Chapter 11

Markov Chains

11.1 Introduction

1. $\mathbf{w}(1) = (.5,\ .25,\ .25)$

$\mathbf{w}(2) = (.4375,\ .1875,\ .375)$

$\mathbf{w}(3) = (.40625,\ .203125,\ .390625)$

2. $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \mathbf{P}^2 = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}, \quad \mathbf{P}^3 = \begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix}.$

$\mathbf{P}^n = \begin{pmatrix} 1 & 0 \\ \frac{2^n-1}{2^n} & \frac{1}{2^n} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$

Whatever the President's decision, in the long run each person will be told that he or she is going to run.

3. $\mathbf{P}^n = \mathbf{P}$ for all n.

4. .7.

5. 1

6. $\mathbf{w}^{(1)} = \mathbf{w}^{(2)} = \mathbf{w}^{(3)} = \mathbf{w}^{(n)} = (.25, .5, .25)$.

7. (a) $\mathbf{P}^n = \mathbf{P}$

(b) $\mathbf{P}^n = \begin{cases} \mathbf{P}, & \text{if } n \text{ is odd,} \\ \mathbf{I}, & \text{if } n \text{ is even.} \end{cases}$

8. $\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} \end{array}.$

9. $p^2+q^2,\ q^2,\ \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} p & q \\ q & p \end{pmatrix} \end{array}$

11. .375

12. (a) $$\mathbf{P} = \begin{array}{c c} & \begin{array}{cccc} P & SL & UL & NS \end{array} \\ \begin{array}{c} P \\ SL \\ UL \\ NS \end{array} & \begin{pmatrix} .64 & .08 & .08 & .2 \\ .16 & .48 & .16 & .2 \\ .2 & .2 & .4 & .2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

(b) .24.

19. (a) 5/6.

(b) The ‘transition matrix’ is

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} 5/6 & 1/6 \\ 1/2 & 1/2 \end{pmatrix} \end{array}.$$

(c) 9/10.

(d) No. If it were a Markov chain, then the answer to (c) would be the same as the answer to (a).

11.2 Absorbing Markov Chains

1. $a = 0$ or $b = 0$

2. H is the absorbing state. Y and D are transient states. It is possible to go from each of these states to the absorbing state, in fact in one step.

3. Examples 11.10 and 11.11

4.
$$N = \begin{array}{c c} & \begin{array}{cc} Gg & gg \end{array} \\ \begin{array}{c} GG \\ gg \end{array} & \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \end{array}.$$

5. The transition matrix in canonical form is

$$P = \begin{array}{c} \\ GG,Gg \\ GG,gg \\ Gg,Gg \\ Gg,gg \\ GG,GG \\ gg,gg \end{array} \begin{array}{c} \begin{array}{cccccc} GG,Gg & GG,gg & Gg,Gg & Gg,gg & GG,GG & gg,gg \end{array} \\ \begin{pmatrix} 1/2 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/8 & 1/4 & 1/4 & 1/16 & 1/16 \\ 0 & 0 & 1/4 & 1/2 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

Thus

$$Q = \begin{array}{c} \\ GG,Gg \\ GG,gg \\ Gg,Gg \\ Gg,gg \\ . \end{array} \begin{array}{c} \begin{array}{cccc} GG,Gg & GG,gg & Gg,Gg & Gg,gg \end{array} \\ \begin{pmatrix} 1/2 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 0 \\ 1/4 & 1/8 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/2 \\ & & & \end{pmatrix} \end{array} ,$$

and

$$N = (I - Q)^{-1} = \begin{array}{c} \\ GG,Gg \\ GG,gg \\ Gg,Gg \\ Gg,gg \end{array} \begin{array}{c} \begin{array}{cccc} GG,Gg & GG,gg & Gg,Gg & Gg,gg \end{array} \\ \begin{pmatrix} 8/3 & 1/6 & 4/3 & 2/3 \\ 4/3 & 4/3 & 8/3 & 4/3 \\ 4/3 & 1/3 & 8/3 & 4/3 \\ 2/3 & 1/6 & 4/3 & 8/3 \end{pmatrix} \end{array} .$$

