α_1 takes values in $L_\zeta^{-1} \otimes (F^* \wedge T_{0,1}^*)$. But since $L_z^1 \otimes L_z^{-1} \cong \mathbb{C}$, the section α_1 takes values in

$$L_\zeta^{-1} \otimes (F^* \wedge T_{0,1}^*) \cong L_z^1 \otimes L_\zeta^{-1} \otimes (L_z^{-1} \otimes F^* \wedge T_{0,1}^*) =$$
$$= L_z^1 \otimes L_\zeta^{-1} \otimes (E^* \wedge T_{0,1}^*).$$

In Paper I, we also apply the general method to line bundles over $\mathbb{P}^n \times \mathbb{P}^m$ and to Stein manifolds. On Stein manifolds we have to modify it a little, since one cannot in general find a section η which is zero only on the diagonal - it might have other zeroes as well. Following [18], one can get around this snag by using a weight which damps out the unwanted singularities in the kernel.

Once we have Koppelman formulas for forms taking values in vector bundles over a manifold, we can use them to investigate the cohomology groups of the vector bundles. If X is a compact manifold such as a Grassmannian, the only obstruction to solving the $\bar{\partial}$-problem is the integral

$$\int_X P \wedge \phi = 0.$$

By examining for which p and q this integral vanishes, we can get vanishing theorems for the cohomology groups. In Paper I we do this for line bundles over $\mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^m$, and in Paper II we do it for line bundles over Grassmannians.

These vanishing theorems are already known, in fact, for a given cohomology group $H^{p,q}(Gr(k,N), L^r)$ one can determine with an algorithm whether it is trivial or not (see [36]). We do not find all groups that are trivial, but on the other hand we get explicit solutions to the $\bar{\partial}$-equation. In $\mathbb{P}^n$ we prove more cases than for general Grassmannians.

6. Paper III

In Paper III we use integral representation to solve division problems involving matrices of polynomials. We first give a background to the problem.

Let $F = (F_1, \dots, F_m)$ be a tuple of polynomials in $\mathbb{C}^n$, and Φ another polynomial in $\mathbb{C}^n$ which vanishes on the common zero set of F. Let $d_1 \geq d_2 \geq \cdots \geq d_m$ be the degrees of the F_i. By Hilbert's Nullstellensatz, there exist $\nu \in \mathbb{N}$ and polynomials $\Psi = (\Psi_1, \dots, \Psi_m)$ such that

$$\Phi^\nu = F \cdot \Psi = F_1\Psi_1 + \cdots + F_m\Psi_m, \tag{16}$$

or if F has no common zeroes in $\mathbb{C}^n$, we can solve the special case

$$1 = F \cdot \Psi. \tag{17}$$

However, this gives us no information on the size of ν or the polynomial degrees of the Ψ_j. These turn out to depend not only on the d_j, but on the zeroes of F and the singularity of F at infinity. Working in $\mathbb{P}^n$ will give us better control of what happens at infinity, so we homogenize F, so that $f_j(\zeta_0, \dots, z_n) = \zeta_0^{d_j} F(\zeta)$ is a section of the line bundle $\mathcal{O}(d_j) \to$

$\mathbb{P}^n$, and define $Z = \{f = 0\} \subseteq \mathbb{P}^n$. If we impose geometric conditions on Z, we can find Ψ with quite low degrees:

Example 13. Assume that $m = n + 1$ and that $Z = \emptyset$. In that case, the classical theorem of Macaulay [26] states that we can solve $F \cdot \Psi = 1$ with $\deg F_j \Psi_j \leq \sum d_j - n$.

Example 14. Assume that Z is discrete and contained in $\mathbb{C}^n$ and that Φ belongs to the ideal (F). Then by Max Nöther's theorem [29] there exists Ψ such that $\Phi = F \cdot \Psi$ with $\deg F_j \Psi_j \leq \deg \Phi$.

On the other hand, in the worst case scenarios the degree of Ψ can be much higher.

Example 15. In [28], Mayr and Meyer construct an F with $m = n+1$ such that $\zeta_1 - \zeta_n \in (F)$, and show that any Ψ satisfying $\zeta_1 - \zeta_n = F \cdot \Psi$ must have $\max \deg \Psi_j > (d-2)^{2^{k-1}}$, where $d = \max \deg F_j$ and $n = 10k$.

A breakthrough in finding an effective version of the Nullstellensatz in the general case was made by Brownawell [13]. Assuming that $Z \subseteq \{\zeta_0 = 0\}$, he proved an estimate

$$C\|f\| \geq \|\zeta_0\|^M \tag{18}$$

by algebraic methods, where $\|f\|^2 = \sum |F_j(\zeta)|^2/|(1+|\zeta|^2)|^{d_j}$ and $\|\zeta_0\|^2 = 1/(1+|\zeta|^2)$. The constant M depends on the d_j. He then used Skoda's L^2-method to obtain an estimate on the degree of the solution of (17). As a corollary, he also obtained a result for the general case (16).

Given an estimate (18), instead of using Skoda's L^2-method one can get precisely the same estimates on Ψ as in [13] by means of the Koszul complex (see [15]), or get estimates which are slightly less accurate by looking at an explicit Ψ given by integral formulas.

The optimal result, slightly better than Brownawell's, was found by Kollár [23], who used purely algebraic methods. We state one version of his result here: Let $d_j = d$ for all j and $\mu = \min(m, n)$. We can then solve (16) with $\nu \leq d^\mu$ and $\deg F_j \Psi_j \leq (\deg \Phi + 1) d^\mu$. See [23] for more details. This result should be compared to Example 15.

As for purely analytic methods, in [3] residue currents on $\mathbb{P}^n$ are used to capture the obstructions to solving (16) with $\nu = 1$. When the residue current vanishes, one gets a solution Ψ together with an estimate of its degree. Explicit solutions are also found by means of integral representation, though one then loses some precision in the degree estimates.

In Paper III, we let F be an $r \times m$ matrix of polynomials, and Φ an r-column of polynomials. If we know that

$$\Phi = F \cdot \Psi,$$

is solvable, where Ψ is an m-column of polynomials, we want to find an explicit Ψ and give an estimate of its degree Ψ. This division problem is treated in [4] by means of residue currents on $\mathbb{P}^n$. We will use weighted integral formulas on $\mathbb{P}^n$ to find explicit solutions, and our weight will be a generalization of the weight in Example 12. Now, recall the Hefer forms which are a component of this weight. A main part of the paper is dedicated to examining the degrees of generalized Hefer forms h, where h is a matrix, which will allow us to find estimates of the degree of Ψ. A major tool for determining the degrees of the Hefer forms is solving the equation $\alpha = \delta_{\zeta - z}\beta$, where $\alpha(\zeta, z)$ is a $\delta_{\zeta-z}$-closed $(l, 0)$-form in $\mathbb{C}^n$ with holomorphic polynomials of degree r for coefficients. We show that one can find a solution β which is a $(l-1, 0)$-form with holomorphic polynomials of degree $r - 1$ for coefficients. If $l = 0$ and $\alpha(\zeta)$ is a holomorphic polynomial, we can instead solve $\alpha(\zeta) - \alpha(z) = \delta_{\zeta-z}\beta$, where β satisfies the same conditions as above.

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WEIGHTED INTEGRAL FORMULAS ON MANIFOLDS

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Abstract. We present a method of finding weighted Koppelman formulas for (p,q)-forms on n-dimensional complex manifolds X which admit a vector bundle of rank n over $X \times X$, such that the diagonal of $X \times X$ has a defining section. We apply the method to $\mathbb{P}^n$ and find weighted Koppelman formulas for (p,q)-forms with values in a line bundle over $\mathbb{P}^n$. As an application, we look at the cohomology groups of (p,q)-forms over $\mathbb{P}^n$ with values in various line bundles, and find explicit solutions to the $\bar{\partial}$-equation in some of the trivial groups. We also look at cohomology groups of $(0,q)$-forms over $\mathbb{P}^n \times \mathbb{P}^m$ with values in various line bundles. Finally, we apply our method to developing weighted Koppelman formulas on Stein manifolds.

Contents

1. Introduction

The Cauchy integral formula provides a decomposition of a holomorphic function in one complex variable in simple rational functions, and is a cornerstone in function theory in one complex variable. The kernel is holomorphic and works for any domain. In several complex variables it is harder to find appropriate representations. The simplest multivariable analog, the Bochner-Martinelli kernel, is not as useful since the kernel is not holomorphic. The Cauchy-Fantappie-Leray formula is a generalization which gives a holomorphic kernel in domains which admit a holomorphic support function. Henkin and Ramirez in [17], [23] obtained holomorphic kernels in strictly pseudoconvex domains G by finding such support functions. Henkin also found solutions to the

$\bar{\partial}$-equation in such domains. This was done by means of a Koppelman formula, which represents a (p,q)-form ϕ defined in some domain D as a sum of integrals

$$\phi(z) = \int_{\partial D} K \wedge \phi + \int_D K \wedge \bar{\partial}\phi + \bar{\partial}_z \int_D K \wedge \phi + \int_D P \wedge \phi,$$

by means of the current K and the smooth form P. If ϕ is a closed form and the first and fourth terms of the right hand side of Koppelman's formula vanish, we get a solution of the $\bar{\partial}$-problem for ϕ. Henkin's result paved the way for the Henkin-Skoda theorem (see [18] and [24]), which provided improved L^1-estimates on ∂G for solutions of the $\bar{\partial}$-equation by weighting the integral formulas.

Andersson and Berndtsson [7] found a flexible method of generating weighted formulas for representing holomorphic functions and solutions of the $\bar{\partial}$-equation. It was further developed by Berndtsson [8] to find solutions to division and interpolation problems. If V is a regular analytic subvariety of some domain D in $\mathbb{C}^n$ and h is holomorphic in V, then Berndtsson found a kernel K such that

$$H(z) = \int_V h(\zeta)K(\zeta, z)$$

is a holomorphic function which extends h to D. If $f = (f_1, \ldots, f_m)$ are holomorphic functions without common zeros, he also found a solution to the division problem $\phi = f \cdot p$ for a given holomorphic function ϕ. Passare [20] used weighted integral formulas to solve a similar division problem, where the f_i's do have common zeros, but the zero sets have a complete intersection. He also proved the duality theorem for complete intersections (also proved independently by Dickenstein and Sessa [14]). Since then weighted integral formulas have been used by a row of authors to obtain qualitative estimates of solutions of the $\bar{\partial}$-equation and of division and interpolation problems, for example sharp approximation by polynomials [25], estimates of solutions to the Bézout equation [5], and explicit versions of the fundamental principle [11]. More examples and references can be found in the book [6]. More recently, Andersson [4] introduced a method generalizing [7] and [8] which is even more flexible and also easier to handle. It allows for some recently found representations with residue currents, applications to division and interpolation problems, and also allows for f to be a matrix of functions.

There have been several attempts to obtain integral formulas on manifolds. Berndtsson [10] gave a method of obtaining integral kernels on n-dimensional manifolds X which admit a vector bundle of rank n over $X \times X$ such that the diagonal has a defining section, but did not consider weighted formulas. Formulas on Stein manifolds were treated first in Henkin and Leiterer [19], where formulas for $(0,q)$-forms are found, then in Demailly and Laurent-Thiébaut [13], where the leading term in a kernel for (p,q)-forms is found, in Andersson [1], which is a generalization of [7] following Henkin and Leiterer, and finally in Berndtsson [10] where the method described therein is applied to Stein manifolds. Formulas on $\mathbb{P}^n$ have been considered in [21], where they were constructed by using known formulas in $\mathbb{C}^{n+1}$, and in [9], where they were constructed directly on $\mathbb{P}^n$. There is also an example at the end of Berndtsson [10], where the method of that article is applied to $\mathbb{P}^n$.

In this article, we begin in Section 2 by developing a method for generating weighted integral formulas on $\mathbb{C}^n$, following [2]. Section 3 describes a similar method which can be used on n-dimensional manifolds X which admit a vector bundle of rank n over $X \times X$ such that the diagonal has a defining section. It has similar results as the method described in [10], but with the added benefit of yielding weighted formulas. The method of Section 3 is applied to complex projective space $\mathbb{P}^n$ in Section 4, where we find a Koppelman formula for differential forms with values in a line bundle over $\mathbb{P}^n$. In the $\mathbb{P}^n$ case we get formulas which coincide with Berndtsson's formulas in [9] in the case $p = 0$, but they are not the same in the general (p,q)-case.

As an application, in Section 5 we look at the cohomology groups of (p,q)-forms over $\mathbb{P}^n$ with values in various line bundles, and find which of them are trivial (though we do not find all the trivial groups). Berndtsson's formulas in [9] give the same result. The trivial cohomology groups of the line bundles over $\mathbb{P}^n$ are, of course, known before, but our method gives explicit solutions of the $\bar\partial$-equations. In Section 6 we look instead at cohomology groups of $(0,q)$-forms over $\mathbb{P}^n \times \mathbb{P}^m$ with values in various line bundles. Finally, in Section 7 we apply the method of Section 3 to finding weighted integral formulas on Stein manifolds, following [19] but also developing weighted formulas.

2. Weighted Koppelman formulas in $\mathbb{C}^n$

As a model for obtaining representations on manifolds, we present the $\mathbb{C}^n$ case in some detail. The material in this section follows the last section of [2]. The article [2] is mostly concerned with representation of holomorphic functions, but in the last section a method of constructing weighted Koppelman formulas in $\mathbb{C}^n$ is indicated. We expand this material and give proofs in more detail. We begin with some motivation from the one-dimensional case:

One way of obtaining a representation formula for a holomorphic function would be to solve the equation

$$\bar\partial u = [z],$$

where $[z]$ is the Dirac measure at z considered as a $(1,1)$-current, since then one would get an integral formula by Stokes' theorem. Less obviously, note that the kernel of Cauchy's integral formula in $\mathbb{C}$ also satisfies the equation

$$\delta_{\zeta-z}u = 1,$$

where $\delta_{\zeta-z}$ denotes contraction with the vector field $2\pi i(\zeta - z)\partial/\partial\zeta$. These two can be combined into the equation

$$\nabla_{\zeta-z}u := (\delta_{\zeta-z} - \bar{\partial})u = 1 - [z]. \tag{1}$$

To find representation formulas for holomorphic functions in $\mathbb{C}^n$, we look for solutions to equation (1) in $\mathbb{C}^n$, where $\delta_{\zeta-z}$ is contraction with

$$2\pi i \sum(\zeta_j - z_j)\frac{\partial}{\partial\zeta_j}.$$

Since the right hand side of (1) contains one form of bidegree $(0,0)$ and one of bidegree (n,n), we must in fact have $u = u_{1,0}+u_{2,1}+\ldots+u_{n,n-1}$, where $u_{k,k-1}$ has bidegree $(k,k-1)$. We can then write (1) as the system of equations

$$\delta_{\zeta-z}u_{1,0} = 1, \qquad \delta_{\zeta-z}u_{1,2} - \bar{\partial}u_{1,0} = 0 \qquad \ldots \qquad \bar{\partial}u_{n,n-1} = [z].$$

In that case, $u_{n,n-1}$ will satisfy $\bar{\partial}u_{n,n-1} = [z]$ and will give a kernel for a representation formula. The advantage of this approach is that it easily allows for weighted integral formulas, as we will see.

To get Koppelman formulas for (p,q)-forms, we need to consider z as a variable and not a constant. If we find $u_{n,n-1}$ such that $\bar{\partial}u_{n,n-1} = [\Delta]$, where $\Delta = \{(\zeta,z) : \zeta = z\}$ is the diagonal of $\mathbb{C}^n_\zeta \times \mathbb{C}^n_z$ and $[\Delta]$ is the current of integration over Δ, then $u_{n,n-1}$ will be the kernel that we seek. In fact, if we let ϕ be a (p,q)-form, and ψ an $(n-p,n-q)$ test form, we have

$$\int_z \left(\int_\zeta \phi(\zeta)\wedge[\Delta]\right)\wedge\psi(z) = \int_{z,\zeta}\phi(\zeta)\wedge\psi(z)\wedge[\Delta] = \int_z \phi(z)\wedge\psi(z)$$

so that $\int_\zeta \phi(\zeta)\wedge[\Delta] = \phi(z)$ in the current sense.

In more detail, then: Let Ω be a domain in $\mathbb{C}^n$ and let $\eta(\zeta,z) = 2\pi i(z-\zeta)$, where $(\zeta,z)\in\Omega\times\Omega$. Note that η vanishes to the first order on the diagonal. Consider the subbundle $E^* = \mathrm{Span}\{d\eta_1,\ldots,d\eta_n\}$ of the cotangent bundle $T^*_{1,0}$ over $\Omega\times\Omega$. Let E be its dual bundle, and let δ_η be an operation on E^*, defined as contraction with the section

$$\sum_1^n \eta_j e_j, \tag{2}$$

where $\{e_j\}$ is the dual basis to $\{d\eta_j\}$. Note that δ_η anticommutes with $\bar{\partial}$.

Consider the bundle $\Lambda(T^*(\Omega\times\Omega)\oplus E^*)$ over $\Omega\times\Omega$. An example of an element of the fiber of this bundle at (ζ,z) is $d\zeta_1\wedge d\bar{z}_2\wedge d\eta_3$. We define

$$\mathcal{L}^m = \bigoplus_p C^\infty(\Omega \times \Omega, \Lambda^p E^* \wedge \Lambda^{p+m} T^*_{0,1}(\Omega \times \Omega)). \tag{3}$$

Note that $\mathcal{L}^m$ is a subset of the space of sections of $\Lambda(T^*(\Omega\times\Omega)\oplus E^*)$. Let $\mathcal{L}^m_{curr}$ be the corresponding space of currents. If $f \in \mathcal{L}^m$ and $g \in \mathcal{L}^k$, then $f \wedge g \in \mathcal{L}^{m+k}$.

We define the operator

$$\nabla = \nabla_\eta = \delta_\eta - \bar\partial,$$

which maps $\mathcal{L}^m$ to $\mathcal{L}^{m+1}$. We see that ∇ obeys Leibniz' rule, that is,

$$\nabla(f \wedge g) = \nabla f \wedge g + (-1)^m f \wedge \nabla g, \tag{4}$$

if $f \in \mathcal{L}^m$. Note that $\nabla^2 = 0$, which means that

$$\dots \xrightarrow{\nabla} \mathcal{L}^m \xrightarrow{\nabla} \mathcal{L}^{m+1} \xrightarrow{\nabla} \dots$$

is a complex. We also have the following useful property: If f is a form of bidegree $(n, n-1)$ and $D \subset \Omega \times \Omega$, then

$$\int_{\partial D} f = -\int_D \nabla f. \tag{5}$$

This follows from Stokes' theorem and the fact that $\int_D \delta_\eta f = 0$. The operator ∇ is defined also for currents, since $\bar\partial$ is defined for currents, and δ_η just amounts to multiplying with a smooth function, which is also defined for a current.

As in the beginning of this section, we want to find a solution to the equation

$$\nabla_\eta u = 1 - [\Delta]. \tag{6}$$

with $u \in \mathcal{L}^{-1}_{curr}$ (since the left hand side lies in $\mathcal{L}^0_{curr}$), so as before, we have $u = u_{1,0} + u_{2,1} + \dots + u_{n,n-1}$, where $u_{k,k-1}$ has degree k in E^* and degree $k-1$ in $T^*_{0,1}$.

Proposition 2.1. *Let*

$$b(\zeta, z) = \frac{1}{2\pi i}\frac{\partial|\eta|^2}{|\eta|^2}$$

and

$$u_{BM} = \frac{b}{\nabla_\eta b} = \frac{b}{1 - \bar\partial b} = b + b \wedge \bar\partial b + \dots b \wedge (\bar\partial b)^{n-1}, \tag{7}$$

where we get the right hand side by expanding the fraction in a geometric series. Then u solves equation (6).

The crucial step in the proof is showing that $\bar{\partial}(b \wedge (\bar{\partial} b)^{n-1}) = [\Delta]$, which is common knowledge, since $b \wedge (\bar{\partial} b)^{n-1}$ is actually the well-known Bochner-Martinelli kernel.

A form u which satifies $\nabla_\eta u = 1$ outside Δ is a good candidate for solving equation (6). The following proposition gives us a criterion for when such a u in fact is a solution:

Proposition 2.2. *Suppose* $u \in \mathcal{L}^{-1}(\Omega \times \Omega \setminus \Delta)$ *solves* $\nabla_\eta u = 1$, *and that* $|u_k| \lesssim |\eta|^{-(2k-1)}$. *We then have* $\nabla_\eta u = 1 - [\Delta]$.

Proof. Let u_{BM} be the form defined by (7), and let u be a form satisfying the conditions in the proposition. We know that $\nabla(u \wedge u_{BM}) = u_{BM} - u$ pointwise outside Δ, in light of (4). We want to show that this also holds in the current sense, i. e.

$$\int \nabla(u \wedge u_{BM}) \wedge \phi = \int (u_{BM} - u) \wedge \phi, \tag{8}$$

where ϕ is a test form in $\Omega \times \Omega$. Using firstly that $u \wedge u_{BM}$ is locally integrable (since $u \wedge u_{BM} = \mathcal{O}(|\eta|^{-(2n-2)})$ near Δ), and secondly (5), we get

$$\int \nabla(u \wedge u_{BM}) \wedge \phi = -\lim_{\epsilon \to 0} \int_{|\eta|>\epsilon} (u \wedge u_{BM}) \wedge \nabla\phi =$$

$$= \lim_{\epsilon \to 0} \left(\int_{|\eta|=\epsilon} u \wedge u_{BM} \wedge \phi + \int_{|\eta|>\epsilon} \nabla(u \wedge u_{BM}) \wedge \phi \right). \tag{9}$$

The boundary integral in (9) will converge to zero when $\epsilon \to 0$, since $u \wedge u_{BM} = \mathcal{O}(|\eta|^{-2n+2})$ and $\mathrm{Vol}(\{|\eta| = \epsilon\} \cap \mathrm{supp}(\phi)) = \mathcal{O}(\epsilon^{2n-1})$. As for the last integral in (9), we get

$$\lim_{\epsilon \to 0} \int_{|\eta|>\epsilon} \nabla(u \wedge u_{BM}) \wedge \phi = \lim_{\epsilon \to 0} \int_{|\eta|>\epsilon} (u_{BM} - u) \wedge \phi = \int (u_{BM} - u) \wedge \phi,$$

since $u_{BM} - u$ is locally integrable, thus $\nabla(u \wedge u_{BM}) = u_{BM} - u$ as currents. It follows that $\nabla u = \nabla u_{BM}$ since $\nabla^2 = 0$, and since u_{BM} satisfies the equation (6), u must also do so. □

Example 1. If s is a smooth $(1,0)$-form in $\Omega \times \Omega$ such that $|s| \lesssim |\eta|$ and $|\delta_\eta s| \gtrsim |\eta|^2$, we can set $u = s/\nabla s$. By Proposition 2.2, u will satisfy equation (6), and

$$u_{n,n-1} = \frac{s \wedge (\bar{\partial} s)^{n-1}}{(\delta_\eta s)^n}$$

is the classical Cauchy-Fantappie-Leray kernel.

We now introduce weights, which will allow us to get more flexible integral formulas:

Definition 1. A form $g \in \mathcal{L}^0(\Omega \times \Omega)$ is a weight if $g_{0,0}(z,z) = 1$ and $\nabla_\eta g = 0$.

The form $1 + \nabla Q$ is an example of a weight, if $Q \in \mathcal{L}^{-1}$. In fact, we have considerable flexibility when choosing weights: if Q is a $(1,0)$-form, $g = 1 + \nabla Q$, and $G(\lambda)$ is a holomorphic function such that $G(0) = 1$, then it is easy to see that

$$G(g) = \sum_0^n G^{(k)}(\delta_\eta Q)(-\bar\partial Q)^k/k!$$

is also a weight. We can now prove the following representation formula:

Theorem 2.3 (Koppelman's formula)**.** *Assume that $D \subset\subset \Omega$, $\phi \in \mathcal{E}_{p,q}(\bar D)$, and that the current K and the smooth form P solve the equation*

$$\bar\partial K = [\Delta] - P. \tag{10}$$

We then have

$$\phi(z) = \int_{\partial D} K \wedge \phi + \int_D K \wedge \bar\partial\phi + \bar\partial_z \int_D K \wedge \phi + \int_D P \wedge \phi, \tag{11}$$

where the integrals are taken over the ζ variable.

Proof. First assume that ϕ has compact support in D, so that the first integral in (11) vanishes. Take a test form $\psi(z)$ of bidegree $(n-p, n-q)$ in Ω. Then we have

$$\begin{aligned}
&\int_z \left(\int_\zeta K \wedge \bar\partial\phi + \bar\partial_z \int_\zeta K \wedge \phi + \int_\zeta P \wedge \phi\right) \wedge \psi = \\
&= \int_{z,\zeta} K \wedge d\phi \wedge \psi + (-1)^{p+q} \int_{z,\zeta} K \wedge \phi \wedge d\psi + \int_{z,\zeta} P \wedge \phi \wedge \psi = \\
&= \int_{z,\zeta} K \wedge d(\phi \wedge \psi) + \int_{z,\zeta} P \wedge \phi \wedge \psi = \\
&= \int_{z,\zeta} dK \wedge \phi \wedge \psi + \int_{z,\zeta} P \wedge \phi \wedge \psi = \int_z \phi \wedge \psi,
\end{aligned}$$

where we use Stokes' theorem repeatedly. If ϕ does not have compact support in D, we can prove the general case e g by replacing ϕ with $\chi_k\phi$, where $\chi_k \to \chi_D$, and let $k \to \infty$. □

It is easy to obtain K and P which solve (10): If we take g to be a weight and u to be a solution of (6), then we can solve the equation

$$\nabla_\eta v = g - [\Delta]$$

by choosing $v = u \wedge g$. This means that $K = (u \wedge g)_{n,n-1}$ and $P = g_{n,n}$ will solve (10).

