MVE051 2017 Lecture 4

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Joint distribution

- Up to now: univariate distribution $\rightarrow$ a single random value.
- Typically we need to consider multivariate distribution, used to model many uncertain values. In this way, we can take into account the dependencies between these quantities.
- We will now focus on bivariate distribution, but generalization to more variables is straightforward.

Joint density function/1

Definition (discrete joint density)

Let X, Y be two discrete random variables. The vector (X, Y) is a bivariate discrete random variable and a function f_{XY} which satisfies

$$f_{XY}(x, y) = \Pr[X = x, Y = y], \text{ for all}(x, y) \in \mathbb{R}^2$$

is called joint density for the vector (X, Y).

Theorem

A function $f(x, y)$ is a discrete joint density if and only if

- $f(x, y) \geq 0$
- $\sum_{all\ (x,y)} f(x, y) = 1$

Definition (discrete marginal density)

Let (X, Y) be a bivariate discrete random vector with joint density f_{XY}. The marginal density f_X for X is given by

$$f_X(x) = \sum_{\text{all } y} f_{XY}(x, y)$$

and similarly the marginal density for Y is

$$f_Y(y) = \sum_{\text{all } x} f_{XY}(x, y)$$

Joint density function/2

Definition (continuous joint density)

Let X, Y be two continuous random variables. The vector (X, Y) is a bivariate continuous random variable and a function f_{XY} which satisfies

- $f(x, y) \geq 0$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\ dy\ dx = 1$
- $\Pr[X \in [a, b], Y \in [c, d]] = \int_a^b \int_c^d f(x, y)\ dy\ dx,$ for all$a, b, c, d \in \mathbb{R}$ is called joint density for the vector (X, Y).

Furthermore, the marginal densities f_X and f_Y for, respectively, X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\ dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\ dx$$

Independence of random variable

- Recall: two events A, B are said to be independent if $\Pr[A \cap B] = \Pr[A]\Pr[B]$.

Definition (independence for random variables)

Two random variables X and Y with joint density f_{XY} and marginal densities f_X, f_Y are independent if and only if

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

for all x and y

Expected value

- In general, the expected value of a function of $H(X, Y)$ is given by
$$\mathbb{E}[H(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y) f_{XY}(x,y)\ dx\ dy$$
if X, Y are discrete and by
$$\mathbb{E}[H(X,Y)] = \sum_{\text{all }(x,y)} H(x,y) f_{XY}(x,y)$$
- Same properties as in the discrete case.
- If X, Y are independent then
$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$$
(the viceversa is not true in general).

Covariance

Definition (Covariance)

Let X and Y be two random variables. The covariance between X and Y is defined as

$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

It holds that

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

- If X, Y are independent $\rightarrow \mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] \rightarrow \mathrm{Cov}(X,Y) = 0$ (the viceversa is not true in general).
- $\mathrm{Cov}(X,Y)$ gives an indication of association between X and Y
- $\mathrm{Cov}(X,Y)$ can be any real value $\rightarrow$ no information about the strength of the dependence.

