THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Mixed moments in II_1-factor Trace-positivity of polynomials in the free group algebra

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Contents

Abstract

We investigate certain matrices composed of mixed, second–order moments of unitaries. The unitaries are taken from C*–algebras with moments taken with respect to traces, or, alternatively, from matrix algebras with the usual trace. These sets are of interest in light of a theorem of E. Kirchberg about Connes' embedding problem. We give also a modification of I. Klep and M. Schweighofer algebraic reformulation of Connes' embedding problem by considering $*$-algebra of the countably generated free group. This allows to consider only quadratic polynomials in unitary generators instead of arbitrary polynomials in self-adjoint generators (see Chapter 1 and 2).

In the third Chapter we study the question when for a given $*$-algebra $\mathcal{A}$ a sequence of cones $C_n \subseteq M_n(\mathcal{A})_{sa}$ can be realized as cones of positive operators in a faithful $*$-representation of $\mathcal{A}$ on a Hilbert space. We present a criterion analogous to Effros-Choi abstract characterization of operator systems. A characterization of operator algebras which are completely boundedly isomorphic to C^*-algebras is given.

Chapter 4 is devoted to the ideals of a C^*-algebra $C^*(\mathcal{B})$ generated by an operator algebra $\mathcal{B}$. A closed ideal $J \subseteq C^*(\mathcal{B})$ is called K-boundary ideal if the restriction of the quotient map on $\mathcal{B}$ has a completely bounded inverse with cb-norm equal to K^{-1}. For $K = 1$ one gets the notion of boundary ideals introduced by Arveson. We study the properties of K-boundary ideals and characterize them in the case when operator algebra λ-norms itself. Several reformulations of Kadison similarity problem are given. In particular, the affirmative answer to this problem is equivalent to the statement that every bounded homomorphism from $C^*(\mathcal{B})$ onto $\mathcal{B}$ which is a projection on $\mathcal{B}$ is completely bounded. In particular, the affirmative answer to this problem is equivalent to the statement that every bounded homomorphism from $C^*(\mathcal{B})$ onto $\mathcal{B}$ which is a projection on $\mathcal{B}$ is completely bounded. Moreover, we prove that Kadison's similarity problem is decided on one particular C^*-algebra which is a completion of the $*$-double of $M_2(\mathbb{C})$.

In the chapter 5 and 6 we introduce multidimensional Schur multipliers and characterise them generalising well known results by Grothendieck and Peller. We define a multidimensional version of the two dimensional operator multipliers studied recently by Kissin and Shulman. The multidimensional operator multipliers are defined as elements of the minimal tensor product of several C*-algebras satisfying certain boundedness conditions. In the case of commutative C*-algebras, the multidimensional operator multipliers reduce to continuous multidimensional Schur multipliers. We show that the multipliers with respect to some given representations of the corresponding C*-algebras do not change if the representations are replaced by approximately equivalent ones. We establish a non-commutative and multidimensional version of the characterisations by Grothendieck and Peller which shows that universal operator multipliers can be obtained as certain weak limits of elements of the algebraic tensor product of the corresponding C*-algebras. The notion of the symbol of an operator multiplier is introduced. We characterise completely compact operator multipliers in terms of their symbol as well as in terms of approximation by finite rank multipliers. We give sufficient conditions for the sets of compact and completely compact multipliers to coincide and characterise the cases where an operator multiplier in the minimal tensor product of two C*-algebras is automatically compact. We give a description of multilinear modular completely compact completely bounded maps defined on the direct product of finitely many copies of the C*-algebra of compact operators in terms of tensor products, generalising results of Saar [85].

Thesis includes the following papers:

Paper I

Matrices of unitary moments,
joint with Ken Dykema

Paper II

Algebraic reformulation of Connes' embedding problem,
joint with S. Popovych

Paper II

Matrix ordered operator algebras.
joint with S. Popovych

Paper III

Ideals of a C^*-algebra generated by an operator algebra.

Paper IV

Multidimensional operator multipliers.
joint with I.G.Todorov, L.Turowska.

Paper VI

Compact operator multipliers.
joint with R. Levene, I.G.Todorov, L.Turowska

Chapter 1

Matrices of unitary moments

1.1 Basic definitions and facts

One fundamental question about operator algebras is Connes' embedding problem, which in its original formulation asks whether every II_1–factor $\mathcal{M}$ embeds in the ultrapower R^ω of the hyperfinite II_1–factor R. This is well known to be equivalent to the question of whether all elements of II_1–factors possess matricial microstates, (which were introduced by Voiculescu [96] for free entropy), namely, whether such elements are approximable in $*$–moments by matrices. Connes' embedding problem is known to be equivalent to a number of different problems, in large part due to a remarkable paper [51] of Kirchberg. (See also the survey [60], and the papers [72], [81], [82], [16], [83], [20], [54], [84], [47] for results with bearing on Connes' embedding problem.)

In Proposition 4.6 of [51], Kirchberg proved that, in order to show that a finite von Neumann algebra $\mathcal{M}$ with faithful tracial state τ embeds in R^ω, it would be enough to show that for all n, all unitary elements $U_1, \ldots, U_n$ in $\mathcal{M}$ and all $\varepsilon > 0$, there is $k \in \mathbf{N}$ and there are $k \times k$ unitary matrices $V_1, \ldots, V_n$ such that $|\tau(U_i^* U_j) - \mathrm{tr}_k(V_i^* V_j)| < \varepsilon$ for all $i, j \in \{1, \ldots, n\}$, where tr_k is the normalized trace on $M_k(\mathbf{C})$. (He also required $|\tau(U_i) - \mathrm{tr}_k(V_i)| < \varepsilon$, but this formally stronger condition is easily satisfied by taking the $n+1$ unitaries $U_1, \ldots, U_n, U_{n+1} = I$ in $\mathcal{M}$ finding $k \times k$ unitaries $\widetilde{V}_1, \ldots, \widetilde{V}_{n+1}$, so that $|\tau(U_i^* U_j) - \mathrm{tr}_k(\widetilde{V}_i^* \widetilde{V}_j)| < \varepsilon$, and letting $V_i = \widetilde{V}_{n+1}^* \widetilde{V}_i$.) It is, therefore, of interest to consider the set of possible second–order mixed moments of unitaries in such $(\mathcal{M}, \tau)$ or, equivalently, of unitaries in C^*–algebras with respect to tracial states. (See also [81], where some similar sets were considered by F. Rădulescu.)

Let $\mathcal{G}_n$ be the set of all $n \times n$ matrices X of the form

$$X = \big(\tau(U_i^* U_j)\big)_{1 \le i,j \le n} \tag{1.1}$$

as $(U_1, \ldots, U_n)$ runs over all n–tuples of unitaries in all C*–algebras A possessing a faithful tracial state τ.

Remark 1. *The set–theoretic difficulties in the phrasing of Definition 1.1 can be evaded by insisting that A be represented on a given separable Hilbert space. Alternatively, let $\mathfrak{A} = \mathbf{C}\langle U_1, \ldots, U_n\rangle$ denote the universal, unital, complex $*$–algebra generated by unitary elements $U_1, \ldots, U_n$. A linear functional φ on $\mathfrak{A}$ is positive if $\varphi(a^*a) \geq 0$ for all $a \in \mathfrak{A}$. By the usual Gelfand–Naimark–Segal construction, any such positive functional φ gives rise to a Hilbert space $L^2(\mathfrak{A}, \varphi)$ and a $*$–representation $\pi_\varphi : \mathfrak{A} \to B(L^2(\mathfrak{A}, \varphi))$. Thus, the set $\mathcal{G}_n$ equals the set of all matrices X as in* (1.1) *as τ runs over all positive, tracial, unital, linear functionals τ on $\mathfrak{A}$.*

Let $\mathcal{F}_n$ be the closure of the set

$$\big\{\big(\operatorname*{tr}_k(V_i^*V_j)\big)_{1\leq i,j\leq n} \mid k \in \mathbf{N},\, V_1, \ldots, V_n \in \mathcal{U}_k\big\},$$

where $\mathcal{U}_k$ is the group of $k \times k$ unitary matrices.

A *correlation matrix* is a complex, positive semidefinite matrix having all diagonal entries equal to 1. Let Θ_n be the set of all $n \times n$ correlation matrices. Clearly, we have

$$\mathcal{F}_n \subseteq \mathcal{G}_n \subseteq \Theta_n\,.$$

Kirchberg's result is that Connes' embedding problem is equivalent to the problem of whether $\mathcal{F}_n = \mathcal{G}_n$ holds for all n.

For each n,

(i) $\mathcal{F}_n$ and $\mathcal{G}_n$ are invariant under conjugation with $n \times n$ diagonal unitary matrices and permutation matrices,

(ii) $\mathcal{F}_n$ and $\mathcal{G}_n$ are compact, convex subsets of Θ_n,

(iii) $\mathcal{F}_n$ and $\mathcal{G}_n$ are closed under taking Schur products of matrices.

Proof. Part (i) is clear. Note that Θ_n is a norm–bounded subset of $M_n(\mathbf{C})$. That $\mathcal{F}_n$ is closed is evident. That $\mathcal{G}_n$ is closed follows from the description in Remark 1 and the fact that a pointwise limit of positive traces on $\mathfrak{A}$ is a positive trace. This proves compactness. Convexity of $\mathcal{F}_n$ follows from by observing that if V is a $k \times k$ unitary and V' is a $k' \times k'$ unitary, then for arbitrary $\ell, \ell' \in \mathbf{N}$,

$$\underbrace{V \oplus \cdots \oplus V}_{\ell \text{ times}} \oplus \underbrace{V' \oplus \cdots \oplus V'}_{\ell' \text{ times}}$$

can be realized as a block–diagonal $(k\ell + k'\ell') \times (k\ell + k'\ell')$ matrix whose normalized trace is

$$\frac{k\ell}{k\ell + k'\ell'} \operatorname*{tr}_k(V) + \frac{k'\ell'}{k\ell + k'\ell'} \operatorname*{tr}_{k'}(V').$$

Convexity of $\mathcal{G}_n$ follows because a convex combination of positive traces on $\mathfrak{A}$ is a positive trace. This proves (ii).

Closedness of $\mathcal{F}_n$ under taking Schur products follows by observing that if V and V' are unitaries as above, then $V \otimes V'$ is a $kk' \times kk'$ unitary whose normalized trace is $\mathrm{tr}_k(V)\,\mathrm{tr}_{k'}(V')$. For $\mathcal{G}_n$, we observe that if U and respectively, U', are unitaries in C*–algebras A and A' having tracial states τ and τ', then the spatial tensor product C*–algebra $A \otimes A'$ has tracial state $\tau \otimes \tau'$ that takes value $\tau(U)\tau'(U')$ on the unitary $U \otimes U'$. This proves (iii). □

Since it is important to decide whether we have $\mathcal{F}_n = \mathcal{G}_n$ for all n, it is interesting to learn more about the sets $\mathcal{F}_n$. A first question is whether $\mathcal{F}_n = \Theta_n$ holds. In Section 1.2, we show that this holds for $n = 3$ but fails for $n \geq 4$. The proof relies on a characterization of extreme points of Θ_n, and it uses also the set $\mathcal{C}_n$ of matrices of moments of commuting unitaries. In Section 1.3 we prove $M_n(\mathbf{R}) \cap \Theta_n \subseteq \mathcal{F}_n$, and some further results concerning $\mathcal{C}_n$. In Section 1.4, we show that $\mathcal{F}_n$ has nonempty interior, as a subset of Θ_n.

1.2 Extreme points of Θ_n and some consequences

The set Θ_n of $n \times n$ correlation matrices is embedded in the affine space consisting of the self–adjoint complex matrices having all diagonal entries equal to 1; it is just the intersection of the set of positive, semidefinite matrices with this space. Every element of Θ_n is bounded in norm by n (*cf* Remark 5), and Θ_n is a compact, convex space. Since, in the space of self–adjoint matrices, every positive definite matrix is the center of a ball consisting of positive matrices, it is clear that the boundary of Θ_n (for $n \geq 2$) consists of singular matrices.

The extreme points of Θ_n and $\Theta_n \cap M_n(\mathbf{R})$ have been studied in [19], [57], [34] and [56]. In this section, we will use an easy characterization of the extreme points of Θ_n to draw some conclusions about matrices of unitary moments. The papers cited above contain the facts about extreme points of Θ_n found below, and have results going well beyond. However, for completeness and for use later in examples, we provide proofs, which are brief.

We also introduce the subset $\mathcal{C}_n$ of $\mathcal{F}_n$, consisting of matrices of moments of commuting unitaries.

This is a convenient place to recall the following standard fact. We include a proof for convenience.

Lemma 2. *The set of all $X \in \Theta_n$ of rank r is the set of all frame operators $X = F^*F$ of frames $F = (f_1, \dots, f_n)$, consisting of n unit vectors $f_j \in \mathbf{C}^r$, where $r = \mathrm{rank}(X)$. If, in addition, $X \in M_n(\mathbf{R})$, then the frame $f_1, \dots, f_n$ can be chosen in $\mathbf{R}^r$.*

Proof. Every frame operator F^*F as above clearly belongs to Θ_n and has rank r.

Recall that the support projection of a Hermitian matrix X is the projection onto the orthocomplement of the nullspace of X. Let P be the support projection of X and let $\lambda_1 \geq \cdots \geq \lambda_r > 0$ be the nonzero eigenvalues of X with corresponding orthonormal eigenvectors $g_1, \dots, g_r \in \mathbf{C}^n$. Let $V : \mathbf{C}^r \to P(\mathbf{C}^n)$ be the isometry defined by $e_i \mapsto g_i$, where $e_1, \dots, e_r$ are the standard basis vectors of $\mathbf{C}^r$. So $P = VV^*$. Then $X = F^*F$, where F is the $r \times n$ matrix

$$F = V^*X^{1/2} = \mathrm{diag}(\lambda_1, \dots, \lambda_r)^{1/2}V^*.$$

If $f_1, \dots, f_n \in \mathbf{C}^r$ are the columns of F, then $\|f_i\| = X_{ii} = 1$ and the linear span of $f_1 \dots, f_n$ is $\mathbf{C}^r$. Thus, $f_1, \dots, f_n$ comprise a frame.

If X is real, then the vectors $g_1, \dots, g_r$ can be chosen in $\mathbf{R}^n$. Then V and $X^{1/2}$ are real matrices and $f_1, \dots, f_n$ are in $\mathbf{R}^r$. □

Lemma 3. *Let $X \in M_n(\mathbf{C})$ be a positive semidefinite matrix and let P be the support projection of X. Then a Hermitian $n \times n$ matrix Y has the property that there is $\varepsilon > 0$ such that $X + tY$ is positive semidefinite for all $t \in (-\varepsilon, \varepsilon)$ if and only if $Y = PYP$.*

Proof. If $X = 0$ then this is trivially true, so suppose $X \neq 0$. After conjugating with a unitary, we may without loss of generality assume $P = \mathrm{diag}(1, \dots, 1, 0, \dots, 0)$ with $\mathrm{rank}(X) = \mathrm{rank}(P) = r$. Then PXP, thought of as an $r \times r$ matrix, is positive definite. By continuity of the determinant, we see that if $Y = PYP$, then Y enjoys the property described above.

Conversely, if $Y \neq PYP$, then we may choose two standard basis vectors e_i and e_j for $i \leq r < j$, such that the compressions of X and Y to the subspace spanned by e_i and e_j are given by the matrices

$$\widehat{X} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \qquad \widehat{Y} = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$$

for some $x > 0$, $a, c \in \mathbf{R}$ and $b \in \mathbf{C}$ with c and b not both zero. But

$$\det(\widehat{X} + t\widehat{Y}) = txc + t^2(ac - |b|^2).$$

If $c \neq 0$, then $\det(\widehat{X}+t\widehat{Y}) < 0$ for all nonzero t sufficiently small in magnitude and of the appropriate sign, while if $c = 0$ then $b \neq 0$ and $\det(\widehat{X} + t\widehat{Y}) < 0$ for all $t \neq 0$. □

Let $n \in \mathbf{N}$, let $X \in \Theta_n$ and let P be the support projection of X. A necessary and sufficient condition for X to be an extreme point of Θ_n is that there be no nonzero Hermitian $n \times n$ matrix Y having zero diagonal and satisfying $Y = PYP$. Consequently, if X is an extreme point of Θ_n, then $\mathrm{rank}(X) \leq \sqrt{n}$.

Proof. X is an extreme point of Θ_n if and only if there is no nonzero Hermitian $n \times n$ matrix Y such that $X + tY \in \Theta_n$ for all $t \in \mathbf{R}$ sufficiently small in magnitude. Now use Lemma 3 and the fact that Θ_n consists of the positive semidefinite matrices with all diagonal values equal to 1.

For the final statement, if $r = \mathrm{rank}(X)$ then the set of Hermitian matrices with support projection under P is a real vector space of dimension r^2, while the space of $n \times n$ Hermitian matrices with zero diagonal has dimension $n^2 - n$. If $r^2 > n$, then the intersection of these two spaces is nonzero. □

Let $X \in \Theta_n$. Suppose $f_1, \ldots, f_n$ is a frame consisting of n unit vectors in $\mathbf{C}^r$, where $r = \mathrm{rank}(X)$, so that $X = F^*F$ with $F = (f_1, \ldots, f_n)$ is the corresponding frame operator. (See Lemma 2.) Then X is an extreme point of Θ_n if and only if the only $r \times r$ self–adjoint matrix Z satisfying $\langle Zf_j, f_j \rangle = 0$ for all $j \in \{1, \ldots, n\}$ is the zero matrix.

Proof. Since F is an $r \times n$ matrix of rank r, the map $M_r(\mathbf{C})_{s.a.} \to M_n(\mathbf{C})_{s.a.}$ given by $Z \mapsto F^*ZF$ is an injective linear map onto $PM_n(\mathbf{C})_{s.a.}P$, where P is the support projection of X. If $Y = F^*ZF$, then $Y_{jj} = \langle Zf_j, f_j \rangle$. Thus, the condition for X to be extreme now follows from the characterization found in Proposition 1.2. □

Let $n \in \mathbf{N}$ and suppose $X \in \Theta_n$ satifies $\mathrm{rank}(X) = 1$. Then X is an extreme point of Θ_n and $X \in \mathcal{F}_n$. Moreover, using the notation introduced in Remark 1, we have

$$\begin{aligned}&\mathrm{conv}\{X \in \Theta_n \mid \mathrm{rank}(X) = 1\} = \\ &= \{\big(\tau(U_i^*U_j)\big)_{1\leq i,j\leq n} \mid \tau : \mathfrak{A} \to \mathbf{C} \text{ a positive trace, } \tau(1) = 1,\ \pi_\tau(\mathfrak{A}) \text{ commutative}\}\end{aligned} \tag{1.2}$$

and this set is closed in Θ_n.

We let $\mathcal{C}_n$ denote the set given in (1.2). Thus, we have $\mathcal{C}_n \subseteq \mathcal{F}_n$. Moreover, (*cf* Remark 1), $\mathcal{C}_n$ is the set of matrices as in (1.1) where $(U_1,\ldots,U_n)$ run over all n–tuples of commuting unitarires in C*–algebras A with faithful tracial state τ.

Proof of Proposition 1.2. By Lemma 2, we have $X = F^*F$ where $F = (f_1,\ldots,f_n)$ for complex numbers f_j with $|f_j| = 1$. Using Proposition 1.2, we see immediately that X is an extreme point of Θ_n. Thinking of each f_j as a 1×1 unitary, we have $X \in \mathcal{F}_n$ and, moreover, $X = \big(\tau(U_i^*U_j)\big)_{1\le i,j\le n}$, where $\tau : \mathfrak{A} \to \mathbf{C}$ is the character defined by $\tau(U_i) = f_i$; in fact, it is apparent that every character on $\mathfrak{A}$ yields a rank one element of Θ_n. Since the set of traces τ on $\mathfrak{A}$ having $\pi_\tau(\mathfrak{A})$ commutative is convex, this implies the inclusion $\subseteq$ in (1.2).

That the left–hand–side of (1.2) is compact follows from Caratheodory's theoem, because the rank one projections form a compact set. If $\tau : \mathfrak{A} \to \mathbf{C}$ is a positive trace with $\tau(1) = 1$ and $\pi_\tau(\mathfrak{A})$ commutative, then $\tau = \psi\circ\pi_\tau$ for a state ψ on the C*–algebra completion of $\pi_\tau(\mathfrak{A})$. Since every state on a unital, commutative C*–algebra is in the closed convex hull of the characters of that C*–algebra, τ is itself the limit in norm of a sequence of finite convex combinations of characters of $\mathfrak{A}$. Thus, $X = \big(\tau(U_i^*U_j)\big)_{1\le i,j\le n}$ is the limit of a sequence of finite convex combinations of rank one elements of Θ_n, and we have $\supseteq$ in (1.2). □

Remark 4. *We see immediately from* (1.2) *that* $\mathcal{C}_n$ *is a closed convex set that is closed under conjugation with diagonal unitary matrices and permutation matrices; also, since the set of rank one elements of* Θ_n *is closed under taking Schur products, so is the set* $\mathcal{C}_n$*. Furthermore, since* $\mathcal{C}_n$ *lies in a vector space of real dimension* $m := n^2 - n$*, by Caratheodory's theorem every element of* $\mathcal{C}_n$ *is a convex combination of not more than* $m+1$ *rank one elements of* Θ_n*.*

An immediate application of Propositions 1.2 and 1.2 is the following. The extreme points of Θ_3 are precisely the rank one elements of Θ_3. Moreover, we have

$$\mathcal{C}_3 = \mathcal{F}_3 = \mathcal{G}_3 = \Theta_3.$$

Remark 5. *Let* $X \in \mathcal{G}_n$ *and take* A*,* τ *and* $U_1,\ldots,U_n$ *as in Definition 1.1 so that* (1.1) *holds, and assume without loss of generality that* τ *is faithful on* A*. If we identify* $M_n(A)$ *with* $A\otimes M_n(\mathbf{C})$*, then we have* $X = n(\tau\otimes \mathrm{id}_{M_n(\mathbf{C})})(P)$, *where P is the projection*

$$P = \frac{1}{n} \begin{pmatrix} U_1^* \\ U_2^* \\ \vdots \\ U_n^* \end{pmatrix} (U_1 \; U_2 \; \cdots \; U_n)$$

in $M_n(A)$. If $c = (c_1, \ldots, c_n)^t \in \mathbf{C}^n$ is such that $Xc = 0$, then this yields $\tau(Z^*Z) = 0$, where $Z = c_1 U_1 + \cdots + c_n U_n$. Since τ is a faithful, we have $Z = 0$.

Let $n \in \mathbf{N}$. If $X \in \mathcal{G}_n$ and $\operatorname{rank}(X) \leq 2$, then $X \in \mathcal{C}_n$.

Proof. If $\operatorname{rank}(X) = 1$, then this follows from Propostion 1.2, so assume $\operatorname{rank}(X) = 2$. Let $\tau : \mathfrak{A} \to \mathbf{C}$ be a positive, unital trace such that $X = \big(\tau(U_i^* U_j)\big)_{1 \leq i,j \leq n}$ and let $\pi_\tau : \mathfrak{A} \to B(L^2(\mathfrak{A}, \tau))$ the the $*$–representation as described in Remark 1. Let $\sigma : \mathfrak{A} \to \pi_\tau(\mathfrak{A})$ be the $*$–representation defined by $\sigma(U_i) = \pi_\tau(U_1)^* \pi_\tau(U_i)$ for each $i \in \{1, \ldots, n\}$ and let $\tau' = \tau \circ \sigma$. Then τ' is a positive, unital trace on $\mathfrak{A}$ and the matrix $\big(\tau'(U_i^* U_j)\big)_{1 \leq i,j \leq n}$ is equal to X. Furthermore, $\pi_{\tau'}(U_1) = I$. Consequently, we may without loss of generality assume $\pi_\tau(U_1) = I$.

Let $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbf{C}^n$. Let $i, j \in \{2, \ldots, n\}$, with $i \neq j$. Since $\operatorname{rank}(X) = 2$, there are $c_1, c_i, c_j \in \mathbf{C}$ with $c_1 \neq 0$ such that $X(c_1 e_1 + c_i e_i + c_j e_j) = 0$. By Remark 5, we have $\pi_\tau(c_1 I + c_i U_i + c_j U_j) = 0$. We do not have $c_i = c_j = 0$, so assume $c_i \neq 0$. If $c_j = 0$, then $\pi_\tau(U_i)$ is a scalar multiple of the identity, while if $c_j \neq 0$, then $\pi_\tau(U_i)$ and $\pi_\tau(U_j)$ generate the same C*–algebra, which is commutative. In either case, we have that the $*$–algebras generated by $\pi_\tau(U_i)$ and $\pi_\tau(U_j)$ commute with each other. Therefore, $\pi_\tau(\mathfrak{A})$ is commutative, and $X \in \mathcal{C}_n$. □

$\mathcal{G}_4 \neq \Theta_4$.

Proof. Combining Proposition 1.2 and Proposition 1.2, we see that $\mathcal{G}_4$ has no extreme points of rank 2. It will suffice to find an extreme point X of Θ_4 with $\operatorname{rank}(X) = 2$. By Proposition 1.2, it will suffice to find four unit vectors $f_1, \ldots, f_4$ spanning $\mathbf{C}^2$ such that the only self–adjoint $Z \in M_2(\mathbf{C})$ satisfying $\langle Z f_i, f_i \rangle = 0$ for all $i = 1, \ldots, 4$ is the zero matrix. It is easily verified that the frame

$$f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad f_4 = \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

does the job, and, with $F = (f_1, f_2, f_3, f_4)$, this yields the matrix

$$X = F^*F = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & \frac{1+i}{2} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1-i}{2} & 1 \end{pmatrix} \in \Theta_4 \backslash \mathcal{G}_4 \,. \tag{1.3}$$

□

Remark 6. *We cannot have $\mathcal{C}_n = \mathcal{F}_n$ for all n, because by an easy a modification of Kirchberg's proof of Proposition 4.6 of [51], this would imply that $M_2(\mathbf{C})$ can be faithfully represented in a commutative von Neumann algebra. (This argument shows that for some n there must be two–by–two unitaries $V_1, \ldots, V_n$ such that the matrix $\big(\mathrm{tr}_2(V_i^* V_j)\big)_{1 \le i,j \le n}$ does not belong to $\mathcal{C}_n$.) In fact, in Proposition 1.3 we will show $\mathcal{F}_6 \neq \mathcal{C}_6$. However, we don't know whether $\mathcal{F}_n = \mathcal{C}_n$ holds or not for $n = 4$ or $n = 5$.*

1.3 Real matrices

The main result of this section is the following, which easily follows from the usual representation of the Clifford algebra.

For every $n \in \mathbf{N}$, we have

$$M_n(\mathbf{R}) \cap \Theta_n \subseteq \mathcal{F}_n \,.$$

We first recall the representation of the Clifford algebra. Let Λ be a linear map from a real Hilbert space H into the bounded, self–adjoint operators $B(\mathcal{K})_{s.a.}$, for some complex Hilbert space $\mathcal{K}$, satisfying

$$\Lambda(x)\Lambda(y) + \Lambda(y)\Lambda(x) = 2\langle x, y\rangle I_H, \qquad (x, y \in H). \tag{1.4}$$

The real algebra generated by range of Λ is uniquely determined by H and called the real Clifford algebra.

Consider a real Hilbert space H of finite dimension r with its canonical basis $\{e_i\}$. Let

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the real Clifford algebra of H has the following representation by $2^r \times 2^r$ matrixes

$$\Lambda(x) = \sum \lambda_i U^{\otimes i-1} \otimes V \otimes I_2^{\otimes(n-i)},$$

where $x = \sum \lambda_i e_i$. It easy to check that the relation (1.4) is satisfied. Moreover if $||x|| = 1$ then $\Lambda(x)$ is symmetry, i.e. $\Lambda(x)^* = \Lambda(x)$ and $\Lambda(x)^2 = I$.

Proof of Theorem 1.3. Let r be the rank of X. By Lemma 2, there are unit vectors $f_1, \ldots, f_n \in \mathbf{R}^r$ such that $X_{i,j} = \langle f_i, f_j \rangle$ for all i and j. Taking Λ as described above, we get $2^r \times 2^r$ unitary matrices $\Lambda(f_i)$ (in fact, they are symmetries), and from (1.4) we have $\mathrm{tr}(\Lambda(f_i)\Lambda(f_j)) = \langle f_i, f_j \rangle$. □

Below is the result for real matrices that is entirely analogous to Proposition 1.2.

Let $n \in \mathbf{N}$, let $X \in M_n(\mathbf{R}) \cap \Theta_n$ and let P be the support projection of X. A necessary and sufficient condition for X to be an extreme point of $M_n(\mathbf{R}) \cap \Theta_n$ is that there be no nonzero Hermitian real $n \times n$ matrix Y having zero diagonal and satisfying $Y = PYP$. Consequently, if X is an extreme point of $M_n(\mathbf{R}) \cap \Theta_n$ and $r = \mathrm{rank}(X)$, then $r(r+1)/2 \leq n$.

Proof. This is just like the proof of Proposition 1.2, the only difference being that the dimension of $PM_n(\mathbf{R})_{s.a.}P$ for a projection P of rank r is $r(r+1)/2$. □

If $n \leq 5$, then

$$M_n(\mathbf{R}) \cap \Theta_n \subseteq \mathcal{C}_n. \tag{1.5}$$

Proof. From Proposition 1.3, we see that every extreme point X of $M_n(\mathbf{R}) \cap \Theta_n$ for $n \leq 5$ has rank $r \leq 2$. But $X \in \mathcal{F}_n \subseteq \mathcal{G}_n$, by Theorem 1.3, so using Proposition 1.2, it follows that all extreme points of $M_n(\mathbf{R}) \cap \Theta_n$ lie in $\mathcal{C}_n$. Since $\mathcal{C}_n$ is closed and convex (see Proposition 1.2), the inclusion (1.5) follows. □

Of course, we also have the result for real matrices (and real frames) that is analogous to Proposition 1.2, which is stated below. The proof is the same. Let $X \in M_n(\mathbf{R}) \cap \Theta_n$. Suppose $f_1, \ldots, f_n$ is a frame consisting of n unit vectors in $\mathbf{R}^r$, where $r = \mathrm{rank}(X)$, so that $X = F^*F$ with $F = (f_1, \ldots, f_n)$ is the corresponding frame operator. (See Lemma 2.) Then X is an extreme point of $M_n(\mathbf{R}) \cap \Theta_n$ if and only if the only real Hermitian $r \times r$ matrix Z satisfying $\langle Zf_j, f_j \rangle = 0$ for all $j \in \{1, \ldots, n\}$ is the zero matrix.

Although Corollary 1.3 shows that every element of $M_n(\mathbf{R}) \cap \Theta_n$ for $n \leq 5$ is in the closed convex hull of the rank one operators in Θ_n, it is not true that every element of $M_n(\mathbf{R}) \cap \Theta_n$ lies in the closed convex hull of rank one operators in $M_n(\mathbf{R}) \cap \Theta_n$, even for $n = 3$, as the following example shows.

Example 7. *Consider the frame*

$$f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

of three unit vectors in $\mathbf{R}^2$. *It is easily verified that the only real Hermitian* 2×2 *matrix* Z *such that* $\langle Z f_i, f_i \rangle = 0$ *for all* $i = 1, 2, 3$ *is the zero matrix. Thus, by Proposition 1.3,*

$$X = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

is a rank–two extreme point of $M_3(\mathbf{R}) \cap \Theta_3$. *However, an explicit decomposition as a convex combination of rank one operators in* Θ_3 *is*

$$X = \frac{1}{2} \begin{pmatrix} 1 & i & \frac{1+i}{\sqrt{2}} \\ -i & 1 & \frac{1-i}{\sqrt{2}} \\ \frac{1-i}{\sqrt{2}} & \frac{1+i}{\sqrt{2}} & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & -i & \frac{1-i}{\sqrt{2}} \\ i & 1 & \frac{1+i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & \frac{1-i}{\sqrt{2}} & 1 \end{pmatrix}.$$

We have

$$M_6(\mathbf{R}) \cap \Theta_6 \not\subseteq \mathcal{C}_6 .$$

Thus, we have $\mathcal{F}_6 \neq \mathcal{C}_6$.

Proof. We construct an example of $X \in (M_6(\mathbf{R}) \cap \Theta_6) \backslash \mathcal{C}_6$. In fact, it will be a rank–three extreme point of $M_6(\mathbf{R}) \cap \Theta_6$.

Consider the frame

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad f_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$f_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad f_5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad f_6 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

of six unit vectors in $\mathbf{R}^3$. It is easily verified that the only real Hermitian 3×3 matrix Z such that $\langle Z f_i, f_i \rangle = 0$ for all $i \in \{1, \ldots, 6\}$ is the zero matrix.

Thus, by Proposition 1.3,

$$X = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 & \frac{1}{2} & \sqrt{\frac{2}{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{2} & 1 & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 1 \end{pmatrix}$$

is a rank–three extreme point of $M_6(\mathbf{R}) \cap \Theta_6$. The nullspace of X is spanned by the vectors

$$\begin{aligned} v_1 &= (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0, -1, 0, 0)^t \\ v_2 &= (0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0, -1, 0)^t \\ v_3 &= (\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, 0, 0, -1)^t. \end{aligned}$$

Suppose, to obtain a contradiction, that we have $X \in \mathcal{C}_6$. Then there is a commutative C*–algebra $A = C(\Omega)$ with a faithful tracial state τ and there are unitaries $I = U_1, U_2, \ldots, U_6 \in A$ such that $X = \big(\tau(U_i^* U_j)\big)_{1 \le i,j \le 6}$. Taking the vectors v_1, v_2 and v_3, above, by Remark 5 we have

$$U_4 = \frac{1}{\sqrt{2}}(U_1 + U_2) \tag{1.6}$$

$$U_5 = \frac{1}{\sqrt{2}}(U_2 + U_3) \tag{1.7}$$

$$U_6 = \frac{1}{\sqrt{3}}(U_1 + U_2 + U_3). \tag{1.8}$$

Fixing any $\omega \in \Omega$, we have that $\zeta_j := U_j(\omega)$ is a point on the unit circle $\mathbf{T}$, $(1 \le j \le 6)$. From (1.6) and $|\zeta_4| = 1$, we get $\zeta_1 = \pm i\zeta_2$ and similarly from (1.7) we get $\zeta_3 = \pm i\zeta_2$. However, from (1.8), we then have

$$\zeta_6 \in \{\frac{1-2i}{\sqrt{3}}\zeta_2, \frac{1}{\sqrt{3}}\zeta_2, \frac{1+2i}{\sqrt{3}}\zeta_2\},$$

which contradicts $|\zeta_6| = |\zeta_2| = 1$. □

1.4 Nonempty interior of $\mathcal{F}_n$

In this section, we show that the interior of $\mathcal{F}_n$ and, in fact, of $\mathcal{C}_n$, is nonempty, when considered as a subset of Θ_n. (Since $\mathcal{C}_n = \Theta_n$ for $n = 1, 2, 3$, this needs proving only for $n \geq 4$.)

Given $X \in \Theta_n$, let

$$\begin{aligned} a_X &= \sup\{t \in [0,1] \mid tX + (1-t)I \in \mathcal{F}_n\} \\ c_X &= \sup\{t \in [0,1] \mid tX + (1-t)I \in \mathcal{C}_n\}. \end{aligned}$$

Of course, $c_X \leq a_X$. We now show that c_X is bounded below by a nonzero constant that depends only on n. In particular, we have that the identity element lies in the interior of $\mathcal{C}_n$, when this is taken as a subset of the affine space of self–adjoint matrices having all diagonal entries equal to 1.

Let $n \in \mathbf{N}$, $n \geq 3$, and let $X \in \Theta_n$. Then

$$c_X \geq \frac{6}{n^2 - n}\,. \tag{1.9}$$

Moreover, if λ_0 is the smallest eigenvalue of X, then

$$c_X \geq \min\left(\frac{6}{(n^2-n)(1-\lambda_0)}, 1\right). \tag{1.10}$$

Proof. We have $X = (x_{ij})_{i,j=1}^n$ with $x_{ii} = 1$ for all $i = 1, \ldots, n$. Denote $G = \{\sigma \in S_n \mid \sigma(1) < \sigma(2) < \sigma(3)\}$. Then

$$\#G = \binom{n}{3}(n-3)!$$

Let $U_\sigma = (u_{ij})$ be the permutation unitary matrix where $u_{ij} = \delta_{i,\sigma(i)}$. Then $U^*XU = (x_{\sigma^{-1}(i)\sigma^{-1}(j)})_{i,j}$. Define the block-diagonal matrix

$$B_\sigma = \begin{pmatrix} 1 & x_{\sigma(1)\sigma(2)} & x_{\sigma(1)\sigma(3)} \\ x_{\sigma(2)\sigma(1)} & 1 & x_{\sigma(2)\sigma(3)} \\ x_{\sigma(3)\sigma(1)} & x_{\sigma(3)\sigma(2)} & 1 \end{pmatrix} \oplus I_{n-3}\,.$$

Using Corollary 1.2 (and Remark 4), we easily see $B_\sigma \in \mathcal{C}_n$.

Let $J_\sigma = \{(\sigma(1), \sigma(2)), (\sigma(1), \sigma(3)), (\sigma(2), \sigma(3))\}$. Put $X_\sigma = U^* B_\sigma U$. Then

$$(X_\sigma)_{k\ell} = \begin{cases} 0, & \text{if } (k,\ell) \notin \{(1,1), \ldots, (n,n)\} \cup J_\sigma, \\ 1, & \text{if } k = \ell \\ x_{k\ell}, & \text{if } (k,\ell) \in J_\sigma\,. \end{cases}$$

Since for any $k < \ell$ we have

$$\#\{\sigma \in G \mid \sigma(1) = k, \sigma(2) = \ell \text{ or } \sigma(1) = k, \sigma(3) = \ell \text{ or } \sigma(2) = k, \sigma(3) = \ell\} = \\ ((n-\ell) + (\ell - k - 1) + (k-1))(n-3)! = (n-2)!$$

it follows that matrix

$$X' = \frac{1}{\#G} \sum_{\sigma \in G} X_\sigma$$

has entries $x'_{ii} = 1$, and $x'_{k\ell} = \frac{6}{n^2-n} x_{k\ell}$ if $k \neq \ell$.

Since $\mathcal{C}_n$ is closed under conjugating with permutation matrices, we have $X_\sigma \in \mathcal{C}_n$ for all $\sigma \in G$. But then the average X' also belongs to $\mathcal{C}_n$. This implies (1.9).

Now (1.10) is an easy consequence of (1.9). Indeed, if $\lambda_0 = 1$, then X is the identity matrix and $c_X = 1$. If $\lambda_0 < 1$, then let $Y = \frac{1}{1-\lambda_0}(X - \lambda_0 I)$. We have $Y \in \Theta_n$, and

$$(1-t)I + tY = (1 - \frac{t}{1-\lambda_0})I + \frac{t}{1-\lambda_0} X.$$

This implies $c_X \geq \min(1, \frac{c_Y}{1-\lambda_0})$. □

Given an $n \times n$ matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, let $\overline{A}$ denote matrix whose (i,j) entry is the the complex conjugate of a_{ij}. If A is self–adjoint, then so is $\overline{A}$, and these two matrices have the same eigenvalues (and multiplicities). Consequently, $A - \overline{A}$ has spectrum that is symmetric about zero.

Lemma 8. *Let $X \in \Theta_n$ and let $d > 0$ be such that*

$$I + d\left(\frac{X - \overline{X}}{2}\right) \in \mathcal{F}_n \,.$$

Then $a_X \geq d/(d+1)$. If $n \leq 5$ and

$$I + d\left(\frac{X - \overline{X}}{2}\right) \in \mathcal{C}_n \,, \tag{1.11}$$

then $c_X \geq d/(d+1)$.

Proof. The matrix $(X + \overline{X})/2$ is real and lies in Θ_n. Using Theorem 1.3, we have $(X + \overline{X})/2 \in \mathcal{F}_n$. Thus, we have

$$\frac{1}{d+1} I + \frac{d}{d+1} X = \frac{1}{d+1}\left(I + d\left(\frac{X - \overline{X}}{2}\right)\right) + \frac{d}{d+1}\left(\frac{X + \overline{X}}{2}\right) \in \mathcal{F}_n \,.$$

If $n \leq 5$ and (1.11) holds, then we similarly apply Corollary 1.3. □

Example 9. *Consider the matrix* X *as in* (1.3)*, from Corollary 1.2. From Proposition 1.4 and closedness of* $\mathcal{F}_n$*, we know* $\frac{1}{2} \le c_X \le a_X < 1$*. It would be interesting to know the precise value of* a_X*, in order to have a concrete example of an element on the boundary of* $\mathcal{F}_4$ *in* Θ_4*.*

Since

$$\frac{X - \overline{X}}{2} = \begin{pmatrix} 0 & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{2} & 0 \end{pmatrix}$$

has norm $\sqrt{3}/2$ *and since it is conjugate by a permutation matrix to an element of* $M_3(\mathbf{C}) \oplus \mathbf{C}$*, using Corollary 1.2 we have that* (1.11) *holds with* $d = 2/\sqrt{3}$*. A slightly better value is obtained by letting* Y *be the result of conjugation of* X *with the diagonal unitary* $\mathrm{diag}(1, 1, 1, e^{-i\pi/4})$*. Then*

$$\frac{Y - \overline{Y}}{2} = \begin{pmatrix} 0 & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 & 0 \\ -\frac{i}{2} & \frac{i}{2} & 0 & 0 \end{pmatrix}$$

which has norm $1/\sqrt{2}$ *and similarly yields* $d = \sqrt{2}$*. Applying Lemma 8 gives* $c_X = c_Y \ge \sqrt{2}/(1 + \sqrt{2}) \approx 0.586$.

Chapter 2

Trace-positivity of quadratic polynomials

Let $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ be a free ultrafilter on $\mathbb{N}$ and R be the hyperfinite II_1-factor with faithful tracial normal state τ. Then the subset I_ω in $l^\infty(\mathbb{N}, R)$ consisting of $(x_1, x_2, \ldots)$ with $\lim_{n\to\omega} \tau(x_n^* x_n) = 0$ is a closed ideal in $l^\infty(\mathbb{N}, R)$ and a quotient algebra $R^\omega = l^\infty(\mathbb{N}, R)/I_\omega$ is a von Neumann II_1-factor called *ultrapower* of R. It is naturally endowed with a faithful tracial normal state

$$\tau_\omega((x_n) + I_\omega) = \lim_{n\to\omega} \tau(x_n).$$

A. Connes' embedding problem asks whether every finite von Neumann algebra with fixed normal faithful tracial state can be embedded into R_ω in a trace-preserving way.

It is well know that Connes' embedding problem is equivalent to the problem whether every finite set $x_1, \ldots, x_n$ of self-adjoint contractions in arbitrary II_1-factor (M, τ) has *matricial microstates*, i.e whether for any $\varepsilon > 0$ and $t \geq 1$ there is $k \in \mathbb{N}$ and self-adjoint contractive $k \times k$-matrices $A_1, \ldots, A_n$ such that $|\operatorname{tr}(w(x_1, \ldots, x_n)) - \tau(w(A_1, \ldots, A_n))| < \varepsilon$ for all words w of length at most t.

In [39] D. Hadwin proved that solving Connes' embedding problem in affirmative is equivalent to proving that there is no polynomial $p(x_1, ..., x_n)$ in non-commutative variables such that

1. $tr_k(p(A_1, \ldots, A_n)) \geq 0$ for every k and self-adjoint contractions $A_1, ..., A_n \in M_k$.

2. $\tau(p(T_1, \ldots, T_n)) < 0$, where $T_1, \ldots, T_n$ are self-adjoint contractive elements in a finite factor with trace τ.

Recently I. Klep and M. Schweighofer established that Connes' embedding problem has the following equivalent algebraic reformulation.

Let $f(X_1, \ldots, X_m)$ be a self-adjoint element in a free associative algebra $\mathbb{K}\langle\overline{X}\rangle$ with countable family of self-adjoint generators $\overline{X} = \{X_1, X_2, \ldots\}$, where $\mathbb{K} = \mathbb{R}$ or $K = \mathbb{C}$. If $\mathrm{tr}(f(A_1, \ldots, A_m)) \geq 0$ for any n and family of self-adjoint contractive matrices $A_1, \ldots, A_m \in M_n(\mathbb{K})$ then f has the property that for every $\varepsilon > 0$ we have $\varepsilon e + f = g + c$ where c is a sum of commutators in $\mathbb{K}\langle\overline{X}\rangle$, g belongs to quadratic module generated by $1 - X_i^2$ and e is the unit in $\mathbb{K}\langle\overline{X}\rangle$. Recall that a *quadratic module* is the smallest subset of $\mathbb{K}\langle\overline{X}\rangle$ containing unit, closed under addition and conjugation $x \to g^*xg$ by arbitrary $g \in \mathbb{K}\langle\overline{X}\rangle$.

In the present paper we consider the group $*$-algebra $\mathcal{F}$ of the countably generated free group $\mathbb{F}_\infty = \langle u_1, u_2, \ldots\rangle$ instead of $\mathbb{K}\langle\overline{X}\rangle$. One reason is that we can use a more standard and well known set of hermitian squares $\{g^*g | g \in \mathcal{F}\}$ instead of quadratic module M and the second that we can bound the degree of polynomials f in the above reformulation by 2. This modification provides the following.

Connes' embedding conjecture is true iff for any self-adjoint $f \in \mathcal{F}$ of the form $f(u_1, \ldots, u_n) = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$ condition

$$Tr(f(V_1, \ldots, V_n)) \geq 0 \tag{2.1}$$

for every $m \geq 1$ and every n-tuple of unitary matrices $V_1, \ldots, V_n \in U(m)$ implies that for every $\varepsilon > 0$, $\varepsilon e + f = g + c$ where c is a sum of commutators and g is a sum of Hermitian squares. We will call f satisfying (2.1) a *trace-positive quadratic polynomial*. Elements of the form $g + c$ with c being a sum of commutators are called *cyclically equivalent* to g (see Section 2.1).

In Section 2.2 we study a subset of covariance matrices of the form $[\mathrm{tr}(U_i^* U_j)]_{ij}$ where $U_1, \ldots, U_n$ runs over n-tuple of unitary matrices and $\mathrm{tr}(U)$ denotes normalized trace of U. Using Clifford algebra methods we show that this set contains all covariance matrices with real coefficients. This implies that all trace-positive quadratic polynomials f with real coefficients do satisfy the property from the above theorem.

The description of the set $\{[\mathrm{tr}(U_i^* U_j)]_{ij} \mid U_1, \ldots, U_n \in U_m(\mathbb{C}), m \geq 1\}$ seems to be unknown even for $n = 3$. In this case it is equivalent to the problem of description of the set of triples $(\mathrm{tr}(U), \mathrm{tr}(V), \mathrm{tr}(UV))$ where U and V are unitary matrices. Note that the lists of possible eigenvalues of U, V and UV can be described by generalization of Horn's inequalities (see [31]) but little is know about possible traces $(\mathrm{tr}(U), \mathrm{tr}(V), \mathrm{tr}(UV))$. The only known connection between these traces seems to be the inequality $\sqrt{1 - |\mathrm{tr}(UV)|^2} \leq \sqrt{1 - |\mathrm{tr}(U)|^2} + \sqrt{1 - |\mathrm{tr}(V)|^2}$ established in [97].

2.1 An algebraic reformulation of Connes' problem.

Let $\mathcal{F}$ be the $*$-algebra of the countably generated free group $\mathbb{F}_\infty$. Let K denote the $\mathbb{R}$-subspace in $\mathcal{F}_{sa}$ generated by the commutators $fg - gf$ $(f, g \in F)$. We will say that f and g in $\mathcal{F}$ are *cyclically equivalent* (denote $f \overset{cyc}{\sim} g$) if $f - g \in K$. Let $\Sigma^2(\mathcal{F})$ denote the set of positive elements of the $*$-algebra $\mathcal{F}$, i.e. elements of the form $\sum_{j=1}^m f_j^* f_j$ with $f_j \in \mathcal{F}$. An element of the form f^*f is called Hermitian square and therefore the cone $\Sigma^2(\mathcal{F})$ is called the cone of Hermitian squares.

Definition 10. *Let C be a subset of the vector space V. An element $v \in C$ is called an algebraic interior point of C if for every $u \in V$ there is $\varepsilon > 0$ in $\mathbb{R}$ s.t. $v + \lambda u \in C$ for all $0 \leq \lambda \leq \varepsilon$.*

Definition 11. *Let A be a unital $*$-algebra with the unit e. Then*
1. An element $a \in \mathcal{A}_{sa}$ is called bounded if there is $\alpha \in \mathbb{R}_+$ such that $\alpha e \pm a \in \Sigma^2(\mathcal{A})$.
2. An element $x = a + ib$ with $a,\ b \in \mathcal{A}_{sa}$ is bounded if the elements a and b are such.
3. The algebra $\mathcal{A}$ is bounded if all its elements are bounded.

It is well known that the set of all bounded elements in $\mathcal{A}$ is a $*$-subalgebra in $\mathcal{A}$ and that an element $x \in A$ is bounded if and only if xx^* is such (see for example [77]). In particular $\mathcal{F}$ is a bounded $*$-algebra. Obviously this implies that the unit of the algebra is an algebraic interior point of $\Sigma^2(\mathcal{F})$.

The following lemma is a modification of Theorem 3.12 in [**?**].

Lemma 12. *Let $f \in \mathcal{F}$ be self-adjoint. If for any II_1 factor M with faithful normal tracial state τ and separable predual and every n-tuple of unitary elements $U_1, \ldots, U_n$ in the unitary group $\mathcal{U}(M)$ of M we have that*

$$\tau(f(U_1, \ldots, U_n)) \geq 0$$

then for every $\varepsilon > 0$, $\varepsilon e + f \sim g$ for some $g \in \Sigma^2(\mathcal{F})$.

Proof. Clearly $\Sigma^2(\mathcal{F}) + K$ is a convex cone in $\mathbb{R}$-space $\mathcal{F}_{sa}$. Since e is an an algebraic internal point of $\Sigma^2(\mathcal{F})$ it is also an algebraic internal point of $\Sigma^2(\mathcal{F}) + K$.

Assume that there is $\varepsilon > 0$ such that $\varepsilon e + f \not\sim g$ for any $g \in \Sigma^2(\mathcal{F})$, i.e. $\varepsilon e + f \notin \Sigma^2(\mathcal{F}) + K$. By Eidelheit-Kakutani separation theorem there is $\mathbb{R}$-linear unital functional $L_0 : \mathcal{F}_{sa} \to \mathbb{R}$ s.t. $L_0(\Sigma^2(\mathcal{F}) + K) \subseteq \mathbb{R}_{\geq 0}$ and

$L_0(\varepsilon e + f) \in \mathbb{R}_{\leq 0}$. Since $-K \subset \Sigma^2(\mathcal{F}) + K$ we have that $L_0(K) = 0$. In particular extending L_0 to $\mathbb{C}$-linear functional on $\mathcal{F}$ we get a tracial functional L. Since L maps $\Sigma^2(\mathcal{F})$ into the non-negative reals it defines a pre-Hilbert space structure on $\mathcal{F}$ by means of sesquilinear for $\langle p, q\rangle = L(q^*p)$, $p, q \in \mathcal{F}$. Let $N = \{p : \langle p, p\rangle = 0\}$. By Cauchy-Schwarz inequality $N = \{p : L(q^*p) = 0 \text{ for all } q \in \mathcal{F}\}$ and hence is a left ideal. Let H_0 be the pre-Hilbert space $\mathcal{F}/N$. Consider the left regular representation $\pi : \mathcal{F} \to L(H_0)$. Since π is a $*$-homomorphism for every $f \in \mathcal{F}$ operator $\pi(f)$ is bounded as a linear combination of unitary operators. Thus $\pi(f)$ can be extended to the bounded operator acting on the Hilbert space H which is the completion of H_0. Thus we have a representation $\pi : \mathcal{F} \to B(H)$ with a cyclic vector $\xi = e + N$ and such that $L(p) = \langle \pi(p)\xi, \xi\rangle$. In particular L is a contractive tracial state on $\mathcal{F}$ and thus defines a tracial state of the universal enveloping C^*-algebra $C^*(\mathcal{F})$. By Banach-Alaoglu and Krein-Milman theorem we can assume that L is an extreme point in the set of all tracial states and thus $\pi(\mathcal{F})$ generates a factor von Neumann algebra M (see [39]). Clearly M is a finite factor. If it is type I then it should be $\mathbb{C}$ (since ξ is a trace vector) and thus can be embedded into any II_1-factor in trace preserving way. Thus we can assume that M is a type II_1-factor. But then condition $L(f) < 0$ is impossible. $\square$

Corollary 13. *If self-adjoint $f \in \mathcal{F}$ has real coefficients and for any real type II_1 von Neumann algebra (M, τ) with normal faithful tracial state τ and every n-tuple of unitary elements $U_1, \ldots, U_n$ in M we have that*

$$\tau(f(U_1, \ldots, U_n)) \geq 0$$

then the same holds for the complex II_1 von Neumann algebras.

Proof. Element f can be written as $f = \alpha + \sum_{w_j} \alpha_{w_j}(w_j + w_j^*)$ with $\alpha_{w_j} \in \mathbb{R}$ and for complex trace τ and $U_1, \ldots, U_n \in U(M)$ we will have $\tau(f) = \alpha + 2\sum_{w_j} \alpha_{w_j} \operatorname{Re}\tau(w_j)$, i.e. $\tau(f) = (\operatorname{Re}\tau)(f)$. To finish the proof note that M can be regarded as a real finite von Neumann algebra with faithful trace $\operatorname{Re}\tau$. $\square$

Lemma 14. *If $f \in \mathbb{R}[\mathbb{F}_\infty]$, $f = f^*$ and for every real type II_1 von Neumann algebra (M, τ) we have that $\tau(f) \geq 0$ then for every $\varepsilon > 0$, $\varepsilon + f \overset{cyc}{\sim} g$ for some $g \in \left\{\sum_{j=1}^m g_j^* g_j \mid m \in \mathbb{N}, g_j \in \mathbb{R}\langle \mathbb{F}_\infty\rangle\right\}$.*

Proof. The proof of this statement can be obtained by obvious modification of the proof of lemma 12. The only nontrivial part is that the unit e is an algebraic internal point but this is equivalent to $\mathbb{R}\langle \mathbb{F}_\infty\rangle$ being bounded $*$-algebra. The proof of the last fact can be found in [94]. $\square$

This lemma gives another proof of corollary 13. In sequel we will need the following lemma.

Lemma 15. *If (M,τ) is a II_1 factor which can be embedded into R^ω and $f \in \mathcal{F}$ is self-adjoint then the condition $\mathrm{tr}(f(V_1,\ldots,V_n)) \geq 0$ for all $m \geq 0$ and all unitary $V_1,\ldots,V_n$ in $M_{m\times m}(\mathbb{C})$ implies that $\tau(f(U_1,\ldots,U_n)) \geq 0$ for all unitary $U_1,\ldots,U_n$ in M.*

Proof. Considering M as a subalgebra in R^ω and τ as a restriction of the trace on R^ω we can find a representing sequences $\left\{u_j^{(k)}\right\}_{j=1}^\infty$ for U_k, $k = 1,\ldots,n$ in $l^\infty(\mathbb{N},R)$ which are unitary elements in von Neumann algebra $l^\infty(\mathbb{N},R)$. This can be done since every unitary in von Neumann algebra R^ω can be lifted to a unitary in von Neumann algebra $l^\infty(\mathbb{N},R)$ with respect to canonical morphism $\pi : l^\infty(\mathbb{N},R) \to R^\omega$. Taking j sufficiently large we can approximate mixed moments of $U_1,\ldots,U_k$ up to order m, i.e. $\tau(U_{s_1}\ldots U_{s_t})$ with $t \leq m$ and $s_1,\ldots,s_t \in \{1,\ldots,n\}$, by the mixed moments of unitary matrices $u_1^{(k)},\ldots,u_n^{(k)}$. □

The following theorem is Proposition 4.6 in [?]

Theorem 16. (E. Kirchberg) *Let $(M,\ \tau)$ be von Neumann algebra with separable predual and faithful normal tracial state τ. If for all $n \geq 1$ and for all unitaries $u_1,\ldots,u_n$ in M and for arbitrary $\varepsilon > 0$ there exists $m \geq 1$ and unitary $m\times m$ matrices $V_1,\ldots,V_n \in U(m)$ s.t. for all i,j:*

$$|\tau(u_i^*u_j) - \frac{1}{m}\,\mathrm{Tr}(V_i^*V_j)| \ < \ \varepsilon, \tag{2.2}$$

$$|\tau(u_j) - \frac{1}{m}\,\mathrm{Tr}(V_j)| \ < \ \varepsilon \tag{2.3}$$

then M can be embedded into R^ω.

Remark 17. *We may drop condition (2.3) since we may take $u_0 = 1, u_1,\ldots,u_n$ and by (2.2) find matrices $W_0,\ldots,W_n$ such that $|\tau(u_i^*u_j) - \frac{1}{m}\mathrm{Tr}(W_i^*W_j)| < \varepsilon$ for all i and j. Thus (2.2) and (2.3) will be satisfied if we take $V_j = W_0^*W_j$.*

The proof of the following theorem is an adaptation of the proof of Proposition 3.17 from [?].

Theorem 18. *Let (M,τ) be II_1-factor with separable predual. If for every self-adjoint element $f \in \mathcal{F}$ of the form $f = \alpha + \sum_{i\neq j}\alpha_{ij}u_i^*u_j$ the condition*

$$\mathrm{Tr}(f(V_1,\ldots,V_n)) \geq 0$$

for all $m \geq 1$ and every n-tuple of unitary matrices $V_1,\ldots,V_n \in U(m)$ implies that $\tau(f(U_1,\ldots,U_n)) \geq 0$ for all unitaries $U_1,\ldots,U_n$ in M then M can be embedded into R^ω.

Proof. Take $n \geq 1$. Consider the finite dimensional vector space $W = \{\alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j | \alpha_{ij} \in \mathbb{C}\}$. Denote by C the convex hull of the set F of the functionals $T \in W^*$ of the form $T(p) = \frac{1}{m} \operatorname{Tr}(p(V_1, \ldots, V_n))$ where $m \geq 1$ and $V_1, \ldots, V_n \in U(m)$. Take arbitrary n-tuple of unitary elements $U_1, \ldots, U_n$ in M and put $L(p) = \tau(p(U_1, \ldots, U_n))$ for $p \in W$. Assume that $L \notin C$. By Hahn-Banach theorem there is $f \in W^{**} = W$ and $c \in \mathbb{R}$ s.t. $Re(L(f)) < c < Re(T(f))$ for all $T \in C$. Since $e \in W$ we can substitute $f - c$ instead of f and thus assume that $c = 0$. Since $T(f^*) = \overline{T(f)}$ for every $T \in C$ and $L(f^*) = \overline{L(f)}$ we have that $L(f + f^*) = 2Re(L(f)) < 0 < 2Re(T(f)) = T(f + f^*)$ which is a contradiction. Thus $L \in C$. Let T be a rational convex combination of elements $T_1, \ldots, T_s$ from F and T_k corresponds to n-tuples $V_{j,k}$. Then $T = \frac{1}{q}(p_1 T_1 + \ldots + p_s T_s)$ for some positive integers $p_1, \ldots, p_s, q$. Taking block-diagonal $V_j = (V_{j,1}^{\otimes p_1} \oplus \ldots \oplus V_{j,s}^{\otimes p_s})$ we see that $T \in F$. Thus each element of C, in particular element L can be approximated by elements of F. By the Kirchberg's Theorem we have that M can be embedded into R^ω.

□

Theorem 19. *Connes' embedding conjecture problem has affirmative solution iff for any self-adjoint $f \in \mathcal{F}$ of the form $f = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$ condition*

$$\operatorname{Tr}(f(V_1, \ldots, V_n)) \geq 0$$

for every $m \geq 1$ and every n-tuple of unitary matrices $V_1, \ldots, V_n \in U(m)$ implies that for every $\varepsilon > 0$, $\varepsilon e + f \sim g$ with $g \in \Sigma^2(\mathcal{F})$.

Proof. If Connes' embedding problem has affirmative solution and quadratic $f \in \mathcal{F}_{sa}$ is such that $\operatorname{Tr}(f(V_1, \ldots, V_n)) \geq 0$ for every $m \geq 1$ and every n-tuple of unitary matrices $V_1, \ldots, V_n \in U(m)$ then by lemma 15 we have $\tau(f(U_1, \ldots, U_n)) \geq 0$ for any unitary $U_1, \ldots, U_n$ in M. Hence by lemma 12, $\varepsilon e + f$ is cyclically equivalent to a sum of Hermitian squares. This proves that the conditions of the theorem are necessary.

If $\varepsilon e + f$ is cyclically equivalent to an element in $\Sigma^2(\mathcal{F})$ for every $\varepsilon > 0$ then clearly $\tau(f(U_1, \ldots, U_n)) \geq 0$ for any unitary $U_1, \ldots, U_n$ in M. Hence the sufficiency of the theorem conditions follows from Theorem 18.

□

2.2 The trace-positive quadratic polynomials.

The results of the preceding section motivate the study of trace-positive self-adjoint quadratic polynomials $f = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$ in unitary generators

$u_1, \dots, u_n$, i.e. polynomials having the property that $\mathrm{Tr}(f(V_1, \dots, V_n)) \geq 0$ for every $m \geq 1$ and every n-tuple of unitary matrices $V_1, \dots, V_n \in U(m)$. If A denotes the matrix

$$\begin{pmatrix} \alpha/n & \alpha_{12} & \dots & \alpha_{1n} \\ \overline{\alpha_{12}} & \alpha/n & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \overline{\alpha_{1n}} & \overline{\alpha_{2n}} & \dots & \alpha/n \end{pmatrix}$$

then $\mathrm{Tr}\, f(U_1, \dots, U_n) \geq 0$ can be expressed as positivity of the sum of all entries of the Schur product $A \circ X$ where $X = [\mathrm{tr}(U_i^* U_j)]_{ij}$.

Thus the trace-positive polynomials f can be characterized as those for which the sum of all entries of $A \circ X$ for all $X \in K_n := \{[\mathrm{tr}(U_i^* U_j)]_{ij} \mid m \geq 1, U_1, \dots, U_n \in U(m)\}$. Thus our primary objective is to describe the sets $K_n \subseteq M_n(\mathbb{C})$. Note that in the case A is positive semidefinite we have $f \in \Sigma^2(\mathcal{F})$. Indeed in this case A is a sum of rank one positive semidefinite matrices $A = \sum_s (\beta_{s,1}, \dots, \beta_{sn})^T (\beta_{s,1}, \dots, \beta_{sn})$ and hence $f = \sum_s (\sum_j \beta_{s,j} u_j)^* (\sum_j \beta_{s,j} u_j)$. We will also be interested in real analog of the sets K_n, i.e. the sets $K_n(\mathbb{R}) = K_n \cap M_n(\mathbb{R})$. Note that the sets of the traces of monomials of unitary operators and their asymptotic properties in the context of Connes' embedding problem also studied in [**?**] and [**?**].

A self-adjoint matrix A such that $f = (u_1^{-1}, \dots, u_n^{-1}) A (u_1, \dots, u_n)^T$ is defined uniquely except for the diagonal entries. This motivates the following definition. We will call A and B *diagonally equivalent* and write $A \overset{d}{\sim} B$ if $A - B$ is a diagonal matrix with vanishing trace.

Definition 20. *Let $S \subseteq M_n(\mathbb{C})$ and $A \in M_n(\mathbb{C})$ be self-adjoint. We say that A is S-positive and denote $A \geq_S 0$ if there is self-adjoint B such that $A \overset{d}{\sim} B$ and*

$$\sum_{ij} b_{ij} s_{ij} \geq 0$$

for all $s \in S$.

The three natural choices for S will be

$$F_n = \{(t_{ij}) | t_{jj} = 1 \text{ and } |t_{ij}| \leq 1 \text{ for all } i, j\},$$

$P_n \subset F_n$ consisting of positive matrices and the set $K_n \subset F_n$. Clearly, an self-adjoint matrix $A = [a_{ij}]$ is K_n-positive iff $f = \sum_i a_{ii} e + \sum_{i \neq j} a_{ij} u_i^* u_j$ is a trace positive quadratic polynomial. Note that if

$$A \geq_{F_n} 0$$

then

$$\operatorname{Tr} A \geq \sum_{i \neq j} |a_{ij}|$$

and hence $A \overset{d}{\sim} B$ for some diagonally dominant matrix B. In this case polynomial $f = (u_1^{-1}, \dots, u_n^{-1})A(u_1, \dots, u_n)^T$ is a sum of hermitian squares. However if $A \geq_{P_n} 0$ then A need not be diagonally equivalent to positive matrix. Note that for the three choices of S mentioned above one can use equality instead of diagonal equivalence since diagonal entries of elements in S equal to 1.

The following lemma gives a description of cyclically equivalent quadratic polynomials.

