

There are 30 total points in the examination. One needs 14 points for grade 3 (to pass), 18 points for grade 4, and 24 points for grade 5.

---

**Problem 1.** The input  $X$  in a binary optical communication system is a random variable with equally likely values 1 and 2. The receiver output  $Y$  is a Poisson random variable which parameter is  $\mu$ , when 1 is transmitted, and  $\nu$  when 2 is transmitted.

- (a) Compute  $E[Y|X]$  and  $E[Y]$ . 1.5p
- (b) Given that the receiver output is equal to 2, find the conditional probability that 1 was sent. 1.5p

*Solution*

(a)

$$E[Y|X = 1] = \mu, \quad E[Y|X = 2] = \nu, \quad E[Y] = E[E[Y|X]] = \frac{\mu}{2} + \frac{\nu}{2}$$

(b)

$$P(X = 1|Y = 2) = \frac{P(Y = 2|X = 1)P(X = 1)}{P(Y = 2)} = \frac{\frac{1}{2} \frac{\mu^2}{2} e^{-\mu}}{\frac{1}{2} \frac{\mu^2}{2} e^{-\mu} + \frac{1}{2} \frac{\nu^2}{2} e^{-\nu}} = \frac{\mu^2 e^{-\mu}}{\mu^2 e^{-\mu} + \nu^2 e^{-\nu}}$$

**Problem 2.** A multiplexer combines  $N$  digital television signals into a common transmission line. Signal  $n$  generates  $X_n$  bits every 33 milliseconds, where  $X_n$  is a Gaussian random variable with mean  $m/N$  and variance  $\sigma^2/\sqrt{N}$ . Suppose that the multiplexer accepts a maximum total of  $T$  bits from the combined sources every 33 ms, and that any bits in excess of  $T$  are discarded. Let the signals be independent and assume that  $T = m + t\sigma$ , where  $t > 0$  is a fixed number. Let  $Y_{Disc}$  be the number of bits discarded per 33-ms period, i.e.,

$$Y_{Disc} = \begin{cases} X - T, & X > T \\ 0, & X \leq T. \end{cases}$$

Compute  $E[Y_{Disc}]$ . What is the result when  $t \rightarrow \infty$ ? 3p

*Solution* Let  $X = X_1 + X_2 + \dots + X_N$  be the total number of bits generated by the combined source.  $X$  is a normal random variable with expected value  $m$  and variance  $\sigma_1^2 = \sqrt{N}\sigma^2$ . Then  $T = m + t_1\sigma_1^2$ , where  $t_1 = t/\sqrt{N}$ . We have

$$Y_{Disc} = \begin{cases} X - T, & \text{if } X > T \\ 0, & \text{if } X \leq T. \end{cases}$$

Thus

$$\begin{aligned}
 E[Y_{Disc}] &= \int_T^\infty (x - T)f_X(x)dx \\
 &= \int_{m+t_1\sigma_1}^\infty xf_X(x)dx - TP(X \geq m + t_1\sigma_1) \\
 &= \frac{1}{\sqrt{2\pi}} \int_{t_1}^\infty (\sigma_1y + m)e^{-\frac{y^2}{2}} dy - (m + t_1\sigma_1)Q(t_1) \\
 &= \frac{\sigma_1}{\sqrt{2\pi}}e^{-\frac{t_1^2}{2}} - t_1\sigma_1Q(t_1)
 \end{aligned}$$

When  $t \rightarrow \infty$  we obtain  $E[Y_{Disc}] \rightarrow 0$ , as expected.

**Problem 3.** Suppose  $Z_1$  and  $Z_2$  are independent standard normal random variables. Define  $X_1 = Z_1$ ,  $X_2 = 3/5Z_1 + 4/5Z_2$ . Compute  $f_{X_2|X_1}(x_2|x_1)$ , the conditional PDF of  $X_2$ , given  $X_1 = x_1$ . 3p

*Solution* The random variables  $Z_1, Z_2$  are jointly Gaussian, and so are  $X_1, X_2$ . We have  $m_{X_1} = m_{X_2} = 0, \sigma_{X_1}^2 = \sigma_{X_2}^2 = 1, \rho_{X_1, X_2} = 3/5$ . Then

$$f_{X_1, X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{1 - (3/5)^2}} \exp \left\{ -\frac{1}{2} \frac{1}{1 - (3/5)^2} [x_1^2 + x_2^2 - 2\frac{3}{5}x_1x_2] \right\}$$

and

$$f_{X_2|X_1}(x_2|x_1) = \frac{f_{X_1, X_2}(x_1, x_2)}{f_{X_1}(x_1)} = \frac{1}{\sqrt{2\pi}4/5} \exp \left\{ -\frac{(x_2 - \frac{3}{5}x_1)^2}{2 \cdot 16/25} \right\}$$

**Problem 4.** Messages arrive in a multiplexer according to a Poisson process with mean  $\lambda = 10$  messages/second. Use the CLT to estimate the probability that more than 650 messages arrive in one minute. 3p

*Solution*

$$P(S_{650} < 60) = P\left(\frac{S_{650} - 605/10}{\sqrt{650}/10} < \frac{60 - 650/10}{\sqrt{650}/10}\right) \approx Q(1.96) = 2.49 \times 10^{-2}$$

