Random Processes with Application 2007/2008

Solutions to optional home work 1

1. a) 80% of 12 = 9.6 $\Rightarrow \boxed{10}$ channels needed.

   b) X = the number of active subscribers.
   $X \sim \text{Bin}(12, 1/5)$
   We look for the minimal n such that
   $P_r\{X \leq n\} \geq 0.8$
   Computations show that
   $P\{X \leq 3\} = 0.7945, \quad P\{X = 4\} = 0.1328 \Rightarrow \boxed{n = 4}$

   c) P{X=5}=0.0531$\Rightarrow \boxed{n = 5}$

2. X = the # of loads in operation; $Y = 10X$
   $\downarrow$ the required power

   $X \sim \text{Bin}(5, 0.25)$

   a) $E[Y] = 10E[X] = 10 * 5 * 0.25 = \boxed{12.5w}$

   b) $\text{Var}(Y) = 100 \ \text{Var}(X) = 100 * 5 * 0.025 * 0.75 = \boxed{93.75w^2}$

   c) $P\{Y > 40\} = P\{X > 4\} = P\{X = 5\} = (0.25)^5 = \boxed{9.77 * 10^{-4}}$

3. $\mathcal{N}$ - the noise; $\mathcal{N} \sim \mathcal{N}(0, 1)$

   a) $P\{\text{false alarm}\} = P\{N > 5\} = Q(5) = \boxed{2.87 * 10^{-7}}$

   b) $P\{\text{detection}\} = P\{N + 8 > 5\} = Q(-3) =$
   $= 1 - Q(3) = 1 - 1.35 * 10^{-3} = \boxed{0.9986}$

   c)

$$f_N(x|N > a) = \frac{f_N(x)}{Q(a)}, \quad x > a$$

$$E[N|N > a] = \frac{1}{Q(a)} \int_a^\infty \frac{1}{\sqrt{2\pi}} x e^{-x^2/2} dx$$

$$= \frac{1}{Q(a)\sqrt{2\pi}} \Big[- e^{-x^2/2} \Big|_a^\infty \Big] = \boxed{\frac{e^{-a^2/2}}{Q(a)\sqrt{2\pi}}}$$

   <u>$a = 5$</u>:
   $E[N|N > a] = \boxed{5.18}$

d) We use part c)

$$E[N+8|N+8>5\} = E[N|N>-3]+8$$
$$= \frac{e^{-9/2}}{(1-Q(3))\sqrt{2\pi}} + 8 = \boxed{4.44 * 10^{-3} + 8}$$

4. $X_i \sim \mathcal{N}(0,\sigma^2), \;\; i = 1,2, \;$ indep.

$$V = \sqrt{X_1^2 + X_2^2} = \sigma \underbrace{\sqrt{(X_1/\sigma)^2 + (X_2/\sigma)^2}}_{R \text{ - the Reyleigh random variable}} = \sigma R$$

$f_R(r) = re^{-r^2/2}, \;\; r > 0$ (see p. 230 of the book)

a) $f_R'(r) = e^{-r^2/2}(r - r^2) = 0 \Rightarrow r = 1.$
The most probable value of R is $r = 1$.
Thus the most probable value of V is $\boxed{v_0 = \sigma}$.

b)

$$E[V] = \sigma E[R] = \sigma \int_0^\infty r^2 e^{-r^2/2} dr$$

$$= \sigma \Big[\underbrace{-re^{-r^2/2}\Big|_0^\infty}_{=0} + \underbrace{\int_0^\infty e^{-r^2/2} dr}_{\sqrt{2\pi}/2} \Big] = \boxed{\sigma\sqrt{\frac{\pi}{2}}}$$

$\downarrow$

$\boxed{\sigma = 4 : E[V] = 5.01}$

c) $\underline{\sigma = 4}$: $P\{V > 10\} = P_r\{R > 2.5\}$

$$= \int_{2.5}^\infty re^{-r^2/2} dr = -e^{-r^2/2}\Big|_{2.5}^\infty = e^{-6.25/2} = \boxed{4.39 * 10^{-2}}$$

5.

$$Z(t) = \cos(100t + \Theta) + \cos(100t + \Psi)$$
$$= \underbrace{2\cos\frac{\Theta - \Psi}{2}}_{A} \cos(100t + \underbrace{\frac{\Theta + \Psi}{2}}_{\Phi})$$

a)

$$P\{A > 1\} = P\{\cos\frac{\Theta - \Psi}{2} > 0.5\}$$
$$= \Big\{ \underbrace{\Big|\frac{\Theta - \Psi}{2}\Big|}_{\pi W} < \underbrace{\arccos 0.5}_{\pi/3} \Big\} = P\{|W| < 1/3\},$$

where $W = \frac{\Theta - \Psi}{2\pi}$ is the triangular r.v. with

$$f_W(w) = \begin{cases} w+1, & -1 \le w \le 0 \\ 1-w, & 0 < w \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Thus

$$P\{A > 1\} = \int_{-1/3}^{1/3} f_W(w)dw = \boxed{\frac{5}{9}}$$

b) $P\{A < 0.5\} = \int_{-1}^{0.5} f_W(w)dw = \boxed{0.34}$