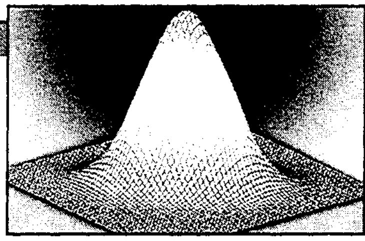
COLLECTION OF FORMULAS

FOR MVE135

RANDOM PROCESSES WITH APPLICATIONS

Summary of Common Random Variables



This appendix provides a quick reference of some of the most common random variables. Special functions that are used in this appendix are defined in the following list.

- Gamma function: $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}\,du,\ Re[\alpha] > 0.$
- Incomplete gamma function: $\gamma(\alpha,\beta) = \int_0^\beta u^{\alpha-1}e^{-u}\,du,\ Re[\alpha] > 0.$
- Beta function: $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$
- Incomplete beta function: $\beta(a,b,x) = \int_0^x u^{a-1}(1-u)^{b-1}du,\ 0 < x < 1.$
- Modified Bessel function of order m: $I_m(x) = \dfrac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\theta)}\cos(m\theta)\,d\theta.$
- Q-function: $Q(x) = \int_x^\infty \frac{1}{2\pi}\exp\left(-\frac{u^2}{2}\right)du.$
- Marcum's Q-function: $Q(\alpha,\beta) = \int_\beta^\infty u\exp\left(-\dfrac{\alpha^2+u^2}{2}\right)I_0(\alpha u)\,du.$

D.1 Continuous Random Variables

D.1.1 Arcsine

For any $b > 0$,

$$f_X(x) = \frac{1}{\pi\sqrt{b^2-x^2}} \quad -b < x < b. \tag{D.1}$$

$$F_X(x) = \begin{cases} 0 & x < -b \\ \dfrac{1}{2} + \dfrac{1}{\pi}\sin^{-1}\left(\dfrac{x}{b}\right) & -b \le x \le b \\ 1 & x > b \end{cases} \quad \text{(D.2)}$$

$$\mu_X = 0, \quad \sigma_X^2 = \frac{b^2}{2}. \quad \text{(D.3)}$$

Note:

(1) Formed by a transformation $X = b\cos(2\pi U + \theta)$, where b and θ are constants and U is a uniform random variable over $[0, 1)$.

D.1.2 Beta

For any $a > 0$ and $b > 0$,

$$f_X(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1. \quad \text{(D.4)}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \dfrac{\beta(a,b,x)}{B(a,b)} & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \quad \text{(D.5)}$$

$$\mu_X = \frac{a}{a+b}, \quad \sigma_X^2 = \frac{ab}{(a+b)^2(a+b+1)}. \quad \text{(D.6)}$$

D.1.3 Cauchy

For any $b > 0$,

$$f_X(x) = \frac{b/\pi}{b^2 + x^2}. \quad \text{(D.7)}$$

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{b}\right). \quad \text{(D.8)}$$

$$\Phi_X(\omega) = e^{-b|\omega|}. \quad \text{(D.9)}$$

Notes:

(1) Both the mean and variance are undefined.
(2) Formed by a transformation of the form $X = b\tan(2\pi U)$, where U is uniform over $[0, 1)$.

D.1.4 Chi-Square

For integer $n > 0$,

$$f_X(x) = \frac{x^{n/2-1}}{2^{n/2}\Gamma(n/2)} e^{-x/2}, \quad x \geq 0. \tag{D.10}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \dfrac{\gamma(n/2, x/2)}{\Gamma(n/2)} & x \geq 0 \end{cases}. \tag{D.11}$$

$$\Phi_X(\omega) = \frac{1}{(1-2j\omega)^{n/2}}. \tag{D.12}$$

$$\mu_X = n, \quad \sigma_X^2 = 2n. \tag{D.13}$$

Notes:

(1) The chi-square random variable is a special case of the gamma random variable.
(2) The parameter n is referred to as the number of degrees of freedom of the chi-square random variable.
(3) The chi-square random variable is formed by a transformation of the form $X = \sum_{k=1}^{n} Z_k^2$, where the Z_k are independent and identically distributed (IID), zero-mean, unit variance Gaussian random variables.

