RANDOM PROCESSES WITH APPLICATIONS 2007

Solution to Optional home work 2

Day assigned: **September 30**

Assignment deadline: **11:45 am, October 12**

Moved deadline: **11:45 am, October 19**

Problem 1. Consider two random variables defined as linear combinations of other random variables,

$$Y = \sum_{1}^{n} a_i Y_i, \quad Z = \sum_{1}^{m} b_j Z_j$$

.

a) Show that
$$Cov(Y,\, Z) = \sum_i \sum_j a_i b_j Cov(Y_i,\, Z_j)(1)$$
(1)

Next, let $X_1,\, X_2,\, ...\, X_n$ be independent observations on the random variable $X \sim N(\mu,\, \sigma^2)$. Consider the sample mean and variance

$$\hat{\mu} = \frac{1}{n}\sum_{1}^{n} X_i, \quad s^2 = \frac{1}{n-1}\sum_{1}^{n}(X_i - \hat{\mu})^2.$$

b) Use a) to show that the sample mean $\hat{\mu}$ and $X_i - \hat{\mu}$ are uncorrelated, $i = 1,\, 2,\, ...,\, n$. (1)

c) Use (b) to show that the sample mean and the sample variance are independent. (1)

Solution

a)

$$\begin{aligned}
Cov(Y,\, Z) = E\left[\sum_1^n a_i Y_i \sum_1^m b_j Z_j\right] - E\left[\sum_1^n a_i Y_i\right] E\left[\sum_1^m b_j Z_j\right] \\
\sum_1^n \sum_1^m a_i b_j E[Y_i Z_j] - \sum_1^n a_i E[Y_i] \sum_1^m b_j E[Z_j] \\
\sum_1^n \sum_1^m a_i b_j E[Y_i Z_j] - \sum_1^n \sum_1^m a_i b_j E[Y_i] E[Z_j] \\
\sum_1^n \sum_1^m a_i b_j \left[E[Y_i Z_j] - E[Y_i]E[Z_j]\right] = \sum_1^n \sum_1^m a_i b_j Cov(Y_i,\, Z_j).
\end{aligned} \tag{1}$$

b)

$$\begin{aligned}
Cov(\hat{\mu},\, X_i - \hat{\mu}) = E[\hat{\mu}(X_i - \hat{\mu})] - E[\hat{\mu}]E[X_i - \hat{\mu}] \\
= E[\hat{\mu}(X_i - \hat{\mu})] = E[\hat{\mu}X_i] - E[\hat{\mu}^2] = 0,
\end{aligned}$$

since from

$$E[\hat{\mu}X_i] = E[\hat{\mu}X_j],$$

it follows

$$E[\hat{\mu}X_i] = \frac{1}{n}\sum_i^n E[\hat{\mu}X_j] = E\left[\hat{\mu}\frac{1}{n}\sum_i^n X_j\right] = E[\hat{\mu}^2] \tag{1}$$

c) It follows from b) that $\hat{\mu}$ and $(X_i - \hat{\mu})^2$ are independent, implying that also $\hat{\mu}$ and s^2 are independent. (1)

Problem 2. Consider the rectangular pulse function $g(t) = u(t) - u(t-1)$ and the random variable $T \sim U(0,\, 1)$. Define the random process

$$Y(t) = g(t - T)$$

a) Find the CDF of $Y(t)$. (1)

b) Find the mean function $\mu_Y(t)$. (1)

c) Find the autocovariance function $C_{YY}(t_1,\ t_2)$. (1)

Solution

a) The possible values of $Y(t)$ are 0 and 1.

$$P(Y(t)=1)=P(0\le t-T\le 1)=P(t-1\le T\le t)$$

Since $T\sim U(0,1)$, the above probability equals zero when $t\le 0$ or $t\ge 2$.

$$\underline{0<t\le 1:}\qquad P(Y(t)=1)=P(t-1<T<t)=P(T<t)=t.$$

$$\underline{1<t<2:}\qquad P(Y(t)=1)P(t-1<T<t)=P(t-1<T)=2-t$$

$$P(Y(t)=1)=\begin{cases}0, & t<0\\ t, & 0<t\le 1\\ 2-t, & 1<t\le 2\\ 0, & t>2\end{cases}$$

(1)

b) $P_Y(t)$ is a Bernoulli random variable, thus

$$\begin{aligned}&m_Y(t)=P(Y(t)=1)\\ &Var(Y(t))=P(Y(t)=1)(1-P(Y(t)=1))\end{aligned}$$

