RANDOM PROCESSES WITH APPLICATIONS 2008

Solution to Optional homework 1
This assignment gives two bonus points to the written examination, when the submitted solution collects 12 points or more.

Day assigned: **September 15**

Assignment deadline: **September 22, 15:00**

Problem 1. Professor Random has taught probability for many years. She has found that 80% of students who do the homework pass the exam, while 10% of students who don't do the homework pass the exam. If 60% of students do the homework, what percent of students who pass the exam did the homework? (2)

Solution. Let H denote the event that a student does the homework, and let E denote the event that a student passes the exam. Then

$$P(E|H) = 0.8, \quad P(E|\overline{H}) = 0.1, \quad P(H) = 0.6.$$

$$P(H|E) = \frac{P(H \cap E)}{P(E)} = \frac{P(E|H)P(H)}{P(E)},$$

$$P(E) = P(E|H)P(H) + P(E|\overline{H})P(\overline{H}) = 0.8 \times 0.6 + 0.1(1 - 0.6) = 0.52$$

$$P(H|E) = \frac{0.8 \times 0.6}{0.52} = 0.92$$

Problem 2. In a radar system, the reflected signal pulses have amplitudes that are Rayleigh distributed. Let the mean value of these pulses be $\sqrt{\pi/2}$. However, the only pulses that are displayed on the radar scope are those for which the pulse amplitude R is greater than some threshold r_0 in order that the effect of system noise can be suppressed.

(a) Determine the conditional PDF $f(r|R > r_0)$ of the displayed pulses. (2)

(b) Compute the expected value of the displayed pulses for $r_0 = 0.5$ (2)

Solution. The PDF of the Rayleigh distribution with parameter σ is given by

$$f_R(r) = \frac{r}{\sigma^2} \exp\{-r^2/2\sigma^2\}$$

Since $E[R] = \sqrt{\pi/2}\,\sigma$, we must have $\sigma = 1$.

(a)

$$F_R(r|R > r_0) = \frac{P(r_0 < R \le r)}{P(R > r_0)} = \frac{F_R(r) - F_R(r_0)}{1 - F_R(r_0)}\, u(r - r_0)$$

$$f_R(r|R > r_0) = \Big(F_R(r|R > r_0)\Big)' = \frac{f_R(r)}{1 - F_R(r_0)}\, u(r - r_0)$$

$$F_R(r_0) = \int_0^{r_0} r \exp\Big\{-\frac{r^2}{2}\Big\} dr = -\exp\Big\{-\frac{r^2}{2}\Big\}\Big|_0^{r_0} = 1 - \exp\Big\{-\frac{r_0^2}{2}\Big\}$$

$$f_R(r|R > r_0) = \exp\Big\{\frac{r_0^2}{2}\Big\} r \exp\Big\{-\frac{r^2}{2}\Big\} u(r - r_0)$$

(b)

$$\begin{aligned} E[R|R > r_0] &= \int_{-\infty}^{\infty} r\ f_R(r|R > r_0)\ dr = \exp\Big\{\frac{r_0^2}{2}\Big\} \int_{r_0}^{\infty} r^2\ \exp\{-r^2/2\}\ dr \\ &= \exp\Big\{\frac{r_0^2}{2}\Big\} \Big[-r\ \exp\{-r^2/2\}\Big|_{r_0}^{\infty} + \int_{r_0}^{\infty} \exp\{-r^2/2\} dr \Big] \\ &= \exp\{r_0^2/2\} \Big[r_0\ \exp\{-r_0^2/2\} + \sqrt{2\pi}\ Q(r_0) \Big] \\ &= r_0 + \exp\{r_0^2/2\} \sqrt{2\pi}\ Q(r_0) \end{aligned}$$

$$E[R|R > 0.5] = 1.376$$

Problem 3. The random variable X is uniformly distributed in $[0, \pi)$. Consider the random variables $V = \cos X$ and $W = \sin X$. Show that V and W are uncorrelated. Are they independent? (2)

Solution. We have

$$E[V] = \frac{1}{\pi} \int_0^{\pi} \cos x\ dx = 0$$

$$E[VW] = \frac{1}{\pi} \int_0^{\pi} \cos x \sin x dx = \frac{1}{4\pi} \int_0^{\pi} \sin 2x\ dx = 0$$

$$Cov(V, W) = E[VW] - E[V]E[W] = 0, \quad \rho_{VW} = 0.$$

The random variables are uncorrelated. However, since $V^2 + W^2 = 1$, they are not independent.