Lemma 21. *For every matrix A the element $(u_1^{-1}, \dots, u_n^{-1})A(u_1, \dots, u_n)^T$ is cyclically equivalent to*

$$\sum_k g_k^{-1}(u_1^{-1}, \dots, u_n^{-1})A_{g_k}(u_1, \dots, u_n)^T g_k \tag{2.4}$$

for any finite collection $g_1, \dots, g_k \in \mathbb{F}_\infty$ and any matrices A_g such that

$$\sum_k A_{g_k} \overset{d}{\sim} A. \tag{2.5}$$

Any element $g \in \mathcal{F}$ such that $g \overset{cyc}{\sim} f$ is of the form (2.4) *for some matrices satisfying* (2.5)*. Moreover for self-adjoint g matrices A_g can also be chosen to be self-adjoint.*

Proof. The lemma follows from the following easy observation. For any w_1 and w_2 in $\mathbb{F}_\infty$ the element $w_1 - w_2$ is a commutator $ab - ba$ for some a, $b \in \mathbb{F}_\infty$ if and only if w_1 and w_2 are conjugated. Hence K consists of finitely supported sums of the form

$$\sum_j \sum_k \alpha_{jk} g_k^{-1} w_j g_k$$

where w_j, g_k belong to $\mathbb{F}_\infty$ and $\sum_k \alpha_{jk} = 0$ for all j. □

2.3 The Clifford Algebras and positive polynomials with real coefficients.

For a real Hilbert space V there is a unique associative algebra $\mathcal{C}(V)$ with a linear embedding $J : V \to \mathcal{C}(V)$ with generating range and such that for all

$x, y \in V$

$$J(x)J(y) + J(y)J(x) = 2\langle x, y\rangle . \tag{2.6}$$

The algebra $\mathcal{C}(V)$ is called Clifford algebras associated to V. Clifford algebra can be realized on a Hilbert space such that for every $x \in V$ with $\|x\| = 1$ operator $J(x)$ is symmetry, i.e. $J(x)^* = J(x)$ and $J(x)^2 = I$. To see this consider Pauli matrices

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly U and Q are self-adjoint unitary matrices and $U^2 = I, Q^2 = I$, $QU + UQ = 0$. Then matrices $Q_j = U \otimes \ldots \otimes U \otimes Q \otimes I \otimes I \ldots$ are symmetries and $\{Q_i, Q_j\} = 2\delta_{ij}I$. Hence operator $J(x) = \sum_j x_j Q_j$ is also a symmetry for unit real vector x. For further properties of Clifford algebras we refer to the books [15] and [70].

Theorem 22. *For every real covariance matrix $P \in M_n(\mathbb{R})$ there is n-tuple of symmetries $S_1, \ldots, S_n$ in finite dimensional real Hilbert space s.t. $P = [\mathrm{tr}(S_i^* S_j)]_{ij}$.*

Proof. Every covariance $n \times n$-matrix P is a Gram matrix for a system of unit vectors $x_1, \ldots, x_n$, i.e. $P = [\langle x_i, x_j\rangle]_{ij}$. Taking Clifford symmetries $S_j = J(x_j)$ as in the paragraph preceding the theorem we see that $P = [\mathrm{tr}(S_i^* S_j)]_{ij}$. □

Proposition 23. *For every $n \geq 1$ the closure $T_n(\mathbb{R})$ of the set of matrices*

$$\{[\tau(U_i^* U_j)]_{ij} | U_1, \ldots, U_n \in \mathcal{U}(M)\}$$

does not depend on real type II_1 von Neumann algebra (M, τ).

If self-adjoint $f(u_1, \ldots, u_n) \in \mathcal{F}$ has real coefficients and possess property that for every n-tuple of unitary matrices $U_1, \ldots, U_n \in U(m)$ we have $tr(f(U_1, \ldots, U_n)) \geq 0$ then for every $\varepsilon > 0$, $\varepsilon e + f \overset{cyc}{\sim} g$ for some $g \in \left\{\sum_{j=1}^m g_j^* g_j | m \in \mathbb{N}, g_j \in \mathbb{R}[\mathbb{F}_\infty]\right\}$.

Proof. Since every II_1 factor contains matrix algebras of arbitrary size we see that $T_n(\mathbb{R})$ coincides with the set of covariance matrices. The last statement follows from Lemma 14. □

Corollary 24. *If quadratic $f \in \mathcal{F}$, $f(u_1, \ldots, u_n) = \alpha + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$ is such that*

$$\mathrm{Tr}(f(U_1, \ldots, U_n)) = 0$$

for all unitary matrices $U_1, \ldots, U_n$ then $f = 0$.

Proof. For every $k \neq j$ and $t \in [0,1]$ matrices $P_1 = I + (E_{kj} + E_{jk})t$ and $P_2 = I + (iE_{kj} - iE_{jk})t$ are covariance matrices. Hence by the theorem $\alpha + (\alpha_{kj} + \alpha_{jk})t = 0$ and $\alpha + (\alpha_{kj} - \alpha_{jk})it = 0$. From which follows that $\alpha = \alpha_{kj} = 0$ and hence $f = 0$. □

Chapter 3

Matrix ordered operator algebras.

3.1 Cones of positive elements in a $*$-algebra.

3.1.1 Basic definitions and facts.

An operator system S is a not necessarily closed subspace in $B(\mathcal{H})$ containing the identity operator $I_{\mathcal{H}}$, such that $x^* \in S$ for all $x \in S$.

In [17] Choi and Effros obtained an abstract characterization of operator systems among $*$-vector spaces. More precisely, a $*$-vector space V is a vector space over $\mathbb{C}$ with a given conjugate-linear map $x \to x^*$ such that $(x^*)^* = x$. A $*$-vector space is called *matrix ordered* if it possesses a sequence of cones C_n with the following properties:

1. For every $n \geq 1$ we have $C_n \subseteq M_n(S)_{sa}$.
2. $C_n \cap (-C_n) = \{0\}$.
3. For all $m, n \geq 1$ and every $A \in M_{n\times m}(\mathbb{C})$ we have $A^* C_n A \subseteq C_m$.

Here $M_n(S)_{sa}$ denotes the set of self-adjoint matrices $x^* = x$.

Two matrix ordered $*$-vector spaces S and S' are called *complete order isomorphic* if there exists linear isomorphism $\varphi\colon S \to S'$ such that $\varphi^{(n)}(C_n) = C'_n$. Here $\varphi^{(n)}((a_{ij})) = (\varphi(a_{ij}))$ for every matrix $(a_{ij}) \in M_n(S)$.

An element $e \in S_{sa}$ is called a *matrix order unit* provided that for every $n \in \mathbb{N}$ and every $x \in M_n(S)_{sa}$ there exists $r > 0$ such that $re_n + x \in C_n$, where $e_n = e \otimes I_n$. A matrix order unit is called *Archimedean matrix order unit* if for all $n \in \mathbb{N}$ the inclusion $re_n + x \in C_n$ for all $r > 0$ implies that $x \in C_n$.

Theorem 25. (Choi-Effros'77) *If S is a matrix ordered $*$-vector space with an Archimedean matrix order unit e. Then there exist a Hilbert space $\mathcal{H}$, an operator system $S_1 \subseteq B(\mathcal{H})$ and a complete order isomorphism $\varphi : S \to S_1$ such that $\varphi(e) = I_{\mathcal{H}}$.*

We refer the reader to Section 2 for the definition of Archimedean matrix order unit.

A $*$-algebra $\mathcal{A}$ is matrix ordered if it is a matrix ordered $*$-vector space and for all n and m and all $A \in M_{n\times m}(\mathcal{A})$, we have that $A^*C_nA \subseteq C_m$. The main result of the paper is the following analog of the above theorem valid for matrix ordered $*$-algebras.

Theorem 26. *If $\mathcal{A}$ is a matrix ordered $*$-algebra with unit e which is Archimedean matrix order unit then there exist Hilbert space $\mathcal{H}$ and a unital $*$-subalgebra $\mathcal{A}_1 \subseteq B(\mathcal{H})$ such that $\mathcal{A}$ and $\mathcal{A}_1$ are complete order $*$-isomorphic.*

Here complete order $*$-isomorphism is a complete order isomorphism between $\mathcal{A}$ and $\mathcal{A}_1$ considered as matrix ordered $*$-vector spaces which is also a unital $*$-homomorphism. The $*$-algebra $\mathcal{A}_1$ is endowed with the matrix order consisting of the cones $M_n(A)_{sa} \cap B(\mathcal{H})^+$ of positive operators. The proof of Theorem 26 will be given in Section 3.

In other words Theorem 26 gives a characterization of the collections of cones $C_n \subseteq M_n(\mathcal{A})$ for which there exist a faithful $*$-representation π of $\mathcal{A}$ on a Hilbert space H such that C_n coincides with the cone of positive operators contained in $\pi^{(n)}(M_n(\mathcal{A}))$. Here $\pi^{(n)}((x_{i,j})) = (\pi(x_{i,j}))$ for every matrix $(x_{i,j}) \in M_n(\mathcal{A})$. Note that we do not assume that $\mathcal{A}$ has any faithful $*$-representation. This follows from the requirements imposed on the cones.

Recall that subspaces of $B(\mathcal{H})$ can be abstractly characterized as L^∞-matrix normed spaces (see [80]). Namely, a space V is called *L^∞-matrix normed space* if we are given norms $\|\cdot\|_{m,n}$ on $M_{m,n}(V)$ such that for all $A \in M_{p,m}(\mathbb{C})$, $X, Y \in M_{m,n}(V)$, $B \in M_{n,q}(\mathbb{C})$ we have

$$\|AXB\| \leq \|A\|\|X\|\|B\| \tag{3.1}$$

and

$$\|X \oplus Y\| = \max\{\|X\|, \|Y\|\} \tag{3.2}$$

It follows from the famous Blecher-Ruan-Sinclair theorem (see [11] and [12]) that in order to obtain an abstract characterization of subalgebras of $B(\mathcal{H})$ we need to allow matrices A and B in (3.1) to have coefficients in algebra V. The motivation of the present paper was to find similar modification of the axioms of matrix ordered $*$-vector space which works for $*$-algebras.

One of the proofs (see [62]) of Ruan's theorem uses reduction to the selfadjoint case and then Effros-Choi theorem. It looks attractive to deduce Blecher-Ruan-Sinclair theorem from Theorem 26.

The key ingredient of the proof of Theorem 26 is the case of one cone $C \subset \mathcal{A}_{sa}$ considered in Section 2. The cones C with property that $a^*Ca \subseteq C$ for all $a \in \mathcal{A}$ were introduced by R. Powers for the study of representations in unbounded operators in [79]. In Theorem 30 we prove that such cones C with the property that unit of the algebra is an Archimedean matrix order unit can be represented as a cones of positive operators. In Section 3 we prove the main result Theorem 26.

Based on the above characterization of $*$-subalgebras in $B(\mathcal{H})$ we study the question when an operator algebra is similar to a C^*-algebra.

Let $\mathcal{B}$ be a unital (closed) operator algebra in $B(\mathcal{H})$. The algebra $M_n(B(\mathcal{H}))$ of $n \times n$ matrices with entries in $B(\mathcal{H})$ has a norm $\|\cdot\|_n$ via the identification of $M_n(B(\mathcal{H}))$ with $B(\mathcal{H}^n)$, where $\mathcal{H}^n$ is the direct sum of n copies of a Hilbert space $\mathcal{H}$. Algebra $M_n(\mathcal{B})$ inherits a norm $\|\cdot\|_n$ via natural inclusion into $M_n(B(\mathcal{H}))$. The norms $\|\cdot\|_n$ are called matrix norms on the operator algebra $\mathcal{B}$. If $\varphi\colon \mathcal{B} \to \mathcal{B}_1$ is a linear bounded map between two operator algebras then $\varphi^{(n)}$ maps $M_n(\mathcal{B})$ into $M_n(\mathcal{B}_1)$ and $\|\varphi\|_{cb} = \sup_n \|\varphi^{(n)}\|$ is called *completely bounded norm* of φ. Map φ is called *completely bounded* if $\|\varphi\|_{cb} < \infty$. Map φ is called *completely isometric* if $\varphi^{(n)}$ is such for all n.

In [55] C. Le Merdy presented necessary and sufficient conditions for $\mathcal{B}$ to be self-adjoint. These conditions involve all completely isometric representations of $\mathcal{B}$ on Hilbert spaces. Our characterization is different in the following respect. If S is a bounded invertible operator in $B(\mathcal{H})$ and $\mathcal{A}$ is a C^*-algebra in $B(\mathcal{H})$ then the operator algebra $S^{-1}\mathcal{A}S$ is not necessarily self-adjoint but only isomorphic to a C^*-algebra via completely bounded isomorphism with completely bounded inverse. By Haagerup's theorem every completely bounded isomorphism π from a C^*-algebra $\mathcal{A}$ to an operator algebra $\mathcal{B}$ has the form $\pi(a) = S^{-1}\rho(a)S$, $a \in \mathcal{A}$, for some $*$-isomorphism $\rho : \mathcal{A} \to B(\mathcal{H})$ and invertible $S \in B(\mathcal{H})$. Thus the question whether an operator algebra $\mathcal{B}$ is completely boundedly isomorphic to a C^*-algebra via isomorphism which has completely bounded inverse, is equivalent to the question if there is bounded invertible operator S such that $S\mathcal{B}S^{-1}$ is a C^*-algebra.

We will present a criterion for an operator algebra $\mathcal{B}$ to be completely boundedly isomorphic to a C^*-algebra in terms of the existence of a collection of cones $C_n \in M_n(\mathcal{B})$ satisfying certain axioms (see def. 33). The axioms are derived from the properties of the cones of positive elements of a C^*-algebra preserved under completely bounded isomorphisms.

3.1.2 Faithful $*$-representation of ordered $*$-algebras.

Let $\mathcal{A}_{sa}$ denote the set of self-adjoint elements in $\mathcal{A}$. A subset $C \subset \mathcal{A}_{sa}$ containing unit e of $\mathcal{A}$ is *algebraically admissible* cone (see [79]) provided that

(i) C is a cone in $\mathcal{A}_{sa}$, i.e. $\lambda x + \beta y \in C$ for all x, $y \in C$ and $\lambda \geq 0$, $\beta \geq 0$, $\lambda, \beta \in \mathbb{R}$;

(ii) $C \cap (-C) = \{0\}$;

(iii) $xCx^* \subseteq C$ for every $x \in \mathcal{A}$;

With a cone C we can associate a partial order $\geq_C$ on the real vector space A_{sa} given by the rule $a \geq_C b$ if $a - b \in C$. It is clear that $(A_{sa}, \leq_C)$ is preordered real vector space. Henceforth we will suppress subscript C if it will not lead to ambiguity. The following lemma is straightforward.

Lemma 27. *For every $x \in A$, $x^*x \in C$. In particular $a^2 \in C$ for $a \in A_{sa}$. If for $a, b \in A_{sa}$, $a \geq b$ then for every $x \in A$, $x^*ax \geq x^*bx$.*

The following lemma is a direct consequence of the above.

Lemma 28. *Let A be a $*$-algebra with algebraically admissible cone C and unit e which is an order unit. The function $\|\cdot\|$ defined as*

$$\|a\| = \inf\{r > 0 : re \pm a \in C\}$$

is a seminorm on the $\mathbb{R}$-space A_{sa}. Moreover $\|x^*ax\| \leq \|x^*x\|\|a\|$ for every $x \in A$ and $a \in A_{sa}$.

Lemma 29. *Let A be a $*$-algebra with algebraically admissible cone C and with unit e which is an Archimedean order unit. For $x \in A$ define $|x| = \sqrt{\|x^*x\|}$. Then*

1. *$|\lambda x| = (\lambda\overline{\lambda})^{1/2}|x|$ for every $\lambda \in \mathbb{C}$ and $x \in A$;*

2. *$|xy| \leq |x||y|$ for every x, y in A;*

3. *$\|a\| \leq |a|$ for every $a \in A_{sa}$.*

Proof. The first statement is trivial. For x, y in A, by Lemma 28, we have $\|(xy)^*xy\| = \|y^*(x^*x)y\| \leq \|y^*y\|\|x^*x\|$. Hence $|xy| \leq |x||y|$. By Lemma 27, $(\|a\|\pm a)^2 \in C$. Thus $-(\|a\|^2 + a^2) \leq 2\|a\|a \leq \|a\|^2 + a^2$. If $a^2 \leq \varepsilon$ then $-(\|a\|^2 + \varepsilon) \leq 2\|a\|a \leq \|a\|^2 + \varepsilon$. Consequently, $\|2 \cdot \|a\| \cdot a\| \leq \|a\|^2 + \varepsilon$. Thus, $\|a\|^2 \leq \varepsilon$. Letting $\varepsilon \searrow \|a^2\|$ we obtain that $\|a\|^2 \leq \|a^2\|$. Therefore, $\|a\| \leq |a|$. $\square$

Theorem 30. *Let $\mathcal{A}$ be a $*$-algebra with unit e and $C \subseteq \mathcal{A}_{sa}$ be a cone containing e. If $xCx^* \subseteq C$ for every $x \in \mathcal{A}$ and e is an Archimedean order unit then there is a unital $*$-representation $\pi : \mathcal{A} \to B(H)$ such that $\pi(C) = \pi(\mathcal{A}_{sa}) \cap B(H)^+$. Moreover*

1. $\|\pi(x)\| = \inf\{r > 0 : r^2 \pm x^*x \in C\}$.
2. $\ker \pi = \{x : x^*x \in C \cap (-C)\}$.
3. *If* $C \cap (-C) = \{0\}$ *then* $\ker \pi = \{0\}$,

$$\|\pi(a)\| = \inf\{r > 0 : r \pm a \in C\} \text{ for all } a \in \mathcal{A}_{sa}$$

and $\pi(C) = \pi(\mathcal{A}) \cap B(H)^+$

Proof. By Lemmas 28 we have that $\|\cdot\| : \mathcal{A}_{sa} \to \mathbb{R}_+$ is a seminorm on $\mathbb{R}$-space $\mathcal{A}_{sa}$. Let us prove that $|x| = \sqrt{\|x^*x\|}$ for $x \in \mathcal{A}$ defines a pre-C^*-norm on $\mathcal{A}$.

First we will prove that $|x^*| = |x|$ for every $x \in A$. For this it suffices to show that $|x^*| \le |x|$. In fact, if this is true then $|x| = |(x^*)^*| \le |x^*|$. By definition $|x^*|^2 = \|xx^*\|$. Since xx^* is self-adjoint, $\|xx^*\| \le |xx^*|$ by Lemma 29. Thus $|x^*|^2 \le |xx^*| \le |x||x^*|$. If $|x^*| = 0$ then $0 \le |x|$ and the required inequality holds, otherwise we have $|x^*| \le |x|$.

For every $x \in A$ by Lemma 29 we have $|x^*x| \le |x||x^*| = |x|^2$ and $|x|^2 = \|x^*x\| \le |x^*x|$. Thus $|x|^2 = |x^*x|$.

Applying the previous equality to a self-adjoint element a we obtain $|a|^2 = |a^*a| = |a^2|$. Thus $|a^2| = |a|^2$.

We will prove that $|x+y| \le |x|+|y|$. For every $x \in A$ one has $\|x^2+x^{*2}\| \le 2\|x^*x\|$. Indeed, since $x + x^*$ is self-adjoint we have $(x + x^*)^2 \ge 0$, i.e

$$x^2 + x^{*2} + xx^* + x^*x \ge 0.$$

From this it follows that $x^2+x^{*2} \ge -\{x, x^*\}$ where $\{x, x^*\} = xx^*+x^*x$. Since $i(x-x^*)$ is also self-adjoint we have $-(x-x^*)^2 \ge 0$. Thus $\{x, x^*\} \ge x^2+x^{*2}$ and therefore $-\{x, x^*\} \le x^2 + x^{*2} \le \{x, x^*\}$. Hence

$$\begin{aligned}\|x^2 + x^{*2}\| &\le \|\{x, x^*\}\| = \|xx^* + x^*x\| \\ &\le \|xx^*\| + \|x^*x\| = |x|^2 + |x^*|^2 \\ &= 2|x|^2 = 2\|xx^*\|.\end{aligned}$$

We will prove the following.

$$\|x^* + x\| \le 2\|x^*x\|^{1/2} = 2|x|. \tag{3.3}$$

Indeed, for self-adjoint a by Lemma 29, $\|a\|^2 \le \|a^2\|$ hence

$$\begin{aligned}\|x+x^*\|^2 &\le \|x^2+x^{*2}+xx^*+x^*x\| \\ &\le \|x^2+x^{*2}\|+\|xx^*+x^*x\| \\ &\le 2\|x^*x\|+\|x^*x\|+\|xx^*\| \\ &= 4\|x^*x\|.\end{aligned}$$

Thus $\|x^*+x\| \le 2|x|$. We will prove that $\|x^*y+y^*x\| \le 2|x||y|$. Indeed, the substitution x^*y instead of x in (3.3) implies $\|x^*y+y^*x\| \le 2|x^*y| \le 2|x||y|$. The inequality $|x+y| \le |x|+|y|$ follows from the following estimates:

$$\begin{aligned}|x+y|^2 &\le \|x^*x\|+\|y^*y\|+\|x^*y+y^*x\| \\ &\le |x|^2+|y|^2+2|x||y| \\ &= (|x|+|y|)^2.\end{aligned}$$

Thus $|\cdot|$ is pre-C^*-norm.

If N denote the null-space of $|\cdot|$ then the completion $\mathcal{B} = \overline{\mathcal{A}/N}$ with respect to this norm is a C^*-algebra and canonical epimorphism $\pi : \mathcal{A} \to \mathcal{A}/N$ is a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$. We can assume without loss of generality that $\mathcal{B}$ is a concrete C^*-algebra in $B(H)$ for some Hilbert space H. Thus $\pi : \mathcal{A} \to B(H)$ can be regarded as a unital $*$-representation. Clearly,

$$\|\pi(x)\| = |x| \text{ for all } x \in \mathcal{A}.$$

This implies (1).

To show (2) take $x \in \ker\pi$ then $\|\pi(x)\| = 0$ and $re \pm x^*x \in C$ for all $r > 0$. Since e is an Archimedean unit we have $x^*x \in C \cap (-C)$. Conversely if $x^*x \in C \cap (-C)$ then $re \pm x^*x \in C$, for all $r > 0$, hence $\|\pi(x)\| = 0$ and (2) holds.

Let us prove that $\pi(C) = \pi(\mathcal{A}_{sa}) \cap B(H)^+$. Let $x \in \mathcal{A}_{sa}$ and $\pi(x) \ge 0$. Then there exists a constant $\lambda > 0$ such that $\|\lambda I_H - \pi(x)\| \le \lambda$, hence $|\lambda e - x| \le \lambda$. Since $\|a\| \le |a|$ for all self-adjoint $a \in \mathcal{A}$, see Lemma 29, we have $\|\lambda e - x\| \le \lambda$. Thus given $\varepsilon > 0$ we have $(\lambda+\varepsilon)e \pm (\lambda e - x) \in C$. Hence $\varepsilon e + x \in C$. Since e is Archimedean $x \in C$.

Conversely, let $x \in C$. To show that $\pi(x) \ge 0$ it is sufficient to find $\lambda > 0$ such that $\|\lambda I_H - \pi(x)\| \le \lambda$. Since $\|\lambda I_H - \pi(x)\| = |\lambda e - x|$ we will prove that $|\lambda e - x| \le \lambda$ for some $\lambda > 0$. From the definition of norm $|\cdot|$ we have the following equivalences:

$$|\lambda e - x| \le \lambda \;\Leftrightarrow\; (\lambda+\varepsilon)^2 e - (\lambda e - x)^2 \in C \text{ for all } \varepsilon > 0 \tag{3.4}$$

$$\Leftrightarrow\; \varepsilon_1 e + x(2\lambda e - x) \ge 0, \text{ for all } \varepsilon_1 > 0. \tag{3.5}$$

By condition (iii) in the definition of algebraically admissible cone we have that $xyx \in C$ and $yxy \in C$ for every $x, y \in C$. If $xy = yx$ then $xy(x + y) \in C$. Since e is an order unit we can choose $r > 0$ such that $re - x \in C$. Put $y = re - x$ to obtain $rx(re - x) \in C$. Hence (3.5) is satisfied with $\lambda = \frac{r}{2}$. Thus $\|\lambda e - \pi(x)\| \leq \lambda$ and $\pi(x) \geq 0$, which proves $\pi(C) = \pi(\mathcal{A}_{sa}) \cap B(H)^+$.

In particular, for $a = a^*$ we have

$$\|\pi(a)\| = \inf\{r > 0 : rI_H \pm \pi(a) \in \pi(C)\}. \tag{3.6}$$

We now in a position to prove claim (3). Suppose that $C \cap (-C) = 0$. Then $\ker \pi$ is a $*$-ideal and $\ker \pi \neq 0$ implies that there exists a self-adjoint $0 \neq a \in \ker \pi$, i.e. $|a| = 0$. Inequality $\|a\| \leq |a|$ implies $re \pm a \in C$ for all $r > 0$. Since e is Archimedean $\pm a \in C$, i.e. $a \in C \cap (-C)$ and, consequently, $a = 0$.

Since $\ker \pi = 0$ the inclusion $rI_H \pm \pi(a) \in \pi(C)$ is equivalent to $re \pm a \in C$, and by (3.6), $\|\pi(a)\| = \inf\{r > 0 : re \pm a \in C\}$. Moreover if $\pi(a) = \pi(a)^*$ then $a = a^*$. Thus we have $\pi(C) = \pi(\mathcal{A}) \cap B(H)^+$. □

Remark 31. *Note that J. Kelly and R. Vaught in 1953 proved that*

$$\sup\|\pi(x)\| = \inf\left\{t \in \mathbb{R}_+ | t^2 - x^*x \in \mathcal{A}_+\right\} \tag{$*$}$$

where $\mathcal{A}_+ = \left\{\sum_{j=1}^n a_j^* a_j, n \in \mathbb{N}, a_j \in \mathcal{A}\right\}$, *$\pi$ runs over all $*$-representations for Banach $*$-algebras $\mathcal{A}$ with isometric involution (see [50]). This is a particular case of claim (1) of Theorem 30 for a special choice of algebraically admissible cone $C = \mathcal{A}_+$. The proof of formula $(*)$ based on the Hahn-Banach theorem for any T^*-algebra (every $x \in \mathcal{A}_{sa}$ is bounded) presented in monograph [61].*

3.2 Operator realizations of matrix-ordered $*$-algebras.

The aim of this section is to give necessary and sufficient conditions on a sequences of cones $C_n \subseteq M_n(\mathcal{A})_{sa}$ for a unital $*$-algebra $\mathcal{A}$ such that C_n coincides with the cone $M_n(\mathcal{A}) \cap M_n(B(H))^+$ for some realization of $\mathcal{A}$ as a $*$-subalgebra of $B(H)$, where $M_n(B(H))^+$ denotes the set of positive operators acting on $H^n = H \oplus \ldots \oplus H$.

We say that a $*$-algebra $\mathcal{A}$ with unit e is a *matrix ordered* if the following conditions hold:

(a) for each $n \geq 1$ we are given a cone C_n in $M_n(\mathcal{A})_{sa}$ and $e \in C_1$,

(b) $C_n \cap (-C_n) = \{0\}$ for all n,

(c) for all n and m and all $A \in M_{n\times m}(\mathcal{A})$, we have that $A^*C_nA \subseteq C_m$,

An element $e \in \mathcal{A}_{sa}$ is called a *matrix order unit* provided that for every $n \in \mathbb{N}$ and every $x \in M_n(\mathcal{A})_{sa}$ there exists $r > 0$ such that $re_n + x \in C_n$, where $e_n = e \otimes I_n$. A matrix order unit is called *Archimedean matrix order unit* provided that for all $n \in \mathbb{N}$ the inclusion $re_n + x \in C_n$ for all $r > 0$ implies that $x \in C_n$.

Let $\pi : \mathcal{A} \to B(H)$ be a $*$-representation. Define $\pi^{(n)} : M_n(\mathcal{A}) \to M_n(B(H))$ by $\pi^{(n)}((a_{ij})) = (\pi(a_{ij}))$.

Theorem 32. *If $\mathcal{A}$ is a matrix-ordered $*$-algebra with a unit e which is Archimedean matrix order unit then there exists a Hilbert space H and a faithful unital $*$-representation $\tau : \mathcal{A} \to B(H)$, such that $\tau^{(n)}(C_n) = M_n(\tau(\mathcal{A}))^+$ for all n. Conversely, every unital $*$-subalgebra $\mathcal{D}$ of $B(H)$ is matrix-ordered by cones $M_n(\mathcal{D})^+ = M_n(\mathcal{D}) \cap B(H)^+$ and the unit of this algebra is an Archimedean order unit.*

Proof. Consider an inductive system of $*$-algebras and unital injective $*$-homomorphisms $\varphi_n : M_{2^n}(\mathcal{A}) \to M_{2^{n+1}}(\mathcal{A})$:

$$\varphi_n(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \text{ for all } n \geq 0, a \in M_{2^n}(\mathcal{A}).$$

Let $\mathcal{B} = \varinjlim M_{2^n}(\mathcal{A})$ be the inductive limit of this system. By (c) in the definition of the matrix ordered algebra we have $\varphi_n(C_{2^n}) \subseteq C_{2^{n+1}}$. We will identify $M_{2^n}(\mathcal{A})$ with a subalgebra of $\mathcal{B}$ via canonical inclusions. Let $C = \bigcup_{n\geq 1} C_{2^n} \subseteq \mathcal{B}_{sa}$ and let e_∞ be the unit of $\mathcal{B}$.

Let us prove that C is an algebraically admissible cone. Clearly, C satisfies conditions (i) and (ii) of definition of algebraically admissible cone. To prove (iii) suppose that $x \in \mathcal{B}$ and $a \in C$, then for some n we have $a \in C_{2^n}$ and $x \in M_{2^n}(\mathcal{A})$. Therefore, by (c), $x^*ax \in C$. Thus (iii) is proved. Since e is an Archimedean matrix order unit we obviously have that e_∞ is also an Archimedean order unit. Thus $*$-algebra $\mathcal{B}$ satisfies the assumptions of Theorem 30 and therefore there is a faithful $*$-representation $\pi : \mathcal{B} \to B(H)$ such that $\pi(C) = \pi(\mathcal{B}) \cap B(H)^+$.

Let $\xi_n : M_{2^n}(\mathcal{A}) \to \mathcal{B}$ be canonical injections ($n \geq 0$). Then $\tau = \pi \circ \xi_0 : \mathcal{A} \to B(H)$ is an injective $*$-homomorphism.

We claim that $\tau^{(2^n)}$ is unitary equivalent to $\pi \circ \xi_n$. By replacing π with π^α, where α is an infinite cardinal, we can assume that π^α is unitary equivalent to π. Since $\pi\circ\xi_n : M_{2^n}(\mathcal{A}) \to B(H)$ is a $*$-homomorphism there exist unique Hilbert space K_n, $*$-homomorphism $\rho_n : \mathcal{A} \to B(K_n)$ and unitary operator $U_n : K_n \otimes \mathbb{C}^{2^n} \to H$ such that

$$\pi \circ \xi_n = U_n(\rho_n \otimes id_{M_{2^n}})U_n^*.$$

For $a \in \mathcal{A}$, we have

$$\begin{aligned} \pi \circ \xi_0(a) &= \pi \circ \xi_n(a \otimes E_{2^n}) \\ &= U_n(\rho_n(a) \otimes E_{2^n})U_n^*, \end{aligned}$$

where E_{2^n} is the identity matrix in $M_{2^n}(\mathbb{C})$. Thus $\tau(a) = U_0\rho_0(a)U_0^* = U_n(\rho_n(a) \otimes E_{2^n})U_n^*$. Let $\sim$ stands for the unitary equivalence of representations. Since $\pi \circ \xi_n \sim \rho_n \otimes id_{M_{2^n}}$ and $\pi^\alpha \sim \pi$ we have that $\rho_n^\alpha \otimes id_{M_{2^n}} \sim \pi^\alpha \circ \xi_n \sim \rho_n \otimes id_{M_{2^n}}$. Hence $\rho_n^\alpha \sim \rho_n$. Thus $\rho_n \otimes E_{2^n} \sim \rho_n^{2^n\alpha} \sim \rho_n$. Consequently $\rho_0 \sim \rho_n$ and $\pi \circ \xi_n \sim \rho_0 \otimes id_{M_{2^n}} \sim \tau \otimes id_{M_{2^n}}$. Therefore $\tau^{(2^n)} = \tau \otimes id_{M_{2^n}}$ is unitary equivalent to $\pi \circ \xi_n$.

What is left to show is that $\tau^{(n)}(C_n) = M_n(\tau(\mathcal{A}))^+$. Note that $\pi \circ \xi_n(M_{2^n}(\mathcal{A})) \cap B(H)^+ = \pi(C_{2^n})$. Indeed, the inclusion $\pi \circ \xi(C_{2^n}) \subseteq M_{2^n}(\mathcal{A}) \cap B(H)^+$ is obvious. To show the converse take $x \in M_{2^n}(\mathcal{A})$ such that $\pi(x) \geq 0$. Then $x \in C \cap M_{2^n}(\mathcal{A})$. Using (c) one can easily show that $C \cap M_{2^n}(\mathcal{A}) = C_{2^n}$. Hence $\pi \circ \xi_n(M_{2^n}(\mathcal{A})) \cap B(H)^+ = \pi(C_{2^n})$. Since $\tau^{(2^n)}$ is unitary equivalent to $\pi \circ \xi_n$ we have that $\tau^{(2^n)}(C_{2^n}) = M_{2^n}(\tau(\mathcal{A})) \cap B(H^{2^n})^+$.

Let now show that $\tau^{(n)}(C_n) = M_n(\tau(\mathcal{A}))^+$. For $X \in M_n(\mathcal{A})$ denote

$$\widetilde{X} = \begin{pmatrix} X & 0_{n\times(2^n-n)} \\ 0_{(2^n-n)\times n} & 0_{(2^n-n)\times(2^n-n)} \end{pmatrix} \in M_{2^n}(\mathcal{A}).$$

Then, clearly, $\tau^{(n)}(X) \geq 0$ if and only if $\tau^{(2^n)}(\widetilde{X}) \geq 0$. Thus $\tau^{(n)}(X) \geq 0$ is equivalent to $\widetilde{X} \in C_{2^n}$ which in turn is equivalent to $X \in C_n$ by (c). □

Theorem 26 is a direct corollary of the above theorem.

3.2.1 Operator Algebras completely boundedly isomorphic to C^*-algebras.

In the sequel all operator algebras will be assumed to be norm closed.

Operator algebras $\mathcal{A}$ and $\mathcal{B}$ are called completely boundedly isomorphic if there is a completely bounded isomorphism $\tau : \mathcal{A} \to \mathcal{B}$ with completely bounded inverse. The aim of this section is to give necessary and sufficient conditions for an operator algebra to be completely boundedly isomorphic to a C^*-algebra. To do this we introduce a concept of $*$-admissible cones which reflect the properties of the cones of positive elements of a C^*-algebra preserved under completely bounded isomorphism.

Definition 33. *Let $\mathcal{B}$ be an operator algebra with unit e. A sequence $C_n \subseteq M_n(\mathcal{B})$ of closed (in the norm $\|\cdot\|_n$) cones will be called $*$-admissible if it satisfies the following conditions:*

1. $e \in C_1$;
2. (i) $M_n(\mathcal{B}) = (C_n - C_n) + i(C_n - C_n)$, *for all* $n \in \mathbb{N}$,
   (ii) $C_n \cap (-C_n) = \{0\}$, *for all* $n \in \mathbb{N}$,
   (iii) $(C_n - C_n) \cap i(C_n - C_n) = \{0\}$, *for all* $n \in \mathbb{N}$;
3. (i) *for all* $c_1, c_2 \in C_n$ *and* $c \in C_n$, *we have that* $(c_1-c_2)c(c_1-c_2) \in C_n$,
   (ii) *for all* n, m *and* $B \in M_{n\times m}(\mathbb{C})$ *we have that* $B^* C_n B \subseteq C_m$;
4. *there is* $r > 0$ *such that for every positive integer* n *and* $c \in C_n - C_n$ *we have* $r\|c\|e_n + c \in C_n$,
5. *there exists a constant* $K > 0$ *such that for all* $n \in \mathbb{N}$ *and* $a, b \in C_n - C_n$ *we have* $\|a\|_n \leq K \cdot \|a + ib\|_n$.

Theorem 34. *If an operator algebra $\mathcal{B}$ has a $*$-admissible sequence of cones then there is a completely bounded isomorphism τ from $\mathcal{B}$ onto a C^*-algebra $\mathcal{A}$. If, in addition, one of the following conditions holds*

(1) there exists $r > 0$ such that for every $n \geq 1$ and $c, d \in C_n$ we have $\|c + d\| \geq r\|c\|$.

(2) there exists $\alpha > 0$ such that

$$\|(x - iy)(x + iy)\| \geq \alpha\|x - iy\|\|x + iy\|$$

for all $x, y \in C_n - C_n$

then the inverse $\tau^{-1} : \mathcal{A} \to \mathcal{B}$ is also completely bounded.

Conversely, if such isomorphism τ exists then $\mathcal{B}$ possesses a $*$-admissible sequence of cones and conditions (1) *and* (2) *are satisfied.*

The proof will be divided into 4 lemmas.

Let $\{C_n\}_{n\geq 1}$ be a $*$-admissible sequence of cones of $\mathcal{B}$. Let $\mathcal{B}_{2^n} = M_{2^n}(\mathcal{B})$, $\varphi_n : \mathcal{B}_{2^n} \to \mathcal{B}_{2^{n+1}}$ be unital homomorphisms given by $\varphi_n(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, $x \in \mathcal{B}_{2^n}$. Denote by $\mathcal{B}_\infty = \varinjlim \mathcal{B}_{2^n}$ the inductive limit of the system $(\mathcal{B}_{2^n}, \varphi_n)$. As all inclusions φ_n are unital $\mathcal{B}_\infty$ has a unit, denoted by e_∞. Since $\mathcal{B}_\infty$ can be considered as a subalgebra of a C^*-algebra of the corresponding inductive limit of $M_{2^n}(B(\mathcal{H}))$ we can define the closure of $\mathcal{B}_\infty$ in this C^*-algebra denoted by $\overline{\mathcal{B}}_\infty$.

Now we will define an involution on $\mathcal{B}_\infty$. Let $\xi_n : M_{2^n}(\mathcal{B}) \to \mathcal{B}_\infty$ be the canonical morphisms. By (3*ii*), $\varphi_n(C_{2^n}) \subseteq C_{2^{n+1}}$. Hence $C = \bigcup_n \xi_n(C_{2^n})$ is a well defined cone in $\mathcal{B}_\infty$. Denote by $\overline{C}$ its completion. By (2*i*) and (2*iii*), for every $x \in \mathcal{B}_{2^n}$, we have $x = x_1 + ix_2$ with unique x_1, $x_2 \in C_{2^n} - C_{2^n}$. By (3*ii*) we have $\begin{pmatrix} x_i & 0 \\ 0 & x_i \end{pmatrix} \in C_{2^{n+1}} - C_{2^{n+1}}$, $i = 1, 2$. Thus for every $x \in B_\infty$ we have unique decomposition $x = x_1 + ix_2$, $x_1 \in C - C$, $x_2 \in C - C$. Hence the mapping $x \mapsto x^\sharp = x_1 - ix_2$ is a well defined involution on $\mathcal{B}_\infty$. In particular, we have an involution on $\mathcal{B}$ which depends only on the cone C_1.

Lemma 35. *Involution on $\mathcal{B}_\infty$ is defined by the involution on $\mathcal{B}$, i.e. for all $A = (a_{ij})_{i,j} \in M_{2^n}(\mathcal{B})$*

$$A^\sharp = (a^\sharp_{ji})_{i,j}.$$

Proof. Assignment $A^\circ = (a^\sharp_{ji})_{i,j}$, clearly, defines an involution on $M_{2^n}(\mathcal{B})$. We need to prove that $A^\sharp = A^\circ$.

Let $A = (a_{ij})_{i,j} \in M_{2^n}(\mathcal{B})$ be self-adjoint $A^\circ = A$. Then $A = \sum_i a_{ii} \otimes E_{ii} + \sum_{i<j}(a_{ij} \otimes E_{ij} + a^\sharp_{ij} \otimes E_{ji})$ and $a^\sharp_{ii} = a_{ii}$, for all i. By (3*ii*) we have $\sum_i a_{ii} \otimes E_{ii} \in C_{2^n} - C_{2^n}$. Since $a_{ij} = a'_{ij} + ia''_{ij}$ for some a'_{ij}, $a''_{ij} \in C_{2^n} - C_{2^n}$ we have

$$\begin{aligned}
a_{ij} \otimes E_{ij} + a^\sharp_{ij} \otimes E_{ji} &= (a'_{ij} + ia''_{ij}) \otimes E_{ij} + (a'_{ij} - ia''_{ij}) \otimes E_{ji} \\
&= (a'_{ij} \otimes E_{ij} + a'_{ij} \otimes E_{ji}) + (ia''_{ij} \otimes E_{ij} - ia''_{ij} \otimes E_{ji}) \\
&= (E_{ii} + E_{ji})(a'_{ij} \otimes E_{ii} + a'_{ij} \otimes E_{jj})(E_{ii} + E_{ij}) \\
&- (a'_{ij} \otimes E_{ii} + a'_{ij} \otimes E_{jj}) \\
&+ (E_{ii} - iE_{ji})(a''_{ij} \otimes E_{ii} + a''_{ij} \otimes E_{jj})(E_{ii} + iE_{ij}) \\
&- (a''_{ij} \otimes E_{ii} + a''_{ij} \otimes E_{jj}) \in C_{2^n} - C_{2^n}.
\end{aligned}$$

Thus $A \in C_{2^n} - C_{2^n}$ and $A^\sharp = A$. Since for every $x \in M_{2^n}(\mathcal{B})$ there exist unique $x_1 = x_1^\circ$ and $x_2 = x_2^\circ$ in $M_{2^n}(\mathcal{B})$, such that $x = x_1 + ix_2$, and unique $x'_1 = x'^\sharp_1$ and $x'_2 = x'^\sharp_2$, such that $x = x'_1 + ix'_2$, we have that $x_1 = x^\sharp_1 = x'_1$, $x_2 = x^\sharp_2 = x'_2$ and involutions $\sharp$ and $\circ$ coincide. □

Lemma 36. *Involution $x \to x^\sharp$ is continuous on $\mathcal{B}_\infty$ and extends to the involution on $\overline{\mathcal{B}}_\infty$. With respect to this involution $\overline{C} \subseteq (\overline{\mathcal{B}}_\infty)_{sa}$ and $x^\sharp \overline{C} x \subseteq \overline{C}$ for every $x \in \overline{\mathcal{B}}_\infty$.*

Proof. Consider a convergent net $\{x_i\} \subseteq \mathcal{B}_\infty$ with the limit $x \in \mathcal{B}_\infty$. Decompose $x_i = x_i' + ix_i''$ with $x_i', x_i'' \in C - C$. By (5), the nets $\{x_i'\}$ and $\{x_i''\}$ are also convergent. Thus $x = a + ib$, where $a = \lim x_i' \in \overline{C - C}$, $b = \lim x_i'' \in \overline{C - C}$ and $\lim x_i^\sharp = a - ib$. Therefore the involution defined on $\mathcal{B}_\infty$ can be extended by continuity to $\overline{\mathcal{B}}_\infty$ by setting $x^\sharp = a - ib$.

Under this involution $\overline{C} \subseteq (\overline{\mathcal{B}}_\infty)_{sa} = \{x \in \overline{\mathcal{B}}_\infty : x = x^\sharp\}$.

Let us show that $x^\sharp c x \in \overline{C}$ for every $x \in \overline{\mathcal{B}}_\infty$ and $c \in \overline{C}$. Take firstly $c \in C_{2^n}$ and $x \in \mathcal{B}_{2^n}$. Then $x = x_1 + ix_2$ for some x_1, $x_2 \in C_{2^n} - C_{2^n}$ and

$$(x_1 + ix_2)^\sharp c(x_1 + ix_2) = (x_1 - ix_2)c(x_1 + ix_2)$$
$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} -x_1 & -ix_2 \\ ix_2 & x_1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} -x_1 & -ix_2 \\ ix_2 & x_1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

By (3i), Lemma 35 and (3ii) $x^\sharp c x \in C_{2^n}$.

Let now $c \in \overline{C}$ and $x \in \overline{\mathcal{B}}_\infty$. Suppose that $c_i \to c$ and $x_i \to x$, where $c_i \in C$, $x_i \in \mathcal{B}_\infty$. We can assume that c_i, $x_i \in B_{2^{n_i}}$. Then $x_i^\sharp c_i x_i \in C_{2^{n_i}}$ for all i and since it is convergent we have $x^\sharp c x \in \overline{C}$. □

Lemma 37. *The unit of $\overline{\mathcal{B}}_\infty$ is an Archimedean order unit and $(\overline{\mathcal{B}}_\infty)_{sa} = \overline{C} - \overline{C}$.*

Proof. Firstly let us show that e_∞ is an order unit. Clearly, $(\overline{\mathcal{B}}_\infty)_{sa} = \overline{C - C}$. For every $a \in \overline{C - C}$, there is a net $a_i \in C_{2^{n_i}} - C_{2^{n_i}}$ convergent to a. Since $\sup_i \|a_i\| < \infty$ there exists $r_1 > 0$ such that $r_1 e_{n_i} - a_i \in C_{2^{n_i}}$, i.e. $r_1 e_\infty - a_i \in C$. Passing to the limit we get $r_1 e_\infty - a \in \overline{C}$. Replacing a by $-a$ we can find $r_2 > 0$ such that $r_2 e_\infty + a \in \overline{C}$. If $r = \max(r_1, r_2)$ then $re_\infty \pm a \in \overline{C}$. This proves that e_∞ is an order unit and that for all $a \in \overline{C - C}$ we have $a = re_\infty - c$ for some $c \in \overline{C}$. Thus $\overline{C - C} \in \overline{C} - \overline{C}$. The converse inclusion, clearly, holds. Thus $\overline{C - C} = \overline{C} - \overline{C}$.

If $x \in (\overline{\mathcal{B}}_\infty)_{sa}$ such that for every $r > 0$ we have $r + x \in \overline{C}$ then $x \in \overline{C}$ since $\overline{C}$ is closed. Hence e_∞ is an Archimedean order unit. □

Lemma 38. $\mathcal{B}_\infty \cap \overline{C} = C$.

Proof. Denote by $\mathcal{D} = \varinjlim M_{2^n}(B(\mathcal{H}))$ the C^*-algebra inductive limit corresponding to the inductive system φ_n and denote $\varphi_{n,m} = \varphi_{m-1} \circ \ldots \circ \varphi_n : M_{2^n}(B(\mathcal{H})) \to M_{2^m}(B(\mathcal{H}))$. For $n < m$ we identify $M_{2^{m-n}}(M_{2^n}(B(\mathcal{H})))$ with $M_{2^m}(B(\mathcal{H}))$ by omitting superfluous parentheses in a block matrix $B = [B_{ij}]_{ij}$ with $B_{ij} \in M_{2^n}(B(\mathcal{H}))$.

Denote by $P_{n,m}$ the operator $diag(I, 0, \ldots, 0) \in M_{2^{m-n}}(M_{2^n}(B(\mathcal{H})))$ and set $V_{n,m} = \sum_{k=1}^{2^{m-n}} E_{k,k-1}$. Here I is the identity matrix in $M_{2^n}(B(\mathcal{H}))$ and $E_{k,k-1}$ is $2^n \times 2^n$ block matrix with identity operator at $(k, k-1)$-entry and all other entries being zero. Define an operator $\psi_{n,m}([B_{ij}]) = diag(B_{11}, \ldots, B_{11})$. It is easy to see that

$$\psi_{n,m}([B_{ij}]) = \sum_{k=0}^{2^{m-n}-1} (V_{n,m}^k P_{n,m}) B (V_{n,m}^k P_{n,m})^*.$$

Hence by $(3ii)$

$$\psi_{n,m}(C_{2^m}) \subseteq \varphi(C_{2^n}) \subseteq C_{2^m}. \tag{3.7}$$

Clearly, $\psi_{n,m}$ is a linear contraction and

$$\psi_{n,m+k} \circ \varphi_{m,m+k} = \varphi_{m,m+k} \circ \psi_{n,m}$$

Hence there is a well defined contraction $\psi_n = \lim_m \psi_{n,m} : \mathcal{D} \to \mathcal{D}$ such that

$$\psi_n|_{M_{2^n}(B(H))} = id_{M_{2^n}(B(\mathcal{H}))},$$

where $M_{2^n}(B(\mathcal{H}))$ is considered as a subalgebra in $\mathcal{D}$. Clearly, $\psi_n(\overline{\mathcal{B}}_\infty) \subseteq \overline{\mathcal{B}}_\infty$ and $\psi_n|_{\mathcal{B}_{2^n}} = id$. Consider C and C_{2^n} as subalgebras in $\mathcal{B}_\infty$. By (3.7) we have $\psi_n : C \to C_{2^n}$.

To prove that $\mathcal{B}_\infty \cap \overline{C} = C$ take $c \in \mathcal{B}_\infty \cap \overline{C}$. Then there is a net c_j in C such that $\|c_j - c\| \to 0$. Since $c \in \mathcal{B}_\infty$, $c \in \mathcal{B}_{2^n}$ for some n, and consequently $\psi_n(c) = c$. Thus

$$\|\psi_n(c_j) - c\| = \|\psi_n(c_j - c)\| \leq \|c_j - c\|.$$

Hence $\psi_n(c_j) \to c$. But $\psi_n(c_j) \in C_{2^n}$ and the latter is closed. Thus $c \in C$. The converse inclusion is obvious. $\square$

Remark 39. *Note that for every $x \in \mathcal{D}$*

$$\lim_n \psi_n(x) = x. \tag{3.8}$$

Indeed, for every $\varepsilon > 0$ there is $x \in M_{2^n}(B(H))$ such that $\|x - x_n\| < \varepsilon$. Since ψ_n is a contraction and $\psi_n(x_n) = x_n$ we have

$$\begin{aligned} \|\psi_n(x) - x\| &\leq \|\psi_n(x) - x_n\| + \|x_n - x\| \\ &= \|\psi_n(x - x_n)\| + \|x_n - x\| \leq 2\varepsilon. \end{aligned}$$

Since $x_n \in M_{2^n}(B(\mathcal{H}))$ also belong to $M_{2^m}(B(\mathcal{H}))$ for all $m \geq n$, we have that $\|\psi_m(x) - x\| \leq 2\varepsilon$. Thus $\lim_n \psi_n(x) = x$.

Proof of Theorem 34. By Lemma 36 and 37 the cone $\overline{C}$ and the unit e_∞ satisfies all assumptions of Theorem 30. Thus there is a homomorphism $\tau : \overline{\mathcal{B}}_\infty \to B(\widetilde{H})$ such that $\tau(a^\sharp) = \tau(a)^*$ for all $a \in \overline{\mathcal{B}}_\infty$. Since the image of τ is a $*$-subalgebra of $B(\widetilde{H})$ we have that τ is bounded by [22, (23.11), p. 81]. The arguments at the end of the proof of Theorem 32 show that the restriction of τ to $\mathcal{B}_{2^n}$ is unitary equivalent to the 2^n-amplification of $\tau|_{\mathcal{B}}$. Thus $\tau|_{\mathcal{B}}$ is completely bounded.

Let us prove that $\ker(\tau) = \{0\}$. By item 3 in Theorem 32 it is sufficient to show that $\overline{C} \cap (-\overline{C}) = 0$. If $c, d \in \overline{C}$ such that $c + d = 0$ then $c = d = 0$. Indeed, for every $n \geq 1$, $\psi_n(c) + \psi_n(d) = 0$. By Lemma 38, we have

$$\psi_n(\overline{C}) \subseteq \overline{C} \cap \mathcal{B}_{2^n} = C_{2^n}.$$

Therefore $\psi_n(c)$, $\psi_n(d) \in C_{2^n}$. Hence $\psi_n(c) = -\psi_n(d) \in C_{2^n} \cap (-C_{2^n})$ and, consequently, $\psi_n(c) = \psi_n(d) = 0$. Since $\|\psi_n(c) - c\| \to 0$ and $\|\psi_n(d) - d\| \to 0$ by Remark 39, we have that $c = d = 0$. If $x \in \overline{C} \cap (-\overline{C})$ then $x + (-x) = 0$, $x, -x \in \overline{C}$ and $x = 0$. Thus τ is injective.

We will show that the image of τ is closed if one of the conditions (1) or (2) of the statement holds.

Assume firstly that operator algebra $\mathcal{B}$ satisfies the first condition. Since $\tau(\overline{\mathcal{B}}_\infty) = \tau(\overline{C}) - \tau(\overline{C}) + i(\tau(\overline{C}) - \tau(\overline{C}))$ and $\tau(\overline{C})$ is exactly the set of positive operators in the image of τ, it is suffices to prove that $\tau(\overline{C})$ is closed. By item 3 in Theorem 30, for self-adjoint (under involution $\sharp$) $x \in \overline{\mathcal{B}}_\infty$ we have

$$\|\tau(x)\|_{B(\widetilde{H})} = \inf\{r > 0 : re_\infty \pm x \in \overline{C}\}.$$

If $\tau(c_\alpha) \in \tau(C)$ is a Cauchy net in $B(\widetilde{H})$ then for every $\varepsilon > 0$ there is γ such that $\varepsilon \pm (c_\alpha - c_\beta) \in \overline{C}$ when $\alpha \geq \gamma$ and $\beta \geq \gamma$. Since $\overline{C} \cap \mathcal{B}_\infty = C$, $\varepsilon \pm (c_\alpha - c_\beta) \in C$. Denote $c_{\alpha\beta} = \varepsilon + (c_\alpha - c_\beta)$ and $d_{\alpha\beta} = \varepsilon - (c_\alpha - c_\beta)$. The set of pairs (α, β) is directed if $(\alpha, \beta) \geq (\alpha_1, \beta_1)$ iff $\alpha \geq \alpha_1$ and $\beta \geq \beta_1$. Since $c_{\alpha\beta} + d_{\alpha\beta} = 2\varepsilon$ this net converges to zero in the norm of $\overline{\mathcal{B}}_\infty$. Thus by assumption 4 in the definition of $*$-admissible sequence of cones, $\|c_{\alpha\beta}\|_{\overline{\mathcal{B}}_\infty} \to 0$. This implies that c_α is a Cauchy net in $\overline{\mathcal{B}}_\infty$. Let $c = \lim c_\alpha$. Clearly, $c \in \overline{C}$. Since τ is continuous $\|\tau(c_\alpha) - \tau(c)\|_{\overline{\mathcal{B}}_\infty} \to 0$. Hence the closure $\overline{\tau(C)}$ is contained in $\tau(\overline{C})$. By continuity of τ we have $\tau(\overline{C}) \subseteq \overline{\tau(C)}$. Hence $\tau(\overline{C}) = \overline{\tau(C)}$, $\tau(\overline{C})$ is closed.

Let now $\mathcal{B}$ satisfy condition (2) of the theorem. Then for every $x \in \overline{\mathcal{B}}_\infty$ we have $\|x^\sharp x\| \geq \alpha \|x\| \|x^\sharp\|$. By [22, theorem 34.3] $\overline{\mathcal{B}}_\infty$ admits an equivalent C^*-norm $|\cdot|$. Since τ is a faithful $*$-representation of the C^*-algebra $(\overline{\mathcal{B}}_\infty, |\cdot|)$ it is isometric. Therefore $\tau(\overline{\mathcal{B}}_\infty)$ is closed.

Let us show that $(\tau|_{\mathcal{B}})^{-1} : \tau(\mathcal{B}) \to \mathcal{B}$ is completely bounded. The image $\mathcal{A} = \tau(\overline{\mathcal{B}}_\infty)$ is a C^*-algebra in $B(\widetilde{H})$ isomorphic to $\overline{\mathcal{B}}_\infty$. By Johnson's theorem (see [45]), two Banach algebra norms on a semi-simple algebra are equivalent, hence, $\tau^{-1} : \mathcal{A} \to \overline{\mathcal{B}}_\infty$ is bounded homomorphism, let $R = \|\tau^{-1}\|$. Let us show that $\|(\tau|_{\mathcal{B}})^{-1}\|_{cb} = R$. Since

$$\tau|_{\mathcal{B}_{2^n}} = U_n(\tau|_{\mathcal{B}} \otimes id_{M_{2^n}})U_n^*,$$

for some unitary $U_n : K \otimes \mathbb{C}^{2^n} \to \widetilde{H}$ we have for any $B = [b_{ij}] \in M_{2^n}(\mathcal{B})$

$$\begin{aligned} \|\sum b_{ij} \otimes E_{ij}\| &\leq R\|\tau(\sum b_{ij} \otimes E_{ij})\| \\ &= R\|U_n(\sum \tau(b_{ij}) \otimes E_{ij})U_n^*\| \\ &= R\|\sum \tau(b_{ij}) \otimes E_{ij}\|. \end{aligned}$$

This is equivalent to

$$\|\sum \tau^{-1}(b_{ij}) \otimes E_{ij}\| \leq R\|\sum b_{ij} \otimes E_{ij}\|,$$

hence $\|(\tau^{-1})^{2^n}(B)\| \leq R\|B\|$. This proves that $\|(\tau|_{\mathcal{B}})^{-1}\|_{cb} = R$.

The converse statement evidently holds with $*$-admissible sequence of cones given by $(\tau^{(n)})^{-1}(M_n(\mathcal{A})^+)$.□

Conditions (1) and (2) were used to prove that the image of isomorphism τ is closed. The natural question one can ask is wether there exists a Banach operator algebra isomorphic to a non-closed self-adjoint operator algebra via bounded isomorphism. The following example gives the affirmative answer to this question.

Example 40. *Consider the algebra $\mathcal{B} = C^1([0,1])$ as an operator algebra in C^*-algebra $\bigoplus_{q \in \mathbb{Q} \cap [0,1]} M_2(C([0,1]))$ via inclusion*

$$f(\cdot) \mapsto \oplus_{q \in \mathbb{Q} \cap [0,1]} \begin{pmatrix} f(q) & f'(q) \\ 0 & f(q) \end{pmatrix}.$$

The induced norm

$$\|f\| = \sup_{q \in \mathbb{Q} \cap [0,1]} \left[\frac{1}{2}(2|f(q)|^2 + |f'(q)|^2 + |f'(q)|\sqrt{4|f(q)|^2 + |f'(q)|^2})\right]^{\frac{1}{2}}$$

satisfies the inequality $\|f\| \geq \frac{1}{\sqrt{2}} \max\{\|f\|_\infty, \|f'\|_\infty\} \geq \frac{1}{2\sqrt{2}}\|f\|_1$ where $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$ is the standard Banach norm on $C^1([0,1])$. Thus $\mathcal{B}$ is a closed operator algebra with isometric involution $f^\sharp(x) = \overline{f(x)}$, $x \in [0,1]$. The identity map $C^1([0,1]) \to C([0,1])$, $f \mapsto f$ is a $*$-isomorphism of $\mathcal{B}$ with non-closed self-adjoint subalgebra of $C([0,1])$.

Chapter 4

Ideals of a C^*-algebra generated by an operator algebra.

4.1 Basic definitions and facts.

An operator algebra is a subalgebra of $B(H)$, the algebra of all bounded operators on a Hilbert space H. The algebra $M_n(B(H))$ of $n \times n$ matrices with entries in $B(H)$ has a norm $\|\cdot\|_n$ via the standard identification of $M_n(B(H))$ with $B(H^n)$, where H^n is the direct sum of n copies of a Hilbert space H. If $\mathcal{A}$ is a subalgebra of $B(H)$ then $M_n(\mathcal{A})$ inherits a norm $\|\cdot\|_n$ via the natural inclusion into $M_n(B(H))$. The norms $\|\cdot\|_n$ are called matrix norms on the operator algebra $\mathcal{A}$. The Blecher-Ruan-Sinclair Theorem [11] abstractly characterize the operator algebras in terms of matrix norms.

A linear mapping $\varphi : \mathcal{A} \to B(H)$ induces a linear mapping

$$\varphi^{(n)} : M_n(\mathcal{A}) \to B(\bigoplus_{i=1}^{n} H)$$

via the formula $\varphi^{(n)}((a_{ij})) = (\varphi(a_{ij}))$ for every $(a_{ij}) \in M_n(\mathcal{A})$. The map φ is called completely bounded if there exists C such that $\|\varphi\|_{cb} = \sup_n \|\varphi^{(n)}\| < C < \infty$, it is called completely contractive (isometric) if $\varphi^{(n)}$ is contractive (corresp. isometric) for every $n \in \mathbb{N}$.

The C^*-envelope of an operator algebra $\mathcal{A}$, denoted by $C^*_e(\mathcal{A})$, is a C^*-algebra generated by $i(\mathcal{A})$ for some completely isometric homomorphism $i : \mathcal{A} \to B(H)$ having the following universal property. For any completely isomorphic homomorphism $\rho : \mathcal{A} \to B(K)$ there exists a unique onto $*$-homomorphism $\pi : C^*(\rho(\mathcal{A})) \to C^*_e(\mathcal{A})$ such that $\pi(\rho(a)) = i(a)$ for every $a \in \mathcal{A}$.

In [1] Arveson defined a noncommutative analog of the Shilov boundary of a uniform algebra. It is known that the Shilov boundary of a uniform algebra A is the closure of the Choquet boundary. The (irreducible) boundary representations of A correspond to points of the Choquet boundary. In the noncommutative setting let $\mathcal{B}$ be an operator algebra in $B(H)$ and $C^*(\mathcal{B})$ be the C^*-algebra generated by $\mathcal{B}$ in $B(H)$. Then a closed (two-sided) ideal J of $C^*(\mathcal{B})$ is called a boundary ideal if the canonical quotient map $q : C^*(\mathcal{B}) \to C^*(\mathcal{B})/J$ is completely isometric on $\mathcal{B}$. The Shilov boundary is a boundary ideal containing all boundary ideals. It can be shown that the C^*-envelope of an operator algebra $\mathcal{B}$ is the quotient of C^*-algebra generated by $\mathcal{B}$ by the Shilov boundary whenever the latter exists. Arveson showed the existence of C^*-envelopes for the *admissible* operator algebras developing the theory of *boundary representations* in [1]. The existence in full generality was shown in [37]. But the question wether there are sufficiently many boundary representations to construct the C^*-envelope was not settled until the works [23] and [2] appeared.

In this paper we study a generalization of boundary ideals. Namely, a closed (two-sided) ideal $J \subset C^*(\mathcal{B})$ will be called K-boundary ideal if a canonical quotient map $q : C^*(\mathcal{B}) \to C^*(\mathcal{B})/J$ restricted to $\mathcal{B}$ has a completely bounded inverse $q|_{\mathcal{B}}^{-1}$ with completely bounded norm equal to K^{-1}. For $K = 1$ we obtain the boundary ideals of [1]. Surprisingly, there is no analog of Shilov boundary for K-boundary ideals for $K \neq 1$. That is in general there is no K-boundary ideal containing every other K-boundary ideal, see Example 43.

As a corollary of [62] and the solution of Halmos problem [71] there exists bounded homomorphism of the disk algebra $A(\mathbb{D})$ into $B(H)$ which is not completely bounded. Adapting this result to the quotient maps in Example 45 we construct an operator algebra $\mathcal{B}$ and an ideal J such that $q|_{\mathcal{B}}^{-1}$ is bounded but not completely bounded.

Further, we prove that if an operator algebra $\mathcal{B}$ λ-norms itself and J is closed ideal of C^*-algebra generated by $\mathcal{B}$ such that the restriction of the quotient map $q : C^*(\mathcal{B}) \to C^*(\mathcal{B})/J$ has continuous inverse $q|_{\mathcal{B}}^{-1}$ then $q|_{\mathcal{B}}^{-1}$ is completely bounded.

In Section 3 we consider the following problem raised by Kadison in 1955. Is any bounded homomorphism π of a C^*-algebra $\mathcal{A}$ into $B(H)$ is similar to a $*$-homomorphism? The similarity above means that there exists an invertible operator $S \in B(H)$ such that $S^{-1}\pi(\cdot)S$ is a $*$-homomorphism. Haagerup [36] showed that π is similar to a $*$-homomorphism if and only if it is completely bounded. Obviously, for this problem it is sufficient to consider only faithful homomorphisms. Pitts [75] proved that every bounded faithful homomorphism of a C^*-algebra has a completely bounded inverse.

In Section 3 we show that for Kadison's similarity problem it is sufficient to consider bounded faithful homomorphisms that have completely contractive inverses. As a corollary of this result we obtain the following equivalence. Every bounded homomorphism of a C^*-algebra is completely bounded if and only if for every operator algebra $\mathcal{B}$ in $B(H)$ and C^*-algebra $C^*(\mathcal{B})$ generated by it every bounded homomorphism from $C^*(\mathcal{B})$ onto $\mathcal{B}$ which is projection on $\mathcal{B}$ is completely bounded.

It was proved in [75] that bounded homomorphism π of a C^*-algebra $\mathcal{A}$ is completely bounded if and only if $\pi(\mathcal{A})$ λ-norms itself. Using this result and Theorem 44 we have a reformulation of Kadison's similarity problem in terms of ideals of a C^*-algebra generated by an operator algebra, see Theorem 46.

In the last section we prove that Kadison's similarity problem is decided on one particular C^*-algebra $\mathcal{D}^*(M_2)$ which is a completion of the $*$-double $\mathcal{D}(M_2)$ of $M_2(\mathbb{C})$. This provides a direct proof of the implication (ii) $\Rightarrow$ (i) in Theorem 48. Moreover, the homomorphisms of this algebra into $B(H)$, with H being a separable Hilbert space, are easy to describe. Each such homomorphism up to a similarity corresponds to an operator $T \in B(\mathbb{C}^2 \otimes H)$. Denoting this homomorphism by τ_T we prove that τ_T is similar to a $*$-representation iff there is an invertible $R \in B(H)$ such that $(I \otimes R)T \geq 0$ (see Theorem 57). It is an open question under what conditions on T the homomorphism τ_T can be extended to a bounded homomorphism on C^*-algebra $\mathcal{D}^*(M_2)$.

