Petter Mostad
Mathematical Statistics
Chalmers and GU

MVE055/MSG810 Mathematical statistics and discrete mathematics
MVE055/MSG810 Matematisk statistik och diskret matematik

Re-exam: 24 August 2016, 14:00 - 18:00
Jour: Roza Maghsood
Allowed aids: A Chalmers-approved calculator and at most one (double-sided) A4 page of personal notes. Some tables of statistical distributions are provided.
Grading: This exam can give a maximum of 30 points. For full points, each answer must be motivated. Points from VLE tests autumn 2015 will be added. To pass, a minimum of 12 points is needed.
Language: You may write your answers in either English or Swedish.

1. (4 points) Out of six machines, three have defects. They are tested in a random order.
   (a) Compute the probability that all three defect machines are found after four tests have been done.
   (b) Compute the probability that, first, two defect machines are found, then, two non-defect machines, and then the last defect machine is found.
2. (2 points) Assume one wants to find out what proportion of fish in a lake has contains levels of some pollutant above some threshold. Out of 23 tested fish, 9 had levels above the threshold. Compute a 95% confidence interval for the true proportion. (2 points)
3. (3 points) A set of 7 observations has average 7.43 and sample variance 0.039.
   (a) Assume the observations are a random sample from a normal distribution with known distribution variance 0.041. Compute a 95% confidence interval and a 99% confidence interval for the mean of the observation.
   (b) Assume now the observations are a random sample from a normal distribution with unknown distribution variance. Compute a 95% confidence interval for the mean of the observation.
4. (6 points) Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $S = 1, 2, 3, 4$ and transition matrix

$$\begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.1 & 0.9 \end{pmatrix}$$

   Assume the distribution over states at $t = 0$ is $(0.1, 0.3, 0, 0.6)$.

(a) Find the distribution over states at $t = 2$.

(b) What is the probability of the event $\{X_0 = 1, X_1 = 2, X_2 = 3\}$?

(c) Is the chain ergodic? Is it absorbing?

(d) Conditionally on $X_1 = 1$, what is the probability that $X_0 = 2$?

5. (4 points) Assume $x_1, \ldots, x_n$ is a random sample from a Poisson random variable with parameter λ. Derive the formula for the Maximum Likelihood estimator for λ given the sample. Is the ML estimator biased or unbiased in this case?

6. (4 points) Define a random variable X with the following probability density function on the interval $[0, 1]$:

$$f(x) = \begin{cases} 0 \le x \le \theta & ax/\theta \\ \theta \le x \le 1 & a(1-x)/(1-\theta) \end{cases}$$

where θ is some parameter and a is a constant.

(a) Compute a.

(b) Find the expectation of X as a function of the parameter θ.

7. (3 points) Assume a random variable X has a uniform distribution on the interval $[0, \pi]$. Compute $\mathbf{E}(\cos X)$, $\mathbf{Var}(\cos X)$, $\mathbf{E}(\sin X)$, and $\mathbf{Var}(\sin X)$.

8. (4 points) A random variable X with the *standard Cauchy distribution* is defined with the density function

$$f(x) = \frac{1}{\pi(1 + x^2)}$$

(a) Prove that the expectation of X does not exist.

(b) For any $n \ge 1$, assume $X_1, \ldots, X_n$ is a random sample from the standard Cauchy distribution and define define $Y_n = (X_1 + X_2 + \cdots + X_n)/n$. The figure below shows four histograms of samples of size 1000 of Y_n, for values $n = 10$, $n = 100$, $n = 1000$ and $n = 10000$. Describe what the histograms would have looked like if we instead of the standard Cauchy distribution had started with a different distribution, say a Beta distribution or a uniform distribution. Why do the histograms look different when we use the Cauchy distribution?