From this we obtain

$$t = Nc = \begin{array}{c} GG,Gg \\ GG,gg \\ Gg,Gg \\ Gg,gg \end{array} \begin{pmatrix} 29/6 \\ 20/3 \\ 17/3 \\ 29/6 \end{pmatrix} ,$$

and

$$\mathbf{B} = \mathbf{NR} = \begin{array}{c} \\ GG,Gg \\ GG,gg \\ Gg,Gg \\ Gg,gg \end{array} \begin{array}{c} \begin{array}{cc} GG,GG & gg,gg \end{array} \\ \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix} \end{array} .$$

6. The canonical form of the transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ N \\ S \\ R \end{array} \begin{array}{c} \begin{array}{ccc} N & S & R \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix} \end{array} ,$$

$$\mathbf{N} = \begin{array}{c c} & \begin{array}{cc} N & S \end{array} \\ \begin{array}{c} N \\ S \end{array} & \begin{pmatrix} 4/3 & 4/3 \\ 2/3 & 8/3 \end{pmatrix} \end{array},$$

$$\mathbf{t} = \mathbf{Nc} = \begin{array}{c} N \\ S \end{array} \begin{pmatrix} 8/3 \\ 10/3 \end{pmatrix},$$

$$\mathbf{B} = \mathbf{NR} = \begin{array}{c} N \\ S \end{array} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Here is a typical interpretation for an entry of $\mathbf{N}$. If it is snowing today, the expected number of nice days before the first rainy day is 2/3. The entries of $\mathbf{t}$ give the expected number of days until the next rainy day. Starting with a nice day this is 8/3, and starting with a snowy day it is 10/3. The entries of $\mathbf{B}$ reflect the fact that we are certain to reach the absorbing state (rainy day) starting in either state N or state S.

7. $$\mathbf{N} = \begin{pmatrix} 2.5 & 3 & 1.5 \\ 2 & 4 & 2 \\ 1.5 & 3 & 2.5 \end{pmatrix}$$

$$\mathbf{Nc} = \begin{pmatrix} 7 \\ 8 \\ 7 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 5/8 & 3/8 \\ 1/2 & 1/2 \\ 3/8 & 5/8 \end{pmatrix}$$

8. The transition matrix in canonical form is

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 0 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{array} & \begin{pmatrix} 0 & 2/3 & 0 & 1/3 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \; cr \end{pmatrix} \end{array},$$

$$\mathbf{N} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 7/5 & 6/5 & 4/5 \\ 3/5 & 9/5 & 6/5 \\ 1/5 & 3/5 & 7/5 \end{pmatrix} \end{array},$$

$$\mathbf{B} = \mathbf{NR} = \begin{array}{c c} & \begin{array}{cc} 0 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 7/15 & 8/15 \\ 3/15 & 12/15 \\ 1/15 & 14/15 \end{pmatrix} \end{array},$$

$$\mathbf{t} = \mathbf{NC} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 17/5 \\ 18/5 \\ 11/5 \end{pmatrix}.$$

9. 2.08

12.

$$\mathbf{P} = \begin{array}{c} \\ ABC \\ AC \\ BC \\ A \\ B \\ C \\ none \end{array} \begin{array}{c} \begin{array}{ccccccc} ABC & AC & BC & A & B & C & none \end{array} \\ \begin{pmatrix} 5/18 & 5/18 & 4/18 & 0 & 0 & 4/18 & 0 \\ 0 & 5/12 & 0 & 5/2 & 0 & 1/12 & 1/12 \\ 0 & 0 & 10/18 & 0 & 5/18 & 2/18 & 1/18 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\mathbf{N} = \begin{pmatrix} 1.385 & .659 & .692 \\ 0 & 1.714 & 0 \\ 0 & 0 & 2.25 \end{pmatrix}$$

$$\mathbf{Nc} = \begin{pmatrix} 2.736 \\ 1.714 \\ 2.25 \end{pmatrix}$$

$$\mathbf{B} = \begin{array}{c} \\ ABC \\ AC \\ BC \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & none \end{array} \\ \begin{pmatrix} .275 & .192 & .440 & .093 \\ .714 & 0 & .143 & .143 \\ 0 & .625 & .25 & .125 \end{pmatrix} \end{array}$$