In this paper all operator algebras are supposed to be unital, closed and for a given operator algebra $\mathcal{A} \subseteq B(H)$ the C^*-algebra generated by $\mathcal{A}$ in $B(H)$ is denoted by $C^*(\mathcal{A})$.

We refer the reader to the books [28], [62] and [71] for precise definitions, basic facts and terminology related to operator algebras, operator spaces and completely bounded maps.

4.2 Boundary Ideals "up to a constant".

In this section we define and investigate a generalization of boundary ideals.

Definition 41. *Let $\mathcal{B}$ be a closed subalgebra of a unital C^*-algebra $\mathcal{A}$ such that $\mathcal{B}$ contains the unit and generates $\mathcal{A}$ as a C^*-algebra. A closed ideal J of $\mathcal{A}$ is called a K-boundary ideal if $K > 0$ is the greatest constant having the following property*

$$K \cdot \|a\|_{M_n(\mathcal{A})} \leq \|a\|_{M_n(\mathcal{A}/J)} \tag{4.1}$$

for every $a \in M_n(\mathcal{B})$ and $n \in \mathbb{N}$. In other words the canonical quotient map

$q : \mathcal{A} \to \mathcal{A}/J$ restricted to $\mathcal{B}$ is injective and has a completely bounded inverse with completely bounded norm equal to K^{-1}.

If $C^*(\mathcal{B})$ is commutative then every ideal which satisfies inequality (5.24) with $n = 1$ and $K > 0$ is automatically boundary ideal, i.e. $K = 1$. It follows from the following observation and the fact that if $\|b\| = 1$ then $\|b^n\| = 1$ for every $b \in C^*(\mathcal{B})$.

Proposition 42. *Let J be a closed ideal of $C^*(\mathcal{B})$ and $0 < K < 1$ be the greatest constant satisfying the inequality*

$$K \cdot \|b\|_{C^*(\mathcal{B})} \leq \|b\|_{C^*(\mathcal{B})/J}$$

for every $b \in \mathcal{B}$. Then there exists $b \in \mathcal{B}$ such that $\|b\|_{C^*(\mathcal{B})} = 1$ and $\|b^n\|_{C^*(\mathcal{B})} \to 0$ when $n \to \infty$.

Proof. Straightforward. □

A boundary ideal is called Shilov boundary if it contains every other boundary ideal. In the following example we present an operator algebra such that the C^*-algebra it generates has K-boundary ideals (for some $K < 1$) but does not have a K-boundary ideal that contains all K-boundary ideals.

Example 43. *Consider*

$$\mathcal{B} = \{\begin{pmatrix} x_1 & x_2 - x_1 \\ 0 & x_2 \end{pmatrix} \oplus x_1 \oplus x_2 \oplus \begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix} \oplus y_1 \oplus y_2 : x_1, x_2, y_1, y_2 \in \mathbb{C}\}.$$

One can easily check that $\mathcal{B}$ is an algebra and the C^*-algebra generated by $\mathcal{B}$ in $M_8(\mathbb{C})$ is $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$. Consider the following ideals

$$J_1 = M_2(\mathbb{C}) \oplus 0 \oplus 0 \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C},$$
$$J_2 = 0 \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus 0 \oplus 0.$$

We have

$$\begin{aligned}
&\|\begin{pmatrix} x_1 & x_2 - x_1 \\ 0 & x_2 \end{pmatrix} \oplus x_1 \oplus x_2 \oplus \begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix} \oplus y_1 \oplus y_2\|_{C^*(\mathcal{B})} \\
&= \max\{\|\begin{pmatrix} x_1 & x_2 - x_1 \\ 0 & x_2 \end{pmatrix}\|, \|\begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix}\|\} \\
&= \max\{\|\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\|, \|\begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix}\|\} \\
&\leq C \cdot \max\{\|\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\|, \|\begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix}\|\} \\
&= C \cdot \|\begin{pmatrix} x_1 & x_2 - x_1 \\ 0 & x_2 \end{pmatrix} \oplus x_1 \oplus x_2 \oplus \begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix} \oplus y_1 \oplus y_2\|_{C^*(\mathcal{B})/J_1}
\end{aligned}$$

Let $q : C^*(\mathcal{B}) \to C^*(\mathcal{B})/J_1$ be the canonical quotient map. Using the inequalities above we have $\|b\| \leq C \cdot \|q(b)\|$ for every $b \in \mathcal{B}$. Thus the restriction of q to $\mathcal{B}$, $q|_{\mathcal{B}}$, is invertible map into $M_8(\mathbb{C})$ and it is easy to see that $\|q|_{\mathcal{B}}^{-1}\| > 1$. By Smith's theorem [87] the completely bounded norm of $\tau := q|_{\mathcal{B}}^{-1}$ equals to the norm 8-th amplification. Therefore $\|\tau\|_{cb} = \|\tau^{(8)}\| \geq \|\tau\| > 1$ and J_1 is K-boundary ideal with $K = \|\tau\|_{cb}^{-1}$.

Similar arguments prove that J_2 is a K-boundary ideal with the same constant K. There is no K-boundary ideal that contains both ideals since the sum of J_1 and J_2 is the whole $C^*(\mathcal{B})$.

Let us note that inequality (5.24) with $n = 1$ may fail even when $q|_{\mathcal{B}}$ is injective. The reason is that the image of a Banach subalgebra of a C^*-algebra $\mathcal{A}$ under the quotient map is not necessarily closed. Consider for example the C^*-algebra $C(\overline{\mathbb{D}}_2)$ of continuous function on the disk $\mathbb{D}_2$ of radius 2 and its Banach subalgebra $A(\mathbb{D}_2)$ of analytic functions on $\mathbb{D}_2$ which have continuous extension on $\overline{\mathbb{D}}_2$. Let $\mathbb{D}_1$ be the disk of radius 1 and the same center as $\mathbb{D}_2$. The restriction of functions to $\mathbb{D}_1$ is a $*$-homomorphism $\pi : C(\overline{\mathbb{D}}_2) \to C(\overline{\mathbb{D}}_1)$ and $\pi(A(\mathbb{D}_2))$ is not closed since there are analytic functions on $\mathbb{D}_1$ which are not extendable to the analytic function on $\mathbb{D}_2$. Thus the quotient map $q : C(\overline{\mathbb{D}}_2) \to C(\overline{\mathbb{D}}_2)/J$, where $J = ker(\pi)$, maps $A(\mathbb{D}_2)$ into non-closed subalgebra in $C(\overline{\mathbb{D}}_2)/J$. An example of a Banach operator algebra which is isomorphic to a non-closed self-adjoint operator algebra via contractive isomorphism can be found in [**?**].

K-boundary ideals can be easily characterized in the case when $\mathcal{B}$ λ-norms itself. Let us recall necessary definitions.

For a given operator algebra $\mathcal{B}$ define norms $|||\cdot|||_n$ on $M_n(\mathcal{B})$ via

$$|||X|||_n = \sup\{\|RXC\| : R \in M_{1,n}(\mathcal{B}), C \in M_{n,1}(\mathcal{B}), \|R\| \leq 1, \|C\| \leq 1\}.$$

Evidently $|||X|||_n \leq \|X\|_n$ for every $X \in M_n(\mathcal{B})$.

An operator algebra $\mathcal{B}$ λ-norms itself, see [76], if there exists $\lambda > 0$ such that

$$\lambda \cdot \|X\|_n \leq |||X|||_n$$

for every $X \in M_n(\mathcal{B})$ and $n \in \mathbb{N}$.

In the following we will need the notion of the maximal enveloping C^*-algebra of an operator algebra. For a given operator algebra $\mathcal{B}$ there exists a C^*-algebra, denoted by $C^*_{max}(\mathcal{B})$, and a completely isometric homomorphism $i : \mathcal{B} \to C^*_{max}(\mathcal{B})$ such that $i(\mathcal{B})$ generates $C^*_{max}(\mathcal{B})$ as a C^*-algebra and has the following universal property (see [12], [14]). If $\pi : \mathcal{B} \to \mathcal{A}$ is a completely contractive homomorphism into a C^*-algebra $\mathcal{A}$ then there exists a unique

$*$-homomorphism $\widetilde{\pi} : C^*_{max}(\mathcal{B}) \to \mathcal{A}$ extending π, i.e. $\widetilde{\pi} \circ i = \pi$. Algebra $C^*_{max}(\mathcal{B})$ is called maximal enveloping C^*-algebra of $\mathcal{B}$. The existence follows from the following construction, see [12] for details.

Let $\mathcal{B}^*$ denote an algebra wich is anti-isomorphic to $\mathcal{B}$, i.e. there is one-to-one map φ from $\mathcal{B}$ onto $\mathcal{B}^*$ which is additive and $\varphi(ab) = \varphi(b)\varphi(a)$, $\varphi(\lambda a) = \overline{\lambda}\varphi(a)$ for all $a, b \in \mathcal{B}$, $\lambda \in \mathbb{C}$. One can check that such an algebra exists and unique up to isomorphism. Moreover, for a homomorphism $\pi\colon \mathcal{B} \to B(H)$ we can define the homomorphism $\pi^*\colon \mathcal{B}^* \to B(H)$ by the rule $\pi^*(b) = \pi(\varphi^{-1}(b))^*$, $b \in \mathcal{B}^*$. Define a semi-norm on the algebraic free product of $\mathcal{B}$ and $\mathcal{B}^*$ by

$$\|a\|_{\mathcal{B}*\mathcal{B}^*} = \sup\{\|(\pi * \pi^*)(a)\| : \pi : \mathcal{B} \to B(H) \text{ is completely contractive homomorphism}\}$$

The null-space of this norm is a two-sided ideal J. Then $\mathcal{B} * \mathcal{B}^*/J$ is pre-C^*-algebra and $C^*_{max}(\mathcal{B})$ is its completion. Uniqueness of $C^*_{max}(\mathcal{B})$ follows from the universal property.

Theorem 44. *If an operator algebra $\mathcal{B}$ λ-norms itself and $i : \mathcal{B} \to B(H)$ is completely isometric homomorphism then for every ideal $J \subset C^*(i(\mathcal{B}))$ such that the inequality*

$$K' \cdot \|b\|_{C^*(i(\mathcal{B}))} \le \|b\|_{C^*(i(\mathcal{B}))/J} \tag{4.2}$$

holds for every $b \in \mathcal{B}$ and some $K' > 0$ we have that J is a K-boundary ideal for some $K \ge \lambda K'$.

Proof. Let $X \in M_n(\mathcal{B})$, $R \in M_{1,n}(\mathcal{B})$, $C \in M_{n,1}(\mathcal{B})$ and $\|R\| \le 1$, $\|C\| \le 1$. Since the canonical quotient map is completely contractive we have

$$\begin{aligned} \|RXC\|_{C^*(i(\mathcal{B}))} &= \|\sum_{i,j} R_i X_{ij} C_j\|_{C^*(i(\mathcal{B}))} \\ &\le \frac{1}{K'} \cdot \|\sum_{i,j} R_i X_{ij} C_j\|_{C^*(i(\mathcal{B}))/J} \\ &= \frac{1}{K'} \cdot \|RXC\|_{C^*(i(\mathcal{B}))/J} \\ &\le \frac{1}{K'} \cdot \|X\|_{M_n(C^*(i(\mathcal{B}))/J)}. \end{aligned}$$

Taking supremum over all R and C we have

$$\lambda K' \cdot \|X\|_{M_n(C^*(i(\mathcal{B})))} \le \|X\|_{M_n(C^*(i(\mathcal{B}))/J)}.$$

Thus J is a K-boundary ideal for some $K \geq \lambda K'$. $\square$

The example of a semi-simple operator algebra which does not λ-norm itself was given in [75]. In the following we present slightly simplified proof of this result and construct an operator algebra and an ideal such that inequality (5.24) is valid for $n = 1$ but not for all $n \geq 1$.

Example 45. *An operator $T \in B(H)$ is called polynomially bounded if there exists a bounded homomorphism $u_T : A(\mathbb{D}) \to B(H)$ such that $u_T(p) = p(T)$ for every polynomial p. An operator T is called completely polynomially bounded if u_T is completely bounded. In [62] it was shown that T is completely polynomially bounded if and only if T is similar to a contraction. There exists a polynomially bounded operator which is not similar to a contraction, see [71]. Thus there is $T \in B(H)$ such that u_T is a bounded but not completely bounded homomorphism. Since T is polynomially bounded we have that $\sigma(T) \subseteq \mathbb{D}$. Let U be a unitary operator such that $\sigma(U) = \mathbb{T}$. Then $u_{T\oplus U}$ is bounded but not completely bounded and $\mathbb{T} \subseteq \sigma(T \oplus U)$. Thus $\|u_{T\oplus U}(f)\| \geq \|f\|$ for every $f \in \mathbb{A}(\mathbb{D})$ and $\mathcal{B} = u_{T\oplus U}(\mathbb{A}(\mathbb{D}))$ is a Banach algebra. Since $u_{T\oplus U}^{-1} : \mathcal{B} \to \mathbb{A}(\mathbb{D})$ acts into a commutative C^*-algebra we have $\|u_{T\oplus U}^{-1}\|_{cb} = \|u_{T\oplus U}^{-1}\| \leq 1$. Let $i : \mathcal{B} \to C^*_{max}(\mathcal{B})$ be the natural embedding of $\mathcal{B}$ into its maximal enveloping C^*-algebra and let $\tau : C^*_{max}(\mathcal{B}) \to C(\overline{\mathbb{D}})$ be the $*$-homomorphism extending $u_{T\oplus U}^{-1} \circ i^{-1} : i(\mathcal{B}) \to \mathbb{A}(\mathbb{D})$. Since $\tau(C^*_{max}(\mathcal{B}))$ is a C^*-algebra generated by $\mathbb{A}(\mathbb{D})$ we have that τ is surjective. Consider the canonical quotient map*

$$q : C^*_{max}(\mathcal{B}) \to C^*_{max}(\mathcal{B}) / \ker(\tau) \simeq C(\overline{\mathbb{D}}).$$

Then there is $\widetilde{K} > 0$ such that

$$\widetilde{K} \|b\|_{C^*_{max}(\mathcal{B})} \leq \|b\|_{C^*_{max}(\mathcal{B})/\ker(\tau)}.$$

Since $q|_{\mathcal{B}}^{-1} = i \circ u_{T\oplus U}$ is not completely bounded we have that $\ker(\tau)$ is not a K-boundary ideal. Thus Theorem 44 implies that $u_{T\oplus U}(A(\mathbb{D}))$ does not λ-norm itself.

4.3 K-boundary Ideals and the Similarity Problem.

Let $\mathcal{A}$ be a C^*-algebra and $\pi : \mathcal{A} \to B(H)$ be a bounded homomorphism. It was shown in [75] that $\pi(\mathcal{A})$ is a Banach algebra. Moreover, if π is injective then it has the completely bounded inverse, see [75], [?]. Define $\breve{\pi} : \mathcal{A} \to B(H \oplus H)$ by the following rule:

$$\breve{\pi}(a) = \pi(a) \oplus \pi(a^*)^*.$$

Evidently π is completely bounded iff $\breve{\pi}$ is such. Thus by the theorem of Haagerup (see [36]) we have that π is similar to a $*$-homomorphism iff $\breve{\pi}$ is such. A simple proof of this fact which does not use the Haagerup's results can be found in [86] (see also [53]).

Let J be a unitary operator. A homomorphism $\pi : \mathcal{A} \to B(H)$ is called J-symmetric if $\pi(a^*) = J\pi(a)^*J^*$ for every $a \in \mathcal{A}$.

Let $J : B(H \oplus H) \to B(H \oplus H)$ be the unitary operator defined by $J(x \oplus y) = y \oplus x$. Then $\breve{\pi}(a^*) = \pi(a^*) \oplus \pi(a)^* = J\breve{\pi}(a)^*J^*$ and $\breve{\pi}$ is J-symmetric.

Theorem 46. *If $\pi : \mathcal{A} \to B(H)$ is bounded injective homomorphism then the inverse of $\breve{\pi}$ is completely contractive.*

Proof. Let $\mathcal{B} = \breve{\pi}(\mathcal{A})$ and $r(a)$ denotes the spectral radius of $a \in \mathcal{A}$. Since $\mathcal{B}$ is a Banach algebra and an isomorphism preserves the spectrum we have $\sigma_{M_n(\mathcal{A})}(a) = \sigma_{M_n(\breve{\pi}(\mathcal{A}))}(\breve{\pi}^{(n)}(a))$ and

$$\begin{aligned}
\|a\|^2 &= \|a^*a\| = r(a^*a) \\
&= r(\breve{\pi}^{(n)}(a^*a)) \leq \|\breve{\pi}^{(n)}(a^*a)\| \\
&\leq \|\breve{\pi}^{(n)}(a^*)\| \cdot \|\breve{\pi}^{(n)}(a)\| \\
&= \|(J \otimes I_n)\breve{\pi}^{(n)}(a)^*(J \otimes I_n)\| \cdot \|\breve{\pi}^{(n)}(a)\| \\
&\leq \|\breve{\pi}^{(n)}(a)\|^2,
\end{aligned}$$

for every $a \in M_n(\mathcal{A})$. Thus the inverse homomorphism $\breve{\pi}^{-1} : \mathcal{B} \to \mathcal{A}$ is completely contractive. □

Assume that $\breve{\pi}$ is as in Theorem 46. Let $i : \mathcal{B} \to C^*_{max}(\mathcal{B})$ be the canonical inclusion of $\mathcal{B}$ into its maximal enveloping C^*-algebra. Replacing $\breve{\pi}$ by $i \circ \breve{\pi}$, which does not effect completely boundedness of $\breve{\pi}$, we have that $\breve{\pi}(\mathcal{A})$ generate its maximal enveloping C^*-algebra. Therefore by universal property of the maximal enveloping C^*-algebra $\breve{\pi}^{-1}$ can be extended to a $*$-homomorphism $\rho : C^*_{max}(\mathcal{B}) \to \mathcal{A}$. Now we have $C^*_{max}(\mathcal{B})/ker(\rho) \simeq \mathcal{A}$ and for every $b \in \mathcal{B}$,

$$\|b\|_{C^*_{max}(\mathcal{B})/\ker(\rho)} = \|\rho(b)\| = \|\breve{\pi}^{-1}(b)\|.$$

Thus

$$\widetilde{K} \cdot \|b\|_{C^*_{max}(\mathcal{B})} \leq \|b\|_{C^*_{max}(\mathcal{B})/ker(\rho)} \leq \|b\|_{C^*_{max}(\mathcal{B})} \tag{4.3}$$

for every $b \in \mathcal{B}$ and $\widetilde{K} = \|\check{\pi}\|^{-1}$.

In the following we present a description of this kernel and some of its properties.

Theorem 47. *The homomorphism $\check{\pi}$ is completely bounded if and only if $ker(\rho)$ is a K-boundary ideal for some $K > 0$. The kernel of ρ is the ideal J generated by $\{\check{\pi}(a) - \check{\pi}(a^*)^* : a \in \mathcal{A}\}$. If a closed ideal J' satisfies (4.3) for some $\widetilde{K} > 0$ and $J \subseteq J'$ then $J = J'$.*

Proof. The first part follows from the construction above.

Since ρ is a $*$-homomorphism $\rho(\check{\pi}(a) - \check{\pi}(a^*)^*) = 0$ and $J \subseteq ker(\rho)$.

Let us prove the converse inclusion. Let $q : C^*_{max}(\mathcal{B}) \to C^*_{max}(\mathcal{B})/J$ be the canonical quotient map. Let $\mathcal{C} = q(\mathcal{B})$. By 4.3 $\mathcal{B}$ and $\mathcal{C}$ are bicontinuously isomorphic.

Assume that there exists $x \in ker(\rho) \setminus J$. Since $\mathcal{B}$ generates $C^*_{max}(\mathcal{B})$ as a C^*-algebra we have that x is the uniform limit of some polynomials $P_k(b_1, \ldots, b_{n_k}, b_1^*, \ldots, b_{n_k}^*)$, where $b_i \in \mathcal{B}$. Thus

$$q(P_k(b_1, \ldots, b_{n_k}, b_1^*, \ldots, b_{n_k}^*)) = P_k(q(b_1), \ldots, q(b_{n_k}), q(b_1)^*, \ldots, q(b_{n_k})^*)$$

converge uniformly in $C^*_{max}(\mathcal{B})/J$. Clearly $q(b_j)^* = q(\widetilde{b}_j)$ for some $\widetilde{b}_j \in \mathcal{B}$ and the elements

$$P_k(q(b_1), \ldots, q(b_{n_k}), q(\widetilde{b}_1), \ldots, q(\widetilde{b}_{n_k})) = q(P_k(b_1, \ldots, b_{n_k}, \widetilde{b}_1, \ldots, \widetilde{b}_{n_k}))$$

converge in $\mathcal{C}$. Since $q : \mathcal{B} \to \mathcal{C}$ is bicontinuous $P_k(b_1, \ldots, b_{n_k}, \widetilde{b}_1, \ldots, \widetilde{b}_{n_k}) \in \mathcal{B}$ converge to some element $y \in \mathcal{B}$. Clearly $x - y \in J$. Hence $y \in \mathcal{B} \setminus J$. Since $J \subseteq ker(\rho)$ and $x \in ker(\rho)$ we have $y \in ker(\rho)$ which contradicts to $y \in \mathcal{B}$. Thus $J = ker(\rho)$.

To prove the second statement of the theorem consider the canonical quotient map $q_{J'} : C^*_{max}(\mathcal{B}) \to C^*_{max}(\mathcal{B})/J'$ and let $x \in J' \setminus J$. Then x is a uniform limit of polynomials $P_k(b_1, \ldots, b_{n_k}, b_1^*, \ldots, b_{n_k}^*)$ and there are $\widetilde{b}_j \in \mathcal{B}$ such that $q(b_j^*) = q(\widetilde{b}_j)$. Then $q_{J'}(P_k(b_1, \ldots, b_{n_k}, b_1^*, \ldots, b_{n_k}^*)) = q_{J'}(P_k(b_1, \ldots, b_{n_k}, \widetilde{b}_1, \ldots, \widetilde{b}_{n_k}))$ uniformly converges to 0. Thus $P_k(b_1, \ldots, b_{n_k}, \widetilde{b}_1, \ldots, \widetilde{b}_{n_k})$ converges to 0 in $\mathcal{B}$. Since

$$q(P_k(b_1, \ldots, b_{n_k}, b_1^*, \ldots, b_{n_k}^*)) = q(P_k(b_1, \ldots, b_{n_k}, \widetilde{b}_1, \ldots, \widetilde{b}_{n_k}))$$

converges to $q(x)$ we have $q(x) = 0$ and consequently $x \in J$. □

Another way to make $\check{\pi}^{-1}$ extendable to a $*$-homomorphism from C^*-algebra generated by $\mathcal{B} = \check{\pi}(\mathcal{A})$ into $\mathcal{A}$ is the following. Since $\|a\| \leq \|\check{\pi}(a)\|$

the embedding of $\mathcal{B}$ into $\mathcal{B} \oplus \mathcal{A}$ via $i : \breve{\pi}(a) \mapsto \breve{\pi}(a) \oplus a$ is a completely isometric isomorphism. Let $\tau = (i \circ \breve{\pi})^{-1}$. Then $\tau(\breve{\pi}(a) \oplus a) = a$ and τ has a contractive extension, $\widetilde{\tau}$, to the C^*-algebra generated by $i(\mathcal{B})$, such that $\widetilde{\tau}(a_1 \oplus a_2) = a_2$ for every $a_1 \oplus a_2 \in C^*(i(\mathcal{B}))$. Since $\widetilde{\tau}$ is unital and contractive we have that $\widetilde{\tau}$ is a $*$-homomorphism.

Now we can summarize our observations in several reformulations of the Kadison's similarity problem.

Theorem 48. *The following are equivalent:*

(i) Kadison's conjecture has affirmative answer,

(ii) for every operator algebra $\mathcal{B}$ and every bounded homomorphism $\rho : C^*(\mathcal{B}) \to \mathcal{B}$, such that $\rho(b) = b$ for every $b \in \mathcal{B}$, ρ is completely bounded.

(iii) if $\mathcal{B}$ is an operator algebra and $\rho : C^*_{max}(\mathcal{B}) \to \mathcal{B}$ is bounded homomorphism such that $\rho(b) = b$ for every $b \in \mathcal{B}$ and the restriction of ρ to $\mathcal{B}^*$ is completely isometric then ρ is completely bounded.

(iv) if an operator algebra $\mathcal{B}$ is isomorphic to a C^*-algebra and $J \subset C^*(\mathcal{B})$ is a closed ideal and some $C > 0$ such that

$$C \cdot \|b\|_{C^*(\mathcal{B})} \leq \|b\|_{C^*(\mathcal{B})/J}$$

for every $b \in \mathcal{B}$, then J is a K-boundary ideal.

Proof. Evidently (i)⇒(ii) and (ii)⇒(iii).

(iii)⇒(i). Let $\pi : \mathcal{A} \to B(H)$ be a bounded injective homomorphism from C^*-algebra $\mathcal{A}$. By Theorem 46 and considerations preceeding Theorem 48 we have the bounded injective J-symmetric homomorphism $\breve{\pi} : \mathcal{A} \to B(H \oplus H)$ and completely isometric homomorphism $i : \breve{\pi}(\mathcal{A}) \to C^*_{max}(\breve{\pi}(\mathcal{A}))$. Let $\rho = i \circ \breve{\pi}$ and $\mathcal{B} = \rho(\mathcal{A})$. Then $\|\rho\|_{cb} \leq 1$ and $\rho^{-1} : \mathcal{B} \to \mathcal{A}$ extends to $*$-homomorphism $\tau : C^*_{max}(\mathcal{B}) \to \mathcal{A}$. Thus we have the bounded homomorphism

$$\rho \circ \tau : C^*_{max}(\mathcal{B}) \to \mathcal{B}$$

and $\rho \circ \tau(b) = b$ for every $b \in \mathcal{B}$. Consider the restriction of $\rho \circ \tau$ to B^*. Let $(b_{ij})_{i,j} \in M_n(\mathcal{B}^*)$ then $b_{ij} = \rho(a_{ij})^*$ for some $a_{ij} \in \mathcal{A}$. Since $\breve{\pi}$ is J-symmetric we have

$$\begin{aligned}
\|\rho \circ \tau|_{B^*}^{(n)}((\rho(a_{ij})^*)_{i,j})\| &= \|(\rho \circ \tau(\rho(a_{ij})^*))_{i,j}\| \\
&= \|(\rho(\tau(\rho(a_{ij}))^*))_{i,j}\| \\
&= \|(\rho(a_{ij}^*))_{i,j}\| = \|(\breve{\pi}(a_{ij}^*))_{i,j}\| \\
&= \|(J \otimes I_n)(\breve{\pi}(a_{ij})^*)_{i,j}(J \otimes I_n)\| \\
&= \|(\breve{\pi}(a_{ij})^*)_{i,j}\| = \|(\breve{\pi}(a_{ji}))_{i,j}\| \\
&= \|(\rho(a_{ji}))_{i,j}\| = \|(\rho(a_{ij})^*)_{i,j}\|.
\end{aligned}$$

Thus $\rho \circ \tau|_{B^*}$ is a complete isometry. By (iii) $\rho \circ \tau$ is completely bounded. Then $\mathcal{B}$ is similar to some C^*-algebra $\mathcal{C}$, i.e. there exists $S \in B(K)$ such that $\mathcal{B} = S\mathcal{C}S^{-1}$. Since $AdS \circ \rho : \mathcal{A} \to \mathcal{C}$ is a bounded isomorphism between two C^*-algebras by Gardner's theorem [32] we have that $AdS \circ \rho$ is similar to a $*$-homomorphism which proves similarity of ρ to $*$-homomorphism.

(iv)⇒(i). Let π be a bounded homomorphism of C^*-algebra $\mathcal{A}$. Then π is completely bounded if and only if $\ker(\tau)$ is K-boundary ideal for some $K > 0$, where $\ker(\tau)$ is the same as in the previos paragraph. Since $\ker(\tau)$ satisfies inequalities of (iv) we have that it is a K-boundary ideal. Thus π is completely bounded.

(i)⇒(iv). In [75] it was proved that if Kadison conjecture has affirmative answer then every Banach operator algebra which is isomorphic to a C^*-algebra λ-norms itself for some $\lambda > 0$. By Proposition 44 we have that ideal of (iv) is a K-boundary ideal. □

Remark 49. *By example 45 we have that in condition (iv) of Theorem 48 it is not enough to require $\mathcal{B}$ be isomorphic to a semi-simple Banach algebra.*

Question. Note that ρ in Theorem 48 (ii) is $\mathcal{B}$-bimodule map. Let $\mathcal{C}$ and $\mathcal{D}$ be C^*-algebras. In [87] Smith proved that if $\tau : \mathcal{C} \to B(H)$ is bounded $\mathcal{D}$-bimodule map and $\mathcal{D}$ has cyclic vector then τ is completely bounded and $\|\tau\|_{cb} = \|\tau\|$. Is Smith's theorem true if $\mathcal{D}$ is an operator algebra with a cyclic vector?

4.4 $*$-Double of $M_2(\mathbb{C})$ and the similarity conjecture.

In the previous section we showed that the similarity conjecture is true if and only if every bounded homomorphism from $C^*(\mathcal{B})$ onto $\mathcal{B}$ which is projection on $\mathcal{B}$ is completely bounded, see Theorem 48 (ii). It is sufficient to verify this condition on a certain example, see Corollary 53 and Proposition 54.

Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$ with a unit. There exists an associative algebra $\mathcal{A}^*$ and anti-isomorphism $\varphi : \mathcal{A} \to \mathcal{A}^*$, i.e. φ is a bijection and for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, $\varphi(ab) = \varphi(b)\varphi(a)$ and $\varphi(\lambda a) = \overline{\lambda} a$, see [78]. It is easy to see that $\mathcal{A}^*$ is unique up to isomorphism.

Definition 50. *The $*$-double of $\mathcal{A}$, denoted by $\mathcal{D}(\mathcal{A})$, is the free product $\mathcal{A} * \mathcal{A}^*$ with an involution defined on the generators by the rule $a^* = \varphi(a)$, $b^* = \varphi^{-1}(b)$ where $a \in \mathcal{A}$, $b \in \mathcal{A}^*$ and $\varphi : \mathcal{A} \to \mathcal{A}^*$ is a fixed anti-isomorphism.*

The $*$-double of $\mathcal{A}$ has the following universal property. There are injective homomorphisms $i : \mathcal{A} \to \mathcal{D}(\mathcal{A})$ and $j : \mathcal{A}^* \to \mathcal{D}(\mathcal{A})$ such that for any $*$-algebra $\mathcal{C}$ and any homomorphism $\pi : \mathcal{A} \to \mathcal{C}$ there exists a unique $*$-homomorphism $\hat{\pi} : \mathcal{D}(\mathcal{A}) \to \mathcal{C}$ such that the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{A} & \overset{i}{\hookrightarrow} & \mathcal{D}(\mathcal{A}) & \overset{j}{\hookleftarrow} & \mathcal{A}^* \\ & \searrow^{\pi} & \downarrow \hat{\pi} & \swarrow^{\pi^*} & \\ & & \mathcal{C} & & \end{array} \tag{4.4}$$

Let A be the D^*-double of $M_2(\mathbb{C})$. Thus as an associative algebra $\mathcal{D}(M_2)$ is just the free product $M_2(\mathbb{C}) * M_2(\mathbb{C})$. If $\mathcal{B}$ denotes the fist factor in the above free product then $\mathcal{B}^*$ is identified with the second one and A is generated by $\mathcal{B}$ as a $*$-algebra. The involution $\sharp$ on A maps a matrix $X \in \mathcal{B}$ to the adjoint matrix X^* considered as an element of $\mathcal{B}^*$.

Let $\mathcal{A}_k$ be a unital C^*-algebra which is the closure of the $*$-algebra $A_k = \langle a_1, \ldots, a_k : a_1^* = a_1, \ldots, a_k^* = a_k \rangle$ generated by a free k-tuple of self-adjoint generators with respect to the norm $\|a\| = \sup\limits_{\|\pi(a_i)\| \le 1, i=1..k} \|\pi(a)\|$, where supremum is taken over all $*$-representation of A_k such that $\|\pi(a_i)\| \le 1$ for every $i = 1..k$.

Let $\{e_{ij}\}_{i,j=1..2}$ be the matrix units of $M_2(\mathbb{C})$. Then a mapping τ into $M_2(\mathcal{A}_2)$ defined by

$$\tau(e_{11}) = \frac{1}{2} \begin{pmatrix} e - a_1 - ia_2 & -e - a_1 - ia_2 \\ -e + a_1 + ia_2 & e + a_1 + ia_2 \end{pmatrix}, \tau(e_{12}) = \frac{1}{2} \begin{pmatrix} e & e \\ -e & -e \end{pmatrix},$$

$$\tau(e_{21}) = \frac{1}{2} \begin{pmatrix} -(a_1 + ia_2)^2 & -(a_1 + ia_2)^2 - 2a_1 - 2ia_2 \\ (a_1 + ia_2)^2 - 2a_1 - 2ia_2 & (a_1 + ia_2)^2 \end{pmatrix},$$

$$\tau(e_{22}) = \frac{1}{2} \begin{pmatrix} a_1 + ia_2 & a_1 + ia_2 \\ -a_1 - ia_2 & -a_1 - ia_2 \end{pmatrix}.$$

extends to a $*$-homomorphism on $\mathcal{D}(M_2(\mathbb{C}))$. To see this one should only check the relations $\tau(e_{ij})\tau(e_{jk}) = \tau(e_{ik})$ which is straightforward. It easy to show that that $\tau(\mathcal{D}(M_2))$ is dense in $M_2(\mathcal{A}_2)$. Since the $*$-double of an operator algebra has a faithful $*$-representation in $B(H)$, see [78], there is a C^*-norm $\| \cdot \|_1$ on $\mathcal{D}(M_2)$. Let us endow $\mathcal{D}(M_2)$ with C^*-norm $\|a\| = \sup\{\|a\|_1, \|\tau(a)\|\}$ and denote the completion by $\mathcal{D}^*(M_2)$. Then τ extends to a $*$-homomorphism on $\mathcal{D}^*(M_2)$, since τ is contractive on $D^*(M_2)$. We will denote this extension also by τ.

The following proposition shows that the category of $*$-representations of $\mathcal{D}^*(M_2)$ is very complex. Such algebras are called $*$-wild, about other classes of $*$-wild algebras see [46], [59].

Proposition 51. *For any finitely generated C^*-algebra $\mathcal{C}$ there exists $n \in \mathbb{N}$ and a $*$-ideal $J \subset \mathcal{D}^*(M_2)$ such that $\mathcal{D}^*(M_2)/J \simeq M_{2n}(\mathcal{C})$.*

Proof. For any $m \in \mathbb{N}$ there exists a surjective $*$-homomorphism $\psi_m : \mathcal{A}_2 \to M_{m+2}(\mathcal{A}_m)$, see [59, Chapter 3]. If $\mathcal{C}$ is generated by m self-adjoint generators then, clearly, $\mathcal{C}$ is a quotient algebra of $\mathcal{A}_m$. Composing ψ_m with the $(m+2)$-amplification of the quotient map $q : \mathcal{A}_m \to \mathcal{C}$ we get a surjective $*$-homomorphism $\varphi : \mathcal{A}_2 \to M_{2(m+2)}(\mathcal{C})$, then $\varphi^{(2)} \circ \tau : \mathcal{D}^*(M_2) \to M_{4(m+2)}(\mathcal{C})$ is required surjection. □

Corollary 52. *If a C^*-algebra $\mathcal{C}$ is generated by m self-adjoint generators then $M_{4(m+2)}(\mathcal{C}) = C^*(\mathcal{N})$ for an operator subalgebra $\mathcal{N}$ that is completely bounded isomorphic to $M_2(\mathbb{C})$.*

Corollary 53. *The Kadison similarity conjecture is true iff every bounded homomorphism of $\mathcal{D}^*(M_2)$ is similar to a $*$-homomorphism.*

Proof. If Kadison's similarity conjecture is true then every bounded homomorphism of $\mathcal{D}^*(M_2)$ is, clearly, similar to a $*$-homomorphism.

Conversely, assume that every bounded homomorphism of $\mathcal{D}^*(M_2)$ is similar to a $*$-homomorphism. Then the same holds true for any finitely generated C^*-algebra. Moreover, there is a factorization pair (d, C) for all finitely generated C^*-algebras, see [73]. If $\mathcal{A}$ is arbitrary C^*-algebra and $n \in \mathbb{N}$ then every element x in $M_n(\mathcal{A})$ belongs to a finitely generated C^*-algebra and hence has a representation $\alpha_0 D_1 \alpha_1 \ldots D_d \alpha_d$ where α_j are matrices with scalar coefficients and D_j are diagonal matrices with coefficients in $\mathcal{A}$ such that $\prod_j \|\alpha_j\| \prod_j \|D_j\| \le C\|x\|$. Hence (d, C) is a factorization pair for $\mathcal{A}$. Since $\mathcal{A}$ is arbitrary Kadison's conjecture is true, see [73].

□

We will fix some faithful representation of $\mathcal{D}^*(M_2)$ on a separable Hilbert space H. Thus $\mathcal{B}$, defined after Definition 50, is identified with a closed subalgebra in $B(H)$. The algebra $\mathcal{B}$ is isomorphic to M_2 and $\mathcal{D}^*(M_2) \simeq C^*(M_2)$.

Proposition 54. *Let $\pi \colon \mathcal{D}(M_2) \to B(H)$ be a unital homomorphism. Then there exist invertible operators S and G in $B(H)$ s.t. denoting $\rho = S^{-1}\pi S$ we have for every $b \in \mathcal{B}$*

$$\rho(b) = b,$$
$$\rho(b^*) = G^{-1} b^* G.$$

Proof. Let $\xi \colon M_2 \to \mathcal{B}$ be an isomorphism. Then $\pi \circ \xi \colon M_2 \to \mathcal{B}$ is completely bounded and similar to a $*$-homomorphism. Hence there exists an invertible $R \in B(H)$ and $*$-isomorphism $\kappa : M_2 \to R^{-1}\mathcal{B}R$. Thus there exists a unitary $U \colon H \to \mathbb{C}^2 \otimes H$ s.t. $UR^{-1}\mathcal{B}RU^{-1} = M_2 \otimes I$. It follows that the mapping

$\psi\colon M_2 \to \mathcal{B}$ given by $\psi(a) = RU^{-1}(a \otimes I)UR^{-1}$ is an isomorphism. Since $\pi \circ \psi\colon M_2 \to B(H)$ is a bounded unital homomorphism of the nuclear C^*-algebra M_2 there is an invertible $T \in B(H)$ such that $T^{-1}\pi \circ \psi T$ is unital $*$-representation of M_2. Hence there exists a unitary $V\colon \mathbb{C}^2 \otimes H \to H$ s.t.

$$V^{-1}T^{-1}\pi \circ \psi TV = id \otimes I.$$

Thus for every $a \in M_2$ we have $\pi(RU^{-1}(a \otimes I)UR^{-1}) = TV(a \otimes I)V^{-1}T^{-1}$. Put $S = TVUR^{-1}$. Then $S^{-1}\pi(b)S = b$ for all $b \in \mathcal{B}$. Reasoning similarly we get that $\pi(b^*) = X^{-1}b^*X$ for some invertible $X \in B(H)$ and every $b \in \mathcal{B}$. Setting $G = XS$ we get the desired claim. □

Proposition 55. *Let $\pi\colon \mathcal{D}(M_2) \to B(H)$ be a unital $*$-representation on a separable Hilbert space H. Then there is a unitary $U\colon H \to \mathbb{C}^2 \otimes H$ and a positive invertible $C \in B(\mathbb{C}^2 \otimes H)$ s.t. $U\pi(\mathcal{D}(M_2))U^*$ is generated as a $*$-algebra by $C^{-1}(M_2 \otimes I)C$.*

Proof. In the proof of preceding proposition we obtain an invertible bounded $S\colon \mathbb{C}^2 \otimes H \to H$ s.t $\pi(\mathcal{B}) = S(M_2 \otimes I)S^{-1}$. Consider polar decomposition $S = WD$ where $D = (S^*S)^{1/2}$ is positive in $B(\mathbb{C}^2 \otimes H)$ and $W\colon \mathbb{C}^2 \otimes H \to H$ is unitary. Setting $C = D^{-1}$ and $U = W^{-1}$ we get $U\pi(\mathcal{B})U^* = C^{-1}(M_2 \otimes I)C$. Since $\mathcal{D}(M_2)$ is generated by $\mathcal{B}$ and $\mathcal{B}^*$ as an algebra and π is $*$-representation we get the claim. □

Corollary 56. *There is an invertible positive operator $C \in B(\mathbb{C}^2 \otimes H)$ s.t. C^*-algebra $\mathcal{D}^*(M_2)$ is $*$-isomorphic to the C^*-algebra generated by $C^{-1}(M_2 \otimes I)C$.*

We will identify $\mathcal{D}^*(M_2)$ with the C^*-algebra generated by $\mathcal{B} = C^{-1}(M_2 \otimes I)C$. Under this identification the dense $*$-subalgebra $\mathcal{D}(M_2)$ is identified with $*$-subalgebra generated by $C^{-1}(M_2 \otimes I)C$.

Theorem 57. *Every unital homomorphism $\pi\colon \mathcal{D}(M_2) \to B(H)$ is similar to a homomorphism τ_T given by*

$$\tau_T(C^{-1}(a \otimes I)C) = a,$$
$$\tau_T(C(a \otimes I)C^{-1}) = T^{-1}aT.$$

where $T \in B(\mathbb{C}^2 \otimes H)$. Moreover τ_T is similar to a $*$-representation iff there is an invertible $R \in B(H)$ s.t.

$$(I \otimes R)T \geq 0.$$

Proof. Every unital homomorphism is associated with some $G \in B(H)$ by Proposition 54. Conjugating it by C^{-1} we obtain the homomorphism τ_T where $T = C^{-1}GC^{-1}$.

Let $S \in B(\mathbb{C}^2 \otimes H)$ be invertible. Then $S^{-1}\tau_T S$ is a $*$-homomorphism iff $S^{-1}(T^{-1}(a^* \otimes I)T)S = (S^{-1}(a \otimes I)S)^*$ for all $a \in M_2$. The latter is equivalent to $T(SS^*) \in (M_2 \otimes I)' = I \otimes B(H)$. From this the claim follows easily. □

Corollary 58. *If $T \in B(\mathbb{C}^2 \otimes H)$ is unitary and $T \notin I \otimes B(H)$ then representation τ_T can not be extended to a bounded homomorphism of $\mathcal{D}^*(M_2)$.*

Proof. If T is unitary then the Banach algebra $\mathcal{C}$ generated by the union of $M_2 \otimes I$ and $T^{-1}(M_2 \otimes I)T$ is, clearly, a C^*-algebra.

Assume that τ_T extends to a bounded homomorphism of $\mathcal{D}^*(M_2)$ then the image of $\mathcal{D}^*(M_2)$ is closed, see [75]. Hence it is a C^*-algebra $\mathcal{C}$. The kernel of this homomorphism is a closed ideal of the C^*-algebra $\mathcal{D}^*(M_2)$ and thus the quotient algebra is a C^*-algebra. From this follows that τ_T induces an isomorphism between the two C^*-algebras. By Gardner's theorem this isomorphism is similar to a $*$-homomorphism which contradicts the criterium stated in Theorem 57.

□

Proposition 59. *If τ_T extends to a continuous homomorphism of $\mathcal{D}^*(M_2)$ then there is an invertible matrix $R \in \mathcal{D} \otimes B(H)$ where $\mathcal{D}$ is the subalgebra of diagonal matrices in M_2 s.t.*

$$RT \geq 0.$$

Proof. The $*$-subalgebra Y generated by $\mathcal{D}$ in $\mathcal{D}(M_2)$ is $*$-isomorphic to $\mathcal{D}(\mathcal{D})$ which is the same as a $*$-algebra generated by one idempotent

$$\langle q, q^* | q^2 = q, q^{*2} = q^* \rangle.$$

This algebra has only one-and two-dimensional irreducible representations. Thus C^*-algebra $\mathcal{Y}$ generated by Y is nuclear. Hence restriction of τ_T on $\mathcal{Y}$ is similar to $*$-representation. From this follows that there exists $S \in B(\mathbb{C}^2 \otimes H)$ s.t. $S^{-1}(\tau_T)|_{\mathcal{Y}} S$ is $*$-representation. As in the previous theorem we get that $T(SS^*) \in \mathcal{Y}'$. It is easy to see that $\mathcal{Y}' = \mathcal{D} \otimes B(H)$ which finishes the proof. □

Chapter 5

Operator multipliers.

5.1 Schur multipliers. Non-commutative generalization.

5.1.1 Basic notions and facts.

A bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is called a Schur multiplier if $(\varphi(i,j)a_{ij})$ is the matrix of a bounded linear operator on ℓ^2 whenever (a_{ij}) is such. The study of Schur multipliers was initiated by Schur in the early 20th century. A characterisation of these objects was given by A. Grothendieck in his *Résumé* [35], where he showed that Schur multipliers are precisely the functions φ of the form $\varphi(i,j) = \sum_{k=1}^{\infty} a_k(i)b_k(j)$, where $a_k, b_k : \mathbb{N} \to \mathbb{C}$ are such that $\sup_i \sum_{k=1}^{\infty} |a_k(i)|^2 < \infty$ and $\sup_j \sum_{k=1}^{\infty} |b_k(j)|^2 < \infty$. Schur multipliers have had many important applications in Analysis, see e.g. [4], [21] and [69]. One of the forms of the celebrated Grothendieck inequality can be given in terms of these objects [69].

A construction underlying many of the developments in Operator Space Theory is the Haagerup tensor product, as well as its weak counterpart, the weak* Haagerup tensor product [10] and its generalisation, the extended Haagerup tensor product [27]. Grothendieck's characterisation can be formulated by saying that the set of Schur multipliers coincides with the extended (or the weak*) Haagerup tensor product $\ell^\infty \otimes_{eh} \ell^\infty$ of the space ℓ^∞ of all bounded complex sequences, with itself.

Schur multipliers are elements of the commutative von Neumann algebra $\ell^\infty(\mathbb{N}\times\mathbb{N})$, or equivalently of the (von Neumann) tensor product of (the commutative von Neumann algebra) ℓ^∞ with itself. Subsequently, they form a commutative algebra themselves. Their quantisation was initiated by Kissin and Shulman in [52]. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras and π and ρ

their representations on H and K, respectively. The Hilbert space tensor product $H \otimes K$ can be naturally identified with the Hilbert space $\mathcal{C}_2(H^{\mathrm{d}}, K)$ of Hilbert-Schmidt operators from the dual H^{d} of H into K. It follows that π and ρ give rise to a representation $\sigma_{\pi,\rho}$ of the minimal tensor product $\mathcal{A} \otimes \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{C}_2(H^{\mathrm{d}}, K)$. Kissin and Shulman call an element $\varphi \in \mathcal{A} \otimes \mathcal{B}$ a π, ρ-multiplier if $\sigma_{\pi,\rho}(\varphi)$ is bounded in norm of $\mathcal{C}_2(H^{\mathrm{d}}, K)$ induced by its inclusion into the algebra $\mathcal{B}(H^{\mathrm{d}}, K)$ of all bounded operators from H^{d} into K. In [52], they study two sets of problems: the dependence of π, ρ-multipliers on π and ρ and the description of the norm of an operator multiplier. Most of their results are established in the more general setting of symmetrically normed ideals.

Assume that $\mathcal{A}$ and $\mathcal{B}$ are commutative, say $\mathcal{A} = C_0(X)$ and $\mathcal{B} = C_0(Y)$, for some locally compact Hausdorff spaces X and Y, and that the representations π and ρ arise from some spectral measures on X and Y. The notion of a π, ρ-multiplier is in this case closely related to double operator integrals. The theory of these integrals was developed by Birman and Solomyak [5, 6, 7, 8] in connection with various problems of Mathematical Physics and in particular of Perturbation Theory. If $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are spectral measures on Hilbert spaces H and K, they defined the double operator integral

$$I_\psi(T) = \int_{X \times Y} \psi(x, y)\, d\mathcal{E}(x) T\, d\mathcal{F}(y)$$

for every bounded measurable function ψ and every operator T from the Hilbert-Schmidt class $\mathcal{C}_2(H, K)$. A function ψ is called a Schur multiplier with respect to $\mathcal{E}$ and $\mathcal{F}$ if I_ψ can be extended to a bounded linear transformer on the space $(\mathcal{B}(H, K), \|\cdot\|_{\mathrm{op}})$ of bounded operators from H to K, that is, if there exists $C > 0$ such that $\|I_\psi(T)\|_{\mathrm{op}} \leq C\|T\|_{\mathrm{op}}$ for all $T \in \mathcal{C}_2(H, K)$. Peller [66] (see also [41]) characterised Schur multipliers with respect to $\mathcal{E}$ and $\mathcal{F}$ in several ways. In particular, he showed that the space of Schur multipliers with respect to $\mathcal{E}$ and $\mathcal{F}$ coincides with the extended Haagerup tensor product $L^\infty(X) \otimes_{eh} L^\infty(Y)$ and the integral projective tensor product $L^\infty(X) \hat{\otimes}_i L^\infty(Y)$.

Several attempts were made to generalise the Birman-Solomyak theory to the case of multiple operator integrals [65, 91, 89]. Such integrals appear, for instance, in the study of differentiability of functions of operators depending on a parameter. A recent definition of multiple operator integrals by Peller in [68] is based on the integral projective tensor product. For some fixed spectral measures $(X_1, \mathcal{E}_1), \ldots, (X_n, \mathcal{E}_n)$ on Hilbert spaces $H_1, \ldots, H_n$, he defines

$$I_\psi(T_1, \ldots, T_{n-1}) = \int_{X_1 \times \ldots \times X_n} \psi(x_1, \ldots, x_n)\, d\mathcal{E}_1(x_1) T_1\, d\mathcal{E}_2(x_2) \ldots T_{n-1}\, d\mathcal{E}_n(x_n),$$

where $\psi \in L^\infty(X_1)\hat{\otimes}_i \ldots \hat{\otimes}_i L^\infty(X_n)$ and $T_1, \ldots, T_{n-1}$ are bounded linear operators, and shows that

$$\|I_\psi(T_1, \ldots, T_{n-1})\|_{\text{op}} \le \|\psi\|_i \|T_1\|_{\text{op}} \ldots \|T_{n-1}\|_{\text{op}},$$

where $\|\psi\|_i$ denotes the integral projective tensor norm of ψ. If the spectral measures are multiplicity free and $T_1, \ldots, T_{n-1}$ are Hilbert-Schmidt operators with kernels $f_1, \ldots, f_{n-1}$, respectively, then $I_\psi(T_1, \ldots, T_{n-1})$ is a Hilbert-Schmidt operator with kernel $S_\psi(f_1, \ldots, f_{n-1}) \in L^2(X_1 \times X_n)$ equal to

$$\int_{X_2 \times \ldots \times X_{n-1}} \psi(x_1, \ldots, x_n) f_1(x_1, x_2) \ldots f_{n-1}(x_{n-1}, x_n)\, d\mathcal{E}_2(x_2) \ldots d\mathcal{E}_{n-1}(x_{n-1}). \tag{5.1}$$

This was the starting point for our definition of multidimensional Schur multipliers in Section 5.1.3. Let (X_i, μ_i), $i = 1, \ldots, n$, be standard σ-finite measure spaces and $\Gamma(X_1, \ldots, X_n) = L^2(X_1 \times X_2) \odot L^2(X_2 \times X_3) \odot \ldots \odot L^2(X_{n-1} \times X_n)$ be the algebraic tensor product of the corresponding L^2-spaces equipped with the projective tensor norm, where each of the L^2-spaces is equipped with its L^2-norm. An element $\psi \in L^\infty(X_1 \times \cdots \times X_n)$ determines a bounded linear map S_ψ from $\Gamma(X_1, \ldots, X_n)$ to $L^2(X_1, X_n)$ given on elementary tensors $f_1 \otimes \ldots \otimes f_n \in \Gamma(X_1, \ldots, X_n)$ by (5.1) (where the integration is now with respect to μ_i instead of $\mathcal{E}_i$). On the other hand, for any measure spaces (X, μ) and (Y, ν), the space $L^2(X \times Y)$ can be identified with the class of all Hilbert-Schmidt operators from $L^2(X)$ to $L^2(Y)$; to each $f \in L^2(X \times Y)$ there corresponds the operator T_f given by $T_f\xi(y) = \int_X f(x, y)\xi(x) d\mu(x)$, $\xi \in L^2(X)$. Using this identification, one can equip the space $L^2(X \times Y)$ with the opposite operator space structure arising from the inclusion of $L^2(X \times Y)$ into $\mathcal{B}(L^2(X), L^2(Y))$. We further equip $\Gamma(X_1, \ldots, X_n)$ with the Haagerup tensor norm $\|\cdot\|_{\text{h}}$, where the L^2-spaces are given their opposite operator space structure described above, and say that an element $\psi \in L^\infty(X_1 \times \ldots \times X_n)$ is a Schur multiplier (with respect to $\mu_1, \ldots, \mu_n$) if there exists $C > 0$ such that

$$\|S_\psi(\Phi)\|_{\text{op}} \le C\|\Phi\|_{\text{h}}, \text{ for all } \Phi \in \Gamma(X_1, \ldots, X_n). \tag{5.2}$$

Using a generalisation of a result of Smith [87] on the complete boundedness of certain bounded bimodule maps to the case of multilinear modular maps, we obtain a characterisation of multidimensional Schur multipliers as elements of the extended Haagerup tensor product $L^\infty(X_1) \otimes_{eh} \ldots \otimes_{eh} L^\infty(X_n)$ (Theorem 63). This generalises Grothendieck's and Peller's characterisations in the case $n = 2$. We show that the integral projective tensor product consists of multipliers and, therefore, $L^\infty(X_1)\hat{\otimes}_i \ldots \hat{\otimes}_i L^\infty(X_n) \subset L^\infty(X_1) \otimes_{eh} \ldots \otimes_{eh} L^\infty(X_n)$. The converse inclusion is true in the case $n = 2$ [66] but remains an open problem for $n > 2$.

In Section 5.1.4 we consider a non-commutative version of multidimensional multipliers following the Kissin-Shulman approach in the two dimensional case. We replace the functions ψ by elements of the minimal tensor product $\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$ of some given C*-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and the measure μ_i by a representation π_i of $\mathcal{A}_i$. We thus obtain a class of operator $\pi_1, \ldots, \pi_n$-multipliers. If each $\mathcal{A}_i$ is a commutative C*-algebra, say $\mathcal{A}_i = C_0(X_i)$ for some locally compact Hausdorff space X_i, and $\pi_i(f)$ is the operator of multiplication by $f \in C_0(X)$ acting on $L^2(X_i, \mu_i)$, then ψ is a $\pi_1, \ldots, \pi_n$-multiplier if and only if ψ is a Schur multiplier with respect to $\mu_1, \ldots, \mu_n$ (Proposition 71). As in the two-dimensional case, we show that the set of $\pi_1, \ldots, \pi_n$-multipliers does not change if we replace each π_i by an approximately equivalent representation (Theorem 74). A consequence of this result is the fact that the class of continuous (multidimensional) Schur multipliers depends only on the supports of the measures μ_i.

In Section 5.3 we study universal mutlipliers, that is, the elements of $\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$ which are $\pi_1, \ldots, \pi_n$-multipliers for all representations π_i of $\mathcal{A}_i$, $i = 1, \ldots, n$. We characterise such multipliers as the elements of a certain weak completion of the algebraic tensor product $\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n$ (Theorem 85). In the case where the C*-algebras are commutative and $n = 2$ this was proved in [52]; the case of arbitrary C*-algebras was left as a conjecture. Our result may be thought of as a non-commutative and multidimensional version of Grothendieck's and Peller's characterisations of Schur multipliers. The key ingredient in the proof is the observation that a universal multiplier determines a completely bounded multilinear modular map from the Cartesian product of the C*-algebras of compact operators into the C*-algebra of compact operators which allows us to use a result by Christensen and Sinclair [18] providing a description of all such mappings.

5.1.2 Non-commutative Schur multipliers.

In this section we collect some preliminary notions and results which will be needed in the sequel.

Let H be a Hilbert space. The dual space H^{d} of H is a Hilbert space and there exists an anti-isometry $\partial : H \to H^{\mathrm{d}}$ given by $\partial(x)(y) = (y, x)$, $x, y \in H$. We set $x^{\mathrm{d}} = \partial(x)$.

If H and K are Hilbert spaces, we let $\mathcal{B}(H, K)$ be the space of all bounded linear operators from H into K, and $\|\cdot\|_{\mathrm{op}}$ be the usual operator norm on $\mathcal{B}(H, K)$. We let $\mathcal{K}(H, K)$ be the subspace of all compact operators, and $\mathcal{C}_2(H, K)$ be the subspace of all Hilbert-Schmidt operators, from H into K. For each $T \in \mathcal{C}_2(H, K)$, we denote by $\|T\|_2$ the Hilbert-Schmidt norm of T. The space $\mathcal{C}_2(H, K)$ is a Hilbert space with respect to the inner product

$(T,S) = \mathrm{tr}(TS^*)$, where S^* denotes the adjoint of the operator S. We let $\mathcal{B}(H) = \mathcal{B}(H,H)$, $\mathcal{K}(H) = \mathcal{K}(H,H)$ and $\mathcal{C}_2(H) = \mathcal{C}_2(H,H)$.

If $T \in \mathcal{B}(H,K)$ we denote by $T^{\mathrm{d}} \in B(K^{\mathrm{d}}, H^{\mathrm{d}})$ the conjugate of T. We have that $\|T^{\mathrm{d}}\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}$ and $T^{\mathrm{d}}x^{\mathrm{d}} = (T^*x)^{\mathrm{d}}$, whenever $x \in H_2$. Another way of expressing the last identity is

$$T^{\mathrm{d}} = \partial T^* \partial^{-1}. \tag{5.3}$$

We also have

$$(T^*)^{\mathrm{d}} = (T^{\mathrm{d}})^* \;\text{ and }\; (\lambda T)^{\mathrm{d}} = \lambda T^{\mathrm{d}}, \quad \lambda \in \mathbb{C}. \tag{5.4}$$

We let $H \otimes K$ be the Hilbert space tensor product of H and K. There exists a unitary operator $\theta : H \otimes K \to \mathcal{C}_2(H^{\mathrm{d}}, K)$ given on elementary tensors $x \otimes y \in H \otimes K$ by

$$\theta(x \otimes y)(z^{\mathrm{d}}) = (x,z)y, \quad z^{\mathrm{d}} \in H^{\mathrm{d}}.$$

If $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$, $x \in H$ and $y \in K$, we have that $\theta((A\otimes B)(x\otimes y)) = B\theta(x\otimes y)A^{\mathrm{d}}$, and hence

$$\theta((A\otimes B)\xi) = B\theta(\xi)A^{\mathrm{d}} \quad \text{for all } \xi \in H\otimes K. \tag{5.5}$$

If $\varphi \in \mathcal{B}(H \otimes K)$, let $\sigma(\varphi) \in \mathcal{B}(\mathcal{C}_2(H^{\mathrm{d}}, K))$ be given by the formula

$$\sigma(\varphi)\theta(\xi) = \theta(\varphi\xi), \quad \xi \in H \otimes K.$$

Then σ implements a unitary equivalence between $\mathcal{B}(H \otimes K)$ and $\mathcal{B}(\mathcal{C}_2\,(H^{\mathrm{d}}, K))$. We will call an element $\varphi \in \mathcal{B}(H\otimes K)$ a concrete (operator) multiplier if there exists $C > 0$ such that $\|\sigma(\varphi)T\|_{\mathrm{op}} \leq C\|T\|_{\mathrm{op}}$, for each $T \in \mathcal{C}_2(H^{\mathrm{d}}, K)$. Suppose that $H = l^2(X)$, $K = l^2(Y)$ for some sets X and Y and φ is the operator on $H \otimes K = \ell^2(X \times Y)$ of multiplication by a function $\varphi \in \ell^\infty(X \times Y)$. The concrete operator multipliers of this form are precisely the classical Schur multipliers on $X \times Y$ (see e.g. [69]).

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. We denote by $\mathcal{A} \otimes \mathcal{B}$ the minimal tensor product of $\mathcal{A}$ and $\mathcal{B}$. Let $\pi : \mathcal{A} \to \mathcal{B}(H)$ (resp. $\rho : \mathcal{B} \to \mathcal{B}(K)$) be a representation of $\mathcal{A}$ (resp. $\mathcal{B}$). Then $\pi \otimes \rho : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}(H \otimes K)$, given on elementary tensors by $(\pi \otimes \rho)(a \otimes b) = \pi(a) \otimes \rho(b)$, is a representation of $\mathcal{A} \otimes \mathcal{B}$. Let $\sigma_{\pi,\rho} = \sigma \circ (\pi \otimes \rho)$; clearly, $\sigma_{\pi,\rho}$ is a representation of $\mathcal{A} \otimes \mathcal{B}$ on $\mathcal{C}_2(H^{\mathrm{d}}, K)$, unitarily equivalent to $\pi \otimes \rho$. We moreover have

$$\sigma_{\pi,\rho}(a \otimes b)T = \rho(b)T\pi(a)^{\mathrm{d}}, \quad a \in \mathcal{A}, b \in \mathcal{B}, T \in \mathcal{C}_2(H^{\mathrm{d}}, K).$$

An element $\varphi \in \mathcal{A} \otimes \mathcal{B}$ is called a π, ρ-multiplier [52] if there exists $C > 0$ such that

$$\|\sigma_{\pi,\rho}(\varphi)T\|_{\mathrm{op}} \leq C\|T\|_{\mathrm{op}}, \quad \text{for each } T \in \mathcal{C}_2(H^{\mathrm{d}}, K), \tag{5.6}$$

in other words, if $(\pi \otimes \rho)(\varphi)$ is a concrete operator multiplier. The set of all π, ρ-multipliers in $\mathcal{A} \otimes \mathcal{B}$ is denoted by $\mathbf{M}_{\pi,\rho}(\mathcal{A}, \mathcal{B})$, and the smallest constant C appearing in (5.6) is denoted by $\|\varphi\|_{\pi,\rho}$. If φ is a π, ρ-multiplier for all representations π of $\mathcal{A}$ and ρ of $\mathcal{B}$ then φ is called a universal multiplier. The set of all universal multipliers is denoted by $\mathbf{M}(\mathcal{A}, \mathcal{B})$; if $\varphi \in \mathbf{M}(\mathcal{A}, \mathcal{B})$ we let $\|\varphi\|_{\mathrm{univ}} = \sup_{\pi,\rho} \|\varphi\|_{\pi,\rho}$. It is not difficult to see that in this case $\|\varphi\|_{\mathrm{univ}} < \infty$ [52].

We now recall some notions from Operator Space Theory. We refer the reader to [9], [28] and [62] for more details. An operator space $\mathcal{E}$ is a closed subspace of $\mathcal{B}(H, K)$, for some Hilbert spaces H and K. If $n, m \in \mathbb{N}$, by $M_{n,m}(\mathcal{E})$ we will denote the space of all n by m matrices with entries in $\mathcal{E}$ and let $M_n(\mathcal{E}) = M_{n,n}(\mathcal{E})$. Note that $M_{n,m}(\mathcal{E})$ can be identified in a natural way with a subspace of $\mathcal{B}(H^m, K^n)$ and hence carries a natural operator norm. If $n = \infty$ or $m = \infty$, we will denote by $M_{n,m}(\mathcal{E})$ the space of all (singly or doubly infinite) matrices with entries in $\mathcal{E}$ which represent a bounded linear operator between the corresponding amplifications of the Hilbert spaces and set $M_\infty(\mathcal{E}) = M_{\infty,\infty}(\mathcal{E})$. We also write $M_{n,m} = M_{n,m}(\mathbb{C})$ and $M_\infty = M_{\infty,\infty}(\mathbb{C})$. If $a = (a_{ij}) \in M_{n,m}(\mathcal{E})$, where $a_{ij} \in \mathcal{E}$, we let $a^{\mathrm{d}} = (a_{ij}^{\mathrm{d}})$; thus $a^{\mathrm{d}} \in \mathcal{B}(K^{\mathrm{d},m}, H^{\mathrm{d},n})$. We also let $a^{\mathrm{t}} = (a_{ji}) \in M_{m,n}(\mathcal{E})$; thus $a^{\mathrm{t}} \in \mathcal{B}(H^n, K^m)$. We have $\|a^{\mathrm{d}}\|_{\mathrm{op}} = \|a^{\mathrm{t}}\|_{\mathrm{op}}$ and $\|a^{\mathrm{d,t}}\|_{\mathrm{op}} = \|a\|_{\mathrm{op}}$. The opposite operator space $\mathcal{E}^o$ of the operator space $\mathcal{E}$ is defined as follows: if $\mathcal{E} \subseteq \mathcal{B}(H, K)$ then $\mathcal{E}^o = \{x^{\mathrm{d}} : x \in \mathcal{E}\} \subseteq \mathcal{B}(K^{\mathrm{d}}, H^{\mathrm{d}})$.

If $\mathcal{E}$ and $\mathcal{F}$ are operator spaces, a linear map $\Phi : \mathcal{E} \to \mathcal{F}$ is called completely bounded if the map $\Phi^{(k)} : M_k(\mathcal{E}) \to M_k(\mathcal{F})$, given by $\Phi^{(k)}((a_{ij})) = (\Phi(a_{ij}))$, is bounded for each $k \in \mathbb{N}$ and $\|\Phi\|_{\mathrm{cb}} \stackrel{def}{=} \sup_k \|\Phi^{(k)}\| < \infty$.