(1)

c) Fix $t_1<t_2$. We know that

$$C_Y(t_1,t_2)=E[Y(t_1)Y(t_2)]-m_Y(t_1)m_Y(t_2)$$

Both terms above equal zero, when $t_1\le 0$ or $t_2\ge 2$. Hence we have to consider $0<t_1<t_2<2$.

$$E[Y(t_1)Y(t_2)] = P(Y(t_1) = 1, Y(t_2) = 1)$$
$$= P(t_1 - 1 \le T < t_1,\ t_2 - 1 \le T < t_2)$$

From the above we find that

$$\underline{0 < t_1 < t_2 \le 1:}$$
$$P(t_1 - 1 \le T < t_1,\ t_2 - 1 \le T < t_2)$$
$$P(t_1 - 1 \le T < t_1,\ t_2 - 1 \le T < t_2) = P(T < t_1) = t_1$$

$$\underline{0 < t_1 \le 1, \quad 1 < t_2 \le 2, \quad t_2 - 1 < t_1:}$$
$$P(t_1 - 1 \le T < t_1,\ t_2 - 1 \le T < t_2)$$
$$= P(t_2 - 1 < T < t_1) = t_1 - t_2 + 1$$

$$\underline{0 < t_1 \le 1, \quad 1 < t_2 \le 2, \quad t_2 - 1 > t_1:}$$
$$P(t_1 - 1 \le T < t_1,\ t_2 - 1 \le T < t_2) = 0$$

$$\underline{1 < t_1 < t_2 \le 2:}$$
$$P(t_1 - 1 \le T < t_1,\ t_2 - 1 \le T < t_2)$$
$$= P(t_2 - 1 < T) = 2 - t_2$$

We have then

$$E[Y(t_1)Y(t_2)] = \begin{cases} t_1, & 0 < t_1 < t_2 \le 1 \\ t - t_2 + 1, & 0 < t_1 \le 1, 1 < t_2 \le 1 + t_1, \\ 0, & 0 < t_1 \le 1, 1 + t_1 < t_2 \le 2, \\ 2 - t_2, & 1 < t_1 < t_2 \le 2 \end{cases}$$

$$C_Y(t_1, t_2) = \begin{cases} t_1(1 - t_2), & 0 < t_1 < t_2 \le 1 \\ t - t_2 + 1 - t_1(2 - t_2), & 0 < t_1 \le 1, 1 < t_2 \le 1 + t_1, \\ -t_1(2 - t_1), & 0 < t_1 \le 1, 1 + t_1 < t_2 \le 2, \\ (2 - t_2)(t_1 - 1), & 1 < t_1 < t_2 \le 2 \end{cases} \tag{1}$$

Problem 3. Let $X(t)$ be a white sense stationary random process with $\mu_X(t) = 0$ that is ergodic in the mean and the autocorrelation, and let $Y(t) = ZX(t)$, where Z is a random variable with expected value zero which is independent of $X(t)$. Answer and explain:

a) Is the process $Y(t)$ ergodic in mean? (1)

b) Is it ergodic in autocorrelation? (1)

Solution

a)

$$\langle Y(t)\rangle = \lim_{t\to\infty} \frac{1}{2T}\int_{-T}^{T} Y(t)dt = \lim_{t\to\infty} \frac{Z}{2T}\int_{-T}^{T} X(t)dt$$
$$= \langle ZX(t)\rangle = Z\langle X(t)\rangle = Zm_X$$

$$E[Y(t)] = E[ZX(t)] = E[Z]m_X$$

Since $\mu_X = 0$, the process is ergodic in mean. (1)

b)

$$\langle Y(t_1)Y(t_2)\rangle = \lim_{t\to\infty} \int_{-T}^{T} \langle Z^2 X(t_1)X(t_2)\rangle = Z^2\langle X(t_1)X(t_2)\rangle = Z^2 R_{XX}(t_1, t_2)$$

$$E[Y(t_1)Y(t_2)] = E[Z^2 X(t_1)X(t_2)] = E[Z^2]R_{XX}(t_1, t_2)$$

If Z is non-degenerate the process is not ergodic in autocorrelation. (1)

Problem 4. A shot noise process with random amplitude is defined by

$$X(t) = \sum_{1}^{\infty} A_i h(t - S_i),$$

where the S_i are the points of occurrences of a Poisson process $N(t)$ of rate λ, and A_i are iid random variables independent of $N(t)$.

a) Compute $\mu_X(t)$. (2)

b) Compute $C_{XX}(t_1, t_2)$. (2)

