1 Mock exam

Grading: The total number of points for the exam is 30, and you need minimum of 12 points to pass the exams. To obtain full points, the answer need to be well motivated. Grades: 3 -[12,17], 4 - [18,24], 5 - [25-30].

1. **(5 points)** Let X be a continuous random variable with density function

$$f_X(x) = \begin{cases} |x|, & \text{for } x \in (-1,1) \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

   and let Y be independent of X and with distribution $Y \sim exp(1)$.

   (a) Compute $\mathbb{E}[X], \mathbb{E}[X^2]$;

   (b) Compute $\mathbb{P}(X \leq 0.3, Y \leq 1)$;

   (c) Let $W = X + Y$, compute $\mathbb{E}[W], \mathrm{Var}[W]$.

2. **(4 points)** Let $X_1, \dots, X_{100}$ be independent and identically distributed normal random variables with mean $\mu = 3$ and variance $\sigma^2 = 1$:

   (a) Compute $\mathbb{P}(X_1 + \cdots + X_{100} \leq 310)$;

   (b) find the smallest integer n such that

$$\mathbb{P}(\sum_{i=1}^{n} X_i \leq 310) \leq 0.975.$$

3. **(5 points)** Two discrete random variables (X, Y) have joint density $f_{XY}(x, y)$ given by the following table

X/Y	-1	0	2	6
-2	1/9	1/27	1/27	1/9
0	2/9	0	1/9	1/9
3	0	0	1/9	4/27

   (a) Compute the marginal density of X and Y. Are X and Y independent?

   (b) Find a couple of random variables $(\hat{X}, \hat{Y})$ with the same marginals of (X, Y) and such that its components are independent.

   (c) Compute the conditional density of X given $Y = 6$.

4. **(5 points)** Let $X_1, ..., X_{10}$ be 10 observations with distribution $N(\mu, \sigma^2)$ such that the sample mean is $\bar{X} = 5.2$ and the sample variance is $s^2 = 4$.

(a) Compute a 95% confidence interval for μ

(b) How many observation are needed to approximately halve the length of the 95% confidence interval?

(c) If the variance is known to be $\sigma^2 = 10$, what is a 95% confidence interval for the mean?

5. **(4 points)** Consider a square with vertices A, B, C, D and suppose that, at time 0, Alice is standing at vertex A and Bob in vertex C (i.e. on opposite vertices of the square). At each time step, both Alice and Bob moves to a vertex adjacent to the one where they are standing independently of each other and with the same probability. Let D_n denote the distance along the square between Alice and Bob. For example, $D_0 = 2$ as Bob would need two moves to reach Alice. They continue moving until they meet each other, i.e. $D_{\hat{n}} = 0$ and $D_n > 0$ for all $n < \hat{n}$.

   (a) Show that D_n is Markov chain;

   (b) Find the transition matrix of $\{D_n\}$.

   (c) Compute the expected time before Alice and Bob end up in the same position.

6. **(7 points)** Recall that for a random variable X, the characteristic function ϕ_X is defined as $\phi_X(t) = \mathbb{E}[e^{itX}]$.

   (a) Is the function $\phi(t) = sin(t)$ the characteristic function of a random variable? Explain your answer. (Hint: compute $\phi(0)$)

   (b) Is the function $\phi(t) = cos(t) = \frac{e^{it}+e^{-it}}{2}$ the characteristic function of a random variable?

   (c) Prove that if $X_1, \ldots, X_n$ are independent and identically distributed random variables with characteristic function $\phi_{X_i}(t) = \phi_X(t)$, for all $i = 1, ..., n$, then the characteristic function of $Y = X_1 + \ldots + X_n$ is given by

$$\phi_Y(t) = (\phi_X(t))^n.$$

   (d) Using (b) and (c), describe the random variable Z such that $\phi_Z(t) = (cos(t))^4$.

2 Solutions

1. (a) $\mathbb{E}[X] = 0$ as we are integrating an odd function over the symmetric interval $[-1, 1]$.

$$\mathbb{E}[X^2] = \int_{-1}^{1} x^2|x| \, dx = -\int_{-1}^{0} x^3 \, dx + \int_{0}^{1} x^3 \, dx = \frac{1}{2}$$

   (b)

$$P(X \leq 0.3) = 1 - P(X > 0.3) = 1 - \int_{0.3}^{1} x \, dx = \frac{109}{200}$$

Y has exponential distribution with parameter 1, so we have $P(Y \leq y) = F(y) = \begin{cases} 1 - e^{-y}, & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$ X and Y are independent, so their joint distribution is given by the product of their distribution functions

$$P(X \leq 0.3, Y \leq 1) = P(X \leq 0.3)P(Y \leq 1) = \frac{109}{200}(1 - e^{-1}).$$

(c) $W = X + Y$ and independence of X and Y imply

$$\mathbb{E}[W] = \mathbb{E}[X] + \mathbb{E}[Y] = 1$$

$$\mathrm{Var}(W) = \mathrm{Var}(X) + \mathrm{Var}(Y) = \frac{3}{2}.$$

2. In general, if we have independent and identically distributed random variables $X_1, \dots, X_n \sim N(3,1)$, then $\frac{\sum_{i=1}^n X_i - 3n}{\sqrt{n}} \sim N(0,1)$

(a)

$$P\left(\sum_{i=1}^{100} X_i \leq 310\right) = P\left(\frac{\sum_{i=1}^{100} X_i - 300}{\sqrt{100}} \leq 1\right) = P(N(0,1) \leq 1) = 0.8413$$

(b)

$$0.975 \geq P(\sum_{i=1}^{n} X_i \leq 310) = P\left(\frac{\sum_{i=1}^{n} X_i - 3n}{\sqrt{n}} \leq \frac{310 - 3n}{\sqrt{n}}\right) = P\left(N(0,1) \leq \frac{310 - 3n}{\sqrt{n}}\right)$$

To ensure that the above inequality holds, we need that $\frac{310-3n}{\sqrt{n}} \leq 1.96$ or equivalently

$$310 - 3n \leq 1.96\sqrt{n}.$$

By using the substitution $t = \sqrt{n}$ we end up with a second order equation in t. This will lead to two solutions, but we need to disregard the negative one due to the fact that $t = \sqrt{n}$ has to be positive. Thus, we obtain $t \geq 9.8439$ or $n \geq 96.9$, i.e. the smallest integer n to satisfy the inequality is $n = 97$.