Let $\mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_n$ be operator spaces. We denote by $\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n$ the algebraic tensor product of $\mathcal{E}_1, \ldots, \mathcal{E}_n$. Let $a_k = (a_{ij}^k) \in M_{m_k,m_{k+1}}(\mathcal{E}_k)$, $k = 1, \ldots, n$. We denote by

$$a^1 \odot \cdots \odot a^n \in M_{m_1,m_{n+1}}(\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n) \tag{5.7}$$

the matrix whose i, j-entry is

$$\sum_{i_2,\ldots,i_n} a_{i,i_2}^1 \otimes a_{i_2,i_3}^2 \otimes \cdots \otimes a_{i_n,j}^n. \tag{5.8}$$

Let $\Phi : \mathcal{E}_1 \times \cdots \times \mathcal{E}_n \to \mathcal{E}$ be a multilinear map and

$$\Phi^{(m)} : M_m(\mathcal{E}_1) \times M_m(\mathcal{E}_2) \times \cdots \times M_m(\mathcal{E}_n) \to M_m(\mathcal{E})$$

be the multilinear map given by

$$\Phi^{(m)}(a^1, \ldots, a^n)_{ij} = \sum_{i_2,\ldots,i_n} \Phi(a^1_{i,i_2}, a^2_{i_2,i_3}, \ldots, a^n_{i_n,j}), \tag{5.9}$$

where $a^k = (a^k_{ij}) \in M_m(\mathcal{E}_k)$, $1 \le i, j \le m$. The map Φ is called completely bounded if there exists $C > 0$ such that for all $m \in \mathbb{N}$ and all elements $a^k \in M_m(\mathcal{E}_k)$, $k = 1, \ldots, n$, we have

$$\|\Phi^{(m)}(a^1, \ldots, a^n)\| \le C\|a^1\| \ldots \|a^n\|.$$

Every completely bounded multilinear map $\Phi : \mathcal{E}_1 \times \cdots \times \mathcal{E}_n \to \mathcal{E}$ gives rise to a completely bounded linear map from the Haagerup tensor product $\mathcal{E}_1 \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{E}_n$ into $\mathcal{E}$. For details on the Haagerup tensor product we refer the reader to [28].

If $R_1, \ldots, R_{n+1}$ are rings, M_i is a R_i-left and R_{i+1}-right module for each $i = 1, \ldots, n$, and M is an R_1, R_{n+1}-module, a multilinear map $\Phi : M_1 \times \cdots \times M_n \to M$ will be called $R_1, \ldots, R_{n+1}$-modular (or simply modular if $R_1, \ldots, R_{n+1}$ are clear from the context) if

$$\Phi(a_1 m_1 a_2, m_2 a_3, m_3 a_4, \ldots, m_n a_{n+1}) = a_1 \Phi(m_1, a_2 m_2, a_3 m_3, \ldots, a_n m_n) a_{n+1},$$

for all $m_i \in M_i$ $(i = 1, \ldots, n)$ and $a_j \in R_j$ $(j = 1, \ldots, n+1)$. If $R_i = \mathcal{A}_i$ are C*-algebras and $M_i = \mathcal{E}_i$ are operator spaces, we let $\mathcal{B}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{E})$ (resp. $CB_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{E})$) denote the spaces of all bounded (resp. completely bounded) $\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}$-modular maps from $\mathcal{E}_1 \times \cdots \times \mathcal{E}_n$ into $\mathcal{E}$.

5.1.3 Multidimensional Schur multipliers.

In this section, we define multidimensional Schur multipliers on the direct product of finitely many measure spaces. The main result of the section is Theorem 63 which characterises multidimensional Schur multipliers generalising the results of Peller [66] and Spronk [90].

Let (X_i, μ_i), $i = 1, 2, \ldots, n$, be standard σ-finite measure spaces. For notational convenience, integration with respect to μ_i will be denoted by dx_i. Direct products of the form $X_{i_1} \times \cdots \times X_{i_k}$ will be equipped with the corresponding product measure. We equip the space $L^2(X_1 \times X_2)$ with an $L^\infty(X_1), L^\infty(X_2)$-module action by letting $(a\xi b)(x, y) = a(x)\xi(x, y)b(y)$. We will denote by M_a the operator of multiplication by the essentially bounded function a acting on the corresponding L^2-space.

Theorem 60. *A multilinear map*

$$S : L^2(X_1 \times X_2) \times L^2(X_2 \times X_3) \times \cdots \times L^2(X_{n-1} \times X_n) \to L^2(X_1 \times X_n)$$

is a bounded modular map if and only if there exists $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$ *such that* $S = S_\varphi$ *where* $S_\varphi(f_1, \ldots, f_{n-1})(x_1, x_n)$ *is defined as*

$$\int_{X_2\times\cdots\times X_{n-1}} \varphi(x_1, \ldots, x_n) f_1(x_1, x_2) f_2(x_2, x_3) \ldots f_{n-1}(x_{n-1}, x_n) dx_2 \ldots dx_{n-1}.$$

Moreover, $\|S_\varphi\| = \|\varphi\|_\infty$.

Proof. We first show that for each φ, the map S_φ is a bounded modular map with norm not exceeding $\|\varphi\|_\infty$. For simplicity, we will assume in this part of the proof that $n = 3$. Fix φ, f_1 and f_2. We have

$$\begin{aligned}
& \|S_\varphi(f_1, f_2)\|_2^2 \\
\leq \ & \int_{X_1\times X_3} \left(\int |\varphi(x_1, x_2, x_3) f_1(x_1, x_2) f_2(x_2, x_3)| dx_2 \right)^2 dx_1 dx_3 \\
\leq \ & \|\varphi\|_\infty^2 \int_{X_1\times X_3} \left(\int |f_1(x_1, x_2) f_2(x_2, x_3)| dx_2 \right)^2 dx_1 dx_3 \\
\leq \ & \|\varphi\|_\infty^2 \int_{X_1\times X_3} \left(\int |f_1(x_1, x_2)|^2 dx_2 \right) \left(\int |f_2(x_2, x_2)|^2 dx_2 \right) dx_1 dx_3 \\
= \ & \|\varphi\|_\infty^2 \|f_1\|_2^2 \|f_2\|_2^2.
\end{aligned}$$

Thus, φ is bounded with $\|S_\varphi\| \leq \|\varphi\|_\infty$; the modularity of S_φ is obvious.

Conversely, let

$$S : L^2(X_1 \times X_2) \times L^2(X_2 \times X_3) \times \cdots \times L^2(X_{n-1} \times X_n) \to L^2(X_1 \times X_n)$$

be a bounded modular map. We first assume that the measures μ_i are finite. Write $K_1 = L^2(X_1 \times X_n)$ and let

$$S_1 : L^2(X_2) \times L^2(X_2) \times L^2(X_3) \times L^2(X_3) \times \cdots \times L^2(X_{n-1}) \times L^2(X_{n-1}) \to K_1$$

be given by

$$S_1(\xi_2, \eta_2, \xi_3, \eta_3, \ldots, \xi_{n-1}, \eta_{n-1}) = S(1 \otimes \xi_2, \eta_2 \otimes \xi_3, \ldots, \eta_{n-1} \otimes 1)$$

(here and in the sequel we denote by 1 the constant function taking value one). The fact that S is modular implies that

$$S_1(\xi_2 a_2, \eta_2, \xi_3 a_3, \ldots, \xi_{n-1} a_{n-1}, \eta_{n-1}) = S_1(\xi_2, a_2\eta_2, \xi_3, \ldots, a_{n-1}\eta_{n-1}),$$

whenever $a_i \in L^\infty(X_i)$, $i = 2, \ldots, n-1$. For fixed $\xi_3, \eta_3, \ldots, \xi_{n-1}, \eta_{n-1}$, let $S_2 : L^2(X_2) \times L^2(X_2) \to K_1$ be given by

$$S_2(\xi_2, \eta_2) = S_1(\xi_2, \eta_2, \xi_3, \eta_3, \ldots, \xi_{n-1}, \eta_{n-1}).$$

For $h \in K_1$, let $S_2^h : L^2(X_2) \times L^2(X_2) \to \mathbb{C}$ be defined by $S_2^h(\xi_2, \eta_2) = (S_2(\xi_2, \eta_2), h)$. Clearly,

$$|S_2^h(\xi_2, \eta_2)| \leq \|h\| \|S\| \prod_{i=2}^{n-1} \|\xi_i\| \|\eta_i\|.$$

Hence there exists a bounded operator $T_2^h : L^2(X_2) \to L^2(X_2)$ such that $S_2^h(\xi_2, \eta_2) = (T_2^h \xi_2, \overline{\eta_2})$, for all $\xi_2, \eta_2 \in L^2(X_2)$ and $\|T_2^h\| \leq \|h\| \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|$. For each $a \in L^\infty(X_2)$ and $\xi_2, \eta_2 \in L^2(X_2)$, we have that

$$\begin{aligned}(T_2^h M_a \xi_2, \overline{\eta_2}) &= S_2^h(a\xi_2, \eta_2) = S_2^h(\xi_2, a\eta_2) \\ &= (T_2^h \xi_2, \overline{a\eta_2}) = (T_2^h \xi_2, M_{\overline{a}} \overline{\eta_2}) = (M_a T_2^h \xi_2, \overline{\eta_2}).\end{aligned}$$

Thus, there exists $\varphi_2^h \in L^\infty(X_2)$ such that $T_2^h = M_{\varphi_2^h}$. Moreover,

$$\|\varphi_2^h\|_\infty \leq \|h\| \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|.$$

For each $f \in L^1(X_2)$, the functional on K_1 given by $h \to \int_{X_2} f(x_2) \varphi_2^h(x_2) dx_2$ is conjugate linear and bounded with norm not exceeding $\|f\|_1 \ \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|$. Hence, there exists $\Phi_2(f) \in K_1$ such that

$$(\Phi_2(f), h) = \int_{X_2} f(x_2) \varphi_2^h(x_2) dx_2,$$

and $\|\Phi_2(f)\|_{K_1} \leq \|f\|_1 \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|$. Thus, the mapping $\Phi_2 : L^1(X_2) \to K_1$ is bounded and $\|\Phi_2\| \leq \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|$. Since Hilbert spaces possess Radon-Nikodym property, the vector valued Riesz Representation Theorem [24, Theorem 5, p. 63] implies that there exists $\varphi_2 \in L^\infty(X_2, K_1)$ ($L^\infty(X_2, K_1)$ being the space of essentially bounded K_1-valued measurable functions on X_2) such that

$$\Phi_2(f) = \int_{X_2} f(x_2) \varphi_2(x_2) dx_2,$$

where the integral is in Bochner's sense. Moreover,

$$\|\varphi_2\|_{L^\infty(X_2, K_1)} = \operatorname*{esssup}_{x_2 \in X_2} \|\varphi_2(x_2)\|_{K_1} = \|\Phi_2\| \leq \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|.$$

For $\xi_2, \eta_2 \in L^2(X_2)$, we have that $\xi_2 \overline{\eta_2} \in L^1(X_2)$ and hence

$$\begin{aligned}(S_2(\xi_2, \eta_2), h) &= (T_2^h \xi_2, \overline{\eta_2}) = \int_{X_2} \varphi_2^h(x_2) \xi_2(x_2) \eta_2(x_2) dx_2 \\ &= \left(\int_{X_2} \varphi_2(x_2) \xi_2(x_2) \eta_2(x_2) dx_2, h \right);\end{aligned}$$

in other words,

$$S_2(\xi_2,\eta_2) = \int_{X_2} \varphi_2(x_2)\xi_2(x_2)\eta_2(x_2)dx_2,$$

where the integral is in Bochner's sense.

We consider φ_2 as a function on $X_1 \times X_2 \times X_n$ by letting $\varphi_2(x_1,x_2,x_n) = \varphi_2(x_2)(x_1,x_n)$. Note that φ_2 depends on $\xi_3,\eta_3,\dots,\xi_{n-1},\eta_{n-1}$; we denote this dependence by $\varphi_2 = \varphi_{2,\xi_3,\eta_3,\dots,\xi_{n-1},\eta_{n-1}}$.

Let $K_2 = L^2(X_1 \times X_2 \times X_n)$. We have

$$\begin{aligned} \|\varphi_2\|_{K_2} &= \int_{X_2}\int_{X_1\times X_n} |\varphi_2(x_2)(x_1,x_n)|^2 dx_1 dx_n dx_2 = \int_{X_2} \|\varphi_2(x_2)\|^2_{K_1} dx_2 \\ &\le \mu_2(X_2)\|\varphi_2\|_{L^\infty(X_2,K_1)}. \end{aligned}$$

It follows that the mapping $S_3 : L^2(X_3) \times L^2(X_3) \to K_2$ given by

$$S_3(\xi_3,\eta_3) = \varphi_{2,\xi_3,\eta_3,\dots,\xi_{n-1},\eta_{n-1}}$$

is well-defined and

$$\|S_3(\xi_3,\eta_3)\|_{K_2} \le \mu_2(X_2)\|S\| \prod_{i=3}^{n-1} \|\xi_i\|\|\eta_i\|.$$

Hence, S_3 is bounded and $\|S_3\| \le \mu_2(X_2)\|S\| \prod_{i=4}^{n-1} \|\xi_i\|\|\eta_i\|$. An argument similar to the above implies the existence of $\varphi_3 \in L^\infty(X_3,K_2)$ with

$$\|\varphi_3\|_{L^\infty(X_3,K_2)} \le \mu_2(X_2)\|S\| \prod_{i=4}^{n-1} \|\xi_i\|\|\eta_i\|$$

such that

$$S_3(\xi_3,\eta_3) = \int_{X_3} \varphi_3(x_3)\xi_3(x_3)\eta_3(x_3)dx_3,$$

where the integral is in Bochner's sense. We may consider φ_3 as a function on $X_1 \times X_2 \times X_3 \times X_n$ by letting $\varphi_3(x_1,x_2,x_3,x_n) = \varphi_3(x_3)(x_1,x_2,x_n)$. We express the dependence of φ_3 on $\xi_4,\dots,\eta_{n-1}$ by writing $\varphi_3 = \varphi_{3,\xi_4,\dots,\eta_{n-1}}$. We have that

$$\begin{aligned} &S_1(\xi_2,\eta_2,\dots,\xi_{n-1},\eta_{n-1}) = \\ &\int_{X_2}\int_{X_3} \varphi_{3,\xi_4,\dots,\eta_{n-1}}(x_1,x_2,x_3,x_n)\xi_2(x_2)\eta_2(x_2)\xi_3(x_3)\eta_3(x_3)dx_3dx_2, \end{aligned}$$

where both integrals are in Bochner's sense.

Continuing inductively, we obtain $\varphi \in L^\infty(X_{n-1}, K_{n-2})$, where $K_{n-2} = L^2(X_1 \times \cdots \times X_{n-2} \times X_n)$, such that

$$S_1(\xi_2, \eta_2, \ldots, \xi_{n-1}, \eta_{n-1}) = \int_{X_2} \cdots \int_{X_{n-1}} \varphi(x_1, \ldots, x_n)\xi_2\eta_2 \ldots \xi_{n-1}\eta_{n-1} dx_{n-1} \ldots dx_2,$$

where the integrals are understood in Bochner's sense and φ is viewed as a function on $X_1 \times \cdots \times X_n$ by letting $\varphi(x_1, \ldots, x_n) = \varphi(x_{n-1})(x_1, \ldots, x_{n-2}, x_n)$.

It is easy to see that if $\psi \in L^1(Y, L^2(Z))$, where Y and Z are finite measure spaces, then $\int_{Y \times Z} |\psi(y)(z)| dydz$ is finite and $\left(\int_Y \psi(y) dy\right)(z) = \int_Y \psi(y)(z) dy$, for almost all $z \in Z$ (the first integral is in Bochner's sense, while the second one is a Lebesgue integral with respect to the variable y). It now follows that the last equality holds when the integrals are interpreted in the sense of Lebesgue.

The modularity of S implies

$$S(a \otimes \xi_2, \eta_2 \otimes \xi_3, \ldots, \eta_{n-1} \otimes b) = \int_{X_2} \int_{X_3} \cdots \int_{X_{n-1}} \varphi(x_1, \ldots, x_n) a \xi_2 \eta_2 \ldots \xi_{n-1} \eta_{n-1} b dx_{n-1} \ldots dx_2,$$

for all $a \in L^\infty(X_1)$, $b \in L^\infty(X_n)$ and $\xi_i, \eta_i \in L^2(X_i)$, $i = 2, \ldots, n-1$. Letting $a = \chi_{\alpha_1}$, $b = \chi_{\alpha_n}$ and $\xi_i = \eta_i = \chi_{\alpha_i}$, $i = 2, \ldots, n-1$, the boundedness of S implies

$$\int_{\alpha_1 \times \cdots \times \alpha_n} |\varphi(x_1, \ldots, x_n)| dx_1 \ldots dx_n \leq \|S\| \mu_1(\alpha_1) \ldots \mu_n(\alpha_n).$$

It follows that the mapping

$$f = \sum_{i=1}^{N} \lambda_i \chi_{\alpha_1^i \times \cdots \times \alpha_n^i} \longrightarrow \int_{X_1 \times \cdots \times X_n} \varphi f,$$

where $\{\alpha_1^i \times \cdots \times \alpha_n^i\}$ is a finite family of disjoint Borel rectangles, is a linear functional on a dense subspace of $L^1(X_1 \times \cdots \times X_n)$ of norm not exceeding $\|S\|$. Therefore, $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$ and $\|\varphi\|_\infty \leq \|S\|$.

We have that the mappings S and S_φ coincide on the tuples of the form $a \otimes \xi_2, \eta_2 \otimes \xi_3, \ldots, \eta_{n-1} \otimes b$; by linearity and continuity, they are equal. By the first part of the proof, $\|S\| \leq \|\varphi\|_\infty$ and hence $\|\varphi\|_\infty = \|S\|$.

Now relax the assumption on the finiteness of μ_i, and let X_i^k, $k \in \mathbb{N}$, be a measurable subset of X_i such that $\mu_i(X_i^k) < \infty$, $X_i^k \subseteq X_i^{k+1}$ and $X_i = \cup_{k=1}^\infty X_i^k$, $i = 1, \ldots, n$. For each $k \in \mathbb{N}$, let

$$S_k : L^2(X_1^k \times X_2^k) \times L^2(X_2^k \times X_3^k) \times \cdots \times L^2(X_{n-1}^k \times X_n^k) \to L^2(X_1^k \times X_n^k)$$

be the map given by $S_k(f_1, \dots, f_{n-1}) = S(\tilde{f}_1, \dots, \tilde{f}_{n-1})$, where $\tilde{f}_i$ coincides with f_i on X_i^k and is equal to zero on the complement of X_i^k. Since

$$\begin{aligned} S_k(f_1, \dots, f_{n-1}) &= S(\chi_{X_1^k}\tilde{f}_1, \dots, \tilde{f}_{n-1}\chi_{X_n^k}) \\ &= \chi_{X_1^k} S(\tilde{f}_1, \dots, \tilde{f}_{n-1})\chi_{X_n^k}, \end{aligned}$$

the map S_k is well-defined and $\|S_k\| \leq \|S\|$. Since S_k is obviously $L^\infty(X_n^k)$, $\dots$, L^∞ (X_1^k)-modular, the above paragraphs imply that there exists $\varphi_k \in L^\infty(X_1^k \times \dots \times X_n^k)$ such that $S_k = S_{\varphi_k}$, for each $k \in \mathbb{N}$. The space $L^2(X_i^k \times X_{i+1}^k)$ can be considered as a subspace of $L^2(X_i^{k+1} \times X_{i+1}^{k+1})$ in a natural way. We have that the restriction of S_{k+1} to $L^2(X_1^k \times X_2^k) \times L^2(X_2^k \times X_3^k) \times \dots \times L^2(X_{n-1}^k \times X_n^k)$ coincides with S_k. This implies that the restriction of φ_{k+1} to $X_1^k \times \dots \times X_n^k$ coincides (almost everywhere) with φ_k. Hence, there exists a function φ defined on $X_1 \times \dots \times X_n$ which coincides with φ_k on $X_1^k \times \dots \times X_n^k$, for each $k \in \mathbb{N}$. Since $\|\varphi_k\|_\infty = \|S_k\| \leq \|S\|$, we have that $\|\varphi\|_\infty \leq \|S\|$. We have that S and S_φ coincide on the union of $L^2(X_1^k \times X_2^k) \times L^2(X_2^k \times X_3^k) \times \dots \times L^2(X_{n-1}^k \times X_n^k)$, $k \in \mathbb{N}$, which is a dense subset of $L^2(X_1 \times X_2) \times L^2(X_2 \times X_3) \times \dots \times L^2(X_{n-1} \times X_n)$. It follows that $S = S_\varphi$, and by the first part of the proof, $\|S\| = \|\varphi\|_\infty$. $\diamondsuit$

Let (Y_1, ν_1) and (Y_2, ν_2) be measure spaces. A subset $E \subset Y_1 \times Y_2$ is called marginally null [3] if $E \subset (A \times Y_2) \cup (Y_1 \times B)$, $\nu_1(A) = \nu_2(B) = 0$. It is well-known that the projective tensor product $L^2(Y_1)\hat{\otimes}L^2(Y_2)$ can be identified with a space of complex-valued functions, defined marginally almost everywhere on $Y_1 \times Y_2$: the element $\sum_{i=1}^\infty f_i \otimes g_i \in L^2(Y_1)\hat{\otimes}L^2(Y_2)$, where $f_i \in L^2(Y_1)$, $g_i \in L^2(Y_2)$, $\sum_{i=1}^\infty \|f_i\|^2 < \infty$ and $\sum_{i=1}^\infty \|g_i\|^2 < \infty$, is identified with the function h given by $h(x, y) = \sum_{i=1}^\infty f_i(x)g_i(y)$ (see e.g. [3]).

Let

$$\Gamma(X_1, \dots, X_n) = L^2(X_1 \times X_2) \odot \dots \odot L^2(X_{n-1} \times X_n).$$

We identify the elements of $\Gamma(X_1, \dots, X_n)$ with functions on

$$X_1 \times X_2 \times X_2 \times \dots \times X_{n-1} \times X_{n-1} \times X_n$$

in the obvious fashion. We equip $\Gamma(X_1, \dots, X_n)$ with two norms; one is the projective norm $\|\cdot\|_{2,\wedge}$, where each of the L^2-spaces is equipped with its L^2-norm, and the other is the Haagerup tensor norm $\|\cdot\|_{\mathrm{h}}$, where the L^2-spaces are given their opposite operator space structure arising from the identification of $L^2(X \times Y)$ with the class of Hilbert-Schmidt operators from $L^2(X)$ into $L^2(Y)$ given by

$$(T_f\xi)(y) = \int_X f(x, y)\xi(x)dx, \qquad f \in L^2(X \times Y), \xi \in L^2(X). \tag{5.10}$$

For each $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$, we consider the linearisation of the map S_φ from Theorem 60 to a map defined on $\Gamma(X_1, \ldots, X_n)$ and taking values in $L^2(X_1 \times X_n)$ and we denote it in the same way. Thus, if $f_1 \otimes \cdots \otimes f_{n-1}$ is in $\Gamma(X_1, \ldots, X_n)$ then $S_\varphi(f_1 \otimes \cdots \otimes f_{n-1})(x_1, x_n)$ is equal to

$$\int_{X_2 \times \cdots \times X_{n-1}} \varphi(x_1, \ldots, x_n) f_1(x_1, x_2) f_2(x_2, x_3) \ldots f_{n-1}(x_{n-1}, x_n) dx_2 \ldots dx_{n-1}.$$

By Theorem 60, S_φ is bounded and $\|S_\varphi\| = \|\varphi\|_\infty$. Hence it extends to a bounded map from $(\Gamma(X_1, \ldots, X_n), \|\cdot\|_{2,\wedge})$ into $(L^2(X_1 \times X_n), \|\cdot\|_2)$.

Definition 61. *Let $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$. We say that φ is a Schur multiplier (relative to the measure spaces $(X_1, \mu_1), \ldots (X_n, \mu_n)$) if there exists $C > 0$ such that $\|S_\varphi(\Phi)\|_{\mathrm{op}} \leq C\|\Phi\|_{\mathrm{h}}$, for all $\Phi \in \Gamma(X_1, \ldots, X_n)$. The smallest constant C with this property will be denoted by $\|\varphi\|_{\mathrm{m}}$.*

Note that in the case where $n = 2$ and the measure spaces are discrete, the definition above reduces to the definition of the classical Schur multipliers. In the case of arbitrary measure spaces and $n = 2$, we obtain the Schur multipliers studied by Peller [66] (see also [90]).

We will present next a characterisation of the n-dimensional Schur multipliers which generalises Grothendieck's and Peller's characterisations. We will need the following generalisation of a result of Smith [87].

Lemma 62. *Let $\mathcal{E}_i \subseteq B(H_i, H_{i+1})$, $i = 1, \ldots, n$ be spaces of operators and $\mathcal{C} \subseteq B(H_1)$, $\mathcal{D} \subseteq B(H_{n+1})$ be C^*-algebras with cyclic vectors. Assume that $\mathcal{E}_1$ is a right $\mathcal{C}$-module and $\mathcal{E}_n$ is a left $\mathcal{D}$-module. Let $\varphi : \mathcal{E}_n \times \cdots \times \mathcal{E}_1 \to B(H_1, H_{n+1})$ be a multilinear $\mathcal{D},\mathcal{C}$-module map (that is, $\varphi(dy, \ldots, xc) = d\varphi(y, \ldots, x)c$, whenever $x \in \mathcal{E}_1$, $y \in \mathcal{E}_n$, $c \in \mathcal{C}$ and $d \in \mathcal{D}$) such that the corresponding linear map from $\mathcal{E}_n \odot \cdots \odot \mathcal{E}_1$ into $B(H_1, H_{n+1})$ is bounded in the Haagerup norm. Then φ is a completely bounded multilinear map.*

Proof. The proof is a straightforward generalisation of the argument given by Smith [87]. We will denote by $\tilde{\varphi}$ the linear map from $\mathcal{E}_n \odot \cdots \odot \mathcal{E}_1$ into $\mathcal{B}(H_1, H_{n+1})$ defined by $\tilde{\varphi}(a_n \otimes \cdots \otimes a_1) = \varphi(a_n, \ldots, a_1)$. By the assumption of the lemma, it is bounded in the Haagerup norm $\|\cdot\|_{\mathrm{h}}$. Assume that $\|\tilde{\varphi}\| = 1$. We will show that $\|\tilde{\varphi}\|_{\mathrm{cb}} = 1$. Suppose, to the contrary, that $\|\tilde{\varphi}\|_{\mathrm{cb}} > 1$. Then there exists $m \in \mathbb{N}$, matrices $x^i = (x^i_{kj}) \in M_m(\mathcal{E}_i)$, $i = 1, \ldots, n$ and column vectors $\xi_0 = (\xi_1, \ldots, \xi_m) \in H_1^m$ and $\eta_0 = (\eta_1, \ldots, \eta_m) \in H_{n+1}^m$ such that $\|\xi_0\| < 1$, $\|\eta_0\| < 1$, all $\|x^i\| < 1$ and

$$|(\varphi^{(m)}(x^n, x^{n-1} \ldots, x^1)\xi_0, \eta_0)| > 1. \tag{5.11}$$

If ξ and η are cyclic vectors for $\mathcal{C}$ and $\mathcal{D}$, respectively, we may moreover assume that $\xi_i = a_i\xi$ and $\eta_j = b_j\eta$, for some $a_i \in \mathcal{C}$ and $b_j \in \mathcal{D}$, where

$i,j = 1,\dots,m$. Let $a = \sum_{i=1}^m a_i^* a_i$ and $b = \sum_{j=1}^m b_j^* b_j$. Assume first that a and b are invertible, and let $c_i = a_i a^{-1/2}$, $d_j = b_j b^{-1/2}$, $\tilde{\xi} = a^{1/2}\xi$ and $\tilde{\eta} = b^{1/2}\eta$. Then $\xi_i = c_i\tilde{\xi}$ and $\eta_j = d_j\tilde{\eta}$. Taking into account (5.9), the left hand side of (5.11) becomes

$$\begin{aligned}
&\left|\sum_{i,j=1}^m (\varphi^{(m)}(x^n, x^{n-1},\dots,x^1)_{ji} c_i\tilde{\xi}, d_j\tilde{\eta})\right| \\
= &\left|\sum_{k_1,\dots,k_{n-1}=1}^m \sum_{i,j=1}^m (\varphi(d_j^* x^n_{jk_{n-1}}, x^{n-1}_{k_{n-1}k_{n-2}},\dots,x^1_{k_1 i}c_i)\tilde{\xi},\tilde{\eta})\right| \\
= &\left|\sum_{k_1,\dots,k_{n-1}=1}^m \left(\varphi\left(\sum_{j=1}^m d_j^* x^n_{jk_{n-1}}, x^{n-1}_{k_{n-1}k_{n-2}},\dots,\sum_{i=1}^m x^1_{k_1,i}c_i\right)\tilde{\xi},\tilde{\eta}\right)\right| \\
\leq &\left\|\sum_{k_1,\dots,k_{n-1}=1}^m \varphi\left(\sum_{j=1}^m d_j^* x^n_{jk_{n-1}}, x^{n-1}_{k_{n-1}k_{n-2}},\dots,\sum_{i=1}^m x^1_{k_1,i}c_i\right)\right\| \|\tilde{\xi}\|\|\tilde{\eta}\| \qquad (5.12)
\end{aligned}$$

We have that

$$\|\tilde{\xi}\| = (a^{1/2}\xi, a^{1/2}\xi) = (a\xi,\xi) = \sum_{k=1}^n \|a_i\xi\|^2 = \sum_{k=1}^n \|\xi_i\|^2 = \|\xi_0\| \leq 1,$$

and similarly $\|\tilde{\eta}\| \leq 1$. Set $d^* = (d_j^*) \in M_{1,m}(\mathcal{D})$, $c = (c_i) \in M_{m,1}(\mathcal{C})$, $u = d^* x^n \in M_{1,m}(\mathcal{E}_n)$ and $v = x^1 c \in M_{m,1}(\mathcal{E}_1)$. It follows from (5.7) and (5.8) that

$$\begin{aligned}
&\left\|\sum_{k_1,\dots,k_{n-1}=1}^m \varphi\left(\sum_{j=1}^m d_j^* x_{jk_{n-1}}, x^{n-1}_{k_{n-1}k_{n-2}},\dots,\sum_{i=1}^m x_{k_1,i}c_i\right)\right\| \\
= &\left\|\sum_{k_1,\dots,k_{n-1}=1}^m \varphi\left(u_{k_{n-1}}, x^{n-1}_{k_{n-1}k_{n-2}},\dots,v_{k_1}\right)\right\| \\
= &\left\|\tilde{\varphi}\left(\sum_{k_1,\dots,k_{n-1}=1}^m u_{k_{n-1}} \otimes x^{n-1}_{k_{n-1}k_{n-2}} \otimes\cdots\otimes v_{k_1}\right)\right\| \\
\leq &\left\|\sum_{k_1,\dots,k_{n-1}=1}^m u_{k_{n-1}} \otimes x^{n-1}_{k_{n-1}k_{n-2}} \otimes\cdots\otimes v_{k_1}\right\|_{\mathrm{h}} \\
= &\|u \odot x^{n-1} \odot\cdots\odot x^2 \odot v\|_{\mathrm{h}} \\
\leq &\|d^*\|\|x^n\|\|x^{n-1}\|\dots\|x^2\|\|x^1\|\|c\|. \qquad (5.13)
\end{aligned}$$

We have that

$$\|d^*\| = \left\| \sum_{j=1}^{m} d_j^* d_j \right\|^{1/2} = \|I\| = 1$$

and, similarly, $\|c\| = 1$. It follows from (5.12) and (5.13) that

$$|(\varphi^{(m)}(x^n, x^{n-1} \dots, x^1)\xi_0, \eta_0)| \leq 1,$$

which contradicts (5.11).

In the case a or b is not invertible, one can again follow [87] and, for each i, consider the matrix $\hat{x}^i \in M_{m+1}(\mathcal{E}_i)$ which has the matrix x^i in its upper left corner and zeros in the last row and column. The vectors ξ_0 and η_0 are replaced with $\hat{\xi}_0 = (\xi_1, \dots, \xi_m, \xi_{m+1})$ and $\hat{\eta}_0 = (\eta_1, \dots, \eta_m, \eta_{m+1})$, where $\xi_{m+1} = \varepsilon\xi$ and $\eta_{m+1} = \varepsilon\eta$, respectively, for ε small enough so that the norms of these vectors remain less than one. Letting $a_{n+1} = b_{n+1} = \varepsilon I$, we have that $a_i\xi = \xi_i$ and $b_i\eta = \eta_i$ for each $i = 1, \dots, m+1$. Finally,

$$(\varphi^{(m)}(x^n, x^{n-1} \dots, x^1)\xi_0, \eta_0) = (\varphi^{(m+1)}(\hat{x}^n, \hat{x}^{n-1} \dots, \hat{x}^1)\hat{\xi}_0, \hat{\eta}_0)$$

and the proof proceeds as before. $\diamondsuit$

The main result of this section is the following

Theorem 63. *Let $\varphi \in L^\infty(X_1 \times \dots \times X_n)$. The following are equivalent:*

(i) φ is a Schur multiplier and $\|\varphi\|_{\mathrm{m}} < 1$;

(ii) there exist essentially bounded functions $a_1 : X_1 \to M_{\infty,1}$, $a_n : X_n \to M_{1,\infty}$ and $a_i : X_i \to M_\infty$, $i = 2, \dots, n-1$, such that, for almost all $x_1, \dots, x_n$ we have

$$\varphi(x_1, \dots, x_n) = a_n(x_n)a_{n-1}(x_{n-1}) \dots a_1(x_1) \quad \text{and} \quad \operatorname*{esssup}_{x_i \in X_i} \prod_{i=1}^{n} \|a_i(x_i)\| < 1.$$

Proof. (i)$\Rightarrow$(ii) Let $\varphi \in L^\infty(X_1 \times \dots \times X_n)$ be a Schur multiplier with $\|\varphi\|_{\mathrm{m}} < 1$. Then the map S_φ induces a map, denoted in the same way, from $L^2(X_1 \times X_2) \times \dots \times L^2(X_{n-1} \times X_n)$ into $L^2(X_1 \times X_n)$. Let $H_i = L^2(X_i)$, $\mathcal{D}_i = \{M_\psi : \psi \in L^\infty(X_i)\}$, $i = 1, \dots, n$, and

$$\hat{S}_\varphi : \mathcal{C}_2(H_1, H_2) \times \dots \times \mathcal{C}_2(H_{n-1}, H_n) \to \mathcal{C}_2(H_1, H_n)$$

be the map defined by $\hat{S}_\varphi(T_{f_1}, \dots, T_{f_n}) = T_{S_\varphi(f_1, \dots, f_n)}$. Since φ is a Schur multiplier, the linearisation of the map $\hat{S}_\varphi$ from $(\mathcal{C}_2(H_1, H_2) \odot \dots \odot \mathcal{C}_2(H_{n-1}, H_n), \|\cdot\|_{\mathrm{h}})$ into $(\mathcal{C}_2(H_1, H_n), \|\cdot\|_{\mathrm{op}})$ is bounded. (Here each of the operator spaces

$\mathcal{C}_2(H_i, H_{i+1})$ is given its opposite operator space structure arising from the inclusion $\mathcal{C}_2(H_i, H_{i+1}) \subseteq \mathcal{B}(H_i, H_{i+1})$.) If $a_i \in L^\infty(X_i)$, $i = 1, \dots, n$, then

$$\begin{aligned}\hat{S}_\varphi(T_{f_1}M_{a_1}, T_{f_2}M_{a_2}, \dots, M_{a_n}T_{f_n}M_{a_{n-1}}) &= \hat{S}_\varphi(T_{f_1a_1}, T_{f_2a_2}, \dots, T_{a_nf_na_{n-1}}) \\ &= T_{S_\varphi(f_1a_1, f_2a_2, \dots, a_nf_na_{n-1})} \qquad (5.14) \\ &= T_{a_nS_\varphi(a_2f_1, a_3f_2, \dots, a_{n-1}f_{n-2}, f_n)a_1} \\ &= M_{a_n}\hat{S}_\varphi(M_{a_2}T_{f_1}, \dots, T_{f_n})M_{a_1}.\end{aligned}$$

By continuity, the map $\hat{S}_\varphi$ has an extension (denoted in the same way)

$$\hat{S}_\varphi : \mathcal{K}(H_1, H_2) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_{n-1}, H_n) \to \mathcal{K}(H_1, H_n)$$

to a map with norm less than one, where the spaces $\mathcal{K}(H_i, H_{i+1})$ are equipped with the operator space structure opposite to their natural operator space structure. It follows from (5.14) that the map

$$\check{S}_\varphi : \mathcal{K}(H_{n-1}, H_n) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_1, H_2) \to \mathcal{K}(H_1, H_n)$$

given by

$$\check{S}_\varphi(T_{n-1} \otimes \cdots \otimes T_1) = \hat{S}_\varphi(T_1 \otimes \cdots \otimes T_{n-1})$$

is modular and bounded when the spaces $\mathcal{K}(H_i, H_{i+1})$ are given their natural operator space structure. By Lemma 62, $\check{S}_\varphi$ is completely bounded. It follows that the second dual

$$\check{S}_\varphi^{**} : \mathcal{B}(H_{n-1}, H_n) \otimes_{\sigma h} \cdots \otimes_{\sigma h} \mathcal{B}(H_1, H_2) \to \mathcal{B}(H_1, H_n)$$

is a weak* continuous map with c.b. norm less than one, which extends the map $\check{S}_\varphi$. (Here $\otimes_{\sigma h}$ denotes the normal Haagerup tensor product, see e.g. [9].)

Denote by $\tilde{S}_\varphi$ the corresponding multilinear map

$$\tilde{S}_\varphi : \mathcal{B}(H_{n-1}, H_n) \times \cdots \times \mathcal{B}(H_1, H_2) \to \mathcal{B}(H_1, H_n).$$

The map $\tilde{S}_\varphi$ is separately weak* continuous and hence modular.

A modification of Corollary 5.9 of [18] now implies that there exist bounded linear operators $V_1 : H_1 \to H_1^\infty$, $V_n : H_n^\infty \to H_n$ and $V_i : H_i^\infty \to H_i^\infty$, $i = 2, \dots, n-1$, such that the entries of V_i belong to $\mathcal{D}_i$ and

$$\tilde{S}_\varphi(T_{n-1}, \dots, T_1) = V_n(T_{n-1} \otimes I)V_{n-1}(T_{n-2} \otimes I) \dots (T_1 \otimes I)V_1.$$

Moreover, the operators V_i can be chosen so that $\prod_{i=1}^n \|V_i\| < 1$. Let $V_1 = (M_{a_1^1}, M_{a_2^1}, \dots)^{\mathrm{t}}$, $V_i = (M_{a_{kl}^i})$ and $V_n = (M_{a_1^n}, M_{a_2^n}, \dots)$, for some $a_1 =$

$(a_1^1, a_2^1, \dots)^{\mathrm{t}} \in L^\infty(X_1, M_{1,\infty})$, $a_n = (a_1^n, a_2^n, \dots) \in L^\infty(X_n, M_{1,\infty})$ and $a_i = (a_{kl}^i) \in L^\infty(X_i, M_\infty)$, $i = 2, \dots, n-1$. Moreover,

$$\operatorname*{esssup}_{x_i \in X_i} \prod_{i=1}^n \|a_i(x_i)\| = \prod_{i=1}^n \|V_i\| < 1.$$

If $\xi \in L^2(X)$ and $\eta \in L^2(Y)$ denote by $\xi \otimes \eta$ the function on $X \times Y$ given by $(\xi \otimes \eta)(x,y) = \xi(x)\eta(y)$; this function gives rise by (5.10) to a rank one operator $T_{\xi\otimes\eta}$. Fix $\xi_i, \eta_i \in H_i$, $i = 1, \dots, n$. Then

$$\begin{aligned}
& \tilde{S}_\varphi(T_{\xi_{n-1}\otimes\eta_n}, \dots, T_{\xi_1\otimes\eta_2})(\eta_1) = V_n(T_{\xi_{n-1}\otimes\eta_n} \otimes I) \dots (T_{\xi_1\otimes\eta_2} \otimes I)V_1(\eta_1) \\
= \; & V_n(T_{\xi_{n-1}\otimes\eta_n} \otimes I) \dots V_2(T_{\xi_1\otimes\eta_2} \otimes I)(a_{k_1}^1 \eta_1)_{k_1} \\
= \; & V_n(T_{\xi_{n-1}\otimes\eta_n} \otimes I) \dots V_2((\int_{X_1} a_{k_1}^1(x_1)\xi_1(x_1)\eta_1(x_1)dx_1)\eta_2)_{k_1} \\
= \; & V_n \dots (T_{\xi_2\otimes\eta_3} \otimes I)((\sum_{k_1=1}^\infty \int_{X_1} a_{k_1}^1(x_1)\xi_1(x_1)\eta_1(x_1)dx_1) a_{k_2,k_1}^2 \eta_2)_{k_2} \\
= \; & V_n \dots V_3((\sum_{k_1=1}^\infty \int_{X_1\times X_2} a_{k_2,k_1}^2(x_2)a_{k_1}^1(x_1)(\xi_1\eta_1)(x_1)(\xi_2\eta_2)(x_2)dx_1dx_2)\eta_3)_{k_2} \\
\\
= \; & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
= \; & \sum_{k_n=1}^\infty (\int_{X_1\times\dots\times X_{n-1}} \sum_{k_1,\dots,k_{n-1}=1}^\infty a_{k_{n-1},k_{n-2}}^{n-1}(x_{n-1}) \dots a_{k_1}^1(x_1) \times \\
\times \; & \xi_1(x_1)\eta_1(x_1) \dots \xi_{n-1}(x_{n-1}))dx_1 \dots dx_{n-1}) M_{a_{k_n}^n} \eta_n.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \tilde{S}_\varphi(T_{\xi_{n-1}\otimes\eta_n}, \dots, T_{\xi_1\otimes\eta_2})(\eta_1)(x_n) \\
= \; & (\int_{X_1\times\dots\times X_{n-1}} \sum_{k_1,\dots,k_n=1}^\infty a_{k_n}^n(x_n) a_{k_{n-1},k_{n-2}}^{n-1}(x_{n-1}) \dots a_{k_1}^1(x_1) \times \\
\times \; & \xi_1(x_1)\eta_1(x_1) \dots \xi_{n-1}(x_{n-1})dx_1 \dots dx_{n-1})\eta_n(x_n).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \tilde{S}_\varphi(T_{\xi_{n-1}\otimes\eta_n}, \dots, T_{\xi_1\otimes\eta_2})(\eta_1)(x_n) = T_{S_\varphi(\xi_1\otimes\eta_2,\dots,\xi_{n-1}\otimes\eta_n)}(\eta_1)(x_n) \\
= \; & (\int_{X_1\times\dots\times X_{n-1}} \varphi(x_1, \dots, x_{n-1}, x_n) \\
\times \; & \xi_1(x_1)\eta_1(x_1) \dots \xi_{n-1}(x_{n-1})dx_1 \dots dx_{n-1})\eta_n(x_n).
\end{aligned}$$

It follows that

$$\varphi(x_1,\dots,x_n) = a_n(x_n)a_{n-1}(x_{n-1})\dots a_1(x_1),$$

for almost all $x_1,\dots,x_n$.

(ii)⇒(i) Assume that φ is given as in (ii), where $a_1 = (a^1_1, a^1_2, \dots)^{\mathrm{t}} \in L^\infty(X_1,\, M_{\infty,1})$, $a_n = (a^n_1, a^n_2, \dots) \in L^\infty(X_n, M_{1,\infty})$ and $a_i = (a^i_{kl}) \in L^\infty(X_i, M_\infty)$, $i = 2,\dots,n-1$. Let $V_1 : H_1 \to H_1^\infty$ be the operator corresponding to the column matrix $V_1 = (M_{a^1_1}, M_{a^1_2}, \dots)^{\mathrm{t}} : H_1 \to H_1^\infty$, $V_n : H_n^\infty \to H_n$ be the operator corresponding to the row matrix $V_n = (M_{a^n_1}, M_{a^n_2}, \dots)$ and $V_i : H_i^\infty \to H_i^\infty$ be the operator corresponding to the matrix $V_i = (M_{a^i_{kl}})$, $i = 2,\dots,n-1$. Then $\prod_{i=1}^n \|V_i\| < 1$. It follows from the first part of the proof that

$$\tilde{S}_\varphi(T_{\xi_{n-1}\otimes\eta_n},\dots,T_{\xi_1\otimes\eta_2}) = V_n(T_{\xi_{n-1}\otimes\eta_n}\otimes I)\dots(T_{\xi_1\otimes\eta_2}\otimes I)V_1,$$

for all $\xi_1 \in H_1$, $\eta_n \in H_n$ and $\xi_i, \eta_i \in H_i$, $i = 2,\dots,n-1$. Since the operator norm is dominated by the Hilbert-Schmidt norm, we conclude that

$$\tilde{S}_\varphi(T_{f_{n-1}},\dots,T_{f_1}) = V_n(T_{f_{n-1}}\otimes I)\dots(T_{f_1}\otimes I)V_1,$$

for all $f_i \in L^2(X_i \times X_{i+1})$, $i = 1,\dots,n-1$.

Let

$$F = F_1 \odot \dots \odot F_{n-1} \in L^2(X_1\times X_2)\odot\dots\odot L^2(X_{n-1}\times X_n),$$

where $F_1 \in M_{1,\infty}(L^2(X_1\times X_2))$, $F_{n-1} \in M_{\infty,1}(L^2(X_{n-1}\times X_n))$ and $F_i \in M_\infty(L^2(X_i\times X_{i+1}))$, $i = 2,\dots,n-2$. Lemma 72 implies that

$$T_{S_\varphi(F)} = V_n(T_{F_{n-1}}\otimes I)\dots(T_{F_1}\otimes I)V_1,$$

where $T_{F_i} = (T_{f^i_{lk}})_{k,l}$ whenever $F_i = (f^i_{kl})_{k,l}$. It follows that

$$\|T_{S_\varphi(F)}\|_{\mathrm{op}} \le \prod_{i=1}^{n-1}\|F_i^t\|_{\mathrm{op}}\prod_{i=1}^{n}\|V_i\|.$$

Taking infimum with respect to all representations of F, we conclude that $\|T_{S_\varphi(F)}\|_{\mathrm{op}} \le \|F\|_{\mathrm{h}}\prod_{i=1}^n\|V_i\|$ and so $\|\varphi\|_{\mathrm{m}} < 1$. ♢

Remark The space of all functions $\varphi(x_1,\dots,x_n)$ satisfying condition (ii) of Theorem 63 can be identified with the extended Haagerup tensor product $L^\infty(X_1)\otimes_{eh} L^\infty(X_2) \otimes_{eh} \dots \otimes_{eh} L^\infty(X_n)$.

The next proposition relates our approach with a recent work of Peller [68] on multiple operator integrals. For some fixed spectral measures, Peller defines a multiple operator integral $I_\varphi(T_1,\dots,T_{n-1})$ of a function φ and $(n-1)$-tuple of operators $(T_1,\dots,T_{n-1})$, and shows that if φ belongs to the integral projective tensor product of the corresponding L^∞-spaces, then $I_\varphi(T_1,\dots,T_{n-1})$ is well-defined and, moreover,

$$\|I_\varphi(T_1,\dots,T_{n-1})\|_{\rm op} \leq \|\varphi\|_i \|T_1\|_{\rm op}\dots\|T_{n-1}\|_{\rm op}.$$

Recall that the integral projective tensor product $L^\infty(X_1)\hat{\otimes}_i\dots\hat{\otimes}_i L^\infty(X_n)$ is the space of all functions φ for which there exists a measure space $(\mathcal{T},\nu)$ and measurable functions g_i on $X_i\times\mathcal{T}$ such that

$$\varphi(x_1,\dots,x_n) = \int_{\mathcal{T}} g_1(x_1,t)\dots g_n(x_n,t)d\nu(t), \tag{5.15}$$

for almost all $x_1\dots,x_n$, where

$$\int_{\mathcal{T}} \|g_1(\cdot,t)\|_\infty \dots \|g_n(\cdot,t)\|_\infty d\nu(t) < \infty.$$

The integral projective norm $\|\varphi\|_i$ of φ is the infimum of the above expressions over all representations of φ of the form (5.15). It was proved by Peller in [66] that in the case where $n=2$ the integral projective tensor product $L^\infty(X_1)\hat{\otimes}_i L^\infty(X_2)$ coinsides with the set of all Schur mulipliers. The next proposition shows that for $n>2$ the integral projective tensor product consists of multipliers. We do not know whether it coincides with the space of all Schur multipliers.

Proposition 64. *Let $\varphi \in L^\infty(X_1)\hat{\otimes}_i\dots\hat{\otimes}_i L^\infty(X_n)$. Then φ is a Schur multiplier and $\|\varphi\|_{\rm m} \leq \|\varphi\|_i$.*

Proof. Suppose that

$$\varphi(x_1,\dots,x_n) = \int_{\mathcal{T}} g_1(x_1,t)\dots g_n(x_n,t)d\nu(t),$$

for almost all $x_1\dots,x_n$, where $(\mathcal{T},\nu)$ is a measure space, g_i is a measurable function on $X_i\times\mathcal{T}$, $i=1,\dots,n$, such that

$$\int_{\mathcal{T}} \|g_1(\cdot,t)\|_\infty \dots \|g_n(\cdot,t)\|_\infty d\nu(t) < \infty.$$

Let $F = F_1\odot\dots\odot F_{n-1}$, where $F_1\in M_{1,k_1}(L^2(X_1\times X_2))$, $F_{n-1}\in M_{k_{n-2},1}$ $(L^2(X_{n-1}\times X_n))$ and $F_i\in M_{k_{i-1},k_i}(L^2(X_i\times X_{i+1}))$, $i=2,\dots,n-2$, and

$\tilde{F}(x_1, x_2 \ldots, x_n) = F(x_1, x_2, x_2, x_3, \ldots, x_n)$. Denoting by $M_{g_i(\cdot,t)}$ the multiplication operator by the function $g_i(\cdot, t)$, and by $M_{g_i(\cdot,t)} \otimes I$ the ampliation of $M_{g_i(\cdot,t)}$ of multiplicity k_i, we have

$$\begin{aligned}
\|S_\varphi(F)\|_{\text{op}} &= \|\int_{X_2\times\cdots\times X_{n-1}} \varphi\tilde{F} dx_2 \ldots dx_{n-1}\|_{\text{op}} \\
&= \|\int_{X_2\times\cdots\times X_{n-1}} \left(\int_{\mathcal{T}} g_1(x_1,t)\ldots g_n(x_n,t)dt\right) \tilde{F} dx_2 \ldots dx_{n-1}\|_{\text{op}} \\
&= \|\int_{\mathcal{T}} \left(\int_{X_2\times\cdots\times X_{n-1}} g_1(x_1,t)\ldots g_n(x_n,t)dx_2\ldots dx_{n-1}\right) \tilde{F} dt\|_{\text{op}} \\
&= \|\int_{\mathcal{T}} (\int_{X_2\times\cdots\times X_{n-1}} M_{g_1(\cdot,t)}F_1(M_{g_2(\cdot,t)}\otimes I)(x_1,x_2)\odot\ldots \\
&\odot\ F_{n-1}M_{g_n(\cdot,t)}(x_{n-1},x_n)dx_2\ldots dx_{n-1})dt\|_{\text{op}}
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathcal{T}} \|\int_{X_2\times\cdots\times X_{n-1}} M_{g_1(\cdot,t)}F_1(M_{g_2(\cdot,t)}\otimes I)(x_1,x_2)\odot\ldots \\
&\odot\ F_{n-1}M_{g_n(\cdot,t)}(x_{n-1},x_n)dx_2\ldots dx_{n-1}\|_{\text{op}}dt \\
&\leq \int_{\mathcal{T}} \|M_{g_1(\cdot,t)}\|\|F_1\|_{\text{op}}^o\|M_{g_2(\cdot,t)}\|\ldots\|F_{n-1}\|_{\text{op}}^o\|M_{g_n(\cdot,t)}\|dt \\
&\leq \|\varphi\|_i\|F_1\|_{\text{op}}^o\ldots\|F_{n-1}\|_{\text{op}}^o.
\end{aligned}$$

where $\|\cdot\|_{\text{op}}^o$ is the opposite operator norm (see Section 5.1.2). The claim follows by taking infimum over all representations $F = F_1 \odot \cdots \odot F_{n-1}$. ♢

Corollary 65. $L^\infty(X_1)\hat{\otimes}_i \ldots \hat{\otimes}_i L^\infty(X_n) \subseteq L^\infty(X_1) \otimes_{eh} \ldots \otimes_{eh} L^\infty(X_n)$.

In the case where $n = 2$, it follows by Peller's characterisation of Schur multipliers [66] that there is an equality in the inclusion of Corollary 65. We do not know whether equality holds in the general case.

We finally point out another interesting open question, namely the one of characterising the class of multipliers defined by using the projective tensor norm instead of the Haagerup tensor norm in (5.2); equivalently, the class of multipliers obtained after replacing (5.2) with the weaker condition

$$\|S_\psi(f_1 \otimes \ldots \otimes f_n)\|_{\text{op}} \leq C\|f_1\|_{\text{op}}\ldots\|f_n\|_{\text{op}} \text{ for all } f_i \in L^2(X_i), i = 1, \ldots, n.$$

5.1.4 Multidimensional operator multipliers: the definition.

In this section we generalise the notion of operator multipliers given by Kissin and Shulman [52] to the multidimensional case.

We recall the mapping $\theta_{K_1,K_2} : K_1 \otimes K_2 \to \mathcal{C}_2(K_1^{\rm d}, K_2)$, where K_1 and K_2 are Hilbert spaces, which is the unitary operator between the Hilbert spaces $K_1 \otimes K_2$ and $\mathcal{C}_2(K_1^{\rm d}, K_2)$ given on elementary tensors by

$$\theta_{K_1,K_2}(\xi_1 \otimes \xi_2)(\eta_1^{\rm d}) = (\xi_1, \eta_1)\xi_2.$$

Note that there is a natural identification of $(K_1 \otimes K_2)^{\rm d}$ and $K_1^{\rm d} \otimes K_2^{\rm d}$. It follows that $\mathcal{C}_2(K_1^{\rm d}, K_2)^{\rm d}$ can be identified with $\mathcal{C}_2(K_1, K_2^{\rm d}) = \mathcal{C}_2((K_1^{\rm d})^{\rm d}, K_2^{\rm d})$; we have that $\theta_{K_1^{\rm d},K_2^{\rm d}}(\xi^{\rm d}) = \theta_{K_1,K_2}(\xi)^{\rm d}$.

Let $H_1, \dots, H_n$ be Hilbert spaces and $H = H_1 \otimes \dots H_n$. For any permutation π of $\{1, \dots, n\}$, we will identify H with the tensor product $H_{\pi(1)} \otimes \dots H_{\pi(n)}$ without explicitly mentioning this. The symbol $\xi_{j_1,\dots,j_k}$ will denote an element of $H_{j_1} \otimes \dots H_{j_k}$.

We define a Hilbert space $HS(H_1, \dots, H_n)$, isometrically isomorphic to H. Let $HS(H_1, H_2) = \mathcal{C}_2(H_1^{\rm d}, H_2)$. In the case where n is even, we let by induction

$$HS(H_1, \dots, H_n) = \mathcal{C}_2(HS(H_2, H_3)^{\rm d}, HS(H_1, H_4, \dots, H_n)),$$

and let

$$\theta_{H_1,\dots,H_n} : H \to HS(H_1, \dots, H_n)$$

be given by

$$\theta_{H_1,\dots,H_n}(\xi_{2,3} \otimes \xi) = \theta_{HS(H_2,H_3),HS(H_1,H_4,\dots,H_n)}(\theta_{H_2,H_3}(\xi_{2,3}) \otimes \theta_{H_1,H_4,\dots,H_n}(\xi)),$$

where $\xi \in H_1 \otimes H_4 \otimes \cdots \otimes H_n$. In particular, we have that

$$\theta_{H_1,\dots,H_n}(\xi_{2,3} \otimes \xi)\theta_{H_2,H_3}(\eta_{2,3})^{\rm d} = (\theta_{H_2,H_3}(\xi_{2,3}), \theta_{H_2,H_3}(\eta_{2,3}))\theta_{H_1,H_4,\dots,H_n}(\xi).$$

In the case where n is odd, we let

$$HS(H_1, \dots, H_n) = HS(\mathbb{C}, H_1, \dots, H_n).$$

If K is a Hilbert space, we will identify $\mathcal{C}_2(\mathbb{C}^{\rm d}, K)$ with K via the map $S \to S(1^{\rm d})$. Thus, $HS(H_1, \dots, H_n)$ can, in the case of odd n, be defined inductively by letting $HS(H_1) = H_1$ and

$$HS(H_1, \dots, H_n) = \mathcal{C}_2(HS(H_1, H_2)^{\rm d}, HS(H_3, \dots, H_n)).$$

The isomorphism $\theta_{H_1,\dots,H_n}$ is in this case given by

$$\theta_{H_1,\dots,H_n}(\xi) = \theta_{\mathbb{C},H_1,\dots,H_n}(1 \otimes \xi).$$

We will usually omit the subscripts and write simply θ, when the corresponding Hilbert spaces are understood.

Lemma 66. *(i) Assume n is even. Let $\xi \in H$ be of the form $\xi = \xi_{1,2} \otimes \dots \otimes \xi_{n-1,n}$. If $\eta_{i,i+1} \in H_i \otimes H_{i+1}$ (i even) then*

$$\theta(\xi)(\theta(\eta_{2,3}^{\mathrm{d}}))\dots(\theta(\eta_{n-2,n-1}^{\mathrm{d}})) = \theta(\xi_{n-1,n})\theta(\eta_{n-2,n-1}^{\mathrm{d}})\dots\theta(\xi_{3,4})\theta(\eta_{2,3}^{\mathrm{d}})\theta(\xi_{1,2}).$$

(ii) Assume n is odd. Let $\xi \in H$ be of the form $\xi = \xi_1 \otimes \xi_{2,3} \dots \otimes \xi_{n-1,n}$. If $\eta_{i,i+1} \in H_i \otimes H_{i+1}$ (i odd) then

$$\theta(\xi)(\theta(\eta_{1,2}^{\mathrm{d}}))(\theta(\eta_{3,4}^{\mathrm{d}}))\dots(\theta(\eta_{n-2,n-1}^{\mathrm{d}})) = \theta(\xi_{n-1,n})\theta(\eta_{n-2,n-1}^{\mathrm{d}})\dots\theta(\eta_{1,2}^{\mathrm{d}})(\xi_1).$$

Proof. (i) Assume first that $\xi_{i-1,i} = \xi_{i-1} \otimes \xi_i$ and $\eta_{i,i+1} = \eta_i \otimes \eta_{i+1}$ (i even). Fix $\eta_1^{\mathrm{d}} \in H_1^{\mathrm{d}}$. The image of η_1^{d} under the operator on the right hand side of the identity in (i) is

$$(\xi_1, \eta_1)(\xi_2, \eta_2)\dots(\xi_{n-1}, \eta_{n-1})\xi_n.$$

On the other hand, the image of η_1^{d} under the operator on the left hand side is

$$\begin{aligned} & (\theta_{H_2,H_3}(\xi_2 \otimes \xi_3), \theta_{H_2,H_3}(\eta_2 \otimes \eta_3)) \\ \times\ & \theta_{H_1,H_4,\dots,H_n}(\xi_1 \otimes \xi_4 \otimes \dots \otimes \xi_n)(\theta(\eta_{4,5})^{\mathrm{d}})\dots(\theta(\eta_{n-2,n-1})^{\mathrm{d}})(\eta_1^{\mathrm{d}}) \\ =\ & (\xi_2, \eta_2)(\xi_3, \eta_3) \\ \times\ & \theta_{H_1,H_4,\dots,H_n}(\xi_1 \otimes \xi_4 \otimes \dots \otimes \xi_n)(\theta(\eta_{4,5})^{\mathrm{d}})\dots(\theta(\eta_{n-2,n-1})^{\mathrm{d}})(\eta_1^{\mathrm{d}}). \end{aligned}$$

By induction, (i) holds in the case of elementary tensors.

By linearity, (i) holds for finite sums of elementary tensors. Using continuity arguments and the fact that the operator norm is dominated by the Hilbert-Schmidt norm, one can easily prove that (i) holds for genral ξ and $\eta_{i,i+1}$. ◇

We define a representation σ_H of $B(H)$ on $HS(H_1, \dots, H_n)$ by letting

$$\sigma_H(A)\theta(\xi) = \theta(A\xi);$$

clearly, σ_H is unitarily equivalent to the identity representation of $B(H)$. If $H_1, \dots, H_n$ are clear from the context we will simply write σ in the place of

σ_H. If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are C*-algebras, $\pi_1, \ldots, \pi_n$ corresponding representations on $H_1, \ldots, H_n$, and $\pi = \pi_1 \otimes \ldots \pi_n$ we let

$$\sigma_\pi = \sigma_H \circ \pi \ ;$$

thus, σ_π is a representation of $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ on $HS(H_1, \ldots, H_n)$, unitarily equivalent to π.