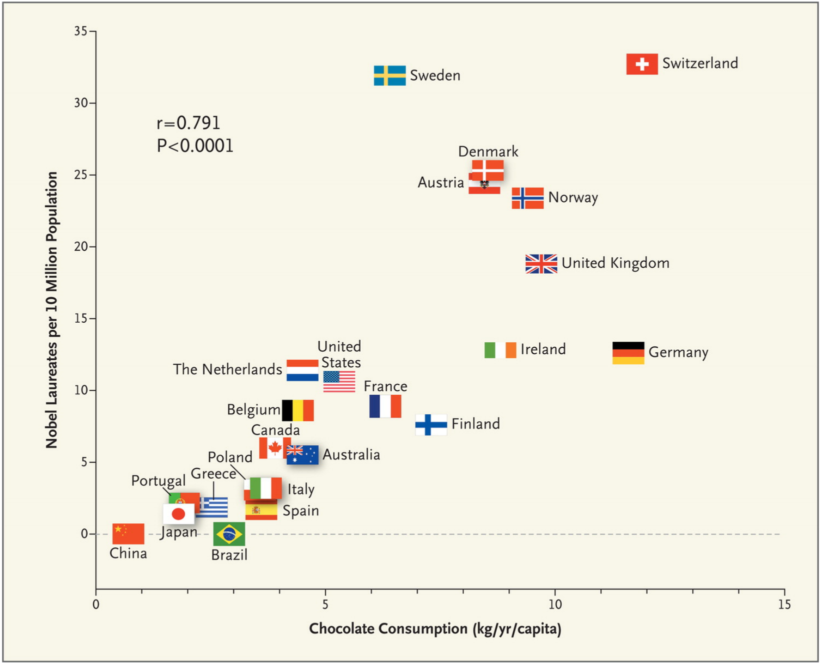
Correlation

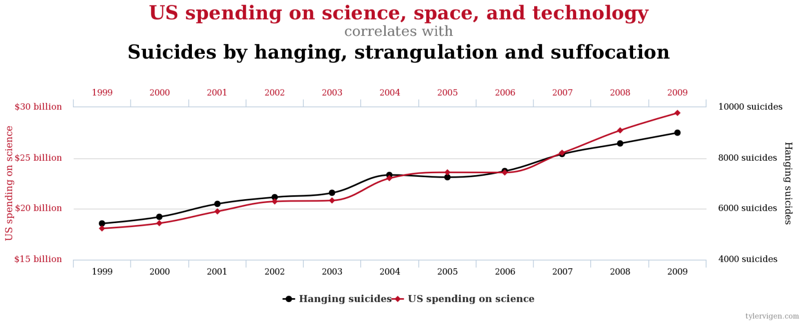
Definition (Correlation)

Let X and Y be two random variables. The correlation between X and Y is defined as

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}}.$$

- ρ_{XY} measures linear dependence between X and Y.
- ρ_{XY} can be any real value between -1 and 1.
- $|\rho_{XY}| = 1$ if and only if $Y = \beta_0 + \beta_1 X$ for some β_0 and $\beta_1 \neq 0$.


r=0.791
P<0.0001
Nobel Laureates per 10 Million Population
Chocolate Consumption (kg/yr/capita)
0
5
10
15
20
25
30
35
Sweden
Switzerland
Denmark
Austria
Norway
United Kingdom
Ireland
Germany
United States
The Netherlands
France
Belgium
Finland
Canada
Australia
Poland
Greece
Portugal
Italy
Spain
Japan
China
Brazil


US spending on science, space, and technology
correlates with
Suicides by hanging, strangulation and suffocation
1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009
$30 billion
$25 billion
$20 billion
$15 billion
US spending on science
10000 suicides
8000 suicides
6000 suicides
4000 suicides
Hanging suicides
Hanging suicides
US spending on science
tylervigen.com

Conditional density

- Recall: given two events A, B (if $\Pr[B] > 0$) we have $\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$.

Definition (conditional density)

Given two random variables X and Y with joint density f_{XY} and marginal densities f_X, f_Y we define the conditional density for X given $Y = y$ as

$$f_{X|y} = \frac{f_{XY}(x,y)}{f_Y(y)}, \text{ if } f_Y(y) > 0$$

Transformation of variables

Theorem

Let (X, Y) be a continuous bivariate vector with density f_{XY}. Moreover, let (U, V) be a continuous bivariate vector with density f_{UV} and

$$(X, Y) = (h_1(U, V), h_2(U, V))$$

where h_1 and h_2 define a one-to-one transformation and have continuous partial derivatives. Then

$$f_{UV}(u, v) = f_{XY}(h_1(u, v), h_2(u, v))|J|$$

where J is the given by

$$J = det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}.$$