Laboratory Exercise 1:

Simulation and analysis of a noisy communication system

Random Processes With Applications (MVE 135)

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1 Introduction

The purpose of this laboratory exercise is to illustrate an important engineering application where random processes play a crucial role, namely a digital communication system. The exercise has been adapted from a previous project in Digital Communications by Erik Ström. The laboratory project is mandatory. To get a pass, you need to hand in a short report with your results and the Matlab code used to run the simulations. The deadline for handing in the report will be posted on the course homepage, but a preliminary date is October 3. You should work in groups of 2 students and send in the report as a pdf-file to `viberg@chalmers.se`.
The task for a communications system is to transfer a signal from some source to a remote user with a sufficient quality. Depending on the nature of the message, the user can tolerate more or less distortion. A speech signal usually puts significantly less demands on the transmission than for example transfer of a file. On the other hand, the speech signal needs to be transferred in real time with a small time delay. In a digital communication system, the message signal is somewhere in the communication link represented by a sequence of binary words with a finite number of bits. If the source signal is analog, this means that it must be sampled and quantized before transmission. This gives rise to distortion, even if the sampling rate is high enough to fulfill the sampling theorem.
The fact that the transmitted signal is represented in digital form means that we can quantify the quality of the transmission by the probability of detecting the wrong bit, i.e. the Bit-Error Rate (BER) P_b. If $P_b = 0$, the system is perfect, whereas if $P_b = 0.5$ it is quite useless.
A communication system consists of a transmitter, a channel and a receiver. The transmitter starts with a source signal, then applies source coding, channel coding and finally modulation to put the signal into a format that fits the channel. The receiver performs the reverse operations, ending with a detector that decides the message bits and presents them to the user in a suitable form. In general, the source signal is of random nature, and can be well modeled as a stochastic process. In a wireless communication system, the channel adds to the randomness, since the propagation paths can be considered random. Finally, the receiver circuitry introduces ”measurement noise”, due to random fluctuations of currents around the ”true” values. In addition, the system may be subject to interference from other users that would also be modeled as random. Thus, modeling and understanding a communication system requires a good

background in stochastic processes. The reader is referred to [1], Chapters 8 and 10 (especially 8.3, 10.1 and 10.5) for some background material.

2 Noise in Electrical Circuits

All electrical conductors contain free electrons. If the temperature of the conductor exceeds 0 degrees Kelvin (K), the free electrons will exhibit random movements (Brownian motion) in the conductor. This introduces a random current which is generally referred to as thermal noise.
In Figure 1, a model of a noisy resistor is shown. In this model, the noisy resistor is replaced by a noiseless resistor and a random voltage source $v(t)$.

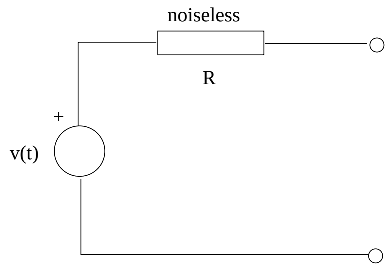

Figure 1: *A model of a noisy resistor, using a random voltage source and an ideal resistor.*

It can be shown that the thermal noise $v(t)$ can be well modeled as a stationary Gaussian stochastic process with zero mean ($E[v(t)] = 0$) and a variance that is proportional to the temperature T (in K):

$$E[v^2(t)] \approx 4RkTB$$

Here, $k \approx 1.38 \times 10^{-23}$ is Boltzman's constant and B is the receiver bandwidth in Hz. The approximation is valid for frequencies up to at least 100 GHz, which is sufficient for most practical applications. Also, the spectrum of the thermal noise is approximately flat in this frequency range, so $v(t)$ can for most practical purposes be modeled as a white noise. This means that $v(t_1)$ and $v(t_2)$ can be considered uncorrelated, provided t_1 and t_2 are not "too close". See e.g. [2] for more details about thermal noise.
A communication receiver consists among others of electrical circuits like filters and amplifiers. Most components in these circuits will add noise. Just like one can replace a noisy resistor with a noiseless one and a random voltage source, it is common to model a communication receiver as an ideal receiver that includes a random noise source. In general, the noise can affect the amplitude and the phase of the receiver output as well as adding a random voltage. However, to first order the noise can always be modeled as purely additive, similar to Figure 1.

