

MVE055 2017 Lecture 11

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Definition

Given a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$, the generating function of the sequence is defined as

$$g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- Generating function can be useful to solve many problems, as we will see.
- We will not be concerned too much with the issue of convergence.

Generating function

Examples of generating functions:

- (Geometric series) let $a_n = c^n$ for some constant c , then

$$g(x) = \sum_{n=0}^{\infty} c^n x^n = \sum_{n=0}^{\infty} (cx)^n = \frac{1}{1 - cx}.$$

- Recall: the binomial coefficient is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \prod_{i=1}^k \frac{n-i+1}{i}$$

for all real numbers n and integers k . and the binomial theorem says that

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k b^{n-k}.$$

Thus

$$\sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n.$$

- Let $a_n = \binom{n+k}{k}$, then

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{(1-x)^{k+1}}$$

Operations on generating functions

Proposition (Addition + Multiplication by a constant)

Addition: Let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ be two sequence with corresponding generating functions $A(x), B(x)$. The sequence $\{c_n\}_{n=0}^{\infty} = \{a_n + b_n\}_{n=0}^{\infty}$ has generating function $C(x) = A(x) + B(x)$.

Multiplication by a constant: Moreover, if p is a constant, then the sequence $\{d_n\}_{n=0}^{\infty} = \{pa_n\}_{n=0}^{\infty}$ has generating function $D(x) = pA(x)$

Proposition (Right shifting + Differentiation)

Right shifting: Let $\{a_n\}_{n=0}^{\infty}$ be a sequence with corresponding generating function $A(x)$. The sequence $\{c_n\}_{n=0}^{\infty} = \{0, 0, \dots, 0, a_0, a_1, a_2, \dots\}$ with $k > 0$ leading zeros has generating function $C(x) = x^k A(x)$.

Differentiation: Moreover, the sequence $\{a_1, 2a_2, \dots, na_n, \dots\}$ has generating function $F(x) = A'(x)$.

Theorem (Convolution rule)

Let $A(x)$ denote the generating function for selecting items from a set A and $B(x)$ the generating function for selecting items from a set B , such that $A \cap B = \emptyset$. Then, the generating function for selecting items from $A \cup B$ is the product $A(x) \cdot B(x)$.

- Very useful!
- The reason why the rule holds lies in the way the product is computed.

Exponential generating function

Definition (Exponential generating function)

Given a sequence $\{a_n\}_{n=0}^{\infty}$ the function

$$E(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

is called exponential generating function for the sequence.

- if $a_n = 1, \forall n$ then $E(x) = e^x$.
- if $a_n = \mathbb{E}[X^n]$ are the moments of a random variable X , then $E(x) = m_X(t)$ is the moment generating function of X .

Characteristic function

- Given a random variable X the characteristic function ϕ_X is defined as

$$\phi_X(t) = \mathbb{E}[e^{itX}],$$

where $i = \sqrt{-1}$ is the imaginary unit.

- Example: if X is a discrete random variable and $a_n = \mathbb{E}[X^n]$ then

$$\phi_X(t) = \sum_{n=0}^{\infty} a_n \frac{(it)^n}{n!}$$

- Has similar properties to the moment generating function, but its definition ensure that it exists for any random variable X .

Chebychev's inequality

Also known as Chebyshev, Chebyshev, Chebyshev, Tchebychev, Tchebycheff, Tschebyschev, Tschebyschef, Tschebyscheff...

Proposition (Chebychev's inequality)

Let X be a random variable such that $\mathbb{E}[X] = \mu$, $\text{Var}(X) = \sigma^2$. If $0 < \sigma^2 < \infty$ then for any $k > 0$ it holds

$$P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$$

or equivalently for any $a > 0$

$$P[|X - \mu| \geq a] \leq \frac{\sigma^2}{a^2}$$