Example 2. Let

$$g(\zeta,z)=1-\nabla\frac{1}{2\pi i}\frac{\bar\zeta\cdot d\eta}{1+|\zeta|^2}=\frac{1+\bar\zeta\cdot z}{1+|\zeta|^2}-\bar\partial\frac{i}{2\pi}\frac{\bar\zeta\cdot d\eta}{1+|\zeta|^2},$$

then g is a weight for all (ζ,z). Take a (p,q)-form $\phi(\zeta)$ which grows polynomially as $|\zeta|\to\infty$. If we let $K=(u\wedge g^k)_{n,n-1}$ and $P=(g^k)_{n,n}$, then

$$\phi(z)=\int_{|\zeta|=R}K\wedge\phi+\int_{|\zeta|\le R}K\wedge\bar\partial\phi+\bar\partial_z\int_{|\zeta|\le R}K\wedge\phi+\int_{|\zeta|\le R}P\wedge\phi.$$

If k is large enough, then the weight will compensate for the growth of ϕ, so that the boundary integral will go to zero when $R\to\infty$. We get the representation

$$\phi(z)=\int K\wedge\bar\partial\phi+\bar\partial_z\int K\wedge\phi+\int P\wedge\phi.$$

Note that if ϕ in (11) is a closed form and the first and fourth terms of the right hand side of Koppelman's formula vanish, we get a solution of the $\bar\partial$-problem for ϕ. Note also that the proof of Koppelman's formula works equally well over $X\times X$, where X is any complex manifold, provided that we can find K and P such that (10) holds. The purpose of the next section is to find such K and P in a special type of manifold.

3. A method for finding weighted Koppelman formulas on manifolds

We will now describe a method which can be used to find integral formulas on manifolds in certain cases, and which is modelled on the one in the previous section. The method is similar to one presented in [10], see Remark 2 at the end of this section for a comparison.

Let X be a complex manifold of dimension n, and let $E\to X_\zeta\times X_z$ be a vector bundle of rank n, such that we can find a holomorphic section η of E that defines the diagonal $\Delta=\{(\zeta,z):\zeta=z\}$ of $X\times X$. In other words, η must vanish to the first order on Δ and be non-zero elsewhere. Let $\{e_i\}$ be a local frame for E, and $\{e_i^*\}$ the dual local frame for E^*. Contraction with η is an operation on E^* which we denote by δ_η; if $\eta=\sum\eta_i e_i$ then

$$\delta_\eta\left(\sum\sigma_i e_i^*\right)=\sum\eta_i\sigma_i.$$

Set

$$\nabla_\eta=\delta_\eta-\bar\partial.$$

Choose a Hermitian metric h for E, let D_E be the Chern connection on E, and D_{E^*} the induced connection on E^*. Consider $G_E=C^\infty(X\times X,\Lambda[T^*(X\times X)\oplus E\oplus E^*])$. If A lies in $C^\infty(X\times X,T^*(X\times X)\otimes E\otimes E^*))$, then we define $\tilde A$ as the corresponding element in G_E, arranged with the differential form first, then the section of E and finally the section of E^*. For example, if $A = dz_1 \otimes e_1 \otimes e_1^*$, then $\tilde{A} = dz_1 \wedge e_1 \wedge e_1^*$.

To define a derivation D on G_E, we first let $Df = \widetilde{D_E f}$ for a section f of E, and $Dg = \widetilde{D_{E^*} g}$ for a section g of E^*. We then extend the definition by

$$D(\xi_1 \wedge \xi_2) = D\xi_1 \wedge \xi_2 + (-1)^{\deg \xi_1} \xi_1 \wedge D\xi_2,$$

where $D\xi_i = d\xi_i$ if ξ_i happens to be a differential form, and $\deg \xi_1$ is the total degree of ξ_1. For example, $\deg(\alpha \wedge e_1 \wedge e_1^*) = \deg \alpha + 2$, where $\deg \alpha$ is the degree of α as a differential form. We let

$$\mathcal{L}^m = \bigoplus_p C^\infty(X \times X, \Lambda^p E^* \wedge \Lambda^{p+m} T^*_{0,1}(X \times X));$$

note that $\mathcal{L}^m$ is a subspace of G_E. The operator ∇ will act in a natural way as $\nabla : \mathcal{L}^m \to \mathcal{L}^{m+1}$. Notice also the analogy with the construction (3) in $\mathbb{C}^n$. As before, if $f \in \mathcal{L}^m$ and $g \in \mathcal{L}^k$, then $f \wedge g \in \mathcal{L}^{m+k}$. We also see that ∇ obeys Leibniz' rule, and that $\nabla^2 = 0$. Let $\mathrm{End}(E)$ denote the bundle of endomorphisms of E.

Proposition 3.1. *If v is a differential form taking values in End(E), and $D_{End(E)}$ is the induced Chern connection on End(E), then*

$$\widetilde{D_{End(E)} v} = D\tilde{v}. \tag{12}$$

Proof. Suppose that $v = f \otimes g$, where f is a section of E and g a section of E^*. We prove first that

$$D_{\mathrm{End}(E)} v = D_E f \otimes g + f \otimes D_{E^*} g. \tag{13}$$

In fact, if s takes values in E, we have

$$(D_{\mathrm{End}(E)} v).s = D_E((g.s)f) - (g.(D_E s))f = d(g.s)f + (g.s)D_E f - (g.(D_E s))f =$$
$$= (g.s)D_E f + (D_{E^*} g.s)f = (D_E f \otimes g + f \otimes D_{E^*} g).s,$$

which proves (13). We have

$$\widetilde{D_{\mathrm{End}(E)} v} = \widetilde{D_E f \otimes g} + \widetilde{f \otimes D_{E^*} g} = Df \wedge g - f \wedge Dg = D\tilde{v}$$

which proves (12). If $v = \alpha \otimes f \otimes g$, where α is a differential form, we would have $D_{\mathrm{End}(E)} v = d\alpha \otimes f \otimes g + (-1)^{\deg \alpha} \alpha \otimes D_{\mathrm{End}(E)}(f \otimes g)$, so the result follows by an application of $\sim$. Since any differential form taking values in $\mathrm{End}(E)$ is a sum of such elements, the result follows by linearity. □

Definition 2. For a form $f(\zeta, z)$ on $X \times X$, we define

$$\int_E f(\zeta, z) \wedge e_1 \wedge e_1^* \wedge \ldots \wedge e_n \wedge e_n^* = f(\zeta, z).$$

Note that if I is the identity on E, then $\tilde{I} = e \wedge e^* = e_1 \wedge e_1^* + \ldots + e_n \wedge e_n^*$. It follows that $\tilde{I}_n = e_1 \wedge e_1^* \wedge \ldots \wedge e_n \wedge e_n^*$ (with the notation $a_n = a^n/n!$), so the definition above is independent of the choice of frame.

Proposition 3.2. *If $F \in G_E$ then*

$$d\int_E F = \int_E DF.$$

Proof. If $F = f \wedge \tilde{I}_n$ we have $d\int_E F = df$ and

$$\int_E DF = \int_E [df \wedge \tilde{I}_n \pm f \wedge D(\tilde{I}_n)].$$

It is obvious that $D_{\mathrm{End}(E)} I = 0$, and by Proposition 3.1 it follows that $D\tilde{I} = 0$, so we are finished. □

We will now construct integral formulas on $X \times X$. As a first step, we find a section σ of E^* such that $\delta_\eta \sigma = 1$ outside Δ. For reasons that will become apparent, we choose σ to have minimal pointwise norm with respect to the metric h, which means that $\sigma = \sum_{ij} h_{ij}\bar{\eta}_j e_i^*/|\eta|^2$. Close to Δ, it is obvious that $|\sigma| \lesssim 1/|\eta|$, and a calculation shows that we also have $|\bar{\partial}\sigma| \lesssim 1/|\eta|^2$. Next, we construct a section u with the property that $\nabla u = 1 - R$ where R has support on Δ. We set

$$u = \frac{\sigma}{\nabla_\eta \sigma} = \sum_{k=0}^{\infty} \sigma \wedge (\bar{\partial}\sigma)^k, \tag{14}$$

note that $u \in \mathcal{L}^{-1}$. By $u_{k,k-1}$ we will mean the term in u with degree k in E^* and degree $k-1$ in $T^*_{0,1}(X \times X)$. It is easily checked that $\nabla u = 1$ outside Δ.

We will need the following lemma:

Lemma 3.3. *If Θ is the Chern curvature tensor of E, then*

$$\nabla_\eta \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right) = 0.$$

Proof. The lemma will follow from the more general statement that if v takes values in $\mathrm{End}(E)$, then $\delta_\eta \tilde{v} = -v.\eta$. In fact, let $v = f \otimes g$, where f is a section of E and g a section of E^*; then we have $\delta_\eta(f \wedge g) = -f \wedge \eta.g = -(f \otimes g).\eta$. Now, note that $\bar{\partial}\tilde{\Theta} = 0$ since D is the Chern connection. We have

$$\nabla_\eta \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right) = -\frac{1}{2\pi i}\left[\bar{\partial} D\eta + \delta_\eta \tilde{\Theta}\right] = -\frac{1}{2\pi i}\left[\Theta\eta - \Theta\eta\right] = 0.$$

In the calculations we use that η is holomorphic and that $\bar{\partial}\theta = \Theta$ where θ is the connection matrix of D_E with respect to the frame e. □

The following theorem yields a Koppelman formula by Theorem 2.3:

Theorem 3.4. *Let $E \to X \times X$ be a vector bundle with a section η which defines the diagonal Δ of $X \times X$. We have*

$$\bar\partial K = [\Delta] - P,$$

where

$$K = \int_E u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_n \quad \text{and} \quad P = \int_E \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_n, \tag{15}$$

and u is defined by (14).

Note that since $D\eta$ contains no e_i's, we have

$$P = \int_E \left(\frac{i\tilde\Theta}{2\pi}\right)_n = \det \frac{i\Theta}{2\pi} = c_n(E),$$

i. e. the n:th Chern class of E.

Proof. We claim that

$$\frac{1}{(2\pi i)^n}\int_E R \wedge (D\eta)_n = [\Delta], \tag{16}$$

where R is defined by $\nabla u = 1 - R$. If this were true, we would have by Lemma 3.3 and Proposition 3.2

$$\bar\partial \int_E u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_n = \int_E \bar\partial \left[u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_n\right] =$$

$$= -\int_E \nabla \left[u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_n\right] =$$

$$= -\int_E \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_n + \frac{1}{(2\pi i)^n}\int_E R \wedge (D\eta)_n = [\Delta] - P.$$

We want to use Proposition 2.2 to prove the claim (16), so we need to express the left hand side of (16) in local coordinates. Since η defines Δ, we can choose $\eta_1, \dots, \eta_n$ together with some functions $\tau_1, \dots, \tau_n$ to form a coordinate system locally in a neighborhood of Δ. We have

$$\frac{1}{(2\pi i)^n}\int_E R \wedge (D\eta)_n = \bar\partial \frac{1}{(2\pi i)^n}\int_E \sigma \wedge (\bar\partial\sigma)^{n-1} \wedge (D\eta)_n,$$

and

$$\int_E \sigma \wedge (\bar\partial\sigma)^{n-1} \wedge (D\eta)_n = s \wedge (\bar\partial s)^{n-1} + A,$$

where $s = \sum \sigma_i d\eta_i$ and A contains only terms which lack some $d\eta_i$, i. e., every term in A will contain at least one η_i. Note that both s and A are now forms in $\mathbb{C}^n$. Recall that we have $|\sigma| \lesssim 1/|\eta|$ and $|\bar\partial\sigma| \lesssim 1/|\eta|^2$

close to Δ (this is why we chose σ to have minimal norm). Thus, by Theorem 2.2 we know that

$$\bar\partial[s \wedge (\bar\partial s)^{n-1}] = [\Delta],$$

so it suffices to show that $\bar\partial A = 0$ in the current sense. But since every term in A contains at least one η_i, the singularities which come from the σ_i's and $\bar\partial\sigma_i$'s will be alleviated, and in fact we have $A = \mathcal{O}(|\eta|^{-2n+2})$. A calculation shows also that $\bar\partial A = \mathcal{O}(|\eta|^{-2n+1})$, and it follows that $\bar\partial A = 0$ (also cf the proof of Proposition 2.2). □

It should be obvious from the proof that instead of $u = \sigma/\nabla\sigma$, we can choose any u such that $\nabla u = 1$ outside Δ and $|u_{k,k-1}| \lesssim |\eta|^{-2k+1}$.

We will obtain more flexible formulas if we use weights:

Definition 3. The section g with values in $\mathcal{L}_0$ is a weight if $\nabla g = 0$ and $g_{0,0}(z,z) = 1$.

Theorem 3.4 goes through with essentially the same proof if we take

$$K = \int_E u \wedge g \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_n \quad \text{and} \quad P = \int_E g \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_n, \tag{17}$$

as shown by the following calculation:

$$\bar\partial K = -\int_E \nabla u \wedge g \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_n = -\int_E (g-R) \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_n = [\Delta] - P,$$

which follows from the proof of Theorem 3.4 and the properties of weights. In the next section we will make use of weighted formulas.

Remark 1. If L is a line bundle over X, let L_ζ denote the line bundle over $X_\zeta \times X_z$ defined by $\pi^{-1}(L)$ where $\pi : X_\zeta \times X_z \to X_\zeta$. If we want to find formulas for (p,q)-forms $\phi(\zeta)$ taking values in some line bundle L over X, we can use a weight g taking values in $L_z \otimes L_\zeta^*$. In fact, then K and P will also take values in $L_z \otimes L_\zeta^*$, so that $\phi \wedge K$ and $\phi \wedge P$ take values in L_z. Integrating over ζ, we obtain $\phi(z)$ taking values in L.

Remark 2. To obtain more general formulas, one can find forms K and P such that

$$dK = [\Delta] - P \tag{18}$$

by setting $\nabla'_\eta = \delta_\eta - D$ and checking that the corresponding Lemma 3.3 and Theorem 3.4 are still valid. The main difference lies in the fact that since $(\nabla')^2 \neq 0$, we do not have $\nabla' u = 1$ outside Δ, but rather

$$\nabla' u = 1 - \frac{\sigma}{(\nabla'\sigma)^2} \wedge (\nabla')^2\sigma.$$

A calculation shows that $(\nabla')^2\sigma = \delta_\sigma(D\eta - \tilde\Theta)$, where δ_σ operates on sections of E. We have $\delta_\sigma(D\eta - \tilde\Theta) \wedge (D\eta - \tilde\Theta)^n = \delta_\sigma(D\eta - \tilde\Theta)^{n+1} = 0$ for degree reasons, so that Theorem 3.4 will still hold with ∇ replaced by ∇'. We can use weights in the same way, if we require that a weight g has the property $\nabla' g = 0$ instead of $\nabla g = 0$. In this article we are interested in applications which only require the formulas obtained by using ∇.

In [10] Berndtsson obtains P and K satisfying (18) by a different means, resulting in the same formulas, but without weights. Also noteworthy is that ∇' is a superconnection in the sense of Quillen [22], and our ∇ is the $(0,1)$-part of this superconnection. Lemma 3.3 for ∇' is a Bianchi identity for the superconnection.

4. Weighted Koppelman formulas on $\mathbb{P}^n$

We will now apply the method of the previous section to $X = \mathbb{P}^n$. We let $[\zeta] \in \mathbb{P}^n$ denote the equivalence class of $\zeta \in \mathbb{C}^{n+1}$. In order to construct the bundle E, we first let $F' = \mathbb{C}^{n+1} \times (\mathbb{P}^n_{[\zeta]} \times \mathbb{P}^n_{[z]})$ be the trivial bundle of rank $n+1$ over $\mathbb{P}^n_{[\zeta]} \times \mathbb{P}^n_{[z]}$. We next let F be the bundle of rank n over $\mathbb{P}^n_{[\zeta]} \times \mathbb{P}^n_{[z]}$ which has the fiber $\mathbb{C}^{n+1}/(\zeta)$ at the point $([\zeta],[z])$; F is thus a quotient bundle of F'. If α is a section of F', we denote its equivalence class in F with $[\alpha]$. We will not always bother with writing out the brackets, since it will usually be clear from the context whether a section is to be seen as taking values in F' or F. Let L^{-1} denote the tautological line bundle of $\mathbb{P}^n$, that is,

$$L^{-1} = \{([\zeta],\xi) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : \xi \in \mathbb{C}\cdot\zeta\}$$

We also define $L^{-k} = (L^{-1})^{\otimes k}$, $L^1 = (L^{-1})^*$ and $L^k = (L^1)^{\otimes k}$. Finally, let $E = F \otimes L^1_{[z]} \to \mathbb{P}^n_{[\zeta]} \times \mathbb{P}^n_{[z]}$. Observe that E is thus a subbundle of $E' = F' \otimes L^1_{[z]}$. It follows that $E^* = F^* \otimes L^{-1}_{[z]}$, where $F^* = \{\xi \in (F')^* : \xi\cdot\zeta = 0\}$. Berndtsson has the same setup in Example 3, page 337 of [10], but does not develop it as much (cf Remark 2 above).

A remark on notation: we will write a differential form $\alpha([\zeta])$ on $\mathbb{P}^n$ that takes values in L^k as a projective form on $\mathbb{C}^{n+1}$ which is k-homogeneous. That is, α will satisfy $\alpha(\lambda\zeta) = \alpha(\zeta)$, where $\lambda \in \mathbb{C}$, and $\delta_\zeta\alpha = \delta_{\bar\zeta}\alpha = 0$, where δ_ζ is contraction with the vector field $\zeta\cdot\partial/\partial\zeta$ and similarly for $\delta_{\bar z}$.

Let $\{e_i\}$ be an orthonormal basis of F'. The section η (cf Section 3) will be $\eta = z\cdot e = z_0e_0 + \dots z_ne_n$. Note that η takes values in $(F')\otimes L^1_{[z]}$, and will thus define an equivalence class in $F \otimes L^1_{[z]} = E$. The section η defines the diagonal since $[\eta(\zeta,\zeta)] = [\zeta\cdot e] = [0]$, so that η vanishes to the first order on Δ.

We will now choose a metric on E. On F' we choose the trivial metric, which induces the trivial metric also on $(F')^*$ and F^*. For $[\omega]$ taking values in $F = F'/(\zeta)$, the metric induced from F' is $\|[\omega]\|_F = \|\omega - \pi\omega\|_{F'}$, where π is the orthogonal projection $F' \to (\zeta)$. We choose the metric on $E = F \otimes L^1_{[z]}$ to be

$$\|\alpha\otimes[\omega]\|_E = \|\omega - \pi\omega\|_{F'}|\alpha|/|z| \tag{19}$$

for $\alpha \otimes [\omega] \in E$. We introduce the notation $\alpha \cdot \gamma := \alpha_1 \wedge \gamma_1 + \ldots \alpha_1 \wedge \gamma_1$, where α and γ are tuples containing differential forms or sections of a bundle.

Proposition 4.1. *Let $\omega \cdot e$ be a section of E. The Chern connection and curvature of E are*

$$D_E(\omega \cdot e) = d\omega \cdot e - \frac{d\zeta \cdot e}{|\zeta|^2} \wedge \bar{\zeta} \cdot \omega - \partial \log |z|^2 \wedge \omega \cdot e \tag{20}$$

$$\tilde{\Theta}_E = \partial\bar{\partial} \log |z|^2 \wedge e^* \cdot e - \bar{\partial} \frac{\bar{\zeta} \cdot e^*}{|\zeta|^2} \wedge d\zeta \cdot e, \tag{21}$$

with respect to the metric (19) and expressed in the frame $\{e_i\}$ for F'.

Proof. We begin with finding D_F. Let $\hat{\omega} \cdot e = (\omega \cdot \bar{\zeta}/|\zeta|^2)\zeta \cdot e$ be the projection of $\omega \cdot e$ onto $(\zeta \cdot e)$. Since the Chern connection $D_{F'}$ on F' is just d, it is easy to show that $D_F[\omega \cdot e] = [d(\omega \cdot e - \hat{\omega} \cdot e)]$. We have

$$D_F[\omega \cdot e] = [d(\omega \cdot e - \hat{\omega} \cdot e)] = [d\omega \cdot e - \frac{d\zeta \cdot e}{|\zeta|^2} \wedge \bar{\zeta} \cdot \omega],$$

since if d does not fall on ζ in the second term we get something that is in the zero equivalence class in F. If $\omega \cdot e$ is projective to start with, so will $d\omega \cdot e$ be, and $d\zeta \cdot e$ is a projective form since $\delta_\zeta(d\zeta \cdot e) = \zeta \cdot e = 0$ in F.

Since the metric on $L^1_{[z]}$ in the local frame z_0 is $|z_0|^2/|z|^2$, the local connection matrix will be $\partial \log(|z_0|^2/|z|^2)$. If ξ takes values in $L^1_{[z]}$, we get

$$D_{L^1_{[z]}}\xi = [d(\xi/z_0) + \partial \log(|z_0|^2/|z|^2)\xi/z_0]z_0 = d\xi - \partial \log |z|^2 \xi.$$

It is easy to see that $d(\xi/z_0) + \partial \log(|z_0|^2/|z|^2)\xi/z_0$ is a projective form, so $d\xi - \partial \log |z|^2 \xi$ is also projective. Combining the contributions from $L^1_{[z]}$ and F, we get (20), from which also (21) follows. □

We want to find the solution σ to the equation $\delta_\eta \sigma = 1$, such that σ has minimal norm in E^*. It is easy to see that the section $\bar{z} \cdot e^*/|z|^2$ is the minimal solution to $\delta_\eta v = 1$ in the bundle $(E')^* = (F')^* \otimes L^{-1}_{[z]}$. The projection of $\bar{z} \cdot e^*/|z|^2$ onto the subspace E^* is

$$s = \frac{\bar{z} \cdot e^*}{|z|^2} - \frac{\bar{z} \cdot \zeta}{|\zeta|^2 |z|^2} \bar{\zeta} \cdot e^*.$$

Since $\bar{z} \cdot e^*/|z|^2$ is minimal in $(F')^* \otimes L^{-1}_{[z]}$, s must be the minimal solution in E^*.

Finally, we normalize to get $\sigma = s/\delta_\eta s$. According to the method of the previous section, we can then set $u = \sigma/\nabla\sigma$ and obtain the forms P and K which will give us a Koppelman formula (see Theorem 3.4).

Remark 3. In local coordinates, for example where $\zeta_0, z_0 \neq 0$, we have

$$|\eta|^2 = \delta_\eta s = \frac{|\zeta|^2|z|^2 - |\bar{z}\cdot\zeta|^2}{|\zeta|^2|z|^2} = \frac{(1+|\zeta'|^2)(1+|z'|^2) - |1+\bar{z}'\cdot\zeta'|^2}{(1+|\zeta'|^2)(1+|z'|^2)},$$

where $\zeta' = (\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0)$ and analogously for z'. For the denominator we locally have $(1+|\zeta'|^2)(1+|z'|^2) \leq C$ for some constant C. As for the numerator, we have

$$\begin{aligned} & (1+|\zeta'|^2)(1+|z'|^2) - |1+\bar{z}'\cdot\zeta'|^2 = \\ = \; & 1+|\zeta'|^2+|z'|^2+|\zeta'|^2|z'|^2 - (1+2Re|\bar{z}'\cdot\zeta'|+|\bar{z}'\cdot\zeta'|^2) = \\ = \; & |z'-\zeta'|^2 + |\zeta'|^2|z'|^2 - |\bar{z}'\cdot\zeta'|^2 \geq |z'-\zeta'|^2 \end{aligned}$$

In all, we have $\delta_\eta s \gtrsim |z'-\zeta'|^2$.

To compute integrals of the type (17), we need the following proposition.

Lemma 4.2. *Let $A \overset{i}{\hookrightarrow} A'$, where A' is a given vector bundle with a given metric and $A = \{\xi$ taking values in $A' : f\cdot\xi = 0\}$ for a fixed f taking values in $(A')^*$. Let s be the dual section to f, and π be the orthogonal projection $\pi : G_{A'} \to G_A$ induced by the metric on A. If $B' \in G_{A'}$, and $B = \pi B'$, then*

$$\int_A B = \int_{A'} f \wedge s \wedge B'.$$

Proof. We can choose a frame for A' so that $e_0 = s$, and then extend it to an ON frame for A', so that $A = \text{Span}(e_1, \ldots, e_n)$. If we set $e_0^* = f$, we have

$$\int_{A'} f\wedge s\wedge B' = \int_{A'} e_0 \wedge e_0^* \wedge \pi B' = \int_A B$$

and we are done, since the integrals are independent of the frame. □

Note that if $E = A\otimes L$, where L is a line bundle, and $B \in G_E$, then $\int_E B = \int_A B$. At least, this is true if we interpret the latter integral to mean that if g is a local frame for L and g^* a local frame for L^*, then g and g^* should cancel out. Since there are as many elements from L as there are from L^*, there will be no line bundle elements left.

We will apply Lemma 4.2 with $A = E$, $A' = E'$ and $f = \zeta\cdot e^*$. We then have

$$P = \int_E \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi}\right)_n = \int_{E'} \frac{\bar{\zeta}\cdot e\wedge\zeta\cdot e^*}{|\zeta|^2} \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi}\right)_n$$

and similarly for K (this makes it easier to write down P and K explicitly).

By Theorem 3.4, we have

$$\bar{\partial}K = [\Delta] - P.$$

(These K and P are also found at the very end of [10].) We will now modify the method slightly, since in the paper [16] we found formulas for $(0,q)$-forms (derived in a slightly different way) which are more appealing than those we have just found, in that we get better results when we use them to solve $\bar\partial$-equations. We would thus like to have formulas for (p,q)-forms that coincide with those of [16] in the $(0,q)$-case.