Problem 4. A common method for detecting a signal in a presence of noise is to establish a threshold value and compare the value of any observation with this threshold. If the threshold is exceeded, it is decided that a signal is present. Sometimes, of course, noise alone will exceed the threshold and this is known as a "false alarm". Usually, it is desired to make the probability of a false alarm very small. At the same time, we would like any observation that does contain a signal plus the noise to exceed the threshold with a large probability. This is the probability of detection and it should be as close to 1 as possible. Suppose we have Gaussian noise with zero mean and a variance of 1 V^2 and we set a threshold level of 5 V.

(a) Find the probability of false alarm. If a signal having a value of 8 V is observed in the presence of this noise, find the probability of detection. (2)

(b) When noise only is present, find the conditional mean value of the noise that exceeds the threshold. (2)

Solution.

$\mathcal{N}$ - the noise; $\mathcal{N} \sim \mathcal{N}(0, 1)$

(a)

$$P\{\text{false alarm}\} = P\{N > 5\} = Q(5) = 2.87 * 10^{-7}$$
$$P\{\text{detection}\} = P\{N + 8 > 5\} = Q(-3) = 1 - Q(3) = 1 - 1.35 * 10^{-3} = 0.9986.$$

(b)

$$f_N(x|N > 5) = \frac{f_N(x)}{Q(5)}\ u(x - 5)$$
$$E[N|N > 5] = \frac{1}{Q(5)} \int_5^\infty \frac{1}{\sqrt{2\pi}} x e^{-x^2/2} dx$$
$$= \frac{1}{Q(5)\sqrt{2\pi}} \Big[- e^{-x^2/2} \Big|_5^\infty \Big] = \frac{e^{-0.125}}{Q(5)\sqrt{2\pi}} = 5.18.$$

Problem 5. Suppose Z_1 and Z_2 are independent standard normal random variables. Define $X_1 = Z_1, \quad X_2 = 3/5\ Z_1 + 4/5\ Z_2$.

(a) Show that X_1 and X_2 are jointly Gaussian. What is $\rho_{X_1X_2}$? (2)

(b) Find $f_{X_2|X_1}(x_2|x_1)$, the conditional PDF of X_2, given $X_1 = x_1$. (2)

Solution.

(a) The inverse transformation and its Jacobian are

$$z_1 = x_1, \quad z_2 = 5/4[x_2 - 3/5 x_1] = \frac{5}{4}x_2 - \frac{3}{4}x_1,$$

$$J = \det \begin{bmatrix} 1 & 0 \\ -3/4 & 5/4 \end{bmatrix} \;=\; 5/4$$

Then

$$f_{X_1,X_2}(x_1, x_2) = f_{Z_1,Z_2}\left(x_1,\ \frac{5}{4}x_2 - \frac{3}{4}x_1\right)\frac{5}{4} =$$
$$= \frac{1}{2\pi\frac{4}{5}} \exp\left\{ -\frac{1}{2}\Big[x_1^2 + \Big(\frac{5}{4}x_2 - \frac{3}{4}x_1\Big)^2\Big]\right\}$$
$$= \frac{1}{2\pi\frac{4}{5}} \exp\left\{ -\frac{1}{2}\Big[x_1^2 + \frac{25}{16}x_2^2 - 2.\frac{15}{16}x_1x_2 + \frac{9}{16}x_1^2\Big]\right\}$$
$$= \frac{1}{2\pi\frac{4}{5}} \exp\left\{ -\frac{1}{2}\frac{25}{16}\Big[x_1^2 + x_2^2 - 2\frac{3}{5}x_1x_2\Big]\right\}$$
$$= \frac{1}{2\pi\sqrt{1 - (3/5)^2}} \exp\left\{ -\frac{1}{2(1 - (3/5)^2)}\Big[x_1^2 + x_2^2 - 2\frac{3}{5}x_1x_2\Big]\right\}$$

Hence X_1 and X_2 are jointly Gaussian with $\rho_{X_1X_2} = 3/5$.

(b) If $X_1 = x_1 \quad (= Z_1)$ then

$$X_2 = \frac{3}{5}x_1 + \frac{4}{5}Z_2 \sim N\Big(\frac{3x_1}{5}, \frac{16}{25}\Big)$$