13. Using timid play, Smith's fortune is a Markov chain with transition matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 0 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 8 \end{array} \\ \begin{pmatrix} 0 & .4 & 0 & 0 & 0 & 0 & 0 & .6 & 0 \\ .6 & 0 & .4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .6 & 0 & .4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .6 & 0 & .4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .6 & 0 & .4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .6 & 0 & .4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .6 & 0 & 0 & .4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

For this matrix we have

$$\mathbf{B} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 8 \end{array} \\ \begin{pmatrix} .98 & .02 \\ .95 & .05 \\ .9 & .1 \\ .84 & .16 \\ .73 & .27 \\ .58 & .42 \\ .35 & .65 \end{pmatrix} \end{array} .$$

For bold strategy, Smith's fortune is governed instead by the transition matrix

$$\mathbf{P} = \begin{array}{c} \\ 3 \\ 4 \\ 6 \\ 0 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 4 & 6 & 0 & 8 \end{array} \\ \begin{pmatrix} 0 & 0 & .4 & .6 & 0 \\ 0 & 0 & 0 & .6 & .4 \\ 0 & .6 & 0 & 0 & .4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} ,$$

with

$$\mathbf{B} = \begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 8 \end{array} \\ \begin{pmatrix} .744 & .256 \\ .6 & .4 \\ .36 & .64 \end{pmatrix} \end{array} .$$

From this we see that the bold strategy gives him a probability .256 of getting out of jail while the timid strategy gives him a smaller probability .1. Be bold!

14. It is the same.

15. (a)

$$\mathbf{P} = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 4 & 5 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 2/3 & 0 & 1/3 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

(b)

$$\mathbf{N} = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 5/3 & 2 & 4/3 \\ 1 & 3 & 2 \\ 2/3 & 2 & 7/3 \end{pmatrix} \end{array} ,$$

$$\mathbf{t} = \begin{array}{c} 3 \\ 4 \\ 5 \end{array} \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} ,$$

$$
\begin{array}{c} \\ \mathbf{B} = \begin{array}{c} 3 \\ 4 \\ 5 \end{array} \end{array}
\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 5/9 & 4/9 \\ 1/3 & 2/3 \\ 2/9 & 7/9 \end{pmatrix} \end{array} .
$$

(c) Thus when the score is deuce (state 4), the expected number of points to be played is 6, and the probability that B wins (ends in state 2) is 2/3.

16.

$$
\begin{array}{c|cccccc}
 & g,GG & G,Gg & g,Gg & G,gg & G,GG & g,gg \\
\hline
g,GG & 0 & 1 & 0 & 0 & 0 & 0 \\
G,Gg & .25 & .25 & .25 & 0 & .25 & 0 \\
g,Gg & 0 & .25 & .25 & .25 & 0 & .25 \\
G,gg & 0 & 0 & 1 & 0 & 0 & 0 \\
G,GG & 0 & 0 & 0 & 0 & 1 & 0 \\
g,gg & 0 & 0 & 0 & 0 & 0 & 1
\end{array} ,
$$

$$
\mathbf{N} = \begin{pmatrix} 1.667 & 2.667 & 1.333 & .333 \\ .667 & 2.667 & 1.333 & .333 \\ .333 & 1.333 & 2.667 & .667 \\ .333 & 1.333 & 2.667 & 1.667 \end{pmatrix} ,
$$

$$
\mathbf{Nc} = \begin{pmatrix} 6 \\ 5 \\ 5 \\ 6 \\ , \end{pmatrix}
$$

$$
\mathbf{B} = \begin{pmatrix} .667 & .333 \\ .667 & .333 \\ .333 & .667 \\ .333 & .667 \end{pmatrix} .
$$

17. For the color-blindness example, we have

$$
\mathbf{B} = \begin{array}{c} \\ g,GG \\ G,Gg \\ g,Gg \\ G,gg \end{array}
\begin{array}{c} \begin{array}{cc} G,GG & g,gg \end{array} \\ \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \\ 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix} \end{array} ,
$$

and for Example 9 of Section 11.1, we have

$$
\mathbf{B} = \begin{array}{c} \\ GG,Gg \\ GG,gg \\ Gg,Gg \\ Gg,gg \end{array}
\begin{array}{c} \begin{array}{cc} GG,GG & gg,gg \end{array} \\ \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix} \end{array} .
$$