The bundle F^* is actually isomorphic to $T^*_{1,0}(\mathbb{P}^n_{[\zeta]})$, and an explicit isomorphism is given by $\beta = d\zeta \cdot e$. In fact, if $\xi \cdot e^*$ takes values in F^*, then $\beta(\xi) = d\zeta \cdot \xi$. Since $\xi \cdot \zeta = 0$, the contraction of $\beta(\xi)$ with the vector field $\zeta \cdot \partial/\partial\zeta$ will be zero, so $\beta(\xi) \in T^*_{1,0}(\mathbb{P}^n_{[\zeta]})$. If v_{e^*} is a form with values in $\Lambda^n E^*$, then it is easy to see that

$$\int_E v_{e^*} \wedge \beta_n = v_{d\zeta}, \tag{22}$$

where we get $v_{d\zeta}$ by replacing every instance of e_i^* in v_{e^*} with $d\zeta_i$. For example, if $v_{e^*} = f(\zeta,z)e_0^* \wedge \ldots \wedge e_n^*$, then $v_{d\zeta} = f(\zeta,z)d\zeta_0 \wedge \ldots \wedge d\zeta_n$. We can use this to construct integral formulas for $(0,q)$-forms with values in $L^{-n}_{[\zeta]}$, by setting

$$K = \int_E u \wedge \beta_n.$$

The formulas we get from this are the same as in [16]. We will now combine these formulas with the ones in (15):

Theorem 4.3. *Let $D \subset \mathbb{P}^n$. If $\phi(\zeta)$ is a (p,q)-form with values in $L^{-n+p}_{[\zeta]}$ and*

$$\begin{aligned} K_p &= \int_E u \wedge \beta_{n-p} \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_p, \\ P_p &= \int_e \beta_{n-p} \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_p, \end{aligned} \tag{23}$$

with $\beta = d\zeta \cdot e^*$, we have the Koppelman formula

$$\phi([z]) = \int_{\partial D} \phi K_p \wedge \phi + \int_D \bar\partial K_p \wedge \phi + \bar\partial_{[z]} \int_D K_p \wedge \phi + \int_D P_p \wedge \phi,$$

where the integrals are taken over the $[\zeta]$ variable.

Proof. We have

$$\int_E \bar\partial u \wedge \beta_{n-p} \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}\right)_p = [\Delta], \tag{24}$$

where $[\Delta]$ should be integrated against sections of L^{-n+p} with bidegree (p,q). This follows from the proof of Theorem 3.4, since the singularity at Δ comes only from u, and is not affected by exchanging $\left(\frac{D\eta}{2\pi i}+\frac{i\tilde{\Theta}}{2\pi}\right)_{n-p}$ for β_{n-p}.

Using (24), we get

$$\begin{aligned} dK_p &= -\int_E \nabla\left[u\wedge\beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\tilde{\Theta}}{2\pi}\right)_p\right] \\ &= -\int_E (\nabla u)\wedge\beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\tilde{\Theta}}{2\pi}\right)_p = [\Delta]-P_p. \end{aligned}$$

The Koppelman formula then follows as in Theorem 2.3. □

To get formulas for other line bundles, we need to use weights (as defined in the previous section). We will use the weight

$$\alpha=\frac{z\cdot\bar{\zeta}}{|\zeta|^2}-2\pi i\bar{\partial}\frac{\bar{\zeta}\cdot e^*}{|\zeta|^2},$$

note that the first term in α takes values in $L^1_{[z]}\otimes L^{-1}_{[\zeta]}$, and the second is a projective form. We then get a Koppelman formula for (p,q)-forms ϕ with values in L^r by using

$$\begin{aligned} K_{p,r} &= \int_E u\wedge\alpha^{n-p+r}\wedge\beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\tilde{\Theta}}{2\pi}\right)_p, \\ P_{p,r} &= \int_e \alpha^{n-p+r}\wedge\beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\tilde{\Theta}}{2\pi}\right)_p. \end{aligned}$$

Remark 4. Let ϕ be a (p,q)-form. Since we cannot raise α to a negative power, how can we get a Koppelman formula if ϕ takes values in L^r where $r<p-n$? In fact, if we look at the proof of the Koppelman formula in Proposition 2.3, we see that the roles of ϕ and ψ are symmetrical: we could just as well use the proof to get a Koppelman formula for the $(n-p,n-q)$-form ψ which takes values in L^{-r}, using the kernels $K_{p,r}$ and $P_{p,r}$ in Theorem 4.3. This is a concrete realization of Serre duality, which in our case says that

$$H^{p,q}(\mathbb{P}^n,L^r)\simeq H^{n-p,n-q}(\mathbb{P}^n,L^{-r}).$$

We will make use of this dual technique when we look at cohomology groups in the next section.

Remark 5. In [9] Berndtsson constructs integral formulas for sections of line bundles over $\mathbb{P}^n$. These formulas coincide with ours in the case $p=0$, but they are not the same in the general (p,q)-case. Nonetheless, they do give the same result as our formulas when used to find the trivial cohomology groups of the line bundles of $\mathbb{P}^n$ (see the next section). More precisely, his formulas can also be used to prove Proposition 5.1 below, but no more, at least not in any obvious way.

5. An application: the cohomology of the line bundles of $\mathbb{P}^n$

Let D in Theorem 4.3 be the whole of $\mathbb{P}^n$; then the boundary integral will disappear. The only obstruction to solving the $\bar\partial$-equation is then the term containing $P_{p,r}$. We will use our explicit formula for $P_{p,r}$ to look at the cohomology groups of (p,q)-forms with values in different line bundles, and determine which of them are trivial. We have

$$\begin{aligned} P_{p,r} = & \int_E \beta_{n-p} \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi} \right)_p \wedge \alpha^{n-p+r} = \\ = & \int_{E'} \frac{\bar\zeta \cdot e \wedge \zeta \cdot e^*}{|\zeta|^2} \wedge (d\zeta \cdot e)_{n-p} \wedge \Bigg(dz \cdot e - \frac{z \cdot \bar\zeta}{|\zeta|^2} \wedge d\zeta \cdot e - \\ - & \frac{\partial |z|^2}{|z|^2} z \cdot e + \omega_z e^* \cdot e - \frac{d\bar\zeta \cdot e^* \wedge d\zeta \cdot e}{|\zeta|^2} \Bigg)_p \wedge \left(\frac{z \cdot \bar\zeta}{|\zeta|^2} - \bar\partial \frac{\bar\zeta \cdot e^*}{|\zeta|^2} \right)^{n-p+r} . \end{aligned}$$

We can now prove:

Proposition 5.1. *From the formula for $P_{p,r}$ just above, it follows that the cohomology groups $H^{p,q}(\mathbb{P}^n, L^r)$ are trivial in the following cases:*

a) $q = p \neq 0, n$ and $r \neq 0$.
b) $q = 0$, $r \leq p$ and $(r,p) \neq (0,0)$.
c) $q = n$, $r \geq p - n$ and $(r,p) \neq (0,n)$.
d) $p < q$ and $r \geq -(n-p)$.
e) $p > q$ and $r \leq p$.

Unfortunately, these are not all the trivial cohomology groups; instead of d) and e) we should ideally get that the groups are trivial for $q \neq 0, n, p$ (cf [12] page 397).

Proof. The general strategy is this: we take a $\bar\partial$-closed form $\phi(z)$ of given bidegree and with values in a given line bundle, and then try to show that $\phi(z)$ is exact by means of the Koppelman formula. One possibility of doing this is proving that $\int_\zeta \phi(\zeta) \wedge P_{p,r}(\zeta, z) = 0$, which can be either because the integrand is zero, or because the integrand is $\bar\partial_\zeta$-exact (since then Stokes' formula can be applied). Another possibility is proving that $P_{p,r}$ is $\bar\partial_z$-exact, since then $\int_\zeta \phi \wedge P_{p,r}$ will be $\bar\partial_z$-exact as well.

Proof of a): Let $r > 0$ and $p = q \neq 0, n$; we must then look at the term in $P_{p,r}$ with bidegree (p,p) in z and $(n-p,n-p)$ in ζ, it is equal to

(25)
$$C\int_{E'}\frac{\bar\zeta\cdot e\wedge\zeta\cdot e^*}{|\zeta|^2}\wedge(d\zeta\cdot e)_{n-p}\wedge(\omega_z\wedge e^*\cdot e)^p\wedge\left(\frac{z\cdot\bar\zeta}{|\zeta|^2}\right)^r\wedge\left(\frac{d\bar\zeta\cdot e^*}{|\zeta|^2}\right)^{n-p},$$

where C is a constant. We will show that (25) is actually $\bar\partial_z$-exact. The factor in (25) which depends on z is $(z\cdot\bar\zeta)^r\omega_z^p$, which is at least a $\bar\partial_z$-closed form. Can we write $(z\cdot\bar\zeta)^r\omega_z^p=\bar\partial_z g(z)$, where g is a projective form? Actually, we have $\bar\partial_z[(\bar\zeta\cdot z)^r\partial|z|^2/|z|^2\wedge\omega_z^{p-1}]=(z\cdot\bar\zeta)^r\omega_z^p$, but $(\bar\zeta\cdot z)^r\partial|z|^2/|z|^2\wedge\omega_z^{p-1}$ is not a projective form. This can be remedied by adding a holomorphic term $(\bar\zeta\cdot z)^{r-1}(\bar\zeta\cdot dz)\wedge\omega_z^{p-1}$, since then we can take

$$g=(\bar\zeta\cdot z)^{r-1}[(\bar\zeta\cdot z)\frac{\partial|z|^2}{|z|^2}-\bar\zeta\cdot dz]\wedge\omega_z^{p-1}.$$

Since (25) is $\bar\partial_z$-exact, we have proved a) when $r>0$. If $-r<0$, by Remark 4 in the previous section we must look at $P_{n-p,r}$, which is again $\bar\partial_z$-exact, and then $\int_z\phi(z)\wedge P_{n-p,r}=0$ by Stokes' Theorem.

Proof of b): Note that here we really want to prove that $\phi=0$, since ϕ cannot be $\bar\partial$-exact. To prove this we again use the dual case in Remark 4. We want to show that $\int_z\phi(z)\wedge P_{n-p,r}(\zeta,z)=0$, when $\phi(z)$ has bidegree $(p,0)$ and takes values in L_z^{-r}. First assume that $p>0$, then we must look at the term in $P_{n-p,r}$ of bidegree $(n-p,n)$ in z. No term in $P_{n-p,r}$ has a higher degree in $d\bar z$ than in dz, so $\int\phi(z)\wedge P_{n-p,r}(\zeta,z)=0$. If $p=0$, then we must look at the term in $P_{n,r}$ with bidegree (n,n) in z and $(0,0)$ in ζ. The z-dependent factor of this term is $(z\cdot\bar\zeta)^r\omega_z^n$, which is $\bar\partial_z$-exact in the same way as in the proof of a). This proves the case $p=0$, $-r<0$, but the proof breaks down when $r=0$, where there is a non-trivial cohomology.

Proof of c): First let $p<n$. There is no term in $P_{p,r}$ with bidegree (p,n) in z, since there are not enough $d\bar z$'s, so $\int_\zeta\phi(\zeta)\wedge P_{p,r}(\zeta,z)=0$. If $p=n$, we look at the term in $P_{p,r}$ with bidegree (n,n) in z and $(0,0)$ in ζ. This is dealt with exactly as the case $p=0$ in the proof of b).

Proof of d) and e): Let $q\neq 0,n,p$. If $p<q$ and $r\geq-(n-p)$, we look at the term in P_r with bidegree (p,q) in z. It is zero, since we cannot have more $d\bar z$'s than dz's, so $\int_\zeta\phi(\zeta)\wedge P_{p,r}=0$. Similarly, if $p>q$ we use the dual method: the term in $P_{n-p,r}$ with bidegree $(n-p,n-q)$ in z is zero when $n-p<n-q$ and $r\geq -p$, again since we cannot have more $d\bar z$'s than dz's. This shows that $\int_z\phi(z)\wedge P_{n-p,r}=0$ for $r\geq-p$, where ϕ takes values in L^{-r} and $-r\leq p$. □

6. Weighted Koppelman formulas on $\mathbb{P}^n \times \mathbb{P}^m$

We will now find integral formulas on $\mathbb{P}^n \times \mathbb{P}^m$. Let $([\zeta], [\tilde{\zeta}], [z], [\tilde{z}])$ be a point in $(\mathbb{P}^n \times \mathbb{P}^m) \times (\mathbb{P}^n \times \mathbb{P}^m)$. The procedure will be quite similar to that of Section 4, but for simplicity we will limit ourselves to the case of $(0,q)$-forms. This corresponds to using only β in the formula (23). According to formula (22), then, we can construct our kernel directly, without any need to refer to the bundle E, in the following way (also see [16]). Let $\eta_\zeta = 2\pi i z \cdot \frac{\partial}{\partial \zeta}$ and $\eta = \eta_\zeta + \eta_{\tilde{\zeta}}$. We take δ_η to be contraction with η and set $\nabla = \delta_\eta - \bar{\partial}$. Note that $\eta = 0$ on Δ. Now set

$$s_\zeta = \frac{\bar{z} \cdot d\zeta}{|z|^2} - \frac{\bar{z} \cdot \zeta}{|z|^2 |\zeta|^2} \bar{\zeta} \cdot d\zeta$$

and then $s = s_\zeta + s_{\tilde{\zeta}}$. Observe that $\delta_\eta s$ is a scalar, which is zero only on Δ.

Proposition 6.1. *If $u = s/\nabla s$, then u satisfies $\nabla u.\phi = (1 - [\Delta]).\phi$, where ϕ is a form of bidegree $(n+m, n+m)$ which takes values in $L^{-n}_{[\zeta]} \otimes L^{-m}_{[\tilde{\zeta}]} \otimes L^{n}_{[z]} \otimes L^{m}_{[\tilde{z}]}$ and contains no $d\zeta_i$'s or $d\tilde{\zeta}_i$'s.*

Proof. The restriction on ϕ is another way of saying that our formulas only will work for $(0,q)$-forms. The proposition will follow from Theorem 4.3 if we integrate in $\mathbb{P}^n_{[\zeta]} \times \mathbb{P}^n_{[z]}$ and $\mathbb{P}^m_{[\tilde{\zeta}]} \times \mathbb{P}^m_{[\tilde{z}]}$ separately. $\square$

To obtain weighted formulas, let

$$\alpha = \frac{z \cdot \bar{\zeta}}{|\zeta|^2} + 2\pi i \partial\bar{\partial} \log |\zeta|^2,$$

and let $\tilde{\alpha}$ be the corresponding form in $([\tilde{\zeta}], [\tilde{z}])$. We have $\nabla\alpha = \nabla\tilde{\alpha} = 0$, so

$$\nabla(\alpha^{n+k} \wedge \tilde{\alpha}^{m+l} \wedge u) = \alpha^{n+k} \wedge \tilde{\alpha}^{m+l} \wedge \nabla u = \alpha^{n+k} \wedge \tilde{\alpha}^{m+l} - [\Delta],$$

where $[\Delta]$ must be integrated against sections of $L^k_{[\zeta]} \otimes L^l_{[\tilde{z}]}$. The following theorem follows from Theorem 2.3.

Theorem 6.2. *If $K = \alpha^{n+k} \wedge \tilde{\alpha}^{m+l} \wedge u$ and $P = \alpha^{n+k} \wedge \tilde{\alpha}^{m+l}$ we get the Koppelman formula*

$$\begin{aligned}\phi([z], [\tilde{z}]) &= \int_{\partial D} \phi([\zeta], [\tilde{\zeta}]) \wedge K + \int_D \bar{\partial}\phi([\zeta], [\tilde{\zeta}]) \wedge K + \\ &+ (\bar{\partial}_z + \bar{\partial}_{\tilde{z}}) \int_D \phi([\zeta], [\tilde{\zeta}]) \wedge K + \int_D \phi([\zeta], [\tilde{\zeta}]) \wedge P\end{aligned}$$

for differential forms $\phi([\zeta], [\tilde{\zeta}])$ on $\mathbb{P}^n \times \mathbb{P}^m$ with bidegree $(0,q)$ which take values in $L^k_{[\zeta]} \otimes L^l_{[\tilde{\zeta}]}$.

Now assume that $\bar\partial\phi = 0$. For which q, k and l is ϕ $\bar\partial$-exact? To show that a particular ϕ is $\bar\partial$-exact, we need to show that the term $\int_{\mathbb{P}^n\times\mathbb{P}^m}\phi([\zeta])\wedge P$ either is zero, or is $\bar\partial$-exact. Since P consists of two factors where one depends only on ζ and the other only on $\tilde\zeta$, we can write

$$(26)\qquad \int_{\mathbb{P}^n\times\mathbb{P}^m}\phi([\zeta],[\tilde\zeta])\wedge P = \int_{\mathbb{P}^m}\left(\int_{\mathbb{P}^n}\phi([\zeta],[\tilde\zeta])\wedge\alpha^{n+k}\right)\wedge\tilde\alpha^{m+l}.$$

We get the following theorem:

Proposition 6.3. *We look at differential forms $\phi([\zeta],[\tilde\zeta])$ on $\mathbb{P}^n_{[\zeta]}\times\mathbb{P}^m_{[\tilde\zeta]}$ with bidegree $(0,q)$, which take values in the line bundle $L^k_{[\zeta]}\otimes L^l_{[\tilde\zeta]}$. The cohomology groups $H^{(0,q)}(\mathbb{P}^n\times\mathbb{P}^m, L^k_{[\zeta]}\otimes L^l_{[\tilde\zeta]})$ are trivial in the following cases:*

a) $q\neq 0, n, m, n+m$
b) $q = 0$ *and* $k<0$ *or* $l<0$
c) $q = n$ *and* $l<0$ *or* $k\geq -n$
d) $q = m$ *and* $k<0$ *or* $l\geq -m$
e) $q = n+m$ *and* $k\geq -n$ *or* $l\geq -m$.

Proof. To determine when (26) is zero, we use Theorem 5.1. Assume that the form ϕ has bidegree $(0,q_1)$ in ζ and $(0,q_2)$ in $\tilde\zeta$ and $q_1+q_2 = q$. If, for some q_1 and k, we know that $H^{(0,q_1)}(\mathbb{P}^n, L^k)$ is trivial, this means either that $\int_{[\zeta]}\phi([\zeta],[\tilde\zeta])\wedge P([\zeta],[z]) = 0$ or that $\int_{[\zeta]}\phi([\zeta],[\tilde\zeta])\wedge P([\zeta],[z]) = \bar\partial_z a([z],[\tilde\zeta])$ for some $a([z],[\tilde\zeta])$. In the first case, it follows that the expression in (26) is also zero. In the second case, we get

$$\int_{\mathbb{P}^m}\left(\int_{\mathbb{P}^n}\phi([\zeta],[\tilde\zeta])\wedge\alpha^{n+k}\right)\wedge\tilde\alpha^{m+l} = \bar\partial_z\int_{\mathbb{P}^m}a([z],[\tilde\zeta])\wedge\tilde\alpha^{m+l} =$$
$$= \bar\partial\int_{\mathbb{P}^m}a([z],[\tilde\zeta])\wedge\tilde\alpha^{m+l}$$

since the integrand is holomorphic in $[\tilde z]$. The same holds if $H^{(0,q_2)}(\mathbb{P}^m, L^l)$ is trivial. The conclusion is that $H^{(0,q_1+q_2)}(\mathbb{P}^n\times\mathbb{P}^m, L^k_{[\zeta]}\otimes L^l_{[\tilde\zeta]}) = 0$ either when q_1 and k are such that $H^{(0,q_1)}(\mathbb{P}^n, L^k) = 0$, or when q_2 and l are such that $H^{(0,q_2)}(\mathbb{P}^m, L^l) = 0$.

Now, we really have a sum

$$\phi = \sum_{q_1+q_2=q}\phi_{q_1,q_2}$$

of terms of the type above. For the cohomology group to be trivial, we must have $\int\phi_{q_1,q_2}\wedge P = 0$ for all of them. We know that $q_2 = q-q_1$. If we have either $0<q_1<n$ or $0<q_2<m$ then $\int\phi_{q_1,q_2}\wedge P = 0$ according

to Theorem 5.1. The only ways to avoid this are if $q = q_1 = q_2 = 0$; if $q = q_1 = n$ and $q_2 = 0$; if $q_1 = 0$ and $q = q_2 = m$ or if $q = n + m$ and $q_1 = n$, $q_2 = m$. Then a) - e) follow from Theorem 5.1. $\square$

7. Weighted Koppelman formulas on Stein manifolds

If X is a Stein manifold it is, in general, impossible to find $E \to X \times X$ and η with the desired properties as described in Section 3. What is possible is to find a section η of a bundle E such that η has good properties close to Δ, but then η will in general have other zeroes as well. It turns out that it is possible to work around this and still construct weighted integral formulas. This section relies on the article [19] by Henkin and Leiterer, where such an η was constructed.

More precisely, let π be the projection from $X_\zeta \times X_z$ to X_ζ, and $E = \pi^*(T_{1,0}(X_\zeta))$. Let $\{e_i\}$ be a local frame for E. By Section 2.1 in [19] we have the result

Theorem 7.1. *There exists a holomorphic section η of E such that $\{\eta = 0\} = \Delta \cup F$, where F is closed and $\Delta \cap F = \varnothing$. Close to Δ we have*

$$\eta(\zeta, z) = \sum_1^n [\zeta_i - z_i + \mathcal{O}(|\zeta - z|^2)] e_i. \tag{27}$$

Moreover, there exists a holomorphic function ϕ such that $\phi(z, z) = 1$ and $|\phi| \lesssim |\eta|$ on a neighborhood of F.

We define δ_η, ∇ etc in the same way as in Section 2. Let $s \in E^*$ be the section satisfying $\delta_\eta s = 1$ outside $\Delta \cup F$ which has pointwise minimal norm, and define $u = s/\nabla s$. If we define

$$K = \int_E \phi^M u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi}\right)_n \quad \text{and} \quad P = \int_E \phi^M \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi}\right)_n,$$

where M is large enough that $\phi^M u$ has no singularities on F, then Theorem 3.4 applies and we have $\bar{\partial} K = [\Delta] - P$. In this way, we recover the formula found in Example 2 of [10], except that our approach also allows for weights. We define weights in the same way as before (note that ϕ is in fact a weight). If g is a weight, we will get a Koppelman formula with

$$K = \int_E \phi^M g \wedge u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi}\right)_n \text{ and } P = \int_E \phi^M g \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi}\right)_n. \tag{28}$$

Note that since E is a pullback of a bundle on X_ζ, the connection and curvature forms of E depend only on ζ. Hence $P = c_n(E)$ is bidegree (n, n) in ζ, and we have $\int_\zeta P(\zeta, z) \wedge \phi(\zeta) = 0$ except in the case where ϕ has bidegree $(0, 0)$. The last term in the Koppelman formula thus presents no obstruction to solving the $\bar{\partial}$-equation on X.

Example 3. In [15] there is an example of weighted formulas on Stein manifolds, which we can reformulate to fit into the present formalism. Let $G \subset X$ be a strictly pseudoconvex domain. By Theorem 9 in [15] we can find a function ψ defined on a neighborhood U of G which embeds G in a strictly convex set $C \subset \mathbb{C}^n$. If σ is the defining function for C, then $\rho = \sigma \circ \psi$ is a strictly plurisubharmonic defining function for G. On U we introduce the weight

$$g(\zeta, z) = \left(1 - \nabla \frac{\frac{\partial\rho(\zeta)}{\partial\zeta} \cdot e^*}{2\pi i \rho(\zeta)}\right)^{-\alpha} = \left(-\frac{v}{\rho} - \omega\right)^{-\alpha}$$

where

$$v = \frac{\partial\rho(\zeta)}{\partial\zeta} \cdot \eta - \rho(\zeta) \quad \text{and} \quad \omega = \bar\partial \left[\frac{\frac{\partial\rho(\zeta)}{\partial\zeta} \cdot e^*}{2\pi i \rho(\zeta)}\right]$$

Note that g is holomorphic in z. If $\operatorname{Re}\alpha$ is large enough, then $g(\cdot, \zeta)$ will be zero on ∂G, since $\sigma(\partial C) = 0$. This implies that if f is a holomorphic function and P is defined by (28), we will have

$$f(z) = \int_G f(\zeta) P,$$

for $z \in G$, by Koppelman's formula. We also have the estimate

$$-\rho(\zeta) - \rho(z) + \epsilon|\zeta - z|^2 \leq 2\operatorname{Re} v(\zeta, z) \leq -\rho(\zeta) - \rho(z) + c|\zeta - z|^2,$$

where ϵ and c are positive and real. By means of this, we can get results in strictly pseudoconvex domains G in Stein manifolds similar to ones which are known in strictly pseudoconvex domains in $\mathbb{C}^n$. For example, one can obtain a direct proof of the Henkin-Skoda theorem which gives L^1-estimates on ∂G for solutions of the $\bar\partial$-equation.

Example 4. We can also solve division problems on X. Let $D \subseteq X$ be a domain, and take $f(\zeta) = (f_1(\zeta), \dots, f_m(\zeta))$ where $f_i \in \mathcal{O}(\overline{D})$. Assume that f has no common zeroes in D. We want to solve the division problem $\psi = f \cdot p$ in D, where ψ is a given holomorphic function, by means of integral formulas. We do this by a variant of the weights used in [8].

By Cartan's Theorem B, we can find $h(\zeta, z) = (h_1(\zeta, z), \dots, h_m(\zeta, z))$, where h_i is a holomorphic section of E^*, such that $\delta_\eta h_i(\zeta, z) = \phi(\zeta, z)(f_i(\zeta) - f_i(z))$. We set

$$g_1(\zeta, z) = (\phi - \nabla(h \cdot \sigma(\zeta))^\mu = (\phi f(z) \cdot \sigma + h \cdot \bar\partial\sigma)^\mu,$$

where $\sigma = \bar f/|f|^2$ and $\mu = \min(m, n+1)$, then g_1 is a weight. Now, $f(z)$ is a factor in g_1, since $(h \cdot \bar\partial\sigma)^\mu = 0$. In fact, we have $(h \cdot \bar\partial\sigma)^{n+1} = 0$ for degree reasons, and $(h \cdot \bar\partial\sigma)^m = 0$ since $f \cdot \sigma = 1$ implies $f \cdot \bar\partial\sigma = 0$, so that $\bar\partial\sigma_1, \dots, \bar\partial\sigma_m$ are linearly dependent.

By the Koppelman formula we have

$$\psi(z) = \int_{\partial D} \psi\phi^M K + \int_D \psi\phi^M P$$

where K and P are defined by (28) using the weight g_1. Since $f(z)$ is a factor in g_1, we have $\psi(z) = f(z) \cdot p(z)$, where $p(z)$ will be holomorphic if D is such that we can find u holomorphic in z (for example if D is pseudoconvex).

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KOPPELMAN FORMULAS ON GRASSMANNIANS

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Abstract. We construct Koppelman formulas on Grassmannians for forms with values in any holomorphic line bundle as well as in the tautological vector bundle and its dual. As a consequence we obtain some vanishing theorems of the Bott-Borel-Weil type. We also relate the projection part of our formulas to the Bergman kernels associated to the line bundles.