3 A Simple Binary Communication System

In this laboratory exercise we will study a simple binary communication system, without a source or channel coder. The source is assumed to be a speech signal, sampled at the rate $f_s = 8$ kHz. Each sample is quantized to 8 bits, which are here represented as ± 1 rather than 1 and 0. Thus, the data rate for the system is $8 \times 8 = 64$ kbit/s, which means that a bit has to be transferred every $T = 1/64 \times 10^3 \approx 15.6$ μs. This is achieved by transmitting a certain waveform $s(t)$ if the information bit is 1, or $-s(t)$ if it is -1. The duration of the waveform is T, so $s(t) = 0$ for $t < 0$ and $t > T$. For simplicity, we will assume that the waveform is just a rectangular pulse

$$s(t) = \begin{cases} 1 & 0 < t < T \\ 0 & t < 0 \text{ or } t > T \end{cases}$$

In practice, there is also a modulation by some carrier frequency, $\cos \omega_c t$. However, this will be ignored in our system model, assuming that the demodulator has perfectly recovered the baseband signal.

Due to the random nature of speech, we can assume that the information signal is a stochastic process that can be considered (weakly) stationary over some observation interval. Thus, the bitstream is modeled by

$$b_n = \begin{cases} 1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$$

It is further assumed that b_n and b_m are uncorrelated for $n \neq m$. Thus, b_n is a discrete-time white noise. The transmitted signal in the interval $nT \leq t < (n+1)T$ is simply the multiplication of b_n and $s(t)$. For a block of N bits $\{b_n\}_{n=0}^{N-1}$, this can be compactly expressed as

$$t_x(t) = A \cdot \sum_{n=0}^{N-1} b_n s(t - nT)$$

where A is a gain introduced by the transmitter amplifier. A realization of such a transmitted signal is shown in Figure 2.

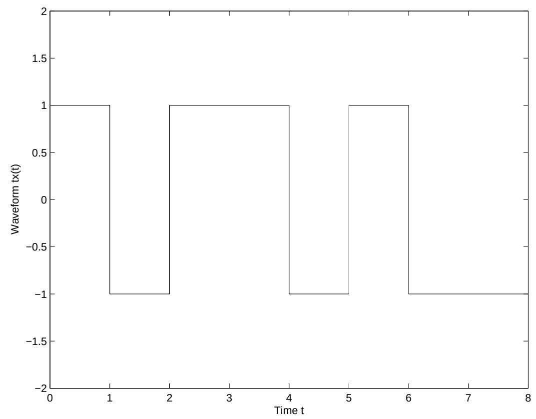

Figure 2: *Transmitted waveform corresponding to the bitstream 10110100 and for A=T=1.*

The signal $t_x(t)$ is transmitted over a so-called AWGN (Additive White Gaussian Noise) channel. The power loss due to the transmission is ignored, so the effect of the channel is assumed to be just a zero-mean additive Gaussian noise $w(t)$. The spectrum of the noise is assumed to be constant over the receiver bandwidth, which means that the noise behaves like a continuous-time white noise. The spectral density of the noise is denoted $N_0/2$, which is a tradition used in most literature. Thus, the auto-correlation function of the continuous-time noise is modeled as

$$r_w(\tau) = E[w(t)w(t-\tau)] = \frac{N_0}{2}\delta(\tau)\,, \tag{1}$$

where $\delta(\tau)$ is Dirac's delta function. In reality, the bandwidth is limited, implying that the autocorrelation function must have a non-zero extension rather than being a perfect impulse function. However, the model given above is sufficient to give accurate performance predictions of the system we are studying.
The received signal is the sum of the transmitted signal and the noise,

$$r_x(t) = t_x(t) + w(t)$$

The task of the receiver is to recover the transmitted bits b_n from the observed signal $r_x(t)$. Due to the presence of noise, the detected bits $\hat{b}_n$ will not all be correct. The BER (Bit-Error Rate) is defined as

$$P_b = Pr(\hat{b}_n \neq b_n)$$

and is assumed to be the same for all n due to stationarity. It can be shown that the optimal receiver in the sense of minimizing P_b consists of a so-called matched filter, which is defined by $h(t) \propto s(T-t)$. Since A is generally not known at the receiver but T is, we normalize the filter to be

$$h(t) = \begin{cases} 1/\sqrt{T} & 0 < t < T \\ 0 & t < 0 \text{ or } t > T \end{cases}$$