In each case the probability of ending up in a state with all G's is proportional to the number of G's in the starting state. The transition matrix for Example 9 is

$$\mathbf{P} = \begin{array}{c} \\ GG,GG \\ GG,Gg \\ GG,gg \\ Gg,Gg \\ Gg,gg \\ gg,gg \end{array} \begin{array}{c} \begin{array}{cccccc} GG,GG & GG,Gg & GG,gg & Gg,Gg & Gg,gg & gg,gg \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1/16 & 1/4 & 1/8 & 1/4 & 1/4 & 1/16 \\ 0 & 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

Imagine a game in which your fortune is the number of G's in the state that you are in. This is a fair game. For example, when you are in state Gg,gg your fortune is 1. On the next step it becomes 2 with probability 1/4, 1 with probability 1/2, and 0 with probability 1/4. Thus, your expected fortune after the next step is equal to 1, which is equal to your current fortune. You can check that the same is true no matter what state you are in. Thus if you start in state Gg,gg, your expected final fortune will be 1. But this means that your final fortune must also have expected value 1. Since your final fortune is either 4 if you end in GG, GG or 0 if you end in gg, gg, we see that the probability of your ending in GG, GG must be 1/4.

18. (a)
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & F & G \end{array} \\ \begin{pmatrix} r & p & 0 & q & 0 \\ 0 & r & p & q & 0 \\ 0 & 0 & r & q & p \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(b) Expected time in second year = 1.09.

Expected time in med school = 3.3 years.

(c) Probability of an incoming student graduating = .67.

19. (a)

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 0 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 0 & 3 \end{array} \\ \begin{pmatrix} 0 & 2/3 & 1/3 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

(b)

$$\mathbf{N} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 9/5 & 6/5 \\ 6/5 & 9/5 \end{pmatrix} \end{array} ,$$

$$\mathbf{B} = \begin{array}{c c} & \begin{array}{cc} 0 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{pmatrix} \end{array},$$

$$\mathbf{t} = \begin{array}{c} 1 \\ 2 \end{array} \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

(c) The game will last on the average 3 moves.

(d) If Mary deals, the probability that John wins the game is 3/5.

20. Consider the Markov chain with state i (for $1 \leq i < k$) the length of the current run, and k an absorbing state. Then when in state $i < k$, the chain goes to $i + 1$ with probability $1/m$ or to 1 with probability $(m - 1)/m$. Thus, starting in state 1, in order to get to state $j + 1$ the chain must be in state j and then move to $j + 1$. This means that

$$N_{1,j+1} = N_{1,j}(1/m) ,$$

or

$$N_{1,j} = mN_{1,j+1} .$$

This will be true also for $j + 1 = k$ if we interpret $N_{1,k}$ as the number of times that the chain enters the state k, namely, 1. Thus, starting with $N_{1,k} = 1$ and working backwards, we see that $N_{1,j} = m^{k-j}$ for $j = 1, \cdots, k$. Therefore, the expected number of experiments until a run of k occurs is

$$1 + m + m^2 + \cdots + m^{k-1} = \frac{m^k - 1}{m - 1} .$$

(The initial 1 is to start the process off.) Putting $m = 10$ and $k = 9$ we see that the expected number of digits in the decimal expansion of π until the first run of length 7 would be about 111 million if the expansion were random. Thus we should not be surprised to find such a run in the first 100,000,000 digits of π and indeed there are runs of length 9 among these digits.

21. The problem should assume that a fraction

$$q_i = 1 - \sum_j q_{ij} > 0$$

of the pollution goes into the atmosphere and escapes.

(a) We note that $\mathbf{u}$ gives the amount of pollution in each city from today's emission, $\mathbf{uQ}$ the amount that comes from yesterday's emission, $\mathbf{uQ}^2$ from two days ago, etc. Thus

$$\mathbf{w}^n = \mathbf{u} + \mathbf{uQ} + \cdots \mathbf{uQ}^{n-1} .$$

(b) Form a Markov chain with **Q**-matrix **Q** and with one absorbing state to which the process moves with probability q_i when in state i. Then

$$\mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \cdots + \mathbf{Q}^{n-1} \to \mathbf{N} ,$$

so

$$\mathbf{w}^{(n)} \to \mathbf{w} = \mathbf{uN} .$$

(c) If we are given **w** as a goal, then we can achieve this by solving $\mathbf{w} = \mathbf{Nu}$ for **u**, obtaining

$$\mathbf{u} = \mathbf{w}(\mathbf{I} - \mathbf{Q}) .$$

22.