Solution

a) Denote by $N(t)$ the Poisson process involved and let λ be the rate of the process. Also, let $E[A_i] = a, \quad E[A_i^2] = b^2$. We have

$$\mu(t) = E[X(t)] = E[E[X(t)|N(t)]]$$

$$E[X(t)|N(t) = n] = E\left[\sum_1^\infty A_i h(t - S_i)|N(t) = n\right]$$

$$E\left[\sum_1^n A_i h(t - S_i)|N(t) = n\right] = E\left[\sum_1^n A_i h(t - X_{(i)})\right]$$

$$= a\sum_1^n E[h(t - X_{(i)})] = aE\left[\sum_1^n h(t - X_{(i)})\right]$$

$$= aE\left[\sum_1^n h(t - X_i)\right] = mn\frac{1}{t}\int_o^t h(t-u)du$$

$$E[X(t)|N(t)] = aN(t)\frac{1}{t}\int_o^t h(t-u)du$$

$$\boxed{\mu(t) = \int_o^t h(t-u)du = a\lambda\int_o^t h(u)du} \tag{2}$$

b) Using the approach presented in the book, p. 308, we compute

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$X(t) \approx \sum_1^\infty A_n V_n h(t - n\Delta), \quad V_n \sim Bernoulli(\lambda\Delta)$$

$$E\left[\sum_0^\infty A_n V_n h(t_1 - n\Delta)\sum_0^\infty A_m V_m h(t_2 - m\Delta)\right]$$

$$= \sum_{n=0}^\infty\sum_{m=0}^\infty E[A_n A_m]E[V_n V_m]h(t_1 - n\Delta)h(t_2 - m\Delta) = \sum\sum_{m=n} + \sum\sum_{m\neq n}$$

$$= b^2\sum_{n=0}^\infty h(t_1 - n\Delta)h(t_2 - n\Delta)\lambda\Delta + a^2\sum\sum_{m\neq n} h(t_1 - n\Delta)h(t_2 - m\Delta)(\lambda\Delta)^2$$

When $\Delta \to 0$, the first term approaches

$$b^2\lambda \int_0^\infty h(t_1-u)h(t_2-u)du$$

The second term can be written as

$$a^2 \sum_{n=0}^\infty \sum_{m=0}^\infty h(t_1-n\Delta)h(t_2-m\Delta)(\lambda\Delta)^2 - a^2 \sum_{n=0}^\infty h(t_1-n\Delta)h(t_2-n\Delta)(\lambda\Delta)^2$$

When $\Delta \to 0$, for the first term above we have

$$a^2 \sum_{n=0}^\infty \sum_{m=0}^\infty h(t_1-n\Delta)h(t_2-m\Delta)(\lambda\Delta)^2$$

$$= a^2\lambda^2 \sum_{n=0}^\infty h(t_1-n\Delta)\Delta \sum_{m=0}^\infty h(t_2-m\Delta)\Delta$$

$$\longrightarrow \quad a^2\lambda^2 \int_0^\infty h(t_1-u)du \int_0^\infty h(t_2-u)du$$

$$= a^2\lambda^2 \int_0^{t_1} h(u)du \int_0^{t_2} h(u)du = \mu_X(t_1)\mu_X(t_2)$$

and for the second term

$$a^2 \sum_{n=0}^\infty h(t_1-n\Delta)h(t_2-n\Delta)(\lambda\Delta)^2$$

$$= a^2\lambda^2\Delta \sum_{n=0}^\infty h(t_1-n\Delta)h(t_2-n)\Delta \longrightarrow 0$$

Thus

$$\boxed{R_{XX}(t_1,t_2) = b^2\lambda \int_0^\infty h(t_1-u)h(t_2-u)du + \mu_X(t_1)\mu_X(t_2)} \tag{2}$$

Problem 5. Consider the second order autoregressive process defined by

$$Y_n = \frac{3}{4}Y_{n-1} - \frac{1}{8}Y_{n-2} + W_n,$$

where W_n is the zero mean white noise process.

a) Show that the unit impulse response of the linear system producing Y is

$$h_n = 2\Big(\frac{1}{2}\Big)^n - \Big(\frac{1}{4}\Big)^n,\ n \geq 0.$$

(1)

b) Find the transfer function of the system. (1)

c) Find the PSD of the process and its autocorrelation function. (1)

Solution

a) The process Y is obtained by passing the white noise process W through a filter T. The unit impulse response function of T, h^T satisfies

$$h_n^T = \frac{3}{4}h_{n-1}^T - \frac{1}{8}h_{n-2}^T + \delta_n,\ \ n = 0,\ \pm 1,\ \pm 2, \ldots.$$

Substituting h_n above we see that the equations are satisfied for $n = 0,\ \pm 1,\ \pm 2, \ldots.$. Hence $h_n^T = h_n$. (1)

