Lösningar till tentamen i Matematisk statistik och diskret matematik D2 (MVE055/MSG810).
Den 28 augusti 2013. These are sketches of the solutions.

1. Lösning:
   Write
   $$\mathbb{P}(U=1,V=1)=\frac{1}{6}\,,\ \ \mathbb{P}(U=-1,V=1)=\frac{1}{3}$$
   $$\mathbb{P}(U=1,V=-1)=\frac{1}{3}\,,\ \ \mathbb{P}(U=-1,V=-1)=\frac{1}{6}$$
   a) x^2+Ux+V has a real root iff $U^2-4V\geq 0$ which means $V=-1$. Clearly, if $V=-1$, then $U^2-4V=5$. So the probability of a real root is $\frac{1}{2}$.
   b) The expected value of the larger root is
   $$\frac{(-1+\sqrt{5})}{2}\mathbb{P}(U=1|V=-1)+\frac{(1+\sqrt{5})}{2}\mathbb{P}(U=-1|V=-1)$$
   $$=\frac{1}{(\mathbb{P}(V=-1)}\Big(\frac{(-1+\sqrt{5})}{2}\frac{1}{3}+\frac{(1+\sqrt{5})}{2}\frac{1}{6}\Big)=\frac{\sqrt{5}}{2}-\frac{1}{6}.$$
   c) x^2+Wx+W has a real root if $W^2-4W\geq 0$. If $W=U+V$, W takes values $2,0,-2$, and the equation has a real root of $W=0$ or -2. Then $\mathbb{P}(W=0)=\frac{2}{3}$ and $\mathbb{P}(W=0 \text{ or } -2)=\frac{5}{6}$.

   Lösning:
2. a) The sample mean $\bar{X}$ has mean μ and variance $\frac{\sigma^2}{n}$. Hence, by Chebyshev's inequality
   $$\mathbb{P}(|\bar{X}-\mu|\geq 2\sigma)\leq\frac{\sigma^2}{n(2\sigma)^2}=\frac{1}{4n}.$$
   Thus $n=25$ is sufficient. If more is known about the distribution of X_i, then a smaller sample size may suffice.
   b) If $X_i\sim N(\mu,\sigma^2)$, than $\bar{X}-\mu\sim N(0,\frac{\sigma^2}{n})$, and probability $\mathbb{P}(|\bar{X}-\mu|\geq\sigma)$ equals
   $$\mathbb{P}\Big(\frac{|\bar{X}-\mu|}{\big(\frac{\sigma^2}{n}\big)^{\frac{1}{2}}}\geq\frac{\sigma}{\big(\frac{\sigma^2}{n}\big)^{\frac{1}{2}}}\Big)=\mathbb{P}(|Z|\geq n^{\frac{1}{2}})$$
   where $Z\sim N(0,1)$. But $\mathbb{P}(|Z|\geq 2.58)=0.99$, and so we require that $n^{\frac{1}{2}}\geq 2.58$, i.e. $n\geq 7$. As we see, knowledge that the distribution is normal allows a much smaller sample size, even to meet a tighter conditions.

3. Lösning:

   a) Label states by A, B, C indicating which player is not playing in a given game. Then the transition matrix is $\{A,B,C\}\times\{A,B,C\}$:
   $$\begin{pmatrix} 0 & \frac{s_C}{(s_B+s_C)} & \frac{s_B}{(s_B+s_C)}\\ \frac{s_C}{(s_A+s_C)} & 0 & \frac{s_A}{(s_A+s_C)}\\ \frac{s_B}{(s_A+s_B)} & \frac{s_A}{(s_A+s_B)} & 0\end{pmatrix}$$
   The process is a Markov chain because the results of the subsequent games are independent.

b) Here, we look for the probability that after three steps the chain returns to a given initial state.
From the symmetry, this probability is the same for any choice of the initial state and is equal to

$$p_{AB}p_{BC}p_{CA} + p_{AC}p_{CB}p_{BA} = \frac{2s_A s_B s_C}{(s_A + s_B)(s_B + s_C)(s_C + s_A)}$$

Lösning:

4. a) As an estimate of μ, we report the sample mean $\bar{T} = \frac{1}{10}\sum_{i=1}^{10} T_i = 1.80$. For a 90% confidence interval, we first obtain the precentile $z_{0.95} = 1.645$ and then compute the interval

$$\left[\bar{T} - z_{0.95}\sqrt{\frac{0.004}{10}},\ \bar{T} + z_{0.95}\sqrt{\frac{0.004}{10}}\right] = [1.767, 1.833].$$

b) We report $S^2 = (\frac{1}{9})\sum_{i=1}^{10}(T_i - 1.80)^2 = 0.0051$ as an estimate of σ^2. For a 90% confidence interval, we obtain percentiles

$$\chi^2_{9,0.95} = 16.919$$
$$\chi^2_{9,0.05} = 3.325.$$

We then compute the interval

$$\left[\frac{(n-1)S^2}{\chi^2_{9,0.95}},\ \frac{(n-1)S^2}{\chi^2_{9,0.05}}\right] = [0.0027, 0.0138].$$