Solution to the written test for examination in MVE135
Random processes with applications, 2007-10-25 Thursday, 14:00 - 18:00, V.

There are 30 total points in the examination. One needs 14 points for grade 3 (to pass), 18 points for grade 4, and 24 points for grade 5.

Problem 1. The input X in a binary optical communication system is a random variable with equally likely values 1 and 2. The receiver output Y is a Poisson random variable which parameter is μ, when 1 is transmitted, and ν when 2 is transmitted.

(a) Compute $E[Y|X]$ and $E[Y]$. 1.5p

(b) Given that the receiver output is equal to 2, find the conditional probability that 1 was sent. 1.5p

Solution

(a)

$$E[Y|X=1]=\mu, \quad E[Y|X=2]=\nu, \quad E[Y]=E[E[Y|X]]=\frac{\mu}{2}+\frac{\nu}{2}$$

(b)

$$P(X=1|Y=2)=\frac{P(Y=2|X=1)P(X=1)}{P(Y=2)}=\frac{\frac{1}{2}\frac{\mu^2}{2}e^{-\mu}}{\frac{1}{2}\frac{\mu^2}{2}e^{-\mu}+\frac{1}{2}\frac{\nu^2}{2}e^{-\nu}}=\frac{\mu^2e^{-\mu}}{\mu^2e^{-\mu}+\nu^2e^{-\nu}}$$

Problem 2. A multiplexer combines N digital television signals into a common transmission line. Signal n generates X_n bits every 33 milliseconds, where X_n is a Gaussian random variable with mean m/N and variance $\sigma^2/\sqrt{N}$. Suppose that the multiplexer accepts a maximum total of T bits from the combined sources every 33 ms, and that any bits in excess of T are discarded. Let the signals be independent and assume that $T=m+t\sigma$, where $t>0$ is a fixed number. Let Y_{Disc} be the number of bits discarded per 33-ms period, i.e.,

$$Y_{Disc}=\begin{cases} X-T, & X>T \\ 0, & X\le T. \end{cases}$$

Compute $E[Y_{Disc}]$. What is the result when $t\to\infty$? 3p

Solution Let $X=X_1+X_2+\ldots+X_N$ be the total number of bits generated by the combined source. X is a normal random variable with expected value m and variance $\sigma_1^2=\sqrt{N}\sigma^2$. Then $T=m+t_1\sigma_1^2$, where $t_1=t/\sqrt{N}$. We have

$$Y_{Dics}=\begin{cases} X-T, & \text{if} \quad X>T \\ 0, & \text{if} \quad X\le T. \end{cases}$$

Thus

$$
\begin{aligned}
E[Y_{Disc}] &= \int_T^\infty (x-T) f_X(x) dx \\
&= \int_{m+t_1\sigma_1}^\infty x f_X(x) dx - TP(X \geq m + t_1\sigma_1) \\
&= \frac{1}{\sqrt{2\pi}} \int_{t_1}^\infty (\sigma_1 y + m) e^{-\frac{y^2}{2}} dy - (m + t_1\sigma_1) Q(t_1) \\
&= \frac{\sigma_1}{\sqrt{2\pi}} e^{-\frac{t_1^2}{2}} - t_1\sigma_1 Q(t_1)
\end{aligned}
$$

When $t \to \infty$ we obtain $E[Y_{Dics}] \to 0$, as expected.

Problem 3. Suppose Z_1 and Z_2 are independent standard normal random variables. Define $X_1 = Z_1, \quad X_2 = 3/5 Z_1 + 4/5 Z_2$. Compute $f_{X_2|X_1}(x_2|x_1)$, the conditional PDF of X_2, given $X_1 = x_1$. 3p

Solution The random variables Z_1, Z_2 are jointly Gaussian, and so are X_1, X_2. We have $m_{X_1} = m_{X_2} = 0, \sigma^2_{X_1} = \sigma^2_{X_2} = 1$, $\rho_{X_1,X_2} = 3/5$. Then

$$
f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sqrt{1-(3/5)^2}} \exp\Big\{ -\frac{1}{2}\frac{1}{1-(3/5)^2}[x_1^2 + x_2^2 - 2\frac{3}{5}x_1x_2]\Big\}
$$

and

$$
f_{X_2|X_1}(x_2|x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)} = \frac{1}{\sqrt{2\pi}4/5} \exp\Big\{ -\frac{\left(x_2 - \frac{3}{5}x_1\right)^2}{2 \cdot 16/25} \Big\}
$$

Problem 4. Messages arrive in a multiplexer according to a Poisson process with mean $\lambda = 10$ messages/second. Use the CLT to estimate the probability that more then 650 messages arrive in one minute. 3p

Solution

$$
P(S_{650} < 60) = P\left(\frac{S_{650} - 605/10}{\sqrt{650}/10} < \frac{60 - 650/10}{\sqrt{650}/10} \right) \approx Q(1.96) = 2.49 \times 10^{-2}
$$

