RANDOM PROCESSES WITH APPLICATIONS 2008
SOLUTION TO HOMEWORK 2

This assignment is optional. It gives two bonus points to the written examination, when the submitted solution collects at least 12 out of 16.

Posted on September 25. Deadline for submission: October 10, 17:00

From October 7 until the deadline for submission I will be away. In this period you can e-mail me your questions regarding the homework. Note that solutions are to be submitted to SIMA SHAHSAVARI.

Rossitza Dodunekova

Problem 1. $X(t)$ is a zero-mean stationary Gaussian process with covariance function $C_X(\tau)$.

(a) Compute $P\{X(t) \geq X(s)\}$ when $C_X(t-s) \neq C_X(0)$ and when $C_X(t-s) = C_X(0)$.
2p

(b) Show that for any t and h

$$E[X(t)X^2(t+h)] = 0.$$

Hint. Let $X_1 = X(t)$ and $X_2 = X(t+h)$. Find a constant α such that $X_1 - \alpha X_2$ and X_2 are independent and use the presentation $X_1 = (X_1 - \alpha X_2) + \alpha X_2$.
2p

Solution.

(a) $P\{X(t) \geq X(s)\} = P\{X(t) - X(s) \geq 0\} = P\{X(t-s) - X(0) \geq 0\}$.
$Var(X(t-s) - X(0)) = Cov(X(t-s) - X(0),\, X(t-s) - X(0)) = 2(C_X(0) - C_X(t-s))$

When $C_X(0) \neq C_X(t-s)$, then $X(t-s) - X(0) \sim \mathcal{N}\big(0,\, 2(C_X(0) - C_X(t-s)\big)$ and

$P\{X(t-s) - X(0) \geq 0\} = \frac{1}{2}$

When $C_X(0) = C_X(t-s)$, then $P\{X(t-s) - X(0) = 0\} = 1$ and
$P\{X(t-s) - X(0) \geq 0\} = P\{X(t-s) - X(0) = 0\} = 1$.

(b) Denote $\sigma^2 = C_X(0)$. X_1 and X_2 are jointly Gaussian with correlation coefficient

$$\rho = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)Var(X_2)}} = \frac{C_X(h)}{\sigma^2}.$$

Then

$$Cov(X_1 - \alpha X_2,\, X_2) = Cov(X_1, X_2) - \alpha Cov(X_2, X_2) = C_X(h) - \alpha\sigma^2 = \sigma^2(\rho - \alpha).$$

Choose $\alpha = \rho$. The random variables $X_1 - \rho X_2$ and X_2 are independent since they are jointly Gaussian and uncorrelated. Thus

$$
\begin{aligned}
E[X_1 X_2^2] &= E[(X_1 - \rho\, X_2 + \rho X_2) X_2^2] \\
&= E[(X_1 - \rho X_2) X_2^2] + \rho E[X_2^3] = E[X_1 - \rho X_2] E[X_2^2] + \rho E[X_2^3] = 0
\end{aligned}
$$

since $E[X_1 - \rho X_2] = 0$ and $E[X_2^3] = \dfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x^3 \exp\{-\frac{x^2}{2\sigma^2}\} dx = 0.$

Problem 2. $X(t)$ is the Poisson process with rate λ. Consider the random process

$$Y(t) = (-1)^{X(t)}, \quad t \geq 0.$$

(a) Find the mean function and the autocorrelation function of $Y(t)$. Is the process wide sense stationary 2p

(b) Let $Z(t) = AY(t)$, where A is a random variable, independent of $Y(t)$ and taking on values ± 1 with equal probabilities. Find the probability mass function of $Z(t)$. Is Z a wide sense stationary process? 2p

Solution. The values $\{-1,\ 1\}$ of the process $Y(t)$ depend on whether the value of $X(t)$ is even or not. Denote $P_{even}(t) = P\{X(t) \text{ is even }\}$. We know (the book, p. 284)

$$P_{even}(t) = \frac{1 + e^{-2\lambda t}}{2}.$$

(a) When $t > 0$ we have

$$P\{Y(t) = 1\} = P_{even}(t), \quad P\{Y(t) = -1\} = 1 - P_{even}(t), \quad E[Y(t)] = 2P_{even}(t) - 1 = e^{-2\lambda t}.$$

Thus

$$\mu_Y(t) = e^{-2\lambda t} \quad \text{when} \quad t > 0 \quad \text{and} \quad \mu_Y(0) = E[1] = 1.$$

For $t \geq 0$ and $\tau \geq 0$ we have

$$R_Y(t,\ t + \tau) = E[Y(t) Y(t + \tau)] = 1 \cdot P_{even}(\tau) + (-1) \cdot (1 - P_{even}(\tau)) = e^{-2\lambda\tau}$$

Thus

$$R_Y(\tau) = e^{-2\lambda|\tau|}, \quad -\infty < \tau < \infty.$$

$Y(t)$ is not WSS, since $m_Y(t)$ is not a constant. The process is known as the *semirandom telegraph signal* because its initial value $Y(0)$ is not random.