Below we use standard techniques for computing the unit impulse response function of a system producing an autoregressive process from the white noise process.

$$Y_n - \frac{3}{4}Y_{n-1} + \frac{1}{8}Y_{n-2} = W_n$$

Taking Z-transform from both sides we obtain

$$Z_Y(z)\Big(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\Big)Z_W(z),\ z > 1$$

$$Z_Y(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}\, Z_W(z)$$

Thus the Z-transform of the filter h producing Y from W is then

$$Z_h(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$= \frac{1}{(1 - z^{-1}/2)(1 - z^{-1}/4)} = \frac{2}{1 - z^{-1}/2} - \frac{1}{1 - z^{-1}/4}$$

$$= 2\sum_0^\infty \left(\frac{z^{-1}}{2}\right)^k - \sum_0^\infty \left(\frac{z^{-1}}{4}\right)^k = \sum_0^\infty \left[2\left(\frac{1}{2}\right)^k - \left(\frac{1}{4}\right)^k\right] z^{-k} = \sum_0^\infty h_k z^{-k}$$

Thus

$$\boxed{h_k = 2\left(\frac{1}{2}\right)^k - \left(\frac{1}{4}\right)^k, \; k \geq 0}$$

b)

$$\begin{aligned} H(f) &= \sum_0^\infty \left(2\left(\frac{1}{2}\right)^k - \left(\frac{1}{4}\right)^k\right) e^{-j2\pi fk} \\ &= 2\,\frac{1}{1 - \frac{1}{2}e^{-j2\pi f}} - \frac{1}{1 - \frac{1}{4}e^{-j2\pi f}} \\ &= \frac{1}{(1 - \frac{1}{2}e^{-j2\pi f})(1 - \frac{1}{4}e^{-j2\pi f})}. \end{aligned} \tag{1}$$

c) Recall that

$$S_Y(f) = |H(f)|^2 \sigma_W^2$$

We compute

$$\begin{aligned} |H(f)|^2 &= \frac{1}{1 + \frac{1}{4} - \cos\, 2\pi f} \cdot \frac{1}{1 + \frac{1}{16} - \frac{1}{2}\cos\, 2\pi f} \\ &= \frac{8}{7}\left[\frac{2}{1 + \frac{1}{4} - \cos\, 2\pi f} - \frac{1}{1 + \frac{1}{16} - \frac{1}{2}\cos\, 2\pi f}\right]. \end{aligned}$$

For $|\alpha| < 1$, the series $\{\alpha^{|k|}\}_{-\infty}^{\infty}$ has a Fourier transform given by

$$\mathcal{F}\left(\{\alpha^{|k|}\}\right)(f) = \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha \cos\, 2\pi f}.$$

Hence

$$\frac{3/4}{1+\frac{1}{4}-\cos 2\pi f} = \mathcal{F}\left(\{(1/2)^{|k|}\}\right)(f),$$
$$\frac{15/16}{1+\frac{1}{16}-\frac{1}{2}\cos 2\pi f} = \mathcal{F}\left(\{(1/4)^{|k|}\}\right)(f),$$

and then

$$|H(f)|^2 = \frac{64}{21}\mathcal{F}\left(\{(1/2)^{|k|}\}\right)(f) + \frac{128}{135}\mathcal{F}\left(\{(1/4)^{|k|}\}\right)(f)$$

Thus

$$S_Y(f) = \frac{64}{21}\Big[\mathcal{F}\left(\{(1/2)^{|k|}\}\right)(f) + \frac{128}{135}\mathcal{F}\left(\{(1/4)^{|k|}\}\right)(f)\Big]\sigma_W^2$$

and

$$\boxed{R_Y(k) = \Big[\frac{64}{21}\Big(\frac{1}{2}\Big)^{|k|} - \frac{128}{105}\Big(\frac{1}{4}\Big)^{|k|}\Big]\sigma_W^2} \tag{1}$$

Below we compute $R_Y(k)$ by help of standard techniques. For convenience, denote $R(k) = R_Y(k)$.