The nth bit is then detected according to

$$y(t) = r_x(t) * h(t) = \int_0^T h(\tau) r_x(t-\tau) d\tau \,, \quad y[n] = y(t)|_{t=nT+T} \,, \quad \hat{b}_n = \text{sgn}\{y[n]\}$$

The full system model is illustrated in Figure 3.

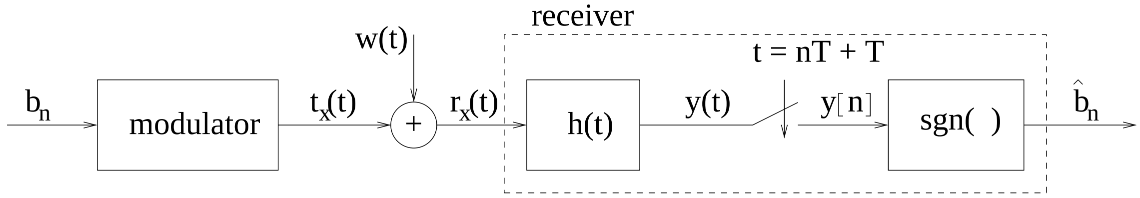

Figure 3: *A model of a binary communication system with transmitter, AWGN channel and receiver.*

Task 1 (Theoretical) *Show that the received sample $y[n]$ can be expressed as*

$$y[n] = \sqrt{E} b_n + w_n$$

where $E = A^2 T$ is the bit energy, and

$$w_n = \frac{1}{\sqrt{T}} \int_{t=nT}^{nT+T} w(t) dt \qquad (2)$$

is the integrated noise over the bit interval.

The next step in the analysis is to derive the BER for our simple communication system. Given that $b_n = -1$ is transmitted in the interval $nT < t < nT + T$, the detected bit will be erroneous if

$$\hat{b}_n = 1 \Leftrightarrow y[n] > 0 \Leftrightarrow -\sqrt{E} + w_n > 0 \Leftrightarrow w_n > \sqrt{E}$$

Clearly, errors happen only when the noise takes on a sufficiently large value, and with an unfavorable sign. The conditional BER, given that $b_n = -1$ is transmitted is therefore

$$P_{b|-1} = Pr(\hat{b}_n = 1 | b_n = -1) = Pr(w_n > \sqrt{E})$$

Similarly, when $b_n = 1$ is transmitted we get

$$P_{b|1} = Pr(\hat{b}_n = -1 | b_n = 1) = Pr(w_n < -\sqrt{E})$$

The total BER is finally given by

$$P_b = p P_{b|1} + (1-p) P_{b|-1}$$

In most cases the noise will be symmetrically distributed, and we see that $P_b = P_{b|1} = P_{b|-1}$. In either case, to compute the BER it is necessary to know the distribution of the noise samples.

Task 2 (Theoretical) *Determine the distribution of the discrete-time noise samples w_n! In particular, find the mean and the auto-correlation function of w_n. Use the expression (2) to compute this from the continuous-time $w(t)$, which is assumed to be Gaussian distributed with zero mean and auto-correlation function given by (1). Once the distribution of w_n is known, derive an expression for the BER P_b in terms of the bit energy and the noise power! What happens when the signal amplitude tends to infinity?*

4 Simulation of a Communication System

For certain simple communication systems we can compute the BER exactly. However, most systems are so complicated that we have to rely on simulations to determine the BER and other performance measures. The principle to determine P_b empirically is simple:

