MVE055 2017 Lecture 11

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Chebychev's inequality

Also know as Chebysheff, Chebychov, Chebyshov, Tchebychev,Tchebycheff, Tschebyschev, Tschebyschef, Tschebyscheff...

Proposition (Chebychev's inequality)

Let X be a random variable such that $\mathbb{E}[X] = \mu, \mathrm{Var}(X) = \sigma^2$. If $0 < \sigma^2 < \infty$ then for any $k > 0$ it holds

$$P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$$

or equivalently for any $a > 0$

$$P[|X - \mu| \geq a] \leq \frac{\sigma^2}{a^2}$$

Law of Large Numbers

Theorem ((Weak) Law of Large Numbers)

Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d) random variables with mean $\mu < \infty$ and variance $0 < \sigma^2 < \infty$. Denote by $S_n = X_1, ..., X_n$ the sum of the n random variables. Then for any $\epsilon > 0$

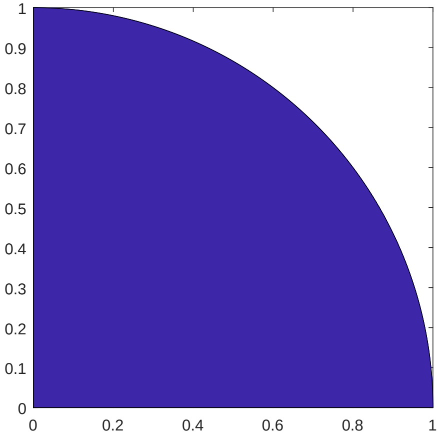
$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \rightarrow 0, \quad as\ n \rightarrow \infty$$

.

- Called "weak" to distinguish it from the "strong" law of large numbers.
- It is NOT valid if the variance is not finite!

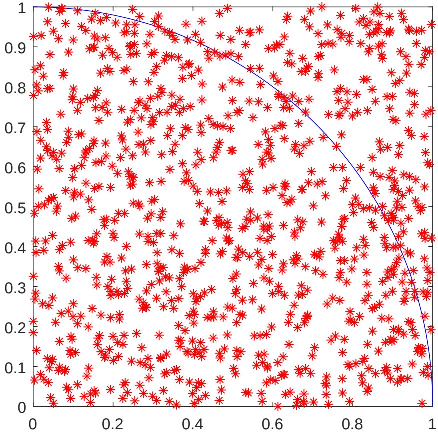
Monte Carlo Method

Problem: how to approximate the area shaded in blue in the figure (which is $\frac{\pi}{4} = 0.7854$:



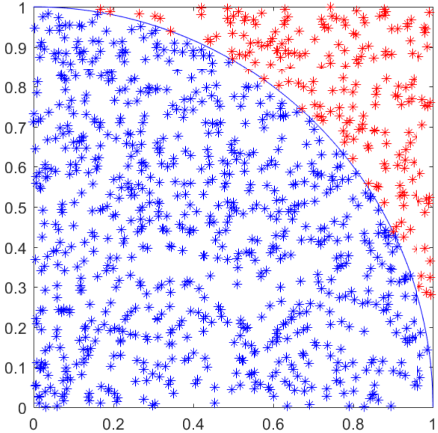
Monte Carlo Method

Idea: Generate points $X_1, ..., X_n$ uniformly in the square $[0,1] \times [0,1]$ for some n and count



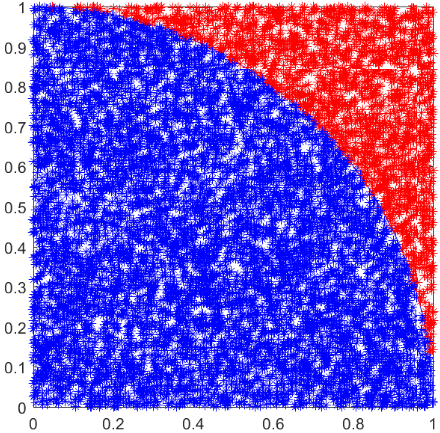
Monte Carlo Method

$n = 1000$ points, estimated area $= 0.7980$



Monte Carlo Method

$n = 10000$ points, estimated area $= 0.7909$



Monte Carlo Method

Assume we want to approximate the area of a set B.

- Generate n independent and uniformly distributed points $X_1, ..., X_n$ in a set A, such that $B \subset A$.
- Count how many of points $X_1, \ldots, X_n$ falls in B. That is, define $Y_i,\ i = 1, ..., n$ by

$$Y_i = \begin{cases} 1, & \text{if } X_i \in B \\ 0, & \text{otherwise} \end{cases}$$

- It follows that $P(Y_i = 1) = P(X_i \in B) = \frac{Area(B)}{Area(A)}$, and $\mathbb{E}[Y_i] = P(Y_i = 1) = \frac{Area(B)}{Area(A)}$.

Monte Carlo Method

- Define $S_n = Y_1 + \cdots + Y_n$. The law of large numbers applied to the Y_i ensure

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \to 0, \quad \text{as } n \to \infty$$

  where $\mu = \mathbb{E}[Y_i] = \frac{Area(B)}{Area(A)}$

- $\frac{S_n}{n}$ should be approximately $\frac{Area(B)}{Area(A)}$.
- Approximate $Area(B) = Area(A)\frac{S_n}{n}$.