**Problem 5.** Let  $X_1, X_2, \dots$  be iid random variables with expected value  $m$  and variance  $\sigma^2$ , and consider the discrete time process  $\{Z_n, n \geq 1\}$  with

$$Z_n = \frac{X_1 + X_2 + \dots + X_n}{n}.$$

(a) Find the autocovariance function of  $Z_n$ . 3p

- (b) Why is this process Markovian? Suppose  $X_1$  is continuous with CDF  $F(x)$  and PDF  $f(x)$ . Compute

$$F_{Z_n|Z_{n-1}}(x|Z_{n-1} = y) = P(Z_n \leq x|Z_{n-1} = y) \quad \text{and} \quad f_{Z_n|Z_{n-1}}(x|Z_{n-1} = y).$$

3p

*Solution*

- (a) Recall that the covariance function of the sum process  $\{S_n, n \geq 1\}$  with  $S_n = X_1 + X_2 + \dots + X_n$  is  $C_S(n, k) = \min(n, k)\sigma^2$ . Then

$$\begin{aligned} C_Z(n, k) &= E[(Z_n - m)(Z_k - m)] = \frac{1}{nk} E[(S_n - nm)(S_k - km)] \\ &= \frac{1}{nk} C_S(n, k) = \frac{1}{nk} \min(n, k)\sigma^2 = \frac{\sigma^2}{\max(n, k)} \end{aligned}$$

- (b) The process has independent increments and is then Markovian.

$$\begin{aligned} F_{Z_n|Z_{n-1}}(x|Z_{n-1} = y) &= P(Z_n \leq x|Z_{n-1} = y) = P(nZ_n \leq nx|(n-1)Z_{n-1} = (n-1)y) \\ &= P(X_n \leq nx - (n-1)y) = F_X(nx - (n-1)y) \\ f_{Z_n|Z_{n-1}}(x|Z_{n-1} = y) &= n f_X(nx - (n-1)y) \end{aligned}$$

**Problem 6.** Consider the short term integration of  $X(t)$

$$Y(t) = \frac{1}{T} \int_{t-T}^t X(u) du,$$

where  $X(t)$  is the white noise process with PSD  $S_X(f) = N_0/2$ .

- (a) Compute  $S_Y(f)$ , the PSD of  $Y(t)$ . 3p
- (b) Compute the average power of  $Y(t)$ . 3p

*Solution* The impulse response is

(a)

$$h(t) = \frac{1}{T} \int_{t-T}^t \delta(x) dx = \frac{1}{T} \left[ \int_{-\infty}^t \delta(x) dx - \int_{-\infty}^{t-T} \delta(x) dx \right] = \frac{1}{T} [u(t) - u(t-T)]$$

and the transfer function is then

$$H(f) = \frac{1}{T} \int_0^T e^{-j2\pi ft} dt = \frac{1}{T} \frac{\sin \pi f T}{\pi f} e^{-j\pi f T}$$

Hence

$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{\sin^2 \pi f T}{T^2 \pi^2 f^2} \cdot \frac{N_0}{2}$$

(b) The inverse Fourier transform gives

$$R_Y(\tau) = \frac{N_0}{2} F^{-1} \left[ \frac{\sin^2 \pi f T}{\pi^2 f^2 T^2} \right] = \frac{N_0}{2T} \text{tri}(\tau T), \quad R_Y(0) = \frac{N_0}{2T}$$

**Problem 7.**  $\{X_n\}$  is a WSS process with autocorrelation function

$$R_X(k) = 4(1/2)^{|k|}, k = 0, \pm 1, \pm 2, \dots$$

Find the optimum linear filter for estimating  $X_n$  from the observations  $X_{n-1}$  and  $X_{n-3}$  and compute the mean-square estimation error. 3p

*Solution*  $\hat{X}_n = h_1 X_{n-1} + h_2 X_{n-3}$ .

$\hat{X}_n$  - optimal  $\Leftrightarrow E[\hat{X}_n X_{n-i}] = E[X_n X_{n-i}], \quad i = 1, 3$

$$\begin{bmatrix} R_X(0) & R_X(2) \\ R_X(2) & R_X(0) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} R_X(1) \\ R_X(3) \end{bmatrix}$$

$$\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1/2 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

$$e^2 = R_X(0) - h_1 R_X(1) = 3$$

**Problem 8.** The spectrum of a stationary stochastic process is to be estimated from the data:

$$x[n] = \{0, 6, -0, 7, 0, 2, 0 : 3\}$$

Due to the small sample support, a simple AR(1)-model is exploited:

$$x[n] + a_1 x[n-1] = e[n].$$

Determine estimates of the AR-parameter  $a_1$  and the white noise variance  $\sigma_e^2$ . Based on these, give a parametric estimate of the spectrum  $P_X(e^{j\omega})$ . 3p

*Solution*

The Yule-Walker method gives the estimate

$$\hat{a}_1 = -\hat{r}_x[0]^{-1} \hat{r}_x[1]$$

With the given data, the sample autocorrelation function is calculated as

$$\hat{r}_x[0] = \frac{1}{4} (0.6^2 + 0.7^2 + 0.2^2 + 0.3^2) = 0.245$$

and

$$\hat{r}_x[1] = \frac{1}{4}(0.6 \times (-0.7) + (-0.7) \times 0.2 + 0.2 \times 0.3) = -0.125$$

Thus, we get

$$\hat{a}_1 = \frac{0.125}{0.245} \approx 0.51.$$

The noise variance estimate follows as

$$\hat{\sigma}_e^2 = \hat{r}_x[0] + \hat{a}_1 \hat{r}_x[1] \approx 0.18.$$