1. Introduction

The Cauchy integral formula in one complex variable is of vast importance in many respects. It provides a way of representing a holomorphic function as a superposition of simple rational functions, and gives an explicit solution to the equation $\bar{\partial}u = f$. Furthermore, it is an important tool in function theory. For our purposes it is convenient to note that Cauchy's formula is equivalent to the current equation $\bar{\partial}u = [z]$, where $u = (2\pi i)^{-1}d\zeta/(\zeta - z)$ is the Cauchy form, and $[z]$ is the Dirac measure at z considered as a $(1,1)$-current. This point of view is well adapted for generating weighted Cauchy formulas. For instance, by computing $\bar{\partial}\big(((1-|\zeta|^2)/(1-z\bar{\zeta}))^{\alpha}u\big)$ in the current sense, one obtains (for suitable α) the weighted representation formula

$$f(z) = \frac{\alpha}{\pi}\int_{\{|\zeta|<1\}} f(\zeta)\frac{(1-|\zeta|^2)^{\alpha-1}}{(1-z\bar{\zeta})^{\alpha+1}}d\lambda(\zeta),$$

for holomorphic functions on the unit disc with certain limited growth at the boundary. The integral kernel is the reproducing kernel for a weighted Bergman space; and this shows that there is a connection between Cauchy kernels and Bergman kernels. Both these kernels are also intimately linked with the symmetry of the disc. Recall that the group

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in M_{22}(\mathbb{C}) |\ |a|^2 - |b|^2 = 1 \right\}$$

acts holomorphically and transitively on the unit disc by $z \mapsto (az+b)/(\bar{b}z + \bar{a})$. The stabilizer of the origin is the subgroup

$$K := \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\} \cong S^1,$$

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and hence the disc can be viewed as the homogeneous space $SU(1,1)/S^1$. The kernels are then invariant under certain actions on functions which are induced from the natural action on the closed disc. From the point of view of representation theory, the Bergman kernels are interesting since the corresponding weighted Bergman spaces form a family of unitary representation spaces for $SU(1,1)$, and moreover, these kernels can be described entirely in terms of the Lie-theoretic structure of the group. This discussion indicates two possible directions of generalizations; namely to domains in $\mathbb{C}^n$, and to complex homogeneous spaces. In the latter case, the class of bounded symmetric domains have been studied extensively from the Lie-theoretic point of view. Hua, [10], computed the Cauchy kernels and Bergman kernels for the classical domains using the explicit description of their symmetry groups. Later, more abstract group theoretic machinery has been used to describe both Bergman kernels (cf. [15]) and the generalized Cauchy-Szeg kernels, [11]. For compact Hermitian symmetric spaces, Bergman kernels for line bundles can be described explicitly in terms of the polynomial models for the spaces of global holomorphic sections, [21].

Complex analysts have mainly been concerned with domains in $\mathbb{C}^n$. The Bochner-Martinelli kernel represents holomorphic functions in any domain but has the drawback of not being holomorphic, a property which is highly useful in applications. The Cauchy-Fantappi-Leray kernel is holomorphic in domains where we can find a holomorphic support function, for example strictly pseudoconvex domains. More flexibility is afforded by using weighted formulas, which was first done in [6], and such formulas have been widely used in applications such as interpolation, division, obtaining estimates for solutions to the $\bar{\partial}$-equation, etc. See, e.g., [1] and [3] and the references therein. Some work has also been done on generalizing integral formulas to complex manifolds, see, e.g., [9], [5], [4]. Of these, the paper [4] by Berndtsson will be of particular importance for us; see below.

More recently, in [1] was introduced a general method for generating weighted formulas for domains in $\mathbb{C}^n$, both for holomorphic functions and (p,q)-forms. For future reference, we will describe this method in the former case in some detail. First, recall that the Cauchy kernel, u, in one variable satisfies $\bar{\partial}u = [z]$, but less obviously, we also have $\delta_{\zeta-z}u = 1$, where $\delta_{\zeta-z}$ denotes contraction with the vector field $2\pi i(\zeta - z)\partial/\partial\zeta$. These equations can be combined into the single equation

$$\nabla_{\zeta-z}u = 1 - [z], \tag{1}$$

where $\nabla_{\zeta-z}$ is the operator

$$\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar{\partial}.$$

To generalize this to $\mathbb{C}^n$, we define $\delta_{\zeta-z}$ as contraction with

$$2\pi i \sum (\zeta_j - z_j)\frac{\partial}{\partial \zeta_j}, \tag{2}$$

and if we construe equation (1) as being in $\mathbb{C}^n$, the right hand side of (1) now contains one form of bidegree $(0,0)$ and one of bidegree (n,n), so we must in fact have $u = u_{1,0} + u_{2,1} + \ldots + u_{n,n-1}$, where $u_{k,k-1}$ has bidegree $(k, k-1)$. We can then write the $\nabla_{\zeta-z}$-equation (1) as the system of equations

$$\delta_{\zeta-z}u_{1,0} = 1, \qquad \delta_{\zeta-z}u_{1,2} - \bar{\partial}u_{1,0} = 0, \qquad \ldots, \qquad \bar{\partial}u_{n,n-1} = [z].$$

In that case, $u_{n,n-1}$ will satisfy $\bar{\partial}u_{n,n-1} = [z]$ and will give a kernel for a representation formula. One advantage of this approach, as opposed to just solving $\bar{\partial}u_{n,n-1} = [z]$, is that it easily allows for weighted integral formulas. We define $g = g_{0,0} + \cdots + g_{n,n}$ to be a weight if $\nabla g = 0$ and $g_{0,0}(z, z) = 1$. It is easy to see that $\nabla(u \wedge g) = g - [\Delta]$, and this yields a representation formula

$$\phi(z) = \int_{\partial D} \phi(\zeta)(u \wedge g)_n + \int_D \phi g_n$$

if $\phi \in \mathcal{O}(\overline{D})$ and $z \in D$. Note that if g_1 and g_2 are weights, then $g_1 \wedge g_2$ is also a weight.

In the case of compact manifolds one is naturally led to consider holomorphic line bundles and representation formulas for holomorphic sections as well as smooth bundle-valued forms. In this setting the integral kernels must be operator valued, and the integrals become superpositions of contributions from all fibres. Our method for achieving this has two crucial components; the above mentioned ∇-formalism, and Berndtsson's method from [4]. Indeed, Berndtsson gave a method for obtaining integral formulas for (p, q)-forms on n-dimensional manifolds X which admit a vector bundle of rank n over $X \times X$ such that the diagonal has a defining section η; and to get formulas for forms with values in bundles the ∇-method is well suited. In fact, by generalizing it to manifolds one realizes that it allows for operator valued weights. We then need something to substitute for the vector field (2), and this is where Berndtsson's assumption comes in: we will use the section η to contract with, and define $\nabla_\eta := \delta_\eta - \bar{\partial}$. It is of independent interest to note that ∇_η in fact is a superconnection in the sense of Quillen, [14]. In the recent article [8] by the first author, this general theory for integral formulas on manifolds has been developed to a large degree, and explicit formulas have been constructed on $\mathbb{CP}^n$ yielding explicit proofs of vanishing theorems for its line bundles. Such proofs could be of interest also for representation theoretic purposes. Indeed, in view of the by now firmly established goal, initiated by the Bott-Borel-Weil theorem and further fortified by the conjecture of Langlands, [12], and Schmid's proof of it, [16], of wanting to realize representations of Lie groups in Dolbeault cohomology (or, rather L^2-cohomology in the non-compact case), (cf. also [19] and [20]), it is our hope that explicit integral formulas could give further insight into the underlying group theory.

In this paper, we extend the method in [8] to the vector bundle setting and we apply the technique to complex Grassmannians, $Gr(k, N)$. We find a suitable vector bundle, with a section η as above, and natural weights for the line bundles and for the tautological k-plane bundle. We thus get Koppelman formulas for (p, q)-forms with values in any holomorphic line bundle as well as in the tautological bundle and its dual. The construction is uniform in the sense that it uses the explicit description of the Picard group of holomorphic line bundles and reduces the problem to that of finding a weight for the generator. The generator in turn, is the determinant of the tautological bundle; by certain algebraic properties of weights, it thus suffices to construct a weight for the tautological bundle. As an application, we give explicit proofs of certain vanishing theorems of Bott-Borel-Weil type [1] for the cohomology groups associated with these line bundles. We also relate the projection part of our Koppelman formulas to Bergman kernels; thus giving a geometric interpretation of the latter ones.

This paper is organized as follows: In Section 2 we recapture the general method for finding weighted Koppelman formulas on manifolds from [8]. The only difference is that we allow for forms with values in vector bundles and state a slightly more general Koppelman formula. The proofs have been omitted since they are straightforward generalizations of the proofs in [8]. Section 3 describes some general operations on weights. In Section 4 we construct the ingredients necessary to generate weighted formulas on Grassmannians according to the general framework. In Section 5 we review the representation theoretic description of the Picard group and we prove a certain invariance property for the weights, which will be useful for the applications. We also prove that the bundle E restricted to the diagonal is equivalent to the holomorphic cotangent bundle over $Gr(k,N)$. In the last section, Section 6, we discuss some applications; we obtain vanishing theorems for the line bundles over Grassmann, and we give a geometric interpretation of the Bergman kernels associated to the line bundles.

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2. A general method for finding weighted Koppelman formulas on manifolds

Let X be a complex manifold of dimension n. We want to find Koppelman formulas for differential forms on X with values in a given vector bundle $H \to X$. The method described in this section is taken from [8], except for the generalization which yields formulas for a general vector bundle H instead of for a line bundle.

We begin by noting that Stokes' theorem holds also for sections of vector bundles, which is easily proved. Let M be any complex manifold, and $G \to M$ a holomorphic Hermitian vector bundle over M. Let D_{G^*} and D_G be the Chern connections for G^* and G respectively. If u is a differential form taking values in G^* and ϕ is a test form with values in G, we have

$$\int_M D_{G^*}u \wedge \phi = (-1)^{\deg u+1} \int_M u \wedge D_G \phi, \tag{3}$$

[1]These are not given in the form including the ρ-shift which is common in representation theory.

where $\wedge$ denotes taking the natural pairing between the factors in G^* and G, and taking the wedge product between the factors which are differential forms. If u is instead a current, we can take (3) as a definition. In the same way, we also have

$$\int_M \bar\partial u \wedge \phi = (-1)^{\deg u+1} \int_M u \wedge \bar\partial \phi. \tag{4}$$

Let Δ be the diagonal in $X_z \times X_\zeta$. Let H_z denote $\pi_z^*(H)$, where π_z is the projection from $X_z \times X_\zeta$ to X_z, and analogously for H_ζ. Let $g_{0,0}$ be a section of $H_z \otimes H_\zeta^* = \mathrm{Hom}(H_\zeta, H_z)$ such that $g_{0,0}(z,z) = \mathrm{Id}$ for all z. If $[\Delta]$ denotes the current of integration over the diagonal and $\omega(\zeta, z)$ is a differential form with values in $H_z^* \otimes H_\zeta$, then we let

$$[\Delta]_{g_{0,0}}(\omega) := [\Delta].((g_{0,0} \otimes \mathrm{Id})\omega),$$

where Id acts on the differential forms in ω, and we take the natural pairing $(H_z^* \otimes H_\zeta) \times (H_z \otimes H_\zeta^*) \to \mathbb{C}$. Note that this does not depend on which $g_{0,0}$ we choose, since the values on the diagonal are the only ones that matter. The reason for the subscript on $g_{0,0}$ will become apparent later on.

Proposition 1 (Koppelman's formula)**.** *Assume that $D \subset X_\zeta$, $\phi \in \mathcal{E}_{p,q}(\bar D, H_\zeta)$, and that the current $K(z,\zeta)$ and the smooth form $P(z,\zeta)$ take values in $H_z \otimes H_\zeta^* = Hom(H_\zeta, H_z)$ and solve the equation*

$$\bar\partial K = [\Delta]_{g_{0,0}} - P. \tag{5}$$

We then have

$$\phi(z) = \int_{\partial D} K \wedge \phi + \int_D K \wedge \bar\partial\phi + \bar\partial_z \int_D K \wedge \phi + \int_D P \wedge \phi, \tag{6}$$

where the integrals are taken over the ζ variable.

The proof of this uses (4) but is otherwise just like the usual proof of the Koppelman formula. Note that if ϕ in (6) is a $\bar\partial$-closed form and the first and fourth terms of the right hand side of Koppelman's formula vanish, we get a solution to the $\bar\partial$-problem for ϕ.

Our purpose now is to find K and P that satisfy (5) in a special type of manifold. To begin with, we will let H be the trivial line bundle. Assume that we can find a holomorphic vector bundle $E \to X_z \times X_\zeta$ of rank n, such that there exists a holomorphic section η of E that defines the diagonal Δ. In other words, η must vanish to the first order on Δ and be non-zero elsewhere. Let $\{e_i\}$ be a local frame for E, and $\{e_i^*\}$ the dual local frame for E^*. Contraction with η is an operation on E^* which we denote by δ_η; if $\eta = \sum \eta_i e_i$ then

$$\delta_\eta \left(\sum \sigma_i e_i^* \right) = \sum \eta_i \sigma_i.$$

We define the operator

$$\nabla_\eta = \delta_\eta - \bar\partial.$$

Choose a Hermitian metric h for E, let D_E be the Chern connection on E, and D_{E^*} the induced connection on E^*. Consider the bundle

$$G_E = \Lambda[T^*(X \times X) \oplus E \oplus E^*] \to X \times X$$

and $\Gamma(X \times X, G_E)$, the space of C^∞ sections of G_E (note the change of notation compared to [8]). If A lies in $\Gamma(X \times X, T^*(X \times X) \otimes E \otimes E^*))$, then we define $\tilde{A}$ as the corresponding element in $\Gamma(X \times X, G_E)$, arranged with the differential form first, then the section of E and finally the section of E^*. For example, if $A = dz_1 \otimes e_1 \otimes e_1^*$, then $\tilde{A} = dz_1 \wedge e_1 \wedge e_1^*$.

To define a derivation D on $\Gamma(X \times X, G_E)$, we first let $Df = \widetilde{D_E f}$ for a section f of E, and $Dg = \widetilde{D_{E^*} g}$ for a section g of E^*. We then extend the definition by

$$D(\xi_1 \wedge \xi_2) = D\xi_1 \wedge \xi_2 + (-1)^{\deg \xi_1} \xi_1 \wedge D\xi_2,$$

where $D\xi_i = d\xi_i$ if ξ_i happens to be a differential form, and $\deg \xi_1$ is the total degree of ξ_1. For example, $\deg(\alpha \wedge e_1 \wedge e_1^*) = \deg \alpha + 2$, where $\deg \alpha$ is the degree of α as a differential form. We let

$$\mathcal{L}^m = \bigoplus_p \Gamma(X \times X, \Lambda^p E^* \wedge \Lambda^{p+m} T^*_{0,1}(X \times X));$$

note that $\mathcal{L}^m$ is a subspace of $\Gamma(X \times X, G_E)$. The operator ∇_η will act in a natural way as $\nabla_\eta \colon \mathcal{L}^m \to \mathcal{L}^{m+1}$. If $f \in \mathcal{L}^m$ and $g \in \mathcal{L}^k$, then $f \wedge g \in \mathcal{L}^{m+k}$. We also see that ∇_η obeys Leibniz' rule, and that $\nabla_\eta^2 = 0$.

Definition 2. For a form $f(z, \zeta)$ on $X \times X$, we define

$$\int_E f(z, \zeta) \wedge e_1 \wedge e_1^* \wedge \ldots \wedge e_n \wedge e_n^* = f(z, \zeta).$$

Note that if I is the identity on E, then $\tilde{I} = e \wedge e^* = e_1 \wedge e_1^* + \ldots + e_n \wedge e_n^*$. It follows that $\tilde{I}_n = e_1 \wedge e_1^* \wedge \ldots \wedge e_n \wedge e_n^*$ (with the notation $a_n = a^n/n!$), so the definition above is independent of the choice of frame. Our derivation D and $\int_E$ interact in the following way:

Proposition 3. *If* $F \in \Gamma(X \times X, G_E)$ *then*

$$d \int_E F = \int_E DF.$$

We will now construct integral formulas on $X \times X$. As a first step, we find a section σ of E^* such that $\delta_\eta \sigma = 1$ outside Δ. For reasons that will become apparent, we choose σ to have minimal pointwise norm with respect to the metric h, which means that $\sigma = \sum_{ij} h_{ij} \bar{\eta}_j e_i^* / |\eta|^2$. Close to Δ, it is obvious that $|\sigma| \lesssim 1/|\eta|$, and a calculation shows that we also have $|\bar{\partial}\sigma| \lesssim 1/|\eta|^2$. Next, we construct a section u with the property that $\nabla_\eta u = 1 - R$ where R is a current with support on Δ. We set

$$u = \frac{\sigma}{\nabla_\eta \sigma} = \sum_{k=0}^{\infty} \sigma \wedge (\bar{\partial}\sigma)^k, \tag{7}$$

and note that $u \in \mathcal{L}^{-1}$. By $u_{k,k-1}$ we will mean the term in u of degree k in E^* and degree $k-1$ in $T^*_{0,1}(X \times X)$. It is easily checked that $\nabla_\eta u = 1$ outside Δ.

The following theorem yields a Koppelman formula by Theorem 1, with the trivial line bundle as H:

Theorem 4. *Let $E \to X \times X$ be a vector bundle with a section η which defines the diagonal Δ of $X \times X$. We have*

$$\bar\partial K = [\Delta] - P,$$

where

$$K = \int_E u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi} \right)_n \quad \text{and} \quad P = \int_E \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi} \right)_n, \tag{8}$$

and u is defined by (7).

Note that since $D\eta$ contains no e_i^*'s, we have

$$P = \int_E (\frac{i\tilde\Theta}{2\pi})_n = \det \frac{i\Theta}{2\pi} = c_n(E),$$

i.e., the nth Chern class of E. The factor

$$\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi}$$

is actually the supercurvature associated with the operator ∇_η if we view ∇_η as a superconnection in the sense of Quillen, [14]. In fact, we have the following Bianchi identity:

$$\nabla_\eta \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi} \right) = 0, \tag{9}$$

for a direct proof see, e.g., [8].

The idea behind the proof of Theorem 4 is that by (9) and Proposition 3 we have

$$\begin{aligned} \bar\partial \int_E u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi} \right)_n &= \int_E \bar\partial \left[u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi} \right)_n \right] = \\ = & -\int_E \nabla_\eta \left[u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi} \right)_n \right] = \\ = & -\int_E \left(\frac{D\eta}{2\pi i} + \frac{i\tilde\Theta}{2\pi} \right)_n + \frac{1}{(2\pi i)^n} \int_E R \wedge (D\eta)_n. \end{aligned} \tag{10}$$

The left hand term in (10) is P. The rest of the proof consists of proving that

$$\frac{1}{(2\pi i)^n}\int_E R\wedge(D\eta)_n = [\Delta], \tag{11}$$

which is proved by choosing local coordinates on X, and reducing the problem to the $\mathbb{C}^n$-case. For details of the proof, see, e.g., [8].

As explained in the introduction, we will obtain more flexible formulas if we use weights.

Definition 5. A section g with values in $\mathcal{L}_0$ is a weight if $\nabla_\eta g = 0$ and $g_{0,0}(z,z) = 1$.

Theorem 4 goes through with essentially the same proof if we take

$$K_g = \int_E u\wedge g\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\tilde{\Theta}}{2\pi}\right)_n \quad \text{and} \quad P_g = \int_E g\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\tilde{\Theta}}{2\pi}\right)_n, \tag{12}$$

as shown by the following calculation:

$$\begin{aligned}\bar\partial K_g &= -\int_E \nabla_\eta u\wedge g\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\tilde{\Theta}}{2\pi}\right)_n = \\ &= -\int_E (g-R)\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\tilde{\Theta}}{2\pi}\right)_n = [\Delta]-P_g,\end{aligned} \tag{13}$$

which follows from the proof of Theorem 4 and the properties of weights.

Finally, we will use weights taking values in $\mathrm{Hom}(H_\zeta, H_z)$ to construct Koppelman formulas for differential forms with values in the vector bundle $H\to X$. We define

$$G_{E,H} = \mathrm{Hom}(H_\zeta,H_z)\otimes\Lambda[T^*(X\times X)\oplus E\oplus E^*]\to X\times X$$

and

$$\mathcal{L}_H^m := \bigoplus_p \Gamma(X\times X, \mathrm{Hom}(H_\zeta,H_z)\otimes[\Lambda^p E^*\wedge\Lambda^{p+m}T^*_{0,1}(X\times X)]). \tag{14}$$

We define δ_η on $\Gamma(X\times X, G_{E,H})$ as $\mathrm{Id}\otimes\delta_\eta$, where Id acts on the factors in $\mathrm{Hom}(H_\zeta,H_z)$ and δ_η on the factors in $\Lambda[T^*(X\times X)\oplus E\oplus E^*]$. We also need to extend the derivation D to $\Gamma(X\times X, G_{E,H})$. If a_1 is a differential form taking values in $\mathrm{Hom}(H_\zeta,H_z)$, and $a_2\in\Gamma(X\times X, G_E)$, then we define

$$D(a_1\wedge a_2) = D_{\mathrm{Hom}(H_\zeta,H_z)}a_1\wedge a_2 + (-1)^{\deg a_1}a_1\wedge Da_2,$$

where $D_{\mathrm{Hom}(H_\zeta,H_z)}$ is the Chern connection on $\mathrm{Hom}(H_\zeta,H_z)$. It is obvious that Leibniz' rule holds for both δ_η and the extended D, with the degree taken as the total degree in E, E^* and $T^*(X\times X)$.

If $F\in\mathcal{L}_H^0$, then in analogy with Proposition 3 we have

$$D_{\mathrm{Hom}(H_\zeta,H_z)}\int_E F = \int_E DF.$$

It follows that we also have $\bar\partial\int_E F = \int_E\bar\partial F$.

Let $g \in \mathcal{L}^0_H$ be such that $\nabla_\eta g = 0$ and $g_{0,0}(z,z) = \mathrm{Id}$. In that case we can use g as a weight just as in (12) and get

$$\bar{\partial} K_g = [\Delta]_{g_{0,0}} - P_g \tag{15}$$

by a calculation similar to (13), and then we get a Koppelman formula by Theorem 1.

Remark 6. To obtain more general formulas, one can find forms K and P such that

$$D_{\mathrm{Hom}(H_\zeta, H_z)} K_g = [\Delta]_{g_{0,0}} - P_g \tag{16}$$

by setting $\nabla^{\mathrm{full}}_\eta = \delta_\eta - D$ and checking that the corresponding equation (9) and Theorem 4 are still valid. See for example [8] for details. This will give the same formulas as in [4], if H is the trivial line bundle. We can use weights just as before, if we require that a weight g has the property $\nabla^{\mathrm{full}}_\eta g = 0$ instead of $\nabla_\eta g = 0$.

3. Algebraic properties of weights

In this section we investigate some general constructions of weights with the purpose of generating weights for a wide class of derived bundles from two given vector bundles and weights for these. This method will be useful later when we focus on line bundles over Grassmannians.

To be more precise, we let H and H' be holomorphic vector bundles over the complex manifold X and assume that X fulfills the requirements of our general setup for constructing Koppelman formulas, i.e., $X \times X$ admits a holomorphic vector bundle E with a holomorphic section defining the diagonal. Assume also that $g \in \Gamma(X \times X, G_{E,H})$ and $g' \in \Gamma(X \times X, G_{E,H'})$ are weights for H and H' respectively. We shall see that one can naturally define weights $g \otimes g'$ and $g \wedge g'$ (when $H = H'$), as well as g^* for the bundles $H \otimes H', H \wedge H$ and H^* respectively. This generalizes the fact, mentioned in the introduction, that the product of weights for the trivial bundle is again a weight.

3.1. Tensor products and exterior products of weights.

For operators $A \in H_z \otimes H^*_\zeta$ and $B \in H_z \otimes (H'_\zeta)^*$ the tensor product $A \otimes B$ defined by

$$A \otimes B(u \otimes v) := A(u) \otimes B(v), u \in H_\zeta, v \in H'_\zeta \tag{17}$$

is a linear operator in $\mathrm{Hom}(H_\zeta \otimes H'_\zeta, H_z \otimes H'_z)$. We can therefore extend the exterior multiplication on the vector space G_E to a linear map (which we still denote by $\otimes$)

$$\otimes : (G_{E,H})_{(z,\zeta)} \otimes (G_{E,H'})_{(z,\zeta)} \to (G_{E,H\otimes H'})_{(z,\zeta)}$$

given by

$$(A \otimes \omega) \otimes (B \otimes \omega') \mapsto (A \otimes B) \otimes (\omega \wedge \omega'),$$

for $\omega, \omega' \in (G_E)_{(z,\zeta)}$. This operation defines a natural fiberwise multiplication on sections.

Lemma 7. *The operator ∇_η acts as a graded derivation with respect to the multiplication, $\otimes$, of sections, i.e.,*

$$\begin{aligned}\nabla_\eta\big((A\otimes\omega)\otimes(B\otimes\omega')\big) &= \nabla_\eta(A\otimes\omega)\otimes(B\otimes\omega')\\ &\quad +(-1)^{deg\,\omega}(A\otimes\omega)\otimes\nabla_\eta(B\otimes\omega'),\end{aligned}$$

where A and B are local smooth sections of $H_z\otimes H_\zeta^*$ and $H'_z\otimes(H'_\zeta)^*$ respectively, and ω and ω' are local smooth sections of G_E.