3. (a) The marginals are summarised in the table below

X/Y	-1	0	2	6	
-2	1/9	1/27	1/27	1/9	8/27
0	2/9	0	1/9	1/9	4/9
3	0	0	1/9	4/27	7/27
	3/9	1/27	7/27	10/27	

X and Y are not independent as, for example, $f_{XY}(3, -1) = 0 \neq f_X(3)f_y(-1)$.

(b) It is enough to consider $\hat{X}$ with density f_X (as above), $\hat{Y}$ with density f_Y and $\hat{X}, \hat{Y}$ independent. The vector $(\hat{X}, \hat{Y})$ has the joint density

X/Y	-1	0	2	6
-2	$\frac{8}{27}\frac{3}{9}$	$\frac{8}{27}\frac{1}{27}$	$\frac{8}{27}\frac{7}{27}$	$\frac{8}{27}\frac{10}{27}$
0	$\frac{4}{9}\frac{3}{9}$	$\frac{4}{9}\frac{1}{27}$	$\frac{4}{9}\frac{7}{27}$	$\frac{4}{9}\frac{10}{27}$
3	$\frac{7}{27}\frac{3}{9}$	$\frac{7}{27}\frac{1}{27}$	$\frac{7}{27}\frac{7}{27}$	$\frac{7}{27}\frac{10}{27}$

(c) $f_{X|y=6} = \frac{f_{XY}(x,6)}{f_Y(6)} = \begin{cases} \frac{3}{10}, & x = -2 \\ \frac{3}{10}, & x = 0 \\ \frac{4}{10}, & x = 3 \\ 0, & \text{otherwise} \end{cases}$

4. $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, $\bar{X} = 5.2, s^2 = 4$

(a) as the variance is unknown, the 95% confidence interval for the mean μ is given by

$$CI = \left[\bar{X} - t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}, \bar{X} + t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}\right]$$

where $\alpha = 0.05$ and $t_{\frac{\alpha}{2}} = 2.262$. Thus

$$CI = [3.7694, 6.6306].$$

(b) The confidence interval has length $L = 2t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$. We want a confidence interval with length $L' = 2t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n'}}$ such that $L' = \frac{L}{2}$. Thus we obtain the system

$$\begin{cases} L' &= \frac{L}{2} = t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \\ L' &= 2t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n'}} \end{cases} \tag{2}$$

which has the solution $n' = 4n$. So to approximately halve the length of the confidence interval we need four times the number of observations.

(c) if $\sigma^2 = 10$ is known, then the 95% confidence interval is given by

$$\left[\bar{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right] = [3.24, 7.16]$$

5. (a) The possible values D_n can assume are 0 and 2, as Alice and Bob starts at distance 2 at time 0 and they can either move to the same vertex, or end up in opposite vertices. Alice and Bob move independently of each other and always with the same probability $\frac{1}{2}$ to any of the adjacent vertices. D_n depends only on the vertices on which Alice and Bob are standing at time n (i.e. either

they are on the same vertex or on opposite ones). Thus, the state of D_{n+1} depends only on the vertices on which Alice and Bob were standing at time n and how they moved from n to $n+1$ (and as said, this move is independent of the past). We conclude that D_n is a Markov chain.

(b) The transition matrix (in canonical form) is given by

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \tag{3}$$

(c) Let t_i denote the average time needed to reach for the first time $D_n = i$ if the starting state is $D_0 = 2$. Then we have the following system

$$\begin{cases} t_0 & = & 0 \\ t_2 & = & 1 + \frac{1}{2}t_2 \end{cases} \tag{4}$$

which gives the solution $t_2 = 2$.

6. (a) In general we have that a characteristic function satisfy $\phi_X(0) = \mathbb{E}[e^{i0X}] = \mathbb{E}[e^0] = \mathbb{E}[1] = 1$ by definition of characteristic function. $\phi(t) = sin(t)$ does not satisfy the above property, as $\phi(0) = 0$, so it is not a characteristic function of any random variable.

(b) $\phi(t) = \frac{e^{it}+e^{-it}}{2} = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$. For a discrete random variable X we have

$$\phi_X(t) = \sum_{\text{all } x} e^{itx} P[X = x].$$

From the above formula for $\phi(t)$ we can conclude that it is the characteristic function of a discrete random variable X with the following density function

$$P[X = x] = \begin{cases} \frac{1}{2}, & x \in \{-1, 1\} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

(c) Using independence (to express the expected value of the product as the product of the expected values) we have

$$\phi_Y(t) = \mathbb{E}\left[e^{itY}\right] = \mathbb{E}\left[e^{it\sum_{j=1}^n X_j}\right] = \mathbb{E}\left[\prod_{j=1}^n e^{itX_j}\right] = \prod_{j=1}^n \mathbb{E}\left[e^{itX_j}\right] = \prod_{j=1}^n \phi_{X_j}(t) = \phi_X(t)^n$$

(d) Z is the sum of 4 independent random variables $X_1, ..., X_4$ with distribution as in point (b).