(b)

$$
\begin{aligned}
P\{Z(t) = 1\} &= P\{A = 1\} P\{Y(t) = 1\} + P\{A = -1\} P\{Y(t) = -1\} \\
&= \frac{1}{2} [P_{even}(t) + P_{odd}(t)] = \frac{1}{2}.
\end{aligned}
$$

Since $E[A] = 0$ and $E[A^2] = 1$ we have

$$E[Z(t)] = E[AY(t)] = E[A] E[Y(t)] = 0$$

$$R_{ZZ}(t_1,\, t_2) = E[AY(t_1) AY(t_2)] = E[A^2] E[Y(t_1) Y(t_2)] = R_Y(|t_2 - t_1| = e^{-2\lambda|t_2 - t_1|}.$$

$Z(t)$ is then WSS. In fact, $Z(t)$ is the random telegraph signal with values ± 1.

Problem 3. $\{X_n\}$ is a WSS random process with autocorrelation function

$$R_X(\tau) = 16e^{-5|\tau|}\cos 2\pi\tau.$$

Compute the power spectral density and the average power of the process. 4p

Solution. The average power of the process is $R_X(0) = 16$. To compute the power spectral density we use the formula

$$\mathcal{F}\{16e^{-5|\tau|}\cos 2\pi\tau\} = \mathcal{F}\{16e^{-5|\tau|}\} * \mathcal{F}\{\cos 2\pi\tau\}\big](f)$$

According to the table on p. 521

$$\mathcal{F}\Big\{\exp\Big\{-\frac{|\tau|}{\tau_0}\Big\}\Big\} = \frac{2\tau_0}{1+(2\pi f\tau_0)^2}$$

and

$$\mathcal{F}\{\cos 2\pi f_0\tau\} = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0).$$

Then

$$S_X(f) = 16\frac{10}{25+4\pi^2 f^2} * \frac{1}{2}\left[\delta(f-1)+\delta(f+1)\right] + = 80\left[\frac{1}{25+4\pi^2(f-1)^2} + \frac{1}{25+4\pi^2(f+1)^2}\right]$$

Problem 4. Y_n is a WSS process defined as

$$Y_n = \frac{1}{2}Y_{n-1} + W_n,$$

where W_n is the white-noise process of average power 1. Show that the unit impulse response of the system producing Y_n is

$$h_n = \begin{cases} 2^{-n}, & n \ge 0, \\ 0, & n < 0. \end{cases}$$

and compute $R_Y(m)$. Write equations for finding a filter producing the best predictor of Y_n from Y_{n-2} and Y_{n-3} 4p

Solution. We have to show that the convolution of W and h satisfies the equation defining Y. Indeed, we have

$$\sum_{k\ge 0} 2^{-k}W_{n-k} = W_n + \sum_{k\ge 1} 2^{-k}W_{n-k}$$
$$= W_n + \frac{1}{2}\sum_{k-1\ge 0} 2^{-(k-1)}W_{(n-1)-(k-1)} = W_n + \frac{1}{2}\sum_{j\ge 0} 2^{-j}W_{(n-1)-j}.$$

From here we obtain

$$R_Y(k) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} 2^{-r}\cdot 2^{-s}R_W(k+r-s)$$
$$= \sum_{r=0}^{\infty} 2^{-r}\cdot 2^{-(k+r)}.1 = 2^{-k}\sum_{r=0}^{\infty} 2^{-2r} = \frac{4}{3}2^{-k}.$$

Then

$$R_Y(k) = \frac{4}{3}\left(\frac{1}{2}\right)^{|k|}, \quad k = 0,\, \pm 1,\, \ldots\,.$$

Let $\hat{Y}_n = z_2 Y_{n-2} + z_3 Y_{n-3}$ be the best predictor. The desired equations follow from the orthogonality condition $Y_n - \hat{Y}_n \perp Y_{n-2}$ and $Y_n - \hat{Y}_n \perp Y_{n-3}$:

$$z_2 R_Y(0) + z_3 R_Y(1) = R_Y(2)$$

$$z_2 R_Y(1) + z_3 R_Y(0) = R_Y(3).$$