Lemma 67. *Let $A_i \in B(H_i)$, $i = 1, \ldots, n$, and $A = A_1 \otimes \cdots \otimes A_n$.*

(i) Assume n is even. Let $\xi_{i-1,i} \in H_{i-1} \otimes H_i$, $\eta_{i,i+1} \in H_i \otimes H_{i+1}$ (i even). If $\xi = \xi_{1,2} \otimes \ldots \xi_{n-1,n}$ then

$$\begin{aligned} & \sigma(A)(\theta(\xi))(\theta(\eta_{2,3}^{\mathrm{d}})) \ldots (\theta(\eta_{n-2,n-1}^{\mathrm{d}})) \\ = \ & A_n \theta(\xi_{n-1,n}) A_{n-1}^{\mathrm{d}} \theta(\eta_{n-2,n-1})^{\mathrm{d}} A_{n-2} \ldots A_2 \theta(\xi_{1,2}) A_1^{\mathrm{d}} \\ = \ & A_n \theta(\xi)(\theta((A_2^* \otimes A_3^*(\eta_{2,3}))^{\mathrm{d}})) \ldots (\theta((A_{n-2}^* \otimes A_{n-1}^*(\eta_{n-2,n-1}))^{\mathrm{d}})) A_1^{\mathrm{d}}. \end{aligned}$$

(ii) Assume n is odd. Let $\xi_1 \in H_1$, $\xi_{i-1,i} \in H_{i-1} \otimes H_i$, $\eta_{i,i+1} \in H_i \otimes H_{i+1}$ (i odd). If $\xi = \xi_1 \otimes \xi_{2,3} \otimes \ldots \xi_{n-1,n}$ then

$$\begin{aligned} & \sigma(A)(\theta(\xi))(\theta(\eta_{1,2}^{\mathrm{d}})) \ldots (\theta(\eta_{n-2,n-1}^{\mathrm{d}})) \\ = \ & A_n \theta(\xi_{n-1,n}) A_{n-1}^{\mathrm{d}} \theta(\eta_{n-2,n-1})^{\mathrm{d}} A_{n-2} \ldots A_2^{\mathrm{d}} \theta(\eta_{1,2}^{\mathrm{d}})(A_1 \xi_1) \\ = \ & A_n \theta(\xi)(\theta((A_1^* \otimes A_2^*(\eta_{1,2}))^{\mathrm{d}})) \ldots (\theta((A_{n-2}^* \otimes A_{n-1}^*(\eta_{n-2,n-1}))^{\mathrm{d}})). \end{aligned}$$

Proof. (i) Let first $n = 2$. If $\eta^{\mathrm{d}} \in H_1^{\mathrm{d}}$ and $\xi = \xi_1 \otimes \xi_2$ then

$$\begin{aligned} \sigma(A)(\theta(\xi))(\eta^{\mathrm{d}}) &= \theta(A_1\xi_1 \otimes A_2\xi_2)(\eta^{\mathrm{d}}) = (A_1\xi_1, \eta) A_2 \xi_2 \\ &= (\xi_1, A_1^*\eta) A_2 \xi_2 = A_2 \theta(\xi_1 \otimes \xi_2)((A_1^*\eta)^{\mathrm{d}}) \\ &= A_2 \theta(\xi_1 \otimes \xi_2) A_1^{\mathrm{d}}(\eta^{\mathrm{d}}) = A_2 \theta(\xi) A_1^{\mathrm{d}}(\eta^{\mathrm{d}}). \end{aligned}$$

It follows by linearity and continuity that $\sigma(A)(\theta(\xi)) = A_2\theta(\xi)A_1^{\mathrm{d}}$, for every $\xi \in H_1 \otimes H_2$. Using Lemma 66 (i) we now obtain

$$\begin{aligned} & \sigma(A)(\theta(\xi))(\theta(\eta_{2,3})^{\mathrm{d}}) \ldots (\theta(\eta_{n-2,n-1}^{\mathrm{d}})) \\ = \ & \theta((A_1 \otimes \ldots A_n)(\xi))(\theta(\eta_{2,3})^{\mathrm{d}}) \ldots (\theta(\eta_{n-2,n-1}^{\mathrm{d}})) = \theta((A_{n-1} \otimes A_n)(\xi_{n-1,n})) \\ & \times \theta(\eta_{n-2,n-1}^{\mathrm{d}}) \ldots \theta((A_3 \otimes A_4)(\xi_{3,4})) \theta(\eta_{2,3}^{\mathrm{d}}) \theta((A_1 \otimes A_2)(\xi_{1,2})) \\ = \ & A_n \theta(\xi_{n-1,n}) A_{n-1}^{\mathrm{d}} \theta(\eta_{n-2,n-1})^{\mathrm{d}} A_{n-2} \ldots A_4 \theta(\xi_{3,4}) A_3^{\mathrm{d}} \theta(\eta_{2,3})^{\mathrm{d}} A_2 \theta(\xi_{1,2}) A_1^{\mathrm{d}} \\ = \ & A_n \theta(\xi)(\theta((A_2^* \otimes A_3^*)(\eta_{2,3}))^{\mathrm{d}})) \ldots (\theta((A_{n-2}^* \otimes A_{n-1}^*)(\eta_{n-2,n-1}))^{\mathrm{d}})) A_1^{\mathrm{d}}. \end{aligned}$$

(ii) By Lemma 66 (ii),

$$\begin{aligned} & \sigma(A)(\theta(\xi))(\theta(\eta_{1,2})^{\mathrm{d}}) \ldots (\theta(\eta_{n-2,n-1})^{\mathrm{d}}) \\ = \ & \theta((A_1 \otimes \ldots A_n)(\xi))(\theta(\eta_{1,2})^{\mathrm{d}}) \ldots (\theta(\eta_{n-2,n-1})^{\mathrm{d}}) \\ = \ & \theta((A_{n-1} \otimes A_n)(\xi_{n-1,n})) \theta(\eta_{n-2,n-1}^{\mathrm{d}}) \ldots \theta(\eta_{1,2}^{\mathrm{d}})(A_1\xi_1) \\ = \ & A_n \theta(\xi_{n-1,n}) A_{n-1}^{\mathrm{d}} \theta(\eta_{n-2,n-1})^{\mathrm{d}} A_{n-2} \ldots A_2^{\mathrm{d}} \theta(\eta_{1,2}^{\mathrm{d}})(A_1\xi_1) \\ = \ & A_n \theta(\xi)(\theta((A_1^* \otimes A_2^*)(\eta_{1,2}))^{\mathrm{d}})) \ldots (\theta((A_{n-2}^* \otimes A_{n-1}^*)(\eta_{n-2,n-1}))^{\mathrm{d}})). \end{aligned}$$

$\diamondsuit$

Let $H_1, \dots, H_n$ be Hilbert spaces. If n is even, we let

$$\Gamma(H_1, \dots, H_n) = (H_1 \otimes H_2) \odot (H_2^{\mathrm{d}} \otimes H_3^{\mathrm{d}}) \odot (H_3 \otimes H_4) \odot \cdots \odot (H_{n-1} \otimes H_n).$$

If n is odd, we let

$$\Gamma(H_1, \dots, H_n) = (H_1^{\mathrm{d}} \otimes H_2^{\mathrm{d}}) \odot (H_2 \otimes H_3) \odot (H_3^{\mathrm{d}} \otimes H_4^{\mathrm{d}}) \odot \cdots \odot (H_{n-1} \otimes H_n).$$

After identifying $\mathbb{C} \otimes H_1$ with H_1, for n odd we have the identification

$$\Gamma(\mathbb{C}, H_1, \dots, H_n) \equiv H_1 \odot \Gamma(H_1, \dots, H_n).$$

Fix $\varphi \in B(H)$. We define a mapping S_φ on $\Gamma(H_1, \dots, H_n)$ taking values in $\mathcal{B}(H_1^{\mathrm{d}}, H_n)$ in the case n is even, and in $\mathcal{B}(H_1, H_n)$, in the case n is odd. Let first n be even. On elementary tensors

$$\zeta = \xi_{1,2} \otimes \eta_{2,3}^{\mathrm{d}} \otimes \xi_{3,4} \otimes \cdots \otimes \xi_{n-1,n} \in \Gamma(H_1, \dots, H_n),$$

we let

$$S_\varphi(\zeta) = \sigma(\varphi)\theta(\xi_{1,2} \otimes \xi_{3,4} \otimes \cdots \otimes \xi_{n-1,n})(\theta(\eta_{2,3}^{\mathrm{d}})) \dots (\theta(\eta_{n-2,n-1}^{\mathrm{d}}))$$

and extend S_φ on the whole of $\Gamma(H_1, \dots, H_n)$ by linearity. Note that the values of S_φ are Hilbert-Schmidt operators. Now assume n is odd. Let $\zeta \in \Gamma(H_1, \dots, H_n)$ and $\xi_1 \in H_1$. Then

$$\xi_1 \otimes \zeta \in H_1 \odot \Gamma(H_1, \dots, H_n) = \Gamma(\mathbb{C}, H_1, \dots, H_n).$$

We let $S_\varphi(\zeta)$ be the operator defined on H_1 by

$$S_\varphi(\zeta)(\xi_1) = S_{1\otimes\varphi}(\xi_1 \otimes \zeta).$$

Note that $S_{1\otimes\varphi}(\xi_1 \otimes \zeta)$ is an element of $\mathcal{C}_2(\mathbb{C}^d, H_n)$, which can be identified with H_n in a natural way. In this way, $S_\varphi(\zeta)(\xi_1)$ can be viewed as an element of H_n. It is clear that the operator $S_\varphi(\zeta) : H_1 \to H_n$ is linear. We moreover claim that $S_\varphi(\zeta)$ is bounded. Let

$$\zeta = \eta_{1,2}^{\mathrm{d}} \otimes \cdots \otimes \xi_{n-1,n} \in \Gamma(H_1, \dots, H_n)$$

and $\xi_1 \in H_1$. Then $S_\varphi(\zeta)$ is a bounded operator and

$$\|S_\varphi(\zeta)\|_{\mathcal{B}(H_1,H_n)} \leq \|\varphi\|_{\mathcal{B}(H)} \|\eta_{1,2}\| \dots \|\eta_{n-2,n-1}\| \|\xi_{2,3}\| \dots \|\xi_{n-1,n}\|. \quad (5.16)$$

In fact, assuming for simplicity that $n = 5$ we have

$$\begin{aligned}
&\|S_\varphi(\zeta)(\xi_1)\| = \|S_{1\otimes\varphi}(\xi_1 \otimes \zeta)\| \\
= \;& \|\sigma(1 \otimes \varphi)\theta((1 \otimes \xi_1) \otimes \xi_{2,3} \otimes \xi_{4,5})(\theta(\eta^{\rm d}_{1,2}))(\theta(\eta^{\rm d}_{3,4}))\| \\
\leq \;& \|\sigma(1 \otimes \varphi)\theta((1 \otimes \xi_1) \otimes \xi_{2,3} \otimes \xi_{4,5})(\theta(\eta^{\rm d}_{1,2}))\|_{\rm op}\|(\theta(\eta^{\rm d}_{3,4}))\| \\
\leq \;& \|\sigma(1 \otimes \varphi)\theta((1 \otimes \xi_1) \otimes \xi_{2,3} \otimes \xi_{4,5})\|_{\rm op}\|\eta_{1,2}\|\|\eta_{3,4}\| \\
\leq \;& \|\varphi\|_{\mathcal{B}(H)}\|\xi_1\|\|\xi_{2,3}\|\|\xi_{4,5}\|\|\eta_{1,2}\|\|\eta_{3,4}\| \\
= \;& \|\varphi\|_{\mathcal{B}(H)}\|\zeta\|_{2,\wedge}\|\xi_1\|.
\end{aligned}$$

Before proceeding, we identify two norms with which the space $\Gamma(H_1, \dots, H_n)$ can be equipped. The first norm on $\Gamma(H_1, \dots, H_n)$ is the projective tensor norm $\|\cdot\|_{2,\wedge}$, where each of the terms $H_i \otimes H_{i+1}$ (resp. $H^{\rm d}_{i-1} \otimes H^{\rm d}_i$) is given its Hilbert space norm. In order to describe the second norm, note that if K_1 and K_2 are Hilbert spaces then $K_1 \otimes K_2$ can be endowed with an operator space structure by letting

$$\|(\xi_{ij})\| = \|\theta(\xi_{ji})\|_{M_m(\mathcal{B}(K^{\rm d}_1, K_2))}, \quad (\xi_{ij}) \in M_m(K_1 \otimes K_2).$$

We write $(K_1 \otimes K_2)^o_{\rm op}$ for this operator space. Note that this is the opposite operator space structure on $\mathcal{C}_2(K^{\rm d}_1, K_2)$, after the identification of $K_1 \otimes K_2$ and $\mathcal{C}_2(K^{\rm d}_1, K_2)$. The norm $\|\cdot\|_{\rm h}$ is the Haagerup norm on $\Gamma(H_1, \dots, H_n)$ when $\Gamma(H_1, \dots, H_n)$ is viewed as the algebraic tensor product of the operator spaces $(H_i \otimes H_{i+1})^o_{\rm op}$ (resp. $(H^{\rm d}_{i-1} \otimes H^{\rm d}_i)^o_{\rm op}$). Thus, the norm $\|u\|_{\rm h}$ of a finite sum $u = \sum_i \xi^i_{1,2} \otimes \ldots \otimes \xi^i_{n-1,n} \in \Gamma(H_1, \dots, H_n)$ of elementary tensors equals the Haagerup norm of the element $\sum_i \theta(\xi^i_{n-1,n}) \otimes \ldots \otimes \theta(\xi^i_{1,2})$.

Remark 68. *For each $\varphi \in B(H)$ and each $\zeta \in \Gamma(H_1, \dots, H_n)$, we have*

$$\|S_\varphi(\zeta)\|_{\rm op} \leq \|\varphi\|_{\mathcal{B}(H)}\|\zeta\|_{2,\wedge}.$$

Proof. In the case where n is odd and ζ is an elementary tensor, the inequality coincides with (5.16). In the case n is even and ζ is an elementary tensor, this is verified similarly. The general case now follows by linearity. $\diamondsuit$

Definition 69. *An element $\varphi \in B(H_1 \otimes \cdots \otimes H_n)$ is called a concrete (operator) multiplier if there exists $C > 0$ such that*

$$\|S_\varphi(\zeta)\|_{\rm op} \leq C\|\zeta\|_{\rm h}, \quad \textit{for each } \zeta \in \Gamma(H_1, \dots, H_n).$$

The smallest such C is denoted by $\|\varphi\|_{\rm m}$.

Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C^*-algebras and $\pi_1, \dots, \pi_n$ be corresponding representations on Hilbert spaces $H_1, \dots, H_n$. An element $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ is called a

$\pi_1, \ldots, \pi_n$-multiplier if $(\pi_1 \otimes \cdots \otimes \pi_n)(\varphi)$ is a concrete multiplier. We denote the set of all $\pi_1, \ldots, \pi_n$-multipliers in $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ by $\mathbf{M}_{\pi_1,\ldots,\pi_n}(\mathcal{A}_1, \ldots \mathcal{A}_n)$. If $\varphi \in \mathbf{M}_{\pi_1,\ldots,\pi_n}(\mathcal{A}_1, \ldots \mathcal{A}_n)$, we let $\|\varphi\|_{\pi_1,\ldots,\pi_n} = \|(\pi_1 \otimes \cdots \otimes \pi_n)(\varphi)\|_{\mathrm{m}}$.

The element $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ is called a universal multiplier if φ is a $\pi_1, \ldots, \pi_n$-multiplier for all representations π_i of $\mathcal{A}_i$, $i = 1, \ldots, n$. We denote by $\mathbf{M}(\mathcal{A}_1, \ldots \mathcal{A}_n)$ the set of all universal multipliers in $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$.

Remark 70. *In the case $n = 2$, Definition 69 reduces to the definition of $\mathcal{C}_\infty$-multipliers studied in [52].*

Next we show that an element $\varphi \in L^\infty(X_1) \otimes \ldots \otimes L^\infty(X_n) \subset L^\infty(X_1 \times \ldots \times X_n)$ is a Schur multiplier as defined in Section 5.1.3 if and only if φ is a $\pi_1, \ldots, \pi_n$-multiplier, where π_i is the canonical representation of $L^\infty(X_i)$ on $L^2(X_i)$ acting by multiplication.

Let $\mathcal{A}$ be a commutative C^*-algebra with maximal ideal space X, acting on a Hilbert space H. It is well-known that, up to unitary equivalence, $H = \oplus_{\gamma\in\Gamma} H_\gamma$, where $H_\gamma = L_2(X, \mu_\gamma)$ is invariant under $\mathcal{A}$ for each $\gamma \in \Gamma$, and an element $f \in \mathcal{A}$ acts as on H_γ by multiplication. Let $j : H \to H$ be given by $\{\xi_\gamma(\lambda)\} \mapsto \{\overline{\xi_\gamma(\lambda)}\}$. Then $V = \partial j$ is a unitary operator from H to H^{d} such that $A^{\mathrm{d}} = VAV^{-1}$ for all $A \in \mathcal{A}$. If K is another Hilbert space then $U(T) = TV$ (resp. $W(S) = V^{-1}S$) is an isometry from $\mathcal{C}_2(H^{\mathrm{d}}, K)$ to $\mathcal{C}_2(H, K)$ (resp. from $\mathcal{C}_2(K, H^{\mathrm{d}})$ to $\mathcal{C}_2(K, H)$).

Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be commutative C^*-algebras and let $\pi_1, \ldots, \pi_n$ be corresponding representations on $H_1, \ldots, H_n$ and $\pi = \pi_1 \otimes \ldots \otimes \pi_n$. Let $V_i : H_i \to H_i^{\mathrm{d}}$ be unitary operator defined above with the property $\pi_i(a_i)^{\mathrm{d}} = V_i \pi_i(a_i) V_i^{-1}$ for each $a_i \in \mathcal{A}_i$, $i = 1, \ldots, n$. Define $U_{i,k} : \mathcal{C}_2(H_i^{\mathrm{d}}, H_k) \to \mathcal{C}_2(H_i, H_k)$ and $W_{i,k} : \mathcal{C}_2(H_i, H_k^{\mathrm{d}}) \to \mathcal{C}_2(H_i, H_k)$ to be $U_{i,k}(T) = TV_i$ and $W_{i,k}(S) = V_k^{-1}S$. Then for $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$, the mapping $S_{\pi(\varphi)}$ can be identified with a mapping $\check{S}_{\pi(\varphi)}$ from $\mathcal{C}_2(H_1, H_2) \odot \mathcal{C}_2(H_2, H_3) \odot \ldots \odot \mathcal{C}_2(H_{n-1}, H_n)$ into $\mathcal{B}(H_1, H_n)$ such that whenever $\varphi = a_1 \otimes \ldots \otimes a_n$ is an elementary tensor then

$$\check{S}_{\pi(\varphi)}(R_1 \otimes \ldots \otimes R_{n-1}) = \pi_n(a_n) R_{n-1} \pi_{n-1}(a_{n-1}) R_{n-2} \ldots R_1 \pi_1(a_1). \quad (5.17)$$

In fact, let $\mathcal{U} = U_{1,2}\theta_{H_1,H_2} \otimes W_{2,3}\theta_{H_2,H_3} \otimes \ldots \otimes U_{n-1,n}\theta_{H_{n-1},H_n}$ if n is even and $\mathcal{U} = W_{1,2}\theta_{H_1,H_2} \otimes U_{2,3}\theta_{H_2,H_3} \otimes \ldots \otimes U_{n-1,n}\theta_{H_{n-1},H_n}$ if n is odd. Then $\mathcal{U}$ maps the space $\Gamma(H_1, H_2 \ldots, H_n)$ onto $\mathcal{C}_2(H_1, H_2) \odot \mathcal{C}_2(H_2, H_3) \odot \ldots \odot \mathcal{C}_2(H_{n-1}, H_n)$ and is an isometry with respect to the norm $\|\cdot\|_{\mathrm{h}}$ (this norm being defined on the algebraic tensor product of the $\mathcal{C}_2$-spaces again as the Haagerup norm where each of the $\mathcal{C}_2$-spaces is equipped with its opposite operator space structure). Let

$$\check{S}_{\pi(\varphi)} = U_{1,n} S_{\pi(\varphi)} \mathcal{U}^{-1}$$

in the case n is even and

$$\check{S}_{\pi(\varphi)} = S_{\pi(\varphi)}\mathcal{U}^{-1}$$

in the case n is odd. Assume that $\varphi = a_1 \otimes \ldots \otimes a_n$. Then, in the case where n is even, we have

$$\begin{aligned}
&\check{S}_{\pi(\varphi)}(R_1 \otimes \ldots \otimes R_{n-1}) \\
&= U_{1,n} S_{\pi(\varphi)} \mathcal{U}^{-1}(R_1 \otimes \ldots \otimes R_{n-1}) \\
&= U_{1,n}(\pi_n(a_n) U_{n-1,n}^{-1}(R_{n-1}) \pi_{n-1}(a_{n-1})^{\mathrm{d}} W_{n-2,n-1}(R_{n-2}) \ldots \pi_1(a_1)^{\mathrm{d}}) \\
&= \pi_n(a_n) R_{n-1} V_{n-1}^{-1} \pi_{n-1}(a_{n-1})^{\mathrm{d}} V_{n-1} R_{n-2} \ldots R_1 V_1^{-1} \pi_1(a_1)^{\mathrm{d}} V_1 \\
&= \pi_n(a_n) R_{n-1} \pi_{n-1}(a_{n-1}) R_{n-2} \ldots R_1 \pi_1(a_1).
\end{aligned}$$

In the case where n is odd one shows in a similar way that (5.17) holds.

Now let $\mathcal{A}_i = L^\infty(X_i)$ and let π_i be the representation of $\mathcal{A}_i$ on $L^2(X_i)$ given by $(\pi_i(f)\xi)(x) = f(x)\xi(x)$, $\xi \in L^2(X_i)$, $i = 1, \ldots, n$.

Suppose n is even. In this case $\check{S}_{\pi(\varphi)}(R_1 \otimes \cdots \otimes R_{n-1})$ is an element of $\mathcal{C}_2(H_1, H_n)$. Using (5.18) and the identification $\psi_{k,l} : f \mapsto T_f$ of $L_2(X_k, X_l)$ with the class of Hilbert-Schmidt operators from $L_2(X_k)$ to $L_2(X_l)$, where

$$(T_f\xi)(y) = \int_{X_k} f(x,y)\xi(x)dx, \quad f \in L_2(X_k \times X_l), \xi \in L^2(X_k), y \in X_l,$$

we obtain that if $f_1 \otimes \ldots \otimes f_{n-1} \in \Gamma(X_1, \ldots, X_n)$ and φ is an elementary tensor then

$$\begin{aligned}
&\psi_{1,n}^{-1}(\check{S}_{\pi(\varphi)}(\psi_{1,2} \otimes \ldots \otimes \psi_{n-1,n})(f_1 \otimes \ldots \otimes f_{n-1}))(x_1, x_n) \qquad (5.18) \\
&= \int_{X_2 \times \ldots \times X_{n-1}} \varphi(x_1, \ldots, x_n) f_1(x_1, x_2) \ldots f_{n-1}(x_{n-1}, x_n) dx_2 \ldots dx_{n-1} \\
&= S_\varphi(f_1 \otimes \ldots \otimes f_{n-1})(x_1, x_n).
\end{aligned}$$

By linearity and continuity, (5.18) holds for any $\varphi \in L^\infty(X_1) \otimes \ldots \otimes L^\infty(X_n)$.

Now assume that n is odd. Let $\xi \in H_1$, $\eta \in H_n$ and $\psi_{0,1} : L^2(X_1) \to \mathcal{C}_2(\mathbb{C}, L^2(X_1))$ be the natural identification. We have that $(S_\varphi(f_1 \otimes \cdots \otimes f_{n-1})\xi, \eta)$ coincides with

$$(\check{S}_{(\mathrm{id}\otimes\pi)(1\otimes\varphi)}(\psi_{0,1} \otimes \cdots \otimes \psi_{n-1,n})((1 \otimes \xi) \otimes f_1 \otimes \cdots \otimes f_{n-1}), \eta)$$

whenever $\varphi \in L^\infty(X_1) \otimes \ldots \otimes L^\infty(X_n)$ is an elementary tensor. By linearity and continuity, we have that $\psi_{1,n}(S_\varphi(f_1 \otimes \cdots \otimes f_{n-1}))$ is equal to

$$\check{S}_{\pi(\varphi)}(\psi_{1,2} \otimes \cdots \otimes \psi_{n-1,n})(f_1 \otimes \cdots \otimes f_{n-1})$$

for all $\varphi \in L^\infty(X_1) \otimes \ldots \otimes L^\infty(X_n)$. In particular, $S_{\pi(\varphi)}$ takes values in $\mathcal{C}_2(H_1, H_n)$. As before, it follows that

$$\begin{aligned}&\psi_{1,n}^{-1}\check{S}_{\pi(\varphi)}(\psi_{1,2} \otimes \ldots \otimes \psi_{n-1,n})(f_1 \otimes \ldots \otimes f_{n-1}))(x_1, x_n) \\ &= S_\varphi(f_1 \otimes \ldots \otimes f_{n-1})(x_1, x_n)\end{aligned} \tag{5.19}$$

for every $\varphi \in L^\infty(X_1) \otimes \ldots \otimes L^\infty(X_n)$. We have thus shown the following

Proposition 71. *An element $\varphi \in L^\infty(X_1) \otimes \ldots \otimes L^\infty(X_n)$ is a Schur multiplier if and only if $\varphi \in \mathbf{M}_{\pi_1,\ldots,\pi_n}(L^\infty(X_1), \ldots, L^\infty(X_n))$.*

Next we want to give a generalisation of Lemma 67 for the case where φ is a sum of elementary tensors. Let V,$V_1, \ldots, V_n$ be vector spaces, $L(V_1, V_2)$ be the space of all linear mappings from V_1 into V_2 and $L(V) = L(V, V)$. Recall that if $f : V_1 \to V_2$ is a linear map, we let $f_{k,l} : M_{k,l}(V_1) \to M_{k,l}(V_2)$ be the mapping given by $f_{k,l}((v_{ij})) = (f(v_{ij}))$, for each $(v_{ij}) \in M_{k,l}(V_1)$. For an element $v = (v_{ij}) \in M_{k,l}(V)$ we denote by $v^{\mathrm{t}} = (v_{ji}) \in M_{l,k}(V)$ the transpose of v. Denote by $d : B(K) \to B(K^{\mathrm{d}})$ the mapping sending A to its dual A^{d}. If $A = (A_{ij}) \in M_{k,l}(B(K))$ let $A^{\mathrm{d}} = (A^{\mathrm{d}}_{ij})$.

We will identify $M_{p,q}(\mathcal{C}_2(K_1, K_2))$ with $\mathcal{C}_2(K_1^q, K_2^p)$. If $\xi \in M_{p,q}(K_1 \otimes K_2)$ then $\theta_{p,q}(\xi) \in M_{p,q}(\mathcal{C}_2(K_1^{\mathrm{d}}, K_2))$; using this identification, we will be considering $\theta_{p,q}(\xi)$ as a Hilbert-Schmidt operator from K_1^q to K_2^p. If $A \in B(K_1, K_2)$ then $A \otimes I_k \in B(K_1^k, K_2^k)$ is the k-fold ampliation of A; under the identification $B(K_1^k, K_2^k) = M_k\ (B(K_1,\ K_2))$, the operator $A \otimes I_k$ has a k by k diagonal matrix, whose every diagonal entry is A.

Lemma 72. *Let $V_1, \ldots, V_n$ be vector spaces, $\mathcal{L}_i \subseteq L(V_i, V_{i+1})$ a subspace, $i = 1, \ldots, n-1$, and*

$$S : (L(V_n) \odot L(V_{n-1}) \odot \cdots \odot L(V_1)) \times (\mathcal{L}_{n-1} \odot \cdots \odot \mathcal{L}_1) \to L(V_1, V_n)$$

be a mapping satisfying

$$S(a_n \otimes \cdots \otimes a_1, \lambda_{n-1} \otimes \cdots \otimes \lambda_1) = a_n \lambda_{n-1} a_{n-1} \ldots \lambda_1 a_1.$$

If $A_1 \in M_{k_1,1}(L(V_1))$, $A_2 \in M_{k_2,k_1}(L(V_2)), \ldots, A_n \in M_{1,k_{n-1}}(L(V_n))$ and $\Lambda_1 \in M_{l_1,1}(\mathcal{L}_1)$, $\Lambda_2 \in M_{l_2,l_1}(\mathcal{L}_2), \ldots, \Lambda_{n-1} \in M_{1,l_{n-2}}(\mathcal{L}_{n-1})$ then

$$S(A_n \odot \cdots \odot A_1, \Lambda_{n-1} \odot \cdots \odot \Lambda_1) = A_n \ldots (\Lambda_2 \otimes I_{k_2})(A_2 \otimes I_{l_1})(\Lambda_1 \otimes I_{k_1})A_1.$$

Proof. "A few moments' thought." $\diamondsuit$

Lemma 73. *Let* $A_1 \in M_{1,k_1}(\mathcal{B}(H_1))$, $A_2 \in M_{k_1,k_2}(\mathcal{B}(H_2)), \ldots, A_n \in M_{k_{n-1},1}(\mathcal{B}(H_n))$ *and* $\varphi = A_1 \odot A_2 \odot \cdots \odot A_n$.

(i) Assume n *is even. Let* $\xi_{1,2} \in M_{1,l_1}(H_1 \otimes H_2)$, $\eta_{2,3} \in M_{l_1,l_2}(H_2^{\mathrm{d}} \otimes H_3^{\mathrm{d}}), \ldots, \xi_{n-1,n} \in M_{l_{n-2},1}(H_{n-1} \otimes H_n)$ *and*

$$\zeta = \xi_{1,2} \odot \eta_{2,3} \odot \cdots \odot \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n).$$

Then

$$S_\varphi(\zeta) = A_n^{\mathrm{t}} \ldots (A_3^{\mathrm{t,d}} \otimes I_{l_2})(\theta_{l_1,l_2}(\eta_{2,3})^{\mathrm{t}} \otimes I_{k_2})(A_2^{\mathrm{t,d}} \otimes I_{l_1})(\theta_{1,l_1}(\xi_{1,2})^{\mathrm{t}} \otimes I_{k_1})A_1^{\mathrm{t,d}}.$$

(ii) Assume n *is odd. Let* $\eta_{1,2} \in M_{1,l_1}(H_1^{\mathrm{d}} \otimes H_2^{\mathrm{d}})$, $\xi_{2,3} \in M_{l_1,l_2}(H_2 \otimes H_3), \ldots, \xi_{n-1,n} \in M_{l_{n-2},1}(H_{n-1} \otimes H_n)$ *and*

$$\zeta = \eta_{1,2} \odot \xi_{2,3} \odot \cdots \odot \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n).$$

Then

$$S_\varphi(\zeta) = A_n^{\mathrm{t}} \ldots (A_3^{\mathrm{t}} \otimes I_{l_2})(\theta_{l_1,l_2}(\xi_{2,3})^{\mathrm{t}} \otimes I_{k_2})(A_2^{\mathrm{t,d}} \otimes I_{l_1})(\theta_{1,l_1}(\eta_{1,2})^{\mathrm{t}} \otimes I_{k_1})A_1^{\mathrm{t}}.$$

Proof. Let $f : V_1 \odot \cdots \odot V_n \to V_n \odot \cdots \odot V_1$ be the flip, namely the map given on elementary tensors by $f(v_1 \otimes \cdots \otimes v_n) = v_n \otimes \cdots \otimes v_1$. Note that if $A_1 \in M_{1,k_1}(V_1)$, $A_2 \in M_{k_1,k_2}(V_2), \ldots, A_n \in M_{k_{n-1},1}(V_n)$ then

$$f(A_1 \odot \cdots \odot A_n) = A_n^{\mathrm{t}} \odot \cdots \odot A_1^{\mathrm{t}}.$$

Let

$$D : B(H_1) \odot B(H_2) \odot \cdots \odot B(H_n) \longrightarrow B(H_n) \odot B(H_{n-1}^{\mathrm{d}}) \odot \cdots \odot B(H_1^{\mathrm{d}})$$

be the map

$$D = f \circ (d \otimes \mathrm{id} \otimes d \otimes \cdots \otimes \mathrm{id}).$$

We have that

$$D(A) = A_n^{\mathrm{t}} \odot A_{n-1}^{\mathrm{t,d}} \odot \cdots \odot A_1^{\mathrm{t,d}}.$$

Define a mapping S from

$$(B(H_n) \odot B(H_{n-1}^{\mathrm{d}}) \odot \cdots \odot B(H_1^{\mathrm{d}})) \times (\mathcal{C}_2(H_{n-1}^{\mathrm{d}}, H_n) \odot \cdots \odot \mathcal{C}_2(H_1^{\mathrm{d}}, H_2))$$

into $\mathcal{C}_2(H_1^{\mathrm{d}}, H_n)$ by

$$S(\psi, \zeta') = S_{D^{-1}(\psi)}(\tilde{\theta}^{-1}(\zeta')),$$

where

$$\tilde{\theta} : \Gamma(H_1, \ldots, H_n) \to \mathcal{C}_2(H_{n-1}^{\mathrm{d}}, H_n) \odot \cdots \odot \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$$

is given on elementary tensors by

$$\tilde{\theta}(\xi_{1,2} \otimes \eta_{2,3} \otimes \cdots \otimes \xi_{n-1,n}) = \theta(\xi_{n-1,n}) \otimes \cdots \otimes \theta(\eta_{2,3}) \otimes \theta(\xi_{1,2}).$$

By Lemma 67 (i), the mapping S satisfies the requirements of Lemma 72 and

$$S_\varphi(\zeta) = S(A_n^{\rm t} \odot A_{n-1}^{\rm t,d} \odot \cdots \odot A_1^{\rm t,d}, \theta_{l_{n-2},1}(\xi_{n-1,n})^{\rm t} \odot \cdots \odot \theta_{1,l_1}(\xi_{1,2})^{\rm t}).$$

The claim now follows from Lemma 72.

The proof of (ii) is similar. $\diamondsuit$

5.2 Multipliers associated with tensor products of representations

It was proved in [52] that the space of all (π, ρ)-multipliers does not change if the representations π and ρ are replaced by approximately equivalent representations. In this section we will prove a corresponding result for multidimensional multipliers. We first recall the notion of approximate equivalence and approximate suborditation introduced by Voiculescu in [95].

Let π and π' be $*$-representations of a C^*-algebra $\mathcal{A}$ on Hilbert spaces H and H', respectively. We say that π' is *approximately subordinate* to π and write $\pi' \overset{a}{\ll} \pi$ if there is a net $\{U_\lambda\}$ of isometries from H' to H such that

$$\|\pi(a)U_\lambda - U_\lambda \pi'(a)\| \to 0 \text{ for all } a \in \mathcal{A}. \tag{5.20}$$

The representations π' and π are said to be *approximately equivalent* if the operators U_λ can be chosen to be unitary; in this case we write $\pi' \overset{a}{\sim} \pi$.

For C*-algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ and corresponding representations $\pi_1, \dots, \pi_n$, we will denote the collection of all $\pi_1, \dots, \pi_n$-multipliers in $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ simply by $\mathbf{M}_{\pi_1,\dots,\pi_n}$, in case there is no danger of confusion.

Theorem 74. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C^*-algebras and π_i and π_i' be representations of $\mathcal{A}_i$ on Hilbert spaces H_i and H_i', respectively, $i = 1, \dots, n$.*

(i) If $\pi_i' \overset{a}{\ll} \pi_i$, $i = 1, \dots, n$, then

$$\mathbf{M}_{\pi_1,\dots,\pi_n} \subseteq \mathbf{M}_{\pi_1',\dots,\pi_n'} \text{ and } \|\varphi\|_{\pi_1',\dots,\pi_n'} \leq \|\varphi\|_{\pi_1,\dots,\pi_n}, \text{ for } \varphi \in \mathbf{M}_{\pi_1,\dots,\pi_n}.$$

(ii) If $\pi_i' \overset{a}{\sim} \pi_i$, $i = 1, \dots, n$, then

$$\mathbf{M}_{\pi_1,\dots,\pi_n} = \mathbf{M}_{\pi_1',\dots,\pi_n'} \text{ and } \|\varphi\|_{\pi_1,\dots,\pi_n} = \|\varphi\|_{\pi_1',\dots,\pi_n'}, \text{ for } \varphi \in \mathbf{M}_{\pi_1,\dots,\pi_n}.$$

Proof. (i) Let first n be even and $\{U_{\lambda_i}\}$ be nets of isometries from H_i' into H_i satisfying

$$\|\pi_i(a_i)U_{\lambda_i} - U_{\lambda_i}\pi_i'(a_i)\| \to 0, \text{ for all } a_i \in \mathcal{A}_i.$$

Set $\pi = \otimes_{i=1}^n \pi_i$, $\pi' = \otimes_{i=1}^n \pi_i'$, $\lambda = (\lambda_1, \dots, \lambda_n)$ and $W_\lambda = U_{\lambda_1} \otimes \dots \otimes U_{\lambda_n}$. Then W_λ are isometries from $\otimes_{i=1}^n H_i'$ to $\otimes_{i=1}^n H_n$ and, for $x \in \mathcal{A}_1 \odot \dots \odot \mathcal{A}_n$, we have

$$\|\pi(x)W_\lambda - W_\lambda \pi'(x)\| \longrightarrow 0.$$

As $\|W_\lambda\| = 1$ for all λ, this holds for all $x \in \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$. By Lemma 67 (i) we have that, for any $\xi \in \otimes_{i=1}^n H_i$,

$$\begin{aligned}&\theta(W_\lambda^*\xi)(\theta(\eta_{2,3}^{\mathrm{d}}))\dots(\theta(\eta_{n-2,n-1}^{\mathrm{d}}))\\ &= U_{\lambda_n}^*\theta(\xi)(\theta((W_{\lambda_2,\lambda_3}\eta_{2,3})^{\mathrm{d}}))\dots(\theta((W_{\lambda_{n-2},\lambda_{n-1}}\eta_{n-2,n-1})^{\mathrm{d}}))(U_{\lambda_1}^*)^{\mathrm{d}}\end{aligned}$$

where $W_{\lambda_k,\lambda_{k+1}} = U_{\lambda_k} \otimes U_{\lambda_{k+1}}$. Therefore, if $\zeta = \xi_{1,2} \otimes (\eta_{2,3})^{\mathrm{d}} \otimes \dots \otimes \xi_{n-1,n}$, then

$$\begin{aligned}&S_{W_\lambda^*\pi(\varphi)W_\lambda}(\zeta) = \\ &= U_{\lambda_n}^* S_{\pi(\varphi)}(W_{\lambda_1,\lambda_2}\xi_{1,2} \otimes (W_{\lambda_2,\lambda_3}\eta_{2,3})^{\mathrm{d}} \otimes \dots \otimes W_{\lambda_{n-1},\lambda_n}\xi_{n-1,n})(U_{\lambda_1}^*)^{\mathrm{d}}.\end{aligned} \tag{5.21}$$

Let $\Gamma_\lambda : \Gamma(H_1', \dots, H_n') \to \Gamma(H_1, \dots, H_n)$ be the linear operator defined on elementary tensors by

$$\Gamma_\lambda(\xi_{1,2} \otimes \eta_{2,3}^{\mathrm{d}} \otimes \dots \otimes \xi_{n-1,n}) = W_{\lambda_1,\lambda_2}\xi_{1,2} \otimes (W_{\lambda_2,\lambda_3}\eta_{2,3})^{\mathrm{d}} \otimes \dots \otimes W_{\lambda_{n-1},\lambda_n}\xi_{n-1,n}.$$

It follows from (5.21) and Remark 68 that if $\varphi \in \mathbf{M}_{\pi_1,\dots,\pi_n}$ and $\zeta \in \Gamma(H_1', \dots, H_n')$ then

$$\begin{aligned}\|S_{\pi'(\varphi)}(\zeta)\|_{\mathrm{op}} &\leq \|S_{W_\lambda^*\pi(\varphi)W_\lambda}(\zeta)\|_{\mathrm{op}} + \|S_{W_\lambda^*\pi(\varphi)W_\lambda - \pi'(\varphi)}(\zeta)\|_{\mathrm{op}}\\ &\leq \|S_{\pi(\varphi)}(\Gamma_\lambda\zeta)\|_{\mathrm{op}} + \|S_{W_\lambda^*\pi(\varphi)W_\lambda - \pi'(\varphi)}(\zeta)\|_{\mathrm{op}}\\ &\leq \|\varphi\|_{\pi_1,\dots,\pi_n}\|\Gamma_\lambda\zeta\|_{\mathrm{h}} + \|W_\lambda^*\pi(\varphi)W_\lambda - \pi'(\varphi)\|_{\mathrm{op}}\|\zeta\|_{2,\wedge}.\end{aligned}$$

Since $\|W_\lambda^*\pi(\varphi)W_\lambda - \pi'(\varphi)\|_{\mathrm{op}} \to 0$, in order to prove that $\varphi \in \mathbf{M}_{\pi_1',\dots,\pi_n'}$, it suffices to show that $\|\Gamma_\lambda\zeta\|_{\mathrm{h}} \leq \|\zeta\|_{\mathrm{h}}$. If $\xi_{i,i+1} \in H_i' \otimes H_{i+1}'$ then $\theta(W_{\lambda_i,\lambda_{i+1}}\xi_{i,i+1}) = U_{\lambda_{i+1}}\theta(\xi_{i,i+1})U_{\lambda_i}^{\mathrm{d}}$. Let $\zeta \in \Gamma(H_1', \dots, H_n')$ be of the form

$$\zeta = \xi_{1,2} \otimes \eta_{2,3}^{\mathrm{d}} \otimes \dots \otimes \xi_{n-1,n}$$

where $\xi_{1,2} \in M_{1,k_2}(H_1' \otimes H_2')$, $\eta_{2,3}^{\mathrm{d}} \in M_{k_2,k_3}((H_2')^{\mathrm{d}} \otimes (H_3')^{\mathrm{d}}), \dots$, and $\xi_{n-1,n} \in M_{k_{n-1},1}(H_{n-1}' \otimes H_n')$ are such that

$$\|\zeta\|_{\mathrm{h}} = \|\theta_{1,k_2}(\xi_{1,2})^{\mathrm{t}}\|_{\mathrm{op}}\|\theta_{k_2,k_3}(\eta_{2,3}^{\mathrm{d}})^{\mathrm{t}}\|_{\mathrm{op}} \dots \|\theta_{k_{n-1},1}(\xi_{n-1,n})^{\mathrm{t}}\|_{\mathrm{op}}.$$

Then

$$\Gamma_\lambda\zeta = W_{\lambda_1,\lambda_2}\xi_{1,2} \odot (W^{*,\mathrm{d}}_{\lambda_2,\lambda_3} \otimes I_{k_2})\eta^{\mathrm{d}}_{2,3} \odot \ldots \odot (W_{\lambda_{n-1},\lambda_n} \otimes I_{k_{n-1}})\xi_{n-1,n}$$

and as

$$\begin{array}{rcl} \theta_{1,k_2}(W_{\lambda_1,\lambda_2}\xi_{1,2}) & = & U_{\lambda_2}\theta_{1,k_2}(\xi_{1,2})(U^{\mathrm{d}}_{\lambda_1} \otimes I_{k_2}), \\ \theta_{k_2,k_3}(((W^*_{\lambda_2,\lambda_3})^{\mathrm{d}} \otimes I_{k_2})\eta^{\mathrm{d}}_{2,3}) & = & (U^{\mathrm{d}}_{\lambda_3} \otimes I_{k_2})\theta_{2,3}(\eta^{\mathrm{d}}_{2,3})(U_{\lambda_2} \otimes I_{k_3}), \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \theta_{k_{n-1},1}((W_{\lambda_{n-1},\lambda_n} \otimes I_{k_{n-1}})\xi_{n-1,n}) & = & (U_{\lambda_n} \otimes I_{k_{n-1}})\theta_{k_{n-1},1}(\xi_{n-1,n})U^{\mathrm{d}}_{\lambda_{n-1}}, \end{array}$$

we get

$$\begin{array}{rcl} \|\Gamma_\lambda\zeta\|_{\mathrm{h}} & \le & \|U_{\lambda_2} \otimes I_{k_2}\|_{\mathrm{op}}\|\theta_{1,k_2}(\xi_{1,2})^{\mathrm{t}}\|_{\mathrm{op}}\|U^{\mathrm{d}}_{\lambda_1}\|_{\mathrm{op}}\ldots \\ & \ldots & \|\theta_{k_{n-1},1}(\xi_{n-1,n})^{\mathrm{t}}\|_{\mathrm{op}}\|U^{\mathrm{d}}_{\lambda_{n-1}} \otimes I_{k_{n-1}}\|_{\mathrm{op}} \\ & = & \|\theta_{1,k_2}(\xi_{1,2})^{\mathrm{t}}\|_{\mathrm{op}}\ldots\|\theta_{k_{n-1},1}(\xi_{n-1,n})^{\mathrm{t}}\|_{\mathrm{op}} = \|\zeta\|_{\mathrm{h}} \end{array}$$

This completes the proof for the case where n is even. Now assume that n is odd and let $\Gamma_\lambda : \Gamma(H'_1, \ldots, H'_n) \to \Gamma(H_1, \ldots, H_n)$ be the linear operator defined on elementary tensors by

$$\Gamma_\lambda(\xi^{\mathrm{d}}_{1,2} \otimes \ldots \otimes \eta_{n-1,n}) = (W_{\lambda_1,\lambda_2}\xi_{1,2})^{\mathrm{d}} \otimes \otimes \ldots \otimes W_{\lambda_{n-1},\lambda_n}\eta_{n-1,n}.$$

An estimate similar to the above shows again that $\|\Gamma_\lambda\zeta\|_{\mathrm{h}} \le \|\zeta\|_{\mathrm{h}}$.

By the definition of the map $S_{\pi'(\varphi)}$ and the arguments above, we obtain

$$\begin{aligned} &\|S_{\pi'(\varphi)}(\zeta)\|_{\mathrm{op}} \le \|S_{W^*_\lambda\pi(\varphi)W_\lambda}(\zeta)\|_{\mathrm{op}} + \|S_{(W^*_\lambda\pi(\varphi)W_\lambda - \pi'(\varphi))}(\zeta)\|_{\mathrm{op}} \\ &= \sup_{\xi_1 \in H'_1, \|\xi_1\|=1} \|S_{1\otimes W^*_\lambda\pi(\varphi)W_\lambda}(\xi_1 \otimes \zeta)\|_{H'_n} + \|S_{(W^*_\lambda\pi(\varphi)W_\lambda - \pi'(\varphi))}(\zeta)\|_{\mathrm{op}} \\ &\le \sup_{\xi_1 \in H'_1, \|\xi_1\|=1} \|S_{1\otimes\pi(\varphi)}(U_{\lambda_1}\xi_1 \otimes \Gamma_\lambda\zeta)\|_{H_n} + \|S_{(W^*_\lambda\pi(\varphi)W_\lambda - \pi'(\varphi))}(\zeta)\|_{\mathrm{op}} \end{aligned}$$

$$\begin{aligned} &\le \sup_{\eta_1 \in H_1, \|\eta_1\|=1} \|S_{1\otimes\pi(\varphi)}(\eta_1 \otimes \Gamma_\lambda\zeta)\|_{H_n} + \|W^*_\lambda\pi(\varphi)W_\lambda - \pi'(\varphi)\|_{\mathrm{op}}\|\zeta\|_{2,\wedge} \\ &= \|S_{\pi(\varphi)}(\Gamma_\lambda\zeta)\|_{\mathrm{op}} + \|W^*_\lambda\pi(\varphi)W_\lambda - \pi'(\varphi)\|_{\mathrm{op}}\|\zeta\|_{2,\wedge} \\ &\le \|\varphi\|_{\pi_1,\ldots,\pi_n}\|\|\Gamma_\lambda\zeta\|_{\mathrm{h}} + \|W^*_\lambda\pi(\varphi)W_\lambda - \pi'(\varphi)\|_{\mathrm{op}}\|\zeta\|_{2,\wedge} \\ &\le \|\varphi\|_{\pi_1,\ldots,\pi_n}\|\|\zeta\|_{\mathrm{h}} + \|W^*_\lambda\pi(\varphi)W_\lambda - \pi'(\varphi)\|_{\mathrm{op}}\|\zeta\|_{2,\wedge}. \end{aligned}$$

As $\|W^*_\lambda\pi(\varphi)W_\lambda - \pi'(\varphi)\|_{\mathrm{op}} \to 0$ we obtain the desired statement.

(ii) is a direct consequence of (i). $\diamondsuit$

For $T \in B(H)$, set $\mathrm{rank}(T) = \overline{\dim(TH)}$. It was proved in [38, Theorem 5.1] that for $*$-representations π and π' of a C^*-algebra $\mathcal{A}$

$$\pi' \overset{a}{\ll} \pi \iff \mathrm{rank}(\pi'(a)) \le \mathrm{rank}(\pi(a)) \text{ for each } a \in \mathcal{A}. \tag{5.22}$$

The next statement is a multidimensional version of [52, Corollary 5.3]. Its proof follows the lines of the proof of the corresponding statement in the two dimensional case and uses Theorem 74 instead of [52, Theorem 5.2].

Corollary 75. *Let π_i, π_i' be representations of separable C^*-algebra $\mathcal{A}_i$, $i = 1, \ldots, n$. Assume that*

$$\min\{\aleph_0, \mathrm{rank}(\pi_i'(a_i))\} \le \min\{\aleph_0, \mathrm{rank}(\pi_i(a_i))\}$$

for each $a_i \in \mathcal{A}_i$ and $i = 1, \ldots, n$.

Then $\mathbf{M}_{\pi_1,\ldots,\pi_n} \subseteq \mathbf{M}_{\pi_1',\ldots,\pi_n'}$ and $\|\varphi\|_{\pi_1',\ldots,\pi_n'} \le \|\varphi\|_{\pi_1,\ldots,\pi_n}$ for $\varphi \in \mathbf{M}_{\pi_1,\ldots,\pi_n}$.

Recall that a $*$-representation π of a C^*-algebra $\mathcal{A}$ has a separating vector if there is a cyclic vector for the commutant $\pi(\mathcal{A})'$.

Lemma 76. *Let $\mathcal{H}, H_1, \ldots, H_n$ be Hilbert spaces, $\pi_1, \ldots, \pi_n$ be representations of the C^*-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ on $H_1, \ldots, H_n$ and $\pi_i \otimes 1$ be the amplification of π_i on $H_i \otimes \mathcal{H}$, respectively. Assume that π_1 and π_n have separating vectors. Then*

$$\mathbf{M}_{\pi_1,\ldots,\pi_n} = \mathbf{M}_{\pi_1\otimes 1,\ldots,\pi_n\otimes 1}$$

and the multiplier norms on these spaces coincide.

Proof. We use ideas from the proofs of [88, Theorem 2.1] and Lemma 62. For simplicity we assume that $n = 3$ and that $\mathcal{H}$ is separable. Let $\varphi \in \mathbf{M}_{\pi_1,\pi_2,\pi_3}$ with $\|\varphi\|_{\pi_1,\pi_2,\pi_3} = 1$ and set $S = S_{(\pi_1\otimes 1)\otimes(\pi_2\otimes 1)\otimes(\pi_3\otimes 1)(\varphi)}$. The mapping S can be regarded as a mapping on

$$\mathcal{C}_2((H_2 \otimes \mathcal{H})^{\mathrm{d}}, H_3 \otimes \mathcal{H}) \odot \mathcal{C}_2((H_1 \otimes \mathcal{H}), (H_2 \otimes \mathcal{H})^{\mathrm{d}}) \tag{5.23}$$

by setting $S(\theta(\xi_{2,3}) \otimes \theta(\eta_{1,2}^{\mathrm{d}})) = S(\eta_{1,2}^{\mathrm{d}} \otimes \xi_{2,3})$ for $\zeta = \eta_{1,2}^{\mathrm{d}} \otimes \xi_{2,3} \in \Gamma(H_1 \otimes \mathcal{H}, H_2 \otimes \mathcal{H}, H_3 \otimes \mathcal{H})$. Similarly, the mapping $S_{\pi_1\otimes\pi_2\otimes\pi_3(\varphi)}$ can be regarded as a mapping on $\mathcal{C}_2(H_2^{\mathrm{d}}, H_3) \odot \mathcal{C}_2((H_1, H_2^{\mathrm{d}})$. It follows from Lemma 73 that $S_{\pi_1\otimes\pi_2\otimes\pi_3(\varphi)}$ is $\pi_3(\mathcal{A}_3)', (\pi_2(\mathcal{A}_2)')^{\mathrm{d}}, \pi_1(\mathcal{A}_1)'$-modular.

Assume that $\|\varphi\|_{\pi_1\otimes 1,\pi_2\otimes 1,\pi_3\otimes 1} > 1$. Then there exists an element $T = (T_1^2, \ldots, T_s^2) \odot (T_1^1, \ldots, T_s^1)^{\mathrm{t}}$ in the space defined in (5.23) with

$$\|\sum (T_i^1)^* T_i^1\| \|\sum T_i^2 (T_i^2)^*\| = 1$$

and vectors $\xi_0 \in H_1 \otimes \mathcal{H}$, $\eta_0 \in H_3 \otimes \mathcal{H}$ of norm less than one such that

$$|(S(T)\xi_0, \eta_0)| > 1.$$

Fix a basis $\{f_l\}$ of $\mathcal{H}$ and denote by P_n the projection onto the space generated by the first n vectors in this basis. Then, as

$$(1_{H_3} \otimes P_n)S(T)(1_{H_1} \otimes P_n) \to S(T),$$

weakly, there exists $n \geq 1$ such that

$$|((1_{H_3} \otimes P_n)S(T)(1_{H_1} \otimes P_n)\xi_0, \eta_0)| > 1.$$

Thus we may assume that $\xi_0 \in H_1 \otimes P_n\mathcal{H}$ and $\eta_0 \in H_3 \otimes P_n\mathcal{H}$, say

$$\xi_0 = (\xi_1, \ldots, \xi_n, 0, \ldots), \eta_0 = (\eta_1, \ldots, \eta_n, 0. \ldots).$$

As $\pi_1(\mathcal{A}_1)'$ and $\pi_3(\mathcal{A}_3)'$ have cyclic vectors, say ξ and η respectively, we may assume that $\xi_i = a_i\xi$, $\eta_i = b_i\eta$ for some $a_i \in \pi_1(\mathcal{A}_1)'$ and $b_i \in \pi_3(\mathcal{A}_3)'$. Let $a = \sum a_i^* a_i$, $b = \sum b_i^* b_i$. Assuming first that a, b are invertible we set $\tilde{a}_i = a_i a^{-1/2}$, $\tilde{b}_i = b_i b^{-1/2}$. Then for $\tilde{\xi} = a^{1/2}\xi$, $\tilde{\eta} = b^{1/2}\eta$ we have $\xi_i = \tilde{a}_i\tilde{\xi}$ and $\eta_i = \tilde{b}_i\tilde{\eta}$. We write $T_i^k = ((T_i^k)_{lm})$, where $(T_i^1)_{lm} = (1_{H_2^{\mathrm{d}}} \otimes P(f_l^{\mathrm{d}}))T_i^1(1_{H_1} \otimes P(f_m))$, $(T_i^2)_{lm} = (1_{H_3} \otimes P(f_l))T_i^2(1_{H_2^{\mathrm{d}}} \otimes P(f_m^{\mathrm{d}}))$, where $P(f)$ is the projection onto the one dimensional space generated by f. Using the modularity of $S_{\pi_1 \otimes \pi_2 \otimes \pi_3(\varphi)}$, we obtain

$$\begin{aligned} |(S(T)\xi_0, \eta_0)| &= \left|\sum_{i=1}^{s}(S(T_i^2 \otimes T_i^1)\xi_0, \eta_0)\right| \\ &= \left|\sum_{i=1}^{s}\sum_{l,m=1}^{n}\sum_{k=1}^{\infty}(S_{\pi_1\otimes\pi_2\otimes\pi_3(\varphi)}((T_i^2)_{lk} \otimes (T_i^1)_{km})\tilde{a}_m\tilde{\xi}, \tilde{b}_l\tilde{\eta})\right| \qquad (5.24) \\ &= \left|\sum_{i=1}^{s}\sum_{l,m=1}^{n}\sum_{k=1}^{\infty}(S_{\pi_1\otimes\pi_2\otimes\pi_3(\varphi)}(\tilde{b}_l^*(T_i^2)_{lk} \otimes (T_i^1)_{km}\tilde{a}_m)\tilde{\xi}, \tilde{\eta})\right|. \end{aligned}$$

The next step is to prove that $\displaystyle\sum_{i=1}^{s}\sum_{k=1}^{\infty}\left(\sum_{l=1}^{n}\tilde{b}_l^*(T_i^2)_{lk}\right) \otimes \left(\sum_{m=1}^{n}(T_i^1)_{km}\tilde{a}_m\right)$ belongs to $\mathcal{K}(H_2^{\mathrm{d}}, H_3) \otimes_{\mathrm{h}} \mathcal{K}(H_1, H_2^{\mathrm{d}})$. Observe first that the row operator

$$R_i = (\sum_{l=1}^{n}\tilde{b}_l^*(T_i^2)_{l1}, \ldots, \sum_{l=1}^{n}\tilde{b}_l^*(T_i^2)_{lk}, \ldots)$$

is equal to the product of the row operator $\tilde{B} = (\tilde{b}_1, \ldots, \tilde{b}_n, 0, \ldots)$ and the Hilbert-Schmidt operator T_i^2. Set $R = (R_1, \ldots, R_s) = (\tilde{B}T_1^2, \ldots, \tilde{B}T_s^2)$.

As each T_i^2 is the operator norm-limit of operators $T_i^2(1_{H_2^{\mathrm{d}}} \otimes P_k)$ as $k \to \infty$, the operator R_i is the uniform limit of the sequence of truncated operators $R_i^k = (\sum_{l=1}^n \tilde{b}_l^*(T_i^2)_{l1}, \ldots, \sum_{l=1}^n \tilde{b}_l^*(T_i^2)_{lk}, 0 \ldots)$. Thus

$$RR^* = \sum_{i=1}^{s} \sum_{k=1}^{\infty} \left(\sum_{l=1}^{n} \tilde{b}_l^*(T_i^2)_{lk} \right) \left(\sum_{l=1}^{n} \tilde{b}_l^*(T_i^2)_{lk} \right)^*,$$

where the series converges uniformly and

$$\begin{aligned}\| \sum_{i=1}^{s} \sum_{k=1}^{\infty} (\sum_{l=1}^{n} \tilde{b}_l^*(T_i^2)_{lk})(\sum_{l=1}^{n} \tilde{b}_l^*(T_i^2)_{lk})^* \| &= \|RR^*\| = \| \sum_{i=1}^{s} R_i R_i^* \| \\ &= \|\tilde{B}(\sum_{i=1}^{s} T_i^2 (T_i^2)^*)\tilde{B}^*\| \le \|\tilde{B}\|^2 \| \| \sum_{i=1}^{s} T_i^2 (T_i^2)^* \| \le 1.\end{aligned}$$

In the same way one shows that the series $\sum_{k=1}^{\infty} (\sum_{m=1}^{n} (T_i^1)_{km}\tilde{a}_m)(\sum_{m=1}^{n} (T_i^1)_{km}\tilde{a}_m)^*$ converges uniformly and

$$\| \sum_{i=1}^{s} \sum_{k=1}^{\infty} (\sum_{m=1}^{n} (T_i^1)_{km}\tilde{a}_m)(\sum_{m=1}^{n} (T_i^1)_{km}\tilde{a}_m)^* \| \le 1.$$

Thus $\sum_{i=1}^{s} \sum_{k=1}^{\infty} (\sum_{l=1}^{n} \tilde{b}_l^*(T_i^2)_{lk}) \otimes (\sum_{m=1}^{n} (T_i^1)_{km}\tilde{a}_m) \in \mathcal{K}(H_1, H_2^{\mathrm{d}}) \otimes_{\mathrm{h}} \mathcal{K}(H_2^{\mathrm{d}}, H_3)$ and

$$\| \sum_{i=1}^{s} \sum_{k=1}^{\infty} (\sum_{l=1}^{n} \tilde{b}_l^*(T_i^2)_{lk}) \otimes (\sum_{m=1}^{n} (T_i^1)_{km}\tilde{a}_m) \|_{\mathrm{h}} \le 1.$$

Next $\|\tilde{\xi}\|^2 = (b^{1/2}\xi, b^{1/2}\xi) = (b\xi, \xi) = \sum_i (b_i\xi, b_i\xi) = \|\xi_0\|^2 < 1$. Similarly, $\|\tilde{\eta}\| < 1$. Since $\|\varphi\|_{\pi_1,\pi_2,\pi_3} = 1$, it now follows from (5.24) that

$$|(S(T)\xi_0, \eta_0)| \le \left\| \sum_{i=1}^{s} \sum_{k=1}^{\infty} \left(\sum_{l=1}^{n} \tilde{b}_l^*(T_i^2)_{lk} \right) \otimes \left(\sum_{m=1}^{n} (T_i^1)_{km}\tilde{a}_m \right) \right\|_{\mathrm{h}} \|\tilde{\xi}\| \|\tilde{\eta}\| \le 1,$$

a contradiction.

If a or b is not invertible, let $\varepsilon > 0$ be such that $\hat{\xi}_0 \stackrel{def}{=} (\xi_1, \ldots, \xi_n, \varepsilon\xi, 0, \ldots)$ and $\hat{\eta}_0 \stackrel{def}{=} (\eta_1, \ldots, \eta_n, \varepsilon\eta, 0, \ldots)$ have norm less than one and $|(S(T)\hat{\xi}_0, \hat{\eta}_0)| > 1$. Choose a_i and b_i in the same way as before, and let $a_{n+1} = \varepsilon I$, $b_{n+1} = \varepsilon I$,

$a = \sum_{i=1}^{n+1} a_i^* a_i$ and $b = \sum_{i=1}^{n+1} b_i^* b_i$. Then a and b are invertible and the proof proceeds in the same fashion.

We have proved that $\mathbf{M}_{\pi_1,\dots,\pi_n} \subseteq \mathbf{M}_{\pi_1\otimes 1,\dots,\pi_n\otimes 1}$ and that $\|\cdot\|_{\pi_1\otimes 1,\dots,\pi_n\otimes 1} \leq \|\cdot\|_{\pi_1,\dots,\pi_n}$. The converse inequality is easy to show, and thus the proof is complete. $\diamondsuit$

Corollary 77. *Let π_i be a representation of the C*-algebra $\mathcal{A}_i$, $i = 1,\dots,n$. Assume that π_1 and π_n have separating vectors. If*

$$\ker(\pi_i) \subseteq \ker(\pi_i'), \text{ for each } i = 1,\dots,n, \tag{5.25}$$

then $\mathbf{M}_{\pi_1,\dots,\pi_n} \subseteq \mathbf{M}_{\pi_1',\dots,\pi_n'}$ *and* $\|\varphi\|_{\pi_1',\dots,\pi_n'} \leq \|\varphi\|_{\pi_1,\dots,\pi_n}$, *for each* $\varphi \in \mathbf{M}_{\pi_1,\dots,\pi_n}$.

Proof. The proof is similar to that of [52, Corollary 5.8]; we include it for completeness. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space of sufficiently large dimension. Then (5.25) implies

$$\mathrm{rank}(\pi_i'(a_i)) \leq \mathrm{rank}(\pi_i(a_i) \otimes 1), \text{ for all } a_i \in \mathcal{A}_i.$$

By (5.22), $\pi_i' \overset{a}{\ll} \pi_i \otimes 1$. Applying now Theorem 74 and then Lemma 76 we obtain the statement. $\diamondsuit$

Using Corollary 77 and results from [52] we will now show that if the C*-algebras $\mathcal{A}_i$ are commutative then the space $\mathbf{M}_{\pi_1,\dots,\pi_n}(\mathcal{A}_1,\dots,\mathcal{A}_n)$ of multipliers depends only on the supports of spectral measures corresponding to the representations π_i.

Assume that $\mathcal{A}_i$ is commutative, $i = 1,\dots,n$ and let X_i be the maximal ideal spaces of $\mathcal{A}_i$; then $\mathcal{A}_i \simeq C_0(X_i)$. Let π_i be a representation of $\mathcal{A}_i$ and $\mathcal{E}_{\pi_i}$ be the spectral measure on X_i corresponding to π_i.

It was proved in [52, Lemma 7.2] that if $f \in C_0(X)$ and the representation π of $C_0(X)$ is such that rank $(\pi(f)) < \infty$ then

$$\mathrm{rank}\ (\pi(f)) = \sum_{x \in S(f,\mathcal{E}_\pi)} \dim(\mathcal{E}_\pi(\{x\})),$$

where $S(f,\mathcal{E}_\pi) = \{x \in \mathrm{supp}\,\mathcal{E}_\pi : f(x) \neq 0\}$. Thus the condition

$$\mathrm{supp}\,\mathcal{E}_{\pi'} \subset \mathrm{supp}\,\mathcal{E}_\pi$$

implies $\ker \pi(f) \subseteq \ker \pi'(f)$. As each representation π of a commutative algebra $C_0(X)$ has a separating vector we have the following

Corollary 78. *Let π_i, π_i' be separable representations of the C*-algebra $\mathcal{A}_i = C_0(X_i)$ and $\mathcal{E}_{\pi_i}$ and $\mathcal{E}_{\pi_i'}$ be the corresponding spectral measures ($i = 1, \ldots, n$). If*

$$\operatorname{supp} \mathcal{E}_{\pi_i'} \subseteq \operatorname{supp} \mathcal{E}_{\pi_i}, \text{ for each } i = 1, \ldots, n,$$

then $\mathbf{M}_{\pi_1,\ldots,\pi_n} \subseteq \mathbf{M}_{\pi_1',\ldots,\pi_n'}$.

Let μ_i be measures on X_i. Let π_i be a representation of $C_0(X_i)$ on $L_2(X_i, \mu_i)$ defined by $(\pi_i(f)h)(x_i) = f(x_i)h(x_i)$. We call $\varphi \in C_0(X_1 \times \ldots \times X_n)$ a $(\mu_1, \ldots, \mu_n)$-*multiplier* if $\varphi \in \mathbf{M}_{\pi_1,\ldots,\pi_n}$ and let $\|\varphi\|_{\mu_1,\ldots,\mu_n} = \|\varphi\|_{\pi_1,\ldots,\pi_n}$.

By Corollary 78, the set of the all $(\mu_1, \ldots, \mu_n)$-multipliers depends only on the supports of measures μ_i. The next statement shows the connection between $(\mu_1, \ldots, \mu_n)$-multipliers and multidimensional Schur multipliers (with respect to discrete measures).

Corollary 79. *Let X_i be locally compact spaces with countable bases and let μ_i be Borel σ-finite measures on X_i with $\operatorname{supp} \mu_i = X_i$. Then $\varphi \in C_0(X_1 \times \ldots \times X_n)$ is a $(\mu_1, \ldots, \mu_n)$-multiplier if and only if φ is a Schur multiplier on $X_1 \times \ldots \times X_n$. Moreover, in this case $\|\varphi\|_{\mu_1,\ldots,\mu_n} = \|S_\varphi\|$.*

Proof. The proof is similar to that of [52, Theorem 7.5]. $\diamondsuit$

5.3 Universal multipliers. Kissin-Shulman conjecture.

The main goal of this section is to give a full description of the multipliers which do not depend on the choice of the representations of the C*-algebras $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$. Recall that an element $\varphi \in \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$ is called a **universal multiplier** if φ is a $\pi_1, \pi_2, \ldots, \pi_n$-multiplier for all representations π_1, π_2,..., π_n of $\mathcal{A}_1$, $\mathcal{A}_2, \ldots, \mathcal{A}_n$, respectively. The set of all universal multipliers in $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ is denoted by $\mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

Along with the universal multipliers, we will describe another class of multipliers which we call projective universal multipliers and define as follows. Let $H_1, \ldots, H_n$ be Hilbert spaces. Equip $\Gamma(H_1, \ldots, H_n)$ with the projective tensor norm $\|\cdot\|_\wedge$, where each of the terms $H_i \otimes H_{i+1}$ (resp. $H_{i-1}^{\mathrm{d}} \otimes H_i^{\mathrm{d}}$) is given its operator norm. We call an element $\varphi \in \mathcal{B}(H_1 \otimes \cdots \otimes H_n)$ a concrete projective multiplier if there exists $C > 0$ such that $\|S_\varphi(\zeta)\|_{\mathrm{op}} \le C\|\zeta\|_\wedge$, for all $\zeta \in \Gamma(H_1, \ldots, H_n)$. If $\mathcal{A}_1, \ldots \mathcal{A}_n$ are C*-algebras, an element $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ will be called a **projective universal multiplier** if $(\pi_1 \otimes \cdots \otimes \pi_n)(\varphi)$ is a concrete projective multiplier for all choices of representations $\pi_1, \ldots, \pi_n$

of $\mathcal{A}_1, \dots, \mathcal{A}_n$, respectively. We denote by $\mathbf{M}^\wedge(\mathcal{A}_1, \dots, \mathcal{A}_n)$ the set of all projective universal multipliers.