Proof. We first observe that

$$\nabla_\eta(A\otimes\omega) = -\bar\partial A\otimes\omega + A\otimes\nabla_\eta\,\omega,$$

and likewise for $B\otimes\omega'$. Hence,

$$\begin{aligned}&\nabla_\eta\big((A\otimes B)\otimes(\omega\wedge\omega')\big)\\ &= -\bar\partial(A\otimes B)\otimes(\omega\wedge\omega') + (A\otimes B)\otimes\nabla_\eta(\omega\wedge\omega')\\ &= -\bar\partial A\otimes(B\otimes(\omega\wedge\omega')) + (A\otimes B)\otimes(\nabla_\eta\,\omega\wedge\omega') -\\ &\quad A\otimes(\bar\partial B\otimes\omega\wedge\omega') + (-1)^{deg\,\omega}(A\otimes B)\otimes(\omega\wedge\nabla_\eta\,\omega')\\ &= (-\bar\partial A\otimes\omega + A\otimes\nabla_\eta\,\omega)\otimes(B\otimes\omega') +\\ &\quad (A\otimes\omega)\otimes(-\bar\partial B\otimes\omega' + (-1)^{deg\,\omega}B\otimes\nabla_\eta\,\omega')\\ &= \nabla_\eta(A\otimes\omega)\otimes(B\otimes\omega') + (-1)^{deg\,\omega}(A\otimes\omega)\otimes\nabla_\eta(B\otimes\omega').\end{aligned}$$

□

Corollary 8. *Given weights g and g' for H and H' respectively, the section*

$$g\otimes g' \in \Gamma(X\times X, G_{E,H\otimes H'})$$

is a weight for $H\otimes H'$.

We next turn to exterior products of a vector bundle. Recall that when A and A' are operators in $\mathrm{Hom}(H_\zeta, H_z)$, $A\wedge A'$ is the operator in $\mathrm{Hom}(\Lambda^2 H_\zeta, \Lambda^2 H_z)$ given by

$$A\wedge A'(u\wedge u') = A(u)\wedge A'(u') - A(u')\wedge A'(u).$$

We can then form the exterior product

$$\wedge\colon (G_{E,H})_{(z,\zeta)}\otimes(G_{E,H})_{(z,\zeta)} \to (G_{E,H\wedge H})_{(z,\zeta)}$$

given by

$$(A\otimes\omega)\otimes(A'\otimes\omega') \mapsto (A\wedge A')\otimes(\omega\wedge\omega').$$

It induces a natural exterior product on sections of $G_{E,H}$. Using the Leibniz identity

$$\bar\partial(A \wedge A') = \bar\partial A \wedge A' + A \wedge \bar\partial A',$$

the following lemma can be proved in the same manner as Lemma 7.

Lemma 9. *The operator ∇_η acts as a graded derivation with respect to the exterior multiplication of sections, i.e.,*

$$\begin{aligned}\nabla_\eta\big((A \otimes \omega) \wedge (A' \otimes \omega')\big) &= \nabla_\eta(A \otimes \omega) \wedge (A' \otimes \omega') + \\ &\quad (-1)^{deg\,\omega}(A \otimes \omega) \wedge \nabla_\eta(A' \otimes \omega'),\end{aligned}$$

where A and A' are local smooth sections of $H_z \otimes H^*_\zeta$, and ω and ω' are local smooth sections of G_E.

In analogy with Corollary 8, we have

Corollary 10. *Given weights g_1 and g_2 for H, the section*

$$g_1 \wedge g_2 \in \Gamma(X \times X, G_{E,H\wedge H})$$

is a weight for $H \wedge H$.

3.2. **Dual weights.** For a local section $A \otimes \omega$ of the bundle $G_{E,H}$, we define the adjoint section

$$(A \otimes \omega)^* := A^* \otimes \omega,$$

where $A^*(z,\zeta)\colon H^*_z \to H^*_\zeta$ is the standard dual operator to $A(z,\zeta)$ given by composing functionals with $A(z,\zeta)$. The relations

$$\begin{aligned}\nabla_\eta(A^* \otimes \omega) &= -\bar\partial A^* \otimes \omega + A^* \otimes \nabla_\eta \omega \\ &= -(\bar\partial A)^* \otimes \omega + (A \otimes \nabla_\eta \omega)^* \\ &= (\nabla_\eta(A \otimes \omega))^*\end{aligned}$$

prove the following lemma.

Lemma 11. *Given a weight g for the bundle H, the section g^* is a weight for the dual bundle H^*.*

4. The necessary constructions on Grassmannians

In this section we construct the ingredients necessary to generate weighted integral formulas on Grassmannians according to the recipe in Section 2. We start by reviewing some elementary facts and introducing some notation. Hereafter, X will denote the Grassmannian $Gr(k,N)$ of complex k-planes in $\mathbb{C}^N$. Just as $\mathbb{CP}^n$, $(= Gr(1,n+1))$, has its tautological line bundle, X has a tautological rank k-vector bundle, which will be denoted by $H \to X$ from now on. We consider H as a subbundle of the trivial rank N-bundle, $\mathbb{C}^N \to X$, and the fiber of H above $p \in X$ is the k-plane in $\mathbb{C}^N$ corresponding to the point p. We will take the standard metric on $\mathbb{C}^N$ and this gives us a Hermitian metric on $H \subset \mathbb{C}^N$. From H we get a natural Hermitian line bundle $L = \det H$, which actually generates the Picard group; see Subsection 5.4. We also get the quotient bundle, $F := \mathbb{C}^N/H$, which is a holomorphic vector bundle of rank $N-k$. As a C^∞-bundle, it is isomorphic to the bundle of orthogonal complements $H^\perp \subset \mathbb{C}^N$ via the mapping $\varphi\colon F \to H^\perp$ defined fiberwise by $\varphi(v+H_z) = v - \pi_{H_z} v$, where π_{H_z} is the orthogonal projection from $\mathbb{C}^N$ onto H_z. (If w is a $\mathbb{C}^N$-valued form we will, for simplicity, also write $\pi_{H_z} w$ for $(\pi_{H_z} \otimes \mathrm{Id})w$.) The mapping φ and the metric on $H^\perp \subset \mathbb{C}^N$ gives us a metric on F.

Let $e = (e_1, \dots, e_N)$ be the standard basis for $\mathbb{C}^N$. The point on X corresponding to the k-plane $\mathrm{Span}\{e_1, \dots, e_k\}$ will be the reference point and denoted by p_0. A local holomorphic chart centered at p_0 can be defined as follows: Let z be a point in $\mathbb{C}^n := \mathbb{C}^{k(N-k)}$ and organize z as an $(N-k)\times k$-matrix, i.e.,

$$z = \begin{pmatrix} z_{11} & \cdots & z_{1k} \\ \vdots & & \vdots \\ z_{N-k,1} & \cdots & z_{N-k,k} \end{pmatrix} \in \mathbb{C}^n.$$

Associate to z the point on X corresponding to the k-plane spanned by the columns of the $N \times k$-matrix

$$\begin{pmatrix} I \\ z \end{pmatrix}, \quad I = I_{k\times k}, \tag{18}$$

with respect to the basis e. This actually gives us an injective map from $\mathbb{C}^n$ onto a dense subset $U \subset X$. We also get natural local holomorphic frames for the bundles H, L, and F over this chart. For $j = 1, \dots, k$, let $\mathfrak{h}_j(z)$ be the jth column of (18), i.e., $\mathfrak{h}_j(z) = e_j + \sum_{i=1}^{N-k} z_{ij} e_{k+i}$. Then $\mathfrak{h}_1, \dots, \mathfrak{h}_k$ are k pointwise linearly independent holomorphic sections of H over U. A natural holomorphic frame for L is thus $\mathfrak{l} = \mathfrak{h}_1 \wedge \cdots \wedge \mathfrak{h}_k$. Also, for $1 \le j \le N-k$, let $\mathfrak{f}_j(z)$ be the equivalence class defined by e_{k+j} in $F = \mathbb{C}^N/H$, in the fiber over z. Then $(\mathfrak{f}_1, \dots, \mathfrak{f}_{N-k})$ is a local holomorphic frame for F over U. The projection $\mathbb{C}^N \to F$, expressed in the e-basis for $\mathbb{C}^N$ and the frame $\mathfrak{f}$ for F, can then be written as the $(N-k)\times N$-matrix

$$\begin{pmatrix} -z & I \end{pmatrix}, \quad I = I_{(N-k)\times(N-k)}. \tag{19}$$

For reference we note some more explicit expressions: As a mapping $\mathbb{C}^N_e \to \mathbb{C}^N_e$ expressed in the e-basis we have

$$\pi_H = \begin{pmatrix} I \\ z \end{pmatrix} (I + z^* z)^{-1} \begin{pmatrix} I & z^* \end{pmatrix}$$

and as a mapping $\mathbb{C}^N_e \to H_{\mathfrak{h}}$,

$$\pi_H = (I + z^* z)^{-1} \begin{pmatrix} I & z^* \end{pmatrix}.$$

The mapping $\varphi\colon F_{\mathfrak{f}} \to \mathbb{C}^N_e$ looks like

$$\varphi = \begin{pmatrix} -(I + z^* z)^{-1} z^* \\ I - z(I + z^* z)^{-1} z^* \end{pmatrix}.$$

We have defined the metric, $\langle\cdot,\cdot\rangle_F$, on F via φ so the Hermitian metric-matrix, h_F, expressed in the frame $\mathfrak{f}$ satisfies $(h_F)_{i,j} = \langle\varphi(\mathfrak{f}_i),\varphi(\mathfrak{f}_j)\rangle_{\mathbb{C}^N}$, (with the convention that $\langle v,w\rangle_F = v^t h_F \bar{w}$). Using the explicit expression for φ, a computation then gives

$$h_F^t(z) = (I + zz^*)^{-1},$$

and so the Chern curvature-matrix of F is

$$\Theta_F = \bar\partial(\bar h_F^{-1}\partial \bar h_F) = \partial\bar\partial \log(I + zz^*),$$

where the last expression should be interpreted in the functional calculus sense. For the bundle H we get

$$h_H^t = I + z^*z, \ \text{ and } \ \Theta_H = \partial\bar\partial\log(I + z^*z)^{-1},$$

expressed in the frame $\mathfrak{h}$.

4.1. **The bundle E and the section η.** We will construct a holomorphic vector bundle $E \to X_z \times X_\zeta$ of rank n $(= k(N-k))$ and a global holomorphic section η of it defining the diagonal. As in Section 2, we let H_z and H_ζ denote the pull-back of the tautological bundle under the projections $X_z \times X_\zeta \to X_z$ and $X_z \times X_\zeta \to X_\zeta$ respectively and we define F_z similarly. However, for convenience we will occasionally abuse this notation and also write, e.g., H_z for the fiber of the bundle $H_z \to X_z \times X_\zeta$ above a point (z,ζ). This ambiguity is (partly) justified since one can identify fibers of $H_z \to X_z \times X_\zeta$ above points (z,ζ) for any ζ. This means also that, e.g, $\{\mathfrak{h}_j(z)\}$ is a local holomorphic frame for $H_z \to X_z \times X_\zeta$ over $U_z \times X_\zeta$.

The bundle E is simply $E = F_z \otimes H_\zeta^*$ and then $\mathfrak{e}_{ij} := \mathfrak{f}_i(z)\otimes\mathfrak{h}_j^*(\zeta)$, $1 \le i \le N-k$, $1 \le j \le k$, is a holomorphic frame for E over $U \times U \subset X \times X$. To define η we start with a vector $v \in H_\zeta$ and via $H_\zeta \subset \mathbb{C}_\zeta^N \cong \mathbb{C}_z^N$ we can identify v with a vector $\tilde v \in \mathbb{C}_z^N$. We then let $\eta(v)$ be the projection of $\tilde v$ on $F_z = \mathbb{C}_z^N/H_z$.

Proposition 12. *The section η of E is holomorphic and defines the diagonal in $X \times X$.*

Proof. It is clear that $\eta(v)$ vanishes if and only if v belongs to the fiber above a point in the diagonal $\Delta \subset X \times X$. Hence, η is a global section of $\mathrm{Hom}(H_\zeta, F_z) \cong E$ and vanishes precisely on Δ. In the coordinates and frames described above, η has the form

$$\eta = \zeta - z.$$

In fact, if $v = \sum_1^k v_j\mathfrak{h}_j(\zeta) \in H_\zeta$ then $\eta(v)$ is the image in F_z of $\sum_1^k v_j e_j + \sum_{i=1}^{N-k}\sum_{j=1}^k \zeta_{ij} v_j e_{k+i}$. By (19) this is equal to $\sum_{i=1}^{N-k}\sum_{j=1}^k(\zeta_{ij} - z_{ij})v_j e_{k+i}$. We thus see that η is holomorphic and vanishes to the first order on Δ. $\square$

4.2. **Bundles and weights.** The bundle $L = \det H$ actually generates the Picard group of holomorphic line bundles; cf. Section 5.3, and [18]. We will construct weights for the line bundles $L^r := L^{\otimes r} \to X$, and for the vector bundle $H \to X$. We start by defining two fundamental sections γ_0 and γ_1 of $\mathrm{Hom}(H_\zeta, H_z)$ and $\mathrm{Hom}(H_\zeta, H_z) \otimes E^* \wedge T^*_{0,1}(X \times X)$ respectively. For $v \in H_\zeta$ we first identify v with the vector $\tilde{v}$ in the trivial bundle $\mathbb{C}^N_z \to X_z \times X_\zeta$ via $H_\zeta \subset \mathbb{C}^N_\zeta \cong \mathbb{C}^N_z$. We then put $\gamma_0(v) = \pi_{H_z}\tilde{v}$. In the $\mathfrak{h}$-frames described above, γ_0 is simply the $k \times k$-matrix

$$\gamma_0 = (I + z^* z)^{-1}(I + z^*\zeta). \tag{20}$$

It is a little bit more complicated to describe γ_1: Let ξ and v be (germs of) smooth sections of E and H_ζ respectively. Since $E = F_z \otimes H^*_\zeta$, $\xi(v)$ defines naturally a smooth section of F_z and hence, $\varphi(\xi(v))$ is a smooth section of $H_z^\perp \subset \mathbb{C}^N_z$. We then put $-\gamma_1(\xi \otimes v) = \pi_{H_z}(\bar{\partial}\,\varphi(\xi(v)))$, which is a smooth section of $H_z \otimes T^*_{0,1}(X \times X)$. We check that γ_1 so defined actually is tensorial. Let f be (a germ of) a smooth function. We then get

$$\begin{aligned}
\gamma_1(f\xi \otimes v) &= -\pi_{H_z}\left(\bar{\partial}\,\varphi(f\xi(v))\right) \\
&= -\pi_{H_z}\left(\varphi(\xi(v)) \otimes \bar{\partial} f + f\bar{\partial}\varphi(\xi(v))\right) \\
&= -\pi_{H_z}\big(\varphi(\xi(v))\big) \otimes \bar{\partial} f + f\gamma_1(\xi \otimes v).
\end{aligned}$$

But $\pi_{H_z}\big(\varphi(\xi(v))\big) = 0$ since $\varphi(\xi(v)) \in H_z^\perp$, and so $\gamma_1(f\xi \otimes v) = f\gamma_1(\xi \otimes v)$. (One could also note that $\gamma_1(\xi \otimes v) = -[\pi_{H_z}, \bar{\partial}]\varphi(\xi(v))$, where $[\pi_{H_z}, \bar{\partial}]$ is the commutator.) Hence, γ_1 defines a section of $H_z \otimes T^*_{0,1}(X_z \times X_\zeta) \otimes E^* \otimes H^*_\zeta \cong \mathrm{Hom}(H_\zeta, H_z) \otimes E^* \wedge T^*_{0,1}(X \times X)$. A computation in the local coordinates shows that

$$\gamma_1 = \sum_{i,j=1}^{k} \mathfrak{h}_i(z) \otimes \mathfrak{h}^*_j(\zeta) \otimes M_{ij}, \tag{21}$$

where M is the $k \times k$-matrix of E^*-valued $(0,1)$-forms

$$M = \bar{\partial}\big((I + z^*z)^{-1}z^*\big) \wedge \mathfrak{e}^*. \tag{22}$$

Here, $\mathfrak{e}^*$ is the matrix with entries $(\mathfrak{e}_{ij})^*$.

Proposition 13. *The section $G := \gamma_0 + \gamma_1 \in \mathcal{L}^0_H$, (cf. (14)), is a weight for the tautological bundle H.*

Proof. We need to check that $\gamma_0(z,z) = \mathrm{Id}$ and that $\nabla_\eta G = 0$. The first equality is obvious from the definition. For the second one we have to verify the two equations $\bar{\partial}\gamma_0 = \delta_\eta\gamma_1$ and $\bar{\partial}\gamma_1 = 0$. Let v be a germ of a holomorphic section of H_ζ. Via $H_\zeta \subset \mathbb{C}^N_\zeta \cong \mathbb{C}^N_z$ we may view v as a holomorphic section of $\mathbb{C}^N_z$ and then we can write

$$\begin{aligned}(\delta_\eta\gamma_1)(v) &= -\pi_{H_z}\big(\bar\partial(\varphi(\eta(v)))\big) = -\pi_{H_z}\big(\bar\partial(\pi_{H_z^\perp}v)\big)\\ &= -\pi_{H_z}\big(\bar\partial(v-\pi_{H_z}v)\big) = \bar\partial_{H_z}(\pi_{H_z}v)\\ &= \bar\partial_{H_z}(\gamma_0(v)).\end{aligned}$$

Hence, $\bar\partial_{H_z}(\gamma_0(v)) = \delta_\eta\gamma_1(v)$ for any germ of holomorphic section v of H_ζ. It follows that $\bar\partial\gamma_0 = \delta_\eta\gamma_1$. Now, let ξ be a germ of a holomorphic section of E. Then $\xi(v)$ is a germ of a holomorphic section of F_z. One can (locally) lift $\xi(v)$ to a germ of a holomorphic section, $\widetilde{\xi(v)}$, of $\mathbb{C}^N$ that projects to $\xi(v)$. We then get

$$\begin{aligned}\bar\partial_{H_z}\gamma_1(\xi\otimes v) &= \bar\partial_{H_z}\big(\pi_{H_z}\bar\partial(\varphi(\xi(v)))\big) = \bar\partial_{H_z}\big(\pi_{H_z}\bar\partial(\widetilde{\xi(v)} - \pi_{H_z}\widetilde{\xi(v)})\big)\\ &= -\bar\partial_{H_z}\big(\pi_{H_z}\bar\partial(\pi_{H_z}\widetilde{\xi(v)})\big) = -\bar\partial^2_{H_z}(\pi_{H_z}\widetilde{\xi(v)})\\ &= 0.\end{aligned}$$

Hence, $\bar\partial_{H_z}\gamma_1(\xi\otimes v) = 0$ for any holomorphic ξ and v, and this finishes the proof. □

By the algebraic properties of weights established in Section 3 we now get that $g := G\wedge\cdots\wedge G$ (the exterior product of G with itself k times) is a weight for L. It is easy to check that

$$g_{0,0} = \gamma_0\wedge\cdots\wedge\gamma_0 = k!\frac{\det(I+z^*\zeta)}{\det(I+z^*z)}$$

in the frame $\mathfrak{l}$ for L. Weights for positive powers of L are then obtained by taking powers of g. By the results at the end of Section 3 we can also get weights for H^* and $L^{-r} = (L^*)^{\otimes r}$ from G. If one wants to construct weights for H^* geometrically, as we have done in this section, it is easier to take $F_\zeta\otimes H_z^*$ as the bundle E. However, our Koppelman formulas have an inherent duality and this gives us weighted formulas for forms with values in H^* and L^{-r} from the weighted formulas for H and L^r.

5. Representation-theoretic interpretations

In this section we describe X and its line bundles in terms of group actions and representations. The purpose of this is threefold. First of all, this point of view gives an easy description of the Picard group of X. Secondly, and more importantly, we prove that the weights we have constructed earlier will all be invariant under a certain group action; a property which will turn out be highly useful in the last section with applications to Bergman kernels. Finally, in this setup, we can fairly easily prove that the restriction of the bundle E to the diagonal is equivalent to the holomorphic cotangent bundle $T^*_{1,0}$ of X.

5.1. **The Grassmannian as a homogeneous space.** The linear action of the group $GL(N,\mathbb{C})$ on $\mathbb{C}^N$ induces an action as holomorphic automorphisms of X, and this action is clearly transitive. Hence, we can describe X as a homogeneous space $X \cong GL(N,\mathbb{C})/P$, where

$$P := \left\{ \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \middle| \det A \det D \neq 0 \right\}$$

is the stabilizer of p_0. One can also restrict the action to the subgroup $SL(N, \mathbb{C})$ and still have a transitive group action; this time exhibiting X as the homogeneous space $SL(N, \mathbb{C})/P'$, where

$$P' := \left\{ \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \middle| \det A \det D = 1 \right\}$$

is the stabilizer of p_0 in $SL(N, \mathbb{C})$. The reason that we mention this realization is that some of the results we refer to later hold only for quotients of semisimple Lie groups. A third realization is given by restricting the $GL(N, \mathbb{C})$-action to the unitary group $U(N)$. The stabilizer of p_0 in this subgroup is

$$\left\{ \left(\begin{array}{cc} A & 0 \\ 0 & D \end{array} \right) \middle| A \in U(k), D \in U(N-k) \right\} \cong U(k) \times U(N-k),$$

and hence we have a third description of X as the quotient space $U(N)/(U(k) \times U(N-k))$.

5.2. **The bundles H, F, and E.** We recall that a vector bundle $\mathcal{V} \to X$ is said to be *homogeneous* under a group G if G acts on it by bundle automorphisms in such a way that the corresponding action on X is transitive. As a consequence, the stabilizer, G_{p_0}, of p_0 in G acts linearly on the fiber $\mathcal{V}_{p_0}$, i.e., $\mathcal{V}_{p_0}$ carries a representation, τ, of G_{p_0}. The vector bundle $\mathcal{V}$ can then be reconstructed from the representation τ as the set of equivalence classes

$$G \times_{G_{p_0}} \mathcal{V}_{p_0} := G \times \mathcal{V}_{p_0} / \sim,$$

where the equivalence relation $\sim$ is defined as $(g, v) \sim (g', v')$ if and only if $(g', v') = (gx^{-1}, \tau(x)v)$ for some x in G_{p_0}. The G-action is then given by $[(g', v)] \overset{g}{\mapsto} [(gg', v)]$, where the brackets denote the equivalence classes of the respective pairs. The holomorphic vector bundles are those associated with holomorphic representations, τ, of G_{p_0}, i.e., $\tau : G_{p_0} \to \operatorname{End}(\mathcal{V}_{p_0})$ is a holomorphic group homomorphism.

Suppose now that $H \subset G$ is a closed subgroup of G which also acts transitively on X. Then we can describe X as a quotient $H/(H \cap G_{p_0})$ and form the H-homogeneous vector bundle $\mathcal{V}^H := H \times_{H \cap G_{p_0}} \mathcal{V}_{p_0}$. This latter bundle is in fact equivalent to the former one via the bundle mapping

$$\begin{array}{rcl} \Psi_H^G : H \times_{H \cap G_{p_0}} \mathcal{V}_{p_0} & \to & G \times_{G_{p_0}} \mathcal{V}_{p_0}, \\ {[(h, v)]_H} & \mapsto & {[(h, v)]_G}, \end{array}$$

where the brackets denote the respective equivalence classes.

For our purposes, this means that we can choose to view $GL(N, \mathbb{C})$-homogeneous vector bundles as $SL(N, \mathbb{C})$-homogeneous ones without any loss of information as long as the corresponding representations of P' are restrictions of P-representations. The Levi subgroup $GL(k,\mathbb{C})\otimes GL(N-k,\mathbb{C})$ is the complexification of $U(k)\times U(N-k)$; and hence any holomorphic P-representation where the standard unipotent subgroup acts trivially is uniquely determined by its restriction to $U(k)\times U(N-k)$. In this case we can regard the corresponding vector bundle as only being $U(N)$-homogeneous without losing any information. In particular, this holds in the case of line bundles.

The group $GL(N,\mathbb{C})$ acts naturally on the trivial bundle $X\times\mathbb{C}^N$ by $(p,v)\overset{g}{\mapsto}(g(p),gv)$. The tautological bundle H is invariant under this action, and is therefore a $GL(N,\mathbb{C})$-homogeneous vector bundle. We let $\tau : P\to \mathrm{End}(\mathbb{C}^k)$ denote the corresponding representation of P on $H_{p_0}\cong\mathbb{C}^k$, namely

$$\tau\begin{pmatrix} A & B\\ 0 & D\end{pmatrix}v = Av,\; v\in\mathbb{C}^k.$$

Since the subbundle H of $\mathbb{C}^N$ is $GL(N,\mathbb{C})$-invariant, there is a well-defined action on the quotient bundle $F=\mathbb{C}^N/H$; i.e., F is also a homogeneous bundle. We can identify the fiber F_{p_0} with $\mathbb{C}^{N-k}$, and we let ρ denote the corresponding P-representation given by

$$\rho\begin{pmatrix} A & B\\ 0 & D\end{pmatrix}v = Dv,\; v\in\mathbb{C}^{N-k}.$$

The bundle $E\to X\times X$ is homogeneous under the product group $GL(N,\mathbb{C})\times GL(N,\mathbb{C})$, and the representation of $P\times P$ on the fiber $(F_z\otimes H^*_\zeta)_{(p_0,p_0)}\cong \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^{N-k})$ is the tensor product representation $\rho\otimes\tau^*$ given by

$$\rho\otimes\tau^*(g_z,g_\zeta)Z = D_zZA_\zeta^{-1},\quad \begin{aligned} g_\zeta &= \begin{pmatrix} A_\zeta & B_\zeta\\ 0 & D_\zeta\end{pmatrix},\\ g_z &= \begin{pmatrix} A_z & B_z\\ 0 & D_z\end{pmatrix},\\ Z &\in M_{N-k,k}(\mathbb{C}).\end{aligned}$$

The trivial bundle $\mathbb{C}^N$ is equipped with the standard Euclidean metric which is $U(N)$-invariant; and the tautological bundle H inherits this metric, thus admitting an isometric action of $U(N)$. Moreover, we recall that the quotient bundle F is smoothly equivalent to the orthogonal complement, $H^\perp$, to the tautological bundle. It should be pointed out that $H^\perp$ is not a holomorphic vector bundle, whereas F is. Since the metric on F is induced from that on $H^\perp$, the $U(N)$-action on F is also isometric. Moreover, the bundle E is equipped with a tensor product metric, and therefore the Cartesian product $U(N)\times U(N)$ acts isometrically on E.