D.1.5 Erlang

For any integer $n > 0$ and any $b > 0$,

$$f_X(x) = \frac{b^n x^{n-1} e^{-bx}}{(n-1)!}, \quad x \geq 0. \tag{D.14}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \dfrac{\gamma(n, bx)}{(n-1)!} & x \geq 0 \end{cases}. \tag{D.15}$$

$$\Phi_X(\omega) = \frac{1}{(1-j\omega/b)^n}. \tag{D.16}$$

$$\mu_X = n/b, \quad \sigma_X^2 = n/b^2. \tag{D.17}$$

Notes:

(1) The Erlang random variable is a special case of the gamma random variable.
(2) The Erlang random variable is formed by summing n IID exponential random variables.
(3) The CDF can also be written as a finite series

$$\frac{\gamma(n,bx)}{(n-1)!} = 1 - e^{bx}\sum_{k=0}^{n-1}\frac{(bx)^k}{k!}, \quad x \geq 0. \tag{D.18}$$

D.1.6 Exponential

For any $b > 0$,

$$f_X(x) = be^{-bx}, \quad x \geq 0. \tag{D.19}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-bx} & x \geq 0 \end{cases}. \tag{D.20}$$

$$\Phi_X(\omega) = \frac{1}{1 - j\omega/b}. \tag{D.21}$$

$$\mu_X = 1/b, \quad \sigma_X^2 = 1/b^2. \tag{D.22}$$

Notes:

(1) The exponential random variable is a special case of the Erlang and gamma random variables.
(2) The exponential random variable possesses the memoryless property,

$$f_X(x|X > a) = f_X(x - a). \tag{D.23}$$

D.1.7 *F*

For any integers $n > 0$ and $m > 0$,

$$f_X(x) = \frac{\left(\frac{n}{m}\right)^{n/2}}{B\left(\frac{n}{2},\frac{m}{2}\right)} x^{\frac{n}{2}-1}\left(1 + \frac{n}{m}x\right)^{-\frac{n+m}{2}}, \quad x > 0. \tag{D.24}$$

$$\mu_X = \frac{m}{m-2} \quad \text{for} \quad m > 2, \qquad \sigma_X^2 = \frac{m^2(2n+2m-4)}{n(m-2)^2(m-4)} \quad \text{for} \quad m > 4. \tag{D.25}$$

Notes:

(1) If U and V are independent chi-square random variables with n and m degrees of freedom, respectively, then $F = (U/n)/(V/m)$ will be an F random variable with n and m degrees of freedom.

D.1.8 Gamma

For any $a > 0$ and $b > 0$,

$$f_X(x) = \frac{b^a x^{a-1} e^{-bx}}{\Gamma(a)}, \quad x \geq 0. \tag{D.26}$$

$$F_X(x) = \frac{\gamma(a, bx)}{\Gamma(a)}. \tag{D.27}$$

$$\Phi_X(\omega) = \frac{1}{(1 - j\omega/b)^a}. \tag{D.28}$$

$$\mu_X = a/b, \quad \sigma_X^2 = a/b^2. \tag{D.29}$$

Note:

(1) The gamma random variable contains the chi-square, Erlang, and exponential random variables as special cases.

D.1.9 Gaussian

For any μ and any $\sigma > 0$,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{D.30}$$

$$F_X(x) = 1 - Q\left(\frac{x-\mu}{\sigma}\right). \tag{D.31}$$

$$\Phi_X(\omega) = \exp\left(j\omega\mu - \frac{1}{2}\omega^2\sigma^2\right). \tag{D.32}$$

$$\mu_X = \mu, \quad \sigma_X^2 = \sigma^2. \tag{D.33}$$

D.1.10 Gaussian-Multivariate

For any n element column vector μ and any valid $n \times n$ covariance matrix C,

$$f_X(x) = \frac{1}{(2\pi)^{n/2}\det(C)} \exp\left(-\frac{1}{2}(X-\mu)^T C^{-1}(X-\mu)\right). \tag{D.34}$$

$$\Phi_X(\omega) = \exp\left(j\mu^T\omega - \frac{1}{2}\omega^T C\omega\right). \tag{D.35}$$

$$E[X] = \mu, \quad E[(X-\mu)(X-\mu)^T] = C. \tag{D.36}$$

D.1.11 Laplace

For any $b > 0$,

$$f_X(x) = \frac{b}{2}\exp(-b|x|). \tag{D.37}$$

$$F_X(x) = \begin{cases} \frac{1}{2}e^{bx} & x < 0 \\ 1 - \frac{1}{2}e^{-bx} & x \geq 0 \end{cases}. \tag{D.38}$$