If $\varphi \in \mathbf{M}(\mathcal{A}_1, \dots, \mathcal{A}_n)$ let

$$\|\varphi\|_{\text{univ}} = \sup_{\pi_1,\pi_2,\dots,\pi_n} \|\varphi\|_{\pi_1,\pi_2,\dots,\pi_n}.$$

Note that $\|\varphi\|_{\text{univ}}$ is finite. In fact, assume that there exist representations $\pi_{1,k}, \dots, \pi_{n,k}$, such that $\|\varphi\|_{\pi_{1,k},\pi_{2,k},\dots,\pi_{n,k}} \to_{k\to\infty} \infty$ and let $\pi_1 = \bigoplus_k \pi_{1,k}$, $\pi_2 = \bigoplus_k \pi_{2,k}, \dots, \pi_n = \bigoplus_k \pi_{n,k}$. Then, by Theorem 74,

$$\|\varphi\|_{\pi_{1,k},\pi_{2,k},\dots,\pi_{n,k}} \leq \|\varphi\|_{\pi_1,\pi_2,\dots,\pi_n},$$

for all $k \in \mathbb{N}$, which contradicts the fact that $\varphi \in \mathbf{M}(\mathcal{A}_1, \dots, \mathcal{A}_n)$.

It is clear that $\mathbf{M}(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is a linear subspace of $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$ containing $\mathcal{A}_1 \odot \dots \odot \mathcal{A}_n$.

Recall that the Haagerup norm on $\mathcal{A}_1 \odot \mathcal{A}_2 \odot \dots \odot \mathcal{A}_n$ is

$$\|\omega\|_{\text{h}} = \inf\{\|\omega_1\|\|\omega_2\| \dots \|\omega_n\| : \omega = \omega_1 \odot \omega_2 \odot \dots \odot \omega_n,$$
$$\omega_1 \in M_{1,i_1}(\mathcal{A}_1), \omega_2 \in M_{i_1,i_2}(\mathcal{A}_2), \dots, \omega_n \in M_{i_{n-1},1}(\mathcal{A}_n), i_1, i_2, \dots, i_{n-1} \in \mathbb{N}\}.$$

A modification of the Haagerup norm on the algebraic tensor product of two C^*-algebras was considered in [44, 52]. We now introduce a natural generalisation of this norm for arbitrary n. Recall the maps $\omega \mapsto \omega^{\text{t}}$ and $\omega \mapsto \omega^{\text{d}}$ on $M_n(\mathcal{A}) = M_n(\mathbb{C}) \otimes \mathcal{A}$ given on elementary tensors by $(a \otimes b)^{\text{t}} = a^{\text{t}} \otimes b$ and $(a \otimes b)^{\text{d}} = a \otimes b^{\text{d}}$ (here $\mathcal{A}$ is a C^*-subalgebra of $B(H)$ for some Hilbert space H). We set

$$\|\omega\|_{\text{ph}} = \inf\{\prod_{0\leq i<\frac{n}{2}} \|\omega^{\text{t}}_{n-2i}\|\|\omega_{n-2i-1}\| : \omega = \omega_1 \odot \omega_2 \odot \dots \odot \omega_n,\ \omega_0 = I,$$
$$\omega_1 \in M_{1,i_1}(\mathcal{A}_1), \omega_2 \in M_{i_1,i_2}(\mathcal{A}_2), \dots, \omega_n \in M_{i_{n-1},1}(\mathcal{A}_n), i_1, i_2, \dots, i_{n-1} \in \mathbb{N}\},$$

In the case $n = 2$, the above norm was denoted in [44] by $\|\cdot\|_{\text{h}'}$. Clearly, if the algebras $\mathcal{A}_i$, $i = 1, \dots, n$, are commutative then the norms $\|\cdot\|_{\text{h}}$ and $\|\cdot\|_{\text{ph}}$ coincide. It was shown in [44] that in general they need not be even equivalent.

Lemma 80. $\|\omega\|_{\text{univ}} \leq \|\omega\|_{\text{ph}}$ *for all* $\omega \in \mathcal{A}_1 \odot \dots \odot \mathcal{A}_n$.

Proof. Let π_i be a representation of $\mathcal{A}_i$, $i = 1, \dots, n$, and let $\omega = \omega_1 \odot \omega_2 \odot \dots \odot \omega_n$, where $\omega_1 \in M_{1,k_1}(\mathcal{A}_1), \omega_2 \in M_{k_1,k_2}(\mathcal{A}_2), \dots, \omega_n \in M_{k_{n-1},1}(\mathcal{A}_n)$ for some $k_1, k_2, \dots, k_{n-1} \in \mathbb{N}$.

Let n be even, $\xi_{1,2} \in M_{1,l_1}(H_1 \otimes H_2)$, $\eta_{2,3} \in M_{l_1,l_2}(H_2^d \otimes H_3^d), \ldots, \xi_{n-1,n} \in M_{l_{n-2},1}(H_{n-1} \otimes H_n)$ and

$$\zeta = \xi_{1,2} \odot \eta_{2,3} \odot \cdots \odot \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n).$$

Letting $\pi = \pi_1 \otimes \ldots \otimes \pi_n$, by Lemma 73 we have

$$S_{\pi(\omega)}(\zeta) = (\mathrm{id}_{\,1,k_{n-1}} \otimes \pi_n)(\omega_n^{\mathrm{t}}) \ldots (\theta_{l_1,l_2}(\eta_{2,3})^{\mathrm{t}} \otimes I_{k_2})$$
$$\times((\mathrm{id}_{\,k_1,k_2} \otimes \pi_2)(\omega_2^{\mathrm{t}}) \otimes I_{l_1})(\theta_{1,l_1}(\xi_{1,2})^{\mathrm{t}} \otimes I_{k_1})(\mathrm{id}_{\,k_1,1} \otimes \pi_1)(\omega_1^{\mathrm{t}})^{\mathrm{d}}.$$

Since $\|(\mathrm{id}_{k_{m-1},k_m} \otimes \pi_m)(\omega_m^{\mathrm{t}})^{\mathrm{d}}\| = \|(\mathrm{id}_{k_{m-1},k_m} \otimes \pi_m)(\omega_m)\|$, we have

$$\begin{aligned} \|S_{\pi(\omega)}(\zeta)\|_{\mathrm{op}} &\leq \|\theta_{1,l_1}(\xi_{1,2})^{\mathrm{t}}\| \ldots \|\theta_{l_{n-2},1}(\xi_{n-1,n})^{\mathrm{t}}\| \\ &\times \prod_{0 \leq i < \frac{n}{2}} \|\omega_{n-2i}^{\mathrm{t}}\| \|\omega_{n-2i-1}\| = \|\omega\|_{\mathrm{ph}} \|\zeta\|_{\mathrm{h}}. \end{aligned}$$

Now let n be odd and

$$\zeta = \eta_{1,2} \odot \xi_{2,3} \odot \cdots \odot \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n),$$

where $\eta_{1,2} \in M_{1,l_1}(H_1^{\mathrm{d}} \otimes H_2^{\mathrm{d}})$, $\xi_{2,3} \in M_{l_1,l_2}(H_2 \otimes H_3), \ldots,$ $\xi_{n-1,n} \in M_{l_{n-2},1}$ $(H_{n-1} \otimes H_n)$. Using the previously obtained inequality, we have

$$\begin{aligned} \|S_{\pi(\omega)}(\zeta)\|_{\mathrm{op}} &= \sup_{\|\xi\| \leq 1} \|S_{\pi(\omega)}(\zeta)(\xi)\|_{H_n} \\ &= \sup_{\|\xi\| \leq 1} \|S_{\mathrm{id} \otimes \pi(1 \otimes \omega)}((1 \otimes \xi) \otimes \zeta)\|_{\mathcal{B}(\mathbb{C}^{\mathrm{d}}, H_n)} \\ &\leq \|\omega\|_{\mathrm{ph}} \|\xi\| \|\zeta\|_{\mathrm{h}}. \end{aligned}$$

The proof is complete. $\diamondsuit$

If $H_1, \ldots, H_n$ are Hilbert spaces, we say that a net $\{\varphi_\nu\} \subseteq B(H_1 \otimes \cdots \otimes H_n)$ converges semi-weakly to an operator $\varphi \in B(H_1 \otimes \cdots \otimes H_n)$ if $(\varphi_\nu \zeta_1, \zeta_2) \to (\varphi \zeta_1, \zeta_2)$ for all $\zeta_1, \zeta_2 \in H_1 \odot \cdots \odot H_n$. Note that if the net $\{\varphi_\nu\}$ is bounded then it converges semi-weakly if and only if it converges weakly.

Let $\mathcal{A}_1 \subseteq B(H_1)$, $\mathcal{A}_2 \subseteq B(H_2), \ldots,$ $\mathcal{A}_n \subseteq B(H_n)$ be C*-algebras and $(\mathcal{A}_1 \odot \mathcal{A}_2 \odot \ldots \odot \mathcal{A}_n)^\sharp$ be the linear space of all $\varphi \in \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \ldots \otimes \mathcal{A}_n$ for which there exists a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \mathcal{A}_2 \odot \ldots \odot \mathcal{A}_n$ converging to φ semi-weakly (as a net of operators in $B(H_1 \otimes H_2 \otimes \ldots \otimes H_n)$) and such that $\sup_\nu \|\varphi_\nu\|_{\mathrm{ph}} < \infty$.

Proposition 81. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \ldots, n$, be C*-algebras. Then $(\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n)^\sharp \subseteq \mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq \mathbf{M}^\wedge(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.*

Proof. Since $\|\zeta\|_h \leq \|\zeta\|_\wedge$ for all $\zeta \in \Gamma(H_1, \ldots, H_n)$ we have $\mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq \mathbf{M}^\wedge(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

Let us first prove that

$$(\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n)^\sharp \subseteq \mathbf{M}_{\pi_1,\ldots,\pi_n}(\mathcal{A}_1, \ldots, \mathcal{A}_n),$$

in the case where $\pi_i = \bigoplus_{\lambda_i} \mathrm{id}$ is the sum of λ_i copies of the identity representation. Let $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n$ be a net converging semi-weakly to φ and such that $D = \sup_\nu \|\varphi_\nu\|_{\mathrm{ph}} < \infty$ and $\pi = \pi_1 \otimes \ldots \otimes \pi_n$. By Lemma 80,

$$\|S_{\pi(\varphi_\nu)}(\zeta)\|_{\mathrm{op}} \leq D\|\zeta\|_{\mathrm{h}}$$

for all ν and $\zeta \in \Gamma(H_1, \ldots, H_n)$.

Suppose first that n is even. To prove that $\|S_{\pi(\varphi)}(\zeta)\|_{\mathrm{op}} \leq D\|\zeta\|_{\mathrm{h}}$, it suffices to show that the net $\{S_{\pi(\varphi_\nu)}(\zeta)\}$ of operators in $B(\widetilde{H}_1^{\mathrm{d}}, \widetilde{H}_n)$ converges weakly to the operator $S_{\pi(\varphi)}(\zeta)$ (here and in the sequel we set $\widetilde{H}_i = \bigoplus_{\lambda_i} H_i$, $i = 1, \ldots, n$). By linearity and the uniform boundedness of the net $\{S_{\pi(\varphi_\nu)}(\zeta)\}$, it is suffices to prove that

$$(S_{\pi(\varphi_\nu)}(\zeta)x^{\mathrm{d}}, y) \to (S_{\pi(\varphi)}(\zeta)x^{\mathrm{d}}, y)$$

for all x^{d} and y which have only one non-zero entry in the corresponding direct sums of H_1^{d} and H_n, respectively.

Fix such x^{d} and y, and let $\zeta = \xi_{1,2} \otimes \eta_{2,3}^{\mathrm{d}} \otimes \ldots \otimes \xi_{n-1,n} \in \Gamma(\widetilde{H}_1, \ldots, \widetilde{H}_n)$. Then

$$(S_{\pi(\varphi_\nu)}(\zeta)x^{\mathrm{d}}, y) = (\pi(\varphi_\nu)(\xi_{1,2} \otimes \ldots \otimes \xi_{n-1,n}), x \otimes \eta_{2,3} \otimes \eta_{4,5} \otimes \ldots \otimes \eta_{n-2,n-1} \otimes y)$$

Indeed, assuming $n = 4$ for the simplicity we get

$$\begin{aligned}
(S_{\pi(\varphi_\nu)}(\zeta)x^{\mathrm{d}}, y) &= (\sigma_\pi(\varphi_\nu)\theta(\xi_{1,2} \otimes \xi_{3,4})(\theta(\eta_{2,3}^{\mathrm{d}})), \theta(x \otimes y))_2 \\
&= (\sigma_\pi(\varphi_\nu)\theta(\xi_{1,2} \otimes \xi_{3,4}), \theta(\theta(\eta_{2,3}) \otimes \theta(x \otimes y)))_2 \\
&= (\sigma_\pi(\varphi_\nu)\theta(\xi_{1,2} \otimes \xi_{3,4}), \theta(x \otimes \eta_{2,3} \otimes y))_2 \\
&= (\pi(\varphi_\nu)(\xi_{1,2} \otimes \xi_{3,4}), x \otimes \eta_{2,3} \otimes y).
\end{aligned}$$

Fix $\varepsilon > 0$ and let $\tilde{\zeta} = \tilde{\xi}_{1,2} \otimes \tilde{\eta}_{2,3}^{\mathrm{d}} \otimes \cdots \otimes \tilde{\xi}_{n-1,n}$ be such that all norms $\|\xi_{1,2} - \tilde{\xi}_{1,2}\|$, $\|\eta_{2,3} - \tilde{\eta}_{2,3}\|$,$\ldots$,$\|\xi_{n-1,n} - \tilde{\xi}_{n-1,n}\|$ are smaller than ε and all vectors $\tilde{\xi}_{1,2}$, $\tilde{\eta}_{2,3}^{\mathrm{d}}$,$\ldots$,$\tilde{\xi}_{n-1,n}$ are finite sums of elementary tensors which have only finitely many non-zero entries in the direct sums of the corresponding Hilbert spaces. Thus, we may assume that $\tilde{\xi}_{1,2} \in H_1^{(k)} \odot H_2^{(k)}$,$\tilde{\eta}_{2,3} \in H_2^{(k)} \odot H_3^{(k)}$ $\ldots$,$\tilde{\xi}_{n-1,n} \in H_{n-1}^{(k)} \odot H_n^{(k)}$, $x^{\mathrm{d}} \in H_1^{(k)}$ and $y \in H_n^{(k)}$ for some $k \in \mathbb{N}$.

It follows from the formula above that there exists ν_0 such that if $\nu \geq \nu_0$ then

$$|(S_{\pi(\varphi_\nu)}(\tilde{\zeta})x^{\mathrm{d}}, y) - (S_{\pi(\varphi)}(\tilde{\zeta})x^{\mathrm{d}}, y)| < \varepsilon.$$

On the other hand,

$$\begin{aligned} & |(S_{\pi(\varphi_\nu)}(\zeta)x^{\mathrm{d}}, y) - (S_{\pi(\varphi_\nu)}(\tilde{\zeta})x^{\mathrm{d}}, y)| \\ \leq \ & D\|x\|\|y\|\|\tilde{\zeta} - \zeta\|_{\mathrm{h}} \leq (C+\varepsilon)^{n-2} D(n-1)\|x\|\|y\|\varepsilon, \end{aligned}$$

for every ν, where $C = \max\{\|\xi_{1,2}\|, \|\eta_{2,3}\| \ldots, \|\xi_{n-1,n}\|\}$. Using Remark 68, we have

$$\begin{aligned} & |(S_{\pi(\varphi)}(\zeta)x^{\mathrm{d}}, y) - (S_{\pi(\varphi)}(\tilde{\zeta})x^{\mathrm{d}}, y)| \\ \leq \ & \|\varphi\|\|x\|\|y\|\|\zeta - \tilde{\zeta}\|_{2,\wedge} \leq \|\varphi\|(C+\varepsilon)^{n-2}(n-1)\|x\|\|y\|\varepsilon. \end{aligned}$$

Thus,

$$\begin{aligned} & |(S_{\pi(\varphi_\nu)}(\zeta)x^{\mathrm{d}}, y) - (S_{\pi(\varphi)}(\zeta)x^{\mathrm{d}}, y)| \\ \leq \ & \varepsilon(1 + (C+\varepsilon)^{n-2}D(n-1)\|x\|\|y\| + \|\varphi\|(C+\varepsilon)^{n-2}(n-1)\|x\|\|y\|) \end{aligned}$$

whenever $\nu \geq \nu_0$. It follows that the net $\{S_{\pi(\varphi_\nu)}(\zeta)\}$ converges weakly to $S_{\pi(\varphi)}(\zeta)$ and hence $\varphi \in \mathbf{M}_{\pi_1,\ldots,\pi_n}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

In the case n is odd, a calculation similar to the one above shows that $(S_{\pi(\varphi_\nu)}(\zeta)x, y)$ is equal to

$$(\pi(\varphi_\nu)(x \otimes \xi_{2,3} \otimes \ldots \otimes \xi_{n-1,n}), \eta_{1,2} \otimes \ldots \otimes \eta_{n-2,n-1} \otimes y),$$

whenever $x \in \widetilde{H}_1$, $y \in \widetilde{H}_n$, $\zeta = \eta^{\mathrm{d}}_{1,2} \otimes \xi_{2,3} \otimes \ldots \otimes \xi_{n-1,n} \in \Gamma(\widetilde{H}_1, \ldots, \widetilde{H}_n)$, and the proof proceeds in a similar fashion.

Now let $\pi_1, \ldots, \pi_n$ be representations of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ on $H_{\pi_1}, \ldots, H_{\pi_n}$ and $\pi = \pi_1 \otimes \ldots \otimes \pi_n$. Then

$$\mathrm{rank}(\pi_i(a_i)) \leq \mathrm{rank}\left(\bigoplus_{dim(H_{\pi_i})} \mathrm{id}(a_i)\right)$$

for all $a_i \in \mathcal{A}_i$ and $i = 1, .., n$. By Theorem 74 (i),

$$\mathbf{M}_{\bigoplus\limits_{\lambda_1}\mathrm{id}, \bigoplus\limits_{\lambda_2}\mathrm{id}, \ldots, \bigoplus\limits_{\lambda_k}\mathrm{id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq \mathbf{M}_{\pi_1,\pi_2,\ldots,\pi_k}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n).$$

The proof is complete. $\diamondsuit$

Assume that n is even. Then the mapping $S_{\mathrm{id}(\varphi)}$ acting on $\Gamma(H_1, \ldots, H_n) = (H_1 \otimes H_2) \odot (H_2^{\mathrm{d}} \otimes H_3^{\mathrm{d}}) \odot \ldots \odot (H_{n-1} \otimes H_n)$ can be regarded as a mapping on the algebraic tensor product

$$HS(H_{n-1}, H_n) \odot HS(H_{n-2}, H_{n-1})^{\mathrm{d}} \odot \ldots \odot HS(H_1, H_2) \tag{5.26}$$

of the corresponding spaces of Hilbert-Schmidt operators by letting

$$S_\varphi(\theta(\xi_{n-1,n}) \otimes \theta(\eta_{n-2,n-1})^{\mathrm{d}} \otimes \theta(\xi_{n-3,n-2}) \otimes \ldots \otimes \theta(\xi_{1,2})) = S_\varphi(\zeta),$$

where $\zeta = \xi_{1,2} \otimes \eta_{2,3}^{\mathrm{d}} \otimes \xi_{3,4} \otimes \ldots \otimes \xi_{n-1,n}$. Denote the space (5.26) by $HS\Gamma$ $(H_1, \ldots, H_n)$. If φ is an elementary tensor then Lemma 73 (i) shows that $S_{\mathrm{id}(\varphi)}$ is $\mathcal{A}'_n, (\mathcal{A}^{\mathrm{d}}_{n-1})', \ldots, \mathcal{A}'_2, (\mathcal{A}_1{}^{\mathrm{d}})'$-modular. It follows by continuity that $S_{\mathrm{id}(\varphi)}$ is $\mathcal{A}'_n, (\mathcal{A}^{\mathrm{d}}_{n-1})', \ldots, \mathcal{A}'_2, (\mathcal{A}_1{}^{\mathrm{d}})'$-modular for every $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. If moreover $\varphi \in \mathbf{M}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ then $S_{\mathrm{id}(\varphi)}$ can be extended to a bounded mapping (denoted in the same way) from the algebraic tensor product

$$\mathcal{K}(H^{\mathrm{d}}_{n-1}, H_n) \odot \mathcal{K}(H^{\mathrm{d}}_{n-2}, H_{n-1})^{\mathrm{d}} \odot \cdots \odot \mathcal{K}(H^{\mathrm{d}}_1, H_2)$$

into $\mathcal{K}(H^{\mathrm{d}}_1, H_n)$. By continuity, this extension is $\mathcal{A}'_n, (\mathcal{A}^{\mathrm{d}}_{n-1})', \ldots, \mathcal{A}'_2, (\mathcal{A}_1{}^{\mathrm{d}})'$-modular.

Similarly, if n is odd and $\varphi \in \mathbf{M}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ then $S_{\mathrm{id}(\varphi)}$ can be regarded as a multilinear $\mathcal{A}'_n, (\mathcal{A}_{n-1}{}^{\mathrm{d}})', \ldots, (\mathcal{A}_2{}^{\mathrm{d}})', \mathcal{A}'_1$-modular map from

$$\mathcal{K}(H^{\mathrm{d}}_{n-1}, H_n) \odot \mathcal{K}(H^{\mathrm{d}}_{n-2}, H_{n-1})^{\mathrm{d}} \odot \cdots \odot \mathcal{K}(H^{\mathrm{d}}_1, H_2)$$

into $\mathcal{B}(H_1, H_n)$. Denote by $\mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ the set of all $(\mathrm{id}, \ldots, \mathrm{id})$-multipliers for which the mapping $S_{\mathrm{id}(\varphi)}$ is completely bounded.

Proposition 82. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \ldots, n$, be von Neumann algebras. Then $\mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq (\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n)^\sharp$.*

Proof. Assume first that n is even. For notational simplicity we assume that H_i is separable, $i = 1, \ldots, n$. Let $\mathrm{id} : \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to \mathcal{B}(H_1 \otimes \cdots \otimes H_n)$ be the identity representation.

Let $\varphi \in \mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Then $S_{\mathrm{id}(\varphi)}$ is a multilinear $\mathcal{A}'_n$, $(\mathcal{A}^{\mathrm{d}}_{n-1})'$, $\ldots$, $\mathcal{A}'_2$, $(\mathcal{A}_1{}^{\mathrm{d}})'$-modular mapping on

$$\mathcal{K}(H^{\mathrm{d}}_{n-1}, H_n) \odot \mathcal{K}(H_{n-2}, H^{\mathrm{d}}_{n-1}) \odot \cdots \odot \mathcal{K}(H^{\mathrm{d}}_1, H_2)$$

taking values in $\mathcal{K}(H^{\mathrm{d}}_1, H_n)$. Let $H^\infty = H \otimes l^2$ and I_∞ be the identity operator on l^2.

Since $S_{\mathrm{id}(\varphi)}$ is completely bounded, it extends to a completely bounded mapping, denoted in the same way, from

$$\mathcal{K}(H^{\mathrm{d}}_{n-1}, H_n) \otimes_{\mathrm{h}} \mathcal{K}(H_{n-2}, H^{\mathrm{d}}_{n-1}) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H^{\mathrm{d}}_1, H_2)$$

into $\mathcal{K}(H_1^{\rm d}, H_n)$. Then the second dual $S^{**}_{{\rm id}(\varphi)}$ is a weak* continuous completely bounded mapping from $\mathcal{B}(H^{\rm d}_{n-1}, H_n) \otimes_{\sigma{\rm h}} \ldots \otimes_{\sigma{\rm h}} \mathcal{B}(H_1^{\rm d}, H_2)$ into $\mathcal{B}(H_1^{\rm d}, H_n)$ and hence gives rise to a weak* continuous completely bounded $\mathcal{A}_n'$, $(\mathcal{A}^{\rm d}_{n-1})'$, …, $\mathcal{A}_2'$, $(\mathcal{A}_1{}^{\rm d})'$-modular multilinear map, denoted in the same way, from

$$\mathcal{B}(H^{\rm d}_{n-1}, H_n) \times \mathcal{B}(H_{n-2}, H^{\rm d}_{n-1}) \times \cdots \times \mathcal{B}(H_1^{\rm d}, H_2)$$

into $\mathcal{B}(H_1^{\rm d}, H_n)$.

It follows from Corollary 5.9 of [18] that there exist bounded linear operators $A_1 : H_1^{\rm d} \to (H_1^{\rm d})^\infty$, $A_j : H_j^\infty \to H_j^\infty$, if j is even, $A_j : (H_j^{\rm d})^\infty \to (H_j^{\rm d})^\infty$ if j is odd ($j = 2, \ldots, n-1$) and $A_n : H_n^\infty \to H_n$ such that the entries of A_j with respect to the corresponding direct sum decomposition belong to $\mathcal{A}_j'' = \mathcal{A}_j$ for even j and to $(\mathcal{A}_j^{\rm d})'' = \mathcal{A}_j^{\rm d}$ for odd j,

$$S_{{\rm id}(\varphi)}(\zeta) = A_n(\theta(\xi_{n-1,n}) \otimes I_\infty)A_{n-1}(\theta(\eta_{n-2,n-1})^{\rm d} \otimes I_\infty)A_{n-2} \ldots A_1,$$

for all

$$\zeta = \theta(\xi_{n-1,n}) \otimes \theta(\eta_{n-2,n-1})^{\rm d} \otimes \ldots \otimes \theta(\xi_{1,2}) \in HS\Gamma(H_1, \ldots, H_n),$$

and

$$\|S_{{\rm id}(\varphi)}\|_{cb} = \prod_{1 \le i \le n} \|A_i\|.$$

Let $P_{m,\nu} = (p^m_{ij})^\infty_{i,j=1}$ be the projection with $p^m_{ij} \in B(H_m)$ (resp. $p^m_{ij} \in B(H_m^{\rm d})$), $p^m_{ii} = I_{H_m}$ (resp. $p^m_{ii} = I_{H_m^{\rm d}}$) if m is even (resp. if m is odd) and $1 \le i \le \nu$, and $p^m_{ij} = 0$ otherwise.

Set $\varphi_\nu = A_1^{{\rm d},{\rm t}} P^{\rm d}_{1,\nu} \odot P_{2,\nu} A_2 P_{2,\nu} \odot P_{3,\nu} A_3^{\rm d} P_{3,\nu} \ldots \odot P_{n,\nu} A_n$. Clearly, $\|\varphi_\nu\|_{\rm ph} \le \prod_{1 \le i \le n} \|A_i\|$ for each ν; it hence suffices to prove that $\{\varphi_\nu\}$ converges semi-weakly to φ.

As $S_{{\rm id}(\varphi_\nu)}(\zeta)$ equals

$$A_n P_{n,\nu}(\theta(\xi_{n-1,n}) \otimes I_\infty) P_{n-1,\nu} A_{n-1} P_{n-1,\nu}(\theta(\eta_{n-2,n-1})^{\rm d} \otimes I_\infty) \ldots P_{1,\nu} A_1$$

and $P_{l,\nu}$ converges strongly to I_{H_l}, we have that $S_{{\rm id}(\varphi_\nu)}(\zeta)$ converges weakly to $S_{{\rm id}(\varphi)}(\zeta)$. By the proof of Proposition 81, if $x^{\rm d} \in H_1^{\rm d}$, $y \in H_n$ and $\psi \in \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$ then $(S_{{\rm id}(\psi)}(\zeta)x^{\rm d}, y)$ equals

$$\begin{aligned} & (\sigma_{\rm id}(\psi)\theta(\xi_{1,2} \otimes \ldots \otimes \xi_{k-1,k}), \theta(x \otimes \eta_{2,3} \otimes \ldots \otimes \eta_{k-2,k-1} \otimes y))_2 \\ = \; & (\psi(\xi_{1,2} \otimes \ldots \otimes \xi_{k-1,k}), x \otimes \eta_{2,3} \otimes \ldots \otimes \eta_{k-2,k-1} \otimes y). \end{aligned}$$

Thus φ_ν converges semi-weakly to φ and therefore $\varphi \in (\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sharp$, giving the inclusion $\mathbf{M}^{cb}_{{\rm id},\ldots,{\rm id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq (\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sharp$.

Now assume that n is odd. In this case $S^{**}_{\mathrm{id}(\varphi)}$ is a weak* continuous completely bounded multilinear $\mathcal{A}'_n$, $(\mathcal{A}^{\mathrm{d}}_{n-1})'$, ..., $(\mathcal{A}^{\mathrm{d}}_2)'$, $\mathcal{A}_1{}'$-modular mapping on

$$\mathcal{B}(H^{\mathrm{d}}_{n-1}, H_n) \times \mathcal{B}(H_{n-2}, H^{\mathrm{d}}_{n-1}) \times \cdots \times \mathcal{B}(H_1, H^{\mathrm{d}}_2)$$

taking values in $\mathcal{B}(H_1, H_n)^{**}$. Let Q be the weak* continuous projection from $\mathcal{B}(H_1, H_n)^{**}$ onto $\mathcal{B}(H_1, H_n)$. Then $Q \circ S^{**}_{\mathrm{id}(\varphi)}$ takes values in $\mathcal{B}(H_1, H_n)$, and coincides with $S_{\mathrm{id}(\varphi)}$ on $HS\Gamma(H_1, \dots, H_n)$. The proof now proceeds as above. $\diamondsuit$

Proposition 83. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \dots, n$, be C^*-algebras. Then* $\mathbf{M}^{\wedge}(\mathcal{A}_1, \dots, \mathcal{A}_n) \subseteq \mathbf{M}^{cb}_{\mathrm{id},\dots,\mathrm{id}}(\mathcal{A}_1, \dots, \mathcal{A}_n)$.

Proof. Let $\varphi \in \mathbf{M}^{\wedge}(\mathcal{A}_1, \dots, \mathcal{A}_n)$. Then there exists a constant $D > 0$ such that

$$\|\sigma_{\pi_1 \otimes \dots \otimes \pi_n}(\varphi)(\zeta)\|_{\mathrm{op}} \le D\|\zeta\|_{\wedge}$$

for all $\zeta \in \Gamma(H_1, \dots, H_n)$ and all representations $\pi_1, \dots, \pi_n$ of $\mathcal{A}_1, \dots, \mathcal{A}_n$, respectively.

Let $k \in \mathbb{N}$. The space $HS\Gamma(H^k_1, \dots, H^k_n)$ is naturally isomorphic to

$$M_k(HS(H_{n-1}, H_n)) \odot M_k(HS(H_{n-2}, H_{n-1})^{\mathrm{d}}) \odot \dots \odot M_k(HS(H_1, H_2)), \tag{5.27}$$

and thus the mapping $S_{(\mathrm{id} \otimes 1_k) \otimes \dots \otimes (\mathrm{id} \otimes 1_k)(\varphi)}$ is well-defined on the space (5.27). One can easily check that

$$S^{(k)}_{\mathrm{id} \otimes \dots \otimes \mathrm{id}(\varphi)}(\Xi_{n-1} \odot \dots \odot \Xi_1) = S_{(\mathrm{id} \otimes 1_k) \otimes \dots \otimes (\mathrm{id} \otimes 1_k)(\varphi)}(\Xi_{n-1} \otimes \dots \otimes \Xi_1), \tag{5.28}$$

where $\Xi_i \in M_k(HS(H_i, H_{i+1}))$ (resp. $\Xi_i \in M_k(HS(H_i, H_{i+1})^{\mathrm{d}})$) if i is even (resp, if i is odd) and $\Xi_i \in M_k(HS(H_i, H_{i+1})^{\mathrm{d}})$ (resp. $\Xi_i \in M_k(HS(H_i, H_{i+1}))$) if i is odd (resp, if i is even). If the matrices Ξ_i are of arbitrary sizes such that the product $\Xi_{n-1} \odot \dots \odot \Xi_1$ is well defined then they may be considered as square matrices, all of the same size, by complementing with zeros, and identity (5.28) will still hold. It follows that

$$\|S^{(k)}_{\mathrm{id} \otimes \dots \otimes \mathrm{id}(\varphi)}(\Xi_1 \odot \dots \odot \Xi_{n-1})\|_{\mathrm{op}} \le D \prod_{1 \le i \le n-1} \|\Xi_i\|_{\mathrm{op}}, \text{ for all } \Xi_1, \dots \Xi_{n-1},$$

and hence the mapping $S_{\mathrm{id} \otimes \dots \otimes \mathrm{id}(\varphi)}$ is completely bounded and φ is an (id, ..., id)-multiplier. $\diamondsuit$

Theorem 84. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \dots, n$, be C^*-algebras. Then* $\mathbf{M}(\mathcal{A}_1, \dots, \mathcal{A}_n) = \mathbf{M}^\wedge(\mathcal{A}_1, \dots, \mathcal{A}_n) = (\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n)^\sharp$.

Proof. By Propositions 81, 82 and 83,

$$\mathbf{M}^{cb}_{\mathrm{id},\dots,\mathrm{id}}(\mathcal{A}_1'', \dots, \mathcal{A}_n'') = (\mathcal{A}_1'' \odot \dots \odot \mathcal{A}_n'')^\sharp.$$

Evidently,

$$\mathbf{M}^{cb}_{\mathrm{id},\dots,\mathrm{id}}(\mathcal{A}_1, \dots, \mathcal{A}_n) \subseteq \mathbf{M}^{cb}_{\mathrm{id},\dots,\mathrm{id}}(\mathcal{A}_1'', \dots, \mathcal{A}_n'') \cap (\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n).$$

Applying Propositions 81, 82 and 83, we obtain

$$\begin{aligned}
(\mathcal{A}_1 \odot \dots \odot \mathcal{A}_n)^\sharp &\subseteq \mathbf{M}(\mathcal{A}_1, \dots, \mathcal{A}_n) \\
&\subseteq \mathbf{M}^\wedge(\mathcal{A}_1, \dots, \mathcal{A}_n) \\
&\subseteq \mathbf{M}^{cb}_{\mathrm{id},\dots,\mathrm{id}}(\mathcal{A}_1, \dots, \mathcal{A}_n) \\
&\subseteq \mathbf{M}^{cb}_{\mathrm{id},\dots,\mathrm{id}}(\mathcal{A}_1'', \dots, \mathcal{A}_n'') \cap (\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n) \\
&= (\mathcal{A}_1'' \odot \dots \odot \mathcal{A}_n'')^\sharp \cap (\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n).
\end{aligned}$$

It hence suffices to show that

$$(\mathcal{A}_1'' \odot \dots \odot \mathcal{A}_n'')^\sharp \cap \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n \subseteq (\mathcal{A}_1 \odot \dots \odot \mathcal{A}_n)^\sharp.$$

Let $\varphi \in (\mathcal{A}_1'' \odot \dots \odot \mathcal{A}_n'')^\sharp \cap (\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n)$. Then there exists a net $\{\varphi_\nu\}_{\nu \in J} \subseteq \mathcal{A}_1'' \odot \dots \odot \mathcal{A}_n''$ with $\sup_\nu \|\varphi_\nu\|_{ph} < \infty$ which converges semi-weakly to φ. Write $\varphi_\nu = A_{1,\nu} \odot \dots \odot A_{n,\nu}$, where $A_{1,\nu} \in M_{1,i_1}(\mathcal{A}_1'')$, $A_{2,\nu} \in M_{i_1,i_2}(\mathcal{A}_2''), \dots, A_{n,\nu} \in M_{i_n,1}(\mathcal{A}_n'')$.

By Kaplansky's Density Theorem for TRO's [40], for each pair (m, ν) there exists a net $\{A_{m,\nu,\tau(m)}\}_{\tau(m)} \subset M_{i_{m-1},i_m}(\mathcal{A}_m)$ converging strongly to $A_{m,\nu}$ and such that $||A_{m,\nu,\tau(m)}|| \leq ||A_{m,\nu}||$ for all $\tau(m)$. Thus if $A_{\nu,\tau} = A_{1,\nu,\tau(1)} \odot A_{2,\nu,\tau(2)} \odot \dots \odot A_{n,\nu,\tau(n)}$, where $\tau = (\tau(1), \dots, \tau(n))$, then the net $\{A_{\nu,\tau}\}_\tau$ converges strongly to φ_ν and $||A_{\nu,\tau}||_{ph} \leq ||\varphi_\nu||_{ph}$.

Let $\mathcal{U}$ be the collection of all weak neighbourhoods of 0 of the form $\{S \in \mathcal{B}(H_1 \otimes \cdots \otimes H_n) : |(S(\zeta_1^j), \zeta_2^j)| < \varepsilon_j, j = 1, \dots, k\}$, where $\zeta_1^j, \zeta_2^j \in H_1 \odot \cdots \odot H_n$ and $\varepsilon_j > 0$, $j = 1, \dots, k$. Note that $\mathcal{U}$ is directed with respect to reverse inclusion. The convergence of the net $\{\varphi_\nu\}_{\nu \in J}$ semi-weakly to φ implies that for every $U \in \mathcal{U}$ there exists $\nu(U)$ such that for every $\lambda \in J$ with $\lambda \geq \nu(U)$, we have that $\varphi_\lambda - \varphi \in U$. The convergence of $\{A_{\nu,\tau}\}_\tau$ to φ_ν implies the existence of $T(\nu(U), U)$ such that for every $\tau \geq T(\nu(U), U)$, we have that $A_{\nu(U),\tau} - \varphi_{\nu(U)} \in U$. Consider the net $A_U = A_{\nu(U),T(\nu(U),U)}$ indexed by $\mathcal{U}$. It is easy to check that A_U converges semi-weakly to φ. The proof is complete. $\diamondsuit$

Note that in Theorem 84 we actually proved that if n is even, $\varphi \in \mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, $\zeta = \xi_{1,2} \otimes \cdots \otimes \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n)$ and

$$S_{\mathrm{id} \otimes \cdots \otimes \mathrm{id}(\varphi)}(\zeta) = A_n(\theta(\xi_{n-1,n}) \otimes I) \ldots (\theta(\xi_{1,2}) \otimes I) A_1^{\mathrm{d}},$$

where A_i for i even (resp. A_i^{d} for i odd) is a bounded block operator matrix with entries in $\mathcal{A}_i''$ (resp. $(\mathcal{A}_i^{\mathrm{d}})''$), then there exists a net $\varphi_\nu = A_1^\nu \odot A_2^\nu \odot \cdots \odot A_n^\nu$, where A_i^ν is a finite block operator matrix with entries in $\mathcal{A}_i$ such that $\varphi_\nu \to \varphi$ semi-weakly, $A_i^\nu \to A_i$ (resp. $A_i^{\nu \mathrm{d}} \to A_i^{\mathrm{d}}$) strongly for i even (resp. for i odd) and all operator norms $\|A_i^\nu\|$, $\|A_i\|$ are bounded by a constant depending only on n. A similar statement holds in the case n is odd.

Denote by $(\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sim$ the set of all $\varphi \in \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$ for which there exists a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$, such that $\sup_\nu \|\varphi_\nu\|_{\mathrm{ph}} < \infty$ and if π_i is an irreducible representation of $\mathcal{A}_i$, $i = 1, \ldots, n$, then $\{(\pi_1 \otimes \ldots \otimes \pi_n)(\varphi_\nu)\}$ converges semi-weakly to $(\pi_1 \otimes \ldots \otimes \pi_n)(\varphi)$. Note that if $\sup_\nu \|\varphi_\nu\|_{min} < \infty$, which holds for example when the norms $\|\cdot\|_{\mathrm{ph}}$ and $\|\cdot\|_{\mathrm{h}}$ are equivalent (see [44]), then in the definition of the space $(\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sim$ the semi-weak convergence can be replaced by the convergence in the weak operator topology.

It follows from [52] that if $\mathcal{A}$ and $\mathcal{B}$ are commutative C*-algebras then $\mathbf{M}(\mathcal{A}, \mathcal{B}) = (\mathcal{A} \odot \mathcal{B})^\sim$. As a corollary of Theorem 84, we show that the same equality holds for an arbitrary number of arbitrary C*-algebras, giving an answer to a problem posed in [52].

Theorem 85. *Let $\mathcal{A}_i$, $i = 1, \ldots, n$, be C*-algebras. Then*

$$\mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \mathbf{M}^\wedge(\mathcal{A}_1, \ldots, \mathcal{A}_n) = (\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sim.$$

Proof. Let $\pi_1 = \bigoplus_{\pi \in IrrRep(\mathcal{A}_1)} \pi, \ldots, \pi_n = \bigoplus_{\pi \in IrrRep(\mathcal{A}_n)} \pi$, where $IrrRep(\mathcal{A}_i)$ is a set whose elements are all inequivalent irreducible representations of $\mathcal{A}_i$. Then

$$\begin{aligned} \mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n) &= (\pi_1 \otimes \ldots \otimes \pi_n)^{-1}(\pi_1(\mathcal{A}_1) \odot \ldots \odot \pi_n(\mathcal{A}_n))^\sharp \\ &\subseteq (\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sim. \end{aligned}$$

Using arguments similar to the ones from the proof of Proposition 81, one can show that

$$(\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sim \subseteq \mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n),$$

which together with Theorem 84 gives the statement of the theorem. $\diamondsuit$

Chapter 6

Compact operator multipliers.

A bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is called a Schur multiplier if $(\varphi(i,j)a_{ij})$ is the matrix of a bounded linear operator on ℓ_2 whenever (a_{ij}) is such. The study of Schur multipliers was initiated by Schur in the early 20th century and since then has attracted considerable attention, much of which was inspired by A. Grothendieck's characterisation of these objects in his *Résumé* [35]. Grothendieck showed that a function φ is a Schur multiplier precisely when it has the form $\varphi(i,j) = \sum_{k=1}^{\infty} a_k(i)b_k(j)$, where $a_k, b_k : \mathbb{N} \to \mathbb{C}$ satisfy the conditions $\sup_i \sum_{k=1}^{\infty} |a_k(i)|^2 < \infty$ and $\sup_j \sum_{k=1}^{\infty} |b_k(j)|^2 < \infty$. In modern terminology, this characterisation can be expressed by saying that φ is a Schur multiplier precisely when it belongs to the extended Haagerup tensor product $\ell_\infty \otimes_{\mathrm{eh}} \ell_\infty$ of two copies of ℓ_∞.

Special classes of Schur multipliers, e.g. Toeplitz and Hankel Schur multipliers, have played an important role in analysis and have been studied extensively (see [69]). Compact Schur multipliers, that is, the functions φ for which the mapping $(a_{ij}) \to (\varphi(i,j)a_{ij})$ on $\mathcal{B}(\ell_2)$ is compact, were characterised by Hladnik [42], who identified them with the elements of the Haagerup tensor product $c_0 \otimes_{\mathrm{h}} c_0$.

A non-commutative version of Schur multipliers was introduced by Kissin and Shulman [52] as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let π and ρ be representations of $\mathcal{A}$ and $\mathcal{B}$ on Hilbert spaces H and K, respectively. Identifying $H \otimes K$ with the Hilbert space $\mathcal{C}_2(H^{\mathrm{d}}, K)$ of all Hilbert-Schmidt operators from the dual space H^{d} of H into K, we obtain a representation $\sigma_{\pi,\rho}$ of the minimal tensor product $\mathcal{A} \otimes \mathcal{B}$ acting on $\mathcal{C}_2(H^{\mathrm{d}}, K)$. An element $\varphi \in \mathcal{A} \otimes \mathcal{B}$ is called a π,ρ-multiplier if $\sigma_{\pi,\rho}(\varphi)$ is bounded in the operator norm of $\mathcal{C}_2(H^{\mathrm{d}}, K)$. If φ is a π,ρ-multiplier for any pair of representations (π, ρ) then φ is called a universal (operator) multiplier.

Multidimensional Schur multipliers and their non-commutative versions were introduced and studied in [48], where the authors gave, in particular, a characterisation of universal multipliers as certain weak limits of elements of the algebraic tensor product of the corresponding C^*-algebras, generalising the corresponding results of Grothendieck and Peller [35], [68] as previously conjectured by Kissin and Shulman in [52]. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C*-algebras. Like Schur multipliers, elements of the set $M(\mathcal{A}_1, \dots, \mathcal{A}_n)$ of (multidimensional) universal multipliers give rise to completely bounded (multilinear) maps. Requiring these maps to be compact or completely compact, we define the sets of compact and completely compact operator multipliers denoted by $M_c(\mathcal{A}_1, \dots, \mathcal{A}_n)$ and $M_{cc}(\mathcal{A}_1, \dots, \mathcal{A}_n)$, respectively. The notion of complete compactness we use is an operator space version of compactness which was introduced by Saar [85] and subsequently studied by Oikhberg [58] and Webster [98]. Our results on operator multipliers rely on the main result of Section 6.2 where we prove a representation theorem for completely compact completely bounded multilinear maps. In [18] Christensen and Sinclair established a representation result for completely bounded multilinear maps which implies that every such map $\Phi : \mathcal{K}(H_2, H_1) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_n, H_{n-1}) \to \mathcal{K}(H_n, H_1)$ (where, for Hilbert spaces H' and H'', we denote by $\mathcal{K}(H', H'')$ the space of all compact operators from H' into H'') has the form

$$\Phi(x_1 \otimes \cdots \otimes x_{n-1}) = A_1(x_1 \otimes 1)A_2 \dots (x_{n-1} \otimes 1)A_n, \tag{6.1}$$

for some index set J and bounded block operator matrices $A_1 \in M_{1,J}(\mathcal{B}(H_1))$, $A_2 \in M_J(\mathcal{B}(H_2)), \dots, A_n \in M_{J,1}(\mathcal{B}(H_n))$. In other words, Φ arises from an element

$$u = A_1 \odot \cdots \odot A_n \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$$

of the extended Haagerup tensor product of $\mathcal{B}(H_1), \dots, \mathcal{B}(H_n)$. Moreover, if Φ is $\mathcal{A}_1', \dots, \mathcal{A}_n'$-modular for some von Neumann algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$, then the entries of A_i can be chosen from $\mathcal{A}_i$. We show in Section 6.2 that a map Φ as above is completely compact precisely when it has a representation of the form (6.1) where

$$u = A_1 \odot \cdots \odot A_n \in \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n).$$

This extends a result of Saar [85] in the two dimensional case. If, additionally, $\mathcal{A}_1, \dots, \mathcal{A}_n$ are von Neumann algebras and Φ is $\mathcal{A}_1', \dots, \mathcal{A}_n'$-modular then u can be chosen from $\mathcal{K}(\mathcal{A}_1) \otimes_{\mathrm{h}} (\mathcal{A}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}_n)$, where $\mathcal{K}(\mathcal{A})$ denotes the ideal of compact elements of a C^*-algebra $\mathcal{A}$. As a consequence of this and a result of Effros and Kishimoto [26] we point out the completely isometric identifications

$$CC(\mathcal{K}(H_2, H_1))^{**} \simeq (\mathcal{K}(H_1) \otimes_{\mathrm{h}} \mathcal{K}(H_2))^{**} \simeq CB(\mathcal{B}(H_2, H_1)),$$

where $CC(\mathcal{X})$ and $CB(\mathcal{X})$ are the spaces of completely compact and completely bounded maps on an operator space $\mathcal{X}$, respectively.

In Section 6.3 we pinpoint the connection between universal operator multipliers and completely bounded maps. This technical result is used in Section 6.4 to define the symbol u_φ of an operator multiplier $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ which, in the case n is even (resp. odd) is an element of $\mathcal{A}_n \otimes_{\rm eh} \mathcal{A}_{n-1}^o \otimes_{\rm eh} \cdots \otimes_{\rm eh} \mathcal{A}_2 \otimes_{\rm h} \mathcal{A}_1^o$ (resp. $\mathcal{A}_n \otimes_{\rm eh} \mathcal{A}_{n-1}^o \otimes_{\rm eh} \cdots \otimes_{\rm eh} \mathcal{A}_2^o \otimes_{\rm h} \mathcal{A}_1$). Here $\mathcal{A}^o$ is the opposite C*-algebra of a C*-algebra $\mathcal{A}$. This notion extends a similar notion that was given in the case of completely bounded masa-bimodule maps by Katavolos and Paulsen in [49]. We give a symbolic calculus for universal multipliers which is used to establish a universal property of the symbol related to the representation theory of the C*-algebras under consideration.

The symbol of a universal multiplier is used in Section 6.5 to single out the completely compact multipliers within the set of all operator multipliers. In fact, we show that $\varphi \in M_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ if and only if

$$u_\varphi \in \begin{cases} \mathcal{K}(\mathcal{A}_n) \otimes_{\rm h} (\mathcal{A}_{n-1}^o \otimes_{\rm eh} \cdots \otimes_{\rm eh} \mathcal{A}_3^o \otimes_{\rm h} \mathcal{A}_2) \otimes_{\rm h} \mathcal{K}(\mathcal{A}_1^o) & \text{if } n \text{ is even} \\ \mathcal{K}(\mathcal{A}_n) \otimes_{\rm h} (\mathcal{A}_{n-1}^o \otimes_{\rm eh} \cdots \otimes_{\rm eh} \mathcal{A}_3 \otimes_{\rm h} \mathcal{A}_2^o) \otimes_{\rm h} \mathcal{K}(\mathcal{A}_1) & \text{if } n \text{ is odd,} \end{cases}$$

which is equivalent to the approximability of φ in the multiplier norm by operator multipliers of finite rank whose range consists of finite rank operators. It follows that a multidimensional Schur multiplier $\varphi \in \ell_\infty(\mathbb{N}^n)$ is compact if and only if $\varphi \in c_0 \otimes_{\rm h} (\ell_\infty \otimes_{\rm eh} \cdots \otimes_{\rm eh} \ell_\infty) \otimes_{\rm h} c_0$.

In Section 6.6 we use Saar's construction [85] of a completely bounded compact mapping which is not completely compact to show that the inclusion $M_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq M_c(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is proper if both $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ contain full matrix algebras of arbitrarily large sizes. However, if both $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ are isomorphic to a c_0-sum of matrix algebras of uniformly bounded sizes then the sets of compact and completely compact multipliers coincide. The case when only one of $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ contains matrix algebras of arbitrary large size remains, however, unsettled. Finally, for $n = 2$, we characterise the cases where every universal multiplier is automatically compact: this happens precisely when one of the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is finite dimensional and the other one coincides with its algebra of compact elements.

6.1 Preliminaries

We start by recalling standard notation and notions from operator space theory. We refer the reader to [9], [28], [62] and [70] for more details.

If H and K are Hilbert spaces we let $\mathcal{B}(H,K)$ (resp. $\mathcal{K}(H,K)$) denote the set of all bounded linear (resp. compact) operators from H into K. If I is a set we let H^I be the direct sum of $|I|$ copies of H and set $H^\infty = H^{\mathbb{N}}$. Writing $H \otimes K$ for the Hilbertian tensor product of two Hilbert spaces, we observe that $H^I = H \otimes \ell_2(I)$ as Hilbert spaces.

An operator space $\mathcal{E}$ is a closed subspace of $\mathcal{B}(H,K)$, for some Hilbert spaces H and K. The opposite operator space $\mathcal{E}^o$ associated with $\mathcal{E}$ is the space $\mathcal{E}^o = \{x^{\mathrm{d}} : x \in \mathcal{E}\} \subseteq \mathcal{B}(K^{\mathrm{d}}, H^{\mathrm{d}})$. Here, and in the sequel, $H^{\mathrm{d}} = \{\xi^{\mathrm{d}} : \xi \in H\}$ denotes the dual of the Hilbert space H, where $\xi^{\mathrm{d}}(\eta) = (\eta, \xi)$ for $\eta \in H$. Note that H^{d} is canonically conjugate-linearly isometric to H. We also adopt the notation $x^{\mathrm{d}} \in \mathcal{B}(K^{\mathrm{d}}, H^{\mathrm{d}})$ for the Banach space adjoint of $x \in \mathcal{B}(H,K)$, so that $x^{\mathrm{d}}\xi^{\mathrm{d}} = (x^*\xi)^{\mathrm{d}}$ for $\xi \in K$. As usual, $\mathcal{E}^*$ will denote the operator space dual of $\mathcal{E}$. If $n, m \in \mathbb{N}$, by $M_{n,m}(\mathcal{E})$ we denote the space of all n by m matrices with entries in $\mathcal{E}$ and let $M_n(\mathcal{E}) = M_{n,n}(\mathcal{E})$. The space $M_{n,m}(\mathcal{E})$ carries a natural norm arising from the embedding $M_{n,m}(\mathcal{E}) \subseteq \mathcal{B}(H^m, K^n)$. Let I and J be arbitrary index sets. If v is a matrix with entries in $\mathcal{E}$ and indexed by $I \times J$, and $I_0 \subseteq I$ and $J_0 \subseteq J$ are finite sets, we let $v_{I_0,J_0} \in M_{I_0,J_0}(\mathcal{E})$ be the matrix obtained by restricting v to the indices from $I_0 \times J_0$. We define $M_{I,J}(\mathcal{E})$ to be the space of all such v for which

$$\|v\| \stackrel{\text{def}}{=} \sup\{\|v_{I_0,J_0}\| : I_0 \subseteq I,\ J_0 \subseteq J \text{ finite}\} < \infty.$$

Then $M_{I,J}(\mathcal{E})$ is an operator space [28, §10.1]. Note that $M_{I,J}(\mathcal{B}(H,K))$ can be naturally identified with $\mathcal{B}(H^J, K^I)$ and every $v \in M_{I,J}(\mathcal{B}(H,K))$ is the weak limit of $\{v_{I_0,J_0}\}$ along the net $\{(I_0,J_0) : I_0 \subseteq I,\ J_0 \subseteq J \text{ finite}\}$. We set $M_I(\mathcal{E}) = M_{I,I}(\mathcal{E})$. For $A = (a_{ij}) \in M_I(\mathcal{E})$, we write $A^{\mathrm{d}} = (a_{ij}^{\mathrm{d}}) \in M_I(\mathcal{E}^o)$.

6.1.1 Completely bounded maps and Haagerup tensor products

If $\mathcal{E}$ and $\mathcal{F}$ are operator spaces, a linear map $\Phi : \mathcal{E} \to \mathcal{F}$ is called completely bounded if the maps $\Phi^{(k)} : M_k(\mathcal{E}) \to M_k(\mathcal{F})$ given by $\Phi^{(k)}((a_{ij})) = (\Phi(a_{ij}))$ are bounded for every $k \in \mathbb{N}$ and $\|\Phi\|_{\mathrm{cb}} \stackrel{\text{def}}{=} \sup_k \|\Phi^{(k)}\| < \infty$.

Given linear spaces $\mathcal{E}_1, \dots, \mathcal{E}_n$, we denote by $\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n$ their algebraic tensor product. If $\mathcal{E}_1, \dots, \mathcal{E}_n$ are operator spaces and $a^k = (a_{ij}^k) \in M_{m_k,m_{k+1}}(\mathcal{E}_k)$, $m_k \in \mathbb{N}$, $k = 1, \dots, n$, we define the multiplicative product

$$a^1 \odot \cdots \odot a^n \in M_{m_1,m_{n+1}}(\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n)$$

by letting its (i,j)-entry $(a^1 \odot \cdots \odot a^n)_{ij}$ be $\sum_{i_2,\dots,i_n} a_{i,i_2}^1 \otimes a_{i_2,i_3}^2 \otimes \cdots \otimes a_{i_n,j}^n$. If $\mathcal{E}$ is another operator space and $\Phi : \mathcal{E}_1 \times \cdots \times \mathcal{E}_n \to \mathcal{E}$ is a multilinear map we let

$$\Phi^{(m)} : M_m(\mathcal{E}_1) \times \cdots \times M_m(\mathcal{E}_n) \to M_m(\mathcal{E})$$

be the map given by

$$\left(\Phi^{(m)}(a^1, \ldots, a^n)\right)_{ij} = \sum_{i_2,\ldots,i_n} \Phi(a^1_{i,i_2}, a^2_{i_2,i_3}, \ldots, a^n_{i_n,j}),$$

where $a^k = (a^k_{s,t}) \in M_m(\mathcal{E}_k)$, $k = 1, \ldots, n$. The multilinear map Φ is called completely bounded if there exists a constant $C > 0$ such that, for all $m \in \mathbb{N}$,

$$\|\Phi^{(m)}(a^1, \ldots, a^n)\| \le C\|a^1\| \ldots \|a^n\|, \quad a^k \in M_m(\mathcal{E}_k),\ k = 1, \ldots, n.$$

Set $\|\Phi\|_{\text{cb}} \stackrel{\text{def}}{=} \sup\{\|\Phi^{(m)}(a^1, \ldots, a^n)\| : m \in \mathbb{N}, \|a^1\|, \ldots, \|a^n\| \le 1\}$. It is well-known (see [28], [64]) that a completely bounded multilinear map Φ gives rise to a completely bounded map on the Haagerup tensor product $\mathcal{E}_1 \otimes_{\text{h}} \cdots \otimes_{\text{h}} \mathcal{E}_n$ (see [28] and [70] for its definition the and basic properties).

The set of all completely bounded multilinear maps from $\mathcal{E}_1 \times \cdots \times \mathcal{E}_n$ into $\mathcal{E}$ will be denoted by $CB(\mathcal{E}_1 \times \cdots \times \mathcal{E}_n, \mathcal{E})$. If $\mathcal{E}_1, \ldots, \mathcal{E}_n$ and $\mathcal{E}$ are dual operator spaces we say that a map $\Phi \in CB(\mathcal{E}_1 \times \cdots \times \mathcal{E}_n, \mathcal{E})$ is normal [18] if it is weak* continuous in each variable. We write $CB^{\sigma}(\mathcal{E}_1 \times \cdots \times \mathcal{E}_n, \mathcal{E})$ for the space of all normal maps in $CB(\mathcal{E}_1 \times \cdots \times \mathcal{E}_n, \mathcal{E})$.

The *extended Haagerup tensor product* $\mathcal{E}_1 \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{E}_n$ is defined [27] as the space of all normal completely bounded maps $u : \mathcal{E}_1^* \times \cdots \times \mathcal{E}_n^* \to \mathbb{C}$. It was shown in [27] that if $u \in \mathcal{E}_1 \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{E}_n$ then there exist index sets $J_1, J_2, \ldots, J_{n-1}$ and matrices $a^1 = (a^1_{1,s}) \in M_{1,J_1}(\mathcal{E}_1)$, $a^2 = (a^2_{s,t}) \in M_{J_1,J_2}(\mathcal{E}_2), \ldots, a^n = (a^n_{t,1}) \in M_{J_{n-1},1}(\mathcal{E}_n)$ such that if $f_i \in \mathcal{E}_i^*$, $i = 1, \ldots, n$, then

$$\langle u, f_1 \otimes \cdots \otimes f_n\rangle \stackrel{\text{def}}{=} u(f_1, \ldots, f_n) = \langle a^1, f_1\rangle \ldots \langle a^n, f_n\rangle, \tag{6.2}$$

where $\langle a^k, f_k\rangle = \left(f_k(a^k_{s,t})\right)$ and the product of the (possibly infinite) matrices in (6.2) is defined to be the limit of the sums

$$\sum_{i_1 \in F_1, \ldots, i_{n-1} \in F_{n-1}} f_1(a^1_{1,i_1}) f_2(a^2_{i_1,i_2}) \ldots f_n(a^n_{i_{n-1},1})$$

along the net $\{(F_1 \times \cdots \times F_{n-1}) : F_j \subseteq J_j \text{ finite}, 1 \le j \le n-1\}$. We may thus identify u with the matrix product $a^1 \odot \cdots \odot a^n$; two elements $a^1 \odot \cdots \odot a^n$ and $\tilde{a}^1 \odot \cdots \odot \tilde{a}^n$ coincide if $\langle a^1, f_1\rangle \ldots \langle a^n, f_n\rangle = \langle \tilde{a}^1, f_1\rangle \ldots \langle \tilde{a}^n, f_n\rangle$ for all $f_i \in \mathcal{E}_i^*$. Moreover,

$$\|u\|_{\text{eh}} = \inf\{\|a^1\| \ldots \|a^n\| : u = a^1 \odot \cdots \odot a^n\}.$$

The space $\mathcal{E}_1 \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{E}_n$ has a natural operator space structure [27]. If $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are dual operator spaces then by [27, Theorem 5.3] $\mathcal{E}_1 \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{E}_n$

coincides with the weak* Haagerup tensor product $\mathcal{E}_1 \otimes_{\mathrm{w^*h}} \cdots \otimes_{\mathrm{w^*h}} \mathcal{E}_n$ of Blecher and Smith [10]. Given operator spaces $\mathcal{F}_i$ and completely bounded maps $g_i : \mathcal{E}_i \to \mathcal{F}_i$, $i = 1, \ldots, n$, Effros and Ruan [27] define a completely bounded map

$$g = g_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} g_n : \mathcal{E}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_n \to \mathcal{F}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{F}_n,$$
$$a^1 \odot \cdots \odot a^n \mapsto \langle a^1, g_1 \rangle \odot \cdots \odot \langle a^n, g_n \rangle$$

where $\langle a^k, g_k \rangle = \big(g_k(a^k_{ij})\big)$. Thus

$$\langle g(u), f_1 \otimes \cdots \otimes f_n \rangle = \langle u, (f_1 \circ g_1) \otimes \cdots \otimes (f_n \circ g_n) \rangle \tag{6.3}$$

for $u \in \mathcal{E}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_n$ and $f_i \in \mathcal{F}_i^*$, $i = 1, \ldots, n$.

The following fact is a straightforward consequence of a well-known theorem due to Christensen and Sinclair [18], and it will be used throughout the exposition.

Theorem 86. *Let H_i be a Hilbert space and $\mathcal{R}_i \subseteq \mathcal{B}(H_i)$ be a von Neumann algebra, $i = 1, \ldots, n$. There exists an isometry γ from $\mathcal{R}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{R}_n$ onto the space of all $\mathcal{R}_1', \ldots, \mathcal{R}_n'$-modular maps in $CB^\sigma(\mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1}), \mathcal{B}(H_n, H_1))$, given as follows: if $u \in \mathcal{R}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{R}_n$ has a representation $u = A_1 \odot \cdots \odot A_n$ where $A_i \in M_J(\mathcal{R}_i) \subseteq \mathcal{B}(H_i \otimes \ell_2(J))$ for some index set J, then*

$$\gamma(u)(T_1, \ldots, T_{n-1}) = A_1(T_1 \otimes I)A_2 \ldots A_{n-1}(T_{n-1} \otimes I)A_n,$$

for all $T_i \in \mathcal{B}(H_{i+1}, H_i)$, $i = 1, \ldots, n-1$, where I is the identity operator on $\ell_2(J)$.

We now turn to the definition of slice maps which will play an important role in our proofs. Given $\omega_1 \in \mathcal{B}(H_1)^*$ we set $L_{\omega_1} = \omega_1 \otimes \mathrm{id}_{\mathcal{B}(H_2)}$. After identifying $\mathbb{C} \otimes \mathcal{B}(H_2)$ with $\mathcal{B}(H_2)$ we obtain a mapping $L_{\omega_1} : \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \to \mathcal{B}(H_2)$ called a left slice map. Similarly, for $\omega_2 \in \mathcal{B}(H_2)^*$ we obtain a right slice map $R_{\omega_2} : \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \to \mathcal{B}(H_1)$. If $u = \sum_{i \in I} v_i \otimes w_i \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2)$ where $v = (v_i)_{i \in I} \in M_{1,I}(\mathcal{B}(H_1))$ and $w = (w_i)_{i \in I} \in M_{I,1}(\mathcal{B}(H_2))$, then

$$L_{\omega_1}(u) = \sum_{i \in I} \omega_1(v_i) w_i \quad \text{and} \quad R_{\omega_2}(u) = \sum_{i \in I} \omega_2(w_i) v_i.$$

Moreover,

$$\langle R_{\omega_2}(u), \omega_1 \rangle = \langle u, \omega_1 \otimes \omega_2 \rangle = \langle L_{\omega_1}(u), \omega_2 \rangle = \sum_{i \in I} \omega_1(v_i) \omega_2(w_i). \tag{6.4}$$

It was shown in [90] that if $\mathcal{E} \subseteq \mathcal{B}(H_1)$ and $\mathcal{F} \subseteq \mathcal{B}(H_2)$ are closed subspaces then, up to a complete isometry,

$$\begin{aligned}\mathcal{E} \otimes_{\rm eh} \mathcal{F} = \{u \in \mathcal{B}(H_1) \otimes_{\rm eh} \mathcal{B}(H_2) : L_{\omega_1}(u) \in \mathcal{F} \text{ and } R_{\omega_2}(u) \in \mathcal{E} \\ \text{for all } \omega_1 \in \mathcal{B}(H_1)_* \text{ and } \omega_2 \in \mathcal{B}(H_2)_*\} \\ = \{u \in \mathcal{B}(H_1) \otimes_{\rm eh} \mathcal{B}(H_2) : L_{\omega_1}(u) \in \mathcal{F} \text{ and } R_{\omega_2}(u) \in \mathcal{E} \\ \text{for all } \omega_1 \in \mathcal{B}(H_1)^* \text{ and } \omega_2 \in \mathcal{B}(H_2)^*\}.\end{aligned} \tag{6.5}$$

Moreover [87],

$$\begin{aligned}\mathcal{E} \otimes_{\rm h} \mathcal{F} = \{u \in \mathcal{B}(H_1) \otimes_{\rm h} \mathcal{B}(H_2) : L_{\omega_1}(u) \in \mathcal{F} \text{ and } R_{\omega_2}(u) \in \mathcal{E} \\ \text{for all } \omega_1 \in \mathcal{B}(H_1)^* \text{ and } \omega_2 \in \mathcal{B}(H_2)^*\}.\end{aligned} \tag{6.6}$$

Thus, $\mathcal{E} \otimes_{\rm h} \mathcal{F}$ can be canonically identified with a subspace of $\mathcal{B}(H_1) \otimes_{\rm h} \mathcal{B}(H_2)$ which, on the other hand, sits completely isometrically in $\mathcal{B}(H_1) \otimes_{\rm eh} \mathcal{B}(H_2)$. These identifications are made in the statement of the following lemma which will be useful for us later.