The Chern connections and curvatures of the three bundles H,F, and E are invariant under the respective group actions since they are associated with invariant metrics. We recall that the invariance of a curvature, $\Theta_{\mathcal{V}}$, of a holomorphic homogeneous vector bundle $\mathcal{V}$ means the invariance as a section of the bundle $\mathrm{End}(\mathcal{V}) \otimes T^*_{1,1}$ with respect to the natural action on sections of this bundle. Concretely, this means that

$$\Theta_{\mathcal{V}}(gp)(u,v)w = g\Theta_{\mathcal{V}}(p)(dg^{-1}(gp)u, dg^{-1}(gp)v)g^{-1}w,$$
$$u \in T^*_{(1,0),gp}, \quad v \in T^*_{(0,1),gp}, \quad w \in \mathcal{V}_{gp}.$$

In particular, it follows that the curvature is determined by its value at a fixed reference point. We shall return to the Chern curvature of E below, and give an explicit formula for it at the point p_0. First, however, we shall undertake a closer study of the restriction of E to the diagonal.

The action of the group $U(N)$ on X defines a fibration $q : U(N) \to X$ given by $q(g) = g(p_0)$ which is $U(N)$-equivariant with respect to left multiplication, $L_g : x \mapsto gx$, on the group itself, and the action on X, i.e., $q(gx) = g(q(x))$ holds for $g, x \in U(N)$. Moreover, the right action $R_l : x \mapsto xl^{-1}$ of the subgroup $U(k) \times U(N-k)$ on $U(N)$ preserves each fiber $q^{-1}(p)$ for $p \in X$, and yields a diffeomorphism $U(k) \times U(N-k) \cong q^{-1}(p)$. This equips $U(N)$ with the structure of a principal $U(k) \times U(N-k)$-bundle over X. Since the right action of $U(k) \times U(N-k)$ commutes with left multiplication, the group $U(N)$ acts equivariantly with respect to the action of $U(k) \times U(N-k)$. Moreover, the embedding of $U(N)$ into $M_N(\mathbb{C})$ induces an Riemannian structure on $U(N)$ by restriction of the trace inner product $(A,B) \mapsto \mathrm{tr}(AB^*)$, and the left multiplication is isometric with respect to this inner product. For any $g \in U(N)$ with $q(g) = p$, we have an orthogonal decomposition

$$T_g(U(N)) = T_g(q^{-1}(p)) \oplus T_g(q^{-1}(p))^{\perp}, \tag{23}$$

and this decomposition is invariant under left multiplication. The restriction of the differential of q to the orthogonal complement $T_g(q^{-1}(p))^{\perp}$ yields an isomorphism

$$dq(g)|_{T_g(q^{-1}(p))^{\perp}} : T_g(q^{-1}(p))^{\perp} \to T_{q(g)}(X).$$

For any $p \in X$ we thus have a family of subspaces parametrized by the set $q^{-1}(p)$ to which the tangent space at p is isomorphic. We therefore define an equivalence relation on the tangent bundle $T(U(N))$ by

$$(g,v) \sim (g',v') \quad \text{iff} \quad (g',v') = (R_l(g), dR_l(g)v), \tag{24}$$

for some $l \in U(k) \times U(N-k)$. By the isometry of the left multiplication, the orthogonal complement bundle $\cup_p T(q^{-1}(p))^{\perp}$ is a $U(N)$-homogeneous vector bundle. Moreover, for any vector in this subbundle, the whole equivalence class lies in the subbundle since also the right action is isometric. It follows that $S := \cup_p T(q^{-1}(p))^{\perp}/\sim$ is a well-defined $U(N)$-homogeneous vector bundle over X. Clearly, S is equivalent to the tangent bundle $T(X)$, and thus it inherits a complex structure.

Proposition 14. *The restriction of E to the diagonal $\Delta(X \times X)$ is equivalent to the holomorphic cotangent bundle $T^*_{1,0}(X)$.*

Proof. We prove that E^* is equivalent to S. Since S is $U(N)$-homogeneous, it is uniquely determined by the corresponding representation of $U(k)\times U(N-k)$ on the fiber S_{p_0}. For the identity element $e \in U(N)$, the tangent space $T_e(U(N))$ is isomorphic to the Lie algebra

$$\mathfrak{u}(N) := \{X \in M_N(\mathbb{C}) | X^* = -X\},$$

and the subspaces in the decomposition (23) are explicitly given by

$$T_e(q^{-1}(p_0)) = \left\{ \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix} \middle| Y^* = -Y, Z^* = -Z \right\}, \tag{25}$$

$$T_e(q^{-1}(p_0))^{\perp} = \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \middle| B \in M_{k,N-k}(\mathbb{C}) \right\}. \tag{26}$$

For $v = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \in T_e(q^{-1}(p_0))^{\perp}$, and $l = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in U(k)\times U(N-k)$,

$$\begin{aligned} dL_l(e)v &= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & AB \\ -DB^* & 0 \end{pmatrix}. \end{aligned}$$

We can represent the equivalence class of this tangent vector by a tangent vector at the identity, namely by

$$\begin{aligned} dR_l(l)dL_l(e)v &= \begin{pmatrix} 0 & AB \\ -DB^* & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & ABD^{-1} \\ -(ABD^{-1})^* & 0 \end{pmatrix}. \end{aligned}$$

Hence, we can identify the representation of $U(k)\times U(N-k)$ on S_{p_0} with the representation on $M_{k,N}(\mathbb{C})$ given by $B \mapsto ABD^{-1}$, i.e., with the representation $\tau\otimes\rho^*$ on $\mathrm{Hom}(\mathbb{C}^{N-k},\mathbb{C}^k)$, which is precisely the $U(k)\times U(N-k)$ representation associated to the restriction of E^* to the diagonal. □

Remark 15. In [4], Berndtsson proves that any appropriate bundle $E \to X\times X$ has to coincide with the holomorphic cotangent bundle on the diagonal. In the case of $\mathbb{CP}^n$, an independent proof of Proposition 14 can be found in the book [7] by Demailly; Proposition 15.7 in Chapter V.

By the identification $T_{p_0}(X)$ with the subspace $T_e(q^{-1}(p_0))^{\perp}$ in (26), we have an explicit realization of its complexification

$$T_{p_0}(X)^{\mathbb{C}} \cong \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| B \in M_{k,N-k}(\mathbb{C}), C \in M_{N-k,k}(\mathbb{C}) \right\}.$$

Consider now the element $\begin{pmatrix} i\frac{N-k}{N}I_k & 0 \\ 0 & -i\frac{k}{N}I_{N-k} \end{pmatrix} \in T_e(q^{-1}(p_0)) \cong \mathfrak{u}(k)\times \mathfrak{u}(N-k)$. Its adjoint action determines the complex structure, J_{p_0}, at p_0 by

$$
\begin{aligned}
J_{p_0}\begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} &:= \left[\begin{pmatrix} i\frac{N-k}{N}I_k & 0 \\ 0 & -i\frac{k}{N}I_{N-k} \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & iB \\ -(iB)^* & 0 \end{pmatrix}.
\end{aligned}
$$

The splitting of $T^{\mathbb{C}}_{p_0}$ into the the $\pm i$-eigenspaces is given by

$$
\begin{aligned}
T_{(1,0),p_0}(X) &\cong \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \middle| Y \in M_{k,N-k}(\mathbb{C}) \right\}, \\
T_{(0,1),p_0}(X) &\cong \left\{ \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} \middle| Z \in M_{N-k,k}(\mathbb{C}) \right\}.
\end{aligned}
$$

We recall that the curvature Θ_E at the point p_0 is given by the formula

$$
\Theta_E(p_0)(Y,Z)W = (\rho \otimes \tau^*)'([Y,Z])(W), \tag{27}
$$

where $(\rho \otimes \tau^*)'$ denotes the differentiated representation of the Lie algebra $\mathfrak{u}(k) \times \mathfrak{u}(N-k)$ given by $(\rho\otimes\tau^*)'(X) := \frac{d}{dt}(\rho\otimes\tau^*)(\exp tX)|_{t=0}$. The explicit expression for (27) is

$$
\begin{aligned}
\Theta_E(p_0)(Y,Z)W &= (\rho\otimes\tau^*)'\begin{pmatrix} YZ & 0 \\ 0 & ZY \end{pmatrix}(W) \\
&= ZYW - WYZ, \quad W \in M_{N-k,k}(\mathbb{C}).
\end{aligned}
$$

5.3. **Invariance of weights.** In this section we study a natural action of $U(N)$ on sections of the bundles $\mathrm{Hom}(L^r_\zeta, L^r_z) \otimes G_E$, and prove that the corresponding weights are invariant under that action.

Recall that for an action of a group, G, on a vector bundle $\mathcal{V} \to M$, a natural action is induced on the space of sections by

$$
(gs)(z) := gs(g^{-1}z), \tag{28}
$$

where the second action on the right hand side refers to the action on the total space of the bundle. The bundles $\mathrm{Hom}(L^r_\zeta, L^r_z) \otimes G_E$ are equipped with the natural $U(N) \times U(N)$ actions given as tensor (and exterior) products of the actions described in the previous section and their duals. In what follows, we will consider the action of $U(N)$ (embedded as the diagonal subgroup of $U(N) \times U(N)$) given by restriction. The actions on the respective total spaces are the obvious ones, and we will therefore use the simple notation from (28) for such an action.

We let $g^r := g^{\otimes r}$ for $r \geq 0$ and $g^r := (g*)^{\otimes r}$ for $r \leq 0$ denote the weight for the line bundle L^r.

Proposition 16. *The weight g^r is a $U(N)$-invariant section of the vector bundle $Hom(L^r_\zeta, L^r_z) \otimes G_E$.*

Proof. It clearly suffices to prove that the section $G = \gamma_0 + \gamma_1$ is an invariant section of $\mathrm{Hom}(H_\zeta, H_\zeta)$; and for this, we prove that γ_0 and γ_1 are invariant separately. We now fix an orthonormal basis, $\{h_1, \ldots, h_k\}$, for H_z. For any $u \in H_\zeta$ and $l \in U(N)$, we have

$$\begin{aligned}(l\gamma_0)(u) = l\gamma_0(l^{-1}u) &= l\sum_{i=1}^{k}\langle l^{-1}u, l^{-1}h_i\rangle l^{-1}h_i \\ &= \sum_{i=1}^{k}\langle u, h_i\rangle h_i \\ &= \gamma_0(u),\end{aligned}$$

which shows the invariance of γ_0. We now consider γ_1, and therefore choose a local section f of F near the point $z \in X$. Then, we have

$$\begin{aligned}(l\gamma_1)(f\otimes u) &= -l(\pi_{H_{l^{-1}z}}(\bar\partial\varphi(l^{-1}f))\otimes l^{-1}u) \\ &= -l(\pi_{H_{l^{-1}z}}(l^{-1}\bar\partial\varphi(f))\otimes l^{-1}u) \\ &= -\pi_{H_z}(\bar\partial\varphi(f)\otimes u) \\ &= \gamma_1(f\otimes u),\end{aligned}$$

where the third equality is completely analogous to the invariance of γ_0. This concludes the proof. □

We now turn our attention to the form P_{g^r} defined in (12) again.

Corollary 17. *The form P_{g^r} is $U(N)$-invariant.*

Proof. First of all, an argument similar to the proof of Proposition 16 shows that the section η is $U(N)$-invariant. Secondly, the Chern connection D_E on E commutes with the $U(N)$-action, and hence $D\eta$ is also $U(N)$-invariant. The curvature Θ is even $U(N)\times U(N)$-invariant; and hence it follows that the form $g\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\tilde\Theta}{2\pi}\right)_n$ is $U(N)$-invariant. We now claim that the operator $\int_E$ is $U(N)$-equivariant. Indeed, the identity section $I\in\mathrm{End}(E)$ is obviously $U(N)$-invariant, and so is therefore also the section $\tilde I_n$ defined in connection with Definition 2. Hence, $\int_E$ is an equivariant operator, and this also finishes the proof. □

The canonical splitting $T^*(X\times X)\cong T^*_z(X)\oplus T^*_\zeta(X)$ of the cotangent bundle of $X\otimes X$ is $U(N)\times U(N)$-invariant, and hence (P_{g^r}) can be decomposed as

$$(P_{g^r}) = \sum_{\substack{p'+p''=n\\ q'+q''=n}} (P_{g^r})_{p',p'',q',q''}, \tag{29}$$

where $(P_{g^r})_{p',p'',q',q''}$ is a section of $\mathrm{Hom}(H_\zeta, H_z)\otimes\Lambda^{p',q'}(T_z^{\mathbb{C}})^*\wedge\Lambda^{p'',q''}(T_\zeta^{\mathbb{C}})^*$, i.e., it is of bidegree (p',q') in the z-variable, and of bidegree (p'',q'') in the

ζ-variable according to the splitting. By the invariance of the splitting, we also have

Corollary 18. *The terms* $(P_{g^r})_{p',p'',q',q''}$ *in the decomposition* (29) *are* $U(N)$*-invariant.*

Only the term $(P_{g^r})_{n,0,n,0}$ which has bidegree (n,n) in the z-variable will contribute to the integral in the Koppelman formula. Later we will examine this term more closely.

Corollary 19. *The current* K_{g^r} *in* (12) *is* $U(N)$*-invariant.*

Proof. It clearly suffices to prove that u in (7) is $U(N)$-invariant; and since the group action commutes with the $\bar{\partial}$-operator and exterior powers, it only remains to prove the invariance of σ. Note that σ can be described by the equation

$$\sigma(v) = \frac{\langle v, \eta\rangle_E}{|\eta|_E^2}, \quad v \in E.$$

The invariance of σ now follows immediately from the invariance of η and from the fact that the action of $U(N)$ preserves the metric. □

5.4. **Line bundles on X.** In this subsection we recapitulate how the Picard group of X can be described in terms of holomorphic characters. All of this is classical theory and well-known, even though the results in their explicit form can be hard to find in the literature. The reason for including it in the paper is rather to give an overview for readers who are not familiar with representation theory of Lie groups.

Suppose now that $\mathcal{L} \to X$ is a $SL(N,\mathbb{C})$-homogenous holomorphic line bundle. The corresponding P'-representation then amounts to a holomorphic character $\chi_{\mathcal{L}} : P' \to \mathbb{C}^*$. Moreover, it is well-known that all holomorphic line bundles over X are in fact $SL(N,\mathbb{C})$-homogeneous (cf. [18]), and hence the Picard group $H^1(X, \mathcal{O}^*)$ is isomorphic to the multiplicative group of holomorphic characters of P'.

Suppose now first that $\chi\colon P \to \mathbb{C}^*$ is a holomorphic character. (This is no restriction, as we shall later see that all holomorphic characters of P' are restrictions of P-characters.) It is well-known that it is then uniquely determined by its restriction to the Levi-subgroup $GL(k) \times GL(N-k)$ realized as

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| \det A \det D \neq 0 \right\}.$$

By restricting to the respective factors, we can uniquely express χ as a product $\chi = \chi_1\chi_2$, where χ_1 and χ_2 are characters of $GL(k,\mathbb{C})$ and $GL(N-k,\mathbb{C})$ respectively. Let $\chi_1' : \mathfrak{gl}(k,\mathbb{C}) \to \mathbb{C}$ denote the differential at the identity of χ_1. Then χ_1' annihilates the commutator ideal in the decomposition

$$\mathfrak{gl}(k,\mathbb{C}) = \mathfrak{z}(\mathfrak{gl}(k,\mathbb{C})) \oplus [\mathfrak{gl}(k,\mathbb{C}), \mathfrak{gl}(k,\mathbb{C})]$$

of $\mathfrak{gl}(k,\mathbb{C})$ as the direct sum of the center and the commutator. More specifically, we have the identity

$$[\mathfrak{gl}(k,\mathbb{C}),\mathfrak{gl}(k,\mathbb{C})]=\mathfrak{sl}(k,\mathbb{C}),$$

from which it follows that the normal subgroup $SL(k,\mathbb{C})$ lies in the kernel of the character χ_1. Hence, χ_1 descends to a character, $\widetilde{\chi_1}$, of the quotient group $GL(k,\mathbb{C})/SL(k,\mathbb{C})$, yielding the commuting diagram

$$\begin{array}{ccc} GL(k,\mathbb{C}) & \xrightarrow{\ \chi_1\ } & \mathbb{C}^* \\ \big\downarrow & \nearrow_{\widetilde{\chi_1}} & \\ GL(k,\mathbb{C})/SL(k,\mathbb{C}) & & \end{array}\ .$$

Moreover, the quotient group is isomorphic to $\mathbb{C}^*$ via the mapping $gSL(k,\mathbb{C})\mapsto\det g$, and hence we have the diagram

$$\begin{array}{ccc} GL(k,\mathbb{C}) & \xrightarrow{\ \chi_1\ } & \mathbb{C}^* \\ \big\downarrow & \nearrow_{\widetilde{\chi_1}} & \big\uparrow \\ GL(k,\mathbb{C})/SL(k,\mathbb{C}) & \hookrightarrow\!\!\!\twoheadrightarrow & \mathbb{C}^*, \end{array}$$

which allows us to identify $\widetilde{\chi_1}$ with a holomorphic character $\mathbb{C}^*\to\mathbb{C}^*$. The latter ones are easily described. Indeed, by holomorphy, any such character is uniquely determined by its restriction to the totally real subgroup $S^1\subset\mathbb{C}^*$, on which it gives a character $S^1\to S^1$. Hence, it is of the form $\zeta\mapsto\zeta^m$, for some integer m. The analogous result holds of course for χ_2. Summing up, we have thus found that

$$\chi\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}=\det A^m\det D^n,$$

for some $m,n\in\mathbb{Z}$.

The line bundle corresponding to the choice $m=1,n=0$ is the determinant of the tautological vector bundle. To study the line bundle corresponding to the parameters $m=0,n=1$, we consider it as a $SL(N,\mathbb{C})$-homogeneous line bundle, which amounts to restricting the corresponding character to the subgroup P' of P. We let χ' denote the differential at the identity of this character. The Lie algebra $\mathfrak{p}'$ admits a decomposition

$$\mathfrak{p}'=\mathfrak{z}(\mathfrak{p}')\oplus[\mathfrak{p}',\mathfrak{p}']$$

as the direct sum of its center and its commutator ideal. These two ideals are given by

$$\begin{aligned}
\mathfrak{z}(\mathfrak{p}') &= \left\{ \left(\begin{array}{cc} c(N-k)I_k & 0 \\ 0 & -ckI_{N-k} \end{array} \right) \middle| \, c \in \mathbb{C} \right\}, \\
[\mathfrak{p}', \mathfrak{p}'] &= \left\{ \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \middle| \, \mathrm{tr} A = \mathrm{tr} D = 0 \right\}.
\end{aligned}$$

On the group level, we have the commutator subgroup

$$[P', P'] = \left\{ \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \middle| \, \det A = \det D = 1 \right\},$$

and the quotient group $P'/[P', P']$ has complex dimension one. In fact, an isomorphism $\Phi : P'/[P', P'] \to \mathbb{C}^*$ is given by

$$\Phi(g\, [P', P']) = \det A,$$

for $g = \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right)$.

If $\mu : P' \to \mathbb{C}^*$ is a holomorphic character, it factors through the projection onto the quotient group just as above, yielding a holomorphic character $\tilde{\mu} : P'/[P', P'] \to \mathbb{C}^*$. Using the isomorphism Φ above, we obtain the commuting diagram

$$\begin{array}{ccc}
P' & \xrightarrow{\ \mu\ } & \mathbb{C}^* \\
\downarrow & \overset{\tilde{\mu}}{\nearrow} & \uparrow \\
P'/[P', P'] & \hookrightarrow\!\!\!\!\rightarrow & \mathbb{C}^*.
\end{array}$$

From this, we conclude that $\mu \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) = \det A^j$, for some $j \in \mathbb{Z}$. In particular, it follows that μ can naturally be extended to a holomorphic character $P \to \mathbb{C}^*$. Moreover, the dual bundle to the determinant of the tautological vector bundle corresponds to the P'-character $\left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \mapsto \det A^{-1} = \det D$, which can be extended to the P-character $\left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \mapsto \det D$. It is easy to see that the $GL(N, \mathbb{C})$-homogeneous line bundle associated with this holomorphic character is isomorphic to the determinant of the quotient bundle $F = \mathbb{C}^N / H$.

5.5. **The Bott-Borel-Weil theorem.** In this subsection we briefly describe some group representations associated with homogeneous vector bundles.

Suppose now that G is a complex Lie group acting transitively and holomorphically on a complex manifold M, so that we can write $M \cong G/T$ for some closed subgroup $T \subseteq G$. Let $\mathcal{V} \to M$ be a G-homogeneous holomorphic vector bundle. Recall that the action of G on $\mathcal{V}$ induces the action on smooth sections given by (28). Since G acts holomorphically on M, there is a natural action on $\mathcal{V}$-valued (p, q)-forms (by taking the pullback composed with inversion). Moreover, the action commutes with the $\overline{\partial}$-operator on $\mathcal{V}$, from which it follows that the action preserves closed forms and exact form; thus inducing an action on the Dolbeault cohomology groups $H^{p,q}(M, \mathcal{V})$. In the case when G is a complexification of some semisimple compact Lie group, $G_{\mathbb{R}}$, the Bott-Borel-Weil theorem (cf. [2], Theorem. 5.0.1) gives a realization of all irreducible representations of $G_{\mathbb{R}}$ as $H^{0,q}(M, \mathcal{L})$ for some homogeneous line bundle, $\mathcal{L}$, over M, and also states the vanishing of the other sheaf cohomology groups associated with $\mathcal{L}$. We shall see examples of it in the context of the vanishing theorems of the next section.

6. Applications

6.1. **Vanishing theorems.** We would like to find vanishing theorems for the bundles L^r and L^{-r} over X by means of the Koppelman formula. This will yield explicit solutions to the $\bar{\partial}$-equation in the cohomology groups which are trivial.

Let D in Theorem 1 be the whole of X, and let $\phi(\zeta)$ be a $\bar{\partial}$-closed form of bidegree (p, q) taking values in L^r_ζ, with $r > 0$. The only obstruction to solving the $\bar{\partial}$-equation is then the term $\int_\zeta \phi(\zeta) \wedge P_{g^r}(\zeta, z)$. We have

$$
\begin{aligned}
(30) \qquad P_{g^r} &= \int_E g^r \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n = \\
&= \int_E \sum_{j=1}^{\min(kr,n)} C_j (g^r)_{j,j} \wedge (D\eta)^j \wedge (\tilde{\Theta}_\zeta + \tilde{\Theta}_z)^{n-j}
\end{aligned}
$$

where $(g^r)_{j,j}$ is the term in g^r which has bidegree $(0, j)$ and takes values in $\Lambda^j E^*$. Note that all the differentials in g are in the z variable; this is because $\bar{\partial}_\zeta$ commutes with π_{H_z}.

Theorem 20. *The cohomology groups $H^{p,q}(X, L^r)$ are trivial in the following cases:*

a) $p \neq q$ *and* $r = 0$.
b) $p > q$ *and* $r > 0$.
c) $p < q$, $rk < q - p$, *and* $r > 0$.
d) $p < q$ *and* $r < 0$.
e) $p > q$, $rk < p - q$, *and* $r < 0$.

Proof. a) If $r = 0$ we do not need a weight, and in that case

$$
P = \int_E \left(\frac{i\tilde{\Theta}}{2\pi} \right)_n = c_n(E),
$$

or the n:th Chern form of E. It is obvious that P consists of terms with bidegree (k, k) in z and $(n - k, n - k)$ in ζ, and thus $\int \phi \wedge P = 0$ if ϕ has bidegree (p, q) with $p \neq q$.

b) Since the only source of antiholomorphic differentials in ζ is $\tilde{\Theta}_\zeta$, which is a $(1, 1)$-form, we can never get more $d\bar{\zeta}_i$:s than $d\zeta_i$:s. This means that

$\int_\zeta \phi(\zeta) \wedge P_{g^r} = 0$ if ϕ has bidegree (p,q) where $p > q$ (since then P_{g^r} would need to have bidegree $(n-p, n-q)$ in ζ with $n-q > n-p$).

c) If $\phi(\zeta)$ has bidegree (p,q), then P_{g^r} needs to have bidegree (p,q) in z. We can take at most p of the $\tilde{\Theta}_z$:s. We will then need at least $q-p$ more $d\bar{z}_i$:s, and these have to come from the factor g^r. But g^r has maximal bidegree $(0, rk)$, so if $rk < q-p$ the obstruction will vanish.

d) By duality, if we have a (p,q)-form ϕ taking values in L^r with $r < 0$, the obstruction is given by $\int_z \phi(z) \wedge P_{g^{-r}}(\zeta, z)$. This is zero unless there is a term in $P_{g^{-r}}$ of bidegree (p,q) in ζ. By the same argument as in the proof of b), the obstruction vanishes if $q > p$.

e) If $\phi(z)$ has bidegree (p,q), then $P_{g^{-r}}$ needs to have bidegree $(n-p, n-q)$ in z, where $n-q > n-p$. The rest follows as in the proof of c). □

Remark 21. In $\mathbb{CP}^n$, we can get rid of the obstruction in more cases, either by proving that P_{g^r} is $\bar{\partial}_\zeta$-exact (since then Stokes' theorem can be applied), or by proving that it is $\bar{\partial}_z$-exact (since then $\int_\zeta \phi \wedge P_{g^r}$ will be $\bar{\partial}_z$-exact as well). See [8] for details.

Part *d)* of the above theorem is the special case of the Bott-Borel-Weil theorem for the parabolic quotient $GL(N, \mathbb{C})/P$. For $r = -1$, all vanishing theorems were proved by le Potier in [13]. He also proved vanishing theorems for exterior and symmetric powers of the tautological bundle and its dual. In [17], Snow gives an algorithm for computing all Dolbeault cohomology groups for all line bundles over Grassmannians. Implementing the algorithm in a computer, Snow obtains various vanishing theorems including ours. It is worth noting that both le Potier and Snow obtain their results by reduction to the Bott-Borel-Weil theorem.