- $Y_n = \frac{3}{4}Y_{n-1} - \frac{1}{8}Y_{n-2} + W_n$

Multiply both sides by Y_{n-k} to get

$$Y_{n-k}Y_n = \frac{3}{4}Y_{n-k}Y_{n-1} - \frac{1}{8}Y_{n-k}Y_{n-2} + Y_{n-k}W_n$$

and take expectation of both sides.

$$R(k) = \frac{3}{4}\,R(k-1) - \frac{1}{8}\,R(k-2) + E[Y_{n-k}\,W_n]$$

$\underline{k=0}$: $\quad R(0) = \frac{3}{4}\,R(1) - \frac{1}{8}\,R(2) + \sigma_W^2$

$\underline{k=1}$: $\quad R(1) = \frac{3}{4}\,R(0) - \frac{1}{8}\,R(1) \qquad (Y_{n-1}$ and W_n are uncorrelated)

$\underline{k=2}$: $\quad R(2) = \frac{3}{4}\,R(1) - \frac{1}{8}\,R(0)$

$\underline{k>2}$: $\quad \boxed{R(k) = \frac{3}{4}\,R(k-1) - \frac{1}{8}\,R(k-2)}$

true for $k \geq 2$

• We first find $R(0)$ and $R(1)$ from the first two equations, substituting there $R(2)$ by $3/4R(1) - 1/8R(0)$.

$$\left| \begin{array}{l} R(0) = \frac{3}{4}R(1) - \frac{1}{8}\left[\frac{3}{4}R(1) - \frac{1}{8}R(0)\right] + \sigma_W^2 \\ \\ R(1) = \frac{3}{4}R(0) - \frac{1}{8}R(1) \end{array} \right.$$

$$\left| \begin{array}{l} \frac{63}{64}R(0) - \frac{21}{32}R(1) = \sigma_W^2 \\ \\ \frac{3}{4}R(0) - \frac{9}{8}R(1) = 0 \end{array} \right.$$

Put $r_i = \dfrac{R(i)}{\sigma_W^2}, \quad i = 0, 1$

$$\left| \begin{array}{l} 63r_0 - 42R_1 = 64 \\ 6r_0 - 9R_1 = 0 \end{array} \right.$$

$$r_0 = \frac{\det \begin{bmatrix} 64 & -42 & = \\ 0 & -9 & \end{bmatrix}}{\det \begin{bmatrix} 63 & -42 \\ 6 & -9 \end{bmatrix}} = \frac{-64 \cdot 9}{-63 \cdot 9 + 6 \cdot 42} = \frac{-576}{-315} = \boxed{\frac{576}{315}}$$

$$R_1 = \frac{\det \begin{bmatrix} 63 & 64 \\ 6 & 0 \end{bmatrix}}{-315} = \boxed{\frac{384}{315}}$$

$$\boxed{R(0) = \frac{576}{315}\sigma_W^2, \quad R(1) = \frac{384}{315}\sigma_W^2}$$

• Next we compute $R(k), \quad k \geq 2$. As we saw, the series $R(k)$ obey the recurrent equations

$$R(k) - \frac{3}{4}R(k-1) - \frac{1}{8}R(k-2) = 0, \quad k \geq 2,$$

called second-order difference equations. The characteristic polynomial of the system is

$$P(\lambda) = \lambda^k - \frac{3}{4}\,\lambda^{k-1} + \frac{1}{8}\,\lambda^{k-2}.$$

We need to find the non-zero roots of this polynomial, i.e., the non-zero Roth's of $\lambda^2 - \frac{3}{4}\lambda + \frac{1}{8}$. Easy to see that these Roth's are

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{4}$$

.

All solutions of the system of difference equations have the form

$$\boxed{R(k) = \alpha\left(\frac{1}{2}\right)^k + \beta\left(\frac{1}{4}\right)^k} \ ,$$

where α and β are some constants. We have already computed $R(0)$ and $R(1)$, thus we must have

$$k = 0: \quad \alpha + \beta = R(0),$$

$$k = 1: \quad \frac{\alpha}{2} + \frac{\beta}{4} = R(1).$$

This gives

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}^{-1} \begin{bmatrix} R(0) \\ R(1) \end{bmatrix}$$

$$= \frac{1}{\det \begin{bmatrix} 1 & 1 \\ 1/2 & 1/4 \end{bmatrix}} \begin{bmatrix} 1/4 & -1 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} \frac{576}{315}\sigma_W^2 \\ \frac{384}{315}\sigma_W^2 \end{bmatrix}$$

$$= \frac{(-4)\sigma_W^2}{315} \begin{bmatrix} 144 - 384 \\ -288 + 384 \end{bmatrix}$$

$$= \frac{-4\sigma_W^2}{315} \begin{bmatrix} -240 \\ +96 \end{bmatrix} = \begin{bmatrix} \frac{64}{21}\sigma_W^2 \\ -\frac{128}{105}\sigma_W^2 \end{bmatrix}$$

and hence

$$\boxed{R(k) = \left[\frac{64}{21}\left(\frac{1}{2}\right)^{|k|} - \frac{128}{105}\left(\frac{1}{2}\right)^{|k|}\right]\sigma_W^2}, \quad k = 0, \pm 1, \pm 2, \ldots$$