MVE051 2017 Lecture 11

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Generating function

Definition

Given a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$, the generating function of the sequence is defined as

$$g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- Generating function can be useful to solve many problems, as we will see.
- We will not be concerned too much with the issue of convergence.

Generating function

Examples of generating functions:

- (Geometric series) let $a_n = c^n$ for some constant c, then

$$g(x) = \sum_{n=0}^{\infty} c^n x^n = \sum_{n=0}^{\infty} (cx)^n = \frac{1}{1-cx}.$$

- Recall: the binomial coefficient is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \prod_{i=1}^{k} \frac{n-i+1}{i}$$

for all real numbers n and integers k. and the binomial theorem says that

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k b^{n-k}.$$

Thus

$$\sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n.$$

- Let $a_n = \binom{n+k}{k}$, then

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{(1-x)^{k+1}}$$

Operations on generating functions

Proposition (Addition + Multiplication by a constant)

Addition: *Let* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *be two sequence with corresponding generating functions* $A(x), B(x)$. *The sequence* $\{c_n\}_{n=0}^{\infty} = \{a_n + b_n\}_{n=0}^{\infty}$ *has generating function* $C(x) = A(x) + B(x)$.

Multiplication by a constant: *Moreover, if* p *is a constant, then the sequence* $\{d_n\}_{n=0}^{\infty} = \{pa_n\}_{n=0}^{\infty}$ *has generating function* $D(x) = pA(x)$

Proposition (Right shifting + Differentiation)

Right shifting: *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence with corresponding generating function* $A(x)$. *The sequence* $\{c_n\}_{n=0}^{\infty} = \{0, 0, ..., 0, a_0, a_1, a_2, ...\}$ *with* $k > 0$ *leading zeros has generating function* $C(x) = x^k A(x)$.

Differentiation: *Moreover, the sequence* $\{a_1, 2a_2, ..., na_n, ...\}$ *has generating function* $F(x) = A'(x)$.

Counting with generating function

Theorem (Convolution rule)

Let $A(x)$ denote the generating function for selecting items from a set A and $B(x)$ the generating function for selecting items from a set B, such that $A \cap B = \emptyset$. Then, the generating function for selecting items from $A \cup B$ is the product $A(x) \cdot B(x)$.

- Very useful!
- The reason why the rule holds lies in the way the product is computed.

Exponential generating function

Definition (Exponential generating function)

Given a sequence $\{a_n\}_{n=0}^{\infty}$ the function

$$E(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

is called exponential generating function for the sequence.

- if $a_n = 1, \forall n$ then $E(x) = e^x$.
- if $a_n = \mathbb{E}[X^n]$ are the moments of a random variable X, then $E(x) = m_X(t)$ is the moment generating function of X.

Characteristic function

- Given a random variable X the characteristic function ϕ_X is defined as
$$\phi_X(t) = \mathbb{E}[e^{itX}],$$
where $i = \sqrt{-1}$ is the imaginary unit.
- Example: if X is a discrete random variable and $a_n = \mathbb{E}[X^n]$ then
$$\phi_X(t) = \sum_{n=0}^{\infty} a_n \frac{(it)^n}{n!}$$
- Has similar properties to the moment generating function, but its definition ensure that it exists for any random variable X.

Chebychev's inequality

Also know as Chebysheff, Chebychov, Chebyshov, Tchebychev,Tchebycheff, Tschebyschev, Tschebyschef, Tschebyscheff...

Proposition (Chebychev's inequality)

Let X be a random variable such that $\mathbb{E}[X] = \mu, \mathrm{Var}(X) = \sigma^2$. If $0 < \sigma^2 < \infty$ then for any $k > 0$ it holds

$$P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$$

or equivalently for any $a > 0$

$$P[|X - \mu| \geq a] \leq \frac{\sigma^2}{a^2}$$