6.2. **Bergman kernels.** We will see that the projection part, P_{g^r}, of our Koppelman formula for L^r basically is the Bergman kernel associated with the space of holomorphic sections of L^{-r}. We begin by examining P_{g^r}. Recall that

$$g^r = \big((\gamma_0 + \gamma_1)^k\big)^{\otimes r} = \big(\sum_{j=0}^{k} \binom{k}{j} \gamma_0^{k-j} \wedge \gamma_1^j\big)^{\otimes r} =: (\gamma_0^k)^{\otimes r} + \tilde{g}^r,$$

where γ_0^k of course is the kth exterior power of γ_0, is our weight for L^r. The projection kernel in our Koppelman formula for L^r is thus

$$\begin{aligned} P_{g^r} &= \int_E g^r \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}_E}{2\pi}\right)_n \\ &= (\gamma_0^k)^{\otimes r} \otimes \int_E \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}_E}{2\pi}\right)_n + \int_E \tilde{g}^r \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}_E}{2\pi}\right)_n \\ &=: P^0_{g^r} + \tilde{P}_{g^r}. \end{aligned}$$

Let $\mathcal{P}_{g^r}$ and $\mathcal{P}^0_{g^r}$ be the parts of P_{g^r} and $P^0_{g^r}$ respectivly, which have bidegree (n,n) in the z-variables. Let us examine $\mathcal{P}_{g^r}$ and $\mathcal{P}^0_{g^r}$ more closely on the set $Z := \{p_0\} \times X_\zeta$. In our local coordinates and frames over $U_z \times U_\zeta$ we have by (20) that $\gamma_0 = (I + z^*z)^{-1}(I + z^*\zeta)$. On Z intersected with $\{p_0\} \times U_\zeta$, denoted Z' below, we thus have $\gamma_0 = I$ expressed in our frames. According to (21) and (22), we see that, as a matrix in our frames for H_z and H_ζ, $\gamma_1 = dz^* \wedge \mathfrak{e}^*$ on Z'. Moreover, a straightforward computation shows that the part of $D\eta$, which does not contain any differentials in the ζ-variables, equals $-\sum_{i,j} dz_{ij} \wedge \mathfrak{e}_{ij}$ on Z'. Also, the part of $\tilde{\Theta}_E$, which does not contain any differentials in the ζ-variables, is $\widetilde{\Theta_{F_z} \otimes \mathrm{Id}}_{H^*_\zeta}$. We thus see that the building blocks for $\mathcal{P}_{g^r}$ and $\mathcal{P}^0_{g^r}$ are independent of ζ on Z' when expressed in our frames. Since both $\mathcal{P}_{g^r}$ and $\mathcal{P}^0_{g^r}$ take values in a line bundle we must have $\mathcal{P}_{g^r} = C\mathcal{P}^0_{g^r}$ on Z'. But Z' is dense in Z and so this equality holds on Z by continuity. Now, by Corollary 18 in Subsection 5.3, it follows that both $\mathcal{P}_{g^r}$ and $\mathcal{P}^0_{g^r}$ are invariant under the diagonal group in $U(N) \times U(N)$ and since Z intersects each orbit under this group we can conclude that $\mathcal{P}_{g^r} = C\mathcal{P}^0_{g^r}$ on all of $X \times X$.

Given any holomorphic section f of L^{-r}, $r > 0$, and any vector v_p in the fiber of L^r above an arbitrary point p, our Koppelman formula now gives

$$f(p).v_p = \int_{X_z} \mathcal{P}_{g^r}(z,p) \wedge v_p \wedge f(z) = C \int_{X_z} \mathcal{P}^0_{g^r}(z,p) \wedge v_p \wedge f(z). \tag{31}$$

It is easy to compute $\mathcal{P}^0_{g^r}$ explicitly, and one gets

$$\mathcal{P}^0_{g^r} = \Big(\frac{i}{2\pi}\Big)^n (\gamma_0^k)^{\otimes r} \otimes \int_E \Big(\widetilde{\Theta_{F_z} \otimes \mathrm{Id}}_{H^*_\zeta}\Big)_n = \Big(\frac{i}{2\pi}\Big)^n (\gamma_0^k)^{\otimes r} \otimes c_{N-k}(\Theta_{F_z})^k.$$

Moreover, Θ_{F_z} is the $U(N)$-invariant curvature of F_z, so it follows that $c_{N-k}(\Theta_{F_z})^k$ is a $U(N)$-invariant (n,n)-form and hence equal to a constant times the invariant volume form dV. We have thus obtained

$$f(p).v_p = C \int_{X_z} f(z).(\gamma_0^k)^{\otimes r} v_p dV(z) \tag{32}$$

for any holomorphic section f of L^{-r}. Modulo a multiplicative constant, one also has that $dV = (c_1(L))^n$, and then the above formula assumes the following form expressed in the frames and coordinates discussed above.

$$f(\zeta) = C \int_{\mathbb{C}^n} f(z) \frac{\det(I + z^*\zeta)^r}{\det(I + z^*z)^r} \big((\partial\bar\partial \log\det(I + z^*z))\big)^n.$$

We will now describe what will be the Bergman kernel. Let $\rho^r \colon L^r_z \to L^{-r}_z$ be the antilinear identification induced by the metric, i.e., $\rho^r(v) = \langle \cdot, v\rangle_{L^r_z}$, and define $K_r(z,\zeta) \colon L^r_\zeta \to L^{-r}_z$ by $K_r(z,\zeta) = \rho^r \circ (\gamma_0^k)^{\otimes r}$. Then one easily checks that $K_r(z,\zeta)$ is a fiberwise antilinear map, which depends antiholomorphically on ζ. To show that it actually depends holomorphically on z we consider the adjoint operator $K_r(z,\zeta)^* \colon L^r_z \to L^{-r}_\zeta$ and the operator

$K_r(\zeta, z)\colon L_z^r \to L_\zeta^{-r}$. We know that the latter operator depends antiholomorphically on z. Note also that since $K_r(z,\zeta)$ is fiberwise antilinear, the adjoint should be defined by $(K_r(z,\zeta)^* u).v = \overline{u.(K_r(z,\zeta)v)}$ for $u \in L_z^r$ and $v \in L_\zeta^r$. It is then straightforward to check that $K_r(z,\zeta)^* = K_r(\zeta, z)$, and so $K_r(z,\zeta)^*$ must depend antiholomorphically on z. It follows that $K_r(z,\zeta)$ depends holomorphically on z. In particular, for any non-zero vector $v \in L_p^r$, the mapping $z \mapsto K_r(z,p)v$ defines a global non-zero holomorphic section of L^{-r}. In fact, these sections generate $H^0(X, L^{-r})$ as we now show. Consider the Bergman space A_r^2 defined as $H^0(X, L^{-r})$ equipped with the norm

$$\|f\|_{A_r^2}^2 := \int_X \|f\|_{L^{-r}}^2 dV, \;\; f \in H^0(X, L^{-r}).$$

We claim that, modulo a multiplicative constant, $K_r(z,\zeta)$ is the Bergman kernel for A_r^2, i.e., that $K_r(z,\zeta)$ is the fiberwise antilinear map $L_\zeta^r \to L_z^{-r}$, which depends holomorphically on z and antiholomorphically on ζ, and has the property that for any $f \in A_r^2$ and any vector $v \in L_\zeta^r$ (in the fiber above ζ) one has

$$f(\zeta).v = \langle f, K_r(\cdot,\zeta)v\rangle_{A_r^2} = \int_X \langle f(z), K_r(z,\zeta)v\rangle_{L_z^{-r}} dV(z).$$

It only remains to verify this last property. But this reproducing property follows directly from (32) after noting the following equality, which basically is the definition of $K_r(z,\zeta)$:

$$u.((\gamma_0^k)^{\otimes r} v) = \langle u, K_r(z,\zeta)v\rangle_{L_z^{-r}}, \;\; \text{for all } u \in L_z^{-r}, \;\; \text{and all } v \in L_\zeta^r.$$

Remark 22. In the case of $\mathbb{CP}^n$ it is not too hard to compute $\mathcal{P}_{g^r}$ directly from its definition. For instance, one can first verify in local or homogeneous coordinates that the part of $\gamma_1 \wedge D\eta$ which contains no $d\zeta$ or $d\bar{\zeta}$ is equal to $-\gamma_0 \otimes \widetilde{\Theta_{F_z} \otimes} \mathrm{Id}_{\mathcal{O}(1)_\zeta}$, cf. Proposition 4.1 and the weight α in [8]. Then, a straightforward computation shows that $\mathcal{P}_{g^r}$ is equal to

$$\binom{n+r}{n} \big(\frac{i}{2\pi}\big)^n \gamma_0^r \otimes \det(\Theta_{F_z}).$$

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EXPLICIT SOLUTIONS OF DIVISION PROBLEMS FOR MATRICES OF POLYNOMIALS

ELIN GÖTMARK

Contents

1. Introduction

Let $P(z) = (P_1(z), \dots, P_m(z))$ be a tuple of polynomials in $\mathbb{C}^n$, and $\Phi(z)$ another polynomial in $\mathbb{C}^n$ which vanishes on the zero set Z of P. By Hilbert's Nullstellensatz, we can find $\nu \in \mathbb{N}$ and polynomials $Q(z) = (Q_1(z), \dots, Q_m(z))$ such that

$$\Phi^\nu = P \cdot Q = P_1Q_1 + \dots + P_mQ_m.$$

There has been much research devoted to finding effective versions of this, i. e. determining bounds on ν and the polynomial degrees of the Q_i's. These degrees depend not only on the degrees of P and Φ, but on Z and the singularity of P at infinity.

A breakthrough in the problem of degree estimates came in [6], where Brownawell found bounds by a combination of algebraic and analytical methods. The optimal degrees, which are slightly better than Brownawell's, were found by Kollár in [11], using purely algebraic methods. There are also classical theorems by M. Nöther and Macaulay which treat simpler cases where one has conditions on the singularity.

In [1], the problem of solving

$$\Phi = P \cdot Q = P_1Q_1 + \dots + P_mQ_m \tag{1}$$

with degree estimates is treated by means of residue currents on $\mathbb{P}^n$ based on the Koszul complex. One can then recover the Nöther and Macaulay theorems. In the same article are explicit solutions Q constructed by means of integral representations; one then has some loss of precision in the degree estimates.

It is natural to look at generalizations of the division problem (1) to the case where P is a matrix. More precisely, we let P be a polynomial mapping $\mathbb{C}^n \to \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^r)$, i. e. an $r \times m$ matrix P where the entries are polynomials in n complex variables. Let $\Phi = (\Phi_1, \dots, \Phi_r)$ be an r-column of polynomials. If we assume that there exists an m-column of polynomials Q such that $\Phi = PQ$, we want to find an explicit solution Q and get an estimate of its degree. We can reduce the case $r > 1$ to the case $r = 1$ by means of the Fuhrmann trick [7] (see Remark 1), however, we will then lose precision in the degree estimates.

The case $r > 1$ is treated in [3], where an estimate of the degree of the solution is obtained by means of residue currents on $\mathbb{P}^n$, based on the Buchsbaum-Rim complex. The aim of this paper is to present explicit integral representations of Q for $r \geq 1$. The paper [2] contains a method of obtaining explicit solutions to division problems $\Phi = PQ$ over open domains $X \subset \mathbb{C}^n$, and we will adapt this method so that we can use it on $\mathbb{P}^n$. Also, we present careful degree estimates of so-called Hefer forms, which make it possible to estimate the degree of the solutions. We will lose some precision in the degree estimates compared to [3], but in return we get explicit solutions. We also point out that in principle, explicit solutions can be obtained which satisfy the same degree estimates as in [3].

2. A division formula for tuples of polynomials

Instead of solving the problem $\Phi = PQ$ in $\mathbb{C}^n$, we will homogenize the expression and look at the corresponding problem in $\mathbb{P}^n$ instead. Let $\zeta = (\zeta_0, \zeta') = (\zeta_0, \zeta_1, \dots \zeta_n)$. Set $\phi = \zeta_0^\rho \Phi(\zeta'/\zeta_0)$, where $\rho = \deg \Phi$; the entries of ϕ will then be ρ-homogeneous holomorphic polynomials in $\mathbb{C}^{n+1}$, and can be interpreted as sections of a line bundle over $\mathbb{P}^n$ (more on this later). Assume that $\deg(P^j) \leq d_j$, where P^j is number j in P, and let $p(\zeta)$ be the matrix with columns $p^j(\zeta) = \zeta_0^{d_j} P^j(\zeta'/\zeta_0)$. We will try to solve the division problem $\phi = p \cdot q$ by means of integral representation. Generally, this is not always possible, so we will get a residue term ϕR^p as well. When ϕ annihilates the residue R^p, we can solve the division problem. Dehomogenizing q will give a solution to our original problem.

We will now find integral representation formulas which solve the division problem in a more general setting, and return to our original problem in Section 4. Let $S = \mathbb{C}[z_0, \dots, z_n]$ be the ring of holomorphic homogeneous polynomials over $\mathbb{C}^{n+1}$ with the ordinary grading, and let $S(-k)$ denote the same ring, but where the grading is given by adding k to the degree of the polynomial. For example, the constants have degree k. Assume that we have a graded complex M of modules over $\mathbb{C}^{n+1}$

$$0 \xrightarrow{f_{N+1}} M_N \xrightarrow{f_N} \cdots \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \to 0,$$

where $M_m = S(-d_m^1) \oplus \dots \oplus S(-d_m^{r_m})$ has rank r_m and basis $e_{m1}, \dots, e_{mr_m}$. We will take $d_0^i = 0$ for all i. Since the complex is graded, f_m must preserve the degree of elements of M_m. To attain this, we must have

$$f_m = \sum_{i=1}^{r_{m-1}} \sum_{j=1}^{r_m} f_m^{ij} e_{m-1,i} \otimes e_{mj}^*, \tag{2}$$

where f_m^{ij} is a $d_{m-1}^i - d_m^j$-homogeneous holomorphic polynomial. Let $M = M_0 + \cdots + M_N$. We define $f : M \to M$ to be such that $f_{|M_m} = f_m$.

Let $\mathcal{O}(-1)$ denote the tautological line bundle over $\mathbb{P}^n$, and if k is positive we let $\mathcal{O}(-k) := \mathcal{O}(-1)^{\otimes k}$ and $\mathcal{O}(k) := (\mathcal{O}(-1)^*)^{\otimes k}$. Note that sections of $\mathcal{O}(k)$ can be seen as k-homogeneous polynomials in $\mathbb{C}^{n+1}$. Our complex over $\mathbb{C}^{n+1}$ has an associated complex E over $\mathbb{P}^n$, namely

$$0 \to E_N \xrightarrow{f_N} \cdots \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \to 0,$$

where

$$E_m = \sum_{i=1}^{r_m} \mathcal{O}(d_m^i) \otimes L_{mi}$$

and the L_{mi} are trivial line bundles with frames e_{mi}. We can then write

$$f_m = \sum_{i=1}^{r_{m-1}} \sum_{j=1}^{r_m} f_m^{ij} e_{m-1,i} \otimes e_{mj}^*,$$

where f_m^{ij} is the section of $\mathcal{O}(d_{m-1}^i - d_m^j)$ corresponding to f_m^{ij} in (2). Note that from E, we can obtain a complex E' of trivial vector bundles over $\mathbb{C}^n$, simply by taking the natural trivialization over the coordinate neighborhood $\{\zeta_0 \neq 0\}$. We have

$$0 \to E_N' \xrightarrow{F_N} \cdots \xrightarrow{F_2} E_1' \xrightarrow{F_1} E_0' \to 0, \tag{3}$$

where $F_m(\zeta') = f_m(1, \zeta')$ for $\zeta' \in \mathbb{C}^n$, and E_m' is a trivial vector bundle of rank r_m over $\mathbb{C}^n$.

We introduce a $\mathbb{Z}_2$-grading on E, by writing $E = E^+ \oplus E^-$ where $E^+ = \bigoplus E_{2k}$ and $E^- = \bigoplus E_{2k+1}$, and we say that sections of E^+ have even degree and sections of E^- have odd degree. This induces a grading $\text{End}E = (\text{End}E)^+ \oplus (\text{End}E)^-$ where a mapping $f : E \to E$ has odd degree if it maps $E^+ \to E^-$ and $E^- \to E^+$, and similarly for even mappings. We also get an induced grading on currents taking values in E, that is, $\mathcal{D}'_{\bullet,\bullet}(\mathbb{P}^n, E)$, by combining the gradings on $\mathcal{D}'_{\bullet,\bullet}(\mathbb{P}^n)$ and E, so that $\alpha \otimes \phi = (-1)^{\deg\alpha \cdot \deg\phi} \phi \otimes \alpha$ if α is a current and ϕ a section of E. Similarly, we get an induced grading on $\mathcal{D}'_{\bullet,\bullet}(\mathbb{P}^n, \text{End}E)$, then $\bar\partial$ is an odd mapping. Since f is odd and holomorphic, it is easy to show that $\bar\partial \circ f = -f \circ \bar\partial$.

Let Z_i be the set where f_i does not attain its highest rank, and set $Z = \bigcup Z_i$. To construct integral formulas, we need to find currents U_k with bidegree $(0, k-1)$ and R_k with bidegree $(0, k)$, with values in $\text{Hom}(E_0, E_k)$, such that if $U = \sum_k U_k$ and $R = \sum_k R_k$, then

$$(f - \bar\partial)U = I_{E_0} - R. \tag{4}$$

For the reader's convenience, we indicate the construction here, but see [5] for details. We first define mappings $\sigma_i : E_{i-1} \to E_i$ defined outside Z_i, such that $f_i\sigma_i = \text{Id}$ on $\text{Im} f_i$, σ_i vanishes on $(\text{Im} f_i)^\perp$, and $\text{Im}\sigma_i \perp \text{Ker} f_i$. Now, set $u_1 = \sigma_1$ and $u_k = \sigma_k \bar\partial \sigma_{k-1} \wedge \cdots \wedge \bar\partial \sigma_1$. We have $(f - \bar\partial)u = \sum_k (f_k u_k - \bar\partial u_k) = f_1\sigma_1 + \sum_k (\bar\partial u_{k-1} - \bar\partial u_k) = \text{Id}_{E_0}$ outside Z. To extend u over Z, we let U be the analytical continuation of $|f|^{2\lambda}u$ to $\lambda = 0$, then we get (4). Note that for degree reasons we necessarily have $U_k = 0$ for $k > \mu + 1$, where $\mu = \min(n, N-1)$.

We will need to recall a proposition about representation of holomorphic sections of line bundles over $\mathbb{P}^n$. See for example [8] for more on this. Let $z = (1, z')$ be a fixed point, and set

$$\eta = 2\pi i \sum_0^n z_i \frac{\partial}{\partial \zeta_i}.$$

If we express a projective form in homogeneous coordinates, we can let contraction with η act on it and get a new projective form. In fact, we will get an operator

$$\delta_\eta : \mathcal{D}'_{l+1,0}(\mathbb{P}^n, \mathcal{O}(k)) \to \mathcal{D}'_{l,0}(\mathbb{P}^n, \mathcal{O}(k-1)),$$

where $\mathcal{D}'_{l,0}(\mathbb{P}^n, \mathcal{O}(k))$ denotes the currents of bidegree $(l, 0)$ taking values in $\mathcal{O}(k)$.

Proposition 2.1. *Let $\phi(\zeta)$ be a section of $\mathcal{O}(\rho)$, and let $g(\zeta, z) = g_{0,0} + \ldots + g_{n,n}$ be a current, where $g_{k,k}$ is of bidegree (k, k) and takes values in $\mathcal{O}(\rho + k)$. Assume that $(\delta_\eta - \bar\partial)g = 0$ and $g_{0,0}(z, z) = \phi(z)$. Then we have*

$$\phi(z) = \int_{\mathbb{P}^n} (g \wedge \alpha^{n+\rho})_{n,n},$$

where

$$\alpha = \alpha_0 + \alpha_1 = \frac{z \cdot \bar\zeta}{|\zeta|^2} - \bar\partial \frac{\bar\zeta \cdot d\zeta}{2\pi i |\zeta|^2}.$$

The idea now is to find a weight g that contains f_1 as a factor, and then apply Proposition 2.1. As components of the weight, we need so-called Hefer forms. We define $f_m^z : E_m \to E_{m-1}$ to be the mapping

$$f_m^z = \sum_{i=1}^{r_{m-1}} \sum_{j=1}^{r_m} \zeta_0^{d^i_{m-1} - d^j_m} f_m^{ij}(z) e_{m-1,i} \otimes e^*_{mj}.$$

Proposition 2.2. *There exist $(k-l, 0)$-form-valued mappings $h^l_k(\zeta, z) : E_k \to E_l$, such that $h^l_k = 0$ for $k < l$, $h^l_l = I_{E_l}$, and*

$$\delta_\eta h^l_k = h^l_{k-1} f_k - f^z_{l+1} h^{l+1}_k$$

Moreover, the coefficients of the form $(h^l_k)_{\alpha\beta}$, that is, the coefficient of $e_{l\alpha} \otimes e^*_{k\beta}$, will take values in $\mathcal{O}(d^\alpha_l - d^\beta_k + k - l)$, and can be chosen so that they are holomorphic polynomials in z' of degree $d^\alpha_l - d^\beta_k - (k - l)$.

We will prove this in the next section. Now we can define our weight:

Proposition 2.3. *If*

$$g = f_1^z \sum h_k^1 U_k + \sum h_k^0 R_k = f_1^z h^1 U + h^0 R,$$

then $g_{0,0}|_\Delta = I_{E_0}$ *and* $\nabla_\eta g = 0$.

The following proof is identical to the one in Section 5 in [2], except that our proof will be in $\mathbb{P}^n$ instead of $\mathbb{C}^n$. We include it for the reader's convenience.

Proof. By definition, we have

$$(5) \qquad \nabla_\eta g = f_1^z[(h^1 f - f^z h^2)U - h^1 \bar\partial U] + (h^0 f - f^z h^1)R - h^0 \bar\partial R.$$

Note that one has to check the total degree of h^1 and h^0 to get all the signs correct. Since $\bar\partial U = fU - R - I_{E_0}$ and $f^2 = 0$, the right hand side of (5) is equal to

$$f_1^z[h^1 fU - h^1(fU - R - I_{E_0})] + h^0(fR - \bar\partial R) - f^z h^1 R.$$

Furthermore, we have $h^1 I_{E_0} = 0$ and $(f - \bar\partial)R = (f - \bar\partial)(R + I_Q) = (f - \bar\partial)^2 U = 0$, so it follows that $\nabla_\eta g = 0$. We also have $g_{0,0} = f^z h_1^1 U_1 = f^z U_1$, so that $g_{0,0}|_\Delta = fU_1 = I_{E_0}$. □

This is the main result of this section:

Theorem 2.4. *Let $\phi(\zeta)$ be a section of $\mathcal{O}(\rho) \otimes E_0$. Fix $z = (1, z')$ and let $\Phi(z') = \phi(1, z')$, and $F_1(z') = f_1(1, z')$. If g is defined as in Proposition 2.3, we have the following decomposition:*

$$\Phi(z') = \int_\zeta g\phi \wedge \alpha^{n+\rho} = F_1(z')Q(z') + \int_\zeta h^0 R\phi \wedge \alpha^{n+\rho},$$

where $Q(z') = \sum_1^{r_1} Q_i(z') e_{1j}$ *with*

$$Q_i(z') = \int_\zeta \zeta_0^{-d_1^i} (h^1 U \phi)_i(\zeta, z) \wedge \alpha^{n+\rho}.$$

If $R\phi = 0$, the second integral will vanish and we get a solution to our division problem. Moreover, we get the estimate $\deg_{z'} F_1 Q \le \max_{k,\beta}(\rho - d_k^\beta)$.

Proof. First, we note that

$$g\phi = \sum_{j=1}^{r_0} (g\phi)_j e_{0j},$$

and according to Proposition 2.3, $(g\phi)_j$ is a weight of the type needed to apply Proposition 2.1. Now we need only check the degree in z'. The term in $(g\phi)_j$ of bidegree (k, k) is $(f_1^z h_{k+1}^1 U_{k+1} \phi)_j$ (if we disregard the terms containing R). We must pair this with $\alpha_0^{\rho+k} \alpha_1^{n-k}$ to get something with full bidegree in z.

The degree in z' of a term in

$$\sum_{k,\alpha,\gamma,\beta}(f_1^z)_{j\alpha}(h_{k+1}^1)_{\alpha\beta}(U_{k+1})_{\beta\gamma}\phi_\gamma \wedge \alpha_0^{\rho+k}\alpha_1^{n-k}$$

is $-d_1^\alpha + (d_1^\alpha - d_{k+1}^\beta - k) + (\rho + k) = \rho - d_k^\beta$. We get the estimate $\deg F_1 Q \le \max_{k,\beta}(\rho - d_{k+1}^\beta)$. □

3. Constructing Hefer forms

In this section we construct Hefer forms for the complexes E and E', and investigate their degrees. We first state and prove a more general theorem, and obtain as a corollary Hefer forms H_k^l with specific degrees for the complex E' over $\mathbb{C}^n$. From the H_k^l we get corresponding Hefer forms h_k^l for the complex E over $\mathbb{P}^n$, i. e. we prove Proposition 2.2 in the previous section.

Let $\delta_{\zeta-z}$ denote contraction with the vector field

$$2\pi i\sum_1^n(\zeta_j - z_j)\frac{\partial}{\partial\zeta_j}.$$

Theorem 3.1. *Let $\phi(\zeta, z)$ be a $(l, 0)$-form with holomorphic polynomials of degree r for coefficients. If $l > 0$ and $\delta_{\zeta-z}\phi = 0$, we have $\phi = \delta_{\zeta-z}\psi$, where ψ is a $(l-1, 0)$-form with holomorphic polynomials of degree $r-1$ for coefficients. If $l = 0$, let $\phi(\zeta)$ be a holomorphic polynomial. We can then write $\phi(\zeta) - \phi(z) = \delta_{\zeta-z}\psi$, where ψ satisfies the same conditions as above.*

For the proof of Theorem 3.1, we need a lemma:

Lemma 3.2. *Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be multiindices and D_R the ball with radius R in $\mathbb{C}^n$. We have*

$$I_{\alpha,\beta,R} = \int_{\partial D_R} w^\alpha \bar{w}^\beta \partial|w|^2 \wedge (\partial\bar\partial|w|^2)^{n-1} = 0 \tag{6}$$

if $\alpha \neq \beta$ and (6) is non-zero and proportional to $R^{2(|\alpha|+n)}$ if $\alpha = \beta$.