(a) The total amount of goods that the ith industry needs to produce \$1 worth of goods is

$$x_1 q_{1i} + x_2 q_{2i} + \cdots + x_n q_{ni} .$$

This is the i'th component of the vector $\mathbf{xQ}$.

(b) By part (a) the amounts the industries need to meet their internal demands is $\mathbf{xQ}$. Thus to meet both internal and external demands, the companies must produce amounts given by a vector **x** satifying the equation

$$\mathbf{x} = \mathbf{xQ} + \mathbf{d} .$$

(c) From Markov chain theory we can always solve the equation

$$\mathbf{x} = \mathbf{xQ} + \mathbf{d}$$

by writing it as

$$\mathbf{x}(\mathbf{I} - \mathbf{Q}) = \mathbf{d}$$

and then using the fact that $(\mathbf{I} - \mathbf{Q})\mathbf{N} = \mathbf{I}$ to obtain

$$\mathbf{x} = \mathbf{dN} .$$

(d) If the row sums of **Q** are all less than 1, this means that every industry makes a profit. A company can rely directly or indirectly on a profit-making company. If for any value of $n, q_{ij}^n > 0$, then i depends at least indirectly on j. Thus *depending upon* is equivalent in the Markov chain interpretation to *being able to reach.* Thus the demands can be met if every company is either profit-making or depends upon a profit-making industry.

(e) Since $\mathbf{x} = \mathbf{dN}$, we see that

$$\mathbf{xc} = \mathbf{dNc} = \mathbf{dt} .$$

24. When the walk is in state i, it goes to $i+1$ with probability p and $i-1$ with probability q. Condition (a) just equates the probability of winning in terms of the current state to the probability after the next step. Clearly, if our fortune is 0, then the probability of winning is 0, and if it is T, then the probability is 1. Here is an instructive way (not the simplest way) to see that the values of $\mathbf{w}$ are uniquely determined by (a), (b), and (c). Let $\mathbf{P}$ be the transition matrix for our absorbing chain. Then these equations state that

$$\mathbf{P}\mathbf{w} = \mathbf{w} \ .$$

That is, the column vector $\mathbf{w}$ is a fixed vector for $\mathbf{P}$. Consider the transition matrix for an arbitrary Markov chain in canonical form and assume that we have a vector $\mathbf{w}$ such that $\mathbf{w} = \mathbf{P}\mathbf{w}$. Multiplying through by $\mathbf{P}$, we see that $\mathbf{P}^2\mathbf{w} = \mathbf{w}$, and in general $\mathbf{P}^n\mathbf{w} = \mathbf{w}$. But

$$\mathbf{P}^n \to \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \ .$$

Thus

$$\mathbf{w} = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \mathbf{w} \ .$$

If we write

$$\mathbf{w} = \begin{pmatrix} \mathbf{w_T} \\ w_A \end{pmatrix} \ ,$$

where T is the set of transient states and A the set of absorbing states, then by the argument above we have

$$\mathbf{w} = \begin{pmatrix} \mathbf{w}_T \\ w_A \end{pmatrix} = \begin{pmatrix} \mathbf{B}\mathbf{w_A} \\ w_A \end{pmatrix} \ .$$

Thus for an absorbing Markov chain, a fixed column vector $\mathbf{w}$ is determined by its values on the absorbing states. Since in our example we know these values are (0,1), we know that $\mathbf{w}$ is completely determined. The solutions given clearly satisfy (b) and (c), and a direct calculation shows that they also satisfy (a).