Lemma 87. *If H_1, H_2, H_3 are Hilbert spaces and $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{B}(H_1)$, $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{B}(H_2)$ and $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{B}(H_3)$ are operator spaces, then*

$$(\mathcal{E}_1 \otimes_{\rm eh} \mathcal{F}_1) \cap (\mathcal{E}_2 \otimes_{\rm h} \mathcal{F}_2) = (\mathcal{E}_1 \cap \mathcal{E}_2) \otimes_{\rm h} (\mathcal{F}_1 \cap \mathcal{F}_2) \quad \textit{and}$$
$$(\mathcal{E}_1 \otimes_{\rm eh} \mathcal{F}_1 \otimes_{\rm eh} \mathcal{G}_1) \cap (\mathcal{E}_2 \otimes_{\rm h} \mathcal{F}_2 \otimes_{\rm h} \mathcal{G}_2) = (\mathcal{E}_1 \cap \mathcal{E}_2) \otimes_{\rm h} (\mathcal{F}_1 \cap \mathcal{F}_2) \otimes_{\rm h} (\mathcal{G}_1 \cap \mathcal{G}_2)$$

completely isometrically.

Proof. Since $\otimes_{\rm eh}$ and $\otimes_{\rm h}$ are both associative, the second equation follows from the first. If $u \in (\mathcal{E}_1 \otimes_{\rm eh} \mathcal{F}_1) \cap (\mathcal{E}_2 \otimes_{\rm h} \mathcal{F}_2) \subseteq \mathcal{B}(H_1) \otimes_{\rm h} \mathcal{B}(H_2)$ then $L_\varphi(u) \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $R_\psi(u) \in \mathcal{E}_1 \cap \mathcal{E}_2$ whenever $\varphi \in \mathcal{B}(H_1)^*$ and $\psi \in \mathcal{B}(H_2)^*$. By (6.6), $u \in (\mathcal{E}_1 \cap \mathcal{E}_2) \otimes_{\rm h} (\mathcal{F}_1 \cap \mathcal{F}_2)$. The converse inclusion follows immediately in light of the injectivity of the Haagerup tensor product. □

6.1.2 Operator multipliers

We now recall some definitions and results from [52] and [48] that will be needed later. Let $H_1, \ldots, H_n$ be Hilbert spaces and $H = H_1 \otimes \cdots \otimes H_n$ be their Hilbertian tensor product. Set $HS(H_1, H_2) = \mathcal{C}_2(H_1^{\rm d}, H_2)$ and let $\theta_{H_1,H_2} : H_1 \otimes H_2 \to HS(H_1, H_2)$ be the canonical isometry given by $\theta(\xi_1 \otimes \xi_2)(\eta^{\rm d}) = (\xi_1, \eta)\xi_2$ for $\xi_1, \eta \in H_1$ and $\xi_2 \in H_2$. When n is even, we inductively define

$$HS(H_1, \ldots, H_n) \stackrel{\rm def}{=} \mathcal{C}_2(HS(H_2, H_3)^{\rm d}, HS(H_1, H_4, \ldots, H_n)),$$

and let $\theta_{H_1,\dots,H_n} : H \to HS(H_1,\dots,H_n)$ be given by

$$\theta_{H_1,\dots,H_n}(\xi_{2,3}\otimes\xi) = \theta_{HS(H_2,H_3),HS(H_1,H_4,\dots,H_n)}(\theta_{H_2,H_3}(\xi_{2,3})\otimes\theta_{H_1,H_4,\dots,H_n}(\xi)),$$

where $\xi_{2,3}\in H_2\otimes H_3$ and $\xi\in H_1\otimes H_4\otimes\cdots\otimes H_n$. When n is odd, we let

$$HS(H_1,\dots,H_n) \stackrel{\text{def}}{=} HS(\mathbb{C},H_1,\dots,H_n).$$

If K is a Hilbert space, we will identify $\mathcal{C}_2(\mathbb{C}^{\text{d}},K)$ with K via the map $S\to S(1^{\text{d}})$. The isomorphism $\theta_{H_1,\dots,H_n}$ in the odd case is given by

$$\theta_{H_1,\dots,H_n}(\xi) = \theta_{\mathbb{C},H_1,\dots,H_n}(1\otimes\xi).$$

We will omit the subscripts when they are clear from the context and simply write θ.

If $\xi\in H_1\otimes H_2$ we let $\|\xi\|_{\text{op}}$ denote the operator norm of $\theta(\xi)$. By $\|\cdot\|_2$ we will denote the Hilbert-Schmidt norm.

Let

$$\Gamma(H_1,\dots,H_n) = \begin{cases} (H_1\otimes H_2)\odot(H_2\otimes H_3)^{\text{d}}\odot\cdots\odot(H_{n-1}\otimes H_n) & n \text{ even},\\ (H_1\otimes H_2)^{\text{d}}\odot(H_2\otimes H_3)\odot\cdots\odot(H_{n-1}\otimes H_n) & n \text{ odd}.\end{cases}$$

We equip $\Gamma(H_1,\dots,H_n)$ with the Haagerup norm $\|\cdot\|_{\text{h}}$ where each of the terms of the algebraic tensor product is given the opposite operator space structure to the one arising from the embedding $H\otimes K\hookrightarrow(\mathcal{C}_2(H^{\text{d}},K),\|\cdot\|_{\text{op}})$. We denote by $\|\cdot\|_{2,\wedge}$ the projective norm on $\Gamma(H_1,\dots,H_n)$ where each of the terms is given its Hilbert space norm.

Suppose n is even. For each $\varphi\in\mathcal{B}(H)$ we let $S_\varphi : \Gamma(H_1,\dots,H_n)\to\mathcal{B}(H_1^{\text{d}},H_n)$ be the map given by

$$S_\varphi(\zeta) = \theta(\varphi(\xi_{1,2}\otimes\xi_{3,4}\otimes\cdots\otimes\xi_{n-1,n}))(\theta(\eta_{2,3}^{\text{d}}))(\theta(\eta_{4,5}^{\text{d}}))\dots(\theta(\eta_{n-2,n-1}^{\text{d}}))$$

where $\zeta = \xi_{1,2}\odot\eta_{2,3}^{\text{d}}\cdots\odot\xi_{n-1,n}\in\Gamma(H_1,\dots,H_n)$ is an elementary tensor. In particular, if $A_i\in\mathcal{B}(H_i)$, $i=1,\dots,n$, and $\varphi = A_1\otimes\dots\otimes A_n$ then

$$S_\varphi(\zeta) = A_n\theta(\xi_{n-1,n})\dots A_3^{\text{d}}\theta(\eta_{2,3}^{\text{d}})A_2\theta(\xi_{1,2})A_1^{\text{d}}.$$

Now suppose that n is odd and let $\zeta\in\Gamma(H_1,\dots,H_n)$ and $\xi_1\in H_1$. Then

$$\xi_1\otimes\zeta\in H_1\odot\Gamma(H_1,\dots,H_n) = \Gamma(\mathbb{C},H_1,\dots,H_n).$$

For $\varphi\in\mathcal{B}(H)$ we let $S_\varphi(\zeta)$ be the operator defined on H_1 by

$$S_\varphi(\zeta)(\xi_1) = S_{1\otimes\varphi}(\xi_1\otimes\zeta).$$

Note that $S_{1\otimes\varphi}(\xi_1 \otimes \zeta) \in \mathcal{C}_2(\mathbb{C}^{\mathrm{d}}, H_n)$; thus, $S_\varphi(\zeta)(\xi_1)$ can be viewed as an element of H_n. It was shown in [48] that $S_\varphi(\zeta) \in \mathcal{B}(H_1, H_n)$. If $\zeta = \eta_{1,2}^{\mathrm{d}} \otimes \xi_{2,3} \otimes \ldots \otimes \xi_{n-1,n}$ and $\varphi = A_1 \otimes \ldots \otimes A_n$ for $A_i \in \mathcal{B}(H_i)$, $i = 1, \ldots, n$ then

$$S_\varphi(\zeta) = A_n \theta(\xi_{n-1,n}) \ldots A_3 \theta(\xi_{2,3}) A_2^{\mathrm{d}} \theta(\eta_{1,2}^{\mathrm{d}}) A_1.$$

As observed in [48, Remark 4.3], for any $\varphi \in \mathcal{B}(H)$ and $\zeta \in \Gamma(H_1, \ldots, H_n)$,

$$\|S_\varphi(\zeta)\|_{\mathrm{op}} \leq \|\varphi\| \, \|\zeta\|_{2,\wedge}. \tag{6.7}$$

On the other hand, an element $\varphi \in \mathcal{B}(H)$ is called a *concrete operator multiplier* if there exists $C > 0$ such that $\|S_\varphi(\zeta)\|_{\mathrm{op}} \leq C\|\zeta\|_{\mathrm{h}}$ for each $\zeta \in \Gamma(H_1, \ldots, H_n)$. When $n = 2$, this is equivalent to $\|S_\varphi(\zeta)\|_{\mathrm{op}} \leq C\|\theta(\zeta)\|_{\mathrm{op}}$ for each $\zeta \in H_1 \otimes H_2$. We call the smallest constant C with this property the concrete multiplier norm of φ.

Now let $\mathcal{A}_i$ be a C*-algebra and $\pi_i : \mathcal{A}_i \to \mathcal{B}(H_i)$ be a representation, $i = 1, \ldots, n$. Set $\pi = \pi_1 \otimes \cdots \otimes \pi_n : \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to \mathcal{B}(H_1 \otimes \cdots \otimes H_n)$ (here, and in the sequel, by $\mathcal{A} \otimes \mathcal{B}$ we will denote the minimal tensor product of the C*-algebras $\mathcal{A}$ and $\mathcal{B}$). An element $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ is called a $\pi_1, \ldots, \pi_n$-*multiplier* if $\pi(\varphi)$ is a concrete operator multiplier. We denote by $\|\varphi\|_{\pi_1,\ldots,\pi_n}$ the concrete multiplier norm of $\pi(\varphi)$. We call φ a *universal multiplier* if it is a $\pi_1, \ldots, \pi_n$-multiplier for all representations π_i of $\mathcal{A}_i$, $i = 1, \ldots, n$. We denote the collection of all universal multipliers by $M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$; from this definition, it immediately follows that

$$\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n \subseteq M(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n.$$

It was observed in [48] that if $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ then

$$\|\varphi\|_{\mathrm{m}} \stackrel{\mathrm{def}}{=} \sup\{\|\varphi\|_{\pi_1,\ldots,\pi_n} : \pi_i \text{ is a representation of } \mathcal{A}_i,\ i = 1, \ldots, n\} < \infty.$$

It is obvious that if $\mathcal{A}_i$ and $\mathcal{B}_i$ are C*-algebras and $\rho_i : \mathcal{A}_i \to \mathcal{B}_i$ is a $*$-isomorphism, $i = 1, \ldots, n$, then

$$(\rho_1 \otimes \cdots \otimes \rho_n)(M(\mathcal{A}_1, \ldots, \mathcal{A}_n)) = M(\mathcal{B}_1, \ldots, \mathcal{B}_n).$$

If φ is an operator, and $\{\varphi_\nu\}$ a net of operators, acting on $H_1 \otimes \cdots \otimes H_n$ we say that $\{\varphi_\nu\}$ converges semi-weakly to φ if $(\varphi_\nu \xi, \eta) \to_\nu (\varphi\xi, \eta)$ for all $\xi, \eta \in H_1 \odot \cdots \odot H_n$. The following characterisation of universal multipliers was established in [48] (see Theorem 6.5, the subsequent remark and the proof of Proposition 6.2) and will be used extensively in the sequel.

Theorem 88. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ be a C*-algebra, $i = 1, \dots, n$, and $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. Suppose that n is even. The following are equivalent:*

(i) $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$;

(ii) there exists a net $\{\varphi_\nu\}$ where $\varphi_\nu = A_1^\nu \odot A_2^\nu \odot \cdots \odot A_n^\nu$ and A_i^ν is a finite block operator matrix with entries in $\mathcal{A}_i$ such that $\varphi_\nu \to \varphi$ semi-weakly, $\|\varphi_\nu\|_{\mathrm{m}} \leq \Pi\|A_{2i}^\nu\|\Pi\|A_{2i+1}^{\nu\mathrm{d}}\|$ and the operator norms $\|A_i^\nu\|$ for i even and $\|A_i^{\nu\mathrm{d}}\|$ for i odd, are bounded by a constant depending only on n.

For every net $\{\varphi_\nu\}$ satisfying (ii) we have that $S_{\varphi_\nu}(\zeta) \to S_\varphi(\zeta)$ weakly for all $\zeta = \xi_{1,2} \otimes \cdots \otimes \xi_{n-1,n} \in \Gamma(H_1, \dots, H_n)$ and $\sup_\nu \|\varphi_\nu\|_{\mathrm{m}}$ is finite.

Moreover, the net φ_ν can be chosen in (ii) so that $A_i^\nu \to A_i$ (resp. $A_i^{\nu\mathrm{d}} \to A_i^{\mathrm{d}}$) strongly for i even (resp. for i odd) for some bounded block operator matrix A_i with entries in $\mathcal{A}_i''$ (resp. $(\mathcal{A}_i^{\mathrm{d}})''$) such that

$$S_{\mathrm{id}\otimes\cdots\otimes\mathrm{id}(\varphi)}(\zeta) = A_n(\theta(\xi_{n-1,n}) \otimes I) \dots (\theta(\xi_{1,2}) \otimes I)A_1^{\mathrm{d}},$$

for all $\zeta = \xi_{1,2} \otimes \cdots \otimes \xi_{n-1,n} \in \Gamma(H_1, \dots, H_n)$.

A similar statement holds if n is odd.

Finally, recall that an element a of a C*-algebra $\mathcal{A}$ is called *compact* if the operator $x \to axa$ on $\mathcal{A}$ is compact. Let $\mathcal{K}(\mathcal{A})$ be the collection of all compact elements of $\mathcal{A}$. It is well known [30], [100] that $a \in \mathcal{K}(\mathcal{A})$ if and only if there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is a compact operator. Moreover, π can be taken to be the reduced atomic representation of $\mathcal{A}$. The notion of a compact element of a C*-algebra will play a central role in Sections 6.5 and 6.6 of the paper.

6.2 Completely compact maps

We start by recalling the notion of a completely compact map introduced in [85] and studied further in [98] and [58]. By way of motivation, recall that if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces then a bounded linear map $\Phi : \mathcal{X} \to \mathcal{Y}$ is compact if and only if for every $\varepsilon > 0$, there exists a finite dimensional subspace $F \subseteq \mathcal{Y}$ such that $\mathrm{dist}(\Phi(x), F) < \varepsilon$ for every x in the unit ball of $\mathcal{X}$.

Now let $\mathcal{X}$ and $\mathcal{Y}$ be operator spaces. A completely bounded map $\Phi : \mathcal{X} \to \mathcal{Y}$ is called *completely compact* if for each $\varepsilon > 0$ there exists a finite dimensional subspace $F \subseteq \mathcal{Y}$ such that

$$\mathrm{dist}(\Phi^{(m)}(x), M_m(F)) < \varepsilon,$$

for every $x \in M_m(\mathcal{X})$ with $\|x\| \leq 1$ and every $m \in \mathbb{N}$. We extend this definition to multilinear maps: if $\mathcal{Y}, \mathcal{X}_1, \dots, \mathcal{X}_n$ are operator spaces and $\Phi : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Y}$ is a completely bounded multilinear map, we call Φ

completely compact if for each $\varepsilon > 0$ there exists a finite dimensional subspace $F \subseteq \mathcal{Y}$ such that

$$\operatorname{dist}(\Phi^{(m)}(x_1, \ldots, x_n), M_m(F)) < \varepsilon,$$

for all $x_i \in M_m(\mathcal{X}_i)$, $\|x_i\| \leq 1$, $i = 1, \ldots, n$, and all $m \in \mathbb{N}$. We denote by $CC(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$ the space of all completely bounded completely compact multilinear maps from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ into $\mathcal{Y}$. A straightforward verification shows the following:

Remark 89. *A completely bounded map $\Phi : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Y}$ is completely compact if and only if its linearisation $\tilde{\Phi} : \mathcal{X}_1 \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{X}_n \to \mathcal{Y}$ is completely compact.*

In view of this remark, we frequently identify the spaces $CC(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$ and $CC(\mathcal{X}_1 \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{X}_n, \mathcal{Y})$. The next result is essentially due to Saar (see Lemmas 1 and 2 of [85]).

Proposition 90. *(i) $CC(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$ is closed in $CB(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$.*

(ii) Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be operator spaces. If $\Phi \in CC(\mathcal{E}, \mathcal{F})$ and $\Psi \in CB(\mathcal{F}, \mathcal{G})$ then $\Psi \circ \Phi \in CC(\mathcal{E}, \mathcal{G})$. If $\Phi \in CC(\mathcal{F}, \mathcal{G})$ and $\Psi \in CB(\mathcal{E}, \mathcal{F})$ then $\Phi \circ \Psi \in CC(\mathcal{E}, \mathcal{G})$.

Let $H_1, \ldots, H_n$ be Hilbert spaces. Recall the isometry

$$\gamma : \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n) \to CB^{\sigma}(\mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1}), \mathcal{B}(H_n, H_1))$$

from Theorem 86. Let us identify a completely bounded map defined on $\mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1})$ with the corresponding completely bounded map defined on

$$\mathcal{B}_{\mathrm{h}} \stackrel{\text{def}}{=} \mathcal{B}(H_2, H_1) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{B}(H_n, H_{n-1}).$$

For $u \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$ we let $\gamma_0(u)$ be the restriction of $\gamma(u)$ to

$$\mathcal{K}_{\mathrm{h}} \stackrel{\text{def}}{=} \mathcal{K}(H_2, H_1) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_n, H_{n-1}).$$

Proposition 91. *The map γ_0 is an isometry from $\mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$ onto $CB(\mathcal{K}_{\mathrm{h}}, \mathcal{B}(H_n, H_1))$.*

Proof. Let $\Phi \in CB(\mathcal{K}_{\mathrm{h}}, \mathcal{B}(H_n, H_1))$. Since Φ is completely bounded, its second dual

$$\Phi^{**} : \mathcal{B}(H_2, H_1) \otimes \cdots \otimes \mathcal{B}(H_n, H_{n-1}) \to \mathcal{B}(H_n, H_1)^{**}$$

is completely bounded (here $\otimes$ denotes the normal Haagerup tensor product [27]). Let $Q : \mathcal{B}(H_n, H_1)^{**} \to \mathcal{B}(H_n, H_1)$ be the canonical projection. The multilinear map

$$\tilde{\Phi} : \mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1}) \to \mathcal{B}(H_n, H_1)$$

corresponding to $Q \circ \Phi^{**}$ is completely bounded and, by (5.22) of [27], weak* continuous in each variable. By Theorem 86, there exists an element $u \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$ such that $\tilde{\Phi} = \gamma(u)$. Hence $\gamma_0(u) = \gamma(u)|_{\mathcal{K}_{\mathrm{h}}} = \tilde{\Phi}|_{\mathcal{K}_{\mathrm{h}}} = \Phi$. Thus γ_0 is surjective.

Fix $u \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$. From the definition of γ_0 we have $\|\gamma_0(u)\|_{\mathrm{cb}} \leq \|\gamma(u)\|_{\mathrm{cb}} = \|u\|_{\mathrm{eh}}$. On the other hand, the restrictions of the maps $Q \circ \gamma_0(u)^{**}$ and $\gamma(u)$ to $\mathcal{K}_{\mathrm{h}}$ coincide, and since both maps are weak* continuous, $\gamma(u) = Q \circ \gamma_0(u)^{**}|_{\mathcal{B}_{\mathrm{h}}}$. Hence,

$$\|u\|_{\mathrm{eh}} \leq \|Q \circ \gamma_0(u)^{**}\|_{\mathrm{cb}} \leq \|\gamma_0(u)^{**}\|_{\mathrm{cb}} = \|\gamma_0(u)\|_{\mathrm{cb}}.$$

Thus, γ_0 is an isometry. □

Theorem 92. *Let $H_1, \dots, H_n$ be Hilbert spaces. The image under γ_0 of the operator space $\mathcal{E} \stackrel{\mathrm{def}}{=} \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$ is $\mathcal{F} \stackrel{\mathrm{def}}{=} CC(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$.*

Proof. We first establish the inclusion $\gamma_0(\mathcal{E}) \subseteq \mathcal{F}$. If $\Phi = \gamma_0(u)$ where $u \in \mathcal{E}$ then, by Proposition 91, Φ is the limit in the cb norm of maps of the form $\gamma_0(v)$, where

$$v = a \odot B \odot b \in \mathcal{K}(H_1) \odot (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \odot \mathcal{K}(H_n),$$

a and b have finite rank and B is a finite matrix with entries in $\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})$. But each such map has finite rank and hence is completely compact. Moreover, every operator in the image of $\gamma_0(v)$ has range contained in the range of a, which is finite dimensional. It follows that Φ takes compact values; it is completely compact by Proposition 90.

To see that $\mathcal{F} \subseteq \gamma_0(\mathcal{E})$, let $\Phi \in \mathcal{F}$. We will assume for technical simplicity that $H_1, \dots, H_n$ are separable. Let $\{p_k\}_k$ (resp. $\{q_k\}_k$) be a sequence of projections of finite rank on H_1 (resp. H_n) such that $p_k \to I$ (resp. $q_k \to I$) in the strong operator topology. Let $\Psi_k : \mathcal{K}(H_n, H_1) \to \mathcal{K}(H_n, H_1)$ be the complete contraction given by $\Psi_k(x) = p_k x q_k$.

Let $\varepsilon > 0$. Since Φ is completely compact there exists a subspace $F \subseteq \mathcal{K}(H_n, H_1)$ of dimension $\ell < \infty$ such that $\mathrm{dist}(\Phi^{(m)}(x), M_m(F)) < \varepsilon$ whenever $x \in M_m(\mathcal{K}_{\mathrm{h}})$ has norm at most one. Denote the restriction of Ψ_k to F by $\Psi_{k,F}$

and let ι be the inclusion map $\iota : F \hookrightarrow \mathcal{K}(H_n, H_1)$. By [28, Corollary 2.2.4], $\|\Psi_{k,F} - \iota\|_{\rm cb} \leq \ell \|\Psi_{k,F} - \iota\|$. Since $F \subseteq \mathcal{K}(H_n, H_1)$, we have that $\Psi_{k,F}(x) \to x$ in norm for each $x \in F$. It follows easily that there exists k_0 such that $\|\Psi_{k,F} - \iota\|_{\rm cb} < \varepsilon$ whenever $k \geq k_0$.

Let $x \in M_m(\mathcal{K}_{\rm h})$ be of norm at most one. Then there exists $y \in M_m(F)$ such that $\|\Phi^{(m)}(x) - y\| < \varepsilon$. Note that

$$\|y\| \leq \|\Phi^{(m)}(x) - y\| + \|\Phi^{(m)}(x)\| \leq \varepsilon + \|\Phi\|_{\rm cb}.$$

Let $\Phi_k = \Psi_k \circ \Phi$. If $k \geq k_0$ then

$$\begin{aligned}\|(\Phi_k^{(m)} - \Phi^{(m)})(x)\| &\leq \|\Phi_k^{(m)}(x) - \Psi_k^{(m)}(y)\| + \|\Psi_k^{(m)}(y) - y\| + \|y - \Phi^{(m)}(x)\| \\ &= \|\Psi_k^{(m)}(\Phi^{(m)}(x) - y)\| + \|(\Psi_{k,F} - \iota)^{(m)}(y)\| + \|y - \Phi^{(m)}(x)\| \\ &\leq 2\varepsilon + \varepsilon(\varepsilon + \|\Phi\|_{\rm cb}).\end{aligned}$$

This shows that $\|\Phi_k - \Phi\|_{\rm cb} \to 0$.

By Proposition 90, it only remains to prove that each Φ_k lies in $\gamma_0(\mathcal{E})$. By Proposition 91, there exists an element

$$u = A_1 \odot A_2 \odot \cdots \odot A_{n-1} \odot A_n \in \mathcal{B}(H_1) \otimes_{\rm eh} \cdots \otimes_{\rm eh} \mathcal{B}(H_n)$$

where $A_1 : H_1^\infty \to H_1$, $A_i : H_i^\infty \to H_i^\infty$, $i = 2, \ldots, n-1$ and $A_n : H_n \to H_n^\infty$ are bounded operators, such that $\Phi = \gamma_0(u)$. Observe that $\Phi_k = \gamma_0(u_k)$ where $u_k = (p_k A_1) \odot A_2 \odot \cdots \odot A_{n-1} \odot (A_n q_k)$. It therefore suffices to show that $u_k \in \mathcal{E}$ for each k. Fix k and let $p = p_k$, $q = q_k$. The operator $pA_1 : H_1^\infty \to H_1$ has finite dimensional range and is hence compact. For $i = 1, \ldots, n$, let $Q_{i,r} : H_i^\infty \to H_i^\infty$ be a projection with block matrix whose first r diagonal entries are equal to the identity operator while the rest are zero. Then by compactness, $(pA_1)Q_{1,r} \to pA_1$ and $Q_{n,r}(A_n q) \to A_n q$ in norm as $r \to \infty$. Let $B = A_2 \odot \cdots \odot A_{n-1}$, $C_r = (pA_1)Q_{1,r} \odot B \odot Q_{n,r}(A_n q)$, $r \in \mathbb{N}$, and $C = (pA_1) \odot B \odot (A_n q)$. Then

$$\begin{aligned}\|C_r - C\|_{\rm eh} &\leq \|C_r - (pA_1)Q_{1,r} \odot B \odot (A_n q)\|_{\rm eh} \\ &\qquad + \|(pA_1)Q_{1,r} \odot B \odot (A_n q) - C\|_{\rm eh} \\ &\leq \|(pA_1)Q_{1,r}\| \, \|B\| \, \|Q_{n,r}(A_n q) - A_n q\| \\ &\qquad + \|(pA_1)Q_{1,r} - pA_1\| \, \|B\| \, \|A_n q\|.\end{aligned}$$

It follows that $\|C_r - C\|_{\rm eh} \to 0$ as $r \to \infty$. Our claim will follow if we show that $C_r \in \mathcal{E}$. To this end, it suffices to show that if $A_1 = [a_1, \ldots, a_r, 0, \ldots]$ and $A_n = [b_1, \ldots, b_r, 0, \ldots]^t$, where a_i, b_i are operators of finite rank, then $A_1 \odot B \odot A_n \in \mathcal{E}$. Let A_1 and A_n be as stated and let $B' = (Q_{2,r}A_2) \odot A_3 \odot$

$\cdots \odot A_{n-2} \odot (A_{n-1}Q_{n,r})$. Then $A_1 \odot B \odot A_n = A_1 \odot B' \odot A_{n+1}$ belongs to the algebraic tensor product $\mathcal{K}(H_1) \odot (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \odot \mathcal{K}(H_n)$ and hence to $\mathcal{E} = \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$. □

Remarks 93. *(i) It follows from Theorem 92 that if $\Phi : \mathcal{K}_{\mathrm{h}} \to \mathcal{K}(H_n, H_1)$ is a mapping of finite rank whose image consists of finite rank operators then there exist finite rank projections p and q on H_1 and H_n, respectively, and $u \in (p\mathcal{K}(H_1)) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} (\mathcal{K}(H_n)q)$ such that $\Phi = \gamma_0(u)$.*
(ii) The identity $\mathcal{E}_1 \otimes_{\mathrm{h}} (\mathcal{E}_2 \otimes_{\mathrm{eh}} \mathcal{E}_3) = (\mathcal{E}_1 \otimes_{\mathrm{h}} \mathcal{E}_2) \otimes_{\mathrm{eh}} \mathcal{E}_3$ does not hold in general; for an example, take $\mathcal{E}_1 = \mathcal{E}_3 = \mathcal{B}(H)$ and $\mathcal{E}_2 = \mathbb{C}$.
(iii) For every $\Phi \in CC(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$ there exist $A_1 \in \mathcal{K}(H_1^{J_1}, H_1)$, $A_i \in \mathcal{B}(H_i^{J_i}, H_i^{J_{i-1}})$, $i = 2, \ldots, n-1$ and $A_n \in \mathcal{K}(H_n, H_n^{J_{n-1}})$ such that

$$\Phi(x_1 \otimes \cdots \otimes x_{n-1}) = A_1(x_1 \otimes 1)A_2 \ldots (x_{n-1} \otimes 1)A_n,$$

whenever $x_i \in \mathcal{K}(H_{i+1}, H_i)$, $i = 1, \ldots, n-1$. Indeed, by Proposition 92, $\Phi(x_1 \otimes \cdots \otimes x_{n-1}) = A_1(x_1 \otimes 1)A_2 \ldots (x_{n-1} \otimes 1)A_n$ for some $A_1 \odot A_2 \odot \cdots \odot A_n \in \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$. Using an idea of Blecher and Smith [10, Theorem 3.1], we can choose $A_1 = [t_j]_{j \in J_1} \in M_{J_1,1}(\mathcal{K}(H_1)) \subseteq \mathcal{B}(H_1^{J_1}, H_1)$ and $A_n = [s_i]_{i \in J_{n-1}} \in M_{1,J_{n-1}}(\mathcal{K}(H_n)) \subseteq \mathcal{B}(H_n, H_n^{J_{n-1}})$ such that the sums $\sum_i s_i s_i^*$ and $\sum_j t_j^* t_j$ converge uniformly. Then A_1 is the norm limit of $A_1^{\mathcal{F}} = [t_j^{\mathcal{F}}]_{i \in J_1}$, where $\mathcal{F}$ is a finite subset of J_1 and $t_j^{\mathcal{F}} = t_j$ if $j \in \mathcal{F}$ and $t_j^{\mathcal{F}} = 0$ otherwise. Therefore $A_1 \in \mathcal{K}(H_1^{J_1}, H)$. Similarly, $A_n \in \mathcal{K}(H_n, H_n^{J_n-1})$.

In the case $n = 2$, Theorem 92 reduces to the following result which was established by Saar (Satz 6 of [85]) using the fact that every completely compact completely bounded map on $\mathcal{K}(H_1, H_2)$ is a linear combination of completely compact completely positive maps.

Corollary 94. *A completely bounded map $\Phi : \mathcal{K}(H_1, H_2) \to \mathcal{K}(H_1, H_2)$ is completely compact if and only if there exist an index set I and families $\{a_i\}_{i \in I} \subseteq \mathcal{K}(H_1)$ and $\{b_i\}_{i \in I} \subseteq \mathcal{K}(H_2)$ such that the series $\sum_{i \in I} b_i b_i^*$ and $\sum_{i \in I} a_i^* a_i$ converge uniformly and*

$$\Phi(x) = \sum_{i \in I} b_i x a_i, \quad x \in \mathcal{K}(H_1, H_2).$$

We note in passing that Theorem 92 together with a result of Effros and Kishimoto [26] yields the following completely isometric identification:

Corollary 95. *$CC(\mathcal{K}(H_2, H_1))^{**} \simeq (\mathcal{K}(H_1) \otimes_{\mathrm{h}} \mathcal{K}(H_2))^{**} \simeq CB(\mathcal{B}(H_2, H_1))$.*

Saar [85] constructed an example of a compact map $\Phi : \mathcal{K}(H) \to \mathcal{K}(H)$ which is not completely compact (see Section 6.6 where we give a detailed account of this construction). We note that a compact completely positive map $\Phi : \mathcal{K}(H) \to \mathcal{K}(H)$ is automatically completely compact. Indeed, the Stinespring Theorem implies that there exist an index set J and a row operator $A = [a_i]_{i\in J} \in \mathcal{B}(H^J, H)$ such that $\Phi(x) = \sum_{i\in J} a_i x a_i^*$, $x \in \mathcal{K}(H)$. The second dual $\Phi^{**} : \mathcal{B}(H) \to \mathcal{B}(H)$ of Φ is a compact map given by the same formula. A standard Banach space argument shows that Φ^{**} takes values in $\mathcal{K}(H)$, and hence $\Phi^{**}(I) \in \mathcal{K}(H)$. This means that $AA^* \in \mathcal{K}(H)$ and so $A \in \mathcal{K}(H^\infty, H)$ which easily implies that Φ is completely compact.

The previous paragraph shows that there exists a compact completely bounded map on $\mathcal{K}(H)$ which cannot be written as a linear combination of compact completely positive maps.

We finish this section with a modular version of Theorem 92. Given von Neumann algebras $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \dots, n$, we let $CC_{\mathcal{A}_1',\dots,\mathcal{A}_n'}(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$ denote the space of all completely compact multilinear maps from $\mathcal{K}_{\mathrm{h}}$ into $\mathcal{K}(H_n, H_1)$ such that the corresponding multilinear map from $\mathcal{K}(H_2, H_1) \times \cdots \times \mathcal{K}(H_n, H_{n-1})$ into $\mathcal{K}(H_n, H_1)$ is $\mathcal{A}_1', \dots, \mathcal{A}_n'$-modular.

Corollary 96. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \dots, n$, be von Neumann algebras. Set $\mathcal{K}'(\mathcal{A}_i) = \mathcal{K}(H_i) \cap \mathcal{A}_i$, for $i = 1$ and $i = n$. Then*

$$\gamma_0\big(\mathcal{K}'(\mathcal{A}_1) \otimes_{\mathrm{h}} (\mathcal{A}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}) \otimes_{\mathrm{h}} \mathcal{K}'(\mathcal{A}_n)\big) = CC_{\mathcal{A}_1',\dots,\mathcal{A}_n'}(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1)).$$

Proof. By Theorems 86 and 92, the image of $\mathcal{K}'(\mathcal{A}_1) \otimes_{\mathrm{h}} (\mathcal{A}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}) \otimes_{\mathrm{h}} \mathcal{K}'(\mathcal{A}_n)$ under γ_0 is contained in $CC_{\mathcal{A}_1',\dots,\mathcal{A}_n'}(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$. For the converse, fix an element $\Phi \in CC_{\mathcal{A}_1',\dots,\mathcal{A}_n'}(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$. By Theorem 92, there exists a unique $u \in \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$ such that $\gamma_0(u) = \Phi$. By Theorem 86, $u \in \mathcal{A}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_n$. Lemma 87 now shows that $u \in \mathcal{K}'(\mathcal{A}_1) \otimes_{\mathrm{h}} (\mathcal{A}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}) \otimes_{\mathrm{h}} \mathcal{K}'(\mathcal{A}_n)$. □

6.3 Complete boundedness of multipliers

Our aim in this section is to clarify the relationship between universal operator multipliers and completely bounded maps, extending results of [48]. We begin with an observation which will allow us to deal with the cases of even and odd numbers of variables in the same manner. We use the notation established in Section 6.1.

Proposition 97. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C^*-algebras and $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$. Let π_i be a representation of $\mathcal{A}_i$ on a Hilbert space H_i, $i = 1, \dots, n$, and*

$\pi = \pi_1 \otimes \cdots \otimes \pi_n$. *The map* $S_{\pi(\varphi)}$ *takes values in* $\mathcal{K}(H_1, H_n)$ *if* n *is odd, and in* $\mathcal{K}(H_1^{\mathrm{d}}, H_n)$ *if* n *is even.*

Proof. For even n, this is immediate as observed in [48]. Let n be odd. Assume without loss of generality that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ and π_i is the identity representation. We call an element $\zeta \in \Gamma(H_1, \dots, H_n)$ thoroughly elementary if

$$\zeta = \eta_{1,2}^{\mathrm{d}} \otimes \xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n}$$

where all $\eta_{j,j+1}^{\mathrm{d}} = \eta_j^{\mathrm{d}} \otimes \eta_{j+1}^{\mathrm{d}}$ and $\xi_{j-1,j} = \xi_{j-1} \otimes \xi_j$ are elementary tensors. The linear span of the thoroughly elementary tensors is dense in the completion of $\Gamma(H_1, \dots, H_n)$ in $\|\cdot\|_{2,\wedge}$. Moreover, the linear span of the elementary tensors $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_n$ is dense in $\mathcal{B}(H_1) \otimes \cdots \otimes \mathcal{B}(H_n)$. By (6.7) and since $S_\varphi(\zeta)$ is linear in both φ and ζ, it suffices to show that $S_\varphi(\zeta)$ is compact when φ is an elementary tensor and ζ is a thoroughly elementary tensor. However, in this case $S_\varphi(\zeta)$ has rank at most 1, since for every $\xi_1 \in H_1$,

$$S_\varphi(\zeta)\xi_1 = \varphi_n \theta(\xi_{n-1,n}) \dots \varphi_2^{\mathrm{d}} \theta(\eta_{1,2}^{\mathrm{d}}) \varphi_1 \xi_1 = \Big(\prod_{j=1}^{n-1} (\varphi_j \xi_j, \eta_j) \Big) \varphi_n \xi_n. \qquad \square$$

We now establish some notation. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C*-algebras and $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. Assume that n is even and let $\pi_1, \dots, \pi_n$ be representations of $\mathcal{A}_1, \dots, \mathcal{A}_n$ on $H_1, \dots, H_n$, respectively. Set $\pi = \pi_1 \otimes \cdots \otimes \pi_n$. Using the natural identifications, we consider the map $S_{\pi(\varphi)} : \Gamma(H_1, \dots, H_n) \to H_1 \otimes H_n$ as a map (denoted in the same way)

$$S_{\pi(\varphi)} : \mathcal{C}_2(H_1^{\mathrm{d}}, H_2) \odot \cdots \odot \mathcal{C}_2(H_{n-1}^{\mathrm{d}}, H_n) \to \mathcal{C}_2(H_1^{\mathrm{d}}, H_n).$$

We let

$$\Phi_{\pi(\varphi)} : \mathcal{C}_2(H_{n-1}^{\mathrm{d}}, H_n) \odot \cdots \odot \mathcal{C}_2(H_1^{\mathrm{d}}, H_2) \to \mathcal{C}_2(H_1^{\mathrm{d}}, H_n)$$

be the map given on elementary tensors by

$$\Phi_{\pi(\varphi)}(T_{n-1} \otimes \cdots \otimes T_1) = S_{\pi(\varphi)}(T_1 \otimes \cdots \otimes T_{n-1}).$$

Note that if $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$ then $\Phi_{\pi(\varphi)}$ is bounded when the domain is equipped with the Haagerup norm and the range with the operator norm. In this case, $\Phi_{\pi(\varphi)}$ has a unique extension (which will be denoted in the same way)

$$\Phi_{\pi(\varphi)} : \big(\mathcal{K}(H_{n-1}^{\mathrm{d}}, H_n) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_1^{\mathrm{d}}, H_2), \|\cdot\|_{\mathrm{h}} \big) \to \big(\mathcal{K}(H_1^{\mathrm{d}}, H_n), \|\cdot\|_{\mathrm{op}} \big).$$

If n is odd then the map $\Phi_{\pi(\varphi)}$ is defined in a similar way. The map $\Phi_{\pi(\varphi)}$ will be used extensively hereafter.

The main result of this section is Theorem 99, where we explain how the complete boundedness of the mappings $\Phi_{\pi(\varphi)}$ relates to the property of φ being a multiplier. We will need the following lemma.

Lemma 98. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ be a C*-algebra, $i = 1, \ldots, n$ and let $k \in \mathbb{N}$. Let $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ and write $\psi = (\mathrm{id}^{(k)} \otimes \cdots \otimes \mathrm{id}^{(k)})(\varphi)$. Suppose that n is even. If $T_i \in M_k(\mathcal{C}_2(H_i^{\mathrm{d}}, H_{i+1}))$ for odd i and $T_i \in M_k(\mathcal{C}_2(H_i, H_{i+1}^{\mathrm{d}}))$ for even i then*

$$\Phi_\varphi^{(k)}(T_{n-1} \odot \cdots \odot T_1) = \Phi_\psi(T_{n-1} \otimes \cdots \otimes T_1),$$

where we identify the operator spaces $M_k(\mathcal{C}_2(H_i^{\mathrm{d}}, H_{i+1}))$ and $\mathcal{C}_2((H_i^{\mathrm{d}})^{(k)}, H_{i+1}^{(k)})$ for odd i, and $M_k(\mathcal{C}_2(H_i, H_{i+1}^{\mathrm{d}}))$ and $\mathcal{C}_2(H_i^{(k)}, (H_{i+1}^{\mathrm{d}})^{(k)})$ for even i. A similar statement holds for odd n.

Proof. To simplify notation, we give the proof for $n = 2$; the general proof is similar. If $\varphi = a_1 \otimes a_2$ is an elementary tensor then $\Phi_\varphi(T) = a_2 T a_1^{\mathrm{d}}$ for $T \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$ and it is easily checked that the statement holds. By linearity, it holds for each $\varphi \in \mathcal{A}_1 \odot \mathcal{A}_2$. Suppose now that $\varphi \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is arbitrary. Let $\{\varphi_m\} \subseteq \mathcal{A}_1 \odot \mathcal{A}_2$ be a sequence converging in the operator norm to φ and $\psi_m = (\mathrm{id}^{(k)} \otimes \mathrm{id}^{(k)})(\varphi_m)$. By (6.7), $\Phi_{\varphi_m}(T) \to \Phi_\varphi(T)$ in the operator norm, for all $T \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$. This implies that if $S \in M_k(\mathcal{C}_2(H_1^{\mathrm{d}}, H_2))$, then $\Phi_{\varphi_m}^{(k)}(S) \to \Phi_\varphi^{(k)}(S)$ in the operator norm of $M_k(\mathcal{C}_2(H_1^{\mathrm{d}}, H_2))$. Since $\psi_m \to \psi$ in the operator norm, we conclude that $\Phi_{\psi_m}(S) \to \Phi_\psi(S)$ in the operator norm of $\mathcal{C}_2((H_1^{\mathrm{d}})^{(k)}, H_2^{(k)})$. It follows that $\Phi_\psi(S) = \Phi_\varphi^{(k)}(S)$. □

Theorem 99. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C*-algebras and $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. The following are equivalent:*

(i) $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$;

(ii) if π_i is a representation of $\mathcal{A}_i$, $i = 1, \ldots, n$, and $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ then the map $\Phi_{\pi(\varphi)}$ is completely bounded;

(iii) there exist faithful representations π_i of $\mathcal{A}_i$, $i = 1, \ldots, n$, such that if $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ then the map $\Phi_{\pi(\varphi)}$ is completely bounded.

Moreover, if the above conditions hold and π is as in (iii) then $\|\varphi\|_{\mathrm{m}} = \|\Phi_{\pi(\varphi)}\|_{\mathrm{cb}}$.

Proof. For technical simplicity only consider the case $n = 3$.

(i)⇒(ii) Let $\varphi \in M(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and $\pi_i : \mathcal{A}_i \to \mathcal{B}(H_i)$ be a representation, $i = 1, 2, 3$. Then $\pi(\varphi) \in M(\pi_1(\mathcal{A}_1), \pi(\mathcal{A}_2), \pi_3(\mathcal{A}_3))$; thus, it suffices to assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ are concrete C*-algebras and that π_i is the identity representation, $i = 1, 2, 3$.

Fix $k \in \mathbb{N}$ and let $\psi = (\mathrm{id}^{(k)} \otimes \mathrm{id}^{(k)} \otimes \mathrm{id}^{(k)})(\varphi)$. Since $\varphi \in M(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, the map

$$\Phi_\psi : \mathcal{K}(H_2^{\mathrm{d}(k)}, H_3^{(k)}) \odot \mathcal{K}(H_1^{(k)}, H_2^{\mathrm{d}(k)}) \to \mathcal{K}(H_1^{(k)}, H_3^{(k)})$$

is bounded with norm not exceeding $\|\varphi\|_{\mathrm{m}}$. By Lemma 98, $\|\Phi_\varphi^{(k)}\| \leq \|\varphi\|_{\mathrm{m}}$. Since this inequality holds for every $k \in \mathbb{N}$, the map Φ_φ is completely bounded.

(ii)⇒(iii) is trivial.

(iii)⇒(i) We may assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ and that π_i is the identity representation, $i = 1, 2, 3$. Let λ be a cardinal number, $\rho_i = \mathrm{id}^{(\lambda)}$ be the ampliation of the identity representation of multiplicity λ, $\psi = (\rho_1 \otimes \rho_2 \otimes \rho_3)(\varphi)$, and $\tilde{H}_i = H_i^\lambda$, $i = 1, 2, 3$. Fix $\varepsilon > 0$ and $\zeta \in \Gamma(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$. Let

$$\tilde{T} = \tilde{T}_2 \odot \tilde{T}_1 \in \mathcal{C}_2(\tilde{H}_2^{\mathrm{d}}, \tilde{H}_3) \odot \mathcal{C}_2(\tilde{H}_1, \tilde{H}_2^{\mathrm{d}})$$

be the element canonically corresponding to ζ. Then there exist $k \in \mathbb{N}$ and canonical projections P_i from $\tilde{H}_i$ onto the direct sum of k copies of H_i such that if $T_0 = (P_3\tilde{T}_2(P_2^{\mathrm{d}} \otimes I)) \odot ((P_2^{\mathrm{d}} \otimes I)\tilde{T}_1 P_1)$ and if ζ_0 is the element of $\Gamma(H_1^{(k)}, H_2^{(k)}, H_3^{(k)})$ corresponding to T_0 then $\|\zeta - \zeta_0\|_{2,\wedge} \leq \varepsilon$.

Set $\psi_0 = (\mathrm{id}^{(k)} \otimes \mathrm{id}^{(k)} \otimes \mathrm{id}^{(k)})(\varphi)$. Arguing as in Lemma 98, we see that $\|\Phi_{\psi_0}(T_0)\|_{\mathrm{op}} = \|\Phi_\psi(T_0)\|_{\mathrm{op}}$. Using (6.7) and Lemma 98 we obtain

$$\begin{aligned}
\|S_\psi(\zeta)\|_{\mathrm{op}} &\leq \|S_\psi(\zeta - \zeta_0)\|_{\mathrm{op}} + \|S_\psi(\zeta_0)\|_{\mathrm{op}} \leq \|S_\psi(\zeta - \zeta_0)\|_{\mathrm{op}} + \|\Phi_\psi(T_0)\|_{\mathrm{op}} \\
&\leq \|\psi\|\|\zeta - \zeta_0\|_{2,\wedge} + \|\Phi_{\psi_0}(T_0)\|_{\mathrm{op}} \leq \varepsilon\|\varphi\| + \|\Phi_\varphi^{(k)}(T_0)\|_{\mathrm{op}} \\
&\leq \varepsilon\|\varphi\| + \|\Phi_\varphi\|_{\mathrm{cb}}\|T_0\|_{\mathrm{h}} \\
&\leq \varepsilon\|\varphi\| + \|\Phi_\varphi\|_{\mathrm{cb}}\|P_3\tilde{T}_2(P_2^{\mathrm{d}} \otimes I)\|_{\mathrm{op}}\|(P_2^{\mathrm{d}} \otimes I)\tilde{T}_1 P_1\|_{\mathrm{op}} \\
&\leq \varepsilon\|\varphi\| + \|\Phi_\varphi\|_{\mathrm{cb}}\|\tilde{T}_2\|_{\mathrm{op}}\|\tilde{T}_1\|_{\mathrm{op}}.
\end{aligned}$$

It follows that $\|\varphi\|_{\mathrm{id}^{(\lambda)},\mathrm{id}^{(\lambda)},\mathrm{id}^{(\lambda)}} \leq \|\Phi_\varphi\|_{\mathrm{cb}}$.

Now let ρ_1, ρ_2, ρ_3 be arbitrary representations of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, respectively. Then there exists a cardinal number λ such that each of the representations ρ_i is approximately subordinate to the representation $\mathrm{id}^{(\lambda)}$ (see [95] for the definition of approximate subordination and [38, Theorem 5.1]). By Theorem 5.1 of [48], $\|\varphi\|_{\rho_1,\rho_2,\rho_3} \leq \|\varphi\|_{\mathrm{id}^{(\lambda)},\mathrm{id}^{(\lambda)},\mathrm{id}^{(\lambda)}}$; now the previous paragraph implies that $\|\varphi\|_{\rho_1,\rho_2,\rho_3} \leq \|\Phi_\varphi\|_{\mathrm{cb}}$. It follows that $\varphi \in M(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and $\|\varphi\|_{\mathrm{m}} \leq \|\Phi_\varphi\|_{\mathrm{cb}}$. As the reversed inequality was already established, we conclude that $\|\varphi\|_{\mathrm{m}} = \|\Phi_\varphi\|_{\mathrm{cb}}$. □

6.4 The symbol of a universal multiplier

Our aim in this section is to generalise the natural correspondence between a function $\varphi \in \ell^\infty \otimes_{\rm eh} \ell^\infty$ and the Schur multiplier S_φ on $\mathcal{B}(\ell^2(\mathbb{N}))$ given by $S_\varphi((a_{ij})) = (\varphi(i,j)a_{ij})$. To each universal operator multiplier we will associate an element of an extended Haagerup tensor product which we call its symbol. This will be used in the subsequent sections to identify certain classes of operator multipliers.

Recall that if $\mathcal{A}$ is a C*-algebra, its opposite C*-algebra $\mathcal{A}^o$ is defined to be the C*-algebra whose underlying set, norm, involution and linear structure coincide with those of $\mathcal{A}$ and whose multiplication $\cdot$ is given by $a \cdot b = ba$. If $a \in \mathcal{A}$ we denote by a^o the element of $\mathcal{A}^o$ corresponding to a. If $\pi : \mathcal{A} \to \mathcal{B}(H)$ is a representation of $\mathcal{A}$ then the map $\pi^{\rm d} : a^o \to \pi(a)^{\rm d}$ from $\mathcal{A}^o$ into $\mathcal{B}(H^{\rm d})$ is a representation of $\mathcal{A}^o$. Clearly, π is faithful if and only if $\pi^{\rm d}$ is faithful. If $\pi_i : \mathcal{A}_i \to \mathcal{B}(H_i)$ are faithful representations, $i = 1, \dots, n$ (n even), then by [27, Lemma 5.4] there exists a complete isometry $\pi_n \otimes_{\rm eh} \pi^{\rm d}_{n-1} \otimes_{\rm eh} \cdots \otimes_{\rm eh} \pi^{\rm d}_1$ from $\mathcal{A}_n \otimes_{\rm eh} \mathcal{A}^o_{n-1} \otimes_{\rm eh} \cdots \otimes_{\rm eh} \mathcal{A}^o_1$ into $\mathcal{B}(H_n) \otimes_{\rm eh} \mathcal{B}(H^{\rm d}_{n-1}) \otimes_{\rm eh} \cdots \otimes_{\rm eh} \mathcal{B}(H^{\rm d}_1)$ which sends $a_n \otimes a^o_{n-1} \otimes \cdots \otimes a^o_1$ to $\pi_n(a_n) \otimes \pi_{n-1}(a_{n-1})^{\rm d} \otimes \cdots \otimes \pi_1(a_1)^{\rm d}$.

Henceforth, we will consistently write $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ and

$$\pi' = \begin{cases} \pi_n \otimes_{\rm eh} \pi^{\rm d}_{n-1} \otimes_{\rm eh} \cdots \otimes_{\rm eh} \pi_2 \otimes_{\rm h} \pi^{\rm d}_1 & \text{if } n \text{ is even,} \\ \pi_n \otimes_{\rm eh} \pi^{\rm d}_{n-1} \otimes_{\rm eh} \cdots \otimes_{\rm eh} \pi^{\rm d}_2 \otimes_{\rm h} \pi_1 & \text{if } n \text{ is odd.} \end{cases}$$

Let $n \in \mathbb{N}$, $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C*-algebras, π_i be a representation of $\mathcal{A}_i$, $i = 1, \dots, n$, and $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$. Assume that n is even. By Theorem 99, the map

$$\Phi_{\pi(\varphi)} : \mathcal{K}(H^{\rm d}_{n-1}, H_n) \otimes_{\rm h} \cdots \otimes_{\rm h} \mathcal{K}(H^{\rm d}_1, H_2) \to \mathcal{K}(H^{\rm d}_1, H_n)$$

is completely bounded. By Proposition 91, there exists a unique element $u^\pi_\varphi \in \mathcal{B}(H_n) \otimes_{\rm eh} \mathcal{B}(H^{\rm d}_{n-1}) \otimes_{\rm eh} \cdots \otimes_{\rm eh} \mathcal{B}(H^{\rm d}_1)$ such that $\gamma_0(u^\pi_\varphi) = \Phi_{\pi(\varphi)}$. For example, if each $\mathcal{A}_i$ is a concrete C*-algebra and $a_i \in \mathcal{A}_i$, $i = 1, \dots, n$, then

$$u^{\rm id}_{a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n} = a_n \otimes a^{\rm d}_{n-1} \otimes \cdots \otimes a_2 \otimes a^{\rm d}_1.$$

If n is odd then we define u^π_φ similarly.

The main result of this section is the following.

Theorem 100. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C*-algebras and $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$. There exists a unique element*

$$u_\varphi \in \begin{cases} \mathcal{A}_n \otimes_{\rm eh} \mathcal{A}^o_{n-1} \otimes_{\rm eh} \cdots \otimes_{\rm eh} \mathcal{A}_2 \otimes_{\rm eh} \mathcal{A}^o_1 & \text{if } n \text{ is even,} \\ \mathcal{A}_n \otimes_{\rm eh} \mathcal{A}^o_{n-1} \otimes_{\rm eh} \cdots \otimes_{\rm eh} \mathcal{A}^o_2 \otimes_{\rm eh} \mathcal{A}_1 & \text{if } n \text{ is odd} \end{cases}$$

with the property that if π_i is a representation of $\mathcal{A}_i$ for $i = 1, \dots, n$ then

$$u_\varphi^\pi = \pi'(u_\varphi). \tag{6.8}$$

The map $\varphi \mapsto u_\varphi$ is linear and if $a_i \in \mathcal{A}_i$, $i = 1, \dots, n$ then

$$u_{a_1 \otimes \cdots \otimes a_n} = \begin{cases} a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_2 \otimes a_1^o & \text{if } n \text{ is even,} \\ a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_2^o \otimes a_1 & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, $\|\varphi\|_{\mathrm{m}} = \|u_\varphi\|_{\mathrm{eh}}$.

Definition 101. *The element u_φ defined in Theorem 100 will be called the symbol of the universal multiplier φ.*

In order to prove Theorem 100 we have to establish a number of auxiliary results.

If $\omega \in \mathcal{B}(H)^*$ we let $\tilde{\omega} \in \mathcal{B}(H^{\mathrm{d}})^*$ be the functional given by $\tilde{\omega}(a^{\mathrm{d}}) = \omega(a)$. Note that if $\omega = \omega_{\xi,\eta}$ is the vector functional $a \mapsto (a\xi, \eta)$ then $\tilde{\omega} = \omega_{\eta^{\mathrm{d}},\xi^{\mathrm{d}}}$.

Lemma 102. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ be a C^*-algebra, $\xi_i, \eta_i \in H_i$ and $\omega_i = \omega_{\xi_i,\eta_i}$, $i = 1, \dots, n$. Suppose that $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$. Then*

$$(\varphi(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n) = \begin{cases} \langle u_\varphi^{\mathrm{id}}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \rangle & n \text{ even,} \\ \langle u_\varphi^{\mathrm{id}}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \omega_1 \rangle & n \text{ odd.} \end{cases} \tag{6.9}$$

Proof. We only consider the case n is even; the proof for odd n is similar. Suppose that φ is an elementary tensor, say $\varphi = a_1 \otimes \cdots \otimes a_n$. Then $u_\varphi^{\mathrm{id}} = a_n \otimes a_{n-1}^{\mathrm{d}} \otimes \cdots \otimes a_1^{\mathrm{d}}$ and thus

$$(\varphi(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n) = \prod_{i=1}^n (a_i\xi_i, \eta_i) = \langle u_\varphi^{\mathrm{id}}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \rangle.$$

By linearity, (6.9) holds for each $\varphi \in \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$.

Now let φ be an arbitrary element of $M(\mathcal{A}_1, \dots, \mathcal{A}_n)$. By Theorem 88, there exists a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$ and representations $u_\varphi^{\mathrm{id}} = A_n \odot \cdots \odot A_1$ and $u_{\varphi_\nu}^{\mathrm{id}} = A_n^\nu \odot \cdots \odot A_1^\nu$, where A_i^ν are finite matrices with entries in $\mathcal{A}_i$ if i is even and in $\mathcal{A}_i^{\mathrm{d}}$ if i is odd, such that $\varphi_\nu \to \varphi$ semi-weakly, $A_i^\nu \to A_i$ strongly and all norms $\|A_i\|, \|A_i^\nu\|$ are bounded by a constant depending only on n. As in (6.2), we have

$$\langle u_\varphi^{\mathrm{id}}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \rangle = \langle A_n, \omega_n \rangle \langle A_{n-1}, \tilde{\omega}_{n-1} \rangle \dots \langle A_1, \tilde{\omega}_1 \rangle. \tag{6.10}$$

Moreover, all norms $\|\langle A_i^\nu, \omega_i\rangle\|$ (for even i) and $\|\langle A_i^\nu, \tilde{\omega}_i\rangle\|$ (for odd i) are bounded by a constant depending only on n, and the strong convergence of A_i^ν to A_i implies that $\langle A_i^\nu, \omega_i\rangle$ converges strongly to $\langle A_i, \omega_i\rangle$. Indeed, it is easy to check that if $\xi, \eta \in H$, $A \in M_I(\mathcal{B}(H)) = \mathcal{B}(H \otimes \ell_2(I))$ and $\zeta \in \ell_2(I)$ for some index set I then

$$\|\langle A, \omega_{\xi,\eta}\rangle\zeta\|^2 = (A(\xi \otimes \zeta), \eta \otimes \langle A, \omega_{\xi,\eta}\rangle\zeta)\,.$$

This implies that $\|\langle A_i - A_i^\nu, \omega_i\rangle\eta\| \le C\|(A_i - A_i^\nu)(\xi_i \otimes \eta)\|$ for some constant $C > 0$, and the strong convergence follows.

Since operator multiplication is jointly strongly continuous on bounded sets, it now follows from (6.10) that

$$\langle u^{\mathrm{id}_{\varphi_\nu}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1\rangle \to \langle u^{\mathrm{id}_\varphi, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1\rangle.}}$$

On the other hand, since $\varphi_\nu \to \varphi$ semi-weakly,

$$(\varphi_\nu(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n) \to (\varphi(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n).$$

The proof is complete. □

Lemma 103. *Let H_i be a Hilbert space and $\mathcal{E}_i \subseteq \mathcal{B}(H_i)$ be an operator space, $i = 1, \ldots, n$. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are closed subspaces of $\mathcal{E}_1$ and $\mathcal{E}_n$, respectively and let $u, v \in \mathcal{E}_1 \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{E}_n$. If*

$$R_\omega(u) \in \mathcal{X} \quad \textit{and} \quad L_{\omega'}(v) \in \mathcal{Y}$$

whenever $\omega = \omega_2 \otimes \cdots \otimes \omega_n$ and $\omega' = \omega_1' \otimes \cdots \otimes \omega_{n-1}'$ where every ω_i, $\omega_i' \in \mathcal{B}(H_i)_*$ is a vector functional, then

$$u \in \mathcal{X} \otimes_{\mathrm{eh}} \mathcal{E}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_n \quad \textit{and} \quad v \in \mathcal{E}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_{n-1} \otimes_{\mathrm{eh}} \mathcal{Y}.$$

Proof. Let $\mathcal{F}_i$ be the span of the vector functionals on $\mathcal{B}(H_i)$. By linearity, $R_\omega(u) \in \mathcal{X}$ for each $\omega \in \mathcal{F}_2 \odot \cdots \odot \mathcal{F}_n$. Now suppose that

$$\omega \in (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n))_* = \mathcal{C}_1(H_2) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{C}_n(H_n).$$

There exists a sequence $(\omega_m) \subseteq \mathcal{F}_2 \odot \cdots \odot \mathcal{F}_n$ such that $\omega_m \to \omega$ in norm. Hence

$$\|R_\omega(u) - R_{\omega_m}(u)\|_{\mathcal{B}(H_1)} \le \|\omega - \omega_m\|\,\|u\|_{\mathrm{eh}} \to 0,$$

whence $R_\omega(u) = \lim_m R_{\omega_m}(u) \in \mathcal{X}$. Spronk's formula (6.5) now implies that $u \in \mathcal{X} \otimes_{\mathrm{eh}} \mathcal{E}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_n$. The assertion concerning v has a similar proof. □

We will use slice maps defined on the minimal tensor product of several C*-algebras as follows. Assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ and $\omega_i \in \mathcal{B}(H_i)^*$, $i = 1, \dots, n$, and let $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. If $1 \le i_1 < \cdots < i_k \le n$ and $\{\ell_1 < \ell_2 < \cdots < \ell_{n-k}\}$ is the complement of $\{i_1, \dots, i_k\}$ in $\{1, \dots, n\}$, let

$$\Lambda_{\omega_{i_1}, \dots, \omega_{i_k}} : \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to \mathcal{A}_{\ell_1} \otimes \cdots \otimes \mathcal{A}_{\ell_{n-k}}$$

be the unique norm continuous linear mapping given on elementary tensors by

$$\Lambda_{\omega_{i_1}, \dots, \omega_{i_k}}(a_1 \otimes \cdots \otimes a_n) = \omega_{i_1}(a_{i_1}) \dots \omega_{i_k}(a_{i_k})\, a_{\ell_1} \otimes \cdots \otimes a_{\ell_{n-k}}.$$

Proposition 104. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \dots, n$, be C*-algebras and let $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$. Then*

$$u_\varphi^{\mathrm{id}} \in \begin{cases} \mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2 \otimes_{\mathrm{eh}} \mathcal{A}_1^{\mathrm{d}} & \text{if } n \text{ is even,} \\ \mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2^{\mathrm{d}} \otimes_{\mathrm{eh}} \mathcal{A}_1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We only consider the case $n = 3$. Let $u = u_\varphi^{\mathrm{id}}$; by definition, $u \in \mathcal{B}(H_3) \otimes_{\mathrm{eh}} \mathcal{B}(H_2^{\mathrm{d}}) \otimes_{\mathrm{eh}} \mathcal{B}(H_1)$. Let $\xi_i, \eta_i \in H_i$ and $\omega_i = \omega_{\xi_i, \eta_i}$, $i = 1, 2, 3$. Then by (6.4) and Lemma 102,

$$\begin{aligned}(R_{\tilde\omega_2 \otimes \omega_1}(u)\xi_3, \eta_3) &= \langle R_{\tilde\omega_2 \otimes \omega_1}(u), \omega_3 \rangle = \langle u, \omega_3 \otimes \tilde\omega_2 \otimes \omega_1 \rangle \\ &= (\varphi(\xi_1 \otimes \xi_2 \otimes \xi_3), \eta_1 \otimes \eta_2 \otimes \eta_3) = (\Lambda_{\omega_1, \omega_2}(\varphi)\xi_3, \eta_3).\end{aligned}$$

Thus

$$R_{\tilde\omega_2 \otimes \tilde\omega_1}(u) = \Lambda_{\omega_1, \omega_2}(\varphi) \in \mathcal{A}_3.$$

Lemma 103 now implies that $u \in \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{B}(H_2^{\mathrm{d}}) \otimes_{\mathrm{eh}} \mathcal{B}(H_1)$.

Let $w = R_{\omega_1}(u)$. By the previous paragraph, $w \in \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{B}(H_2^{\mathrm{d}})$. By (6.4) and Lemma 102,

$$\begin{aligned}(L_{\omega_3}(w)\eta_2^{\mathrm{d}}, \xi_2^{\mathrm{d}}) &= \langle L_{\omega_3}(w), \tilde\omega_2 \rangle = \langle R_{\omega_1}(u), \omega_3 \otimes \tilde\omega_2 \rangle \\ &= \langle u, \omega_3 \otimes \tilde\omega_2 \otimes \omega_1 \rangle = (\Lambda_{\omega_1, \omega_3}(\varphi)\xi_2, \eta_2) = (\Lambda_{\omega_1, \omega_3}(\varphi)^{\mathrm{d}}\eta_2^{\mathrm{d}}, \xi_2^{\mathrm{d}}).\end{aligned}$$

Hence $L_{\omega_3}(w) = \Lambda_{\omega_1, \omega_3}(\varphi)^{\mathrm{d}} \in \mathcal{A}_2^{\mathrm{d}}$ and, by Lemma 103, $w \in \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{A}_2^{\mathrm{d}}$. Applying this lemma again shows that $u \in \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{A}_2^{\mathrm{d}} \otimes_{\mathrm{eh}} \mathcal{B}(H_1)$. Continuing in this fashion we see that $u \in \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{A}_2^{\mathrm{d}} \otimes_{\mathrm{eh}} \mathcal{A}_1$. □

Lemma 105. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C*-algebras and let*

$$\rho_i : \mathcal{A}_i \to \mathcal{B}(K_i), \qquad \theta_i : \rho_i(\mathcal{A}_i) \to \mathcal{B}(H_i)$$

be representations, $i = 1, \dots, n$. Suppose that *(i) for any cardinal number κ, the representations $\theta_i^{(\kappa)} : \rho_i(\mathcal{A}_i) \to \mathcal{B}(H_i^\kappa)$ are strongly continuous, and*

(ii) whenever $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and $\{\varphi_\nu\}$ is a net in $\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$ such that $\rho(\varphi_\nu) \to \rho(\varphi)$ semi-weakly and $\sup_\nu \|\varphi_\nu\|_m < \infty$ then $\Phi_{\theta\circ\rho(\varphi_\nu)} \to \Phi_{\theta\circ\rho(\varphi)}$ pointwise weakly.