$$\Phi_X(\omega) = \frac{1}{1+(\omega/b)^2}. \tag{D.39}$$

$$\mu_X = 0, \quad \sigma_X^2 = 2/b^2. \tag{D.40}$$

D.1.12 Log-Normal

For any μ and any $\sigma > 0$,

$$f_X(x) = \frac{1}{x\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right), \quad x > 0. \tag{D.41}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - Q\left(\frac{\ln(x)-\mu}{\sigma}\right) & x \geq 0 \end{cases}. \tag{D.42}$$

$$\mu_X = \exp\left(\mu + \frac{\sigma^2}{2}\right), \quad \sigma_X^2 = [\exp(\sigma^2) - 1]\exp(2\mu + \sigma^2). \tag{D.43}$$

Notes:

(1) The log-normal random variable is formed by a transformation of the form $X = \exp(Z)$, where Z is a Gaussian random variable with mean μ and variance σ^2.
(2) It is common to find instances in the literature where σ is referred to as the standard deviation of the log-normal random variable. This is a misnomer. The quantity σ is not the standard deviation of the log-normal random variable, but rather is the standard deviation of the underlying Gaussian random variable.

D.1.13 Nakagami

For any $b > 0$ and $m > 0$,

$$f_X(x) = \frac{2m^m}{\Gamma(m)b^m} x^{2m-1} \exp\left(-\frac{m}{b}x^2\right), \quad x \geq 0. \tag{D.44}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \dfrac{\gamma\left(m, \frac{m}{b}x^2\right)}{\Gamma(m)} & x \geq 0 \end{cases}. \tag{D.45}$$

$$\mu_X = \frac{\Gamma(m + 1/2)}{\Gamma(m)}\sqrt{\frac{b}{m}}, \quad \sigma_X^2 = b - \mu_X^2. \tag{D.46}$$

D.1.14 Rayleigh

For any $\sigma > 0$,

$$f_X(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x \geq 0. \tag{D.47}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\dfrac{x^2}{2\sigma^2}\right) & x \geq 0 \end{cases}. \tag{D.48}$$

$$\mu_X = \sqrt{\frac{\pi\sigma^2}{2}}, \quad \sigma_X^2 = \frac{(4-\pi)\sigma^2}{2}. \tag{D.49}$$

Notes:

(1) The Rayleigh random variable arises when performing a Cartesian to polar transformation of two independent, zero-mean Gaussian random variables.

That is, if Y_1 and Y_2 are independent zero mean Gaussian random variables with variances of σ^2, then $X = \sqrt{Y_1^2 + Y_2^2}$ follows a Rayleigh distribution.

(2) The Rayleigh random variable is a special case of the Rician random variable.

D.1.15 Rician

For any $a \geq 0$ and any $\sigma > 0$,

$$f_X(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2 + a^2}{2\sigma^2}\right) I_o\left(\frac{ax}{\sigma^2}\right), \quad x \geq 0. \tag{D.50}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - Q\left(\frac{a}{\sigma}, \frac{x}{\sigma}\right) & x \geq 0 \end{cases}. \tag{D.51}$$

$$\mu_X = \sqrt{\frac{\pi\sigma^2}{2}} \exp\left(-\frac{a^2}{4\sigma^2}\right) \left[\left(1 + \frac{a^2}{2\sigma^2}\right) I_o\left(\frac{a^2}{4\sigma^2}\right) + \frac{a^2}{2\sigma^2} I_1\left(\frac{a^2}{4\sigma^2}\right)\right]. \tag{D.52}$$

$$\sigma_X^2 = 2\sigma^2 + a^2 - \mu_X^2. \tag{D.53}$$

Notes:

(1) The Rician random variable arises when performing a Cartesian to polar transformation of two independent Gaussian random variables. That is, if Y_1 and Y_2 are independent Gaussian random variables with means of μ_1 and μ_2, respectively, and equal variances of σ^2, then $X = \sqrt{Y_1^2 + Y_2^2}$ follows a Rician distribution, with $a = \sqrt{\mu_1^2 + \mu_2^2}$.