Proof. (of Lemma 3.2) The integral is unchanged if we take the pullback of the integrand by the function $A(w) = (\lambda_1 w_1, \dots \lambda_n w_n)$, where $\lambda_i \in \mathbb{C}$ and $|\lambda| = |(\lambda_1, \dots \lambda_n)| = 1$. We have

$$I_{\alpha,\beta,R} = \int_{\partial D_R}(Aw)^\alpha(\overline{Aw})^\beta\partial|Aw|^2 \wedge (\partial\bar\partial|Aw|^2)^{n-1} = \lambda^\alpha\bar\lambda^\beta I_{\alpha,\beta,R}.$$

It is now clear that $I_{\alpha,\beta,R} = 0$ if $\alpha \neq \beta$. A similar calculation gives

$$I_{\alpha,\alpha,R} = \int_{\partial D}(Rw)^\alpha(R\bar w)^\alpha\partial|Rw|^2 \wedge (\partial\bar\partial|Rw|^2)^{n-1} = R^{2(|\alpha|+n)}I_{\alpha,\alpha,1},$$

which shows that $I_{\alpha,\alpha,R}$ is proportional to $R^{2(|\alpha|+n)}$. □

Proof. (of Theorem 3.1) The setup here is the same as in Section 4 in [2]. We view $E = T^*_{1,0}(\mathbb{C}^n)$ as a rank n vector bundle over $\mathbb{C}^n$ and let $f(\zeta) = 2\pi i(\zeta - z)\cdot e^*$, where $z \in \mathbb{C}^n$ is fixed, be a section of $E^* = T_{1,0}(\mathbb{C}^n)$. We can now express the contraction $\delta_{\zeta - z}$ as δ_f operating on E. Take a section $\phi(\zeta, z)$ of $\Lambda^l E$ with holomorphic polynomials of degree r for coefficients. We want to show that $\phi = \delta_f \psi$, where ψ is a section of $\Lambda^{l-1}E$ with holomorphic polynomials of degree $r-1$ for coefficients.

Let $D \subset \mathbb{C}^n_w$ be the unit ball. We let $h = dw \wedge e^*$, then $\delta_{w-\zeta} h = 2\pi i(w-\zeta)\cdot e^* = f(w) - f(\zeta)$, so that h will work as a Hefer form. Let

$$\sigma = \frac{\bar{w}\cdot e}{2\pi i(|w|^2 - \bar{w}\cdot z)},$$

then $\delta_{f(w)}\sigma = 1$ if $w \neq z$. Set $u = \sum \sigma \wedge (\bar\partial\sigma)^k$, note that u depends holomorphically on z. Since u is integrable, we let U be the trivial extension over z. Moreover, we have $(\delta_{f(w)} - \bar\partial)U = 1 - R$, where $R_k = 0$ for $k < n$ and $(\delta_h)_n R_n = [z]$ (this follows, for example, from Theorem 2.2 in [4]). Let $\phi(\zeta, z)$ be a section of $\Lambda^l E$ with $l > 0$ which satisfies $\delta_{f(\zeta)}\phi = 0$. We set

$$g(w,\zeta) = \delta_{f(\zeta)} \sum \delta_h^{k-1}(U_k \phi(w,z)) + \sum \delta_h^k (R^k \phi(w,z));$$

a calculation shows that $(\delta_{w-\zeta} - \bar\partial)g = 0$ and $g_{0,0}(\zeta,\zeta) = \phi(\zeta, z)$. Note that the second sum is actually zero since $R \wedge \phi = 0$.

Now define

$$g_1(w,\zeta) = \chi_D - \bar\partial\chi_D \wedge \frac{s}{\nabla_{w-\zeta}s} = \chi_D - \bar\partial\chi_D \wedge \sum s \wedge (\bar\partial s)^k$$

where

$$s = \frac{1}{2\pi i}\frac{\bar{w}\cdot dw}{|w|^2 - \bar{w}\cdot\zeta}.$$

Note that $(\delta_{w-\zeta} - \bar\partial)g_1 = 0$ and $(g_1)_{0,0}(\zeta,\zeta) = 1$. Also, g_1 depends holomorphically on ζ.

We can now take a current $u = u_{1,0} + \ldots + u_{n,n-1}$ that is smooth outside z and such that $(\delta_{w-\zeta} - \bar\partial)u = 1 - [z]$, where $[z]$ denotes the (n,n)-current point evaluation at z. We have $(\delta_{w-\zeta} - \bar\partial)(u \wedge g \wedge g_1) = g \wedge g_1 - [z] \wedge g \wedge g_1 = g \wedge g_1 - \phi(\zeta, z)[z]$, so that $\bar\partial(u \wedge g \wedge g_1)_{n,n-1} = \phi(\zeta,z)[z] - (g \wedge g_1)_{n,n}$. Stokes' theorem gives us $\phi(\zeta, z) = \delta_{f(\zeta)} T\phi$ where

$$T\phi = \int_{\partial D_w} \sum_1^{n-l} \frac{\bar{w}\cdot e \wedge (\delta_h)_{k-1}[(d\bar{w}\cdot e)^{k-1} \wedge \phi] \wedge \partial|w|^2 \wedge (\frac{i}{2\pi}\partial\bar\partial|w|^2)^{n-k}}{2\pi i(1 - \bar{w}\cdot z)^k (1 - \bar{w}\cdot\zeta)^{n-k+1}}.$$

This is seen by noting that $\delta_h \sigma \wedge s = 0$ and that

$$\sigma \wedge \bar\partial\sigma = \sigma \wedge \frac{d\bar{w}\cdot e}{|w|^2 - \bar{w}\cdot z}.$$

First, note that $T\phi$ is holomorphic in ζ and z and takes values in $\Lambda^{l+1}E$. To see that the coefficients are polynomials of degree $r-1$, we write

$$\frac{1}{1-\bar{w}\cdot\zeta} = \sum_0^\infty (\bar{w}\cdot\zeta)^k$$

and similarly for $1/(1-\bar{w}\cdot z)$. The only place in $T\phi$ where w's occur is in ϕ, and according to Lemma 3.2 we need to match those up with a corresponding number of $\bar{w}$'s. These can come either from the factor $\bar{w}\cdot e$, or from the geometric expansions. It is clear that if we start with a term in $\phi(w,z)$ which is of degree r in w, after we integrate we shall get a term which is of combined degree $r-1$ in ζ and z. It is easy to see that the same thing is also true if we start with a term which is of combined degree r in z and w. The case when $\phi(w)$ takes values in the trivial bundle is proved in a similar way. □

We will now find Hefer forms for the complex E' by means of Theorem 3.1.

Proposition 3.3. *There exist $(k-l,0)$-form-valued mappings $H_k^l : E_k \to E_l$, where the coefficients are holomorphic polynomials in ζ and z, such that $H_k^l = 0$ for $k < l$, $H_l^l = I_{E_l}$, and*

$$\delta_{\zeta-z} H_k^l = H_{k-1}^l F_k(\zeta) - F_{l+1}(z) H_k^{l+1}. \tag{7}$$

Moreover, the polynomial degree of $(H_k^l)_{\alpha\beta}$ (that is, the coefficient of $e_{l\alpha}\otimes e^*_{k\beta}$) is $d_l^\alpha - d_k^\beta - (k-l)$.

Proof. The proof will be by induction, and by application of Theorem 3.1. We begin by proving that the right hand side of (7) is $\delta_{\zeta-z}$-closed. We have

$$\delta_{\zeta-z}(H_{k-1}^l F_k(\zeta) - F_{l+1}(z) H_k^{l+1}) =$$
$$= (H_{k-2}^l F_{k-1}(\zeta) - F_{l+1}(z) H_{k-1}^{l+1}) F_k(\zeta) - F_{l+1}(z)(H_{k-1}^{l+1} F_k(\zeta) - F_{l+2}(z) H_k^{l+2}) = 0,$$

since E' is a complex. For $k = l+1$ we must solve $\delta_{\zeta-z} H_{l+1}^l = F_{l+1}(\zeta) - F_{l+1}(z)$. We can apply Theorem 3.1 to $(H_{l+1}^l)_{\alpha\beta}$, and it follows that $(H_{l+1}^l)_{\alpha\beta}$ is a $(1,0)$-form whose coefficients are holomorphic polynomials of degree $d_l^\alpha - d_{l+1}^\beta - 1$. Assume that the proposition holds for H_k^l with $k-l = i$, and regard the equation

$$\delta_{\zeta-z}(H_k^l)_{\alpha\beta} = (H_{k-1}^l F_k(\zeta) - F_{l+1}(z) H_k^{l+1})_{\alpha\beta} =$$
$$= \sum_{\gamma=1}^{r_{k-1}} (H_{k-1}^l)_{\alpha\gamma} (F_k(\zeta))_{\gamma\beta} - \sum_{\gamma=1}^{r_{l+1}} (F_{l+1}(z))_{\alpha\gamma} (H_k^{l+1})_{\gamma\beta}.$$

where $k-l = i+1$. The right hand side is a form of bidegree $(k-l-1,0)$, and it is easy to see by examining the sums that the coefficients are polynomials of degree $d_l^\alpha - d_k^\beta - (k-l-1)$. It follows from Theorem 3.1 that $(H_k^l)_{\alpha\beta}$ is a form of bidegree $(k-l,0)$ whose coefficients are holomorphic polynomials of degree $d_l^\alpha - d_k^\beta - (k-l)$. □

We will now prove Proposition 2.2 by adapting the Hefer forms H_k^l to the complex E on $\mathbb{P}^n$. From now on, we change notation so that $\zeta = (\zeta_0, \zeta') \in \mathbb{C}^{n+1}$ and $z = (1, z') \in \mathbb{C}^{n+1}$.

Proof. (of Proposition 2.2) The form $(H_k^l)_{\alpha\beta}$ has bidegree $(k-l, 0)$, so we have

$$(H_k^l)_{\alpha\beta} = \sum_{|I|=k-l} a_I d(\zeta')_I.$$

We set

$$(h_k^l)_{\alpha\beta}(\zeta, z') = \zeta_0^{d_l^\alpha - d_k^\beta + (k-l)} \sum_{|I|=k-l} a_I(\zeta'/\zeta_0, z') d(\zeta'/\zeta_0)_I.$$

It is clear that $(h_k^l)_{\alpha\beta}$ is a projective form, since we are multiplying with a high enough degree of ζ_0, and that it takes values in the right line bundle. Now, note that

$$\delta_\eta d(\zeta_j/\zeta_0) = z_j/\zeta_0 - \zeta_j/\zeta_0^2.$$

We have

$$\delta_\eta (h_k^l)_{\alpha\beta} = \zeta_0^{d_l^\alpha - d_k^\beta + (k-l-1)} \sum_{|I|=k-l} \sum_{j\in I} \pm a_I(\zeta'/\zeta_0, z')(z_j - \zeta_j/\zeta_0) d(\zeta'/\zeta_0)_{I\setminus j} =$$

$$= \zeta_0^{d_l^\alpha - d_k^\beta + (k-l-1)} \delta_{\zeta'/\zeta_0 - z'} (H_k^l)_{\alpha\beta}(\zeta'/\zeta_0, z') =$$

$$= \zeta_0^{d_l^\alpha - d_k^\beta + (k-l-1)} (H_{k-1}^l(\zeta'/\zeta_0, z') f_k(\zeta'/\zeta_0) - f_{l+1}(z') H_k^{l+1}(\zeta'/\zeta_0, z'))_{\alpha\beta} =$$

$$= (h_{k-1}^l f_k - f_{l+1}^z h_k^{l+1})_{\alpha\beta},$$

so we are finished. □

4. The Buchsbaum-Rim complex

We will now introduce the Buchsbaum-Rim complex, which is a special case of our complex E over $\mathbb{P}^n$. For more details about this complex see e. g. [3]. Let $P : \mathbb{C}^n \to \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^r)$ be a generically surjective polynomial mapping and Z the set where P is not surjective. The mapping P will have the role of F_1 in the complex (3). Assume that $\deg(P^j) \leq d_j$, where P^j is number j in P, and $d_1 \geq d_2 \geq \cdots$. Let $p(\zeta)$ be the matrix with columns $p^j(\zeta) = \zeta_0^{d_j} P^j(\zeta'/\zeta_0)$. The mapping p will have the role of f_1 in (3). If $L_1, \dots, L_m$ are trivial line bundles over $\mathbb{P}^n$, with frames $e_1, \dots, e_m$, we can define the rank m bundle

$$E_1 = (\mathcal{O}(-d_1) \otimes L_1) \oplus \cdots \oplus (\mathcal{O}(-d_n) \otimes L_m).$$

Let E_0 be a trivial vector bundle of rank r, with the frame $\{\epsilon_j\}$, and let p_{ij} be the i:th element in the column p^j. We can then view

$$p = \sum_{i=1}^{m} \sum_{j=1}^{r} p_{ij} \epsilon_i \otimes e_j^*$$

as a mapping from E_1 to E_0. Contraction with p acts as a mapping $\delta_p : E_1 \otimes E_0^* \to \mathbb{C}$.

Now, note that we can also write $p = \sum p_i \otimes \epsilon_i$ where $p_i = \sum_j p_{ij} e_j^*$ is a section of E_1^*. Let

$$\det p = p_1 \wedge \ldots \wedge p_r \otimes \epsilon_1 \wedge \ldots \wedge \epsilon_r.$$

We now get a complex

$$0 \to E_{m-r+1} \xrightarrow{\delta_p} \cdots \xrightarrow{\delta_p} E_3 \xrightarrow{\delta_p} E_2 \xrightarrow{\det p} E \xrightarrow{p} E_0 \to 0$$

over $\mathbb{P}^n$, where for $k \geq 2$ we have

$$E_k = \Lambda^{k+r-1} E_1 \otimes S^{k-2} E_0^* \otimes \det E_0^*,$$

and $S^l E_0^*$ is the subbundle of $\bigotimes E_0^*$ consisting of symmetric l-tensors of E_0^*. By $f : E \to E$ we mean the mapping which is either p, $\det p$ or δ_p, depending on which E_k we restrict it to. We assume that r is odd, so that $\bar\partial \circ \det f = -\det f \circ \bar\partial$ and consequently $\bar\partial \circ f = -f \circ \bar\partial$. If r is even, one has to insert changes of sign at some places.

Next, we can construct currents U_k with bidegree $(0, k-1)$ and R_k with bidegree $(0, k)$, with values in $\mathrm{Hom}(E_0, E_k)$, such that $(f - \bar\partial)U = I_{E_0} - R$, or in other words

$$pU_1 = I_{E_0}, \quad (\det p)U_2 - \bar\partial U_1 = -R_1, \quad \text{and} \quad \delta_p U_{k+1} - \bar\partial U_k = -R_k$$

for $k \geq 2$. See [3] for details of the construction.

Now, recall that we want to solve the division problem $\Phi = PQ$, where $\Phi = (\Phi_1, \ldots, \Phi_r)$ is an r-column of polynomials of degree ρ. The homogenized version of this is $\phi = pq$, where ϕ takes values in $\mathcal{O}(\rho) \otimes E_0$, so that we can apply Theorem 2.4. To do that, we have to determine the d_k^j of the Buchsbaum-Rim complex.

For E_0 we have $r_0 = r$ and $d_0^j = 0$ for all j, and for E_1 we have $r_1 = m$ and $d_1^j = -d_j$. Furthermore, for E_k with $k \geq 2$ we have $r_k = \binom{m}{k+r-1} r^{k-2}/(k-2)!$, where $\binom{m}{k+r-1}$ is the dimension of $\Lambda^{k+r-1}E_1$ and $r^{k-2}/(k-2)!$ is the dimension of $S^{k-2}E_0^*$. However, it is only the factor $\Lambda^{k+r-1}E_1$ that contributes with any non-trivial line bundles, and and so we will not be interested in the factors in $S^{k-2}E_0^* \otimes \det E_0^*$. We then have $d_k^I = -\sum_{i \in I} d_i$ where $|I| = k + r - 1$, by which we mean that the coefficient of the basis vector e_I in $\Lambda^{k+r-1}E_1$ will take values in $\mathcal{O}(d_k^I) = \mathcal{O}(-\sum_{i \in I} d_i)$.

Recall that $f_k = \delta_p : E_k \to E_{k-1}$ for $k \geq 2$. Let f_k^{IJ} be the coefficent of $e_I \otimes e_J^*$, where $|I| = k + r - 2$, $|J| = k + r - 1$ and $I \subset J$. Then f_k^{IJ} takes values in

$$\mathcal{O}(d_{k-l}^I - d_k^J) = \mathcal{O}(\sum_{i \in I} d_i - \sum_{i \in J} d_i) = \mathcal{O}(d_j)$$

where $\{j\} = J \setminus I$. In the same way, $H_k^l : E_k \to E_l$, and for $k, l \geq 2$, we have that $(H_k^l)^{IJ}$ is a $(k-l)$-form whose coefficients take values in

$\mathcal{O}(\sum_{i\in J\setminus I} d_i + k - l)$ and are holomorphic polynomials in z of degree $\sum_{i\in J\setminus I} d_i - (k-l)$. Also, $(H^l_k)^{IJ} = 0$ if $I \not\subset J$.

Theorem 4.1. *Let $P(\zeta') : \mathbb{C}^n \to Hom(\mathbb{C}^m, \mathbb{C}^r)$ be a generically surjective polynomial mapping where the columns P^j have degree d_j, and $\Phi(\zeta') = (\Phi_1, \dots, \Phi_r)$ a tuple of polynomials with degree ρ. We have the following decomposition:*

$$\Phi(z') = P(z')Q(z') + \int_\zeta H^0 R\phi \wedge \alpha^{n+\rho},$$

where $Q(z') = \sum_1^m Q_i(z')e_j$ with

$$Q_i(z') = \int_\zeta \zeta_0^{d_i} (h^1 U\phi)_i \wedge \alpha^{n+\rho}.$$

If $R\phi = 0$, the second integral will vanish and we get a solution to the division problem. Moreover, the polynomial degree of PQ is $\rho + d_1 + \dots + d_{\mu+r}$, where $\mu = \min(n, m-r)$.

Proof. From Theorem 2.4, we have

$$\deg PQ \le \max_{k,I}(\rho - d^I_k).$$

Since $d^I_k = -\sum_{i\in I} d_i$ with $|I| = k + r - 1$, the degree is obviously largest when k is maximal and when we choose $I = (d_1, \dots, d_{k+r-1})$. We recall that $U_k = 0$ if $k > \mu + 1$, where $\mu = \min(n, m-r)$, and the theorem follows. □

Example 1. If $r = 1$, the Buchsbaum-Rim complex will simply be the Koszul complex. To obtain Hefer forms, we first find a $(1,0)$-form $h(\zeta, z') = \sum h_{ij}\epsilon_i \otimes e^*_j$ taking values in $\mathcal{O}(1) \otimes \text{Hom}\,(E_1, E_0)$, such that $\delta_\eta h = p - p^z$. It is then easy to show that $h^l_k = (\delta_h)^{k-l}$ will act as a Hefer form. We get the decomposition

$$\Phi(z') = P(z')Q(z') + \int_{\mathbb{P}^n} \sum_k \delta^k_h (R^k\phi(\zeta)) \wedge \alpha^{n+\rho},$$

where $Q(z') = \sum_1^m Q_i(z')e_j$ with

$$Q_i(z') = \int_\zeta \zeta_0^{d_i} \sum_k \delta_h^{k-1} U_k \phi(\zeta) \wedge \alpha^{n+\rho}.$$

Furthermore, $\deg PQ \le \rho + d_1 + \cdots + d_{\mu+1}$, where $\mu = \min(m-1, n)$.

Remark 1. Following Hergoualch [10], who uses an idea of Fuhrmann [7], we can reduce the case $r > 1$ to the case $r = 1$, but we will then lose precision in the degree estimates. If we wish to solve $PQ = \Phi$, as before, we can instead set $p' = p_1 \wedge \ldots \wedge p_r$ and solve $p' \cdot q'_j = \phi_j$. Note that we will then get a solution q'_j taking values in $E'_1 = \Lambda^r E_1$. Since the vector space E'_1 has dimension $m' = \binom{m}{r}$, the previous example will give $\deg P'Q'_j \le \rho + d'_1 + \cdots + d'_{\mu'+1}$, where $\mu' = \min(\binom{m}{r} - 1, n)$.

We have $d_1' = d_1 + \cdots + d_r$, so we can make the estimate $\deg P'Q_j' \leq \rho + (\mu' + 1)(d_1 + \cdots + d_r)$.

Now, we can set

$$q_j = (-1)^j \delta_{p_r} \cdots \delta_{p_{j+1}} \delta_{p_{j-1}} \cdots \delta_{p_1} q_j'.$$

Note that q_j takes values in E_1 and that $p_j q_j = \phi_j$. Let $q = q_1 + \cdots q_r$, then we have $PQ = \Phi$, with $\deg PQ \leq \rho + (\mu' + 1)(d_1 + \cdots + d_r)$.

Remark 2. It is possible to get a slightly better result than Theorem 4.1 by means of solving successive $\bar{\partial}$-equations in the Buchsbaum-Rim complex, as is done in [3]. We will sketch this approach. First, we modify the Buchsbaum-Rim complex by taking tensor products with $\mathcal{O}(\rho)$:

$$0 \to E_{m-r+1} \otimes \mathcal{O}(\rho) \xrightarrow{\delta_p} \cdots \xrightarrow{\det p} E_1 \otimes \mathcal{O}(\rho) \xrightarrow{p} E_0 \otimes \mathcal{O}(\rho) \to 0,$$

If ϕ is a section of $E_0 \otimes \mathcal{O}(\rho)$ and $R^f \phi = 0$, it is clear that $v = \phi U$ solves $\nabla_p v = \phi$. As in the classical Koszul complex method of solving division problems, we need to solve a succession of equations $\bar{\partial} w_k = v_k + \delta_p w_{k+1}$. The holomorphic solution will then be $v_1 + \delta_p w_2$. In order to solve the $\bar{\partial}$-equations, we need to assume that

$$m \leq n + r - 1 \quad \text{or} \quad \rho \geq \sum_1^{n+r} d_j - n,$$

where $\deg \Phi \leq \rho$. With this method, one proves that there exist polynomials $Q_1, \ldots, Q_m$ such that $\deg PQ \leq \rho$ (Theorem 1.8 in [3]).

Actually, by Proposition 5.1 in [8] we can solve $\bar{\partial}$-equations for $(0, q)$-forms taking values in $\mathcal{O}(l)$ explicitly. It is thus possible to obtain explicit solutions with these better degree estimates as well, although these solutions will be more complicated than the ones from Theorem 4.1, since they will involve μ integrations. We do not know if it is possible to find a weight such that the method in the present article yields explicit solutions $Q_1, \ldots, Q_m$ such that $\deg PQ \leq \rho$.

Remark 3. In the case of the Buchsbaum-Rim complex we can at least partially find explicit Hefer forms. As in the example above, we find h such that $\delta_\eta h = p - p^z$, and we let $h_k^l = \delta_h^{k-l}$ for $k > l \geq 2$. We also have $h_1^0 = h$ and

$$h_2^1 = \sum_{k=1}^r (-1)^{k+1} p^1 \wedge \ldots \wedge p^{k-1} \wedge h^k \wedge (p^{k+1})^z \wedge \ldots \wedge (p^r)^z \otimes \epsilon_1 \wedge \ldots \epsilon_r,$$

where h^k is the k:th column of h. We refrain from trying to find explicit expressions for h_k^0 for $k \geq 1$ and h_k^1 for $k \geq 2$.

We can get several corollaries to Theorem 4.1, that give conditions which ensure that $R^f \phi = 0$, and state what the estimate of the degree is in that case. First, there is a classical theorem by Macaulay [12] which says that if $r = 1$, $\Phi = 1$ and $p^1, \dots, p^m$ lack common zeroes in $\mathbb{P}^n$, then we have the estimate $\deg P^i Q_i \leq \sum_1^{n+1} d_j - n$. By means of residue currents, one can get a generalization to the case $r > 1$:

Proposition 4.2. *Assume that Z is empty. Then there exists a matrix Q of polynomials such that $PQ = I_r$, and $\deg PQ \leq \sum_1^{n+r} d_j - n$.*

This is Corollary 1.9 in [3]. By using integral representations, we can get instead

Corollary 4.3. *Assume that Z is empty. Then Theorem 4.1 yields an explicit matrix Q of polynomials such that $PQ = I_r$, and $\deg PQ \leq \sum_1^{n+r} d_j$.*

Proof. We first note that necessarily $\mu = n$ since $\operatorname{codim} Z = m - r + 1$. The result now easily follows by several application of Theorem 4.1 where one chooses the columns of I_r as Φ. $\square$

If Z is not empty, we would like some condition which guarantees that $R^f \phi = 0$. We define the pointwise norm $\|\phi\|^2 = \det(pp^*)|p^*(pp^*)^{-1}\phi|^2$.

Corollary 4.4. *If*

$$\|\phi\| \leq C \det(pp^*)^{\min(n, m-r+1)}, \tag{8}$$

then Theorem 4.1 yields explicit Q such that $PQ = \Phi$ with $\deg PQ \leq \rho + d_1 + \ldots + d_{\mu+r}$, where $\mu = \min(n, m-r)$.

This follows from Proposition 1.3 in [3], which says that if (8) holds, then $R^f \phi = 0$. As for the previous corollary, this is not the optimal result, which states that there exists a solution Q such that $\deg PQ \leq \rho$ (see Corollary 1.10 in [3]).

Remark 4. Another possible application for Theorem 2.4 is the Eagon-Northcott complex. This is used if we want to solve the equation $(\det p) \cdot q = \phi$, where ϕ is a scalar function. The complex is given by

$$\cdots \xrightarrow{\delta_p} E_3 \xrightarrow{\delta_p} E_2 \xrightarrow{\delta_p} E_1 \xrightarrow{\det p} \mathbb{C} \to 0$$

where

$$E_k = \Lambda^{k+r-1} E_1 \otimes S^{k-1} E_0^* \otimes \det E_0^*.$$

We have already solved this equation in Remark 1, but by applying Theorem 2.4 to the Eagon-Northcott complex we can obtain a sharper degree estimate. Note that in this division problem we actually have $r = 1$ and so we could solve it by means of the Kozsul complex, but we can improve the degree estimate by choosing a complex that takes advantage of the fact that our mapping $\det p$ is constructed from another mapping p.

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