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Applied Mathematics and Statistics
Chalmers and GU

MVE055 / MSG810 Matematisk statistik och diskret matematik

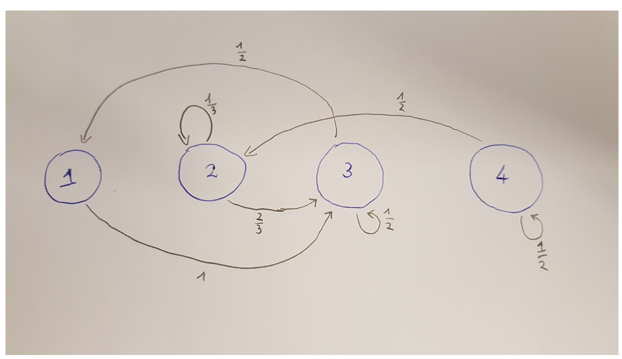
Exam 19 December 2017, 14:00 - 18:00

Allowed aids: Chalmers-approved calculator
and one (two-sided) A4 sheet of paper with your own notes.
Total number of points: 30. To pass, at least 12 points are needed.
Note: All answers should be motivated.

1 Solutions

1. (a) P_1 is not a transition matrix as one of its element is negative, thus it cannot represent a transition probability. P_3 has the second row which sums up to 2, thus it cannot be a transition matrix. P_2 is the transition matrix of a Markov Chain as the rows sums up to 1 and all its elements are between 0 and 1.

   (b) P_2 is the transition matrix of a Markov chain with four transient states as summarized in the following figure



   (c) The distribution of the chain after two steps is given by

$$\pi P_2^2 = [0.2, 0.1, 0.45, 0.25] \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = [0.2458, 0.1153, 0.5764, 0.0625]$$

2. $P(A) = 0.6, P(B) = 0.5$.

(a) Assuming that $P(A \cap B) = 0.05$ we obtain

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.5 - 0.05 = 1.05$$

which can't be as a probability is always smaller or equal to 1.

(b) Using the independence of A and B and DeMorgan's law:

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) = \\ &= 1 - P(A) - P(B) + P(A)P(B) = 1 - P(A) - P(B)[1 - P(A)] = \\ &= (1 - P(A))(1 - P(B)) = P(A^c)P(B^c) \end{aligned} \tag{1}$$

that is, we can factorize the probability of the intersection of A^c, B^c, thus they are independent.

3. X has exponential distribution with parameter $\lambda_X = 1$ and Y has exponential distribution with parameter $\lambda_Y = 2$.

(a) We can describe Z as the minimum between the waiting time for an earthquake in any of the two emispheres, that is $Z = \min\{X, Y\}$. Using the inpendence of X and Y and the fact that they are exponentially distributed

$$\begin{aligned} F_Z(z) = Prob(Z \leq z) &= 1 - Prob(Z > z) = 1 - Prob(\min\{X, Y\} > z) = 1 - Prob(X > z, Y > z) = \\ &= 1 - Prob(X > z)Prob(Y > z) = 1 - e^{-z}e^{-2z} = 1 - e^{-3z}. \end{aligned} \tag{2}$$

Thus, we can conclude that Z has exponential distribution with parameter $\lambda_Z = 3$.

(b) $P(Z > 1) = 1 - F_Z(1) = e^{-3}$

4. (a) $m_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(a+bX)}] = e^{at}\mathbb{E}[e^{(bt)X}] = e^{at}m_X(bt)$

(b) By definition, the characteristic function of a random variable X satisfy $\phi_X(0) = 1$. Thus, it cannot be possible that $\phi_X(t) = 2$ for all t.

(c) $\phi_X(t) = 1$ for all t is the characteristic function of a random variable X such that $P(X = 0) = 1$, thus $P(X = 5) = 0$.

5. We have a sample size $n = 13$ with sample mean $\overline{X} = 197$ and sample standard deviation $s = 5.6$.

(a) A 95% confidence interval for σ^2 is given by

$$\left[(n-1)\frac{s^2}{\chi^2_{\frac{\alpha}{2}}}, (n-1)\frac{s^2}{\chi^2_{1-\frac{\alpha}{2}}}\right]$$

where $\alpha = 0.05$, which gives us

$$\left[12\frac{31.36}{23.3}, 12\frac{31.36}{4.4}\right] = [16.15, 85.53]$$

(b) If we use all available information, since the two brands P and Q have the same variance σ^2, we can estimate σ^2 by the pooled variance

$$S_p^2 = \frac{12 \cdot 5.6^2 + 6 \cdot 4.1^2}{13 + 7 - 2} = 26.51$$

and the 95% confidence interval is

$$\left[(13 + 7 - 1)\frac{S_p^2}{\chi^2_{\frac{\alpha}{2}}}, (13 + 7 - 1)\frac{S_p^2}{\chi^2_{1-\frac{\alpha}{2}}}\right] = [15.31, 56.53]$$

(c) To compute the confidence interval for the difference in the expected weights of the contents of brands P and Q we use the pooled variance S_p^2 and

$$\left[(\overline{X}_Q - \overline{X}_P) - t_{\alpha/2}\sqrt{S_p^2\left(\frac{1}{n_P} + \frac{1}{n_Q}\right)}\right] = \left[6 - 2.101\sqrt{26.51 \cdot 0.22}, 6 + 2.101\sqrt{26.51 \cdot 0.22}\right] = [3.59, 8.41]$$

where $n_P = 13, n_Q = 7, \overline{X}_P = 197, \overline{X}_Q = 203$.

(d) To test if the expected weight of brand Q is greater than the same of brand P we consider the test

$$H_0 : \mu_Q = \mu_P$$
$$H_1 : \mu_Q > \mu_P$$

where μ_Q and μ_P denote the expected weights of the contents of brands Q and P respectively. We consider the test statistic

$$\frac{\overline{X}_Q - \overline{X}_P}{\sqrt{S_p^2\left(\frac{1}{n_P} + \frac{1}{n_Q}\right)}} \sim T_{n_P+n_Q-2}$$

In our case we obtain a value of the statistic T_{18} of 2.486, and the critical value at the 5% significance level is 1.734. Thus, we conclude that the expected weight of brand Q is in fact higher than that of brand P. When formulating the test, we assumed that the variance of the weight of the two brands P and Q was the same (as we used the pooled variance) and the distribution of the weights was normal. Moreover, we assumed the samples to be independent.

6.

$$0.7413 = P(z \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq z),$$

thus solving the equation with respect to $P(Z \leq z)$ and looking in the Table for $P(Z \leq 1) = 0.8413$ we obtain

$$P(Z \leq z) = P(Z \leq 1) - 0.7413 = 0.8413 - 0.7413 = 0.1$$

So we find z from the Table as the value such that $P(Z \leq z) = 0.1$, which gives us approximately $z = -1.28$.