26. Again, it is easy to check that the proposed solution $f(x) = x(n-x)$ satisfies conditions (a) and (b). The hard part is to prove that these equations have a unique solution. As in the case of Exercise 23, it is most instructive to consider this problem more generally. We have a special case of the following situation. Consider an absorbing Markov chain with transition matrix $\mathbf{P}$ in canonical form and with transient states T and absorbing states A. Let $\mathbf{f}$ and $\mathbf{g}$ be column vectors that satisfy the following system of equations

$$\begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{f}_A \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{g}_A \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_A \\ \mathbf{0} \end{pmatrix} \ ,$$

where $\mathbf{g}_A$ is given and it is desired to determine $\mathbf{f}_A$. In our example, $\mathbf{g}_A$ has all components equal to 1. To solve for $\mathbf{f}_A$ we note that these equations are equivalent to

$$\mathbf{Q}\mathbf{f}_A + \mathbf{g}_A = \mathbf{f}_A ,$$

or

$$(\mathbf{I} - \mathbf{Q})\mathbf{f}_A = \mathbf{g}_A .$$

Solving for $\mathbf{f}_A$, we obtain

$$\mathbf{f}_A = \mathbf{N}\mathbf{g}_A .$$

Thus $\mathbf{f}_A$ is uniquely determined by $\mathbf{g}_A$.

27. Use the solution to Exercise 24 with $\mathbf{w} = \mathbf{f}$.

28. Using the program **Absorbing Chain** for the transition matrix corresponding to the pattern HTH, we find that

$$\mathbf{t} = \begin{array}{c} HT \\ H \\ \emptyset \end{array} \begin{pmatrix} 6 \\ 8 \\ 10 \end{pmatrix} .$$

Thus $E(T) = 10$. For the pattern HHH the transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ HHH \\ HH \\ H \\ \emptyset \end{array} \begin{array}{c} \begin{array}{cccc} HHH & HH & H & \emptyset \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ .5 & 0 & 0 & .5 \\ 0 & .5 & 0 & .5 \\ 0 & 0 & .5 & .5 \end{pmatrix} \end{array} .$$

Solving for $\mathbf{t}$ for this matrix gives

$$\mathbf{t} = \begin{array}{c} HH \\ H \\ \emptyset \end{array} \begin{pmatrix} 8 \\ 12 \\ 14 \end{pmatrix} .$$

Thus for this pattern E(T) = 14.

29. For the chain with pattern HTH we have already verified that the conjecture is correct starting in HT. Assume that we start in H. Then the first player will win 8 with probability 1/4, so his expected winning is 2. Thus $E(T|H) = 10-2 = 8$, which is correct according to the results given in the solution to Exercise 28. The conjecture can be verified similarly for the chain HHH by comparing the results given by the conjecture with those given by the solution to Exercise 28.

30. T must be at least 3. Thus when you sum the terms

$$P(T > n) = 2P(T = n + 1) + 8P(T = n + 3),$$

the coefficients of the 2 and the 8 just add up to 1 since they are all possible probabilities for T. Let T be an integer-valued random variable. We write

$$\begin{aligned} E(T) &= P(T = 1) + P(T = 2) + P(T = 3) + \cdots \\ &\qquad\qquad\qquad\quad + P(T = 2) + P(T = 3) + \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad\; + P(T = 3) + \cdots \end{aligned}$$

If we add the terms by columns, we get the usual definition of expected value; if we add them by rows, we get the result that

$$E(T) = \sum_{n=0}^{\infty} P(T > n) \ .$$

That the order does not matter follows from the fact that all the terms in the sum are positive.

31. You can easily check that the proportion of G's in the state provides a harmonic function. Then by Exercise 27 the proportion at the starting state is equal to the expected value of the proportion in the final aborbing state. But the proportion of 1s in the absorbing state GG, GG is 1. In the other absorbing state gg, gg it is 0. Thus the expected final proportion is just the probability of ending up in state GG, GG. Therefore, the probability of ending up in GG, GG is the proportion of G genes in the starting state.(See Exercise 17.)

32. The states with all squares the same color are absorbing states. From any non-absorbing state it is possible to reach any absorbing state corresponding to a color still represented in the state. To see that the game is fair, consider the following argument. In order to decrease your fortune by 1 you must choose a red square and then choose a neighbor that is not red. With the same probability you could have chosen the neighbor and then the red square and your fortune would have been increased by 1. Since it is a fair game, if at any time a proportion p of the squares are red, for example, then p is also the probability that we end up with all red squares.

33. In each case Exercise 27 shows that

$$f(i) = b_{iN} f(N) + (1 - b_{iN}) f(0) \ .$$

Thus

$$b_{iN} = \frac{f(i) - f(0)}{f(N) - f(0)} \ .$$

Substituting the values of f in the two cases gives the desired results.