Problem 5. Let X_1, X_2, ... be iid random variables with expected value m and variance σ^2, and consider the discrete time proces $\{Z_n,\, n \geq 1\}$ with

$$
Z_n = \frac{X_1 + X_2 + \ldots + X_n}{n}.
$$

(a) Find the autocovariance function of Z_n. 3p

(b) Why is this process Markovian? Suppose X_1 is continuous with CDF $F(x)$ and PDF $f(x)$. Compute

$$F_{Z_n|Z_{n-1}}(\,x|\,Z_{n-1}=y) = P(Z_n \leq x|\,Z_{n-1}=y) \quad \text{and} \quad f_{Z_n|Z_{n-1}}(\,x|\,Z_{n-1}=y).$$

3p

Solution

(a) Recall that the covariance function of the sum process $\{S_n,\, n \geq 1\}$ with $S_n = X_1 + X_2 + \,...\, + X_n$ is $C_S(n,k) = \min(n,k)\sigma^2$. Then

$$C_Z(n,k) = E\left[(Z_n - m)(Z_k - m)\right) = \frac{1}{nk}E[(S_n - nm)(S_k - km)]$$
$$= \frac{1}{nk}C_S(n,k) = \frac{1}{nk}\min(n,k)\sigma^2 = \frac{\sigma^2}{\max(n,k)}$$

(b) The process has independent increments and is then Markovian.

$$F_{Z_n|Z_{n-1}}(\,x|\,Z_{n-1}=y) = P(Z_n \leq x|Z_{n-1}=y) = P(nZ_n \leq nx|\,(n-1)Z_{n-1} = (n-1)y)$$
$$= P(X_n \leq nx - (n-1)y) = F_X(nx - (n-1)y)$$

$$f_{Z_n|Z_{n-1}}(\,x|\,Z_{n-1}=y) = nf_X(nx - (n-1)y)$$

Problem 6. Consider the short term integration of $X(t)$

$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(u)du,$$

where $X(t)$ is the white noise process with PSD $S_X(f) = N_0/2$.

(a) Compute $S_Y(f)$, the PSD of $Y(t)$. 3p

(b) Compute the average power of $Y(t)$. 3p

Solution The impulse responce is

(a)

$$h(t) = \frac{1}{T}\int_{t-T}^{t}\delta(x)dx = \frac{1}{T}\left[\int_{\infty}^{t}\delta(x)dx - \int_{\infty}^{t-T}\delta(x)d\right] = \frac{1}{T}[u(t) - u(t-T)]$$

and the transfer functuion is then

$$H(t) = \frac{1}{T}\int_{0}^{T} e^{-j2\pi ft}dt = \frac{1}{T}\frac{\sin\pi fT}{\pi f}e^{-j\pi fT}$$

Hence

$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{\sin^2\pi fT}{T^2\pi^2 f^2}\cdot\frac{N_0}{2}$$

(b) The inverse Fourier transform gives

$$R_Y(\tau) = \frac{N_0}{2} F^{-1}\left[\frac{\sin^2 \pi f T}{\pi^2 f^2 T^2}\right] = \frac{N_0}{2T}\,\mathrm{tri}(\tau T), \quad R_Y(0) = \frac{N_0}{2T}$$

Problem 7. $\{X_n\}$ is a WSS process with autocorrelation function

$$R_X(k) = 4(1/2)^{|k|}, k = 0, \pm 1, \pm 2, \ldots.$$

Find the optimum linear filter for estimating X_n from the observations X_{n-1} and X_{n-3} and compute the mean-square estimation error. 3p

Solution $\hat{X}_n = h_1 X_{n-1} + h_2 X_{n-3}$.

$\hat{X}_n$ – optimal $\Leftrightarrow E[\hat{X}_n X_{n-i}] = E[X_n X_{n-i}], \quad i = 1, 3$

$$\begin{bmatrix} R_X(0) & R_X(2) \\ R_X(2) & R_X(0) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} R_X(1) \\ R_X(3) \end{bmatrix}$$

$$\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1/2 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

$e^2 = R_X(0) - h_1 R_X(1) = 3$

Problem 8. The spectrum of a stationary stochastic process is to be estimated from the data:

$$x[n] = \{0, 6, \ -0, 7, \ 0, 2, \ 0:3\}$$

Due to the small sample support, a simple AR(1)-model is exploited:

$$x[n] + a_1 x[n-1] = e[n].$$

Determine estimates of the AR-parameter a_1 and the white noise variance σ_e^2. Based on these, give a parametric estimate of the spectrum $P_X(e^{j\omega})$. 3p

Solution

The Yule-Walker method gives the estimate

$$\hat{a_1} = -\hat{r_x}[0]^{-1}]\hat{r_x}[1]$$

With the given data, the sample autocorrelation function is calculated as

$$\hat{r}_x[0] = \frac{1}{4}(0.6^2 + 0.7^2 + 0.2^2 + 0.3^2) = 0.245$$

and

$$\hat{r}_x[1] = \frac{1}{4}(0.6 \times (-0.7) + (-0.7) \times 0.2 + 0.2 \times 0.3) = -0.125$$

Thus, we get

$$\hat{a}_1 = \frac{0.125}{0.245} \approx 0.51.$$

The noise variance estimate follows as

$$\hat{\sigma}_e^2 = \hat{r}_x[0] + \hat{a}_1 \hat{r}_x[1] \approx 0.18.$$