1. Create a realization $\{b_n\}_{n=0}^{N-1}$ of the transmitted bit stream.
2. Generate a realization $\{w_n\}_{n=0}^{N-1}$ of the discrete-time noise process.
3. Sum the above to get the sampled match-filter output: $y[n] = \sqrt{E} b_n + w_n,\ n = 0, 1, \ldots, N-1$.
4. Apply the detector to get the bit estimates $\hat{b}_n = \text{sgn}\{y[n]\}$.
5. Compute the error indicator function $\{g[n]\}$ as

$$g[n] = |\hat{b}_n - b_n|/2 = \begin{cases} 1\,, & \hat{b}_n \neq b_n \\ 0\,, & \hat{b}_n = b_n \end{cases}$$

6. Estimate the BER by the empirical error rate over the data block:

$$\hat{P}_b = \frac{1}{N} \sum_{n=0}^{N-1} g[n]$$

Note that only an estimate $\hat{P}_b$ of the true P_b is obtained, and the quality of the estimate depends on how many transmitted bits (N) that are used. Since the data in the simulation are generated exactly according to the "true" model, the error indicator function is a Bernoulli random variable ([1], p. 31)

$$Pr(g[n] = 1) = p\,, \quad Pr(g[n] = 0) = 1 - p$$

Also, $g[n]$ and $g[m]$ are independent if $n \neq m$. Equipped with these insights we can analyze the quality of the estimate $\hat{P}_b$ of P_b.

Task 3 (Theoretical) *Show that $E[\hat{P}_b] = P_b$, i.e. $\hat{P}_b$ is an unbiased estimator. Then determine the estimation error variance*

$$\sigma^2 = E[(\hat{P}_b - P_b)^2]$$

A confidence interval for P_b with the confidence $1 - \alpha$ is an interval (L, U) such that

$$Pr(L \leq P_b \leq U) \geq 1 - \alpha$$

We would of course like to have a confidence interval for P_b centered at the point estimate $\hat{P}_b$. To get this, we must know the distribution of $\hat{P}_b$. It is possible to get the exact distribution in this case, but for reasonably large N it is much simpler to invoke the Central Limit Theorem (CLT). According to the CLT, $\hat{P}_b$ is distributed as $N(P_b, \sigma^2)$ for large N, where σ^2 is derived in Task 3. Based on this result it is easy to determine the required confidence interval.

5 Simulation Task

Through listening tests it has been found that P_b must be less than 10^{-3} in order for the speech quality to be acceptable.

Task 4 (Theoretical) *Determine analytically the required amplitude A to give the BER $P_b < 10^{-3}$ when the channel noise has the spectral density $N_0/2 = 2 \times 10^{-8}$.*

Task 5 (Computer Simulation) *Simulate the communication system using the previously computed value of A and $N_0/2 = 2 \times 10^{-8}$. Use a data length N which is long enough that the error in the estimated BER $\hat{P}_b$ is less than $0.5 \cdot 10^{-3}$ with 99% probability. Give the point estimate $\hat{P}_b$ along with a 99% confidence interval, and also the number of transmitted bits N used in the simulation.*

It can be difficult to compute the smallest N to achieve the desired accuracy. However, this is not necessary here. Just make sure that N is big enough, for example by plotting the achieved confidence intervals versus N in Matlab, assuming $\hat{P}_b = 10^{-3}$.

6 Selected Answers

Task 2

The noise sample w_n is Gaussian distributed with zero mean and autocorrelation function

$$r_w[n] = \begin{cases} \frac{N_0}{2} & n = m \\ 0 & n \neq m \end{cases}$$

The error probability can be expressed as

$$P_b = Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

whrer the function $Q(x)$ is the tail area of the $N(0,1)$ distribution:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\tau^2/2} d\tau$$

Task 4

The required amplitude is

$$A = \sqrt{\frac{N_0}{2T}} Q^{-1}(P_b) \approx 0.111$$

where $Q^{-1}(x)$ is the inverse function of $Q(x)$.

References

[1] S.L. Miller and D.G. Childers, *Probability and Random Processes With Applications to Signal Processing and Communications*, Elsevier Academic Press, Burlington, MA, 2004.

[2] D.M. Pozar, *Microwave and RF Deisgn of Wireless Systems*, John Wiley & Sons, Inc., New York, 2001.