(2) The ratio a^2/σ^2 is often referred to as the Rician parameter or the Rice factor. As the Rice factor goes to zero, the Rician random variable becomes a Rayleigh random variable.

D.1.16 Student t

For any integer $n > 0$,

$$f_X(x) = \frac{1}{B(n/2, 1/2)\sqrt{n}} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}. \tag{D.54}$$

$$\mu_X = 0, \quad \sigma_X^2 = \frac{n}{n-2} \quad \text{for} \quad n > 2. \tag{D.55}$$

Notes:

(1) This distribution was first published by W. S. Gosset in 1908 under the pseudonym "A. Student." Hence, this distribution has come to be known as the student's t-distribution.
(2) The parameter n is referred to as the number of degrees of freedom.
(3) If X_i $i = 1, 2, \ldots, n$ is a sequence of IID Gaussian random variables and $\hat{\mu}$ and $\widehat{s^2}$ are the sample mean and sample variance, respectively, then the ratio $T = (\hat{\mu} - \mu)/\sqrt{\widehat{s^2}/n}$ will have a t-distribution with $n - 1$ degrees of freedom.

D.1.17 Uniform

For any $a < b$,

$$f_X(x) = \frac{1}{b-a}, \quad a \le x < b. \tag{D.56}$$

$$F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}. \tag{D.57}$$

$$\Phi_X(\omega) = \frac{e^{jb\omega} - e^{ja\omega}}{j\omega(b-a)}. \tag{D.58}$$

$$\mu_x = \frac{a+b}{2}, \quad \sigma_X^2 = \frac{(b-a)^2}{12}. \tag{D.59}$$

D.1.18 Weibull

For any $a > 0$ and any $b > 0$,

$$f_X(x) = abx^{b-1}\exp(-ax^b), \quad x \ge 0. \tag{D.60}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp(-ax^b) & x \ge 0 \end{cases}. \tag{D.61}$$

$$\mu_X = \frac{\Gamma\left(1 + \frac{1}{b}\right)}{a^{1/b}}, \quad \sigma_X^2 = \frac{\Gamma\left(1 + \frac{2}{b}\right) - \left[\Gamma\left(1 + \frac{1}{b}\right)\right]^2}{a^{2/b}}. \tag{D.62}$$

Note:

(1) The Weibull random variable is a generalization of the Rayleigh random variable and reduces to a Rayleigh random variable when $b = 2$.

D.2 Discrete Random Variables

D.2.1 Bernoulli

For $0 < p < 1$,

$$P_X(k) = \begin{cases} 1-p & k=0 \\ p & k=1 \\ 0 & \text{otherwise} \end{cases}. \tag{D.63}$$

$$H_X(z) = 1 - p(1-z) \quad \text{for all } z. \tag{D.64}$$

$$\mu_X = p, \quad \sigma_X^2 = p(1-p). \tag{D.65}$$

D.2.2 Binomial

For $0 < p < 1$ and any integer $n > 0$,

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}. \tag{D.66}$$

$$H_X(z) = (1 - p(1-z))^n \quad \text{for any } z. \tag{D.67}$$

$$\mu_x = np, \quad \sigma_X^2 = np(1-p). \tag{D.68}$$

Note:

(1) The binomial random variable is formed as the sum of n independent Bernoulli random variables.

D.2.3 Geometric

For $0 < p < 1$,

$$P_X(k) = \begin{cases} (1-p)p^k & k \geq 0 \\ 0 & k < 0 \end{cases}. \tag{D.69}$$

$$H_X(z) = \frac{1-p}{1-pz} \quad \text{for} \quad |z| < 1/p. \tag{D.70}$$

$$\mu_X = \frac{p}{1-p}, \quad \sigma_X^2 = \frac{p}{(1-p)^2}. \tag{D.71}$$

D.2.4 Pascal (or Negative Binomial)