Then $u_\varphi^{\theta\circ\rho} = \theta'(u_\varphi^\rho)$, for each $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

Proof. We suppose that n is even, the proof for odd n being similar. If $\varphi = a_1 \otimes \cdots \otimes a_n$ is an elementary tensor, then $u_\varphi^\rho = \rho'(a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_1^o)$, so

$$u_\varphi^{\theta\circ\rho} = (\theta \circ \rho)'(a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_1^o) = \theta'(u_\varphi^\rho).$$

By linearity, the claim also holds for $\varphi \in \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$.

If $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is arbitrary then $\rho(\varphi) \in M(\rho(\mathcal{A}_1), \ldots, \rho(\mathcal{A}_n))$ and by Theorem 88 and Proposition 104, there exist a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$ such that $\rho(\varphi_\nu) \to \rho(\varphi)$ semi-weakly, a representation $u_\varphi^\rho = A_n \odot \cdots \odot A_1$, where $A_i \in M_\kappa(\rho_i(\mathcal{A}_i)) \subseteq \mathcal{B}(K_i^\kappa)$ if i is even and $A_i \in M_\kappa(\rho_i^{\mathrm{d}}(\mathcal{A}_i^o)) \subseteq \mathcal{B}(K_i^\kappa)^{\mathrm{d}}$ if i is odd (κ being a suitable index set), whose operator matrix entries belong to $\rho_i(\mathcal{A}_i)$ if i is even and to $\rho_i^{\mathrm{d}}(\mathcal{A}_i^o)$ if i is odd, and representations $u_{\varphi_\nu}^\rho = A_n^\nu \odot \cdots \odot A_1^\nu$ where the A_i^ν are finite matrices with operator entries in $\rho_i(\mathcal{A}_i)$ if i is even and $\rho_i^{\mathrm{d}}(\mathcal{A}_i^o)$ if i is odd such that $A_i^\nu \to A_i$ strongly and all norms $\|A_i^\nu\|$, $\|A_i\|$ are bounded.

Now $\theta'(u_\varphi^\rho) = \tilde{A}_n \odot \cdots \odot \tilde{A}_1$ and $\theta'(u_{\varphi_\nu}^\rho) = \tilde{A}_n^\nu \odot \cdots \odot \tilde{A}_1^\nu$ where $\tilde{A}_i$ and $\tilde{A}_i^\nu$ are the images of A_i and A_i^ν under $\theta_i^{(\kappa)}$ or $(\theta_i^{\mathrm{d}})^{(\kappa)}$ according to whether i is even or odd. By assumption (i),

$$\gamma_0(\theta'(u_{\varphi_\nu}^\rho))(T_{n-1} \otimes \cdots \otimes T_1) \to \gamma_0(\theta'(u_\varphi^\rho))(T_{n-1} \otimes \cdots \otimes T_1) \qquad (6.11)$$

weakly for all $T_{n-1} \in \mathcal{C}_2(H_{n-1}^{\mathrm{d}}, H_n)$,. . . ,$T_1 \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$. On the other hand, assumption (ii) and the first paragraph of the proof show that

$$\gamma_0(\theta'(u_{\varphi_\nu}^\rho)) = \gamma_0(u_{\varphi_\nu}^{\theta\circ\rho}) = \Phi_{\theta\circ\rho(\varphi_\nu)} \to \Phi_{\theta\circ\rho(\varphi)} = \gamma_0(u_\varphi^{\theta\circ\rho})$$

pointwise weakly. Using (6.11) we conclude that $\gamma_0(u_\varphi^{\theta\circ\rho}) = \gamma_0(\theta'(u_\varphi^\rho))$; since γ_0 is injective we have that $u_\varphi^{\theta\circ\rho} = \theta'(u_\varphi^\rho)$. ☐

Proof of Theorem 100. We will only consider the case n is even. Let $\rho_i : \mathcal{A}_i \to \mathcal{B}(K_i)$ be the universal representation of $\mathcal{A}_i$, $i = 1, \ldots, n$. Set $\rho = \rho_1 \otimes \cdots \otimes \rho_n$ and $\rho' = \rho_n \otimes \rho_{n-1}^{\mathrm{d}} \otimes \cdots \otimes \rho_1^{\mathrm{d}}$. By Proposition 104, u_φ^ρ lies in the image of ρ'; we define $u_\varphi = (\rho')^{-1}(u_\varphi^\rho)$.

Let κ be a nonzero cardinal number and let $\sigma_i = \rho_i^{(\kappa)}$. If $\theta_i = \mathrm{id}_{\rho_i(\mathcal{A}_i)}^{(\kappa)} = \sigma_i \circ \rho_i^{-1}$ then it follows from the proof of Proposition 6.2 of [48] that the hypotheses of Lemma 105 are satisfied, so

$$u_\varphi^\sigma = u_\varphi^{\theta\circ\rho} = \theta'(u_\varphi^\rho) = (\theta'\circ\rho')(u_\varphi) = \sigma'(u_\varphi).$$

Now let π_i be an arbitrary representation of $\mathcal{A}_i$. It is well known (see e.g. [92]) that π_i is unitarily equivalent to a subrepresentation of $\sigma_i = \rho_i^{(\kappa)}$ for some κ. Hence there exist unitary operators v_i, $i = 1,\dots,n$ (acting between appropriate Hilbert spaces) and subspaces H_i of K_i^κ, such that if $\tau_i(x) = v_i x v_i^*|_{H_i}$ then $\pi_i = \tau_i \circ \sigma_i$. Examining the proof of Proposition 6.2 of [48], we see that τ satisfies the hypotheses of Lemma 105, so

$$u_\varphi^\pi = u_\varphi^{\tau\circ\sigma} = \tau'(u_\varphi^\sigma) = (\tau\circ\sigma)'(u_\varphi) = \pi'(u_\varphi).$$

The uniqueness of u_φ follows from the injectivity of γ_0. The linearity of the map $\varphi \mapsto u_\varphi$ and its values on elementary tensors are straightforward. The fact that $\|\varphi\|_{\mathrm{m}} = \|u_\varphi\|_{\mathrm{eh}}$ follows from Proposition 91 and Theorem 99. □

Remarks. *(i) Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1,\dots,n$ be concrete C^*-algebras of operators. Taking π_i to be the identity representation for $i = 1,\dots,n$ and writing $\mathrm{id} = \pi_1 \otimes \cdots \otimes \pi_n$ gives $u_\varphi = u^{\mathrm{id}_\varphi}$ if we identify $\mathcal{A}_i^o$ with $\mathcal{A}_i^{\mathrm{d}}$.*
(ii) Theorem 100 implies that if $\mathcal{A}_i$, $i = 1,\dots,n$, are concrete C^*-algebras then the entries of the block operator matrices A_i appearing in the representation of φ in Theorem 88 can be chosen from $\mathcal{A}_i$, $i = 1,\dots,n$.

6.5 Completely compact multipliers

In this section we introduce the class of completely compact multipliers and characterise them within the class of all universal multipliers using the notion of the symbol introduced in Section 6.4. We will need the following lemma.

Lemma 106. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ be a C^*-algebra, $i = 1,\dots,n$, $a \in \mathcal{A}_1$, $b \in \mathcal{A}_n$ and $\varphi \in M(\mathcal{A}_1,\dots,\mathcal{A}_n)$. Let $\psi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ be given by*

$$\psi = \begin{cases} (a \otimes I \otimes \cdots \otimes I \otimes b)\varphi & \text{if } n \text{ is even,} \\ (I \otimes \cdots \otimes I \otimes b)\varphi(a \otimes I \otimes \cdots \otimes I \otimes I) & \text{if } n \text{ is odd.} \end{cases}$$

Then $\psi \in M(\mathcal{A}_1,\dots,\mathcal{A}_n)$ and

$$\Phi_\psi(x) = \begin{cases} b\Phi_\varphi(x)a^{\mathrm{d}} & \text{if } n \text{ is even,} \\ b\Phi_\varphi(x)a & \text{if } n \text{ is odd.} \end{cases} \tag{6.12}$$

Proof. For technical simplicity, we will only consider the case $n = 2$. Let $a_i \in \mathcal{A}_i$, $i = 1, 2$, and $\varphi = a_1 \otimes a_2$. In this case $\psi = (aa_1) \otimes (ba_2)$ so

$$\Phi_\psi(T) = ba_2T(aa_1)^{\mathrm{d}} = ba_2Ta_1^{\mathrm{d}}a^{\mathrm{d}} = b\Phi_\varphi(T)a^{\mathrm{d}}.$$

By linearity, (6.12) holds whenever $\varphi \in \mathcal{A}_1 \odot \mathcal{A}_2$.

Assume that $\varphi \in M(\mathcal{A}_1, \mathcal{A}_2)$ is arbitrary. Fix an operator $T \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$. By Theorem 88, there exists a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \mathcal{A}_2$ such that $\varphi_\nu \to \varphi$ semi-weakly, $\sup_\nu \|\varphi_\nu\|_{\mathrm{m}} < \infty$ and $\Phi_{\varphi_\nu}(T) \to \Phi_\varphi(T)$ weakly.

Let $\psi_\nu = (a \otimes b)\varphi_\nu$; then $\psi_\nu \to \psi$ semi-weakly. Clearly, $\psi_\nu \in \mathcal{A}_1 \odot \mathcal{A}_2$; in particular $\psi_\nu \in M(\mathcal{A}_1, \mathcal{A}_2)$. By the previous paragraph, $\Phi_{\psi_\nu}(\cdot) = b\Phi_{\varphi_\nu}(\cdot)a^{\mathrm{d}}$ and hence $\Phi_{\psi_\nu}(T) \to b\Phi_\varphi(T)a^{\mathrm{d}}$ weakly. If $\varphi_\nu = B_1^\nu \odot B_2^\nu$ then $\psi_\nu = (aB_1^\nu) \odot ((b \otimes I)B_2^\nu)$. It follows from Theorem 88 that $\psi \in M(\mathcal{A}_1, \mathcal{A}_2)$ and that $\Phi_{\psi_\nu}(T) \to \Phi_\psi(T)$ weakly. Thus $\Phi_\psi(T) = b\Phi_\varphi(T)a^{\mathrm{d}}$. □

Given faithful representations $\pi_1, \dots, \pi_n$ of the C*-algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$, respectively, we define

$$M_{cc}^{\pi}(\mathcal{A}_1, \dots, \mathcal{A}_n) = \{\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n) : \Phi_{\pi(\varphi)} \text{ is completely compact}\}$$

$$M_{ff}^{\pi}(\mathcal{A}_1, \dots, \mathcal{A}_n) = \{\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n) : \text{the range of } \Phi_{\pi(\varphi)} \text{ is a finite dimensional space of finite-rank operators}\}.$$

Theorem 107. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ be a C*-algebra, $i = 1, \dots, n$, and $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$. The following are equivalent:*

(i) $\varphi \in M_{cc}^{\mathrm{id}(\mathcal{A}_1,\dots,\mathcal{A}_n)}$;

(ii)

$$u_\varphi^{\mathrm{id}} \in \begin{cases} (\mathcal{K}(H_n) \cap \mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2) \otimes_{\mathrm{h}} (\mathcal{K}(H_1^{\mathrm{d}}) \cap \mathcal{A}_1^{\mathrm{d}}) & n \text{ even,} \\ (\mathcal{K}(H_n) \cap \mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2^{\mathrm{d}}) \otimes_{\mathrm{h}} (\mathcal{K}(H_1) \cap \mathcal{A}_1) & n \text{ odd;} \end{cases}$$

(iii) there exists a net $\{\varphi_\alpha\} \subseteq M_{ff}^{\mathrm{id}(\mathcal{A}_1,\dots,\mathcal{A}_n)}$ such that $\|\varphi_\alpha - \varphi\|_{\mathrm{m}} \to 0$.

Proof. We will only consider the case n is even.

(i)⇒(ii) Theorem 92 implies that

$$u_\varphi^{\mathrm{id} \in \mathcal{K}(H_n) \otimes_{\mathrm{h}} (\mathcal{B}(H_{n-1}^{\mathrm{d}}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2)) \otimes_{\mathrm{h}} \mathcal{K}(H_1^{\mathrm{d}})}$$

while, by Proposition 104,

$$u_\varphi^{\mathrm{id} \in \mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2 \otimes_{\mathrm{eh}} \mathcal{A}_1^{\mathrm{d}}}.$$

The conclusion now follows from Lemma 87.

(ii)⇒(i) By Theorem 92, $\Phi_\varphi = \gamma_0(u_\varphi^{\mathrm{id}})$ is completely compact.

(ii)⇒(iii) Let $p \in \mathcal{B}(H_1)$ (resp. $q \in \mathcal{B}(H_n)$) be the projection onto the span of all ranges of operators in $\mathcal{K}(H_1) \cap \mathcal{A}_1$ (resp. $\mathcal{K}(H_n) \cap \mathcal{A}_n$), and let $\{p_\alpha\} \subseteq \mathcal{K}(H_1) \cap \mathcal{A}_1$ (resp. $\{q_\alpha\} \subseteq \mathcal{K}(H_n) \cap \mathcal{A}_n$) be a net of finite rank projections which tends strongly to p (resp. q). It is easy to see that $\Phi_\varphi(T_{n-1} \otimes \cdots \otimes T_1) = q\Phi_\varphi(T_{n-1} \otimes \cdots \otimes T_1)p^{\mathrm{d}}$, for all $T_1 \in \mathcal{K}(H_1^{\mathrm{d}}, H_2), \ldots, T_{n-1} \in \mathcal{K}(H_{n-1}^{\mathrm{d}}, H_n)$. Let $\varphi_\alpha = (p_\alpha \otimes I \otimes \cdots \otimes I \otimes q_\alpha)\varphi$. By Lemma 106, $\varphi_\alpha \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and $\Phi_{\varphi_\alpha}(\cdot) = q_\alpha \Phi_\varphi(\cdot)p_\alpha^{\mathrm{d}}$; hence $\varphi_\alpha \in M_{ff}^{\mathrm{id}(\mathcal{A}_1,\ldots,\mathcal{A}_n)}$. We have already seen that Φ_φ is completely compact, and it follows from the proof of Theorem 92 that $\Phi_{\varphi_\alpha} \to \Phi_\varphi$ in the cb norm. By Theorem 99, $\|\varphi - \varphi_\alpha\|_{\mathrm{m}} \to 0$.

(iii)⇒(i) is immediate from Proposition 90 and Theorem 99 and the fact that finite rank maps are completely compact. □

Now consider the sets

$$M_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \bigcup_\pi M_{cc}^\pi(\mathcal{A}_1, \ldots, \mathcal{A}_n)$$
$$M_{ff}(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \bigcup_\pi M_{ff}^\pi(\mathcal{A}_1, \ldots, \mathcal{A}_n)$$

where the unions are taken over all $\pi = \pi_1 \otimes \cdots \otimes \pi_n$, each π_i being a faithful representation of $\mathcal{A}_i$. We refer to the first of these as the set of completely compact multipliers.

Lemma 108. *If ρ_i is the reduced atomic representation of $\mathcal{A}_i$, $i = 1, \ldots, n$, and $\rho = \rho_1 \otimes \cdots \otimes \rho_n$ then $M_{ff}(\mathcal{A}_1, \ldots, \mathcal{A}_n) = M_{ff}^\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.*

Proof. Again, we give the proof for the even case only. We must show that $M_{ff}^\pi(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq M_{ff}^\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ whenever $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ where each π_i is a faithful representation of $\mathcal{A}_i$. Without loss of generality, we may assume that each π_i is the identity representation of $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$. Let $\varphi \in M_{ff}^\pi(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ so that the range of Φ_φ is finite dimensional and consists of finite rank operators. By Remark 93 (i) there exist finite rank projections p and q on H_1^{d} and H_n, respectively, such that u_φ^{id} lies in the intersection of

$$(q\mathcal{K}(H_n)) \otimes_{\mathrm{h}} (\mathcal{B}(H_{n-1}^{\mathrm{d}}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2)) \otimes_{\mathrm{h}} (\mathcal{K}(H_1^{\mathrm{d}})p)$$

and $\mathcal{A}_n \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_1^{\mathrm{d}}$. By Lemma 87, u_φ^{id} lies in

$$(q\mathcal{K}(H_n) \cap \mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{B}(H_{n-1}^{\mathrm{d}}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2)) \otimes_{\mathrm{h}} (\mathcal{K}(H_1^{\mathrm{d}})p \cap \mathcal{A}_1^{\mathrm{d}}).$$

Hence there exists a representation $u_\varphi^{\mathrm{id}} = A_n \odot \cdots \odot A_1$ of u_φ^{id} such that $A_n = qA_n$ and $A_1 = A_1 p$. Suppose that $A_n = [b_1, b_2, \ldots]$, where $b_j \in \mathcal{A}_n$ for each j, and let q_j be the orthogonal projection onto the range of b_j. Setting $Q_m = \bigvee_{j=1}^m q_j$ we see that $\{Q_m\}$ is an increasing sequence of projections in $\mathcal{A}_n$ dominated by q. It follows that $\bigvee_{m=1}^\infty Q_m \in \mathcal{A}_n$. We may thus assume that $q \in \mathcal{A}_n$. Similarly, we may assume that $p \in \mathcal{A}_1^{\mathrm{d}}$. Now

$$\rho'(u_\varphi) = (\rho_n(q)\rho_n(A_n)) \odot \cdots \odot (\rho_1(A_1)\rho_1(p)).$$

By [100], $\rho_n(q)$ and $\rho_1(p)$ have finite rank. By Lemma 106, $\varphi \in M_{ff}^\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. □

We are now ready to prove the main result of this section.

Theorem 109. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C*-algebras and $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. The following are equivalent:*

(i) $\varphi \in M_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$;

(ii) $u_\varphi \in \begin{cases} \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}_1^o) & \text{if } n \text{ is even,} \\ \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2^o) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}_1) & \text{if } n \text{ is odd;} \end{cases}$

(iii) there exists a net $\{\varphi_\alpha\} \subseteq M_{ff}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ *such that* $\|\varphi_\alpha - \varphi\|_{\mathrm{m}} \to 0$.

Proof. We will only consider the case n is even.

(i)⇒(ii) Choose $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ such that $\varphi \in M_{cc}^\pi(\mathcal{A}_1, \ldots, \mathcal{A}_n)$; after identifying $\mathcal{A}_i$ with its image under π_i, we may assume that each π_i is the identity representation of a concrete C*-algebra $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$. By Theorem 107, u_φ^{id} lies in

$$(\mathcal{K}(H_n) \cap \mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2) \otimes_{\mathrm{h}} (\mathcal{K}(H_1^{\mathrm{d}}) \cap \mathcal{A}_1^o).$$

The conclusion follows from the fact that $\mathcal{K}(H_i) \cap \mathcal{A}_i \subseteq \mathcal{K}(\mathcal{A}_i)$ for $i = 1, n$.

(ii)⇒(i) Let ρ_i be the reduced atomic representation $\mathcal{A}_i \to \mathcal{B}(H_i)$ for $i = 1, \ldots, n$. Since ρ' is an isometry, $u_\varphi^\rho = \rho'(u_\varphi)$ lies in

$$\rho_n(\mathcal{K}(\mathcal{A}_n)) \otimes_{\mathrm{h}} (\rho_{n-1}^{\mathrm{d}}(\mathcal{A}_{n-1}^o) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \rho_2(\mathcal{A}_2)) \otimes_{\mathrm{h}} \rho_1^{\mathrm{d}}(\mathcal{K}(\mathcal{A}_1^o)).$$

By Theorem 7.5 of [99], $\mathcal{K}(H_i) \cap \rho_i(\mathcal{A}_i) = \rho_i(\mathcal{K}(\mathcal{A}_i))$. By Theorem 107, $\varphi \in M_{cc}^\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

(i)⇒(iii) is immediate from Theorem 107.

(iii)⇒(i) Suppose that $\{\varphi_\alpha\} \subseteq M_{ff}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is a net such that $\|\varphi_\alpha - \varphi\|_{\mathrm{m}} \to 0$. By Lemma 108, $\{\varphi_\alpha\} \subseteq M_{ff}^\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, where ρ is the tensor product of the reduced atomic representations of $\mathcal{A}_1, \ldots, \mathcal{A}_n$. By Theorem 107, $\varphi \in M_{cc}^\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq M_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. □

In the next theorem we show that in the case $n = 2$ one more equivalent condition can be added to those of Theorem 109.

Theorem 110. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and $\varphi \in M(\mathcal{A},\mathcal{B})$. The following are equivalent:*

(i) $\varphi \in M_{cc}(\mathcal{A},\mathcal{B})$;

(ii) there is a sequence $\{\varphi_k\}_{k=1}^{\infty} \subseteq \mathcal{K}(\mathcal{A}) \odot \mathcal{K}(\mathcal{B})$ such that $\|\varphi_k - \varphi\|_{\mathrm{m}} \to 0$ as $k \to \infty$.

Proof. (i)⇒(ii) By Theorem 109, $u_\varphi \in \mathcal{K}(\mathcal{B}) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}^o)$; thus $u_\varphi = \sum_{i=1}^{\infty} b_i \otimes a_i^o$ where $a_i^o \in \mathcal{K}(\mathcal{A}^o)$, $b_i \in \mathcal{K}(\mathcal{B})$, $i \in \mathbb{N}$, and the series $\sum_{i=1}^{\infty} b_i b_i^*$ and $\sum_{i=1}^{\infty} a_i^{o*} a_i^o$ converge in norm. Let $\varphi_k = \sum_{i=1}^{k} a_i \otimes b_i \in \mathcal{A} \odot \mathcal{B}$. By Theorem 100, $u_{\varphi_k} = \sum_{i=1}^{k} b_i \otimes a_i^o$ and $\|\varphi - \varphi_k\|_{\mathrm{m}} = \|u_\varphi - u_{\varphi_k}\|_{\mathrm{eh}} \to 0$ as $k \to \infty$.

(ii)⇒(i) Assume that $\mathcal{A}$ and $\mathcal{B}$ are represented concretely. It is clear that $\varphi_k \in M_{cc}(\mathcal{A},\mathcal{B})$. By Theorem 99, $\|\Phi_{\mathrm{id}(\varphi)} - \Phi_{\mathrm{id}(\varphi_k)}\|_{\mathrm{cb}} = \|\varphi - \varphi_k\|_{\mathrm{m}}$. Proposition 90 now implies that $\Phi_{\mathrm{id}(\varphi)}$ is completely compact, in other words, $\varphi \in M_{cc}(\mathcal{A},\mathcal{B})$. □

6.6 Compact multipliers

In this section we compare the set of completely compact multipliers with that of compact multipliers. We exhibit sufficient conditions for these two sets of multipliers to coincide, and show that in general they are distinct. Finally, we address the question of when any universal multiplier in the minimal tensor product of two C*-algebras is automatically compact. We show that this happens precisely when one of the C*-algebras is finite dimensional while the other coincides with the set of its compact elements.

6.6.1 Automatic complete compactness

We will need the following result complementing Theorem 92. Notation is as in Section 6.2.

Proposition 111. *If $\Phi : \mathcal{K}_{\mathrm{h}} \to \mathcal{K}(H_n, H_1)$ is a compact completely bounded map then $\gamma_0^{-1}(\Phi) \in \mathcal{K}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1}) \otimes_{\mathrm{eh}} \mathcal{K}(H_n)$.*

Proof. Fix $\varepsilon > 0$. By compactness, there exist $y_1, \dots, y_\ell \in \mathcal{K}(H_n, H_1)$ such that $\min_{1\le i\le \ell} \|\Phi(x) - y_i\| < \varepsilon$ for each $x \in \mathcal{K}_{\mathrm{h}}$ with $\|x\| \le 1$.

Let $\{p_\alpha\}$ (resp. $\{q_\alpha\}$) be a net of finite rank projections in $\mathcal{K}(H_1)$ (resp. $\mathcal{K}(H_n)$) such that $p_\alpha \to I$ (resp. $q_\alpha \to I$) strongly and let $\Phi_\alpha : \mathcal{K}_{\mathrm{h}} \to \mathcal{K}(H_n, H_1)$ be the map given by $\Phi_\alpha(x) = p_\alpha \Phi(x) q_\alpha$. Let $u = \gamma_0^{-1}(\Phi)$ and $u_\alpha = \gamma_0^{-1}(\Phi_\alpha)$. Since each y_i is compact there exists α_0 such that $\|p_\alpha y_i q_\alpha - y_i\| < \varepsilon$ for

$i = 1, \ldots, \ell$ and $\alpha \geq \alpha_0$. Moreover, for any $x \in \mathcal{K}_{\mathrm{h}}$, $\|x\| \leq 1$ and $\alpha \geq \alpha_0$, we have

$$\begin{aligned}\|\Phi_\alpha(x) - \Phi(x)\| &\leq \min_{1\leq i\leq \ell}\{\|\Phi_\alpha(x) - p_\alpha y_i q_\alpha\| + \|p_\alpha y_i q_\alpha - y_i\| + \|y_i - \Phi(x)\|\} \\ &\leq \min_{1\leq i\leq \ell}\{2\|\Phi(x) - y_i\| + \|p_\alpha y_i q_\alpha - y_i\|\} \leq 3\varepsilon,\end{aligned}$$

so $\|\Phi_\alpha - \Phi\| \to 0$. Remark 93 (i) shows that $u_\alpha \in \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$; it follows that for every $\omega \in (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1}) \otimes_{\mathrm{eh}} \mathcal{B}(H_n))_*$ we have $R_\omega(u_\alpha) \in \mathcal{K}(H_1)$.

Suppose that $\xi_i, \eta_i \in H_i$ and let $\omega_i = \omega_{\xi_i,\eta_i}$ be the corresponding vector functional. Lemma 102 and a straightforward verification shows that if $v \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$ has a representation of the form $v = A_1 \odot \ldots \odot A_n$ and $\omega = \omega_2 \otimes \cdots \otimes \omega_n$ then

$$(R_\omega(v)\xi_1, \eta_1) = \langle v, \omega_1 \otimes \ldots \otimes \omega_n\rangle = (\gamma_0(v)(\zeta)\xi_n, \eta_1), \tag{6.13}$$

where

$$\zeta = \big((\eta_2^* \otimes \xi_1) \otimes (\eta_3^* \otimes \xi_2) \otimes \cdots \otimes (\eta_{n-1}^* \otimes \xi_{n-2}) \otimes (\eta_n^* \otimes \xi_{n-1})\big) \in \mathcal{K}_{\mathrm{h}}$$

is an elementary tensor whose components are rank one operators.

Since $\gamma_0(u_\alpha) \to \gamma_0(u)$ in norm, (6.13) implies that $R_\omega(u_\alpha) \to R_\omega(u)$ in the operator norm of $\mathcal{K}(H_1)$. Since $R_\omega(u_\alpha) \in \mathcal{K}(H_1)$, we obtain $R_\omega(u) \in \mathcal{K}(H_1)$. By Lemma 103, $u \in \mathcal{K}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$. Similarly we see that $u \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{K}(H_n)$; the conclusion now follows. □

Remark. *The converse of Proposition 111 does not hold, even for $n = 2$. Indeed, let $\{p_i\}_{i=1}^\infty$ be a family of pairwise orthogonal rank one projections on a Hilbert space H and let $u = \sum_{i=1}^\infty p_i \otimes p_i$. Then $u \in \mathcal{K}(H) \otimes_{\mathrm{eh}} \mathcal{K}(H)$ and the range of $\gamma_0(u)$ consists of compact operators, but $\gamma_0(u)(p_i) = p_i$ for each i, so $\gamma_0(u)$ is not compact.*

Given C*-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$, we let $M_c(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be the collection of all $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ for which there exist faithful representations $\pi_1, \ldots, \pi_n$ of $\mathcal{A}_1, \ldots, \mathcal{A}_n$, respectively, such that if $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ then the map $\Phi_{\pi(\varphi)}$ is compact. We call the elements of $M_c(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ *compact multipliers.*

As a consequence of the previous result we obtain the following fact.

Proposition 112. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C*-algebras and let $\varphi \in M_c(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Then*

$$u_\varphi \in \begin{cases} \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2 \otimes_{\mathrm{eh}} \mathcal{K}(\mathcal{A}_1^o) & \textit{if } n \textit{ is even,} \\ \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2^o \otimes_{\mathrm{eh}} \mathcal{K}(\mathcal{A}_1) & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. We only consider the case n is even. We may assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ is a concrete non-degenerate C*-algebra, $i = 1, \ldots, n$, and that Φ_φ is compact. By Propositions 104 and 111, u_φ^{id} belongs to

$$\big(\mathcal{K}(H_n) \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1}^{\mathrm{d}}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \otimes_{\mathrm{eh}} \mathcal{K}(H_1^{\mathrm{d}})\big) \cap \big(\mathcal{A}_n \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_1^{\mathrm{d}}\big).$$

Since $\mathcal{K}(H_n) \cap \mathcal{A}_n \subseteq \mathcal{K}(\mathcal{A}_n)$ and $\mathcal{K}(H_1^{\mathrm{d}}) \cap \mathcal{A}_1^{\mathrm{d}} \subseteq \mathcal{K}(\mathcal{A}_1^{\mathrm{d}})$, an application of (6.5) shows that $u_\varphi^{\mathrm{id}} \in \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2 \otimes_{\mathrm{eh}} \mathcal{K}(\mathcal{A}_1^{\mathrm{d}})$. □

If $\{\mathcal{A}_j\}_{j\in J}$ is a family of C*-algebras, we will denote by $\bigoplus_{j\in J}^{c_0} \mathcal{A}_j$ and $\bigoplus_{j\in J}^{\ell_\infty} \mathcal{A}_j$ their c_0- and ℓ_∞-direct sums, respectively.

Theorem 113. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C*-algebras, and suppose that $\mathcal{K}(\mathcal{A}_1)$ is isomorphic to $\bigoplus_{j\in J}^{c_0} M_{m_j}$ and $\mathcal{K}(\mathcal{A}_n)$ is isomorphic to $\bigoplus_{j\in J}^{c_0} M_{n_j}$ where J is some index set and $\sup_{j\in J} m_j$ and $\sup_{j\in J} n_j$ are finite. Then*

$$M_c(\mathcal{A}_1, \ldots, \mathcal{A}_n) = M_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n).$$

Proof. We give the proof for $n = 3$; the case of a general n is similar. Let $m = \sup\{m_j, n_j : j \in J\}$. By hypothesis, $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_3)$ may both be embedded in the C*-algebra $\mathcal{C} \stackrel{\mathrm{def}}{=} \bigoplus_{j\in J}^{c_0} M_m$ for some $m \in \mathbb{N}$; without loss of generality, we may assume that this embedding is an inclusion and that $\mathcal{A}_i$ is represented faithfully on some Hilbert space H_i such that H_1 and H_3 both contain the Hilbert space $H = \bigoplus_{j\in J} \mathbb{C}^m$. Given $\varphi \in M_c(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, Proposition 112 implies that the symbol u_φ of φ can be written in the form $u_\varphi = A_3 \odot A_2 \odot A_1$, where the entries of A_3 and A_1 belong to $\mathcal{C}$. Let us write $\{e_{ij} : i, j = 1, \ldots, m\}$ for the canonical matrix unit system of M_m and let $P_k = \bigoplus_{j\in J} e_{kk} \in \bigoplus_{j\in J}^{\ell_\infty} M_m$, $k = 1, \ldots, m$. For $k, \ell, s, t = 1, \ldots, m$, we set $A_3^{k,\ell} = P_k A_3 (P_\ell \otimes I)$ and $A_1^{s,t} = (P_s \otimes I) A_1 P_t$ and define

$$u_{k,\ell,s,t} = A_3^{k,\ell} \odot A_2 \odot A_1^{s,t} \quad \text{and} \quad \Phi_{k,\ell,s,t} = \gamma_0(u_{k,\ell,s,t}).$$

Then $\gamma_0(u_\varphi) = \Phi = \sum_{k,\ell,s,t} \Phi_{k,\ell,s,t}$ so it suffices to show that each of the maps $\Phi_{k,\ell,s,t}$ is completely compact. Now

$$\Phi_{k,\ell,s,t}(T_2 \otimes T_1) = P_k \Phi(P_\ell T_2 \otimes T_1 P_s) P_t = A_3^{k,\ell}((P_\ell T_2) \otimes I) A_2 ((T_1 P_s) \otimes I) A_1^{s,t}.$$

Thus, $\Phi_{k,\ell,s,t}$ can be considered as a completely bounded multilinear map from $\mathcal{K}(H_2^{\mathrm{d}}, P_\ell H) \times \mathcal{K}(P_s H, H_2^{\mathrm{d}})$ into $\mathcal{K}(P_t H, P_k H)$. Since Φ is compact, it follows that $\Phi_{k,\ell,s,t}$ is compact.

Take a basis $\{e_i^j : i = 1, \ldots, m, j \in J\}$ of $H = \bigoplus_{j\in J} \mathbb{C}^m$, where for each $j \in J$, the standard basis of the j-th copy of $\mathbb{C}^m$ is $\{e_i^j : i = 1, \ldots, m\}$. Let

$U_k : P_k H \to P_1 H$ be the unitary operator defined by $U_k e_k^j = e_1^j$. Consider the mapping $\Psi : \mathcal{K}(H_2^{\rm d}, P_1 H) \times \mathcal{K}(P_1 H, H_2^{\rm d}) \to \mathcal{K}(P_1 H, P_1 H)$ given by

$$\Psi(T_2 \otimes T_1) = U_k \Phi_{k,\ell,s,t}(U_\ell T_2 \otimes T_1 U_s) U_t.$$

To show that $\Phi_{k,\ell,s,t}$ is completely compact it suffices to show that Ψ is. Let $\mathcal{C}_0 = P_1 \mathcal{C} P_1$; then $\mathcal{C}_0$ is isomorphic to c_0 and its commutant $\mathcal{C}_0'$ has a cyclic vector. Moreover, Ψ is a $\mathcal{C}_0'$-modular multilinear map. Let $\{p_\alpha\}$ be a net of finite rank projections belonging to $\mathcal{C}_0$, such that $\text{s-lim}\, p_\alpha = I_{P_1 H}$. Consider the completely bounded multilinear maps $\Psi_\alpha(x) = p_\alpha \Psi(x) p_\alpha$. Since the range of each p_α's is finite dimensional, Ψ_α has finite rank, so is completely compact. Since Ψ is compact, we may argue as in the proof of Proposition 111 to show that $\|\Psi_\alpha - \Psi\| \to 0$. Now the maps Ψ and Ψ_α are $\mathcal{C}_0'$-modular and $\mathcal{C}_0'$ has a cyclic vector, so by the generalisation [48, Lemma 3.3] of a result of Smith [88, Theorem 2.1],

$$\|\Psi_\alpha - \Psi\|_{\rm cb} = \|\Psi_\alpha - \Psi\| \to 0.$$

Proposition 90 now implies that Ψ is completely compact. □

The following corollary extends Proposition 5 of [42] to the case of multidimensional Schur multipliers. Let $n \geq 2$ be an integer. We recall from [48] that with every $\varphi \in \ell_\infty(\mathbb{N}^n)$ we associate a mapping $S_\varphi : \ell_2(\mathbb{N}^2) \odot \cdots \odot \ell_2(\mathbb{N}^2) \to \ell_2(\mathbb{N}^2)$ which extends the usual Schur multiplication in the case $n = 2$. We equip the domain of S_φ with the Haagerup norm where each of the terms is given its operator space structure arising from its embedding into the corresponding space of Hilbert-Schmidt operators endowed with the operator norm.

Corollary 114. *Let $n \geq 2$ and let $\varphi \in \ell_\infty(\mathbb{N}^n)$. The following are equivalent:*
(i) S_φ is compact;
(ii) $\varphi \in c_0 \otimes_{\rm h} (\underbrace{\ell_\infty \otimes_{\rm eh} \cdots \otimes_{\rm eh} \ell_\infty}_{n-2}) \otimes_{\rm h} c_0$.

Proof. Assume first that S_φ is compact. It follows from [48, Section 3] that the map S_φ induces a completely bounded compact map

$$\hat{S}_\varphi : \mathcal{C}_2 \times \ldots \times \mathcal{C}_2 \to \mathcal{C}_2$$

defined by $\hat{S}_\varphi(T_{f_1}, \ldots, T_{f_n}) = T_{S_\varphi(f_1,\ldots,f_n)}$, where T_f is the Hilbert-Schmidt operator with kernel f. By Proposition 111, $\varphi = \gamma_0^{-1}(\hat{S}_\varphi) \in \mathcal{K}(\ell_2) \otimes_{\rm eh} \mathcal{B}(\ell_2) \otimes_{\rm eh} \ldots \otimes_{\rm eh} \mathcal{B}(\ell_2) \otimes_{\rm eh} \mathcal{K}(\ell_2)$. Since S_φ is bounded, φ is a Schur multiplier and by [48, Theorem 3.4], $\varphi \in \ell_\infty \otimes_{\rm eh} \ldots \otimes_{\rm eh} \ell_\infty$. Hence $\varphi \in c_0 \otimes_{\rm eh} \ell_\infty \otimes_{\rm eh} \ldots \otimes_{\rm eh} \ell_\infty \otimes_{\rm eh} c_0$. We may now argue as in the last paragraph of the preceding proof to show that $\varphi \in c_0 \otimes_{\rm h} (\ell_\infty \otimes_{\rm eh} \cdots \otimes_{\rm eh} \ell_\infty) \otimes_{\rm h} c_0$. □

Our next aim is to show that if both $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ contain full matrix algebras of arbitrarily large sizes then the completely compact multipliers form a proper subset of the compact multipliers. Saar [85] has provided an example of a compact completely bounded map on $\mathcal{K}(H)$ (where H is a separable Hilbert space) which is not completely compact. It turns out that Saar's example also shows that the sets of compact and completely compact multipliers are distinct, in the case under consideration.

We will need some preliminary results. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Recall that a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is called symmetric (or hermitian) if $\Phi = \Phi^*$ where $\Phi^* : \mathcal{A} \to \mathcal{B}$ is the map given by $\Phi^*(a) = (\Phi(a^*))^*$. By $S_{\mathcal{A}}$ we denote the unit ball of $\mathcal{A}$ and set $S^h_{\mathcal{A}} = \{a \in S_{\mathcal{A}} : a = a^*\}$. The following lemma is a special case of Satz 6 of [85]. We include a direct proof for the convenience of the reader.

Lemma 115. *Let H be a Hilbert space. If $\Phi : \mathcal{A} \to \mathcal{K}(H)$ is a symmetric, completely compact linear map with $\|\Phi\|_{\mathrm{cb}} \leq 1$, then there exists a positive operator $c \in \mathcal{K}(H)$ such that $\Phi^{(n)}(a) \leq c \otimes 1_n$ for all $a \in S^h_{M_n(\mathcal{A})}$ and all $n \in \mathbb{N}$. Moreover, c can be chosen to have norm arbitrarily close to one.*

Proof. We first show that for a given $\varepsilon > 0$ there exists a finite rank projection p on H such that

$$\|\Phi^{(n)}(a) - (p \otimes 1_n)\Phi^{(n)}(a)(p \otimes 1_n)\| \leq \varepsilon \quad \text{for any } a \in S_{M_n(\mathcal{A})}. \tag{6.14}$$

Since Φ is completely compact, there exists a finite dimensional subspace $F \subset \mathcal{K}(H)$ such that $\operatorname{dist}(\Phi^{(n)}(a), M_n(F)) \leq \varepsilon/3$ for any $a \in M_n(\mathcal{A})$, $\|a\| \leq 1$ and any $n \in \mathbb{N}$. Let $S_{F,1+\varepsilon} = \{x \in F : \|x\| \leq 1 + \varepsilon\}$ and let $k = \dim F$. Choose a finite rank projection $p \in \mathcal{K}(H)$ such that

$$\|x - pxp\| < \frac{\varepsilon}{k(3+\varepsilon)} \quad \text{for all } x \in S_{F,1+\varepsilon}$$

and let $\Psi : F \to \mathcal{K}(H)$ be defined by $\Psi(x) = x - pxp$. By [28, Corollary 2.2.4], Ψ is completely bounded and $\|\Psi\|_{\mathrm{cb}} \leq k\|\Psi\|$. This implies that

$$\|\Psi^{(n)}(y)\| \leq k\|\Psi\|\,\|y\| \leq \frac{\varepsilon}{3+\varepsilon}\|y\| \leq \frac{\varepsilon}{3}$$

for all $y \in M_n(F)$ with $\|y\| \leq 1 + \varepsilon/3$.

Now for $a \in S^h_{M_n(\mathcal{A})}$ let $y \in M_n(F)$ be such that $\|\Phi^{(n)}(a) - y\| \leq \varepsilon/3$. Then $\|y\| \leq \|\Phi^{(n)}(a)\| + \varepsilon/3 \leq 1 + \varepsilon/3$. Hence

$$\begin{aligned}
&\|\Phi^{(n)}(a) - (p \otimes 1_n)\Phi^{(n)}(a)(p \otimes 1_n)\| \\
&\quad \leq \|\Phi^{(n)}(a) - y\| + \|\Psi^{(n)}(y)\| + \|(p \otimes 1_n)(y - \Phi^{(n)}(a))(p \otimes 1_n)\| \\
&\quad \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,
\end{aligned}$$

proving (6.14). Next we fix $\varepsilon > 0$ and choose a finite rank projection q_1 on H such that

$$\|\Phi^{(n)}(a) - (q_1 \otimes 1_n)\Phi^{(n)}(a)(q_1 \otimes 1_n)\| \leq \frac{\varepsilon}{2}, \quad a \in M_n(\mathcal{A}),\ \|a\| \leq 1,\ n \in \mathbb{N}.$$

Let $r_1 : \mathcal{A} \to \mathcal{K}(H)$ be the mapping given by $r_1(a) = \Phi(a) - q_1\Phi(a)q_1$, $a \in \mathcal{A}$. Then $r_1 = \Psi \circ \Phi$, where $\Psi : \mathcal{K}(H) \to \mathcal{K}(H)$ is the completely bounded map given by $\Psi(x) = x - q_1 x q_1$. By Proposition 90, r_1 is completely compact. Moreover, $\|r_1\|_{\text{cb}} \leq \varepsilon/2$ and $\Phi(a) = q_1\Phi(a)q_1 + r_1(a)$, $a \in \mathcal{A}$. Proceeding by induction, we can find sequences of finite rank projections q_i and completely compact symmetric mappings r_i such that $\|r_i\|_{\text{cb}} \leq \varepsilon/2^i$ and

$$\Phi(a) = q_1\Phi(a)q_1 + \sum_{i=1}^{\infty} q_{i+1} r_i(a) q_{i+1}, \quad a \in \mathcal{A}.$$

Let $c = q_1 + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} q_{i+1}$. We have that $\Phi^{(n)}$ and $r_i^{(n)}$ are symmetric and

$$\Phi^{(n)}(a) = (q_1 \otimes 1_n)\Phi^{(n)}(a)(q_1 \otimes 1_n) + \sum_{i=1}^{\infty}(q_{i+1} \otimes 1_n) r_i^{(n)}(a)(q_{i+1} \otimes 1_n),$$

for each $a \in \mathcal{A}$. Now

$$\Phi^{(n)}(a) \leq (q_1 \otimes 1_n)\|\Phi\|_{\text{cb}} + \sum_{i=1}^{\infty}(q_{i+1} \otimes 1_n)\|r_i\|_{\text{cb}} \leq (q_1 + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} q_{i+1}) \otimes 1_n = c \otimes 1_n$$

for all $a \in S^h_{M_n(\mathcal{A})}$. By construction, c is compact and $\|c\| \leq 1 + \varepsilon$. □

Let H be an infinite dimensional separable Hilbert space and $\{q_k\}_{k \in \mathbb{N}}$ be a family of pairwise orthogonal projections in $\mathcal{B}(H)$ with rank $q_k = k$ and $\sum_{k=1}^{\infty} q_k = I$. Set $p_n = \sum_{k=1}^{n} q_k$, $n \in \mathbb{N}$. Let $\Phi_k : \mathcal{B}(q_k H) \to \mathcal{B}(q_k H)$, $k \in \mathbb{N}$, be symmetric linear maps such that

$$\|\Phi_k\|_{\text{cb}} = 1, \quad \|\Phi_k\| \to 0 \text{ as } k \to \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} \|\Phi_k\|_2^2 < \infty, \tag{6.15}$$

where $\|\Phi_k\|_2$ denotes the norm of the mapping Φ_k when $\mathcal{B}(q_k H) \simeq \mathcal{C}_2(q_k H)$ is equipped with the Hilbert-Schmidt norm. Identifying $\mathcal{B}(q_k H)$ with $q_k \mathcal{B}(H) q_k$, let $\Phi : \mathcal{K}(H) \to \mathcal{B}(H)$ be the map given by the norm-convergent sum

$$\Phi(x) = \sum_{k=1}^{\infty}{}^{\oplus} \Phi_k(q_k x q_k), \quad x \in \mathcal{K}(H). \tag{6.16}$$

An example of such a map is obtained by taking $\Phi_k = k^{-1}\tau_k$ where τ_k is the transposition map $\mathcal{B}(q_kH) \simeq M_k \to M_k \simeq \mathcal{B}(q_kH)$, which is symmetric and an isometry for both the operator and the Hilbert-Schmidt norm. It is well known [70, p. 419] that $\|\tau_k\|_{\mathrm{cb}} = k$ and hence conditions (6.15) are satisfied.

The next lemma is a straightforward extension of [85, pp. 32–34].

Lemma 116. *If Φ is a map satisfying (6.15) and (6.16) then the range of Φ consists of compact operators. Moreover, Φ is completely contractive and compact but not completely compact.*

Proof. Fix $x \in \mathcal{K}(H)$. Since $\|\Phi_k\| \to 0$ as $k \to \infty$, we have $p_n\Phi(x)p_n \to \Phi(x)$ in norm, so $\Phi(x) \in \mathcal{K}(H)$. Each of the maps $x \mapsto \Phi_k(q_kxq_k)$ is completely contractive, so Φ is completely contractive.

Next, note that Φ maps the unit ball of $\mathcal{K}(H)$ into $U \stackrel{\mathrm{def}}{=} U_1 \oplus U_2 \oplus \cdots$, where U_k is the ball of radius $\|\Phi_k\|$ in $q_k\mathcal{B}(H)q_k$. Since U is compact, the map Φ is compact.

If Φ were completely compact then by Lemma 115, there would exist a positive compact operator c on H such that

$$\Phi^{(k)}(x) \leq c \otimes 1_k \text{ for all } x \in S^h_{M_k(\mathcal{K}(H))} \text{ and all } k \in \mathbb{N}.$$

Hence for every $k \in \mathbb{N}$ and $x \in S^h_{M_k(\mathcal{K}(H))}$,

$$\Phi_k^{(k)}((q_k \otimes 1_k)x(q_k \otimes 1_k)) = (q_k \otimes 1_k)\Phi^{(k)}(x)(q_k \otimes 1_k) \leq q_kcq_k \otimes 1_k.$$

However, $\|\Phi_k^{(k)}\| = \|\Phi_k\|_{\mathrm{cb}} = 1$ by [87], so

$$\|q_kcq_k\| = \|q_kcq_k \otimes 1_k\| \geq \sup\{\|\Phi_k^{(k)}(x)\| : x \in S^h_{M_k(q_k\mathcal{K}(H)q_k)}\} \geq \tfrac{1}{2},$$

which is impossible since c is compact. □

Lemma 117. *Given a map Φ be as above, let $\mathcal{C} = \bigoplus^{c_0}_{k\in\mathbb{N}} \mathcal{B}(q_kH) \subseteq \mathcal{K}(H)$. There exists a universal multiplier $\varphi \in M(\mathcal{C}^{\mathrm{d}}, \mathcal{C})$ with $\Phi = \Phi_{\mathrm{id}(\varphi)}$.*

Proof. Let $\varphi_k \in \mathcal{B}(q_kH)^{\mathrm{d}} \otimes \mathcal{B}(q_kH)$ be such that $\Phi_{\mathrm{id}(\varphi_k)} = \Phi_k$, $k \in \mathbb{N}$, where the family $\{\Phi_k\}_{k=1}^\infty$ satisfies (6.15). Then $\|\varphi_k\|_{\min} = \|\Phi_k\|_2$. Let $\psi_n = \sum_{k=1}^n \varphi_k$. If $n < m$ then $\|\psi_m - \psi_n\|_{\min} = \|\sum_{k=n+1}^m \Phi_k\|_2$ so

$$\|\psi_m - \psi_n\|_{\min} \leq \Big(\sum_{k=n+1}^m \|\Phi_k\|_2^2\Big)^{1/2}.$$

By (6.15), the sequence $\{\psi_n\}$ converges to an element $\varphi \in \mathcal{C}^{\mathrm{d}} \otimes \mathcal{C}$. Moreover, for every $x \in \mathcal{C}_2(H)$ we have

$$\Phi_{\mathrm{id}(\varphi)}(x) = \lim_{n\to\infty} p_n \Phi_{\mathrm{id}(\varphi)}(x) p_n = \lim_{n\to\infty} \Phi_{\mathrm{id}(\psi_n)}(x) = \Phi(x),$$

where the limits are in the operator norm. So $\Phi_{\mathrm{id}(\varphi)} = \Phi$ which is completely contractive by Lemma 116, so $\varphi \in M(\mathcal{C}^{\mathrm{d}}, \mathcal{C})$ by Theorem 99. □

Given C*-algebras $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \ldots, n$, and $\psi = c_2 \otimes \cdots \otimes c_{n-1} \in \mathcal{A}_2 \odot \cdots \odot \mathcal{A}_{n-1}$, we may define a bounded linear map $\mathcal{A}_1 \otimes \mathcal{A}_n \to \mathcal{B}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n$, where $\mathcal{B}_1 = \mathcal{A}_1$ if n is even and $\mathcal{B}_1 = \mathcal{A}_1^{\mathrm{d}}$ if n is odd, by

$$a \otimes b \mapsto \begin{cases} a \otimes \psi \otimes b & \text{if } n \text{ is even,} \\ a^{\mathrm{d}} \otimes \psi \otimes b & \text{if } n \text{ is odd.} \end{cases}$$

We write ι_ψ for the restriction of this map to $M(\mathcal{A}_1, \mathcal{A}_n)$.

Lemma 118. *(i) The range of ι_ψ is contained in $M(\mathcal{B}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$.*

(ii) $\iota_\psi(M_c^{\mathrm{id}(\mathcal{A}_1,\mathcal{A}_n)) \subseteq M_c^{\mathrm{id}(\mathcal{B}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)}}$.

(iii) Suppose that n is even and $\omega \in (\mathcal{B}(H_{n-1}^{\mathrm{d}}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2))_*$. Writing

$$M_\omega : \mathcal{B}(H_n) \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1}^{\mathrm{d}}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \otimes_{\mathrm{eh}} \mathcal{B}(H_1^{\mathrm{d}}) \to \mathcal{B}(H_n) \otimes_{\mathrm{eh}} \mathcal{B}(H_1^{\mathrm{d}})$$

for the "middle slice map" $M_\omega = R_\omega \otimes_{\mathrm{eh}} \mathrm{id}_{\mathcal{B}(H_1^{\mathrm{d}})}$, we have

$$M_\omega(u_{\iota_\psi(\varphi)}) = \omega(\tilde{\psi}) u_\varphi$$

where $\tilde{\psi} = c_{n-1}^{\mathrm{d}} \otimes \cdots \otimes c_2$. The same is true, mutatis mutandis, if n is odd.

Proof. Let $\varphi \in M(\mathcal{A}_1, \mathcal{A}_n)$. By Theorem 88, there exist a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \mathcal{A}_n$ and representations $u_{\varphi_\nu}^{\mathrm{id}} = A_2^\nu \odot A_1^\nu$ and $u_\varphi^{\mathrm{id}} = A_2 \odot A_1$, where A_i^ν are finite matrices with entries in $\mathcal{A}_1^{\mathrm{d}}$ if $i = 1$ and in $\mathcal{A}_n$ if $i = 2$, such that $\varphi_\nu \to \varphi$ semi-weakly, $A_i^\nu \to A_i$ strongly and $\sup_{i,\nu} \|A_i^\nu\| < \infty$.

(i) It is easy to see that $\iota_\psi(\varphi_\nu)$ satisfies the boundedness conditions of Theorem 88 and converges semi-weakly to $\iota_\psi(\varphi)$, which is therefore a universal multiplier.

(ii) Suppose that n is even and let $\iota = \iota_\psi$. It is immediate to check that if $\varphi \in \mathcal{A}_1 \odot \mathcal{A}_n$ and $T_1 \in \mathcal{K}(H_1^{\mathrm{d}}, H_2)$, $\ldots$, $T_{n-1} \in \mathcal{K}(H_{n-1}^{\mathrm{d}}, H_n)$ then

$$\Phi_{\iota(\varphi)}(T_{n-1} \otimes \ldots \otimes T_1) = \Phi_\varphi(T_{n-1} c_{n-1}^{\mathrm{d}} \ldots c_2 T_1).$$

Note that this equation holds for any $\varphi \in M(\mathcal{A}_1, \mathcal{A}_n)$ since $\Phi_{\varphi_\nu}(T) \to \Phi_\varphi(T)$ and $\Phi_{\iota(\varphi_\nu)}(T_{n-1} \otimes \ldots \otimes T_1) \to \Phi_{\iota(\varphi)}(T_{n-1} \otimes \ldots \otimes T_1)$ weakly for any T,

$T_1, \ldots, T_{n-1}$. Since $\Phi_{\iota(\varphi)}$ is the composition of the bounded mapping $X_{n-1} \otimes \ldots \otimes X_1 \mapsto X_{n-1} c^{\mathrm{d}}_{n-1} \ldots c_2 X_1$ with Φ_φ, it follows that if φ is a compact operator multiplier then so is $\iota(\varphi)$.

(iii) We have that

$$\begin{aligned}\Phi_{\iota(\varphi_\nu)}(T_{n-1} \otimes \ldots \otimes T_1) &= A^\nu_2 (T_{n-1} \otimes 1)(c^{\mathrm{d}}_{n-1} \otimes 1) \ldots (c_2 \otimes 1)(T_1 \otimes 1) A^\nu_1 \\ &\to A_2 (T_{n-1} \otimes 1)(c^{\mathrm{d}}_{n-1} \otimes 1) \ldots (c_2 \otimes 1)(T_1 \otimes 1) A_1\end{aligned}$$

weakly. On the other hand, $\Phi_{\iota(\varphi_\nu)}(T_{n-1} \otimes \ldots \otimes T_1) \to \Phi_{\iota(\varphi)}(T_{n-1} \otimes \ldots \otimes T_1)$ which implies that $u_{\iota(\varphi)} = A_2 \odot (c^{\mathrm{d}}_{n-1} \otimes 1) \odot \ldots \odot (c_2 \otimes 1) \odot A_1$. It follows that $M_\omega(u_{\iota(\varphi)}) = \omega(\tilde{\psi}) u_\varphi$. □

Theorem 119. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C*-algebras with the property that both $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ contain full matrix algebras of arbitrarily large sizes. Then the inclusion $M_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq M_c(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is proper.*

Proof. We may assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \ldots, n$ for some Hilbert spaces $H_1, \ldots, H_n$. First suppose that $n = 2$. By hypothesis, we may assume that there is an infinite dimensional separable Hilbert space H with $H^{\mathrm{d}} \subseteq H_1$ and $H \subseteq H_2$, and a C*-algebra $\mathcal{C} = \bigoplus^{c_0}_{k \in \mathbb{N}} M_k \subseteq \mathcal{K}(H)$ as in Lemma 117 with $\mathcal{C}^{\mathrm{d}} \subseteq \mathcal{A}_1$ and $\mathcal{C} \subseteq \mathcal{A}_2$. By the injectivity of the minimal tensor product of C*-algebras, $\mathcal{C}^{\mathrm{d}} \otimes \mathcal{C} \subseteq \mathcal{A}_1 \otimes \mathcal{A}_2$.

Let $\varphi \in \mathcal{C}^{\mathrm{d}} \otimes \mathcal{C}$ be given by Lemma 117. It follows from Lemma 116 that $\varphi \in M_c(\mathcal{A}_1, \mathcal{A}_2) \setminus M^{\mathrm{id}(\mathcal{A}_1, \mathcal{A}_2)}_{cc}$. Since faithful representations of $\mathcal{A}_1$ and $\mathcal{A}_2$ restrict to representations of $\mathcal{C}$ containing the identity subrepresentation up to unitary equivalence, we have that $\varphi \in M_c(\mathcal{A}_1, \mathcal{A}_2) \setminus M_{cc}(\mathcal{A}_1, \mathcal{A}_2)$.

Suppose now that n is even. Let $\varphi \in M_c(\mathcal{A}_1, \mathcal{A}_n) \setminus M_{cc}(\mathcal{A}_1, \mathcal{A}_n)$, fix any non-zero $\psi = c_2 \otimes \cdots \otimes c_{n-1} \in \mathcal{A}_2 \odot \cdots \odot \mathcal{A}_{n-1}$ and let us write $\iota = \iota_\psi$. Suppose that $\iota(\varphi)$ is a completely compact multiplier. By Theorem 109, $u_{\iota(\varphi)} \in \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}^o_{n-1} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}^o_1)$.

Let $\tilde{\psi} = c^{\mathrm{d}}_{n-1} \otimes \cdots \otimes c_2 \in \mathcal{A}^{\mathrm{d}}_{n-1} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2$ and fix $\omega \in (\mathcal{B}(H^{\mathrm{d}}_{n-1}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2))_*$ such that $\omega(\tilde{\psi}) \neq 0$. By Lemma 118 (iii), $M_\omega(u_{\iota(\varphi)}) = \omega(\tilde{\psi}) u_\varphi$ and hence $u_\varphi \in \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}^o_1)$ which by Theorem 109 contradicts the assumption that φ is not a completely compact multiplier.

If n is odd then the same proof works with minor modifications. □

Remark 120. *We do not know whether the sets $M_{cc}(\mathcal{A}, \mathcal{B})$ and $M_c(\mathcal{A}, \mathcal{B})$ are distinct if $\mathcal{K}(\mathcal{A})$ contains matrix algebras of arbitrarily large sizes, while $\mathcal{K}(\mathcal{B})$ does not (and vice versa). Let $\mathcal{C}$ be the C*-algebra defined in Lemma 117. To show that the inclusion $M_{cc}(\mathcal{C}, c_0) \subseteq M_c(\mathcal{C}, c_0)$ is proper it would suffice to exhibit mappings $\Phi_k : M_k \to M_k$ which satisfy (6.15) and are left D_k-modular (where D_k is the subalgebra of all diagonal matrices of M_k). This modularity* *condition would enable us to find $\varphi_k \in M_k^{\mathrm{d}} \otimes D_k$ such that $\Phi_k = \Phi_{\mathrm{id}(\varphi_k)}$ using the method of Lemma 117 and we could then conclude from Lemma 116 that $M_{cc}(\mathcal{C}, c_0) \subsetneq M_c(\mathcal{C}, c_0)$. However, we do not know if such mappings Φ_k exist.*

This prompts the following question: if $\mathcal{D}$ is a masa in $\mathcal{B}(H)$, does there exist a constant C such that whenever $\Phi : \mathcal{K}(H) \to \mathcal{K}(H)$ is a bounded and left $\mathcal{D}$-modular map then $\|\Phi\|_{\mathrm{cb}} \leq C\|\Phi\|$? If such a version of Smith's automatic complete boundedness result holds then it would follow that $M_{cc}(\mathcal{C}, c_0) = M_c(\mathcal{C}, c_0)$.

6.6.2 Automatic compactness

We now turn to the question of when every universal multiplier is automatically compact. We will restrict to the case $n = 2$ for the rest of the paper. We will first establish an auxiliary result in a different but related setting. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are commutative C*-algebras and assume that $\mathcal{A} = C_0(X)$ and $\mathcal{B} = C_0(Y)$ for some locally compact Hausdorff spaces X and Y. The C*-algebra $C_0(X) \otimes C_0(Y)$ will be identified with $C_0(X \times Y)$ and $M(\mathcal{A}, \mathcal{B})$ with a subset of $C_0(X \times Y)$. Elements of the Haagerup tensor product $C_0(X) \otimes_{\mathrm{h}} C_0(Y)$, as well as of the projective tensor product $C_0(X)\hat{\otimes}C_0(Y)$, will be identified with functions in $C_0(X \times Y)$ in the natural way. Note that, by Grothendieck's inequality, $C_0(X) \otimes_{\mathrm{h}} C_0(Y)$ and $C_0(X)\hat{\otimes}C_0(Y)$ coincide as sets of functions.

Proposition 121. *Let X and Y be infinite, locally compact Hausdorff spaces. Then $C_0(X) \otimes_{\mathrm{h}} C_0(Y) \subseteq M(C_0(X), C_0(Y))$ and this inclusion is proper.*

Proof. The inclusion $C_0(X) \otimes_{\mathrm{h}} C_0(Y) \subseteq M(C_0(X), C_0(Y))$ follows from Corollary 6.7 of [52]. To show that this inclusion is proper, suppose first that X and Y are compact. By Theorem 11.9.1 of [33], there exists a sequence $(f_i)_{i=1}^{\infty} \subseteq C(X) \otimes_{\mathrm{h}} C(Y)$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathrm{h}} < \infty$, converging uniformly to a function $f \in C(X \times Y) \setminus C(X) \otimes_{\mathrm{h}} C(Y)$. By Corollary 6.7 of [52], $f \in M(C(X), C(Y))$. The conclusion now follows.

Now assume that both X and Y are locally compact but not compact (the case where one of the spaces is compact while the other is not is similar). Let $\tilde{X} = X \cup \{\infty\}$ and $\tilde{Y} = Y \cup \{\infty\}$ be the one point compactifications of X and Y. Then $C(\tilde{X}) = C_0(X) + \mathbb{C}1$ and $C(\tilde{Y}) = C_0(Y) + \mathbb{C}1$, where 1 denotes the constant function taking the value one. Moreover, it is easy to see that

$$C(\tilde{X}) \otimes C(\tilde{Y}) = C_0(X \times Y) + C_0(X) + C_0(Y) + \mathbb{C}1$$

and

$$C(\tilde{X})\hat{\otimes}C(\tilde{Y}) = C_0(X)\hat{\otimes}C_0(Y) + C_0(X) + C_0(Y) + \mathbb{C}1. \tag{6.17}$$

By the first part of the proof, there exists $\varphi \in M(C(\tilde{X}), C(\tilde{Y})) \setminus C(\tilde{X}) \otimes_{\mathrm{h}} C(\tilde{Y})$. Write $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ where $\varphi_1 \in C_0(X \times Y)$, $\varphi_2 \in C_0(X)$, $\varphi_3 \in C_0(Y)$ and $\varphi_4 \in \mathbb{C}1$. Suppose that $\varphi_1 \in C_0(X) \otimes_{\mathrm{h}} C_0(Y)$. By (6.17), $\varphi \in C(\tilde{X}) \hat{\otimes} C(\tilde{Y})$, a contradiction. □

Theorem 122. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. The following are equivalent:*
(i) either $\mathcal{A}$ is finite dimensional and $\mathcal{K}(\mathcal{B}) = \mathcal{B}$, or $\mathcal{B}$ is finite dimensional and $\mathcal{K}(\mathcal{A}) = \mathcal{A}$;
(ii) $M_c(\mathcal{A}, \mathcal{B}) = M(\mathcal{A}, \mathcal{B})$;
(iii) $M_{cc}(\mathcal{A}, \mathcal{B}) = M(\mathcal{A}, \mathcal{B})$.

Proof. (i)⇒(iii) Suppose that $\mathcal{A}$ is finite dimensional and $\mathcal{K}(\mathcal{B}) = \mathcal{B}$, and that $\mathcal{A} \subseteq \mathcal{B}(H_1)$ and $\mathcal{B} \subseteq \mathcal{B}(H_2)$ for some Hilbert spaces H_1 and H_2 where H_1 is finite dimensional. Fix $\varphi \in M(\mathcal{A}, \mathcal{B})$. Then φ is the sum of finitely many elements of the form $a \otimes b$ where a has finite rank and $b \in \mathcal{K}(H_2)$; such elements are completely compact multipliers by Theorem 109.

(iii)⇒(ii) is trivial.

(ii)⇒(i) Assume that both $\mathcal{A}$ and $\mathcal{B}$ are infinite dimensional and are identified with their image under the reduced atomic representation. If either $\mathcal{K}(\mathcal{A})$ or $\mathcal{K}(\mathcal{B})$ is finite dimensional then there exists an elementary tensor $a \otimes b \in (\mathcal{A} \odot \mathcal{B}) \setminus (\mathcal{K}(\mathcal{A}) \odot \mathcal{K}(\mathcal{B}))$. By Proposition 112, $a \otimes b \notin M_c(\mathcal{A}, \mathcal{B})$. We can therefore assume that both $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{B})$ are infinite dimensional. Then, up to a $*$-isomorphism, c_0 is contained in both $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{B})$. By Proposition 121, there exists $\varphi \in M(c_0, c_0) \setminus (c_0 \otimes_{\mathrm{h}} c_0)$. Then $\varphi \in M(\mathcal{A}, \mathcal{B})$ and $\Phi_{\mathrm{id}(\varphi)}$ is not compact by Hladnik's characterisation [42]. Since the restrictions to c_0 of any faithful representations of $\mathcal{A}$, $\mathcal{B}$ contain representations unitarily equivalent to the identity representations, we see that φ is not a compact multiplier.

Thus at least one of the C*-algebras $\mathcal{A}$ and $\mathcal{B}$ is finite dimensional; assume without loss of generality that this is $\mathcal{A}$. Suppose that $\mathcal{B} \neq \mathcal{K}(\mathcal{B})$ and fix an element an $b \in \mathcal{B} \setminus \mathcal{K}(\mathcal{B})$. Let $a \in \mathcal{A}$ be a non-zero element. By Proposition 112, the elementary tensor $a \otimes b$ is not a compact multiplier. □

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