For $0 < q < 1$ and any integer $n > 0$,

$$P_X(k) = \begin{cases} 0 & k < n \\ \binom{k-1}{n-1}(1-q)^n q^{k-n} & k = n, n+1, n+2, \ldots \end{cases}. \tag{D.72}$$

$$H_X(z) = \left(\frac{(1-q)z}{1-qz}\right)^n, \quad \text{for} \quad |z| < 1/q. \tag{D.73}$$

$$\mu_X = \frac{n}{1-q}, \quad \sigma_X^2 = \frac{nq}{(1-q)^2}. \tag{D.74}$$

D.2.5 Poisson

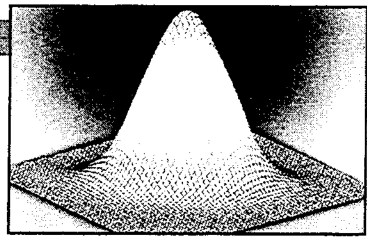
For any $b > 0$,

$$P_X(k) = \begin{cases} \frac{b^k}{k!}e^{-b} & k \geq 0 \\ 0 & k < 0 \end{cases}. \tag{D.75}$$

$$H_X(z) = \exp(b(z-1)), \quad \text{for all } z. \tag{D.76}$$

$$\mu_X = b, \quad \sigma_X^2 = b. \tag{D.77}$$

Mathematical Tables



E.1 Trigonometric Identities

$$\sin^2(x) + \cos^2(x) = 1. \tag{E.1}$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y). \tag{E.2}$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y). \tag{E.3}$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y). \tag{E.4}$$

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y). \tag{E.5}$$

$$\sin(x)\cos(y) = \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y). \tag{E.6}$$

$$\exp(jx) = \cos(x) + j\sin(x). \tag{E.7}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}. \tag{E.8}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}. \tag{E.9}$$

E.2 Series Expansions

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1. \tag{E.10}$$

$$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k, \quad \text{for all } x. \tag{E.11}$$

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=n}^{\infty} \binom{k}{n} x^{k-n} = \sum_{k=0}^{\infty} \binom{k+n}{n} x^k, \quad \text{for } |x| < 1. \tag{E.12}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \quad \text{for all } x, y. \tag{E.13}$$

$$\exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k, \quad \text{for all } x. \tag{E.14}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \quad \text{for all } x. \tag{E.15}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \text{for all } x. \tag{E.16}$$

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{1}{k} x^k, \quad \text{for } |x| < 1. \tag{E.17}$$

$$Q(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k! 2^k (2k+1)} x^{2k+1}, \quad \text{for all } x. \tag{E.18}$$

$$I_m(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m}, \quad \text{for all } x. \tag{E.19}$$

E.3 Some Common Indefinite Integrals

Note: For each of the indefinite integrals, an arbitrary constant may be added to the result.

$$\int x^n dx = \begin{cases} \dfrac{x^{n+1}}{n+1} & n \neq -1 \\ \ln(x) & n = -1 \end{cases}. \tag{E.20}$$

$$\int b^x dx = \frac{b^x}{\ln(b)} \quad b \neq 1. \tag{E.21}$$

$$\int \ln(x)dx = x\ln(x) - x. \tag{E.22}$$

$$\int \sin(x)dx = -\cos(x). \tag{E.23}$$

$$\int \cos(x)dx = \sin(x). \tag{E.24}$$

$$\int \tan(x)dx = -\ln(|\cos(x)|). \tag{E.25}$$

$$\int \sinh(x)dx = \cosh(x). \tag{E.26}$$

$$\int \cosh(x)dx = \sinh(x). \tag{E.27}$$

$$\int \tanh(x)dx = \ln(|\cosh(x)|). \tag{E.28}$$

$$\int e^{ax}\sin(bx)dx = e^{ax}\left(\frac{a\sin(bx) - b\cos(bx)}{a^2+b^2}\right). \tag{E.29}$$

$$\int e^{ax}\cos(bx)dx = e^{ax}\left(\frac{b\sin(bx) + a\cos(bx)}{a^2+b^2}\right). \tag{E.30}$$

$$\int x^n e^{bx}dx = e^{bx}\sum_{k=0}^{n}\frac{(-1)^k}{b^{k+1}}\frac{n!}{(n-k)!}x^{n-k} \quad (n \geq 0). \tag{E.31}$$

$$\int x^n \ln(bx)dx = x^{n+1}\left(\frac{\ln(bx)}{n+1} - \frac{1}{(n+1)^2}\right) \quad (n \neq -1). \tag{E.32}$$

$$\int \frac{1}{x^2+b^2}dx = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) \quad (b > 0). \tag{E.33}$$

$$\int \frac{1}{\sqrt{b^2-x^2}}dx = \sin^{-1}\left(\frac{x}{b}\right) \quad (b > 0). \tag{E.34}$$

$$\int \frac{1}{\sqrt{x^2+b^2}}dx = \log(x+\sqrt{x^2+b^2}) = \sinh^{-1}\left(\frac{x}{b}\right) \quad (b > 0). \tag{E.35}$$

$$\int \frac{1}{\sqrt{x^2-b^2}}dx = \log\left|x+\sqrt{x^2-b^2}\right| = \cosh^{-1}\left(\frac{x}{b}\right) \quad (b > 0). \tag{E.36}$$

$$\int \frac{1}{ax^2+bx+c}dx = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & b^2 > 4ac \\ \dfrac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right) & b^2 < 4ac \end{cases}. \tag{E.37}$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \begin{cases} \frac{1}{\sqrt{a}} \ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| & a>0 \\ \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{-2ax-b}{\sqrt{b^2-4ac}}\right) & a<0 \end{cases}. \tag{E.38}$$

E.4 Some Common Definite Integrals

$$\int_0^\infty x^n e^{-x}dx = \Gamma(n+1) = n! \quad \text{for integer } n \geq 0. \tag{E.39}$$

$$\int_{-\infty}^\infty e^{-x^2}dx = \int_0^\infty x^{-1/2}e^{-x}dx = \Gamma(1/2) = \sqrt{\pi}. \tag{E.40}$$

$$\int_0^\infty x^{n-1/2}e^{-x}dx = \Gamma(n+1/2) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}, \quad \text{for integer } n \geq 1. \tag{E.41}$$

$$\int_{-\infty}^\infty \text{sinc}(x)dx = \int_{-\infty}^\infty \text{sinc}^2(x)dx = 1. \tag{E.42}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^n(x)dx = \frac{1}{2\pi}\int_0^{2\pi} \sin^n(x)dx = \begin{cases} 0 & n \text{ odd} \\ \binom{n}{n/2}\frac{1}{2^n} & n \text{ even} \end{cases}. \tag{E.43}$$

$$\int_{-\infty}^\infty \frac{1}{x^2+b^2}dx = 2\int_0^\infty \frac{1}{x^2+b^2}dx = \frac{\pi}{b}, \quad b>0. \tag{E.44}$$

$$\int_{-b}^b \frac{1}{\sqrt{b^2-x^2}}dx = 2\int_0^b \frac{1}{\sqrt{b^2-x^2}}dx = \pi, \quad b>0. \tag{E.45}$$

E.5 Definitions of Some Common Continuous Time Signals

$$\text{Step function: } u(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}. \tag{E.46}$$

$$\text{Rectangle function: } \mathrm{rect}(x) = \begin{cases} 1 & |x| < 1/2 \\ 0 & |x| > 1/2 \end{cases}. \tag{E.47}$$

$$\text{Triangle function: } \mathrm{tri}(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases}. \tag{E.48}$$

$$\text{Sinc function: } \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \tag{E.49}$$

E.6 Fourier Transforms

Table E.1 Common Fourier Transform Pairs

Signal (time domain)	Transform (frequency domain)
$\mathrm{rect}(t/t_o)$	$t_o\mathrm{sinc}(ft_o)$
$\mathrm{tri}(t/t_o)$	$t_o\mathrm{sinc}^2(ft_o)$
$\exp\left(-\frac{t}{t_o}\right)u(t)$	$\frac{t_o}{1+j2\pi ft_o}$
$\exp\left(-\frac{\lvert t\rvert}{t_o}\right)$	$\frac{2t_o}{1+(2\pi ft_o)^2}$
$\mathrm{sinc}(t/t_o)$	$t_o\mathrm{rect}(ft_o)$
$\mathrm{sinc}^2(t/t_o)$	$t_o\mathrm{tri}(ft_o)$
$\exp(j2\pi f_o t)$	$\delta(f-f_o)$
$\cos(2\pi f_o t+\theta)$	$\frac{1}{2}\delta(f-f_o)e^{j\theta}+\frac{1}{2}\delta(f+f_o)e^{-j\theta}$
$\delta(t-t_o)$	$\exp(-j2\pi ft_o)$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\exp(-(t/t_o)^2)$	$\sqrt{\pi t_o^2}\exp(-(\pi ft_o)^2)$

E.7 z-Transforms

Table E.2 Common z-Transform Pairs

Signal	Transform	Region of convergence
$\delta[n]$	1	all z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert > 1$
$n^3u[n]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\lvert z\rvert > 1$
$b^nu[n]$	$\frac{1}{1-bz^{-1}}$	$\lvert z\rvert > \lvert b\rvert$
$nb^nu[n]$	$\frac{bz^{-1}}{(1-bz^{-1})^2}$	$\lvert z\rvert > \lvert b\rvert$
$n^2b^nu[n]$	$\frac{bz^{-1}(1+bz^{-1})}{(1-bz^{-1})^3}$	$\lvert z\rvert > \lvert b\rvert$
$b^n\cos[\Omega_o n]u[n]$	$\frac{1-b\cos(\Omega_o)z^{-1}}{1-2b\cos(\Omega_o)z^{-1}+bz^{-2}}$	$\lvert z\rvert > \lvert b\rvert$
$b^n\sin[\Omega_o n]u[n]$	$\frac{b\sin(\Omega_o)z^{-1}}{1-2b\cos(\Omega_o)z^{-1}+bz^{-2}}$	$\lvert z\rvert > \lvert b\rvert$
$\frac{u[n-1]}{n}$	$\ln\left(\frac{1}{1-z^{-1}}\right)$	$\lvert z\rvert > 1$
$\binom{n+m}{m}b^nu[n]$	$\frac{1}{(1-bz^{-1})^{m+1}}$	$\lvert z\rvert > \lvert b\rvert$
$\frac{b^n}{n!}u[n]$	$\exp(bz^{-1})$	all z

E.8 Laplace Transforms

Table E.3 Common Laplace Transform Pairs

Function	Transform	Region of convergence
$u(t)$	$1/s$	$\mathrm{Re}[s] > 0$
$\exp(-bt)u(t)$	$\frac{1}{s+b}$	$\mathrm{Re}[s] > -b$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$	$\mathrm{Re}[s] > 0$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$	$\mathrm{Re}[s] > 0$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$	$\mathrm{Re}[s] > -a$
$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$	$\mathrm{Re}[s] > -a$
$\delta(t)$	1	all s
$\frac{d}{dt}\delta(t)$	s	all s
$t^n u(t), \quad n \geq 0$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}[s] > 0$
$t^n e^{-bt} u(t), \quad n \geq 0$	$\frac{n!}{(s+b)^{n+1}}$	$\mathrm{Re}[s] > -b$

E.9 Q-Function

Table E.4 lists values of the function $Q(x)$ for $0 \leq x < 4$ in increments of 0.05. To find the appropriate value of x, add the value at the beginning of the row to the value at the top of the column. For example, to find $Q(1.75)$, find the entry from the column headed by 1.00 and the row headed by 0.75 to get $Q(1.75) = 0.04005916$.

Table E.4 Values of $Q(x)$ for $0 \leq x < 4$ (in increments of 0.05)

$Q(x)$	**0.00**	**1.00**	**2.00**	**3.00**
0.00	0.50000000	0.15865525	0.02275013	0.00134990
0.05	0.48006119	0.14685906	0.02018222	0.00114421
0.10	0.46017216	0.13566606	0.01786442	0.00096760
0.15	0.44038231	0.12507194	0.01577761	0.00081635
0.20	0.42074029	0.11506967	0.01390345	0.00068714
0.25	0.40129367	0.10564977	0.01222447	0.00057703
0.30	0.38208858	0.09680048	0.01072411	0.00048342
0.35	0.36316935	0.08850799	0.00938671	0.00040406
0.40	0.34457826	0.08075666	0.00819754	0.00033693
0.45	0.32635522	0.07352926	0.00714281	0.00028029
0.50	0.30853754	0.06680720	0.00620967	0.00023263
0.55	0.29115969	0.06057076	0.00538615	0.00019262
0.60	0.27425312	0.05479929	0.00466119	0.00015911
0.65	0.25784611	0.04947147	0.00402459	0.00013112
0.70	0.24196365	0.04456546	0.00346697	0.00010780
0.75	0.22662735	0.04005916	0.00297976	0.00008842
0.80	0.21185540	0.03593032	0.00255513	0.00007235
0.85	0.19766254	0.03215677	0.00218596	0.00005906
0.90	0.18406013	0.02871656	0.00186581	0.00004810
0.95	0.17105613	0.02558806	0.00158887	0.00003908